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ON EUCLIDEAN SPACE AND FUNCTION SPACE CONTROLLABILITY OF CONTROL SYSTEMS WITH DELAY

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ABSTRACT

ON EUCLIDEAN SPACE AND FUNCTION SPACE CONTROLLABILITY OF CONTROL SYSTEMS WITH DELAY

by

Ronald Barry Zmood

Controllability is a concept which has assumed a position of importance in a number of areas in control engineering. For control systems described by the vector differential-difference equation

$$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t)$$

$$y(t) = Dx(t),$$

we study two notions of controllability in this dissertation. These are termed controllability in euclidean space and function space controllability. For both these notions of controllability it is necessary to distinguish between complete controllability and null controllability.

Most of the results obtained in the dissertation are based on a new form for the fundamental solution of a differential-difference equation. This new solution form has the advantage that it is rather concise and that many of the methods used in studying controllability of ordinary differential equations can be used in studying controllability of differential-difference equations. By the use of

the new form for the fundamental solution of the differential-difference equation given above, new algebraic necessary and sufficient conditions for complete controllability in euclidean space are obtained. These conditions are shown to reduce to the known algebraic conditions of Kalman, and Kirillova and Curakova. In order to be able to discuss null controllability the concept of pointwise completeness is introduced. A new algebraic necessary and sufficient condition for pointwise completeness, involving the matrices A and B, is obtained. These conditions are shown to reduce to the only previously known conditions, namely that if the matrices satisfy the condition AB = BA, then the above differential-difference equation is pointwise complete. The algebraic conditions for pointwise completeness are also used to characterize those systems which are not pointwise complete. Using this fact a new algebraic criterion for null controllability of differential-difference equations is obtained.

The question of complete function space controllability is also considered. The approach used has been motivated by the geometric approach used for controllability in euclidean space. A function space analogue of the criterion for controllability in euclidean space is obtained. The two special cases where A = 0, and where u and y are scalar valued are considered. In each case a simple algebraic criterion for function space controllability is obtained. For the general case the function space condition for function space controllability of the control system with delay is transformed into a problem concerning the uniqueness of the solutions of a two point boundary value problem for an ordinary differential equation with certain subsidiary conditions. Some recent results of Silverman and Payne on the invertibility of differential systems are applied to the two point boundary value problem to obtain new algebraic necessary and sufficient conditions for complete function space controllability.

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CONTENTS

CHAPTER 1

INTRODUCTION

Controllability is a concept which has proved of great significance in a number of areas of control engineering. In this chapter we will try to outline why this concept is important. As well we will discuss differential-difference equations rather briefly; give a brief survey of the literature on controllability; and finally summarize the results on controllability that will be presented in later chapters.

The usual approach in seeking a solution to an optimal control problem is to use necessary conditions such as the maximum principle [1] to find extremal controls. It is well known that if there exists an optimal control then it is also an extremal control. One approach to determining the optimal control from among the extremal controls is to use a sufficient condition for optimality. However the hypotheses required to prove sufficient conditions are usually quite strong, and hence many problems of interest fail to satisfy them. For this reason many investigators have been motivated to seek results guaranteeing the existence of a solution to an optimal control problem; these results can often be obtained under quite weak hypotheses, which encompass many problems of interest. A usual assumption of these existence theorems is that the system is controllable [37]; that is, there exists a control 'steering' the system from its initial condition to its target, and satisfying all constraints.

While the notion of controllability arises naturally in the theory of optimal control, it was first introduced by Kalman [2],[3] in discussing the existence of a control for the regulator problem, and in relating the state equation representation of a dynamical system to the transfer function representation. It is apparent that Kalman also recognized the relationship of this concept to optimal control. In [2] he showed that the control obtained gave the minimum cost for the linear quadratic problem. Kalman [2] also obtained an algebraic criterion for the controllability of the differential equation

$$\frac{dx}{dt} = A\ x(t) + Bu(t).$$

Prior to the introduction of the concept by Kalman, Pontryagin [4] had used the same algebraic criterion in his discussion of the linear time optimal control problem; but he used it simply as a technical requirement. Kalman [2] has stated that Pontryagin failed to recognize the geometric interpretation of the algebraic criterion. In [5] Kalman discusses some of the philosophical aspects of the notion of controllability and suggests that "controllability is basically an algebraic concept". While this statement seems to have validity for the controllability of ordinary differential equations, it seems that the geometric interpretation of controllability is of greater significance for the hereditary systems to be discussed later.

In recent years considerable interest has been shown in time lag control systems (see Reference [6] for extensive bibliography). Equations depending on the past history arise quite naturally in some physical systems. For example Driver [7],[8] has shown that a differential-difference equation arises in the consideration of a two body problem of classical electro-dynamics, while Volterra[9] has demonstrated that a class of hereditary equations somewhat more general than differential-difference equations has application in such diverse fields as elasticity and biology. Of course in control systems the existence of systems having time lags has been recognized for many years. Such examples as systems with transportation delay, and signal propagation delay have been studied extensively. However it is only within the last two decades that a real interest has been shown in the optimal control of such systems.

It is well known that the past behaviour of the trajectory of an ordinary differential equation can be summarized in the initial conditions. For time lag systems, such is not the case, and the past history of the system can only be summarized by a part or the whole of the past trajectory of the system. Perhaps the simplest type of differential equation exhibiting this time lag effect is the differential-difference equation. For these equations the past behaviour is summarized by a segment of the past trajectory, which is usually called the initial function. Our principal concern in this thesis will be a discussion of the controllability of such differential-difference equations.

1.1 Differential-Difference Equations

As mentioned above, probably the simplest form of hereditary or time lag control system is one described by the differential-difference equation

$$(1.1.1) \qquad \dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t) \qquad t > 0,$$

where $x(t)$ is an n dimensional vector, $u(t)$ is an r dimensional control vector, and A,B and C are matrices with orders commensurate with $x(t)$ and $u(t)$. Such equations have been discussed by a number of authors including Myshkis [10], Krasovskii [11], Pinney [12], Bellman and Cooke [13], Elsgoltz [14], Oguztoreli [6], and Halanay [15].

It will be observed that in order to solve (1.1.1) as an initial value problem it is necessary to know $x(t)$, $t \in [-1,0]$. Since, loosely speaking, the state is the minimum amount of information required to be able to compute the future solution of (1.1.1) we see that for such equations the state space is a function space. This leads to two ways of considering the solution of (1.1.1). The first or classical way of considering the solution is as a trajectory in euclidean space. Secondly, since the state space is a function space it is also natural, by analogy with ordinary differential equations, to consider the evolution of the solution as a trajectory in function space. This approach was first used by Krasovskii [11]. Later authors such as Hale [16],[17] and Shimanov [18] have shown the power of this approach by developing a geometric theory for equations such as (1.1.1).

Myshkis [10] and Elsgoltz [14] have proved local existence and uniqueness theorems for equations such as (1.1.1). However there is a significant difference between the type of existence results obtained by Myshkis and Elsgoltz, and the results usually obtained for ordinary differential equations. For ordinary differential equations under suitable assumptions it is shown that the solution exists and is unique in a neighborhood of the initial point (x_o, t_o); that is the solution exists to both the left and the right of the initial point. For equation such as (1.1.1) it is only possible, under assumptions similar to those for ordinary differential equations, to show that the solution exists and is unique to the right of t_o. Hastings [19] who has derived results guaranteeing the backward existence and uniqueness of solutions of differential-difference equations, has also shown examples where either no backward solution exists or it is not unique.

1.2 Survey of Literature on Controllability

Over the last decade there has grown a considerable body of literature on the controllability of dynamical systems. An excellent survey of this subject has been given by Weiss [20], so we will restrict our attention to discussing the controllability of differential- difference equations.

As was mentioned in the previous section there are two ways of considering the solution of a differential-difference equation. The first way is to consider the solution as a trajectory in euclidean space while the second is to consider it as a trajectory in a function space. At first glance this duality may not seem significant, but we will show by considering the regulator problem that in fact this dual way of considering the solution of a differential-difference equation leads to two notions of controllability.

Let us suppose that for (1.1.1) we are given a non-zero initial function. We can pose the following question: Does there exist a control u such that the trajectory x(t) passes through zero in R^n at some finite time? This problem leads to the concept of euclidean space controllability* which is of importance for problems where we are interested in controlling a system to a target and are not interested in what happens to the system after it reaches the target. It can be seen that this notion does not give an answer to the regulator problem which is: Does there exist a control u such that the trajectory x(t) equals zero for all time t greater than some finite time T? For this problem we see that it is not sufficient to just be able to control the system to zero in R^n for only one instant T, as the future behaviour of the differential-difference equation also depends on the trajectory $x(t), t \in [T-1, T]$. Hence we are

*It should be pointed out that the terminology used in referring to the two notions of controllability mentioned above is not yet well established in the literature. For example euclidean space controllability is termed R^n controllability by Weiss [20], and relative controllability by Kirillova and Curakova [21].

led to a second concept, that of function space controllability. It should be noted in passing that for ordinary differential equations the two concepts are equivalent.

Complete euclidean space controllability was first considered by Chyung and Lee [22] in 1966 where they generalized the integral criterion of Kalman [3] for the case of differential-difference equations. In fact the proof is essentially the same as that originally given by Kalman [3]. Due to certain technical difficulties, for differential-difference equations, it is necessary to differentiate between null controllability and complete controllability. Weiss [23] was able to show, under the hypothesis of a new concept called pointwise completeness, that the condition of Chyung and Lee was also a necessary and sufficient condition for euclidean space null controllability.

All the above results necessitate a knowledge of the system solution. Kirillova and Curakova [21] in 1967 presented for the first time algebraic criteria for euclidean space null controllability of (1.1.1). They were able to obtain both a necessary condition and a sufficient condition, but were unable to show that either of the conditions was necessary and sufficient except for a special case of (1.1.1). However in a recent thesis Johnson [24] was able to obtain, by an involved development, an algebraic criterion for complete euclidean space controllability of (1.1.1) which was both necessary and sufficient. Recently, Weiss [25] obtained an algebraic sufficient condition for controllability of time-varying differential-difference equations, by a method which is a natural generalization of the method of Kirillova and Curakova. The result of Weiss also includes the result of Buckalo [26] as a special case.

The concept of pointwise completeness as introduced by Weiss [23] has to the author's knowledge only been investigated by Popov [27]. He showed that if the matrix B in (1.1.1) has rank one then (1.1.1) is pointwise complete. Popov also presented an example of a time invariant differential-difference equation which was not pointwise complete, thus refuting a conjecture of Weiss [23] that all time invariant differential-difference equations are pointwise complete.

To date no algebraic criterion for function space controllability of (1.1.1) has been presented in the literature. Both Weiss [23] and Buckalo [26] have obtained conditions which necessitate some knowledge of the system solution. An interesting algebraic result for (1.1.1) with A = 0 has been obtained by Kirillova and Curakova [21]; however their method of proof does not appear to offer much hope for generalization. To the author's knowledge the only other criterion for function space controllability is the ingenious result of Popov [28]. This result which is both necessary and sufficient, involves investigating certain properties of the transfer function of (1.1.1). However the approach used by Popov does not seem to offer the possibility of generalization to time varying differential-difference equations.

1.3 Outline of Contents of Thesis

The contents of this thesis may be divided into two parts. In chapter 2 we consider the euclidean space controllability of differential-difference equations. We first present a new algebraic necessary and sufficient condition for complete euclidean space controllabliity, from which we are able to deduce known results for some special cases. We next consider the problem of pointwise completeness as this concept is important for our later discussion of null controllability. We show for the first time that it is possible to determine an algebraic criterion for pointwise completeness, and also that a fairly large class of differential-difference equations are in fact pointwise complete. Finally in this chapter we present a new algebraic necessary and sufficient condition for euclidean space null controllability. The most significant aspect of this chapter was the discovery of an especially useful form for the solution of a differential-difference equation. With this form of the solution in hand the results of the chapter follow straightforwardly.

Chapter 3 is devoted solely to obtaining an algebraic criterion for complete function space controllability. It was discovered that the problem of function space controllability of a differential-difference equation could be transformed into an equivalent two point boundary value problem for an ordinary differential equation. It is then possible to apply the Silverman inversion algorithm [29] to obtain an algebraic necessary and sufficient condition for complete function space controllability.

Finally chapter 4 summarizes the results, and suggests further areas of research.

CHAPTER 2

CONTROLLABILITY TO POINTS IN EUCLIDEAN SPACE

In this chapter we obtain results for euclidean space controllability of constant coefficient differential-difference equations. For such equations it is of interest to obtain algebraic conditions for controllability in terms of the matrices which characterize them. Some progress in this direction has recently been made by Johnson [24]; but it is felt by the author that the approach presented here gives greater insight into the question of controllability than is obtained from Johnson's methods. Futhermore it enables us to give a complete theory for euclidean space controllability of differential-difference equations.

For ordinary differential equations it is well known that the concepts of null controllability and complete controllability are equivalent; but for differential-difference equations this equivalence is not true in general. For this reason the null controllability and complete controllability of differential-difference equations have to be investigated separately. However under the assumption of pointwise completeness, a notion introduced by Weiss [23], the concepts of null controllability and complete controllability are equivalent.

The key to the approach presented here is the realization of a particularly simple form for the fundamental solution of an autonomous differential-difference equation. Utilizing this form for the fundamental solution algebraic necessary and sufficient conditions for complete controllability follow in a straightforward manner. Again using the fundamental solution we develop necessary and sufficient conditions for pointwise completeness. The criterion that we obtain for pointwise completeness also enables us to characterize those systems which are not pointwise complete, and thence to obtain necessary and sufficient conditions for them to be null controllable.

2.1 The Representation of the Solution of a Differential-Difference Equation

In this section we introduce the basic systems that we will consider throughout this chapter. We then go on to consider the representation of the solutions of these systems, and develop an explicit and particularly useful form for the fundamental solution.

The basic system to be considered in this chapter is the following constant coefficient differential-difference equation,

(2.1.1) $\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t)$ for $t \in (0,T]$,

(2.1.2) $x(t) = \varphi(t)$ for $t \in [-1,0]$,

(2.1.3) $y(t) = Dx(t)$,

where $x(t)$ is an n-dimensional vector; $u(t)$ is an r-dimensional vector, and $u(\cdot)$ is an admissible control (that is, it is contained in the space of square integrable functions, L^2, on every finite interval); and $y(t)$ is an m-dimensional vector. A,B,C and D are $n\times n$, $n\times n$, $n\times r$, and $m\times n$ matrices, respectively. We will assume that the initial functions $\varphi(t), t\in[-1,0]$ are contained in $C([-1,0];R^n)$, the space of continuous functions whose range is in R^n. For each admissible control $u(\cdot)$, and each initial function $\varphi\in C([-1,0];R^n)$ there exists a unique solution to (2.1.1) and (2.1.2), where the solution is continuous on the interval $[-1,0]$ and absolutely continuous on the interval $(0,T]$ (see References [6],[14]).

In a recent paper, Banks [30] has obtained a variation of parameters formula for the solution of (2.1.1) and (2.1.2), namely

$$x(T) = x(T,\varphi) + \int_0^T X(T-s)\,Cu(s)\,ds, \tag{2.1.4}$$

where

$$x(T,\varphi) = X(T)\varphi(0) + \int_{-1}^{0} X(T-\alpha-1)B\varphi(\alpha)\,d\alpha\ . \tag{2.1.5}$$

$X(t)$ is the unique $n\times n$ matrix solution of

$$\frac{d}{dt}X(t) = AX(t) + BX(t-1), \qquad t>0 \tag{2.1.6}$$

where

$$X(t) = \begin{cases} I & t = 0 \\ 0 & t\in[-1,0). \end{cases}$$

We will term $X(t)$ the fundamental solution of the homogeneous differential-difference equation (2.1.6).

It will be noted that the second term of (2.1.4) constitutes the solution of (2.1.1) with a zero initial function, while (2.1.5) is the solution of the homogeneous differential-difference equation related to (2.1.1).

We are now interested in obtaining an explicit form for the fundamental solution of (2.1.6). The first approach that we will consider is what Elsgoltz [14] termed the "method of steps". Let us consider how this method works. For $t \in (0,1]$ the term $X(t-1)$ in (2.1.6) is defined by (2.1.7), hence we may solve (2.1.6) as an ordinary differential equation over this interval, and the solution is

(2.1.8) $$X(t) = e^{At} \quad \text{for } t \in (0,1].$$

On the interval $t \in (1,2]$, we again have an ordinary differential equation, as $X(t-1)$ is simply the solution of (2.1.6) over the interval $(0,1]$, and the initial condition is $X(1) = e^{A}$. By induction, on the interval $(k,k+1]$, $k = 0,1,\ldots.$, we find

(2.1.9) $$X(t) = e^{At} + \int_1^t e^{A(t-s_1)} B e^{A(s_1-1)} ds_1 + \ldots + \int_k^t e^{A(t-s_k)} B ds_k \ldots \int_1^{s_1} e^{A(s_2-s_1)} \times B e^{A(s_1-1)} ds_1 \quad .$$

The form of the fundamental solution given by (2.1.8) and (2.1.9) will prove useful in a few special cases in our later discussion. However it gives us very little insight into what properties the fundamental solution $X(t)$ may have. It is our intention to now present a new form for the fundamental solution $X(t)$ which will prove of greater utility in our future discussion.

