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NONLINEAR REACTOR KINETICS

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BASED ON LECTURES DELIVERED AT
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I.1 INTRODUCTION

I would like to review the derivation of the reactor kinetics equations for the purpose of determining whether they are applicable for very fast transients, and to see if they can be used for studies of stability. By studies of stability, I mean the derivation of criteria which can be verified experimentally and which can be used to determine whether the reactor power is stable or unstable.

In order to study the dynamics of a reactor, or for that matter the dynamics of any physical system, one has to take four interrelated steps. The steps will not necessarily be listed in their order of importance but rather in a sequence appropriate to this discussion.

The first step is to decide which variables will be used to characterize the system. For example, a system of a battery and a resistor is usually characterized by a voltage, a current, and a value for the resistance. In the same sense some variable must be chosen to characterize a nuclear reactor.

In making this choice there are several conflicting factors that have to be accommodated. At one extreme one could consider the position and velocity of every single particle in the reactor. Thus, one could have a complete characterization of the system but a characterization which is impossible to analyze. At the other extreme one could consider the reactor as a black box and characterize it by only one variable, say the average power. This may result in a simpler analytical problem but it does not distinguish either between hot and cold spots or between

flux peaks and flux valleys etc. Consequently, the choice must be such as to preserve a reasonable balance between the desirable simplicity of characterization and the required detail of characterization. Also, whatever the choice of variables, it must be borne in mind that the results of any analysis must be such that they can be verified experimentally. No theory, however elegant, is of any value unless it can be verified or implemented experimentally.

A variable of primary importance to reactor dynamics is the energy stored in the reactor because it is this energy which might melt the fuel or change the thermodynamic state of the reactor and thus influence its operating condition. However, the stored energy is not a practical variable because it is not directly measurable. Some measurable variables are the neutron density, the temperature, the coolant flow etc. at different points in the reactor. Of course, from temperature, coolant flow and neutron density measurements one can infer the behavior of the stored energy. For a reactor then, neutron density, temperatures etc. are appropriate variables to characterize the system.

The second step in the process of understanding reactor dynamics is to write balance equations which interrelate the chosen variables. Actually since we are great believers in conserving energy, momentum, number of particles etc., this step is readily dispensed with.

The third and perhaps the most difficult step is to solve the balance equations. Even for relatively simple systems, the

establishment of general and exact solutions is practically impossible. For this reason, one is forced to consider approximations to the equations and/or to the solutions. Approximate solutions are very practical provided they are used in their range of applicability. Approximations and evaluations of their range of applicability require a thorough understanding of the physics of the system.

The fourth step, of course, which underlies the previous three steps, is to end up with results which can be interpreted and/or verified experimentally. Here also there are serious difficulties, particularly for problems in dynamics which involve nonlinearities. Little is known, for example, about the identification problem of nonlinear systems, not only in the reactor field but also in system dynamics in general.

I.2 BALANCE RELATIONS

In the light of the previous remarks let us now examine how we derive a balance equation for, say, the neutron density.

The conservation of neutrons in a reactor may be expressed analytically by means of transport theory. Admittedly, transport theory equations do not necessarily lead to specific and explicit results. However, these equations do provide a vehicle for setting up a frame of reference for what is to follow.

The variable which seems appropriate in writing a balance equation for neutrons is the neutron density $N(\underline{r},E,\underline{\Omega},t)$ as a function of position, $\underline{r}$, energy, E, the solid angle in which the neutrons move, $\underline{\Omega}$, and time, t. Since we are not interested in

specific solutions at this point, we will write down a very general equation describing the time rate of change of N at every point in the reactor and for every energy and every solid angle. In fact we can be so general as to write the equation on one line only:

$$\left[\begin{array}{c}\text{Production}\\ \text{Operator}\end{array} - \begin{array}{c}\text{Destruction}\\ \text{Operator}\end{array}\right] N = \frac{\partial N}{\partial t} \quad . \qquad (1)$$

There is, of course, some ambiguity as to what constitutes a production or destruction mechanism since we can change one to the other merely by changing its sign. We won't belabor this point any further except to say that as a consequence of this ambiguity there are certain reactor parameters (such as lifetime for example) which cannot be uniquely defined.

To recall what is involved in these production and destruction operators consider the number of prompt neutrons produced by fission. We take the neutron density, N, multiply it by the fission cross section, Σ_f, and the relative neutron velocity, v, to find the differential reaction rate. This reaction rate is then multiplied by the average number of neutrons per fission, $\nu(E)$, and by the fraction which appear as prompt neutrons, $(1-\beta)$. The result is integrated over all energies and all solid angles to obtain the total number of neutrons produced. To account for only those neutrons which appear per unit energy in the energy range E to E+dE the integral must be multiplied by the prompt neutron spectrum, $f_o(E)$. Similarly, in order to account for only those neutrons which are in a unit solid angle

in the angle range $\underline{\Omega}$ to $\underline{\Omega} + d\underline{\Omega}$, the integral is also multiplied by the neutron emission angular spectral function or by $1/4\pi$ if the emission is assumed isotropic. Thus the prompt neutron production term is

$$\frac{1}{4\pi} f_o(E) \int d\underline{\Omega}' \int dE' (1-\beta) \nu(E') v' \Sigma_f(E', t) N(\underline{r}, E', \underline{\Omega}', t). \tag{2}$$

Similar terms can be written for scattering, total absorption, leakage, delayed neutron production etc., but we will forego these terms and keep the symbolic representation of Equation (1).

It should be pointed out that if E' in Equation (2) is regarded as the relative energy between targets and neutrons, then one has to include a probability distribution in the integral either for the relative energy or for the target velocities. Another way of looking at this problem might be through consideration of cross sections that are averaged over the velocity distribution of target nuclei.

Returning to Equation (1) we recognize that the difficulties encountered in trying to solve it stem from the fact that seven independent variables are involved and from the fact that the macroscopic cross sections depend on the very quantity, N, which we want to find. This dependence arises because N determines the energy released in the reactor and thus is coupled to the cross sections.

One way to simplify the problem might be to attempt to reduce Equation (1) into a form which does not depend, at least explicitly, on all seven independent variables. This is

essentially what is formally achieved when we reduce Equation (1) (and the associated delayed neutron precursor equations) into the form of the point reactor kinetics equations.

I.3 REDUCTION TO THE POINT KINETICS EQUATIONS AND EFFECTS OF SHAPE CHANGES

The reduction can be achieved by a large variety of methods. We will describe only three of these methods.

For the first method suppose that we write

$$N = G(t)N_g(\underline{r},E,\underline{\Omega},t) \tag{3}$$

with the restriction that G(t), which we might call the growth function, is so selected that $N_g(\underline{r},E,\underline{\Omega},t)$ remains always bounded for any and all values of its arguments. N_g is in a sense a shape function because it is related to the relative distribution of neutrons in the phase space of the reactor as opposed to G(t), which presumably accounts for any overall growth or decay tendencies of the neutron population.

