

Hamilton–Jacobi counterterms for Einstein–Gauss–Bonnet gravity

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Abstract

The on-shell gravitational action and the boundary stress tensor are essential ingredients in the study of black hole thermodynamics. We employ the Hamilton–Jacobi method to calculate the boundary counterterms necessary to remove the divergences and allow the study of the thermodynamics of Einstein–Gauss–Bonnet black holes.

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1. Introduction

The AdS/CFT conjecture has led to a renewed interest in the study of black hole thermodynamics. In this new framework, the thermal properties of an AdS black hole configuration are dual to that of the finite temperature CFT. An important example is the Hawking–Page phase transition [1] for black holes in AdS which corresponds to a deconfinement transition in the dual field theory [2]. In order to study black hole thermodynamics, it is standard to evaluate the on-shell gravitational action and the boundary stress tensor. The on-shell value of the action (which we denote as Γ) is identified with the thermodynamic potential Ω according to $\Gamma = \beta\Omega$. Moreover, for static backgrounds with the time-like Killing vector $\partial/\partial t$, the energy E is given by the ADM mass, extracted from the tt component of the boundary stress tensor. Though one expects that the thermodynamical laws are satisfied in general, an important complication is that both Ω and E are divergent quantities and require regularization.

An approach to regularization suggested in [3] is to subtract the action of a reference spacetime from the action for the spacetime of interest. Under appropriate matching conditions, the divergences in both actions will cancel, thus leading to finite quantities of interest. Although this approach is useful in many cases, it becomes problematic when the appropriate reference

background cannot be found, or when there is a potential ambiguity in the matching conditions. In the framework of AdS/CFT, an alternative method for removing infinities was developed in [4–6]. Inspired by renormalization in the dual CFT, this method involves the addition of a set of covariant boundary counterterms that remove all power-law divergences from the on-shell action. While the values of the counterterms were originally chosen simply to remove divergences, a subsequent refinement of holographic renormalization came about when it was realized that the Hamilton–Jacobi formalism may be used to determine the structure and normalization of these counterterms [7].

Black hole thermodynamics in pure Einstein gravity with a cosmological constant has been extensively studied, especially in light of the AdS/CFT correspondence. In its simplest form, this correspondence relates the $\mathcal{N} = 4$ $SU(N)$ super-Yang–Mills theory in four dimensions to the IIB string theory on $AdS_5 \times S^5$. In the limit of large N and infinite 't Hooft coupling, the gravitational dual simply reduces to $\mathcal{N} = 8$ gauged supergravity in five dimensions. Motivated by this AdS/CFT picture, the Hamilton–Jacobi formalism [7–9] was employed in [10] to study the thermodynamics of asymptotically AdS black holes in various dimensions, $d = 4, 5, 6, 7$.

From an AdS/CFT perspective, it is of natural interest to examine the finite 't Hooft coupling corrections to the familiar infinite coupling results. These corrections originate from higher derivative terms in the α' expansion of the string effective action; in the gravitational sector, they take the form $\alpha'^m R^{m+1}$ where R corresponds to the Riemann tensor and its contractions. While the first corrections in the maximally supersymmetric (i.e. type II) theories do not enter until the $\alpha'^3 R^4$ order, generically the first non-trivial terms show up at the curvature-squared level

$$e^{-1} \delta \mathcal{L} = \alpha_1 R^2 + \alpha_2 R_{\mu\nu}^2 + \alpha_3 R_{\mu\nu\rho\sigma}^2. \quad (1.1)$$

By making an appropriate field redefinition of the form $g_{\mu\nu} \rightarrow g_{\mu\nu} + a R_{\mu\nu} + b g_{\mu\nu} R$, we may shift the coefficients α_1 and α_2 to arbitrary values. Thus only α_3 may affect physical observables. This allows us to take the Gauss–Bonnet combination

$$e^{-1} \delta \mathcal{L} = \alpha (R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2), \quad (1.2)$$

which is the unique combination of curvature-squared terms which does not propagate ghosts [11, 12]. Of course, the presence of ghosts (whose effects do not show up until the string scale) is not a major concern at the effective supergravity level, where the complete string theory serves as an appropriate UV completion. Nevertheless, the Gauss–Bonnet combination is particularly amenable to holographic renormalization and the study of boundary field theories as it admits a well-defined Cauchy problem for radial evolution.

It is the purpose of our present work to apply holographic renormalization to the Einstein–Gauss–Bonnet action, and in particular to apply the Hamilton–Jacobi method to derive a set of universal counterterms renormalizing this action. Local counterterms for higher derivative gravities, including the Gauss–Bonnet combination, have previously been considered in [13–18], and more recently in [19–21]. In addition, a complementary ‘Kounterterm’ regularization scheme was developed in [22–27], which involves the introduction of boundary counterterms built out of the extrinsic curvature tensor. (This approach is more naturally associated with a variational principle where the extrinsic curvature is kept fixed on the boundary.)

We organize our work as follows. In the following section, we review the Hamiltonian formulation (for radial evolution) and we evaluate the Hamiltonian of the Einstein–Gauss–Bonnet theory to first order in α , the coefficient of the Gauss–Bonnet term in the bulk action. In section 3, we derive the Hamilton–Jacobi counterterms, and in section 4 we compare our results to previous investigations of Gauss–Bonnet black hole thermodynamics. Finally, we conclude with a discussion in section 5.

2. Einstein–Gauss–Bonnet gravity

It is well known that, in general, higher curvature gravitational actions lead to potentially undesirable features such as ghosts as well as difficulties in formulating the Cauchy problem because of the appearance of higher order derivatives of the metric. However, as shown by Lovelock, these difficulties may be avoided by taking particular combinations of the higher curvature terms corresponding to d -dimensional continuations of the lower dimensional Euler densities [11]. The family of Lovelock actions then take the form

$$S_{\text{bulk}} = \sum_{k=0}^{d/2} \alpha_k S_{\text{bulk}}^{(k)}, \quad (2.1)$$

where [11, 28]

$$S_{\text{bulk}}^{(k)} = -\frac{1}{2k!} \int_{\mathcal{M}} d^d x \sqrt{-g} \delta^{[\mu_1 \dots \mu_{2k}]_{[\nu_1 \dots \nu_{2k}]}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \dots R^{\nu_{2k-1} \nu_{2k}}_{\mu_{2k-1} \mu_{2k}}. \quad (2.2)$$

Note that we have included $k = 0$, corresponding to a possible cosmological constant. In particular, the first few terms in the expansion of the Lovelock action give

$$S_{\text{bulk}} = - \int_{\mathcal{M}} d^d x \sqrt{-g} [\alpha_0 + \alpha_1 R + \alpha_2 (R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2) + \dots]. \quad (2.3)$$

Truncating the Lovelock theory at this level gives what may be referred to as Einstein–Gauss–Bonnet gravity.