Apart from the advantages of the new form of the fundamental solution in our discussions it seems to offer a computational advantage. By examining (2.1.8) and (2.1.9) we see that to evaluate the fundamental solution $X(t)$ a number of integrations need to be performed. However in the form of the solution $X(t)$, to be presented, only algebraic operations are involved in its evaluation, which potentially offers a considerable saving in computation time.

Let us now consider the homogeneous matrix differential-difference equation (2.1.6) and (2.1.7). We introduce the following notation, by defining $X_k(\tau) = X(\tau+k)$ for $\tau \in [0,1]$ and $k = 0,1,2,\ldots$. By direct substitution in (2.1.6) and (2.1.7) we obtain

$$\text{(2.1.10)} \begin{cases} \frac{d}{d\tau} X_0(\tau) = AX_0(\tau) & X_0(0) = I \\ \frac{d}{d\tau} X_1(\tau) = BX_0(\tau) + AX_1(\tau) & X_1(0) = X_0(1) \\ \quad\vdots & \quad\vdots \\ \frac{d}{d\tau} X_k(\tau) = BX_{k-1}(\tau) + AX_k(\tau) & X_k(0) = X_{k-1}(1), \end{cases}$$

so that the solution of (2.1.6) and (2.1.7) over the interval $t \in [k,k+1]$ is given by $X(t) = X_k(t-k)$.

Letting $Z_k(\tau) = [X_0^T(\tau),\ldots,X_k^T(\tau)]^T$, (2.1.10) can be written more concisely as

(2.1.11) $$\frac{d}{d\tau} Z_k(\tau) = A_k Z_k(\tau) \quad \text{for } \tau \in [0,1],$$

(2.1.12) $$X_k(\tau) = E_k Z_k(\tau),$$

where $Z_k(\tau)$ is an $n(k+1)\times n$ matrix, and

$$A_k = \begin{bmatrix} A & 0 & 0\ldots 0 & 0 \\ B & A & 0\ldots 0 & 0 \\ 0 & B & A\ldots 0 & 0 \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot & \cdot \quad \cdot & \cdot \\ 0 & 0 & 0\ldots B & A \end{bmatrix}, \quad E_k = [0,\ldots\ldots,0,I].$$

A_k and E_k are $n(k+1)\times n(k+1)$ and $n\times n(k+1)$ matrices respectively.

As is well known, the unique solution of (2.1.11) is given by

(2.1.13) $$Z_k(\tau) = e^{A_k\tau} Z_k(0),$$

and hence

(2.1.14) $$X_k(\tau) = E_k e^{A_k} Z_k(0).$$

There remains now the problem of finding the initial conditions for (2.1.11). It is clear from (2.1.10) and the definition of $Z_k(\tau)$ that

(2.1.15) $$Z_o(0) = I.$$

We will show by induction that

$$Z_k(0) = \begin{bmatrix} I \\ \cdots\cdots\cdots\cdots \\ e^{A_{k-1}} Z_{k-1}(0) \end{bmatrix}, \text{ for } k = 1,2,\ldots .$$

From the definition of $Z_i(\tau)$ and (2.1.10)

$$Z_i(0) = \begin{bmatrix} X_o(0) \\ X_1(0) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_k(0) \end{bmatrix} = \begin{bmatrix} X_o(0) \\ X_o(1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ X_{k-1}(1) \end{bmatrix}$$

$$= \begin{bmatrix} X_o(0) \\ \cdots\cdots \\ Z_{i-1}(1) \end{bmatrix} = \begin{bmatrix} I \\ \cdots\cdots\cdots \\ e^{A_{i-1}} Z_{i-1}(0) \end{bmatrix}$$

where the last equality follows from (2.1.7) and (2.1.13).

From the statement following (2.1.10) we can write an explicit relation for $X(t)$, $t \in [k,k+1]$; namely

$$X(t) = X_k(t-k) = E_k e^{A_k(t-k)} Z_k(0). \tag{2.1.17}$$

It should be observed that the fundamental matrix solution $X(t)$ may be singular for some t, in contradistinction to the case for ordinary differential equations. It is this very property of $X(t)$ which leads to the necessity of drawing a distinction between null controllability and complete controllability, in euclidean space, for differential-difference equations. The following example exhibits this property quite clearly.

<u>Example 2.1.1</u> Consider the following differential-difference equation studied by Popov [27]

$$(2.1.18)\quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \\ x_3(t-1) \end{bmatrix}$$

From (2.1.17) and some simple calculations we see

$$(2.1.19)\quad X(2) = E_2 Z_2(0) = \begin{bmatrix} 2 & 4 & -4 \\ 1 & 1 & -2 \\ 0 & 2 & 0 \end{bmatrix},$$

which is a singular matrix.

2.2 Complete Controllability

In this section we define the concept of complete output controllability, and derive algebraic necessary and sufficient conditions for constant coefficient differential-difference equations to have this property. From these general results we derive the usual algebraic criterion for the output controllability of ordinary differential equations. We further obtain a rather simple algebraic criterion for the controllability of a restricted class of differential-difference equations, and also show that the scalar differential-difference equation is controllable. Finally we present an algebraic characterization of the reachable set, as defined below, for those systems which are not completely output controllable. This algebraic characterization of the reachable set will also prove of great use in our later discussion of null controllability.

DEFINITION 2.2.1. The control system (2.1.1), (2.1.2), (2.1.3) is said to be completely output controllable at time $T>0$ if for every $\varphi \in C([-1,0];R^n)$ and for every $y_1 \in R^m$ there exists an admissable control $u(t)$, $t \in [0,T]$ such that $y(T) = y_1$.

We have defined <u>complete output controllability</u> above rather than <u>complete controllability</u>. The reason for this is that the former definition of controllability in no way complicates the proof of the necessary and sufficient conditions, while the results for complete controllability can simply be obtained by setting $D = I$ in any of the results given below.

DEFINITION 2.2.2. Consider the control system (2.1.1), (2.1.2), (2.1.3). The reachable set $\mathcal{R}(T)$ is defined as

$$\mathcal{R}(T) = \{y \in R^m \mid y = y(T),\ u(\cdot) \in L^2,\ \varphi = 0\} .$$

The reachable set is the set of points in R^m which can be 'reached' by the output of the control system (2.1.1), (2.1.2), (2.1.3) starting from zero initial conditions. It is relatively easy to see that the reachable set $\mathcal{R}(T)$ is a linear subspace of R^m.

From (2.1.3) and (2.1.4)

$$(2.2.1) \qquad y(T) = Dx(T,\varphi) + D\int_0^T X(T-s)Cu(s)ds,$$

which can be re-written as

$$(2.2.2) \qquad y(T) - Dx(T,\varphi) = D\int_0^T X(T-s)Cu(s)ds.$$

We see that for each admissible control u the right hand side of (2.2.2) is an element of the reachable set $\mathcal{R}(T)$. Now suppose that the system (2.1.1),(2.1.2),(2.1.3) is completely output controllable. Then for each $y_1 \in R^m$ and $\varphi \in C([-1,0];R^n)$ there exists an admissible control $\tilde{u}$ such that

$$y_1 - Dx(T,\varphi) = D\int_0^T X(T-s)C\tilde{u}(s)ds \in \mathcal{R}(T).$$

Therefore the system (2.1.1),(2.1.2),(2.1.3) is completely output controllable at time T if and only if $\mathcal{R}(T) = R^m$.

THEOREM 2.2.1. The control system (2.1.1),(2.1.2),(2.1.3) is completely output controllable at time T if and only if $\eta^T DX(T-s)C = 0$ for a.e.$s \in [0,T]$ implies $\eta = 0$.

<u>Proof</u>. From above the system (2.1.1),(2.1.2),(2.1.3) is completely output controllable at time T if and only if $\mathcal{R}(T) = R^m$.

Now $\mathcal{R}(T) \neq R^m$ if and only if there exists a non-zero $\eta \in R^m$ such that $\eta^T y = 0$ for every $y \in \mathcal{R}(T)$. This in turn is equivalent to

$$\int_0^T \eta^T DX(T-s)Cu(s)ds = 0, \tag{2.2.3}$$

for all controls $u(\cdot) \in L^2$.

If there exist a non-zero $\eta \in R^m$ such that $\eta^T DX(T-s)C = 0$ for a.e.$s \in [0,T]$, then (2.2.3) is true for all $u(\cdot) \in L^2$. On the other hand if there exists a non-zero $\eta \in R^m$ such that (2.2.3) is true for all $u(\cdot) \in L^2$, then

(2.2.3) is true for

(2.2.4) $$u(s) = [\eta^T DX(T-s)C]^T.$$

Substituting (2.2.4) in (2.2.3) we obtain

(2.2.5) $$\int_0^T \eta^T DX(T-s)CC^T X^T(T-s)D^T \eta \, ds = 0,$$

and therefore conclude

$$\eta^T DX(T-s)C = 0 \quad \text{for a.e.}\, s \in [0,T].$$

We thus have succeeded in proving that $\mathcal{R}(T) \neq R^m$ if and only if there exists a non-zero $\eta \in R^m$ such that $\eta^T DX(T-s)C = 0$ for a.e. $s \in [0,T]$.

COROLLARY 2.2.1. The control system (2.1.1),(2.1.2),(2.1.3) is completely output controllable at time T if and only if

$$W(T,0) = \int_0^T DX(T-s)CC^T X^T(T-s)D^T ds$$

is positive definite.

<u>Proof</u>. Obviously $W(T,0)$ is positive semi-definite. It is easy to see that $\eta^T W(T,0) = 0$ for some non-zero $\eta \in R^m$ if and only if $\eta^T DX(T-s)C = 0$ for a.e. $s \in [0,T]$.

Corollary 2.2.1 was originally obtained by Chyung and Lee [22] and McClamroch [31] for $D = I$.

From the result of Theorem 2.2.1 and the fundamental solution (2.1.22) we now obtain an algebraic necessary and sufficient condition for complete output controllability at time T of the control system (2.1.1), (2.1.2), (2.1.3).

To minimize the notation let us define $D_k = DE_k$ and $C_k = Z_k(0)C$ for $k=0,1,2,\ldots$.

THEOREM 2.2.2. A necessary and sufficient condition for the control system (2.1.1), (2.1.2), (2.1.3) to be completely output controllable at time $T \in (k,k+1]$, $k = 0,1,2,\ldots$ is that the matrix

$$Q(T) = [D_oC_o,\ldots,D_oA_o^{n-1}C_o,\ldots,D_kC_k,\ldots,D_kA_k^{n(k+1)-1}C_k]$$

has rank m. Q(T) is an $m\times(k+1)(k+2)nr$ matrix.

<u>Proof.</u> From Theorem 2.2.1 a necessary and sufficient condition for complete output controllability is that $\eta^T DX(T-s)C = 0$ for a.e. $s \in [0,T]$ implies $\eta = 0$, where $\eta \in R^m$.

This statement is equivalent to:

(2.2.7) $$\eta^T DX(T-s) = 0 \quad \text{for } s \in (T-1,T]$$

(2.2.8) $$\eta^T DX(T-s) = 0 \quad \text{for } s \in (T-2,T-1]$$

$$\vdots$$

(2.2.9) $$\eta^T DX(T-s) = 0 \quad \text{for } s \in (0,T-k]$$

implies $\eta = 0$.

From (2.1.17) we obtain by direct substitution,

(2.2.10) $$\eta^T D_o e^{A_o(T-s)} C_o = 0 \ , \ s \in (T-1,T]$$

(2.2.11) $$\eta^T D_1 e^{A_1(T-s-1)} C_1 = 0 \ , \ s \in (T-2,T-1]$$

$$\vdots$$

$$(2.2.12) \qquad \eta^T D_k e^{A_k(T-k-s)} C_k = 0, \quad s \in (0, T-k]$$

implies $\eta = 0$.

We now only need to show that (2.2.10),(2.2.11) and (2.2.12) implies $\eta = 0$ if and only if the rank of $Q(T)$ equals m.

Suppose the rank of $Q(T) < m$, then there exists a non-zero $\eta \in R^m$ such that

$$(2.2.13) \qquad \eta^T D_i C_i = \dots\dots\dots = \eta^T D_i A_i^{n(i+1)} C_i = 0$$

for i = 0,1,....,k. From the Cayley-Hamilton theorem and (2.2.13) we find

$$(2.2.14) \qquad \eta^T D_i A_i^{n(i+1)} C_i = 0,$$

for i = 0,1,....,k. It can be shown by induction that

$$\eta^T D_i A_i^{n(i+1)+j} C_i = 0,$$

for j = 0,1,2,...., i = 0,1,....,k.

Using the power series expansion of the exponential matrix, we find

$$\eta^T D_o e^{A_o(T-s)} C_o = 0, \quad s \in (T-1, T]$$

$$\eta^T D_1 e^{A_1(T-s-1)} C_1 = 0, \quad s \in (T-2, T-1]$$

$$\vdots$$

$$\eta^T D_k e^{A_k(T-s-k)} C_k = 0, \quad s \in (0, T-k].$$

This is a contradiction of the statement: (2.2.10),(2.2.11)

(2.2.12) implies $\eta = 0$.

Suppose on the other hand there exists a non-zero $\eta \in R^m$ such that the expressions (2.2.10), (2.2.11), (2.2.12) are true. Then by successive differentiation of (2.2.10) and setting $s = T$, we find

$$\eta^T D_o C_o = \eta^T D_o A_o C_o = \ldots\ldots = \eta^T D_o A_o^{n-1} C_o = 0. \tag{2.2.16}$$

Treating (2.2.11) and (2.2.12) in a similar manner we obtain

$$\eta^T D_i A_i^j C_i = 0,$$

for $i = 0,1,\ldots,k$; $j = 0,1,\ldots,n(i+1)-1$. It is thus obvious that $\eta^T Q(T) = 0$ which implies that the rank of $Q(T)$ is less than m.

An interesting property of differential-difference equations will be observed from Theorem 2.2.2. Namely the system may become controllable only after some non-zero time interval has elapsed. In the example below we will present a system which is not completely output controllable for time $T \leqq 1$, but is completely output controllable for all time $T > 1$.

Example 2.2.1. Consider the following differential-difference equation,

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_2(t-1) \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \tag{2.2.17}$$

$$(2.2.18) \qquad \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}.$$

Suppose that $T \in (0,1]$, then we find

$$Q(T) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

and the rank of $Q(T) = 1 < 2$. Now suppose $T > 1$, then

$$Q(T) = \begin{bmatrix} 1 & 1 & 1+e^1 & \ldots\ldots \\ 1 & 1 & e^1 & \ldots\ldots \end{bmatrix},$$

and we note the rank of $Q(T) = 2$. Hence the system (2.2.17), (2.2.18) is not completely output controllable for $T \in (0,1]$, but is completely output controllable for all time thereafter.

COROLLARY 2.2.1. If the system (2.1.1),(2.1.2),(2.1.3) is completely output controllable at time T_1, then it is completely output controllable for all time $T \geq T_1$.

Proof. Suppose $T \geq T_1$ then from Theorem 2.2.2 we can partition $Q(T)$, as

$$Q(T) = [Q(T_1), \tilde{Q}].$$

It can now be seen that if $Q(T_1)$ has rank m, then $Q(T)$ must also have rank m.

The following result is the usual algebraic criterion for complete output controllability of ordinary differential equations. This result was originally proved by Kreindler and Sarachik [32], and for the case D = I by Kalman [2].

COROLLARY 2.2.2. Suppose that in the system (2.1.1), (2.1.2),(2.1.3) the matrix B = 0. Then a necessary and sufficient condition for complete output controllability at time T is that the matrix

$$Q_A(T) = [DC, DAC, \ldots\ldots, DA^{n-1}C]$$

has rank m.

<u>Proof</u>. Suppose $Q_A(T)$ has rank m, then Q(T) has rank m for any T. This follows from the fact that

$$(2.2.19) \qquad D_oC_o = DC, \ldots, D_oA_o^{n-1}C_o = DA^{n-1}C,$$

and hence Q(T) has m linearly independant columns for any T > 0.

Suppose the rank of $Q_A(T) < m$, then there exists a non-zero $\eta \in R^m$ such that

$$(2.2.20) \qquad \eta^T DC = \eta^T DAC = \ldots. = \eta^T DA^{n-1}C = 0.$$

Applying the Cayley-Hamilton theorem we can show

$$\eta^T DA^n C = 0.$$

It can then be shown by induction that

$$(2.2.21) \qquad \eta^T DA^{n+\ell} C = 0 \quad \text{for } \ell = 0,1,\ldots.$$

The general term in $\eta^T Q(T)$ is given by

$$\eta^T D_i A_i^{\ell} C_i \quad \text{for} \quad i = 0,1,\ldots,k; \; \ell = 0,1,\ldots,n(i+1)-1.$$

Substituting for D_i, A_i and C_i the general term becomes

$$\eta^T D_i A_i^{\ell} C_i = \eta^T D[0,\ldots,0,I] \begin{bmatrix} A^{\ell} & 0 & \cdots & 0 \\ 0 & A^{\ell} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A^{\ell} \end{bmatrix} \begin{bmatrix} I \\ e^{A} \\ \vdots \\ e^{iA} \end{bmatrix} C$$

$$= \eta^T D A^{\ell} e^{iA} C \; .$$

Expanding e^{iA} in a power series, and applying (2.2.20) and (2.2.21) to the general term

$$\eta^T D_i A_i^{\ell} C_i = \eta^T D A^{\ell} [I + iA + \frac{1}{2!}(iA)^2 + \ldots.] \; C = 0$$

Hence $\eta^T Q(T) = 0$ and the rank of $Q(T) < m$.