Next assume that we have an arbitrary critical reference reactor in which the adjoint density is $N^*(\underline{r},E,\underline{\Omega})$. The equation describing N* is

$$\left[\begin{array}{lcl} \text{Steady State} & & \text{Steady State} \\ \text{Production} & - & \text{Destruction} \\ \text{Operator} & & \text{Operator} \end{array}\right] N^* = 0 \ . \tag{4}$$

Multiply Equation (1) by N*, multiply Equation (4) by N, subtract the two resulting equations, and then integrate over all $\underline{r}$, E, $\underline{\Omega}$, to obtain an equation of the form:

$$\frac{dG}{dt} = \frac{\rho-\bar{\beta}}{\Lambda} G + \sum_i \lambda_i C_i - \left[\frac{\partial}{\partial t} \ln \int d^3\underline{r}\, dE\, d\underline{\Omega}\, N^* N_g\right] G \,. \tag{5}$$

Similar equations can be obtained for the equivalent delayed neutron precursor concentrations, $C_i(t)$:

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} G - \lambda_i C_i + \left[\frac{\partial}{\partial t} \ln \int d^3\underline{r}\, dE\, d\underline{\Omega}\, N^* N_g\right] C_i \,. \tag{6}$$

For the exact definitions of ρ, β_i and Λ see Henry's paper(1). The difference between this formulation and that of Henry's is the retention of the terms in the square brackets. It is evident that Equations (5) and (6) are, at least in form, the same as the ordinary point reactor kinetics equations except for the logarithmic rate terms.

The quantities ρ, $\bar{\beta}$ and Λ are short hand notations for many things which we quite often do not know how to express analytically or measure experimentally. They are in general time dependent and they involve integrals over all independent variables except t. The integrands depend on N*, N_g, changes in cross sections etc. For example, a typical term in ρ is of the form

$$\rho \sim \int d^3\underline{r}\, dE\, d\underline{\Omega}\, (\Delta\Sigma_a) N_g N^* \,. \tag{7}$$

1. A. F. Henry, Nuc. Sci. and Eng., 3, 52 (1958).

The formal definitions of ρ , $\bar{\beta}$ and Λ involve a normalization factor which is completely arbitrary. Therefore, these quantities cannot in general be interpreted as physically meaningful. In other words it is only the ratios ρ/Λ or $\bar{\beta}/\Lambda$ which may be measured (when they can be measured!) experimentally and which may thus warrant a physical definition.

Let us now consider how we use Equations (5) and (6) to study reactor transients. Suppose that we are able to ascertain from experience that N_g is a weak function of time, i.e. the variation of N_g is small over a long period of time (long compared to the time of interest of the reactor transient in question). Then if we approximate N_g by a time independent function, the values of ρ/Λ and $\bar{\beta}/\Lambda$ that are thus calculated will be good approximations at least to second order. The logarithmic term in Equations (5) and (6) can be neglected altogether without appreciable errors, and thus G(t) is a fair approximation to the average reactor power.

On the other hand, if we follow exactly the same procedure in cases where N_g varies a little but over a short period of time then the calculations of ρ/Λ and $\bar{\beta}/\Lambda$ are still good to second order, but Equations (5) and (6) involve serious errors because of the neglect of the logarithmic term. To appreciate this fact, note that the logarithmic term enters the equations in the same manner as ρ/Λ and, therefore, it may be considered as a reactivity term. "Back of the envelope" type calculations show that it can be of the order of ½ to 1 dollar for transients which occur over 10 milliseconds in

shape as indicated by the solid line with a corresponding measured current and inferred power. If the shape changes so that the slope in the vicinity of the counter remains the same, as indicated by the dashed line, then the reading of the counter will remain the same while the power will be entirely different.

In summary then the growth factor, G(t), has a practical interpretation only when the shape function is either constant with respect to time or changes by a small amount over a long period of time.

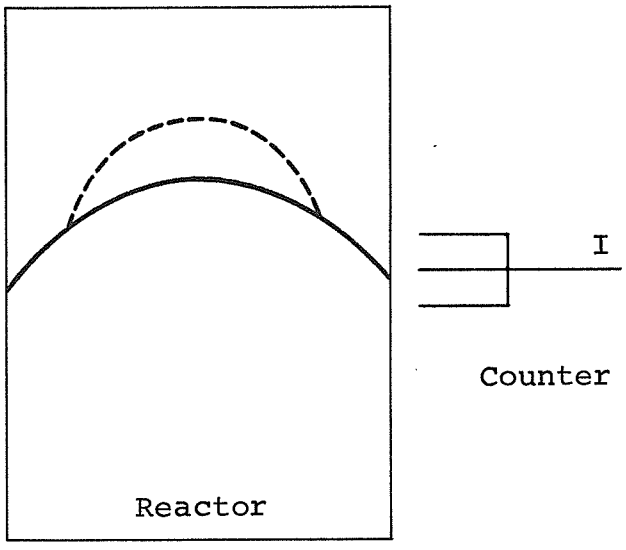
This brings us to the second way of reducing the transport theory equations. If we were really interested in the reactor power we should not have been weighting these equations with the adjoint density but rather with the fission cross section. If we repeat the same steps as before except that we use Σ_f instead of N* we obtain equations which look similar to Equations (5) and (6), and they are practically identical when there are no shape changes. The two sets of equations are not comparable when shape changes occur, however, because one describes power and the other describes the growth factor.

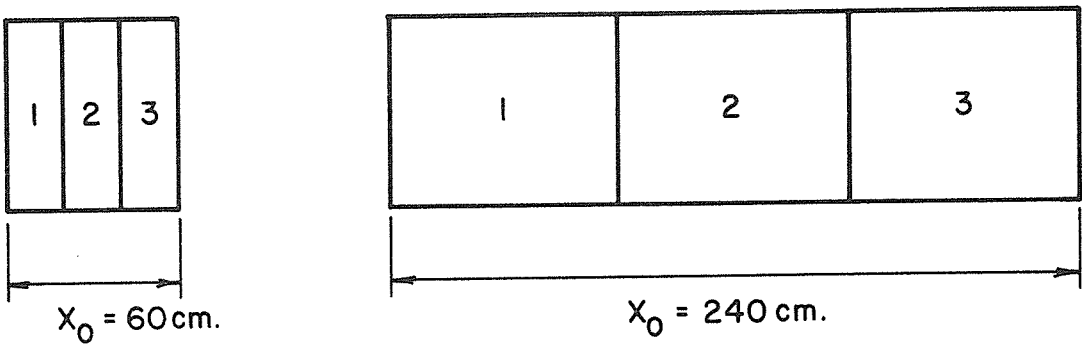
A third possibility of deriving kinetic equations from transport theory is by multiplying the latter by the cross section of a counter and integrating over all variables except time. Again equations which have the same form as the kinetic equations are derived. These have equivalent ρ, $\bar{\beta}$, Λ, etc., but have as a variable not G(t) but the counter readings C(t). Although these equations describe the experimental readings of a counter they have the disadvantage that they cannot be used

reactors which have lifetimes in the range of 10^{-4} - 10^{-5} sec.