Since the Lovelock theory gives rise to equations of motion involving no higher than second derivatives of the metric, it is possible to formulate a well-defined variational principle by adding to (2.1) a set of generalized Gibbons–Hawking surface terms

$$S_{\text{GH}} = \sum_{k=1}^{d/2} \alpha_k S_{\text{GH}}^{(k)}. \quad (2.4)$$

In particular

$$S_{\text{GH}}^{(1)} = -2 \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{-h} K \quad (2.5)$$

is the usual Gibbons–Hawking term and

$$S_{\text{GH}}^{(2)} = 4 \int_{\partial\mathcal{M}} d^{d-1} x \sqrt{-h} [2\mathcal{G}_{ab} K^{ab} + \frac{1}{3} (K^3 - 3K K_{ab}^2 + 2K_a^b K_b^c K_c^a)] \quad (2.6)$$

is a generalized Gibbons–Hawking term [28, 29]. (Of course no boundary term is needed for the $k = 0$ cosmological constant term.) Since we are focused on holographic renormalization, we single out a ‘radial’ coordinate r so that the boundary is reached as $r \rightarrow \infty$, in which the case $1/r$ will ultimately be usable as an asymptotic expansion parameter. In particular, we choose a foliation of spacetime by constant r hypersurfaces, orthogonal to a spacelike unit normal n^μ . The boundary metric is then given by $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, and $K_{\mu\nu}$ is the extrinsic curvature tensor defined by $K_{\mu\nu} = \nabla_{(\mu} n_{\nu)}$. In addition, \mathcal{G}_{ab} is the Einstein tensor constructed from the boundary metric, $\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2} h_{ab} \mathcal{R}$.

2.1. Hamiltonian formulation

Consistent with the foliation of spacetime with constant r hypersurfaces, we may take the above Lovelock action and derive the corresponding Hamiltonian for radial evolution. This was in fact done in [28] for the case of time evolution (which is easily related to radial evolution

by an appropriate analytic continuation). Although the following results are contained in [28], we nevertheless provide some details for clarity of exposition.

To derive the Hamiltonian, we first use the Gauss–Codacci equations for the r -foliation to rewrite the action (2.1) in terms of invariants built from the intrinsic and extrinsic curvatures $\mathcal{R}^a{}_{bcd}$ and K_{ab} . For hypersurfaces specified by a spacelike normal, the relevant Gauss–Codacci equation is

$$\bar{R}_{\mu\nu\rho\sigma} \equiv h_{\mu}^{\mu'} h_{\nu}^{\nu'} h_{\rho}^{\rho'} h_{\sigma}^{\sigma'} R_{\mu'\nu'\rho'\sigma'} = \mathcal{R}_{\mu\nu\rho\sigma} - K_{\mu\rho} K_{\nu\sigma} + K_{\mu\sigma} K_{\nu\rho}. \quad (2.7)$$

In this case, we find

$$\begin{aligned} S_{\text{bulk}}^{(1)} + S_{\text{GH}}^{(1)} &= \int_{\mathcal{M}} d^d x \sqrt{-g} [\mathcal{R} + K^2 - K_{ab}^2], \\ S_{\text{bulk}}^{(2)} + S_{\text{GH}}^{(2)} &= \int_{\mathcal{M}} d^d x \sqrt{-g} \left[(\mathcal{R} + K^2 - K_{ab}^2)^2 - 4(\mathcal{R}_{ab} + K K_{ab} - K_{ac} K_b^c)^2 \right. \\ &\quad + (\mathcal{R}_{abcd} + K_{ac} K_{bd} - K_{ad} K_{bc})^2 - \frac{4}{3} K^4 + 8K^2 K_{ab}^2 \\ &\quad \left. - \frac{32}{3} K K_a^b K_b^c K_c^a - 4(K_{ab}^2)^2 + 8K_a^b K_b^c K_c^d K_d^a \right], \end{aligned} \quad (2.8)$$

where we have introduced a shorthand notation that when tensor quantities are squared, they are to be contracted with themselves without any rearrangement of indices. Note, in particular, that the original surface terms (2.4) are absorbed after the Gauss–Codacci rewriting of the action.

It is now straightforward, at least in principle, to derive the conjugate momenta π^{ab} for radial evolution. Noting that $K_{ab} = \frac{1}{2} \mathcal{L}_n h_{ab}$ (where \mathcal{L}_n is the Lie derivative along the spacelike normal n^μ), we may use

$$\pi^{ab} = \frac{1}{2\sqrt{-g}} \frac{\delta S}{\delta K_{ab}}, \quad (2.9)$$

to obtain the expansion

$$\pi_{ab} = \sum_{k=1}^{d/2} \alpha_k \pi_{ab}^{(k)}, \quad (2.10)$$

where [28]

$$\begin{aligned} \pi_{ab}^{(1)} &= K_{ab} - h_{ab} K, \\ \pi_{ab}^{(2)} &= -2[h_{ab}(\mathcal{R}K - 2\mathcal{R}_{cd}K^{cd}) \\ &\quad - \mathcal{R}K_{ab} - 2\mathcal{R}_{ab}K + 4\mathcal{R}_{(a}^c K_{b)c} + 2\mathcal{R}_{acbd}K^{cd} \\ &\quad + \frac{1}{3}h_{ab}(-K^3 + 3K K_{cd}^2 - 2K_c^d K_d^e K_e^c) \\ &\quad + K^2 K_{ab} - 2K K_a^c K_{bc} - K_{ab} K_{cd}^2 + 2K_a^c K_c^d K_{bd}]. \end{aligned} \quad (2.11)$$