The following result, originally proved by Kirillova and Curakova [21] for $D = I$, gives a simple algebraic criterion for complete output controllability for the system (2.1.1), (2.1.2), (2.1.3) where $A = 0$.

COROLLARY 2.2.3. Suppose in system (2.1.1), (2.1.2), (2.1.3) the matrix $A = 0$. Then a necessary and sufficient condition for complete output controllability at time $T \in (k,k+1]$ is that the matrix

$$Q_B(T) = [DC, \ldots, DB^k C]$$

has rank m.

If $T > n-1$ then a necessary and sufficient condition for complete controllability at time T is that the matrix

$$Q_c(T) = [DC,\ldots,DB^{n-1}C]$$

has rank m.

<u>Proof</u>. Suppose the rank of $Q_B(T) < m$, then there exists a non-zero $\eta \in R^m$ such that

$$\eta^T Q_B(T) = \eta^T[DC,\ldots,DB^kC] = 0. \tag{2.2.22}$$

The general term in $\eta^T Q(T)$ is given by

$$\eta^T D_i A_i^{\ell} C$$

for $i = 0,1,\ldots,k$; $\ell = 0,1,\ldots,n(i+1)-1$. Substituting for D_i, A_i and C_i the general term becomes

$$\eta^T D_i A_i^{\ell} C_i = D[0,\ldots,0,I]\begin{bmatrix} 0 & 0 & 0.\,.0 & 0 \\ B & 0 & 0.\,.0 & 0 \\ 0 & B & 0.\,.0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0.\,.B & 0 \end{bmatrix}^{\ell} \times \begin{bmatrix} I \\ I \\ I+B \\ \vdots \\ I+(i-1)+\ldots+\frac{1}{(i-1)!}B^{i-1} \end{bmatrix} C. \tag{2.2.23}$$

For $\ell = 0$, (2.2.23) becomes

$$\eta^T D[I+(i-1)B+\ldots\ldots+\frac{1}{(i-1)!}B^{i-1}]C.$$

From (2.2.22) this term equals zero. For $\ell=1$, (2.2.23) becomes

$$\eta^T D[I+(i-2)B+\ldots+\frac{1}{(i-2)!}B^{i-2}]C.$$

which is again zero by (2.2.22). Similarly $\eta^T D_i A_i^{\ell} C_i = 0$ for $1 < \ell < i$. For $\ell = i$, expansion of (2.2.23) leads to $\eta^T DB^i C$ which is zero from (2.2.22). For $\ell > i$ we see that $A_i^{\ell} = 0$, hence $\eta^T D_i A_i^{\ell} C_i = 0$ for $\ell > i$. From the above discussion we may finally conclude that $\eta^T D_i A_i^{\ell} C_i = 0$ for $\ell = 0,1,\ldots\ldots,n(i+1)-1$. From the above comments we see that $\eta^T Q(T) = 0$, and the rank of $Q(T) < m$.

For sufficiency suppose $Q_B(T)$ has rank m. We recognize from the general term (2.2.23) and the discussion following it, that $D_i A_i^i C_i = DB^i C$. Hence, indicating only the important terms the matrix $Q(T)$ has the form

$$[DC,\ldots,DBC,\ldots,DB^2C,\ldots\ldots,DB^kC]$$

and so it must also have rank m.

Let us suppose $T > n-1$. Now if $Q_C(T)$ has rank m, then since it constitutes the first n sub-matrices of $Q_B(T)$, $T > n$, we see that $Q_B(T)$ has rank m.

Now suppose $Q_C(T)$ has rank less than m, then there exists a non-zero $\eta \in R^m$ such that

(2.2.24) $$\eta^T[DC,\ldots\ldots\ldots,DB^{n-1}C] = 0.$$

Applying the Cayley-Hamilton theorem and an inductive argument we find

$$\eta^T DB^{n+\ell}C = 0$$

for $\ell = 0,1,\ldots\ldots$. Hence we find the rank of $Q_B(T)$ is also less than m.

In Corollary 2.2.3 it was shown that for $T > n$ it is necessary and sufficient to investigate the rank of $Q_C(T)$ rather than the rank of $Q_B(T)$ which leads one to conjecture that a similar situation arises in Theorem 2.2.2. The author believes this conjecture to be true but up to the present time has only been able to prove it for some special cases.

Let us now consider the scalar differential-difference equation

$$x^{(n)}(t) + \sum_{i=0}^{n-1} a_i x^{(i)}(t) + \sum_{i=0}^{n-1} b_i x^{(i)}(t-1) = cu(t), \tag{2.2.25}$$

where x(t) and u(t) are scalar functions of time t, and the parameters a_i, b_i and c are constant. We denote the i'th derivative of x(t) by $x^{(i)}(t)$. We will now transform (2.2.25) into the form of (2.1.1),(2.1.3) and thus show that (2.2.25) is controllable.

Defining $x_1 = x^{(0)}$, $x_2 = x^{(1)}, \ldots\ldots, x_n = x^{(n-1)}$, we can write (2.2.25) as

$$\left\{\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= x_3(t) \\ &\vdots \\ \dot{x}_n(t) &= -\sum_{i=0}^{n-1} a_i x_{i+1}(t) - \sum_{i=0}^{n-1} b_i x_{i+1}(t-1) + cu(t). \end{aligned}\right. \tag{2.2.26}$$

This equation can be written in the matrix form given in (2.1.1),(2.1.3) where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{n-1} \end{bmatrix}$$

(2.2.27)

$$C = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad D = I.$$

COROLLARY 2.2.4. The scalar control system (2.2.25) is completely output controllable for every time $T > 0$.

Proof. The proof is an immediate consequence of (2.2.27) and Theorem 2.2.2.

We now present an algebraic characterization of the reachable set $\mathcal{R}(T)$ for those systems which are not completely output controllable.

THEOREM 2.2.3. The reachable set $\mathcal{R}(T)$ equals the range of the matrix $Q(T)Q^T(T)$, where $Q(T)$ is the matrix defined in Theorem 2.2.2.

Proof. By examining the proof of Theorem 2.2.2 we see that the orthogonal complement, $\mathcal{R}^{\perp}(T)$, of $\mathcal{R}(T)$ is given by

$$(2.2.28) \qquad \mathcal{R}^{\perp}(T) = \{\eta \in R^m \mid \eta^T Q(T) = 0\} \quad .$$

It is clear that (2.2.28) can be re-written as

$$\mathcal{R}^{\perp}(T) = \{\eta \in R^m \mid Q^T(T)\eta = 0\} = \{\eta \in R^m \mid Q(T)Q^T(T)\eta = 0\}$$

$$= \text{null}(Q(T)Q^T(T)) = \text{range}(Q(T)Q^T(T))^{\perp}.$$

The last two equalities follow from the basic properties of linear transformations, where $\text{null}(Q(T)Q^T(T))$ denotes the null space of $Q(T)Q^T(T)$, and $\text{range}(Q(T)Q^T(T))$ denotes the range of $Q(T)^TQ(T)$. Hence $\mathcal{R}(T) = \text{range}(Q(T)Q^T(T))$.

2.3 Pointwise Completeness

In order to be able to discuss the question of null output controllability of differential-difference equations it is necessary to introduce the concept of output pointwise completeness. In this section we will define the concept of output pointwise completeness, and obtain algebraic necessary and sufficient conditions for the system (2.1.1),(2.1.2),(2.1.3) to be pointwise complete. Some further results will exhibit a large class of systems which are output pointwise complete. Using these results we will present an example of a system which is not pointwise complete. Finally for those systems which are not pointwise complete we will present an algebraic characterization of the set $\mathcal{P}(T)$, defined below.

Let us consider the control system (2.1.1),(2.1.2), (2.1.3) with the control $u \equiv 0$,

$$\dot{x}(t) = Ax(t) + Bx(t-1), \quad t \in (0,T] \tag{2.3.1}$$

$$x(t) = \varphi(t), \qquad t \in [-1,0] \tag{2.3.2}$$

$$y(t) = Dx(t). \tag{2.3.3}$$

We now introduce the following:

DEFINITION 2.3.1. The system (2.3.1),(2.3.2),(2.3.3) is said to be output pointwise complete at time T if for every $y_1 \in R^m$ there exists a $\varphi \in C([-1,0];R^n)$ such that $y(T) = y_1$.

The notion of pointwise completeness as introduced by Weiss [23] is equivalent to our definition if we take $D = I$ and so $y(t) = x(t)$. Weiss gave no explicit conditions under which a system is pointwise complete, but recently Popov [27] has shown that if the matrix $B = b^T g$, where b and g are n-dimensional vectors, then the system (2.3.1),(2.3.2),(2.3.3) is pointwise complete.

Let us recall (2.1.5) which we will write again for convenience

$$x(T,\varphi) = X(T)\varphi(0) + \int_{-1}^{0} X(T-s-1)B\varphi(s)ds. \tag{2.1.5}$$

It is clear that (2.1.5) can be re-written as

$$x(T,\varphi) = \int_{-1}^{0} \left[d_s\left(\int_{0}^{s+1} X(T-\alpha)Bd\alpha + X(T)H(s)\right)\right] \varphi(s), \tag{2.3.4}$$

where
$$H(s) = \begin{cases} 1 & s = 0 \\ 0 & s \in [-1,0). \end{cases}$$

To simplify the notation let us define

$$U(T,s) = \int_0^{s+1} X(T-\alpha)B\,d\alpha + X(T)H(s), \tag{2.3.5}$$

where we note $U(T,-1) = 0$ and $U(T,s)$ is a function of bounded variation in s. Hence (2.3.4) becomes

$$x(T,\varphi) = \int_{-1}^{0} [d_s U(T,s)]\varphi(s). \tag{2.3.6}$$

Finally the output $y(T)$ which we will write as $y(T,\varphi)$ to emphasize its dependance on φ is

$$y(T,\varphi) = \int_{-1}^{0} [d_s DU(T,s)]\varphi(s). \tag{2.3.7}$$

We note (2.3.7) is a linear operator mapping the space of continuous functions $C([-1,0];R^n)$ into the euclidean space R^m.

DEFINITION 2.3.2. The range of the operator (2.3.7) is defined as

$$\rho(T) = \{ y \in R^m \mid y = y(T,\varphi),\ \varphi \in C([-1,0];R^n) \}$$

It can easily be shown that $\rho(T)$ is a linear subspace of R^m.

It is obvious that a necessary condition for output pointwise completeness is that the matrix D has rank m. For completeness we include the following result due to Popov [27].

THEOREM 2.3.1. Let us suppose that D has rank m. If the system (2.3.1), (2.3.2), (2.3.3) is not output pointwise complete at time T* then it is not output pointwise complete for all time $T \geq T^*$.

Proof. Suppose the system is not output pointwise complete at time T*, then there exists a non-zero $\eta \in R^m$ such that

$\eta^T y(T^*,\varphi) = 0$ for every $\varphi \in C([-1,0];R^n)$. Let us suppose $x(t,\varphi)$ is the solution of (2.3.1),(2.3.2) corresponding to $y(t,\varphi)$; that is $y(t,\varphi) = Dx(t,\varphi)$. Now for all $s>0$, the function $\tilde{x}(t) = x(t+s,\varphi)$ is also a solution of (2.3.1) for some initial condition $\tilde{\varphi} \in C([-1,0];R^n)$. Hence $\eta^T y(T^*,\tilde{\varphi})=0$; but $y(T^*+s,\varphi) = \tilde{y}(T^*,\varphi)$ and so

$$\eta^T y(T^*+s,\varphi) = 0.$$

Since s is arbitary, this completes the proof.

We now turn our attention to establishing a necessary and sufficient condition for output pointwise completeness.

THEOREM 2.3.2. A necessary and sufficient condition for the system (2.3.1),(2.3.2),(2.3.3) to be output pointwise complete is that for every non-zero $\eta \in R^m$,

i) $\eta^T DX(T-\alpha)B \not\equiv 0$ for $\alpha \in [0,1]$

or

ii) $\eta^T DX(T) \neq 0$.

Proof. Let us suppose (2.3.1),(2.3.2),(2.3.3) is not output pointwise complete, then since the range of (2.3.7) is a linear subspace of R^m there exists a non-zero $\eta \in R^m$ such that $\eta^T y = 0$ for every $y \in \mathcal{P}(T)$. Hence we see

$$\eta^T \int_{-1}^{0} [d_s DU(T,s)]\varphi(s) = 0, \qquad (2.3.8)$$

for every $\varphi \in C([-1,0];R^n)$. We note (2.3.8) is a continuous linear functional mapping $C([-1,0];R^n)$ into R. By the Riesz representation theorem such linear functionals mapping $C([-1,0];R^n)$ into R are uniquely represented by Stieltjes integrals of the form (2.3.8). Since the functional equals zero for every continuous function φ, we conclude the variation,

$$\operatorname*{Var}_{s=-1}^{0}\left[\eta^T DU(T,s)\right] = 0, \tag{2.3.9}$$

where $\operatorname*{Var}_{s=-1}^{0}\left[\eta^T DU(T,s)\right] = \sup\sum_{j=1}^{k}\left\|\eta^T DU(T,s_{j+1})-\eta^T DU(T,s)\right\|_{R^n}$,

and the supremum is over all possible finite partitions of the interval $[-1,0]$.

Since $U(T,s)$ is absolutely continuous on the interval $s\in[-1,0)$ and $U(T,-1) = 0$ we see that (2.3.9) implies,

$$\eta^T DU(T,s) \equiv 0 \tag{2.3.10}$$

for $s\in[-1,0]$. Substituting (2.3.5) into (2.3.10) we obtain

$$\eta^T D\int_0^{s+1} X(T-\alpha)Bd\alpha + \eta^T DX(T)H(s) = 0,\ s\in[-1,0]. \tag{2.3.11}$$

Hence we find that $\eta^T DX(T-\alpha)B = 0$ for $\alpha\in[0,1]$ and $\eta^T DX(T) = 0$ which contradicts the conclusion of the theorem.

The necessity of this result can be shown by assuming there exists a non-zero $\eta\in R^m$ such that $\eta^T DX(T-\alpha)B = 0$ for $\alpha\in[-1,0]$ and $\eta^T DX(T) = 0$, and then simply reversing the arguments given above. It is clear that we have

$$\eta^T DU(T,s) = 0 \tag{2.3.12}$$

for $s\in[0,1]$. From (2.3.12) we see that (2.3.8) is true for every $\varphi \in C([-1,0];R^n)$ which implies that $\mathcal{O}(T) \neq R^m$, and so (2.3.1),(2.3.2),(2.3.3) is not output pointwise complete.

COROLLARY 2.3.1. Every system (2.3.1),(2.3.2),(2.3.3) is output pointwise complete for all $T \in [0,2)$ if D has rank m.

Proof. For $T \in [0.1]$ we have from (2.1.8) that $X(T) = e^{AT}$. Hence $DX(T)$ has rank m, and so for every non-zero $\nu \in R^m$ we have $\nu^T DX(T) \neq 0$, contradicting Theorem 2.3.2, ii)

Let us now consider the case $T \in [1,2)$. Suppose (2.3.1),(2.3.2),(2.3.3) is not output pointwise complete. Then there exists a non-zero $\nu \in R^m$ such that $\nu^T DX(T-\alpha)B = 0$ for $\alpha \in [0,1]$ and $\nu^T DX(T) = 0$. Therefore

$$\nu^T DX(T-\alpha)B = 0 \tag{2.3.13}$$

for $T-1 < \alpha < 1$, and substituting for $X(T-\alpha)$ from (2.1.8) we have

$$\nu^T De^{A(T-\alpha)}B = 0 \tag{2.3.14}$$

for $T-1 < \alpha < 1$. Since e^{At} is an analytic function of t we see $\nu^T De^{At}B = 0$ for every t. We now examine condition ii) of Theorem 2.3.2. Substituting (2.1.9) for $X(T), 1 < T \leq 2$ we obtain

$$\nu^T D\left[e^{AT} + \int_1^T e^{A(T-s_1)}Be^{A(s_1-1)}ds\right] = 0. \tag{2.3.15}$$

Since $\nu^T De^{A(T-s_1)}B \equiv 0$ for all s, (2.3.15) implies $\nu^T De^{AT} = 0$. Finally, De^{AT} has rank m so we conclude $\nu = 0$ which contradicts the assumption $\nu \neq 0$.

In the proof of Corollary 3.1 we have used the form of the fundamental solution X(t) given by (2.1.9). It could equally well have been proved using (2.1.16) and (2.1.17), but it is slightly more convenient to use (2.1.9).

COROLLARY 2.3.2 If in the system (2.3.1), (2.3.2), (2.3.3) the matrices A and B commute, that is AB = BA, and D has rank m then it is output pointwise complete for every time $T \geq 0$.

Proof. Since AB = BA some simple calculations show (2.1.9) simplifies to

$$X(t) = \sum_{i=0}^{k} \frac{(t-i)^i}{i!} e^{A(t-i)} B^i \tag{2.3.16}$$

for $t \in [k,k+1)$.