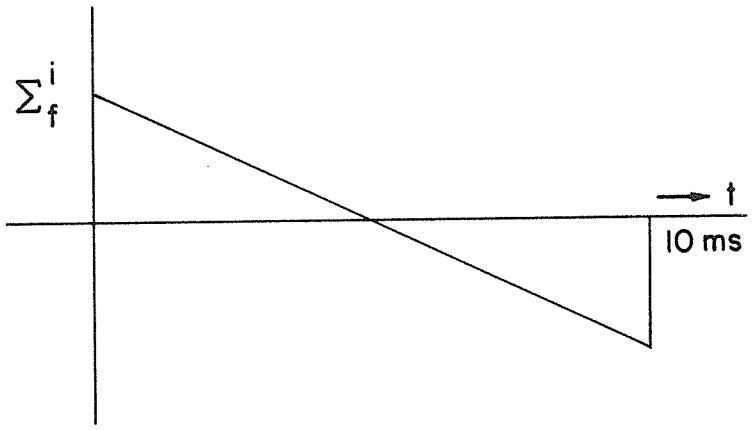
From another point of view, it may be argued that the logarithmic term may be made equal to a constant (and its rate of change equal to zero) by selecting an appropriate variable, N_g. If this variable, N_g, is used in the calculations of ρ/Λ and $\bar{\beta}/\Lambda$ then of course there will be no approximation stemming from N_g. If we assume that this can be done in a practical situation then the resulting equations would describe the time dependent behavior of the component of N which has the same shape as N*. This is fine if we are interested in following the amplitude of the fundamental mode. But if we are interested in comparing the solution, G(t), of the equations with the output of a counter then clearly when there are flux shape changes during the transient this approach would be of little practical value.

As an example of the effect of shape changes consider the sketch below. Assume that the reactor originally has a flux





x-direction is subdivided in twelve mesh points. A transient is introduced by changing the fast and thermal fission cross sections of region 1 of each reactor in a manner shown below.



With the help of the computer code WIGLE, the fast and thermal fluxes $\phi_1(x,t)$ and $\phi_2(x,t)$ are calculated as functions of position and time. Since the time of interest here is only 10 msec, no delayed neutrons are considered.

On the basis of the knowledge of ϕ_1 and ϕ_2 one can now compute the G(t) function and the corresponding ρ/Λ assuming that the logarithmic term is constant (the rate term equal to

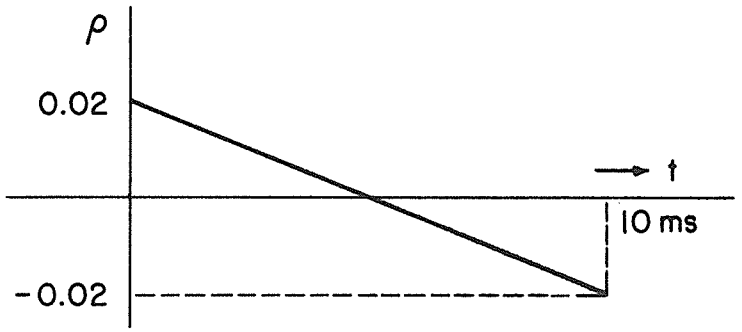
for the description of the power when shape changes occur. This is indeed a disadvantage, since one performs dynamic experiments in order to determine power variations and thus calculate feedback effects.

The results of this discussion indicate that in order to describe transients involving fast shape changes we will either have to modify the concept of reactivity or abandon it entirely and attack the problem as a space-time dependent problem. A lot of work has been done in space-time dynamics. One of the main disadvantages of this work is that the results are not suggestive of specific experiments. For example we know that when the definition of reactivity is applicable, one can either put a reactor on an asymptotic period or use oscillator experiments to measure ρ/Λ . Thus the results of computations can be readily verified. Unfortunately this is not true for many of the space-time techniques that have been developed. Thus it seems to me that we must develop space-time models which are amenable to direct experimental verification.

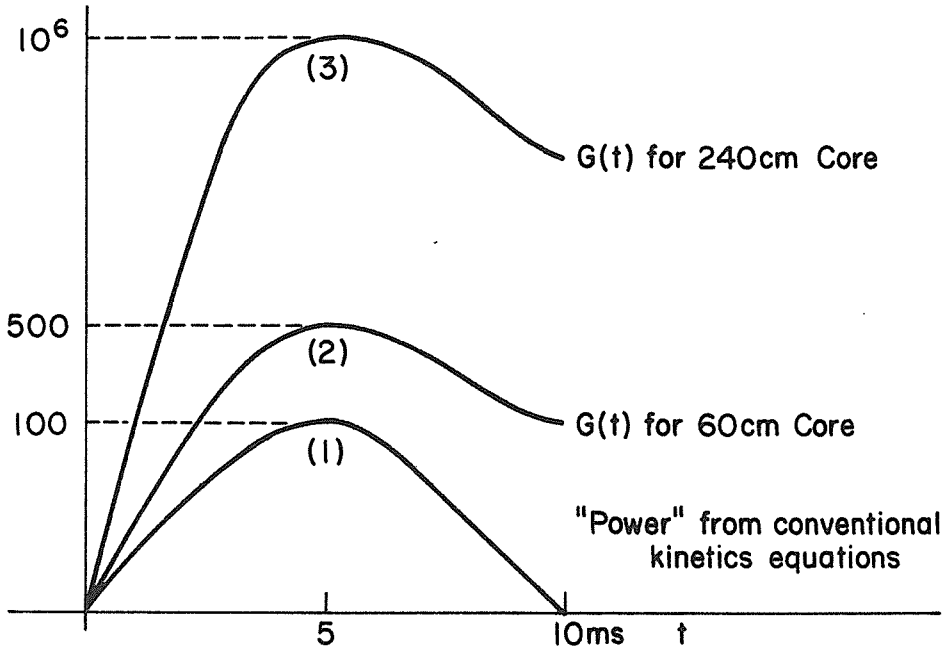
I.4 AN EXAMPLE OF THE EFFECT OF SHAPE CHANGES

An example of some of the effects caused by shape changes is demonstrated by the computer experiment performed, at my request, by Al Henry and his group at Westinghouse. Two reactors sketched on the next page were considered. The reactors are specified in terms of their transverse bucklings and the material properties of each of the three regions 1, 2, 3, on the basis of a two-group diffusion theory model. The

zero). Given the material properties of the two reactors, one can also calculate the lifetime ($\Lambda = 10^{-5}$ sec), the conventional reactivity and the "power" that is derived from the "conventional" kinetics equations. The reactivity is computed assuming no shape change with respect to the critical reference reactor. This reactivity for both reactors is shown in the sketch below. In other words it behaves similarly to the variation of the