The above expressions for the conjugate momenta allow us to derive the Hamiltonian density for radial evolution

$$\mathcal{H} = 2\sqrt{-g} \pi^{ab} K_{ab} - \mathcal{L}. \quad (2.12)$$

The result is especially simple when written in terms of the projected bulk curvature $\bar{R}_{\mu\nu\rho\sigma}$. Following [28], we find

$$\mathcal{H} = \sum_{k=0}^{d/2} \alpha_k \mathcal{H}^{(k)}, \quad (2.13)$$

where

$$\begin{aligned}\mathcal{H}^{(0)} &= \sqrt{-g}, \\ \mathcal{H}^{(1)} &= \sqrt{-g} \bar{R}, \\ \mathcal{H}^{(2)} &= \sqrt{-g} (\bar{R}^2 - 4\bar{R}_{\mu\nu}^2 + \bar{R}_{\mu\nu\rho\sigma}^2).\end{aligned}\quad (2.14)$$

Using the Gauss–Codacci equation (2.7), this is equivalent to

$$\begin{aligned}\mathcal{H}^{(0)} &= \sqrt{-g}, \\ \mathcal{H}^{(1)} &= \sqrt{-g} (\mathcal{R} - K^2 + K_{ab}^2), \\ \mathcal{H}^{(2)} &= \sqrt{-g} [(\mathcal{R} - K^2 + K_{ab}^2)^2 - 4(\mathcal{R}_{ab} - K K_{ab} + K_a^c K_{bc})^2 \\ &\quad + (\mathcal{R}_{abcd} - K_{ac} K_{bd} + K_{ad} K_{bc})^2].\end{aligned}\quad (2.15)$$

Ultimately, the Hamiltonian ought to be written in terms of the canonical variables h_{ab} and π^{ab} . To accomplish this, we must invert the relation between π^{ab} and K^{ab} given by (2.10) and (2.11). It is at this stage that the individual Lovelock terms, parameterized by α_i , end up mixing with each other, as the inversion is in general a nonlinear problem involving all the various $\pi_{ab}^{(k)}$ simultaneously.

In order to proceed, we restrict our attention to the Einstein–Gauss–Bonnet theory given by the first three terms of (2.3), which we rewrite as

$$S_{\text{bulk}} = - \int_{\mathcal{M}} d^d x \sqrt{-g} [R + (d-1)(d-2)g^2 + \alpha(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2)], \quad (2.16)$$

where we have set $16\pi G_d = 1$, and where we have parameterized the cosmological constant α_0 in (2.3) by an inverse length scale g . In this case, the conjugate momentum of (2.10) may be written as

$$\pi_{ab} = K_{ab} - h_{ab}K + \alpha\pi_{ab}^{(2)}. \quad (2.17)$$

A simple rearrangement gives the useful expression

$$K_{ab} = \pi_{ab} - \frac{1}{(d-2)}h_{ab}\pi - \alpha\left(\pi_{ab}^{(2)} - \frac{1}{d-2}h_{ab}\pi^{(2)}\right), \quad (2.18)$$

which allows us to obtain a perturbative solution for K_{ab} in terms of π_{ab} . In particular, inserting the zeroth-order expression $K_{ab} = \pi_{ab} - h_{ab}\pi/(d-2) + \mathcal{O}(\alpha)$ into (2.11) gives

$$\begin{aligned}\pi_{ab}^{(2)} &= 2\left[2h_{ab}\left(\mathcal{R}_{cd}\pi^{cd} - \frac{1}{(d-2)}\mathcal{R}\pi\right) + \mathcal{R}\pi_{ab} + \frac{4}{d-2}\mathcal{R}_{ab}\pi - 4\mathcal{R}_{(a}^c\pi_{b)c}\right. \\ &\quad - 2\mathcal{R}_{acbd}\pi^{cd} + \frac{2}{3}h_{ab}\left(\pi_c^d\pi_d^e\pi_e^c - \frac{3}{d-2}\pi\pi_{cd}^2 + \frac{2}{(d-2)^2}\pi^3\right) \\ &\quad \left. - 2\pi_{ac}\pi^{cd}\pi_{db} + \frac{4}{d-2}\pi_a^c\pi_{bc}\pi + \pi_{ab}\pi_{cd}^2 - \frac{d}{(d-2)^2}\pi_{ab}\pi^2\right] + \mathcal{O}(\alpha).\end{aligned}\quad (2.19)$$

We now work out the Hamiltonian to first order in α . Using (2.15) and (2.18), we write

$$\begin{aligned}\mathcal{H} &= \sqrt{-g}[\mathcal{R} + (d-1)(d-2)g^2 - K^2 + K_{ij}^2] + \alpha\mathcal{H}^{(2)} \\ &= \sqrt{-g}\left[\mathcal{R} + (d-1)(d-2)g^2 + \pi_{ab}^2 - \frac{1}{d-2}\pi^2\right. \\ &\quad \left.+ \alpha\left(-2\pi^{ab}\pi_{ab}^{(2)} + \frac{2}{d-2}\pi\pi^{(2)} + \frac{\mathcal{H}^{(2)}}{\sqrt{-g}}\right) + \mathcal{O}(\alpha^2)\right].\end{aligned}\quad (2.20)$$

The lowest order expression for $\pi_{ab}^{(2)}$ is given in (2.19), while $\mathcal{H}^{(2)}$ may be obtained from (2.15):

$$\begin{aligned} \mathcal{H}^{(2)} = \sqrt{-g} & \left[\left(\mathcal{R} + \pi_{ab}^2 - \frac{1}{d-2} \pi^2 \right)^2 - 4 \left(\mathcal{R}_{ab} + \pi_a^c \pi_{bc} - \frac{1}{d-2} \pi \pi_{ab} \right)^2 \right. \\ & + \left(\mathcal{R}_{abcd} - (\pi_{ac} \pi_{bd} - \pi_{ad} \pi_{bc}) + \frac{1}{d-2} (h_{ac} \pi_{bd} + h_{bd} \pi_{ac} - h_{ad} \pi_{bc} \right. \\ & \left. \left. - h_{bc} \pi_{ad}) - \frac{1}{(d-2)^2} \pi^2 (h_{ac} h_{bd} - h_{ad} h_{bc}) \right)^2 + \mathcal{O}(\alpha) \right]. \end{aligned} \quad (2.21)$$