Let us suppose (2.3.1), (2.3.2), (2.3.3) is not output pointwise complete, then from Theorem 2.3.2, there exists a non-zero $\nu \in R^m$ such that

$$\text{i)} \quad \nu^T D X(T-\alpha) B \equiv 0 \quad \alpha \in [0,1], \tag{2.3.17}$$

and

$$\text{ii)} \quad \nu^T D X(T) = 0 \; . \tag{2.3.18}$$

Let us suppose that $k \leq T < k+1$, where k is an arbitrary integer. Substituting (2.3.16) into (2.3.17) we obtain

$$\nu^T D \sum_{i=0}^{k} \frac{(T-\alpha-i)^i}{i!} e^{A(T-\alpha-i)} B^{i+1} = 0 \tag{2.3.19}$$

for $T-1 < T-2 < k$.

We can rewrite (2.3.19) as

$$\left[\sum_{i=0}^{k} \nu^T DB^{i+1} e^{-iA} \frac{(T-\alpha-i)^i}{i!}\right] e^{A(T-\alpha)} = 0 \tag{2.3.20}$$

for $T-1 < T-\alpha < k$. We note that (2.3.20) consists of the term in brackets multiplied by a non singular matrix, hence

$$\sum_{i=0}^{k} \nu^T DB^{i+1} e^{-iA} \frac{(T-\alpha-i)}{i!} = 0 \tag{2.3.21}$$

for $T-1 < T-\alpha < k$. Since (2.3.21) is a polynomial in α, we conclude that

$$\nu^T DB^i = 0 \tag{2.3.22}$$

for $i = 1,\ldots,k+1$. Substituting (2.3.16) into (2.3.18) we find

$$\nu^T D \sum_{i=0}^{k+1} B^i \frac{(T-i)^i}{i!} e^{A(T-i)} = 0. \tag{2.3.23}$$

Therefore from (2.3.22), (2.3.23) becomes

$$\nu^T De^{AT} = 0, \tag{2.3.24}$$

and since D has rank m we conclude $\nu = 0$. This contradicts our initial assumption and so the system is output pointwise complete for $T \in [k,k+1)$. Since k was an arbitrary integer we see the result is true for all k, and so the proof is complete.

Two particular cases of interest are when $A = 0$, and when $B = 0$. For both these cases (2.3.1),(2.3.2),(2.3.3) is output pointwise complete for all $T \geq 0$, provided D has rank m.

We will now obtain an algebraic necessary and sufficient condition for output pointwise completeness of the system (2.3.1),(2.3.2),(2.3.3). To simplify the notation let us define $F_k = Z_k(0)B$.

THEOREM 2.3.3. The system (2.3.1),(2.3.2),(2.3.3) is output pointwise complete at time $T \in (k,k+1), k = 0,1,2,\ldots,$ if and only if for every non-zero $\nu \in R^m$,

$$\text{i)} \quad \nu^T[D_kF_k,\ldots,D_kA_k^{n(k+1)-1}F_k] \neq 0$$

or

$$\text{ii)} \quad \nu^T[D_{k-1}F_{k-1},\ldots,D_{k-1}A_{k-1}^{nk-1}F_{k-1}] \neq 0$$

or

$$\text{iii)} \quad \nu^T D_k e^{A_k(T-k)} Z_k(0) \neq 0$$

Further (2.3.1),(2.3.2),(2.3.3) is output pointwise complete at time $T = k$, $k = 0,1,2,\ldots$ if and only if for every non-zero $\nu \in R^m$,

$$\text{i')} \quad \nu^T [D_{k-1}F_{k-1},\ldots,D_{k-1}A_{k-1}^{nk-1}F_{k-1}] \neq 0$$

or

$$\text{ii')} \quad \nu^T D_k Z_k(0) \neq 0 .$$

<u>Proof</u>. Let us first consider the case $T \in (k,k+1)$, $k = 0,1,\ldots$ Suppose the system (2.3.1),(2.3.2),(2.3.3) is not output pointwise complete at time T, then from Theorem 2.3.2 there exists a non-zero $\nu \in R^m$ such that

$$\nu^T DX(T-\alpha)B = 0 \tag{2.3.25}$$

for $\alpha \in [0,1]$, and

$$\nu^T DX(T) = 0 . \tag{2.3.26}$$

Substituting (2.1.17) in (2.3.25) and (2.3.26) we obtain

(2.3.27) $$\nu^T D_{k-1} e^{A_{k-1}(T-\alpha-k+1)} F_{k-1} = 0$$

for $T-1 < T-\alpha < k$,

(2.3.28) $$\nu^T D_k e^{A_k(T-\alpha-k)} F_k = 0$$

for $k < T-\alpha < T$, and

(2.3.29) $$\nu^T D_k e^{A_k(T-k)} Z_k(0) = 0.$$

Since the terms on the left of the equality in (2.3.27) and (2.3.28) are analytic functions of α, we see that they are identically zero for all $\alpha \in R$. By successive differentiation of (2.3.27) and (2.3.28) and setting $\alpha = T-k+1$ and $\alpha = T-k$ in (2.3.27) and (2.3.28) respectively, we obtain,

(2.3.30) $$\nu^T D_{k-1} A_{k-1}^i F_{k-1} = 0,$$

for $i = 0,1,\ldots,nk-1$, and

(2.3.31) $$\nu^T D_n A_k^i F_k = 0,$$

for $i = 0,1,\ldots,n(k+1)-1$. Hence (2.3.29),(2.3.30),(2.3.31) lead to a contradiction which completes the proof of sufficiency.

To prove necessity suppose there exists a non-zero $\nu \in R^m$ such that

(2.3.32) $$\nu^T [D_k F_k, \ldots\ldots, D_k A_k^{n(k+1)-1} F_k] = 0,$$

(2.3.33) $$\nu^T [D_{k-1} F_{k-1}, \ldots, D_{k-1} A_{k-1}^{nk-1} F_{k-1}] = 0,$$

and

(2.3.34) $$\nu^T D_k e^{A_k(T-k)} Z_k(0) = 0.$$

From the Cayley-Hamilton theorem and (2.3.32) we obtain

(2.3.35) $$\nu^T D_k A_k^{n(k+1)} F_k = 0.$$

Then by induction

(2.3.36) $$\nu^T D_k A_k^{n(k+1)+\ell} = 0$$

for $\ell = 0,1,\ldots$. Using the power series expansion of the exponential matrix we find

(2.3.37) $$\nu^T D_k e^{A_k(T-\alpha-k)} F_k = 0$$

for $T-\alpha \in (k,T)$. Using a similar argument with respect to (2.3.33) we also find

(2.3.38) $$\nu^T D_{k-1} e^{A_{k-1}(T-\alpha-k+1)} F_{k-1} = 0$$

for $T-\alpha \in (T-1,k)$. Hence from (2.3.34), (2.3.37) and (2.3.38) we have shown

$$\nu^T DX(T-\alpha)B = 0$$

for $\alpha \in [0,1]$, and

$$\nu^T DX(T) = 0,$$

which is a contradiction.

Let us now consider the case where T is an integer, k, say. In this case it is easy to see that it is necessary and sufficient if for every non-zero $\nu \in R^m$, (2.3.39) $$\nu^T D_k e^{A_{k-1}(1-\alpha)} F_{k-1} \not\equiv 0$$

for $\alpha \in [0,1]$, or

(2.3.40) $$\nu^T D_k Z_k(0) \neq 0.$$

For (2.3.39) we use the same arguments as above thus leading to i') in Theorem 2.3.3. This completes the proof.

COROLLARY 2.3.3. A necessary and sufficient condition for the system (2.3.1), (2.3.2), (2.3.3) to be output pointwise complete at time $T \in (k,k+1)$, $k = 0,1,\ldots$ is that the matrix

$$M(T) = [D_k F_k, \ldots, D_k A_k^{n(k+1)-1} F_k, \ldots, D_{k-1} A_{k-1}^{nk-1} F_{k-1}, D_k e^{A_k(T-k)} Z_k(0)],$$

has rank m.

For T equal to an integer, k say, a necessary and sufficient condition for the system (2.3.1), (2.3.2), (2.3.3) to be output pointwise complete is that the matrix

$$M'(T) = [D_{k-1} F_{k-1}, \ldots, D_{k-1} A_{k-1}^{nk-1} F_{k-1}, D_k Z_k(0)]$$

has rank m.

Proof. The proof is an immediate consequence of Theorem 2.3.3.

Corollaries 2.3.1 and 2.3.2 could have been proved using Theorem 2.3.3 rather than Theorem 2.3.2. However it proved possible to give a more direct proof using Theorem 2.3.2 and the fundamental solution given by (2.1.8) and (2.1.9).

It is obvious that if the matrix D in (2.3.3) does not have rank m, then the system (2.3.1),(2.3.2),(2.3.3) is certainly not output pointwise complete. This may be termed the trivial case. For the non-trivial case, that is when D has rank m, the possibility of a system not being pointwise complete is not so clear. Popov [27] recently gave the following example and we will use it to demonstrate that not all systems (2.3.1),(2.3.2),(2.3.3) are output pointwise complete, as well as demonstrating an application of Theorem 2.3.3.

<u>Example 2.3.1</u>. Let us consider the differential-difference equation given in Example 2.1.1, and let us suppose $y(t) = x(t)$, that is $D = I$. We will examine the system defined above to see if it is pointwise complete at time $T = 2$. From Corollary 2.3.3 and some straightforward calculations it may be shown that

$$[D_1F_1,\ldots,D_1A_1^{n-1}F_1,D_2Z_2(0)] = \begin{bmatrix} 2 & -2 & 0 & 2 & -4 & 0 & 0 & -4 & 0 \\ 1 & -2 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & -4 & 0 \end{bmatrix}.$$

It may easily be seen that this matrix has rank less than three. In fact taking $\nu^T = [1\ -2\ -1]$ it follows that

$$\nu^T[D_1F_1,\ldots\ldots,D_1A_1^{n-1}F_1,D_2Z_2(0)] = 0.$$

Hence this system is not output pointwise complete at time $T = 2$. Then from Theorem 2.3.1. due to Popov we see the system is not output pointwise complete for all time $T \geq 2$.

To complete this section we discuss an algebraic characterization of the set $\rho(T)$, where $\rho(T)$ is given by Definition 2.3.2.

THEOREM 2.3.4. The set $\rho(T)$ equals the range of the matrix $M(T)M^T(T)$ for T equal to a non-integer, and $\rho(T)$ equals the range of the matrix $M'(T)\ (M'(T))^T$ when T is an integer. The matrices M(T) and M'(T) are defined in Corollary 2.3.3.

Proof. By examining the proof of Theorem 2.3.3 we see that the orthogonal complement, $\rho^{\perp}(T)$, of $\rho(T)$ is given by

$$\rho^{\perp}(T) = \{ \nu \in R^m \mid \nu^T M(T) = 0 \}$$

for T equal to a non-integer. A similar relation holds when T equals an integer, where M(T) is replaced by M'(T). The proof then follows along the same lines as that of Theorem 2.2.3.

2.4 Null Controllability

In this final section we define null output controllability for differential-difference equations, and obtain algebraic necessary and sufficient conditions under which these equations are null output controllable. We find that for null controllability the concept of pointwise completeness plays a significant role; this is especially so for the result given by Corollary 2.3.3.

DEFINITION 2.4.1. The control system (2.1.1),(2.1.2), (2.1.3) is said to be null output controllable at time $T > 0$

if for every $\varphi \in C([-1,0];R^n)$ there exists a control $u(t)$, $0 \leq t \leq T$ such that $y(T) = 0$.

From (2.1.3) and (2.1.4)

$$y(T) = Dx(T,\varphi) + D\int_0^T X(T,s)Cu(s)ds, \tag{2.4.1}$$

and this can be re-written as

$$y(T) - Dx(T,\varphi) = D\int_0^T X(T,s)Cu(s)ds. \tag{2.4.2}$$

Now since we wish the output $y(T) = 0$ for any initial function $\varphi \in C([-1,0];R^n)$, (2.4.2) becomes

$$-y(T,\varphi) = D\int_0^T X(T,s)Cu(s)ds \tag{2.4.3}$$

for some admissable control $u(t)$, $0 \leq t \leq T$, where $y(T,\varphi) = Dx(T,\varphi)$, as in (2.3.7). From Definition 2.3.2 it immediately follows that the control system is null output controllable if and only if $\mathcal{P}(T) \subset \mathcal{R}(T)$.

If a system is output pointwise complete at time T then we know that $\mathcal{P}(T) = R^m$, and hence null output controllability is equivalent to complete output controllability. For this case the results of section 2 can be used for investigating whether the system is null output controllable. Therefore for null output controllability the principal interest lies with those systems which are not output pointwise complete at time T, that is $\mathcal{P}(T) \neq R^m$.

THEOREM 2.4.1. The control system (2.1.1),(2.1.2),(2.1.3) is null output controllable at time $T \in (k,k+1)$, $k = 0,1,...$ if and only if the matrices $M(T)$ and $Q(T)$ defined in

Corollary 2.3.3 and Theorem 2.2.2 have the following property: The rank of $[M(T)M^T(T), Q(T)Q^T(T)]$ equals the rank of $Q(T)Q^T(T)$.

For T equal to an integer, k say, the control system (2.1.1), (2.1.2), (2.1.3) is null output controllable if and only if the rank of $[M'(T)(M'(T))^T, Q(T)Q^T(T)]$ equals the rank of $Q(T)Q^T(T)$.

<u>Proof</u>. From the discussion preceding the theorem and the results of Theorem 2.2.3 and Theorem 2.3.4 we see that the system (2.1.1), (2.1.2), (2.1.3) is null output controllable if and only if range $(M(T)M^T(T)) \subset$ range $(Q(T)Q^T(T))$.

Let us suppose range $(M(T)M^T(T)) \subset$ range $(Q(T)Q^T(T))$, then the columns of $M(T)M^T(T)$ are linearly dependant on the columns of $Q(T)Q^T(T)$, and so the rank of $[M(T)M^T(T), Q(T)Q^T(T)]$ equals the rank of $Q(T)Q^T(T)$. On the other hand suppose the rank of $[M(T)M^T(T), Q(T)Q^T(T)]$ equals the rank of $Q(T)Q^T(T)$, then the columns of $M(T)M^T(T)$ are linearly dependant on the columns of $Q(T)Q^T(T)$ and hence the range $(M(T)M^T(T)) \subset$ range $(Q(T)Q^T(T))$.

For the case where T is an integer we replace $M(T)$ by $M'(T)$ and repeat the arguments given above.

CHAPTER 3

CONTROLLABILITY IN FUNCTION SPACE

In the last chapter we discussed the controllability in euclidean space of differential-difference equations. The present chapter is devoted to the study of function space controllability of differential-difference equations. The necessity for such a discussion is not apparent until one realizes that the state space for these equations is a function space, whereas for ordinary differential equations the state space is a euclidean space. As was indicated in the Introduction the state space for differential-difference equations consists of functions on the interval $[-1,0]$ whose range is the euclidean space R^n, where n denotes the dimension of x(t) in equation (1.1.1).

For ordinary differential equations we know that controllability in euclidean space gives a complete answer to the regulator problem posed by Kalman [2]. In particular we can decompose the regulator problem into two parts: The first part involves controlling the system from its initial condition to zero, that is, a euclidean space controllability problem, while the second part is involved with keeping the state equal to zero once it has been 'steered' there. However as was shown in the Introduction, controllability in euclidean space does not give a complete answer to the regulator problem for differential-difference equations.

This follows from the fact that while we can still decompose the problem into two parts, the first part now involves the question of controllability to the zero function.

In this chapter we will consider the question of complete function space controllability of differential-difference equations. In the first section we introduce the notion of complete output function space controllability and then obtain a function space criterion for it. From this result we are able to obtain simple algebraic conditions for complete output function space controllability of two special classes of differential-difference equations.

In the rest of the chapter we will be concerned with the development of an algebraic condition for the complete output function space controllability of differential-difference equations which do not fit into either of the classes of equations mentioned above. In the second section we transform the function space condition for complete output function space controllability into an equivalent two point boundary value problem. The two point boundary value problem is closely related to the problem of invertability of ordinary differential equations. So in the third section we present the structure or inversion algorithm of Silverman and Payne [33] and those of their results that are required in our later discussion. Finally in the fourth section we present an algebraic necessary and sufficient condition for complete output function space controllability.

3.1 A Function Space Criterion for Complete Output Function Space Controllability

In this section we will introduce the basic control system that will be considered throughout this chapter. We then define the concept of complete output function space controllability, and give a lemma which proves to be of fundamental importance to all our later discussions. From this result we are able to deduce a function space criterion for function space controllability of these systems. Finally we present results for two special classes of systems where the function space criterion for controllability reduces to simple algebraic conditions.

The basic control system to be considered in this chapter is the same as that considered in the previous one, and is the following constant coefficient differential-difference equation

(3.1.1) $$\dot{x}(t) = Ax(t) + Bx(t-1) + Cu(t)$$

for $t \in (0,T]$,

(3.1.2) $$x(t) = \varphi(t)$$

for $t \in [-1,0]$,

(3.1.3) $$y(t) = Dx(t),$$

where the assumptions concerning (3.1.1),(3.1.2),(3.1.3) are the same as those given in Chapter 2 for the system (2.1.1),(2.1.2),(2.1.3).