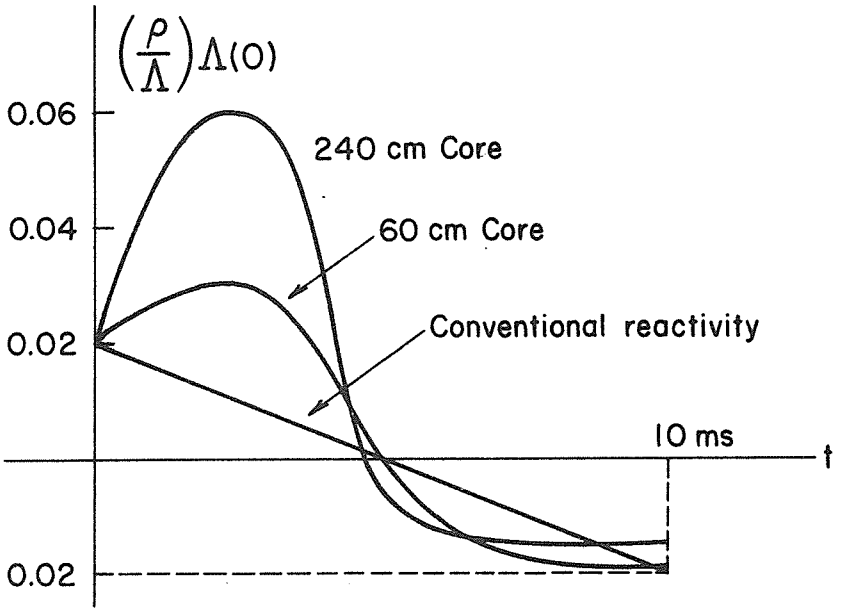


fission cross sections. On the basis of this reactivity and the $\Lambda = 10^{-5}$ sec., the "power" should behave as curve (1) below for both reactors. The computed G(t) values,however,are as shown by curves (2) and (3).



Presumably all these 3 curves should coincide. This, however, is not the case.

The ρ/Λ that must be used in order to have the computed G(t) as a solution of the kinetics equation is shown in the sketch below.



This figure illustrates how different a "reactivity" than the conventional reactivity must be used in order to reproduce the growth factor.

The important point to be learned here is that even in relatively small reactors, flux shape changes that occur over short periods of time make the usefulness of kinetics equations very doubtful.

In the next lecture I will discuss the concept of a space dependent transfer function. By approaching the problem from the general point of view that I have developed so far, I will prove quite rigorously that even though there may be transfer

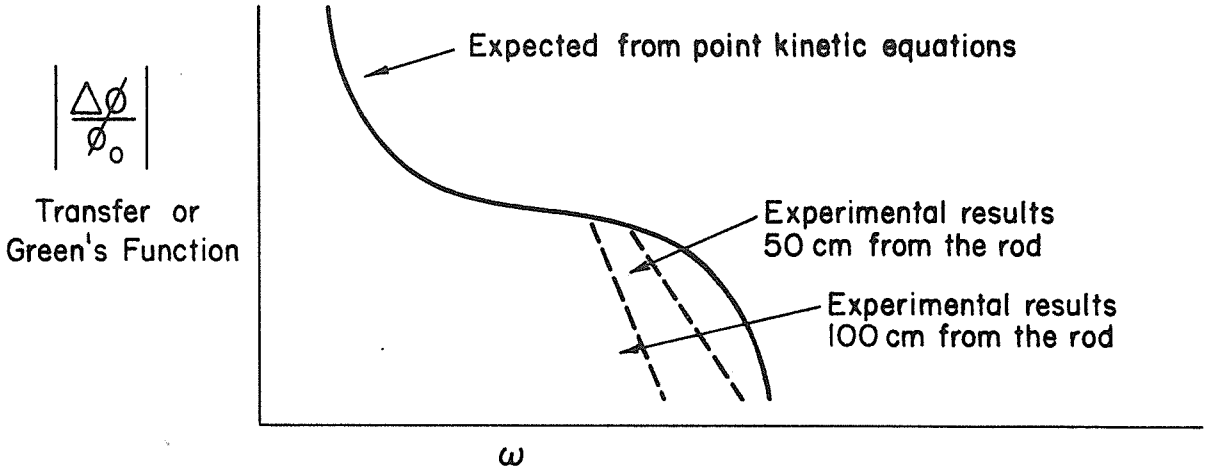
functions which are space dependent there cannot be space dependent transfer functions between neutron fluctuations and reactivity or average reactor power and reactivity. Such transfer functions, between flux and reactivity or power and reactivity, are always space independent.

II.1 CONSISTENT TRANSFER FUNCTIONS

The purpose of the first lecture was to show that point reactor kinetic equations are derived by averaging over various reactor properties. The averaging procedures lead to such quantities as reactivity, lifetime, and effective delayed neutron precursor fractions which can be meaningfully correlated with experiment only when there are no appreciable or very rapid flux shape changes. As illustrated by the computer experiment, difficulties are encountered if one tries to understand space-time reactor problems by looking only at one point (i.e. by using the point kinetic equations).

Today we will consider an example of an experiment which indicates not so much an error as a need for a consistent interpretation of results. The experiment was performed by P. T. Hansson and L. R. Foulke[2] on the Nora Reactor in Norway. A rod was oscillated sinusiodally in the center of the reactor in order to produce a sinusoidal oscillation in the absorption cross section. The neutron fluctuations, as a function of oscillator frequency, were measured at various distances from the rod. Results of the type sketched on the next page were obtained. Similar results were found to hold true for the phase angle.

2. Hansson, P. T. and L. R. Foulke, NSE, December, 1963.



As is shown, the difference between the experimental dependence of the fluctuations on position and the "theoretical curve" increases, for relatively high frequencies, as the distance between detector and oscillator increases.

In order to understand this phenomena, the authors solved a set of equations, such as those given by Weinberg and Wigner for wave effects, on an analog computer and found that under certain geometry conditions the sinusoidal oscillation of a localized absorber will result in neutron fluctuations with position dependent amplitude and phase. They correlated, to a good degree of approximation, the experimental results with this theoretical prediction.

They then took a third step and implied that here is a case of a space dependent transfer function. Their argument was that if the oscillated cross section is averaged with the steady state importance, or flux squared, then the resultant reactivity will also vary sinusiodally and in phase with the change in the absorption cross section. Therefore the experi-

mental results on neutron fluctuation are also proportional to the reactor transfer function and they indicate a space dependent transfer function.

If one defines reactivity as a quantity which is proportional to $\Delta\Sigma_a$, then this interpretation is correct. However, if the definition of reactivity given in the first lecture is adhered to then this interpretation is not acceptable. In order to have a consistent interpretation of the experiments, one must derive an analytical model which is appropriate to the prevailing experimental conditions and thus define the appropriate reactivity which corresponds to the experiment.