The resulting Hamiltonian, valid to linear order in α , then takes the form

$$\begin{aligned} \mathcal{H} = \sqrt{-g} & \left[\mathcal{R} + (d-1)(d-2)g^2 + \pi_{ab}^2 - \frac{1}{d-2} \pi^2 + \alpha \left(\mathcal{R}^2 - 4\mathcal{R}_{ij}^2 + \mathcal{R}_{ijkl}^2 \right. \right. \\ & - \frac{16}{d-2} \mathcal{R}_{ab} \pi \pi^{ab} + \frac{2d}{(d-2)^2} \mathcal{R} \pi^2 - 2\mathcal{R} \pi_{ab}^2 + 8\mathcal{R}_{ab} \pi^{bc} \pi_c^a + 4\mathcal{R}_{abcd} \pi^{ac} \pi^{bd} \\ & + 2\pi_a^b \pi_b^c \pi_c^d \pi_d^a - (\pi_{ab}^2)^2 - \frac{16}{3(d-2)} \pi \pi_a^b \pi_b^c \pi_c^a + \frac{2d}{(d-2)^2} \pi^2 \pi_{ab}^2 \\ & \left. \left. - \frac{3d-4}{3(d-2)^3} \pi^4 \right) + \mathcal{O}(\alpha^2) \right]. \end{aligned} \quad (2.22)$$

We will use this result in the next section when deriving the Hamilton–Jacobi counterterms which renormalize the original action (2.16).

3. Hamilton–Jacobi counterterms

The Einstein–Gauss–Bonnet action (2.16) admits solutions which are asymptotically anti-de Sitter, with an effective ‘inverse AdS radius’ g_{eff} given by

$$g^2 = g_{\text{eff}}^2 [1 - \alpha(d-3)(d-4)g_{\text{eff}}^2]. \quad (3.1)$$

It is well known that the on-shell action evaluated on such a background is divergent. In particular, assuming that the metric is asymptotically given by

$$ds^2 \sim -(1 + g_{\text{eff}}^2 r^2) dt^2 + \frac{dr^2}{1 + g_{\text{eff}}^2 r^2} + r^2 d\Omega_{d-2}^2, \quad (3.2)$$

the leading divergence is of a power-law form, $S \sim r^{d-1}$, with subleading divergences falling by a factor of $1/r^2$ at each order.

The divergences of the on-shell action may be removed by holographic renormalization [4–6]. This involves the introduction of a counterterm action of the form

$$S_{\text{ct}} = \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-h} (W + C\mathcal{R} + D\mathcal{R}^2 + E\mathcal{R}_{ab}^2 + F\mathcal{R}_{abcd}^2 + \dots), \quad (3.3)$$

so the renormalized action

$$\Gamma = S - S_{\text{ct}} \quad (3.4)$$

remains finite on-shell. The terms in (3.3) are organized as an expansion in powers of the inverse metric. Since examination of (3.2) indicates that $h_{ab} \sim r^2$, we see that W may be chosen to cancel the leading r^{d-1} divergence, C to cancel the r^{d-3} divergence, and so on.

A particularly elegant method of obtaining the coefficients in the counterterm action is to apply the Hamilton–Jacobi equation along with diffeomorphism invariance of the theory [7]. In

the last section, we have derived the Hamiltonian \mathcal{H} for radial evolution in the Einstein–Gauss–Bonnet theory. As this corresponds to reparameterizations of r , diffeomorphism invariance constrains the Hamiltonian to vanish:

$$\mathcal{H}[\pi^{ab}, h_{ab}] = 0. \quad (3.5)$$

To obtain the Hamilton–Jacobi equation one rewrites this Hamiltonian constraint in terms of the functional derivatives of the on-shell action. In particular, since the on-shell action is a functional of the bulk fields evaluated at the boundary $\partial\mathcal{M}$, the momenta can be written as

$$\pi^{ab} = \frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}}. \quad (3.6)$$

By replacing the momenta appearing in the Hamiltonian with this functional derivative, we obtain the Hamilton–Jacobi equation

$$\mathcal{H}\left[\frac{1}{\sqrt{-h}} \frac{\delta S}{\delta h_{ab}}, h_{ab}\right] = 0. \quad (3.7)$$

Using the Hamilton–Jacobi equation, we can obtain a set of counterterms that will remove power-law divergences from the on-shell action. In particular, given the renormalized action (3.4), the Hamilton–Jacobi equation takes the form [7, 8, 30]

$$\mathcal{H}[Z^{ab} + P^{ab}, h_{ab}] = 0, \quad (3.8)$$

where

$$Z^{ab} = \frac{1}{\sqrt{-h}} \frac{\delta \Gamma}{\delta h_{ab}}, \quad P^{ab} = \frac{1}{\sqrt{-h}} \frac{\delta S_{\text{ct}}}{\delta h_{ab}}. \quad (3.9)$$

The reason this is useful is that since Z^{ab} is derived from the renormalized action, any terms in (3.8) involving Z^{ab} are finite, or at most logarithmically divergence. Thus all power-law divergences are fully captured by the modified Hamilton–Jacobi equation

$$\mathcal{H}[P^{ab}, h_{ab}] = 0. \quad (3.10)$$

The momentum P^{ab} associated with the counterterm action (3.3) may be organized in an inverse metric expansion

$$P^{ab} = P_{(0)}^{ab} + P_{(1)}^{ab} + P_{(2)}^{ab} + \dots, \quad (3.11)$$

where

$$\begin{aligned} P_{(0)}^{ab} &= \frac{1}{2} h^{ab} W, \\ P_{(1)}^{ab} &= -C \mathcal{G}^{ab}, \\ P_{(2)}^{ab} &= \frac{1}{2} h^{ab} (D\mathcal{R}^2 + E\mathcal{R}_{cd}^2 + F\mathcal{R}_{cdef}^2) - 2D\mathcal{R}\mathcal{R}^{ab} + (2D + E + 2F)\mathcal{D}^a\mathcal{D}^b\mathcal{R} \\ &\quad - (2D + \frac{1}{2}E)h^{ab}\mathcal{D}^2\mathcal{R} - (E + 4F)\mathcal{D}^2\mathcal{R}^{ab} - 2(E + 2F)\mathcal{R}^{abcd}\mathcal{R}_{cd} \\ &\quad + 4F\mathcal{R}_c^a\mathcal{R}^{bc} - 2F\mathcal{R}^{acde}\mathcal{R}_{cde}^b. \end{aligned} \quad (3.12)$$