DEFINITION 3.1.1. The control system (3.1.1),(3.1.2),(3.1.3) is said to be completely output function space controllable at time $T \geq 1$ if for every $\varepsilon > 0$, for every φ in the space of continuous functions $C([-1,0];R^n)$, and for every y_1 in the space of square integrable functions, $L^2([T-1,T];R^m)$, there exists an admissible control $u(t)$, $0 \leq t \leq T$ such that

$$\| y_1 - y_T(\varphi,u) \| < \varepsilon .$$

$\| \cdot \|$ denotes the usual norm in the space $L^2([T-1,T];R^m)$, and $y_T(\varphi,u)$ denotes the output function $y(t)$, $T-1 \leq t \leq T$, given by (3.1.3).

It will be noted that we have defined <u>complete output function space controllability</u> above rather than <u>complete function space controllability</u>. Using the former definition rather than the latter does not complicate the proofs given below, while the results for complete function space controllability can easily be obtained by setting $D = I$ in any of the theorems below.

The definition given above is closely related to the idea expressed by Antosiewicz [34] in his definition of approximate controllability, and the work of Fattorini [35].

In the definition given above it will be noted that we have taken the 'target function' y_1 to be in the space of square integrable functions. We have some latitude in the choice of the class of functions in which the function y_1 may be contained, and we could equally take the class of continuous functions $C([T-1,T];R^m)$ or the class of absolutely continuous functions $AC([T-1,T];R^m)$ in place of the class of square integrable functions $L^2([T-1,T];R^m)$ in Definition 3.1.1. The reason for the choice of the class of square integrable functions is that it simplifies some of the proofs given below; but it appears that the proofs are still valid if either class of functions mentioned above is used in Definition 3.1.1. in place of the class of square integrable functions.

DEFINITION 3.1.2. Consider the control system (3.1.1), (3.1.2), (3.1.3). We define the reachable set $\mathcal{R}(T)$ as

$$\mathcal{R}(T) = \{y \in L^2([T-1,T];R^m) \mid y = y_T(0,u), u(\cdot) \in L^2(0,T),\ \varphi \equiv 0\}.$$

In contrast with the definition of the reachable set given in Definition 2.2.2 the reachable set presented here is a subset of a function space. In fact it is the set of functions defined on the interval $[T-1,T]$ which can be 'reached' by the output of the control system (3.1.1), (3.1.2) (3.1.3) starting from zero initial conditions. This abuse of the term 'reachable set' should cause no difficulty to the reader as the notion given by Definition 2.2.2 is never used in the present chapter. It is relatively easy to see that the reachable set $\mathcal{R}(T)$ is a linear subspace of $L^2([T-1,T];R^m)$.

From (2.1.4) and (3.1.3)

$$y(t) = Dx(t,\varphi) + D\int_0^t X(t-s)Cu(s)ds \tag{3.1.4}$$

for $t \in [T-1,T]$. Denoting $x(t,\varphi)$, $T-1 \le T$ by $x_T(\varphi)$ and the integral operator in (3.1.4) on the interval $t \in [T-1,T]$ by $\Lambda_T(u)$, (3.1.4) can be re-written, with a slight abuse of notation, as

$$(3.1.5) \qquad y_T(\varphi,u) = Dx_T(\varphi) + D\Lambda_T(u).$$

For each admissible control u it can be seen from (3.1.4) that $D\Lambda_T(u)$ is an element of the reachable set $\mathcal{R}(T)$. Suppose that the system (3.1.1),(3.1.2), (3.1.3) is completely output function space controllable at time T. Then for every $\varepsilon > 0$, each $y_1 \in L^2([T-1,T];R^m)$ and $\varphi \in C([T-1,T];R^n)$ there exists an admissible control $\tilde{u}$ such that

$$(3.1.6) \qquad \|y_1 - y_T(\varphi,\tilde{u})\| < \varepsilon.$$

From (3.1.5) and (3.1.6) we have

$$(3.1.7) \qquad \|y_1 - Dx_T(\varphi) - D\Lambda_T(\tilde{u})\| < \varepsilon.$$

Since y_1 may be any square integrable function it is easy to see (3.1.7) implies that the reachable set $\mathcal{R}(T)$ is dense in $L^2([T-1,T];R^m)$. Since the converse is trivially true from Definition 3.1.1, we see (3.1.1),(3.1.2),(3.1.3) is completely output function space controllable at time T if and only if $\mathcal{R}(T)$ is dense in $L^2([T-1,T];R^m)$.

The following lemma was stated without proof by Fattorini [35]. Since this lemma proves to be fundamental to all our future discussions, and the author has been unable to find a proof of it in the literature, we will present a proof in the Appendix. This result is actually a generalization to a function space of the well known result that a linear subspace, M, of the euclidean space R^n equals R^n if and only if the orthogonal complement, $M^{\perp}$, is empty excepting for the zero vector.

DEFINITION 3.1.3. If X is a Banach space, and M is a subset of X, then the set

$$M^{\perp} = \{x' \in X' \mid (x', x) = 0, \text{ for every } x \in M\}$$

is called the orthogonal complement of M. X' denotes the dual space of the Banach space X, and (x',x) denotes the functional x' evaluated for $x \in M$.

LEMMA 3.1.1. Suppose the set M is a linear subspace of the Banach space X. Then M is dense in X, with the norm topology, if and only if the orthogonal complement $M^{\perp}$ is empty excepting for the zero vector.

The next result gives a function space criterion for the system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable. A careful examination of this result shows that it is the function space analogue of Theorem 2.2.1.

THEOREM 3.1.1. The control system (3.1.1),(3.1.2),(3.1.3) is completely output function space controllable at time $T \geq 1$ if and only if the only square integrable function $f(\theta)$, $\theta \in [T-1, T]$ for which

$$\int_{T-1}^{T} f^T(\theta) D X(\theta - s) C \, d\theta = 0$$

for a.e. $s \in [0,T]$ is the zero function $f(\theta) = 0$ for a.e. $\theta \in [T-1,T]$.

Proof. We recall from the discussion following Definition 3.1.2 that the system (3.1.1), (3.1.2), (3.1.3) is completely output function space controllable if and only if the reachable set $\mathcal{R}(T)$ is dense in $L^2([T-1,T];R^m)$.

Suppose $\mathcal{R}(T)$ is not dense in $L^2([T-1,T];R^m)$, then from Lemma 3.1.1 we see that there exists a non-zero functional $\eta' \in (L^2([T-1,T];R^m))'$ such that for every $y \in \mathcal{R}(T)$, $(\eta',y) = 0$. By the Reisz representation theorem there exists a function $f \in L^2([T-1,T];R^m)$ such that

$$(\eta',x) = \int_{T-1}^{T} f^T(\theta)y(\theta)d\theta \tag{3.1.8}$$

Hence from (3.1.8) and Lemma 3.1.1, there exists a non-zero function $f \in L^2([T-1,T];R^m)$ such that for every $y \in \mathcal{R}(T)$,

$$\int_{T-1}^{T} f^T(\theta)y(\theta)d\theta = 0. \tag{3.1.9}$$

From (2.1.4), (2.1.5) and (3.1.3) we see that $y \in \mathcal{R}(T)$ if and only if

$$y(t) = D\int_0^t X(t-s)Cu(s)ds, \quad t \in [T-1,T], \tag{3.1.10}$$

for some admissible control u. Since $X(t) = 0$ for $t \in [-1,0]$, (3.1.10) becomes: $y \in \mathcal{R}(T)$ if and only if

$$y(t) = D\int_0^T X(T-s)Cu(s)ds, \quad t \in [T-1,T], \tag{3.1.11}$$

for some admissible control u. Substituting (3.1.11) into (3.1.9) we find there exists a non-zero function $f \in L^2([T-1,T];R^m)$ such that for every admissible control u

$$\int_{T-1}^{T} f^T(\theta) \left\{ \int_0^T DX(\theta-s)Cu(s)ds \right\} d\theta = 0. \tag{3.1.12}$$

By the Fubini theorem we may interchange the order of integration in (3.1.12) thus obtaining

$$\int_0^T \left\{ \int_{T-1}^{T} f^T(\theta)DX(\theta-s)Cd\theta \right\} u(s)ds = 0. \tag{3.1.13}$$

Since (3.1.13) is true for all admissible controls u, it must be true for

$$u(s) = \left\{ \int_{T-1}^{T} f^T(\theta)DX(\theta-s)Cd\theta \right\}^T. \tag{3.1.14}$$

Substituting (3.1.14) into (3.1.13) we find

$$\int_0^T \left\{ \int_{T-1}^{T} f^T(\theta_1)DX(\theta_1-s)Cd\theta_1 \right\} \times \left\{ \int_{T-1}^{T} f^T(\theta_2)DX(\theta_2-s)Cd\theta_2 \right\}^T ds = 0, \tag{3.1.15}$$

and therefore

$$\int_{T-1}^{T} f^T(\theta)DX(\theta-s)Cd\theta = 0 \tag{3.1.16}$$

for a.e. $s \in [0,T]$. This is a contradiction.

For necessity, let us assume $\mathcal{R}(T)$ is dense in $L^2([T-1,T];R^m)$. Suppose there exists a non-zero function $f \in L^2([T-1,T];R^m)$ such that

$$\int_{T-1}^{T} f^T(\theta)DX(\theta-s)Cd\theta = 0 \tag{3.1.17}$$

for a.e. $s \in [0,T]$. Then from (3.1.17) we see

$$\int_0^T \left\{ \int_{T-1}^{T} f^T(\theta)DX(\theta-s)Cd\theta \right\} u(s)ds = 0 \tag{3.1.18}$$

for every admissible u, i.e. for every $u \in L^2(0,T)$. By the Fubini theorem we may interchange the order of integration in (3.1.18) to obtain

$$(3.1.19) \qquad \int_{T-1}^{T} f^T(\theta)\left\{\int_0^T DX(\theta-s)Cu(s)ds\right\}d\theta = 0.$$

It will be observed from (3.1.11) that the term $\{\cdot\}$ in the integrand of (3.1.19) is an arbitrary element of $\mathcal{R}(T)$. Again using the Riesz representation theorem and Lemma 3.1.1 we find $\mathcal{R}(T)$ is not dense in $L^2([T-1,T];R^m)$.

Utilizing the fundamental solution given in (2.1.17) we can obtain a more useful form for Theorem 3.1.1.

COROLLARY 3.1.1. A necessary and sufficient condition for the control system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable at time $T \in [i,i+1)$, $i = 1,2,\dots.$ is that the only square integrable function $f(\theta)$, $\theta \in [T-1,T]$ for which

$$\int_s^T C_0^T e^{A_0^T(\theta-s)} D_0^T f(\theta)d\theta = 0,$$

$$\int_{T-1}^{s} C_{k-1}^T e^{A_{k-1}^T(\theta-s-1)} D_{k-1}^T f(\theta)d\theta + \\ + \int_s^T C_k^T e^{A_k^T(\theta-s)} D_k^T f(\theta)d\theta = 0,$$

for $s \in [T-1,T]$ and $k = 1,\dots,i-1$, and

$$\int_{T-1}^{s} C_{i-1}^T e^{A_{i-1}^T(\theta-s+1)} D_{i-1}^T f(\theta)d\theta + \\ + \int_s^T C_i^T e^{A_i^T(\theta-s)} D_i^T f(\theta)d\theta = 0$$

for $s \in [i,T]$ is the zero function $f(\theta) = 0$ for a.e. $\theta \in [T-1,T]$, where $C_k = Z_k(0)C$, $D_k = DE_k$, for all $k = 0,1,2,\ldots$. (for definitions of $Z_k(0)$ and E_k see Section 2.1).

Proof. Let us recall from (2.1.17) that the fundamental matrix solution for $t \in [k,k+1]$, $k = 0,1,\ldots$. is

$$X(t) = E_k e^{A_k(t-k)} Z_k(0). \tag{3.1.20}$$

Substituting (3.1.20) in the integral of Theorem 3.1.1 it is immediately clear that

$$\int_s^T f^T(\theta) D_o e^{A_o(\theta-s)} C_o \, d\theta = 0 \tag{3.1.21}$$

for $s \in [T-1,T]$,

$$\int_{T-1}^{s+k} f^T(\theta) D_{k-1} e^{A_{k-1}(\theta-s-k+1)} C_{k-1} \, d\theta + \int_{s+k}^{T} f^T(\theta) D_k e^{A_k(\theta-s+k)} C_k \, d\theta = 0 \tag{3.1.22}$$

for $s \in [T-1,T]$, $k = 1,\ldots,i$, and

$$\int_{T-1}^{s+i} f^T(\theta) D_{i-1} e^{A_{i-1}(\theta-s-i+1)} C_{i-1} \, d\theta + \int_{s+i}^{T} f^T(\theta) D_i e^{A_i(\theta-s+i)} C_i \, d\theta = 0 \tag{3.1.23}$$

for $s \in [0,T-i]$. By making the change of variable $s' = s+k$, $k = 1,\ldots,i$ in (3.1.22) and (3.1.23) and taking the transpose of (3.1.21),(3.1.22),(3.1.23) the result immediately follows.

We will now examine the necessary and sufficient conditions for complete output function space controllability of two special cases of the control system (3.1.1), (3.1.2),(3.1.3). It will in fact be shown that Corollary 3.1.1 reduces to a simple algebraic criterion in each case.

Our first result will be seen to be equivalent to that of Corollary 2.2.3 for $T \geq n$. It will further be noted that when $D = I$, our result is equivalent to the result obtained by Kirillova and Curakova [21]. The reader should be aware however that Kirillova and Curakova discussed function space null controllability while in the present work we are discussing complete function space controllability.

THEOREM 3.1.2. Suppose for the control system (3.1.1), (3.1.2),(3.1.3) the matrix $A = 0$. Then a necessary and sufficient condition for complete output function space controllability at time $T \in [k,k+1)$, $k = 1,2,....$ is that the matrix

$$\tilde{Q}_B(T) = [DC,\ldots\ldots,DB^{k-1}C]$$

has rank m.

If $T \geq n$, then a necessary and sufficient condition for complete output function space controllability at time T is that the matrix

$$Q_C(T) = [DC,\ldots.DB^{n-1}C]$$

has rank m.

<u>Proof</u>. In Corollary 3.1.1 we note that the integrands

contain terms of the form

$$C_i^T e^{A_i^T t} D_i^T \tag{3.1.24}$$

for $i = 0,1,\ldots,k$. Since we have assumed in this theorem that the matrix $A = 0$, we find by simple calculations that (3.1.24) becomes

$$C^T[I + B^T(t+i-1) + (B^T)^2 \frac{(t+i-2)^2}{2!} + \ldots + (B^T)^i \frac{t^i}{i!}]D^T \tag{3.1.25}$$

for $i = 0,1,\ldots,k$.

To prove sufficiency let us assume matrix $\tilde{Q}_B(T)$ has rank m. From Corollary 3.1.1 and (3.1.25) we have

$$\int_s^T C_o^T e^{A_o^T(\theta-s)} D_o^T f(\theta)d\theta = \int_s^T C^T D^T f(\theta)d\theta = 0. \tag{3.1.26}$$

Differentiating (3.1.26) once, we find

$$C^T D^T f(\theta) = 0 \tag{3.1.27}$$

for a.e. $\theta \in [T-1,T]$. Let us now suppose

$$C^T (B^T)^{\ell} D^T f(\theta) = 0 \tag{3.1.28}$$

for a.e. $\theta \in [T-1,T]$, $\ell = 0,\ldots,i-1$. Then from (3.1.25) and (3.1.28)

$$\int_{T-1}^{s} C_{i-1}^T e^{A_{i-1}^T(\theta-s+1)} D_{i-1}^T f(\theta)d\theta = \tag{3.1.29}$$

$$= \int_{T-1}^{s} C^T[I + B^T(\theta-s+i)+\ldots+ (B^T)^{i-1} \frac{(\theta-s+1)^{i-1}}{(i-1)!}]D^T f(\theta)d\theta$$

$$= 0.$$

Hence we find from (3.1.25),(3.1.28),(3.1.29) and Corollary 3.1.1 that

$$(3.1.30)\qquad 0 = \int_s^T C_i^T e^{A_i^T(\theta-s)} D_i^T f(\theta)d\theta$$

$$= \int_s^T C^T[I + B^T(\theta-s) + \ldots + (B^T)^{i-1}\frac{(\theta-s+1)^{i-1}}{(i-1)!} + $$

$$+ (B^T)^i \frac{(\theta-s)^i}{i!}]Df(\theta)d\theta$$

$$= \int_s^T C^T(B^T)^i D^T \frac{(\theta-s)^i}{i!} f(\theta)d\theta.$$

Differentiating (3.1.30) i+1 times, we find

$$(3.1.31)\qquad C^T(B^T)^i D^T f(\theta) = 0$$

for a.e. $\theta \in [T-1,T]$. Therefore by induction we have

$$(3.1.32)\qquad C^T(B^T)^{\ell} D^T f(\theta) = 0$$

for a.e. $\theta \in [T-1,T]$, and $\ell = 0,1,\ldots,k-1$, and

$$(3.1.33)\qquad C^T(B^T)^k D^T f(\theta) = 0$$

for a.e. $\theta \in [k,T]$. It can be seen that (3.1.32) and (3.1.33) can be written as

$$(3.1.34)\qquad f^T(\theta)[DC,\ldots,DB^{k-1}C] = 0$$

for a.e. $\theta \in [T-1,k]$ and

$$(3.1.35)\qquad f^T(\theta)[DC,\ldots,DB^kC] = 0$$

for a.e. $\theta \in [k,T]$. Since the matrix $\tilde{Q}_B(T)$ has rank m we conclude that $f(\theta) = 0$ for a.e. $\theta \in [T-1,T]$, thus showing by Corollary 3.1.1 that the system (3.1.1),(3.1.2),(3.1.3) is completely output function space controllable.