Specifically, in the experiment under consideration the flux is measured at one position as a result of a perturbation introduced at a different position. To correlate the results of this type of experiment one should start with some space-time dependent equations, for example the wave propagation model as given by Weinberg and Wigner, and then reduce these equations to some others which are time dependent only and which relate the measurement and the perturbation. Since a local measurement is being made, the reduction must be performed by multiplying the space-time dependent equations by the cross section of the counter and then integrating over all space. Upon performing this reduction, one finds that the resulting reactivity is neither in phase with, nor directly proportional to the cross section change, $\Delta\Sigma_a$. Instead, the relation between reactivity and absorption cross section involves an additional factor which has a position dependent amplitude and phase. In addition,

it is very easy to show that if the neutron fluctuations were divided by ρ/Λ calculated in this consistent manner, the result does not depend on space and is equal to what is considered as the space independent transfer function.

II.2 LIFETIME

In the last lecture we considered the concept of a neutron lifetime and noted that except for cases when the reactor is on an asymptotic period (flux separable) the concept of lifetime or that of reactivity are not completely meaningful. Both ρ and Λ have a time dependence which can be quite strong, particularly during fast shape changes. In spite of this, we do use the lifetime as an indication of the type of reactor under consideration and in fact the value of this parameter is one way of distinguishing the group attending these lectures, $\Lambda \sim 10^{-7}$ to 10^{-8}, from other groups interested in water reactors, $\Lambda \sim 10^{-4}$.

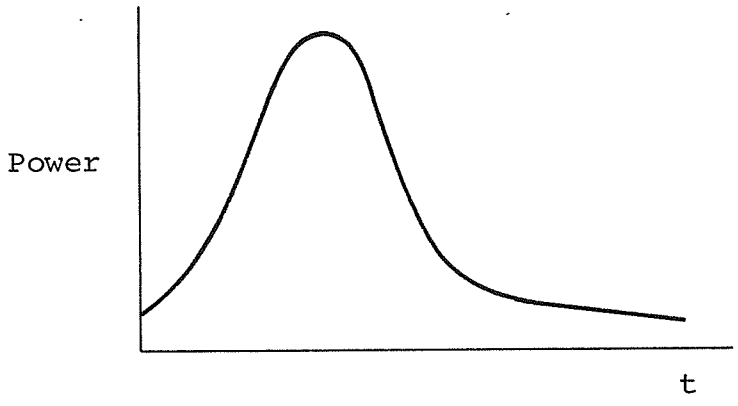
As an exercise, let us now pose the following question. Assume that we have a choice between two types of reactors, thermal and fast. Both reactors have a negative Doppler coefficient of reactivity and both are designed not to melt. Considering only these criteria, which of the two reactors is safer? The usual answer is that a thermal reactor is safer since it is more sluggish. To see whether this is indeed true, let us look at the following arguments.

First, let us consider the controllability of a power reactor, that is, the ability of a reactor to change power level

by 10 or 20% etc. in order to adjust to load requirements. This is a slow type of change. Consequently, it does not depend on the neutron lifetime because in this regime of operation the delayed neutrons play a predominant role and define the "time constant" of the system. This time constant is of the order of 0.1 sec, i.e. several orders of magnitude longer than the thermal or fast lifetime.

If we consider the power level at which loss of linear stability occurs (i.e. the power level at which the "linear approximation" to the point kinetic equations with feedback becomes unstable), then for lagging feedback transfer functions the fast reactor is preferable since the shorter the lifetime the higher the power level at which we have loss of linear stability. Of course this statement presumes that all other parameters are equal for the thermal and fast reactors in question.

Finally, let us consider the question of safety with regard to large step reactivity insertions which are compensated by the Doppler effect. Under these conditions the power burst is as shown in the sketch below. The quantity of interest



from the safety point of view is the integral under the burst curve or the total energy released and stored in the fuel. If it is assumed that the negative Doppler feedback reactivity varies as the n-th power of the energy stored in the reactor, then the total energy is independent of the lifetime. In other words, under the same conditions of step excitation and feedback strength, the same amount of energy will be released and stored in the reactor during a burst, regardless of the value of the lifetime. Thus again the lifetime is not a quantity which can be used to characterize the course of this hazardous condition.

With regard to loss of coolant, fast reactor accidents with core meltdown and reassembly, it turns out also that the energy released during the accident is not a very strong function of the lifetime.

The reason for the preceding exercise and the accompanying arguments and remarks is the following. Historically there has always been some fear associated with the safety features of fast reactors because of their short neutron lifetime. It is evident that this fear is not justified since the lifetime is not necessarily the most important parameter which characterizes serious accidents.

II.3 APPLICABILITY OF THE KINETICS EQUATIONS TO STABILITY STUDIES

Before considering the problem of stability itself, let us have another look at the point kinetics equations.

In the first lecture we assumed that the neutron density could be separated into a product of two functions one of which,

N_g, is bounded for all values of its arguments. The shape function N_g enters into the definitions of ρ, $\bar{\beta}$ and Λ and the assumed boundedness of N_g is consistent with the intuitive interpretation that we give these quantities. In other words since it takes a finite time for neutrons to move around the reactor, Λ is bounded. Also since there is a finite number of neutrons per fission, the excess of neutrons which cause the neutron density to grow or decay is finite and therefore ρ is bounded etc.

In view of these remarks we may argue that the kinetics equations can be used for stability studies of the growth function G(t) by assigning to the coefficients ρ/Λ and $\bar{\beta}/\Lambda$ a variety of bounded variations either as functions of time or as functionals of G(t). Of course, by treating a large variety of changes in the coefficients, one hopes to cover a large number of, if not all, conceivable stable or unstable behaviors of G(t) and establish conditions for overall reactor stability or instability.

Of course, it must be emphasized that stability analyses of this type are by themselves not adequate to understand a reactor dynamically. The reason is that G(t) may be stable and yet the neutron density may involve such localized peaks that certain fuel elements melt. Therefore in a large reactor, even when it is stable, one must make an explicit space-time analysis.

At this point, one may raise the justified question, "Since it is necessary to analyze the space-time behavior of the reactor why bother with the point kinetics?"

One answer to this question is that in many cases it is helpful to know beforehand whether the overall power is bounded or not. The reason is that usually the exact space-time solution cannot be found and instead it is approximated by a series expansion. A series expansion approximation may introduce mathematical restrictions for the convergence of the series. Such restrictions may not necessarily reflect the physics of the problem or the properties of the exact solution of the problem. One example of the kind of difficulty that I am talking about is the function $1/(1+x)$, defined for all $x > 0$. This function can be expanded as:

$$1/(1+x) = 1 - x + x^2 - x^3 + \ldots \qquad (8)$$

Note that the series expansion is valid only for $x < 1$. The series diverges for $x \geq 1$. The function $1/(1+x)$ is very well behaved for all $x \geq 0$ and yet its series expansion is not. Of course in this case we can combine the terms of the series and have the function in closed form. This, however, is not always possible. In summary, because series expansions can introduce unnecessary stability conditions, it is often practical to develop stability requirements by examining the behavior of the overall function rather than the behavior of some series expansion of the function.