The resulting Hamiltonian $\mathcal{H}[P^{ab}, h_{ab}]$ may likewise be expanded in powers of the inverse metric

$$\mathcal{H} = \mathcal{H}_{(0)} + \mathcal{H}_{(1)} + \mathcal{H}_{(2)} + \dots. \quad (3.13)$$

We then demand that each term $\mathcal{H}_{(i)}$ vanishes individually. In this fashion, we end up with a set of ‘descent equations’ [8] for the coefficients of the divergent terms in the counterterm action (3.3).

Substituting the momenta P^{ab} of (3.12) into the Einstein–Gauss–Bonnet Hamiltonian (2.22) gives

$$\begin{aligned}
\mathcal{H}_{(0)} &= (d-1)(d-2)g^2 - \frac{d-1}{4(d-2)}W^2 - \alpha \frac{(d-1)(d-3)(d-4)}{48(d-2)^3}W^4, \\
\mathcal{H}_{(1)} &= \mathcal{R} \left[1 - \frac{d-3}{2(d-2)}WC + \alpha \frac{(d-3)(d-4)}{2(d-2)^2}W^2 \left(1 - \frac{d-3}{6(d-2)}WC \right) \right], \\
\mathcal{H}_{(2)} &= \mathcal{R}^2 \left[-\frac{d-5}{2(d-2)}WD - \frac{d-1}{4(d-2)}C^2 + \alpha \left(1 + \frac{(d-1)(d-4)}{(d-2)^2}WC \right. \right. \\
&\quad \left. \left. - \frac{(d-3)(d-4)(d-5)}{12(d-2)^3}W^3D - \frac{(d-1)(d-3)(d-4)}{8(d-2)^3}W^2C^2 \right) \right] \\
&\quad + \mathcal{R}_{ab}^2 \left[-\frac{d-5}{2(d-2)}WE + C^2 + \alpha \left(-4 - \frac{4(d-4)}{d-2}WC \right. \right. \\
&\quad \left. \left. - \frac{(d-3)(d-4)(d-5)}{12(d-2)^3}W^3E + \frac{(d-3)(d-4)}{2(d-2)^2}W^2C^2 \right) \right] \\
&\quad + \mathcal{R}_{abcd}^2 \left[-\frac{d-5}{2(d-2)}WF + \alpha \left(1 - \frac{(d-3)(d-4)(d-5)}{12(d-2)^3}W^3F \right) \right] \\
&\quad + \mathcal{D}^2\mathcal{R} \left(2(d-2)WD + \frac{d-1}{2}WE + 2WF \right) \left[\frac{1}{d-2} + \alpha \frac{(d-3)(d-4)}{6(d-2)^3}W^2 \right].
\end{aligned} \tag{3.14}$$

Starting with $\mathcal{H}_{(0)} = 0$, we find

$$W = -2(d-2)g \left[1 - \frac{1}{6}\alpha(d-3)(d-4)g^2 \right], \tag{3.15}$$

which is valid to linear order in α . This solution for W may then be inserted into the expression for $\mathcal{H}_{(1)}$. In this way, we may solve $\mathcal{H}_{(1)} = 0$ to obtain

$$C = -\frac{1}{(d-3)g} \left[1 + \frac{3}{2}\alpha(d-3)(d-4)g^2 \right]. \tag{3.16}$$

Working out the next-order terms is somewhat more involved. After solving $\mathcal{H}_{(2)} = 0$, we find

$$\begin{aligned}
D &= \frac{d-1}{4(d-2)(d-3)^2(d-5)g^3} \left[1 - \alpha g^2 \left(\frac{4(d-2)(d-3)^2}{d-1} + \frac{7(d-3)(d-4)}{2} \right) \right], \\
E &= -\frac{1}{(d-3)^2(d-5)g^3} \left[1 - \alpha g^2 \left(4(d-3)^2 + \frac{7(d-3)(d-4)}{2} \right) \right], \\
F &= -\frac{1}{(d-5)g^3}(\alpha g^2).
\end{aligned} \tag{3.17}$$

Inserting these coefficients into the counterterm action (3.3) gives

$$\begin{aligned}
S_{ct} &= -\int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-h} \left[2(d-2)g \left(1 - \frac{1}{6}\alpha(d-3)(d-4)g^2 \right) \right. \\
&\quad + \frac{1}{(d-3)g} \left(1 + \frac{3}{2}\alpha(d-3)(d-4)g^2 \right) \mathcal{R} \\
&\quad + \frac{1}{(d-3)^2(d-5)g^3} \left(1 - \frac{7}{2}\alpha(d-3)(d-4)g^2 \right) \left(\mathcal{R}_{ab}^2 - \frac{d-1}{4(d-2)}\mathcal{R}^2 \right) \\
&\quad \left. + \frac{\alpha}{(d-5)g} (\mathcal{R}^2 - 4\mathcal{R}_{ab}^2 + \mathcal{R}_{abcd}^2) + \dots \right].
\end{aligned} \tag{3.18}$$

Note that this is an expansion both in α (of which we have kept only up to the linear term) and powers of the inverse metric h^{ab} . The explicit counterterms given above are sufficient to cancel all power-law divergences in the Einstein–Gauss–Bonnet theory up to $d = 7$. However, the $\mathcal{O}(\mathcal{R}^3)$ terms, which we have not computed, will yield a finite contribution in $d = 7$ which is necessary for maintaining diffeomorphism invariance in the renormalized theory [30]. At linear order, these counterterms agree with the full nonlinear expressions obtained in [19–21] by imposing divergence cancellation of the gravitational action.