To prove necessity let us assume that the rank of $\tilde{Q}_B(T) < m$, and $T > k$. Hence there exists a non-zero $\eta \in R^m$ such that

$$C^T D^T \eta = \dots = C^T (B^T)^{k-1} D^T \eta = 0.$$

We now define the function $f(\theta)$, $T-1 \leq \theta \leq T$ as follows:

$$(3.1.36) \qquad f(\theta) = \begin{cases} \eta & T-1 \leq \theta < k \\ 0 & k \leq \theta \leq T \end{cases}.$$

From (3.1.25) and the definition of the non-zero function $f(\theta)$, $T-1 \leq \theta \leq T$ given by (3.1.36) we see that the integrals in Corollary 3.1.1 are zero. For $T = k$ we define $f(\theta) = \eta$, $T-1 \leq \theta \leq T$, and again see that the integrals in Corollary 3.1.1 are zero. Hence for both cases considered above there exists a non-zero function $f(\theta)$, $T-1 \leq \theta \leq T$ such that the integrals in Corollary 3.1.1 are zero, and so (3.1.1),(3.1.2),(3.1.3) is not completely output function space controllable.

For $T \geq n$, we will show the matrix $\tilde{Q}_B(T)$ has rank m if and only if $Q_C(T)$ has rank m. Since for $T \geq n$, $Q_C(T)$ is a submatrix of $\tilde{Q}_B(T)$ we see that the rank of $Q_C(T)$ is less than the rank of $\tilde{Q}_B(T)$. Hence if the matrix $\tilde{Q}_B(T)$ has rank less than m, then matrix $Q_C(T)$ has rank less than m.

Now suppose matrix $Q_C(T)$ has rank less than m, then there exists a non-zero vector $\eta \in R^m$ such that $\eta^T Q_C(T) = 0$.

Using the Cayley-Hamilton theorem and an argument by induction we can show that $\eta^T DB^i C = 0$ for $i = n+1, \ldots, k-1$. Hence the matrix $\tilde{Q}_B(T)$ has rank less than m.

COROLLARY 3.1.2. Suppose in the control system (3.1.1), (3.1.2), (3.1.3) the matrix $A = 0$. Then the control system is completely output function space controllable at time $T \geq n$ if and only if it is completely output controllable at time $T \geq n$ (see Definition 2.2.1).

Proof. The proof is immediate from Corollary 2.2.3 and Theorem 3.1.2.

The final class of systems that we will examine in this section is where (3.1.1), (3.1.2), (3.1.3) is a single input-single output system. Again we can obtain an algebraic criterion for complete output function space controllability.

THEOREM 3.1.3 Suppose in the control system (3.1.1), (3.1.2), (3.1.3) C and D are nx1 and 1xn matrices which we will denote by c and d^T respectively. Then a necessary and sufficient condition for complete output function space controllability at time $T \in [k, k+1)$, $k = 1, 2, \ldots$ is that the matrix

$$Q_D(T) = [d_o^T c_o, \ldots, d_o^T A_o^{n-1} c_o, d_1^T c_1, \ldots, d_{k-1}^T A_{k-1}^{nk-1} c_{k-1}]$$

has rank 1.

Proof. We now prove sufficiency. Let us suppose $d_o^T c_o = \ldots$ $\ldots = d_\ell^T A_\ell^{j-1} c_\ell = 0$ and $d_\ell^T A_\ell^j c_\ell \neq 0$, where $\ell \leq k-1$ and $j \leq n-1$.

Now $d_o^T c_o = \ldots = d_o^T A_o^{n-1} c_o = 0$ implies by the Cayley-Hamilton theorem and an argument by induction, that $d_o^T A_o^i c_o = 0$ for all $i = n, n+1, \ldots$. Similarly we can show

$$d_m^T A_m^p c_m = 0 \tag{3.1.37}$$

for $m = 1, \ldots, \ell-1; p = 0, 1, \ldots$. Hence from (3.1.37) and the definition of the exponential matrix we find

$$d_m^T e^{A_m t} c_m = 0, \tag{3.1.38}$$

for $m = 1, \ldots, \ell-1$, and $t \in R$. From Corollary 3.1.1 we have

$$\int_{T-1}^{s} c_{\ell-1}^T e^{A_{\ell-1}^T(\theta-s+1)} d_{\ell-1} f(\theta) d\theta + \int_{s}^{T} c_\ell^T e^{A_\ell^T(\theta-s)} d_\ell f(\theta) d\theta = 0 \tag{3.1.39}$$

for $s \in [T-1, T]$. Now from (3.1.38) we see that the first integral in (3.1.39) is identically zero, and hence (3.1.39) becomes

$$\int_{s}^{T} c_\ell^T e^{A_\ell^T(\theta-s)} d_\ell f(\theta) d\theta = 0 \tag{3.1.40}$$

for $s \in [T-1, T]$. Taking the (j+1)'th derivative of (3.1.40) we obtain

$$\int_{s}^{T} c_\ell^T e^{A_\ell^T(\theta-s)} (A_\ell^T)^{j+1} d_\ell f(\theta) d\theta + c_\ell^T (A_\ell^T)^j d_\ell f(s) = 0 \tag{3.1.41}$$

for $s \in [T-1, T]$. Since $c_\ell^T (A_\ell^T)^j d_\ell \neq 0$ we can divide (3.1.41) by $c_\ell^T (A_\ell^T)^j d_\ell$ to obtain

$$(3.1.42)\quad f(s) + \frac{1}{c_\ell^T (A_\ell^T)^j d_\ell} \int_s^T c_\ell^T e^{A_\ell^T(\theta - s)} (A_\ell^T)^{j+1} d_\ell f(\theta) d\theta = 0$$

for $s \in [T-1,T]$. By the well known fact that a Volterra integral equation has a unique solution [36], we conclude that $f(\theta) = 0$ for a.e. θ $[T-1,T]$ in (3.1.42). Hence from Corollary 3.1.1 the control system is completely output function space controllable.

For necessity suppose that the matrix $Q_D(T)$ has rank zero. Hence by using a similar argument to the one used to obtain (3.1.38) we can show

$$(3.1.43)\qquad d_m^T e^{A_m t} c_m = 0$$

for $m = 1,\ldots\ldots,k-1$ and $t \in R$. We now define $f(\theta), T-1 \leq \theta \leq T$ as follows:

$$(3.1.44)\quad f(\theta) = \begin{cases} \gamma & T-1 \leq \theta < k \\ 0 & k \leq \theta \leq T \end{cases},$$

where γ is any non-zero constant. Substituting the function $f(\theta)$, $T-1 \leq \theta \leq T$ given by (3.1.44) in Corollary 3.1.1, and utilizing (3.1.43) we see that all the integrals in the corollary are zero. Hence we have shown there exists a non-zero function $f(\theta)$, $T-1 \leq \theta \leq T$ such that the conditions of Corollary 3.1.1 are satisfied.

3.2 Transformation of Function Space Criterion

In this section we will be concerned with the transformation of the function space criterion given in Corollary 3.1.1 into an equivalent two point boundary value problem.

To attain this objective it is necessary to introduce some new notation.

In Corollary 3.1.1 let us define

(3.2.1) $$v_o(t) = \int_t^T C_o^T e^{A_o^T(\theta-t)} D_o^T f(\theta)d\theta,$$

and

(3.2.2) $$v_k(t) = \int_{T-1}^t C_{k-1}^T e^{A_{k-1}^T(\theta-t+1)} D_{k-1}^T f(\theta)d\theta + \\ + \int_t^T C_k^T e^{A_k^T(\theta-t)} D_k^T f(\theta)d\theta,$$

for $k = 1,\ldots,i$ and $t\in[T-1,T]$.

Let us further define

(3.2.3) $$z_o(t) = \int_t^T e^{A_o^T(\theta-t)} D_o^T f(\theta)d\theta,$$

(3.2.4) $$\hat{z}_k(t) = \int_{T-1}^t e^{A_{k-1}^T(\theta-t+1)} D_{k-1} f(\theta)d\theta,$$

and

(3.2.5) $$z_k(t) = \int_t^T e^{A_k^T(\theta-t)} D_k f(\theta)d\theta,$$

for $k = 1,\ldots,i$ and $t\in[T-1,T]$. From (3.2.3),(3.2.4) and (3.2.5) we see that (3.2.1) and (3.2.2) become

(3.2.6) $$v_o(t) = C_o^T z_o(t)$$

and

(3.2.7) $$v_k(t) = C_{k-1}^T \hat{z}_{k-1}(t) + C_k^T z_k(t)$$

for $k = 1,\ldots,i$ and $t\in[T-1,T]$.

We further observe that (3.2.3),(3.2.4) and (3.2.5) satisfy the following ordinary differential equations with the given boundary conditions,

$$(3.2.8) \qquad \dot{z}_o(t) = -A_o^T z_o(t) - D_o^T f(t) \qquad z_o(T) = 0,$$

$$(3.2.9) \qquad \dot{\hat{z}}_k(t) = -A_{k-1}^T \hat{z}_k(t) - e^{A_{k-1}^T} D_{k-1}^T f(t) \qquad \hat{z}_k(T-1) = 0,$$

and

$$(3.2.10) \qquad z_k(t) = -A_k^T z_k(t) - D_k^T f(t) \qquad z_k(T) = 0,$$

for $k = 1,\ldots\ldots,i$, and $t \in [T-1,T]$.

Letting $z(t) = [z_o^T(t), \hat{z}_1^T(t), z_1^T(t),\ldots\ldots,z_i^T(t)]^T$, and $v^j(t) = [v_o^T(t),\ldots\ldots,v_j^T(t)]^T$, $j = i-1, i$, (3.2.6) to (3.2.10) can be written more concisely as

$$(3.2.11) \qquad \dot{z}(t) = Pz(t) + Qf(t),$$

$$(3.2.12) \qquad v^{i-1}(t) = R^{i-1}z(t),$$

and

$$(3.2.13) \qquad v^i(t) = R^i z(t)$$

for $t \in [T-1,T]$, where $z(t)$, $v^{i-1}(t)$, and $v^i(t)$ are $(i+1)^2 n$, ir, and $(i+1)r$ dimensional vectors respectively. Also

$$P = \begin{bmatrix} -A_o^T & 0 & 0.\,.\,.0 & 0 \\ 0 & -A_o^T & 0.\,.\,.0 & 0 \\ 0 & 0 & -A_2^T\,.\,.0 & 0 \\ \vdots & \vdots & \vdots \qquad \vdots & \vdots \\ 0 & 0 & 0 \quad -A_{i-1}^T & 0 \\ 0 & 0 & 0 \qquad 0 & -A_i^T \end{bmatrix} \qquad Q = \begin{bmatrix} -D_o^T \\ e^{A_o^T} D_o^T \\ -D_1^T \\ \vdots \\ e^{A_{i-1}^T} D_{i-1}^T \\ -D_i^T \end{bmatrix}$$

$$R^{i-1} = \begin{bmatrix} C_o^T & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & C_o^T & C_1^T & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_{i-2}^T & C_{i-1}^T & 0 & 0 \end{bmatrix},$$

(3.2.14)

$$R^{i} = \begin{bmatrix} C_o^T & 0 & 0 & \cdots & 0 & 0 \\ 0 & C_o^T & C_1^T & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & C_{i-1}^T & C_i^T \end{bmatrix},$$

where P, Q, R^{i-1} and R^i are $(i+1)^2n \times (i+1)^2n$, $(i+1)^2n \times m$, $ir \times (i+1)^2n$ and $(i+1)r \times (i+1)^2n$ matrices respectively. Finally let us define the subspaces U and V, both contained in $R^{(i+1)^2n}$, as

(3.2.15) $$U = \{z \in R^{(i+1)^2n} \mid z = [z_o^T, 0, z_1^T, \ldots\ldots, z_i^T]^T,$$
$$z_{\ell-1} \in R^{\ell n}, \ell = 1, \ldots, i+1\}$$

and

(3.2.16) $$V = \{z \in R^{(i+1)^2n} \mid z = [0, \hat{z}_1^T, 0, \ldots\ldots, \hat{z}_i^T, 0]^T,$$
$$\hat{z}_{\ell-1} \in R^{\ell n}, \ell = 2, \ldots, i+1\}.$$

We may summarize the results of this section in the following two theorems: The first being for T equal to an integer, and the second for non integer values of T.

THEOREM 3.2.1. A necessary and sufficient condition for the the control system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable at time $T = i+1$, $i = 0,1,\ldots$ is that the only function $f(t)$, $T-1 \leq t \leq T$ for which $v^i(t) = 0$, $T-1 \leq t \leq T$ and the two point boundary value problem

$$\dot{z}(t) = Pz(t) + Qf(t)$$

for $t \in [T-1,T]$, $z(T-1) \in U$ and $z(T) \in V$, is satisfied, is the zero function, that is $f(t) = 0$, for a.e. $t \in [T-1,T]$.

THEOREM 3.2.2. A necessary and sufficient condition for the control system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable at time $T \in (i,i+1)$, $i = 1, 2,\ldots$ is that the only function $f(t)$, $T-1 \leq t \leq T$ for which $v^{i-1}(t) = 0$, $t \in [T-1,i)$, $v^i(t) = 0$, $t \in [i,T]$ and the two point boundary value problem

$$\dot{z}(t) = Pz(t) + Qf(t)$$

for $t \in [T-1,T]$, $z(T-1) \in U$ and $z(T) \in V$, is satisfied, is the zero function, that is $f(t) = 0$ for a.e. $t \in [T-1,T]$.

By a change of variable we can shift the solution of the two point boundary value problem to the interval $[0,1]$, rather than on the interval $[T-1,T]$.

3.3 The Silverman Structure Algorithm

In the preceding two sections we have presented function space criteria for complete output function space controllability. We were able to show that for some special cases of the control system (3.1.1),(3.1.2),(3.1.3) these function space conditions reduce to algebraic conditions. In the last section we have shown that the conditions for complete output function space controllability reduced to investigating certain properties of a two point boundary value problem.

We will show in the following section that the structure algorithm of Silverman and Payne [33] is intimately related to the present work. Meanwhile in this section we will present their structure algorithm, as well as those of their results which will be used in the succeeding section to develop an algebraic criterion for complete output function space controllability. Since these results will soon be appearing in the literature we will present them without proof.

Let us consider the system of differential equations defined in Theorem 3.2.1

$$\dot{z}(t) = Pz(t) + Qf(t), \tag{3.3.1}$$

$$v^i(t) = R^i z(t), \tag{3.3.2}$$

where we recall that z(t) is an $(i+1)^2 n$ dimensional vector; f(t) is an m dimensional vector; and $v^i(t)$ is an $(i+1)r$ dimensional vector. We can also develop the algorithm in a similar fashion for $v^{i-1}(t) = R^{i-1}z(t)$. Since in the next section we will be considering both cases we will use i-1 or i either as a superscript or a subscript to differentiate explicitly between the two cases.

In this section we will let $p = (i+1)^2 n$ and $u = (i+1)r$ to simplify the notation. We will also denote the vector $v^i(t)$ by $v_o^i(t)$, the matrix R^i by R_o^i and the system (3.3.1),(3.3.2) by Σ_o^i to conform with the notation of Silverman and Payne [33]. We see that P,Q and R_o^i are p×p, p×m, and u×p matrices, respectively.

Following Silverman and Payne [33] we define a sequence of systems Σ_k^i, k = 1,2,.... inductively, where Σ_k^i denotes the k'th system. Let us assume Σ_k^i has the form

$$\dot{z}(t) = Pz(t) + Qf(t) \tag{3.3.3}$$

$$v_k^i(t) = R_k^i z(t) + S_k^i f(t) \tag{3.3.4}$$

where S_k^i has its last $u-q_k^i$ rows equal to zero and we partition v_k^i, R_k^i, and S_k^i as follows:

$$v_k^i(t) = \begin{bmatrix} \bar{v}_k^i \\ \tilde{v}_k^i \end{bmatrix}, \quad R_k^i = \begin{bmatrix} \bar{R}_k^i \\ \tilde{R}_k^i \end{bmatrix}, \quad S_k^i = \begin{bmatrix} \bar{S}_k^i \\ 0 \end{bmatrix},$$

and $\bar{S}_k^i$ has rank q_k^i, and q_k^i rows; $\bar{v}_k^i$ and $\bar{R}_k^i$ have q_k^i rows; and $\tilde{v}_k^i$ and $\tilde{R}_k^i$ have $u - q_k^i$ rows.