II.4 SOME STABILITY CRITERIA

In attempting to derive stability criteria on the basis of the ordinary kinetics equations we approach the problem in the following ways:

a) First we consider the form of these equations with the coefficients of $G(t)$ and $C_i(t)$ as bounded functions or functionals of $G(t)$. Many useful properties concerning $G(t)$ can be thus derived. For example, it can be shown that no physical reactor admits a finite escape time. This is simply a consequence of the bounded nature of the coefficients of the kinetics equations.

b) Next we may consider that the kinetics equations do not contain any logarithmic terms and that the reactivity can be expressed as

$$\rho = \rho_{ex} + \int_0^t f(t-\tau)\left[G(\tau) - G_0\right]d\tau \quad . \tag{9}$$

Thus we may attempt to establish conditions which must be satisfied by $f(t)$ so that the reactor is stable.

c) Finally we may assume that the logarithmic term is present and thus try to see what kind of conditions must be satisfied in order for the reactor to be stable.

In what follows we will observe that in both cases (b) and (c) the conditions that we will derive will pertain to the linear properties of the system. This is desirable because we do not as yet have experimental means for identifying nonlinear system characteristics.

Suppose we write the kinetics equations in the form:

$$\frac{dG}{dt} = \frac{\rho - \bar{\beta}}{\Lambda} G + \sum_{i}^{m} \lambda_i C_i$$

$$\frac{dC_i}{dt} = \frac{\beta_i}{\Lambda} G - \lambda_i C_i \; ; \quad i = 1, 2, \ldots m \, ,$$

(10)

and then consider briefly what exists in the nuclear literature about the stability of the solutions of equations of this type if ρ is of the form given by Equation (9).

To begin with, the form of Equation (9) has the following implications:

1) Whatever the feedback effects are, they are decoupled (i.e. moderator temperature changes result in a moderator reactivity effect but do not affect the reactivity effects of the fuel etc.).

2) The reactivity effects are linear. In other words, they are related to their causes by a linear relationship. For example, if the void changes by a certain amount, the void reactivity effect changes directly proportionally to the void change etc.

3) The equations which relate the reactivity effect causes to the energy stored in the reactor are linear. In other words, the fuel temperature is a linear function of the energy stored in the fuel etc.

In 1955 Welton established a condition on the feedback kernel, f(t), which is sufficient to insure that any solutions of the nonlinear problem described by Equations (9) and (10) are asymptotically stable. This condition may be stated as

$$\mathrm{Re}\left\{ F(s)\Big|_{s=j\omega} \right\} \leq 0 , \tag{11}$$

where F(s) is the Laplace transform of f(t).

The appealing aspect of inequality (11) is that asymptotic stability for solutions of a nonlinear system is guaranteed by imposing restrictions on the linear properties of the system (properties which can also be established experimentally, i.e. through transfer function measurements).

There are, of course, drawbacks to Welton's criterion.

1) Most of the operating reactors do not satisfy Equation (11). Yet all of these reactors are operating stably and safely.

2) The implication of the criterion is that the reactor is asymptotically stable regardless of the value of the operating power level. This is too severe a restriction to be placed on the design of any reactor since material problems limit the operating power level anyway.

3) The criterion does not include the effect of delayed neutrons. It can be shown that stability requirements derived without delayed neutrons may be either overconservative or sometimes less restrictive than necessary. Therefore, delayed neutrons must be considered.

4) By satisfying Equation (11) one automatically guarantees asymptotic stability. However, this may not be necessary. It may be as good and as practical to design a reactor to admit simply bounded solutions for a given range of operating power levels (Lagrangian Stability). It can be shown that when Lagrangian Stability is acceptable, the conditions which must be satisfied by the reactor parameters are not as restrictive as those necessary for asymptotic stability.

On the basis of these motivating criticisms I will discuss methods of evaluating other stability criteria either for asymptotic or Lagrangian stability in the next lecture.

III.1 CONTINUATION OF STABILITY ANALYSIS

So far in this series of lectures we have considered the neutron balance problem from a general point of view and we have reduced the pertinent equations into a form which at least apparently is simpler than the original one. We also looked at the necessity for meaningful experimental interpretation of results and found that difficulties can occur in the case of fast or large transients. We then looked at the applicability of the kinetic equations to stability studies.

In view of some of the previous comments we would like to know how, if the point kinetics equations are to be used for stability studies, the results of the analysis can be implemented experimentally. We suppose for a moment that we are not very much concerned about the physical meaning of the growth function etc. Instead, suppose that we will try to manipulate the dynamic equations in the hope that we can arrive at conclusions which guarantee stability or boundedness and which can be implemented experimentally.

In finding stability criteria we will use the kinetic equations in their ordinary form along with the following guidelines. 1) We will not try to design a reactor which can withstand an infinite power since we are limited by materials considerations regardless of the results of the analysis. 2) We will try, insofar as possible, to introduce the properties of the delayed neutrons into the stability criteria so that we can either take advantage of their damping effects or to account for their destabilizing effects.

3) We will try to arrive at results which guarantee either asymptotic stability or Lagrangian stability.

Let us now return to the general kinetics Equations (5-6) which are:

$$\frac{dG}{dt} = \frac{\rho-\bar{\beta}}{\Lambda} G + \sum_i \lambda_i C_i - \left[\frac{\partial}{\partial t} \ln \int d^3\underline{r} dE d\underline{\Omega} N^* N_g \right] G$$

$$\frac{dC_i}{dt} = \frac{\bar{\beta}_i}{\Lambda} G - \lambda_i C_i + \left[\frac{\partial}{\partial t} \ln \int d^3\underline{r} dE d\underline{\Omega} N^* N_g \right] C_i \quad . \tag{12}$$

We may note in passing that the logarithmic term may be interpreted as a measure of the rate of change of the lifetime Λ.

The quantities $\bar{\beta}_i$ and Λ are, in general, functions of time and/or the average reactor power. We can, however, write them as a sum of some steady state or asymptotic values corresponding to an operating level G_o and some increments. The increments we are talking about in this case are not necessarily small but rather deviations from a convenient reference. Thus Equations (12) can be lumped into one equation of the form

$$\frac{dg}{dt} = f_o(g(\tau < t)) \left[g + G_o \right] - \int_o^t d(t-\tau) g(\tau) d\tau$$

$$d(t) = \sum_i (\beta_{io}/\Lambda_o)(\delta(t) - \lambda_i e^{-\lambda_i t}) \tag{13}$$

$$G(t) = g(t) + G_o \quad , \tag{14}$$

where β_{io}, Λ_o are the asymptotic values of $\bar{\beta}_i$ and Λ at G_o and the functional $f_o(g(\tau < t))$ contains all the increments arising from $\bar{\beta}_i$ and Λ as well as the expressions for ρ and the logarithmic term. The argument $\tau < t$ implies that the functional f_o depends only on the past history of g. Note that the contribution of delayed neutrons under asymptotic conditions (invariant Λ, $\bar{\beta}_i$) is represented by the convolution integral with the kernel d(t).