4. Gauss–Bonnet black holes

In the previous section, we have derived the counterterm action (3.18) which may be combined with the bulk action (2.16) and the generalized Gibbons–Hawking term

$$S_{\text{GH}} = -2 \int_{\partial\mathcal{M}} d^{d-1}x \sqrt{-h} \left[K - 2\alpha \left(2\mathcal{G}_{ab} K^{ab} + \frac{1}{3} (K^3 - 3K K_{ab}^2 + 2K_a^b K_b^c K_c^a) \right) \right], \quad (4.1)$$

to obtain the total renormalized action (3.4)

$$\Gamma = S_{\text{bulk}} + S_{\text{GH}} - S_{\text{ct}}. \quad (4.2)$$

This action may be identified with the thermodynamic potential of the system through $\Omega = \Gamma/\beta$ where $\beta = 1/T$ is the inverse temperature. Furthermore, we may define the boundary stress tensor

$$T^{ab} = \frac{2}{\sqrt{-h}} \frac{\delta\Gamma}{\delta h_{ab}}. \quad (4.3)$$

Comparing this with (3.6), we see that

$$T^{ab} = 2\pi^{ab} - 2P^{ab}, \quad (4.4)$$

where π^{ab} are given by (2.10) and (2.11) and P^{ab} are given by (3.12). The boundary stress tensor allows us to define the conserved momentum (and in particular the energy) of the spacetime.

The above results allow us to investigate the thermodynamics of Gauss–Bonnet black holes [17, 31–39]. Before proceeding, however, we note that it is straightforward to include a canonically normalized Maxwell field, so that the bulk action (2.16) becomes

$$S_{\text{bulk}} = - \int_{\mathcal{M}} d^d x \sqrt{-g} \left[R + (d-1)(d-2)g^2 - \frac{1}{4}F_{\mu\nu}^2 + \alpha(R^2 - 4R_{\mu\nu}^2 + R_{\mu\nu\rho\sigma}^2) \right]. \quad (4.5)$$

To obtain an electrically charged black hole, we take

$$A = \frac{Q}{(d-3)r^{d-3}} dt \Rightarrow F = \frac{Q}{r^{d-2}} dt \wedge dr, \quad (4.6)$$

as well as the metric ansatz

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2 d\Sigma_{d-2,k}^2, \quad (4.7)$$

where k denotes the curvature of the manifold $\Sigma_{d-2,k}$ ($k = 1, 0, -1$). Working out the ‘angular’ components of the Einstein equation, we find the first-order equation

$$0 = rf' + (d-3)(f-k) - (d-1)g^2 r^2 + \frac{Q^2}{2(d-2)r^{2(d-3)}} - \frac{\alpha}{r^2}(d-3)(d-4)(f-k)[(d-5)(f-k) + 2rf'], \quad (4.8)$$

which may be solved to yield [17, 31–33, 35]

$$f = k + \frac{r^2}{2\tilde{\alpha}} \left[1 \mp \sqrt{1 + 4\tilde{\alpha} \left(\frac{\mu}{r^{d-1}} - g^2 - \frac{Q^2}{2(d-2)(d-3)r^{2(d-2)}} \right)} \right], \quad (4.9)$$

where $\tilde{\alpha} = \alpha(d-3)(d-4)$. Here μ is a non-extremality parameter related to the black hole mass. Note that relative simplicity of the equation of motion (4.8) and its black hole solution is a general feature of the Lovelock actions.

While the above black hole solution is exact in the Gauss–Bonnet parameter α , our derivation of the Hamilton–Jacobi counterterms was restricted to linear order in α . We therefore expand f to first order in α

$$f = k + g^2 r^2 - \frac{\mu}{r^{d-3}} + \frac{Q^2}{2(d-2)(d-3)r^{2(d-3)}} + \frac{\tilde{\alpha}}{r^2} \left(g^2 r^2 - \frac{\mu}{r^{d-3}} + \frac{Q^2}{2(d-2)(d-3)r^{2(d-3)}} \right)^2 + \dots \quad (4.10)$$

Note that we have taken the ‘negative’ branch of (4.9), as it is the one which has a well-behaved $\alpha \rightarrow 0$ limit. In what follows all expressions should be understood to be taken only to linear order in α .

In order to parameterize the Gauss–Bonnet black hole thermodynamics, we introduce the horizon location r_+ , defined by $f(r_+) = 0$. A simple rearrangement of (4.10) then allows us to write μ in terms of r_+ as

$$\mu = g^2 r_+^{d-1} + k r_+^{d-3} + \frac{Q^2}{2(d-2)(d-3)r_+^{d-3}} + \alpha k^2 (d-3)(d-4)r_+^{d-5}. \quad (4.11)$$

This will be useful in what follows. For example, the temperature may be obtained from $T = f'(r_+)/4\pi$, which comes from the requirement of avoiding a conical singularity at the horizon. Taking a derivative of (4.10) and using (4.11) to eliminate μ gives

$$T = \frac{1}{4\pi r_+} \left[(d-1)g^2 r_+^2 + (d-3)k - \frac{Q^2}{2(d-2)r_+^{2(d-3)}} + \alpha k (d-3)(d-4) \left(-2(d-1)g^2 - (d-1)\frac{k}{r_+^2} + \frac{Q^2}{(d-2)r_+^{2(d-2)}} \right) \right]. \quad (4.12)$$

This matches the exact expression for the Hawking temperature [17, 34–36] when expanded to linear order in α (as it must, since the calculation is identical).

Turning next to the entropy, it is well known that the area expression $S = \mathcal{A}_h/4G_d$ gets modified in higher derivative gravity. In this case, we may instead use the Wald entropy formula [40–42]

$$S = -2\pi \int_h E^{\mu\nu\rho\sigma} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} d^{d-2}x, \quad (4.13)$$

where

$$E^{\mu\nu\rho\sigma} = \frac{\delta S_{\text{bulk}}}{\delta R_{\mu\nu\rho\sigma}} \Big|_{g_{\mu\nu} \text{ fixed}}, \quad (4.14)$$

and where $\epsilon_{\mu\nu}$ is the binormal to the horizon h . Taking the action (4.5), we find

$$E^{\mu\nu\rho\sigma} = -\sqrt{-g} \left[\frac{1}{2} (g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) + \alpha \left((g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho}) R - 2(g^{\mu\rho} R^{\nu\sigma} + g^{\nu\sigma} R^{\mu\rho} - g^{\mu\sigma} R^{\nu\rho} - g^{\nu\rho} R^{\mu\sigma}) + 2R^{\mu\nu\rho\sigma} \right) \right]. \quad (4.15)$$