Let us introduce the matrix differential operator

$$M_k^i = \begin{bmatrix} I_{q_k^i} & 0 \\ 0 & I_{u-q_k^i}\frac{d}{dt} \end{bmatrix}, \tag{3.3.5}$$

and observe that

$$(3.3.6) \qquad M_k^i v_k^i(t) = \begin{bmatrix} \bar{R}_k^i \\ \tilde{R}_k^i P \end{bmatrix} z(t) + \begin{bmatrix} \bar{S}_k^i \\ \tilde{R}_k^i Q \end{bmatrix} f(t).$$

We note that (3.3.6) is well defined since $\tilde{v}_k^i = \tilde{R}_k^i z(t)$, and hence is differentiable. In (3.3.6) let

$$H_{k+1}^i = \begin{bmatrix} \bar{R}_k^i \\ \tilde{R}_k^i P \end{bmatrix}, \qquad J_{k+1}^i = \begin{bmatrix} \bar{S}_k^i \\ \tilde{R}_k^i Q \end{bmatrix},$$

and let q_{k+1}^i equal the rank of J_{k+1}^i. If $\bar{S}_{k+1}^i$ is the matrix formed from the first q_{k+1}^i linearly independant rows of J_{k+1}^i then there exists a non singular u×u matrix G_{k+1}^i such that

$$(3.3.7) \qquad G_{k+1}^i J_{k+1}^i = \begin{bmatrix} \bar{S}_{k+1}^i \\ 0 \end{bmatrix}.$$

The system Σ_{k+1}^i is then defined by

$$(3.3.8) \qquad \dot{z}(t) = Pz(t) + Qf(t),$$

$$(3.3.9) \qquad v_{k+1}^i z(t) = R_{k+1}^i z(t) + S_{k+1}^i f(t),$$

where $v_{k+1}^i(t) = G_{k+1}^i M_k^i v_k^i(t)$, $R_{k+1}^i = G_{k+1}^i H_{k+1}^i$ and $S_{k+1}^i =$

$G_{k+1}^i J_{k+1}^i$. Hence $v_k^i(t) = N_k^i v^i(t)$ where

$$(3.3.10) \qquad N_k^i = \prod_{j=0}^{k} G_{k-j}^i M_{k-j-1}^i, \quad k = 1,2,\ldots \quad (M_{-1} \triangleq I)$$

is a sequence of non-singular matrix differential operators.

We define $\bar{N}_k^i$ and $\tilde{N}_k^i$ by $\bar{v}_k^i = \bar{N}_k^i v^i$ and $\tilde{v}_k^i = \tilde{N}_k^i v^i$.

It is obvious that, in general, the matrices G_k^i are not unique; but we will present a useful method of constructing them. We define G_{k+1}^{i*} as the unique elementary matrix such that $G_{k+1}^{i*} J_{k+1}$ has for its first q_{k+1}^i rows, the first q_{k+1}^i linearly independant rows of J_{k+1}^i with their relative order unchanged, and its last $u-q_{k+1}^i$ rows are the remaining rows of J_{k+1}^i, again with their order unchanged. Therefore

$$(3.3.11) \qquad G_{k+1}^{i*} = \begin{bmatrix} I_{q_k^i} & 0 \\ 0 & \bar{Y}_{k+1}^i \\ 0 & \tilde{Y}_{k+1}^i \end{bmatrix},$$

where $\bar{Y}_{k+1}^i$ has $q_{k+1}^i - q_k^i$ rows, and $\tilde{Y}_{k+1}^i$ has $u-q_{k+1}^i$ rows. We note that

$$(3.3.12) \qquad Y_{k+1}^i = \begin{bmatrix} \bar{Y}_{k+1}^i \\ \tilde{Y}_{k+1}^i \end{bmatrix}$$

is an elementary matrix. Let us define

$$(3.3.13) \qquad \bar{S}_{k+1}^i = \begin{bmatrix} \bar{S}_k^i \\ \bar{Y}_{k+1}^i \tilde{R}_k^i Q \end{bmatrix}$$

and

$$(3.3.14) \qquad K_{k+1}^{i*} = \tilde{Y}_{k+1}^i \tilde{R}_{k+1}^i Q \, \bar{S}_{k+1}^{i\dagger},$$

where

$$(3.3.15) \qquad \bar{S}_{k+1}^{i\dagger} = \bar{S}_{k+1}^{iT} (\bar{S}_{k+1}^i \bar{S}_{k+1}^{iT})^{-1}.$$

Then we can define G^i_{k+1} as

$$(3.3.16) \qquad G^i_{k+1} = \begin{bmatrix} I_{q^i_{k+1}} & 0 \\ -K^{i*}_{k+1} & I_{u-q^i_{k+1}} \end{bmatrix} G^{i*}_{k+1} .$$

It will be observed that G^i_{k+1} as defined by (3.3.11) to (3.3.16) achieves the desired transformation of the matrix J^i_{k+1}, and that

$$(3.3.17) \qquad \bar{R}^i_{k+1} = \begin{bmatrix} \bar{R}^i_k \\ \bar{Y}^i_{k+1}\tilde{R}^i_k P \end{bmatrix} ,$$

and

$$(3.3.18) \qquad \tilde{R}^i_{k+1} = \tilde{Y}^i_{k+1}\tilde{R}^i_k P - K^{i*}_{k+1}\bar{R}^i_{k+1} .$$

Let $\alpha_i \leq p$ be the first integer such that $q_{\alpha_i} = q^i_p$, and for the matrix

$$(3.3.19) \qquad L^i_k = [R^{i^T}_o, \tilde{R}^{i^T}_1, \ldots, \tilde{R}^{i^T}_{k-1}]^T$$

let $\beta_i \leq p$ be the first integer such that rank L_{β_i} = rank L_{β_i+1}.

We now present some results given in Reference [33].

LEMMA 3.3.1. There exists an integer δ_i, $\beta_i \leq \delta_i \leq p$ and matrices T^i_k, $k = 0,1,\ldots,\delta_i-1$ such that

$$\tilde{R}^i_\delta = \sum_{k=0}^{\delta_i} T^i_k \left(\prod_{j=k+1}^{\delta_i} Y^i_j\right)\tilde{R}^i_k$$

The proof of this lemma is fairly straightforward, and is presented in [33].

Our next step is to present two fundamental results which give information about the functions which lie in the range of the output of (3.3.1),(3.3.2). To accomplish this task we introduce the following differential operators

$$M^1 = \left(\frac{d^{\delta_i}}{dt^{\delta_i}} - \sum_{k=0}^{\delta_i-1} T_k \frac{d^k}{dt^k}\right)\prod_{j=0}^{\alpha_i} \tilde{Y}^i_j \tag{3.3.20}$$

and

$$M^2 = \sum_{j=0}^{\delta_i}\left(\prod_{\ell=j+1}^{\alpha_i}\tilde{Y}^i_\ell\right) K^i_j \frac{d^{\delta_i-j}}{dt^{\delta_i-j}} - \sum_{j=0}^{\delta_i-1} T^i_j \sum_{k=0}^{j}\left(\prod_{\ell=k+1}^{\alpha_i}\tilde{Y}^i_\ell\right) K^i_k \frac{d^{j-k}}{dt^{j-k}} . \tag{3.3.21}$$

THEOREM 3.3.1. (Range Theorem) A u dimensional vector function $w(t)$ defined for $t \in [T-1,T]$ is in the range of the system (3.3.1),(3.3.2) with initial condition $z(T) = z_o$ if and only if

i) $w \in \mathcal{P}_{\delta_i} = \{v \mid N_{\delta_i} v \in L^2([T-1,T])\}$,

ii) $\tilde{N}^i_j w(t)\big|_{t=T} = \tilde{R}^i_j z_o$ for $j = 0,\ldots,\beta_i-1$, and

iii) $(M^1 - M^2\bar{N}_{\alpha_i})w(t) = 0$ for a.e. $t \in [T-1,T]$.

We can further characterize the function $f(t), T-1 \leq t \leq T$ which will cause the output $v^i(t), T-1 \leq t \leq T$ of (3.3.1) (3.3.2) to equal a function $w(t)$, $T-1 \leq t \leq T$, in the range of (3.3.1),(3.3.2) by an inverse system.

THEOREM 3.3.2. Suppose the function $w(t)$, $T-1 \leq t \leq T$ is in the range of the system (3.3.1),(3.3.2), then the output $v^i(t) = w(t)$ for $t \in [T-1,T]$ if and only if the function $f(t)$ is equal to the output of

$$\dot{y}(t) = (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + Q\bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + QK^i_Q u(t)$$

$$f(t) = -\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t) + \bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + K^i_Q u(t),$$

for a.e. $t \in [T-1,T]$, where $y(T) = z_o$, $u(t)$, $T-1 \leq t \leq T$ is an arbitrary function in $L^2(T-1,T)$, and K^i_Q is a matrix whose columns form a basis for the null space of $\bar{S}_{\alpha_i}$.

From Theorems 3.3.1 and 3.3.2 we immediately deduce the following result.

THEOREM 3.3.3. There exists a function f in $L^2(T-1,T)$ such that in (3.3.1),(3.3.2) $v(t) \equiv 0$, $t \in [T-1,T]$ if and only if z_o is an element of the null space of L_β, and if such a function f exists it is equal to the output of

$$\dot{y}(t) = (P - Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + QK^i_Q u(t), \quad y(T) = z_o$$

$$f(t) = \bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t) + K^i_Q u(t)$$

for a.e. $t \in [T-1,T]$, where u is any element of $L^2([T-1,T])$.

Finally we present the following lemma. Although this result is implied in the work of Silverman and Payne, we present it explicitly as it will be used in the next section.

LEMMA 3.3.2. If in the inverse system defined in Theorem 3.3.2 the initial condition $y(T) = z_o$, the initial condition for the system (3.3.1),(3.3.2), then the solution

$y(t)$, $T-1 \leq t \leq T$ of the inverse system equals the solution $z(t)$, $T-1 \leq t \leq T$ for every function $w(t)$, $T-1 \leq t \leq T$ in the range of (3.3.1),(3.3.2), provided $f(t)$, $T-1 \leq t \leq T$ in (3.3.1),(3.3.2) is set equal to the output of the inverse system.

Proof. From Theorem 3.3.2 the inverse system of (3.3.1) (3.3.2) is

$$(3.3.22) \quad \dot{y}(t) = (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + Q\bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + + QK^{i}_{Q}u(t) , \quad y(t) = z_o$$

and

$$(3.3.23) \quad f(t) = -\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t) + \bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + K^{i}_{Q}u(t) .$$

Let us define $s(t) = y(t) - z(t)$. Therefore from (3.3.22) and (3.3.1)

$$(3.3.24) \quad \dot{s}(t) = \dot{y}(t) - \dot{z}(t)$$

$$= (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + Q\bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + + QK^{i}_{Q}u(t) - Pz(t) + Qf(t).$$

Substituting (3.3.23) into (3.3.24) we obtain

$$(3.3.25) \quad \dot{s}(t) = (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + Q\bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + + QK^{i}_{Q}u(t) - [\, Pz(t) - Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t) + + Q\bar{S}^{\dagger}_{\alpha_i}\bar{N}_{\alpha_i}w(t) + QK^{i}_{Q}u(t)]$$

$$= Ps(t).$$

Since $s(T) = y(T) - z(T) = 0$, we conclude $s(t) \equiv 0$, $T-1 \leq t \leq T$. Hence $y(t) = z(t)$ for $t \in [T-1,T]$.

3.4 An Algebraic Criterion For Complete Output Function Space Controllability

In this section we present an algebraic criterion for complete output function space controllability. The approach to be used is based on the results of Silverman and Payne presented in the previous section, and the two point boundary value problem given in Theorem 3.2.1. For convenience we treat the problem for two separate cases: namely for $T = i+1$, $i = 0,1,\ldots,$ and for $T \in (i,i+1)$, $i = 1, 2,\ldots$.

THEOREM 3.4.1. A necessary and sufficient condition for the control system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable at time $T = i+1$, $i = 0, 1,\ldots$ is that $q_{\alpha_i} = m$ and

$$U \cap e^{-\Xi_i} W = \{0\}$$

where $W = \text{null}\,[L_{\beta_i}] \cap V$. U and V are the linear subspaces defined in Section 2; L_{β_i} is the matrix defined in Section 3; and $\Xi_i = P - Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}$ (see Sections 2 and 3 for definitions of $P, Q, \bar{S}^{\dagger}_{\alpha_i}$ and $\bar{\bar{R}}_{\alpha_i}$).

Proof. We first prove the necessity of the above result. Let us suppose $q_{\alpha_i} < m$. We will now show there exists a non-zero function $f(t)$, $T-1 \leq t \leq T$ such that $v^i(t) = 0$, $T-1 \leq t \leq T$, and the two point boundary value problem in

Theorem 3.2.1 is satisfied, thus contradicting the fact that the system (3.1.1),(3.1.2),(3.1.3) is completely output function space controllable.

Let us take $z(T) = 0$. From Theorem 3.3.3, since $z(T) \in$ null $[L_{\beta_i}]$, we see that the class of functions $f(t)$, $T-1 \leq t \leq T$ which result in $v^i(t) = 0$, $T-1 \leq t \leq T$, is given by all the solutions of

$$(3.4.1) \quad \dot{y}(t) = (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) + QK^i_Q u(t), \quad y(T) = z(T) = 0$$

$$(3.4.2) \quad f(t) = \bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t) + K^i_Q u(t),$$

where $u(t)$, $T-1 \leq t \leq T$ is any square integrable function. To simplify the notation let us define $\Gamma_i = \bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}$, $\Pi_i = QK^i_Q$ and $\Psi_i = K^i_Q$, and then (3.4.1),(3.4.2) become

$$(3.4.3) \quad \dot{y}(t) = \Xi_i y(t) + \Pi_i u(t), \qquad y(T) = 0$$

$$(3.4.4) \quad f(t) = \Gamma_i y(t) + \Psi_i u(t).$$

We now apply the Silverman algorithm to (3.4.3),(3.4.4) to determine the class of functions $u(t)$, $T-1 \leq t \leq T$ for which $f(t) = 0$, $T-1 \leq t \leq T$. We recall that $u(t)$ is an $(m-q_{\alpha_i})$ dimensional vector, $\Psi_i = K^i_Q$ has rank $(m-q_{\alpha_i})$, and $y(T) = 0 \in \mathrm{null}[L'_{\beta_i}]$ where L'_{β_i} denotes the L_{β_i} matrix for (3.4.3),(3.4.4). Hence from Theorem 3.3.3 the function $u(t)$, $T-1 \leq t \leq T$ must equal the output of

$$(3.4.5) \quad \dot{y}'(t) = (\Xi_i - \Pi_i \bar{\Psi}^{\dagger}_{\alpha'_i}\bar{\Gamma}_{\alpha'_i})y'(t) \qquad y'(T) = 0$$

$$u(t) = \bar{\Psi}^{\dagger}_{\alpha_i'} \bar{\Gamma}_{\alpha_i'} y'(t), \tag{3.4.6}$$

for a.e. $t \in [T-1,T]$, where the prime (') denotes the inverse system with respect to the system (3.4.3),(3.4.4). Therefore we conclude that the only function $u(t)$, $T-1 \leq t \leq T$ for which $f(t) = 0$, $T-1 \leq t \leq T$ in (3.4.3), (3.4.4) is when $u(t) = 0$, $T-1 \leq t \leq T$.

Now since $q_{\alpha_i} < m$, we know $K_Q^i \neq 0$, and hence by the use of a non-singular transformation T, we can decompose (3.4.3) into a completely controllable part and a completely uncontrollable part, namely

$$\dot{\bar{y}}_1(t) = \Xi_{11}\bar{y}_1(t) + \Xi_{12}\bar{y}_2(t) + \Pi_1 u_1(t) \tag{3.4.7}$$

$$\dot{\bar{y}}_2(t) = \Xi_{22}\bar{y}_2(t) \tag{3.4.8}$$

where $\bar{y}(t) = [\bar{y}_1^T(t), \bar{y}_2^T(t)]^T = Ty(t)$, and $u(t) = [u_1^T(t), u_2^T(t)]^T$. Hence there exists a function $u(t)$, $T-1 \leq t \leq T$ not identically zero such that $y(T-1) = y(T) = 0$. If (3.4.3) is completely uncontrollable then the choice of the non-zero function $u(t)$, $T-1 \leq t \leq T$ is arbitrary. Since $u(t)$, $T-1 \leq t \leq T$ is not identically zero we see from above that $f(t)$, $T-1 \leq t \leq T$ is not identically zero. Hence we have constructed a non-zero function $f(t)$, $T-1 \leq t \leq T$ such that $v^i(t) = 0$, $T-1 \leq t \leq T$, $z(T) = 0$ and from Lemma 3.3.2 $z(T-1) = 0$. Hence we obtain a contradiction of Theorem 3.2.1.