The functional f_o can also be written as:

$$f_0 = f_1 + f_2(g(\tau<t)) \tag{15}$$

$$f_1 = \int_0^t f(t-\tau)g(\tau)\,d\tau \quad . \tag{16}$$

In other words f_1 represents the linear feedback under invariant $\bar{\beta}_i$ and Λ, while f_2 accounts for all nonlinearities. The exact functional relation between f_o and $g(\tau < t)$ for different types of transients is difficult to identify. However, since there is a limited number of neutrons per fission there are some properties of the functional f_o which can be stated without actually ever stating the exact functional relation. For example, for very small changes of g(t) compared to G_o, f_o can be approximated by the linear convolution integral f_1 (Equation 16). This is the analytical model which leads to oscillation tests. When the changes in g(t) are large and/or fast, then the functional f_2 also becomes important, but regardless of its

exact value it must vary with f_1 so that f_o remains bounded for large values of $|f_1|$ (as shown in Figure 4). The meaning of this figure is as follows. If the functional f_o were always linear, then its behavior vs. f_1 would be as indicated by the dashed line. Since f_o is related to the excess number of neutrons per fission (which make g(t) grow or decay) when $|f_1|$ is large, f_o must saturate. Therefore for f_1 large positive, f_2 must be negative etc. There is, of course, some gray area where linear theory applies and where f_o saturates in which we must know the exact value of the functional f_o in order to be able to compare to f_1.

With the preceding assumptions and definitions and after some elementary algebra, Equation (13) can be manipulated to yield:

$$\frac{d}{dt}\left[\int_0^t d\tau \int_0^\tau k(\tau-\lambda)g(\tau)g(\lambda)d\lambda\right] = g^2(t)\int_0^t f(t-\tau)g(\tau)d\tau + g(t)(g(t)+G_0)f_2 + g(t)I_0(t)\,, \tag{17}$$

where k(t) is the inverse Laplace transform of $(s+D(s)-G_oF(s))$, D(s) and F(s) are the Laplace transforms of d(t) and f(t) respectively and $I_o(t)$ depends on the initial conditions. Note that:

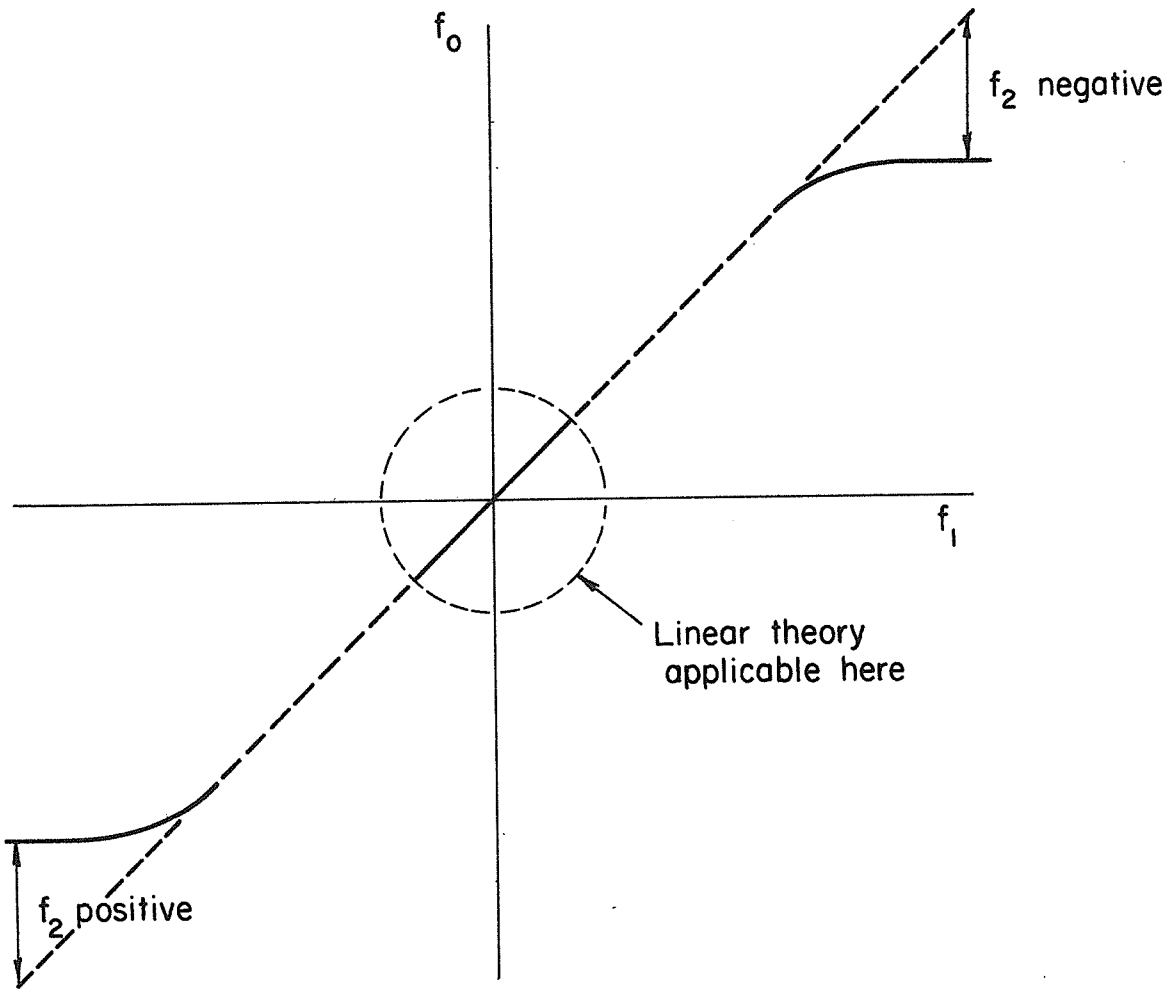

f_o
f_2 negative
f_1
Linear theory
applicable here
f_2 positive

FIGURE 4

$H(s)$ = reactor transfer function at power G_o =

$$= \left[s+D(s)-G_oF(s) \right]^{-1} .$$

If we can guarantee something about the stability of the reactor by looking only at the properties of the function $k(t)$ then we will have achieved the goals we set out to reach at the beginning of this discussion. Indeed $k(t)$ depends on the delayed neutrons and the operating power level and in addition is related to linear properties of the system which are experimentally identifiable without hazardous experiments.

To this end suppose that we require that

a) $k(t)$ be a positive definite function

b) $f(t) < 0$.

The definition of a positive definite function according to KyFan[3] is that the double integral in Equation (17) is positive regardless of the values of $g(t)$. In fact if $k(t)$ is positive definite then its Laplace transform is a positive real function. In other words, the phase shift of $\left\{ \mathcal{L}k(t) \right\}_{s=j\omega}$ is less than $\pm 90^o$ or the real part, $\mathrm{Re} \left[H(s) \right]^{-1}_{s=j\omega}$, is positive.