Working out the curvature components and integrating $E^{\bar{i}\bar{j}\bar{k}}$ (where the overlines indicate tangent space components) over the horizon gives the simple entropy expression

$$S = 4\pi\omega_{d-2,k}r_+^{d-2} \left[1 + 2\frac{\alpha}{r_+^2}k(d-2)(d-3) \right], \quad (4.16)$$

where $\omega_{d-2,k}$ is the volume of $\Sigma_{d-2,k}$ (so that $\omega_{d-2,k}r_+^{d-2}$ is simply the ‘horizon area’). Since we are working in units of $16\pi G_d = 1$, the leading term in S indeed reproduces the standard area expression. Note that this entropy expression is universal for spherically symmetric Gauss–Bonnet black holes in that the equations of motion were not needed for its derivation.

The Gauss–Bonnet black hole entropy was previously computed in [34] by solving the free energy expression $F = E - TS$ for the entropy (where F was computed from the Euclidean action) and in [35] by integrating the first law $dE = TdS$. Both of these computations are in agreement with the Wald entropy formula result (4.16). Furthermore, we may see that the linearized expression (4.16) is in fact exact in α .

We now work out the renormalized action for the Gauss–Bonnet black hole. Substituting in the metric ansatz (4.7) as well as the gauge field (4.6), we find that the bulk action (4.5) may be expressed as a total r derivative. Integrating this from the horizon to a cutoff r_0 gives

$$S_{\text{bulk}} = \beta\omega_{d-2,k} \left[-(d-2)g^2r^{d-1} + r^{d-3}((d-2)(f-k) + rf') + \frac{Q^2}{2(d-3)r^{d-3}} - \alpha(d-2)(d-3)r^{d-5}(f-k)((d-4)(f-k) + 2rf') \right]_{r_+}^{r_0}, \quad (4.17)$$

where $\beta = 1/T$ is the period of the timelike circle. Using the explicit form of f given in (4.10) as well as relation (4.11), we obtain

$$S_{\text{bulk}} = \beta\omega_{d-2,k} \left[2g^2r_0^{d-1}(1 - d(d-3)\alpha g^2) - 2\mu(1 - 2(d-3)\alpha g^2) + 2kr_+^{d-3} + 2\alpha k(d-3) \left(-2(d-1)g^2r_+^{d-3} - 2kr_+^{d-5} + \frac{Q^2}{(d-2)r_+^{d-1}} \right) \right]. \quad (4.18)$$

This clearly exhibits the leading power-law divergence $I_{\text{bulk}} \sim 2g^2r_0^{d-1}$. The generalized Gibbons–Hawking term is evaluated at the cutoff surface $r = r_0$. From (4.1), we find

$$\begin{aligned} S_{\text{GH}} &= \beta\omega_{d-2,k} \left[-r^{d-3}(2(d-2)f + rf') \right. \\ &\quad \left. + 2\alpha(d-2)(d-3)r^{d-5} \left((d-4)f \left(\frac{2}{3}f - 2k \right) + (f-k)rf' \right) \right]_{r=r_0} \\ &= \beta\omega_{d-2,k} \left[-2(d-1)g^2r_0^{d-1} \left(1 + \frac{1}{3}(d-3)(d-8)\alpha g^2 \right) \right. \\ &\quad - 2(d-2)kr_0^{d-3} \left(1 + \frac{2}{3}(d-3)(d-4)\alpha g^2 \right) \\ &\quad \left. - \frac{8}{3}\alpha(d-2)(d-3)(d-4)k^2r_0^{d-5} + (d-1)\mu \left(1 + \frac{4}{3}(d-3)(d-5)\alpha g^2 \right) \right]. \quad (4.19) \end{aligned}$$

Adding together S_{bulk} and S_{GH} , we see that the power-law divergences are given by r_0^{d-1} , r_0^{d-3} and r_0^{d-5} (assuming d is sufficiently large, of course). These will be canceled by the counterterm action (3.18).

Recall that the derivation of the counterterm action involved an expansion in powers of the inverse metric h^{ab}

$$S_{\text{ct}} = S_{(0)} + S_{(1)} + S_{(2)} + \dots, \quad (4.20)$$

where the leading divergence of $S_{(k)}$ is of the form r_0^{d-2k} . Therefore, for arbitrary dimension d , we would need at least the first three counterterms to cancel the divergences of $S_{\text{bulk}} + S_{\text{GH}}$. For the Gauss–Bonnet black hole, we obtain from (3.18)

$$\begin{aligned} S_{\text{ct}} = & -\beta\omega_{d-2,k}(d-2)[2g^2r_0^{d-1}(1+\frac{1}{3}(d-3)(d-4)\alpha g^2) \\ & + 2kr_0^{d-3}(1+\frac{2}{3}(d-3)(d-4)\alpha g^2) + \frac{8}{3}\alpha(d-3)(d-4)k^2r_0^{d-5} + \\ & -\mu(1+\frac{4}{3}(d-3)(d-4)\alpha g^2) + \dots]. \end{aligned} \quad (4.21)$$

Note that here we have taken the dimensional continuation approach of [30]. In particular, the dimension-dependent poles in (3.18) are canceled by zeros in the boundary curvature expressions. This allows, for example, $S_{(2)}$ to generate a finite counterterm in $d = 5$ dimensions. In the two-derivative theory, this finite contribution removes the ‘Casimir energy’ of global AdS₅ and at the same time restores full diffeomorphism invariance of the renormalized theory [30].

Adding together (4.18), (4.19) and (4.21) finally yields the renormalized thermodynamic potential

$$\Omega = \omega_{d-2,k} \left[-\mu + 2kr_+^{d-3} + 2\alpha k(d-3) \left(-2(d-1)g^2r_+^{d-3} - 2kr_+^{d-5} + \frac{Q^2}{(d-2)r_+^{d-1}} \right) \right], \quad (4.22)$$

where $\Omega = \Gamma/\beta$, and where μ is given in (4.11). This expression for the thermodynamic potential agrees (at linear order in α) with the free energy calculations using background subtraction to regulate the Euclidean action [17, 34, 36] and derive through $F = E - TS$ [34]. This provides a welcome check on the counterterm coefficients in (3.18), which involved a fair bit of manipulation to extract from the Einstein–Gauss–Bonnet action. We wish to stress that the inclusion of the finite counterterm was necessary in order to obtain agreement with the previous free energy results.