Let us now suppose $q_{\alpha_i} = m$, but $U \cap e^{-\Xi_i}W \neq \{0\}$. Hence there exists a non-zero vector $x \in R^p$ such that $x \in \text{null}[L_{\beta_i}]$, $x \in V$ and $e^{-\Xi_i}x \in U$. Since $x \in \text{null}[L_{\beta_i}]$, from Theorem 3.3.3, there exists a function $f(t)$, $T-1 \leq t \geq T$ so that in Theorem 3.2.1, the function $v^i(t) = 0$, $T-1 \leq t \leq T$. According to Theorem 3.3.3 this function $f(t)$ is equal to the output of

(3.4.9) $$\dot{y}(t) = \Xi_i y(t) \qquad y(T) = z(T) = x \in V,$$

(3.4.10) $$f(t) = \Gamma_i y(t)$$

for a.e. $t \in [T-1,T]$. From Lemma 3.3.2, $z(t) = y(t)$ for $t \in [T-1,T]$ and hence $z(T-1) = y(T-1) = e^{-\Xi_i}x \in U$. We now show that the function $f(t)$, $T-1 \leq t \leq T$ given by (3.4.9), (3.4.10) is not identically equal to zero. Suppose $f(t) = 0$, $T-1 \leq t \leq T$, and recall that $z(T) = x \neq 0$. Then we see from (3.2.16) that for some $j = 1,\ldots,i$ the component $\hat{z}_j(T)$ of $z(T)$ does not equal zero. Hence from (3.2.9)

(3.4.11) $$\frac{d}{dt}\hat{z}_j(t) = -A_{j-1}^T\hat{z}_j(t) \qquad \hat{z}_j(T) \neq 0,$$

and therefore $\hat{z}_j(T-1) \neq 0$. Hence $z(T-1) \notin U$ and so the boundary conditions of the two point boundary value problem are not satisfied which is a contradiction. Thus we have again constructed a non-zero function $f(t)$, $T-1 \leq t \leq T$ such that $v^i = 0$, $T-1 \leq t \leq T$ and the boundary conditions of the two point boundary value problem are satisfied, which contradicts Theorem 3.2.1.

Let us now turn to the proof of sufficiency. Since $q_{\alpha_i} = m$, and in order that $v^i(t) = 0$, $T-1 \leq t \geq T$ as in Theorem 3.2.1, we find from Theorem 3.3.3 that we must have $z(T) \in \text{null}[L_{\beta_i}]$ and $f(t)$ equal to the output of

$$(3.4.12) \qquad \dot{y}(t) = (P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i})y(t) \qquad y(T) = z(T),$$

$$(3.4.13) \qquad f(t) = \bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}y(t)$$

for a.e. $t \in [T-1,T]$. Now suppose $z(T) = z_o$, where $z_o \in V$ is a non-zero vector. Then from the condition $U \cap e^{-\Xi_i}W = \{0\}$ we see either $z_o \notin \text{null}[L_{\beta_i}]$ or $z_o \notin e^{-\Xi_i}U$. If $z_o \notin \text{null}[L_{\beta_i}]$ then $v^i(t)$, $T-1 \leq t \leq T$ is not identically zero, which is a contradiction. Also if $z_o \notin e^{-\Xi_i}U$ then then the two point boundary value problem is not satisfied, which is again a contradiction. Hence $z(T) = 0$ is the only admissible boundary condition. Therefore from (3.4.12),(3.4.13) we find that $f(t) = 0$, $T-1 \leq t \leq T$, and so the system (3.1.1), (3.1.2),(3.1.3) is completely output function space controllable.

We now state without proof the result for $T \in (i,i+1)$, $i = 1,2,\ldots$.

THEOREM 3.4.2. A necessary and sufficient condition for the control system (3.1.1),(3.1.2),(3.1.3) to be completely output function space controllable at time $T \in (i,i+1)$, $i = 1, 2,\ldots$ is that $q_{\alpha_{i-1}} = m$ and

$$U \cap e^{-\Xi_i(i-T+1)}\left\{\text{null}[L_{\beta_i}] \cap e^{-\Xi_{i-1}(T-i)}(\text{null}[L_{\beta_{i-1}}] \cap V)\right\} = \{0\}.$$

U and V are the linear subspaces defined in Section 2; $L_{\beta_{i-1}}$ and L_{β_i} are the matrices defined in Section 3, for the outputs $v^{i-1}(t)$ and $v^i(t)$, $T-1 \leq t \leq T$, respectively; and $\Xi_{i-1} = P-Q\bar{S}^{\dagger}_{\alpha_{i-1}}\bar{R}_{\alpha_{i-1}}$ and $\Xi_i = P-Q\bar{S}^{\dagger}_{\alpha_i}\bar{R}_{\alpha_i}$ (see Sections 2 and 3 for definitions of P, Q, $\bar{S}^{\dagger}_{\alpha_{i-1}}$, $\bar{S}^{\dagger}_{\alpha_i}$, $\bar{R}_{\alpha_{i-1}}$, and $\bar{R}_{\alpha_i}$).

Proof. The proof follows along similar lines to that of Theorem 3.4.1 excepting for some obvious modifications and so will not be repeated here.

3.5 Examples

In this section we present two examples to illustrate the application of the results presented in this chapter. Both these examples arise from population growth models used in demography where a control variable has been included.

Example 3.5.1. Let us consider the following single sex population model

$$(3.5.1) \qquad \dot{x}(t) = (\beta-\delta)x(t) - \beta e^{-\delta}x(t-1) + u(t),$$

where x(t) is the number of births at time t, β is the birth-rate and δ is the death-rate. A question of interest is whether (3.5.1) is completely function space controllable? From Theorem 3.1.3 it is immediately obvious that (3.5.1) is completely function space controllable for all $T \geq 1$, since the coefficient multiplying the control is unity.

Example 3.5.2. In this example we consider the following two sex female dominant population model

$$(3.5.2)\quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} \beta_1-\delta_1 & 0 \\ \beta_2 & -\delta_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} - \begin{bmatrix} \beta_1 e^{-\delta_1} & 0 \\ \beta_2 e^{-\delta_1} & 0 \end{bmatrix} \begin{bmatrix} x_1(t-1) \\ x_1(t-1) \end{bmatrix} + \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

where $x_1(t)$, $x_2(t)$ are the number of female births and male births at time t respectively; β_1 and β_2 are the female and male birth rates respectively; and δ_1 and δ_2 are the female and male death rates respectively. For this equation we wish to investigate whether it is completely function space controllable at time $T = 2$. It is clear that we have to use the general results presented in Theorem 3.4.1. Since $T = 2$ we see that $i = 1$ in Theorem 3.4.1. Hence we may write the two point boundary value problem in Theorem 3.2.1, as

$$(3.5.3)\qquad \dot{z}(t) = Pz(t) + Qf(t),$$

$$(3.5.4)\qquad v^1(t) = R^1 z(t)$$

where

$$P = \begin{bmatrix} A^T & 0 & 0 & 0 \\ 0 & A^T & 0 & 0 \\ 0 & 0 & A^T & B^T \\ 0 & 0 & 0 & A^T \end{bmatrix}, \quad Q = \begin{bmatrix} -I \\ e^{AT} \\ 0 \\ -I \end{bmatrix}, \quad R^1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & I & e^{AT} \end{bmatrix},$$

$$B = \begin{bmatrix} \beta_1 e^{-\delta_1} & 0 \\ \beta_2 e^{-\delta_1} & 0 \end{bmatrix}, \text{ and } A = \begin{bmatrix} \beta_1 - \delta_1 & 0 \\ \beta_2 & -\delta_1 \end{bmatrix}.$$

Applying the Silverman algorithm we find $q_{\alpha_1} = 2 = m$. However in order to test the second condition of Theorem 3.4.1 we need to find L_{β_1}. The procedure becomes quite involved, and it proves very difficult to derive any general conclusions about (3.5.2) by this procedure.

However by the following indirect approach we can come to some interesting conclusions without resort to Theorem 3.4.1. The first equation in (3.5.2) is not coupled to the second and hence is completely function space controllable at any time $T \geq 1$, by Theorem 3.1.3. The second equation is coupled to the first equation by the term

$$\beta_2 x_1(t) - \beta_2 e^{-\delta_1} x_1(t-1). \tag{3.5.5}$$

We can decompose the control $u_2(t)$ into two parts:

$$u_2(t) = u_2'(t) + u_2''(t)$$

where the first term $u_2'(t)$ is used to cancel the term (3.5.5), and the second term $u_2''(t)$ is used to control what is now an ordinary differential equation. Hence by Theorem 3.1.3 the second equation in (3.5.2) is also completely function space controllable for any $T \geq 1$, and so (3.5.2) is complelely function space controllable at any time $T \geq 1$.

Hence we see that by the use of this indirect approach we can obtain some definite conclusions about the function space controllability of (3.5.2). If however we were to constrain $u_2(t)$ to be identically zero there seems to be no other recourse but to use Theorem 3.4.1 directly.

CHAPTER 4

SUMMARY AND CONCLUSIONS

As indicated in the Introduction we have to distinguish between the notions of controllability in euclidean space and controllability in function space for differential-difference equations. In this thesis we have given a full discussion of euclidean space controllability of constant coefficient differential-difference equations, and a somewhat less than complete discussion of function space controllability of these equations. In this final chapter we will discuss and summarize the results obtained and present some problems for further research.

4.1. Euclidean Space Controllability

The new form for the fundamental solution of a differential-difference equation has been used to obtain an algebraic necessary and sufficient condition for the euclidean space controllability of these equations. By the use of the fundamental solution we were able to obtain algebraic necessary and sufficient conditions for pointwise completeness. The attractive feature of the new form for the fundamental solution is that it is expressed in terms of an exponential matrix, which enables us to obtain results which have a similar form to those for ordinary differential equations.

The main new results are the algebraic condition for complete controllability in euclidean space and the algebraic condition for pointwise completeness. The condition obtained for complete controllability in euclidean space is closely related to the result obtained by Johnson [24]. In fact the author has been able to show that when the differential-difference equation is second order then Johnson's results and our results are equivalent. However for higher order differential-difference equations the author has been unable, up to the present time, to find a simple proof for the equivalence of Johnson's and our results. The main advantage of using our form for the fundamental solution of constant coefficient differential-difference equations is the simplicity with which we may present the results and the ease of proof. It was necessary for Johnson to introduce a large amount of new notation from automata theory in order to be able to write the requisite products of non-commutating matrices in a concise fashion. Our condition for controllability in euclidean space was also shown to reduce to the Kalman condition [2] for ordinary differential equations and the result of Kirillova and Curakova [21] for the case where $A = 0$ in (1.1.1).

For pointwise completeness there seems to be no other alternative but to use the new form for the fundamental solution in order to obtain an algebraic criterion. We were also able to show that if the matrices A and B in (1.1.1) commute, i.e. $AB = BA$, then the differential-difference equation is pointwise complete; a result which has also recently been obtained by Brooks and Schmitt [39]. Of course this includes the well known cases of ordinary differential equations, and when A = 0 in (1.1.1). Finally we were able to characterize the set of all points which could be 'reached' from the class of initial functions, by using the algebraic criterion for pointwise completeness. This enabled us to obtain for the first time an algebraic necessary and sufficient condition for null controllability in euclidean space.

There are a number of directions in which these results may be extended. It seems that for multiple delays the extension is relatively straight forward when the delays are rationally related. When the delays are irrationally related the results should still be able to be extended, although the notation would become rather cumbersome. A somewhat more interesting extension is to the case of time-varying differential-difference equations. For this case it seems almost essential to use the new form for the fundamental solution. It seems probable that a necessary and sufficient condition for euclidean space controllability can be obtained when the coefficients in (1.1.1) are analytic. For pointwise completeness it also seems possible to extend our results for the case of multiple delays, and for time-varying differential-difference equations. It is readily apparent that the conditions for controllability that we have presented have much algebraic structure. Hence it seems that a fruitful line of enquiry may be the investigation of the algebraic structure of these conditions along the lines of the recent work of Kalman [5]. Finally it should be possible to extend these results to the class of problems where we have discrete delays in the control of the differential-difference equations.

4.2. Function Space Controllability

Algebraic necessary and sufficient conditions for the complete output function space controllability of differential-difference equations have been obtained by transforming the problem into a problem concerning the uniqueness of the solution of a two point boundary value problem. These results are related to the work of Kirillova and Curakova [21], and in a sense generalize their results. For instance we are able to obtain, for the case $A = 0$ in (1.1.1), a result which is formally the same as their condition. However their condition is necessary and sufficient for function space null controllability, while our result is for complete function space controllability.

Whereas for euclidean space controllability the notion is fairly clear-cut and natural, in the case of complete function space controllability there are a number of suitable definitions. A cursory consideration should convince the reader that the class of functions in which the system (1.1.1) is to be controlled cannot be arbitrary if we wish to be able to control to every function in the class. Popov [28] resolved this problem by restricting the class of functions to be sufficiently differentiable. Our approach has been somewhat different and has been principally motivated by the desire to retain the geometric ideas which have proved so useful in euclidean space, albeit in a function space setting. With this desire in mind, we have been able to arrive at a new function space condition for complete output function space controllability which is formally analogous to the function space condition for euclidean space controllability.

From this function space condition for complete output function space controllability we are able to obtain simple algebraic condtions for the case where A = 0 in (3.1.1),(3.1.2),(3.1.3), and for the case where (3.1.1), (3.1.2),(3.1.3) has a scalar input and a scalar output. For the first case the condition is formally the same as that of Kirillova and Curakova [21], as was mentioned above. For the second case the result has not appeared in the literature, to the author's knowledge, and is believed to be entirely new. It will be noted however that it is formally similar to the invertability condition given by Brockett [38].

This observation prompted us to investigate whether in fact an algebraic condition could also be obtained for the more general case, where (3.1.1),(3.1.2),(3.1.3) is not a single input-single output system. It has in fact been shown in Chapter 3 that such an algebraic condition can be obtained. The approach taken was to transform the function space condition for complete output function space controllability into an equivalent two point boundary value problem, with certain subsidiary conditions. Utilizing some recent results of Silverman and Payne [33] on the input-output structure of linear systems we are able to obtain the main results of Chapter 3, which are new algebraic necessary and sufficient conditions for complete output function space controllability.

There are still a number of outstanding problems in function space controllability which have yet to be solved. Probably the most important problem is that of function space null controllability. It is felt by the author that a similar approach to that of Chapter 3 may yield an answer to this problem. Also it would be of interest to extend these results to the case of multiple delays, and to the case of time-varying differential-difference equations. Finally there is the computational problem! An examination of the algorithm given in Chapter 3, indicates that even for a quite modest problem, the matrices that have to be handled soon become enormous. However these matrices are rather sparse, and it may prove possible to take advantage of this property to simplify computations.

APPENDIX

PROOF OF LEMMA 3.1.1.

In this appendix we present a proof of Lemma 3.1.1. As was indicated in Chapter 3 Fattorini [35] stated the lemma without proof.

For convenience let us repeat:

DEFINITION 3.1.2. If X is a Banach space, and M is a subset of X, then the set

$$M^{\perp} = \{x' \in X' \mid (x',x) = 0, \text{ for every } x \in M\}$$

is called the orthogonal complement of M. X' denotes the dual space of the Banach space X.

LEMMA 3.1.1. Suppose the set M is a linear subspace of the Banach space X. Then M is dense in X, with the norm topology, if and only if the orthogonal complement is empty excepting for the zero vector.

<u>Proof</u>. We will prove the necessity of the above result by contradiction. Suppose M is dense in X, then for every $\varepsilon > 0$, and for every $x_0 \in X$, there exists a $u \in M$ such that $\|x_0 - u\| < \varepsilon$. Further let us suppose $M^{\perp}$ contains a non-zero vector $u' \in X'$; that is there exists a non-zero vector $u' \in X'$ such that $(u',u) = 0$ for every $u \in M$. Therefore we find

(A.1) $|(u',x_o)| = |(u',x_o-u)| \leq \|u'\|\|x_o-u\| < \varepsilon\|u'\|$,

for every $x_o \in X$. Now since ε is an arbitrary positive number, we see that $|(u',x_o)| = 0$ for every x_o. Therefore we conclude $u' = 0$, which contradicts the assumption $M^{\perp}$ contains a non-zero vector.

We also prove the sufficiency of the above result by contradiction. Suppose M is not dense in X, then there exists an $\varepsilon_o > 0$, and an $x_o \in X-M$ such that for every $u \in M$, $\|x_o-u\| \geq \varepsilon_o$. We now construct a non-zero functional u' such that u' is an element of M', hence showing $M^{\perp}$ contains a non-zero vector.

Since for every $u \in M$, $\|x_o-u\| \geq \varepsilon_o$, we see that

(A.2) $$\inf_{u \in M}\|x_o-u\| \geq \varepsilon_o > 0.$$

Let us define $\inf_{u \in M}\|x_o-u\| = \rho$. We now construct a new linear subspace, M_o, of X by defining

(A.3) $M_o = \{y \in X \,|\, y = \lambda x_o + \bar{x},\ \bar{x} \in M, \text{ and } \lambda \text{ is a scalar}\}$.

On the space M_o we define the following bounded linear functional u_o':

$$(u_o',x) = \lambda d, \text{ for every } x \in M_o.$$

It is clear that u_o' is a linear functional. We now show that u_o' is bounded. For every $x \in M_o$ such that $x = \lambda x_o + \bar{x}$ we have

(A.4) $$|(u_o',x)| = |\lambda|\,\rho = |\lambda|\inf_{u \in M}\|x_o-u\|$$

$$\leq |\lambda| \| x_0 - (-\tfrac{1}{\lambda})\bar{x} \| = \| x \| .$$

By the Hahn Banach theorem there exists a continuous linear functional $u' \in X'$ defined on X such that u' is an extension of u'_0. That is for every $x \in M_0$, $(u',x) = (u'_0,x)$, where $u' \in X'$. Since $(u'_0,x) = 0$ for every $x \in M$, and $(u'_0,x_0) = \rho$, we see that $u' \in X'$ is a non-zero functional and $u' \in M^{\perp}$, hence showing that $M^{\perp}$ contains a non-zero vector.

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