The final quantity we are interested in is the energy of the system. For the energy, we focus on the tt component of the boundary stress tensor. Using (4.4) as well as

$$\begin{aligned} \sqrt{-h}\pi^{\bar{t}\bar{t}} &= -(d-2)r^{d-3}f - 2\alpha r^{d-5}(d-2)(d-3)(d-4)f(k - \frac{1}{3}f), \\ \sqrt{-h}P^{\bar{t}\bar{t}} &= \frac{1}{2}\mathcal{L}_{\text{ct}} \end{aligned} \quad (4.23)$$

(where \mathcal{L}_{ct} is the counterterm Lagrangian of (3.18), and where this expression holds for the constant curvature boundary geometry $S^1 \times \Sigma_{d-2,k}$), we obtain a simple expression for the energy

$$E = \omega_{d-2,k}(d-2)\mu. \quad (4.24)$$

We may now see that the free energy and energy are related by the standard expression

$$\Omega = E - TS - \mathcal{Q}\Phi \quad (4.25)$$

where $\mathcal{Q} = \omega_{d-2,k}Q$ is the normalized electric charge, and

$$\Phi = A_t(r_+) - A_t(\infty) = \frac{Q}{(d-3)r_+^{d-3}} \quad (4.26)$$

is the electric potential at the horizon.

5. Summary

The calculation of the on-shell action and boundary stress tensor is an important aspect of the study of black hole thermodynamics. Such quantities are generally divergent and require

renormalization. While various approaches, including background subtraction, have been developed, holographic renormalization using the Hamilton–Jacobi formalism is particularly elegant and useful in the study of the thermodynamics of black holes in asymptotically AdS spacetimes. This approach generates the appropriate boundary counterterms needed to remove all divergences of the on-shell action for R -charged AdS black holes in various dimensions.

In this paper, we have focused on the Einstein–Gauss–Bonnet system with a negative cosmological constant and used the Hamiltonian–Jacobi approach to evaluate the counterterms up to linear order in α , the coupling of the Gauss–Bonnet term. In general, this linear order in α is all that is physically relevant when considering the R^2 corrections in the expansion of the full higher derivative effective action. However, it is noteworthy that the Gauss–Bonnet form of the R^2 action admits exact R -charged black hole solutions. Because of this, exact expressions may be obtained for the thermodynamic quantities calculated in the previous section. In particular, the temperature and thermodynamic potential take the form [17, 34–36]

$$\begin{aligned} T &= \frac{1}{4\pi r_+} \frac{1}{r_+^2 + 2\tilde{\alpha}k} \left((d-1)g^2 r_+^4 + (d-3)kr_+^2 + (d-5)\tilde{\alpha}k^2 - \frac{Q^2}{2(d-2)r_+^{2(d-4)}} \right), \\ \Omega &= -\frac{\omega_{d-2,k} r_+^{d-5}}{d-4} \left((d-2)(3g^2 r_+^4 + kr_+^2 - \tilde{\alpha}k^2) - \frac{Q^2}{2(d-3)r_+^{2(d-4)}} - 8\pi r_+^3 T \right). \end{aligned} \quad (5.1)$$

It would be of interest to see if the Hamilton–Jacobi method can be extended to capture the nonlinear terms, such as those derived in [19–21], as well. We recall, however, that the main reason we had linearized in α was so we could invert relations (2.11) for the conjugate momenta in order to derive the Hamiltonian (2.22). Obtaining the exact Hamiltonian through a nonlinear inversion of $\pi_{ab} \leftrightarrow K_{ab}$ looks to be a challenge.

The main reason exact solutions of the Einstein–Gauss–Bonnet theory are available is that, while this is a higher derivative gravitational system, the equations of motion arising from the Gauss–Bonnet combination do not involve higher than two derivative of the metric. For this reason, the Cauchy problem for radial evolution remains well defined when conventional Dirichlet conditions are imposed on the boundary. Other curvature combinations such as the Weyl tensor squared combination, which naturally arises in the higher derivative corrections to five-dimensional $\mathcal{N} = 2$ supergravity [43], do not admit a well-posed Dirichlet problem nor an appropriate generalization of the Gibbons–Hawking term [29]. This appears to be a major obstruction to generalizing the Hamilton–Jacobi approach to holographic renormalization to other theories with higher curvature terms.

Finally, we note that, while a main objective of holographic renormalization is the removal of divergences of the on-shell AdS action, the Hamilton–Jacobi method introduces the additional framework of diffeomorphism invariance to the construction of the counterterm action S_{ct} . In practice, this provides no additional information for the counterterms that remove power-law divergences in the action. However, the Hamilton–Jacobi method *does* naturally determine the finite counterterms which would otherwise be free (and related to different renormalization schemes in the dual CFT). For theories involving scalars, the leading counterterm determined by the Hamilton–Jacobi method looks like an effective superpotential [7], and in this fashion, the finite part of S_{ct} is necessary to maintain the supersymmetry of the boundary theory [10, 44]. We have not included any scalars in the present analysis, although we expect the generalization to be straightforward.

Even in the absence of scalars, we have been careful to take into account the finite counterterm which arises (in odd dimensions d) through the dimensional continuation of the $R^{(d-1)/2}$ terms in S_{ct} [30]. For spherically symmetric configurations of the ordinary two-derivative Einstein theory, this finite counterterm removes the ‘Casimir energy’ of the AdS background. Since this is simply a constant, the physical effect of this subtraction is rather

minimal (at least from the AdS/CFT point of view). However, this subtraction appears to be more important in higher curvature theories, as the $\mathcal{O}(\alpha)$ contribution to the finite counterterm can no longer be interpreted as a simple shift in the Casimir energy. We thus feel it is most natural to adhere to a diffeomorphism-invariant renormalization scheme, which is naturally accomplished through the Hamilton–Jacobi method.

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