Note that the requirement that $k(t)$ be a positive definite function is less restrictive than Welton's criterion, provided that we consider only a limited range of G_o. The reason for this is that $\mathrm{Re} \left[s+D(s) \right]_{s=j\omega} > 0$ and so $\mathrm{Re}F(j\omega)$ does not

3. KyFan, M., "Les Fonctions Definies Positives et les Fonctions Completement Monotones, Leurs Applications au Calcul des Probabilites et a la Theorie des Espaces Distancies." Memorial des Sciences Mathematiques, Fascicule CXIV, Paris (1950).

have to be negative for all values of ω in order for $H(j\omega)$ to be positive.

Note also that if $k(t)$ is positive definite then it can be thought of as the input impedance of a passive electrical network. If $g(t)$ is the input current, then the double integral in the left hand side of Equation (17) can be thought of as the energy supplied to the network and therefore is a positive quantity.

Now if the time rate of change of the double (positive) integral is negative for large values of $g(t)$, then $g(t)$ will be Lagrange stable. It turns out that this is indeed the case for $f(t) < 0$. The proof is as follows:

We know that $g(t)$ has a lower negative bound (i.e. $g(t)$ cannot be less than $-G_o$ because this would imply negative power). On the other hand for $f(t) < 0$ there is some level g_1 and a time T such that when $g > g_1$ and $t > T$

$$f_1 = \int_0^t f(t-\tau)g(\tau)d\tau < 0 \ .$$

For such a f_1, f_2 varies more slowly than f_1. Therefore the right hand side of Equation (17) has the sign of the first term, namely it is negative. Therefore $g(t)$ is bounded both from above and below, provided that the two stipulated conditions are satisfied.

Incidentally, note that we have also proved asymptotic stability in the small since the reactor transfer function at power G_o is a positive real function.

Also, depending on the exact form of f(t) the Lagrangian stability considered above may be equivalent to asymptotic stability.

Actually, the assumption $f(t) < 0$ implies Lagrangian stability regardless of whether k(t) is positive definite or not. In order to prove this the kinetics equations must be written in a slightly different form but we will not pursue this point any further.

It must be emphasized that the derived conditions on k(t) and f(t) are only sufficient. Others could be derived by rearranging the equations in a different manner. However, the derived conditions account for the delayed neutrons, for the finite range of power level and can tolerate feedback functions which do not satisfy Welton's criterion.[4]

III.2 EQUILIBRIUM STATES AT INFINITY

Another method for studying stability is based on the properties of equilibrium states at infinity. The technique is probably applicable to any system which can be described by ordinary differential equations.

We will assume that we have succeeded in approximately describing the behavior of a physical system by means of a set of ordinary autonomous differential equations of the form:

$$\frac{dx_i}{dt} = X_i (X_1, X_2, \ldots X_n) ; \quad i = 1, 2, \ldots n, \tag{18}$$

4. This work was done under Project SIFTOR. The first volume of this work should be released by December 1964 by MIT Press.

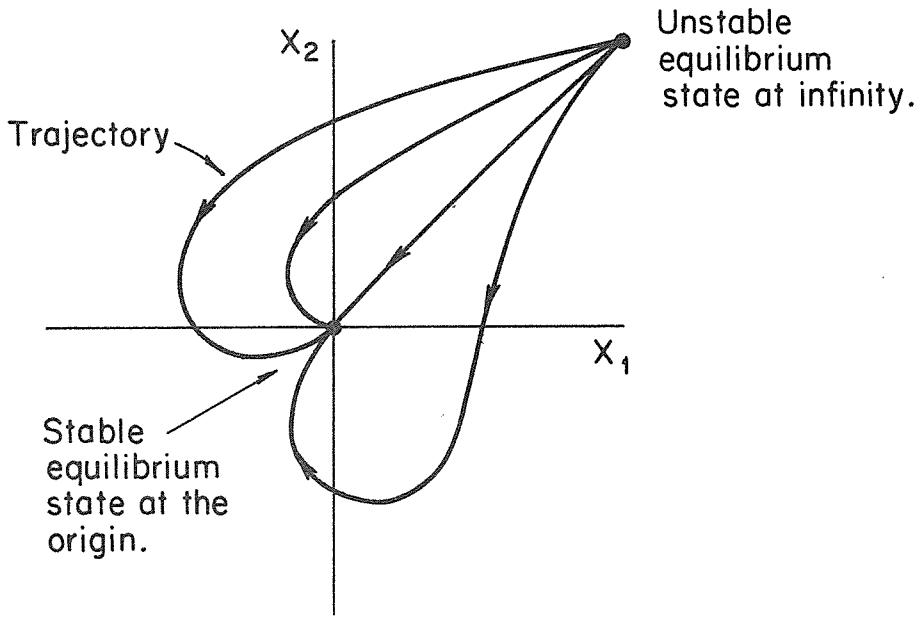
where x_i are the variables of interest and X_i is some nonlinear function of these variables. As was pointed out before, these equations may be applicable only over a limited range of the variables.

The usual method of approach to the problem of stability is to find the equilibrium states by looking for solutions of the set of equations

$$X_i(X_1, X_2, \ldots X_n) = 0 \; ; \quad i = 1, 2, \ldots n) \tag{19}$$

and then to study the stability of these equilibrium states, some of which may be stable, others unstable, others may correspond to limit cycles, etc.

For the moment let us restrict ourselves to a two dimensional system that has only one equilibrium state which, with no loss of generality, we will choose at the origin of phase space. Let all the trajectories described by the solutions of the system converge to the origin as shown in the sketch below.



These trajectories must start from somewhere. If there are no other equilibrium states in the finite phase space, there must be some at infinity. (Equilibrium states at infinity can be disclosed by a transformation of variables into homogeneous coordinates). If we assume for the sake of discussion that there is only one equilibrium state at infinity, then the complete portrait of the trajectories of the system must be as shown in the sketch. Since trajectories originate from the equilibrium state at infinity, this state must be, at the least, locally unstable.

The importance of the preceding oversimplified remarks must now be evident. Given a system of differential equations, if it can be shown that all its equilibrium states at infinity are locally unstable, then all the solutions of the system are bounded. In other words, the conditions for all equilibrium states at infinity to be locally unstable or stable constitute the necessary and sufficient conditions for the solutions to be bounded or unbounded respectively, regardless of initial conditions.

This simple thought has been up to now explicitly excluded from investigation because it was argued that one should not consider solutions for large values of the variables since the equations are (physically) valid only over a limited range of the variables. This statement is true. However, if one can determine the behavior of the solutions in the finite space by looking at conditions on the solutions at infinity, the results are just as acceptable as if they had been derived by any other

method. The reason is that the equations are mathematically valid over the entire range of the variables, regardless of whether they are physically correct over the same range or not.

As we already mentioned, examination of equilibrium states at infinity requires transformation into homogeneous coordinates, i.e. x_i is replaced by x_i/z etc. and then z is allowed to approach zero etc.

An example of this method will be discussed next time.

IV.1 REVIEW

Last time we attempted to answer some questions of stability by formulating the problem in terms of the function k(t) (see Equation 17). The double integral in the left hand side of Equation (17) was interpreted as an "energy" function of an equivalent passive electrical network. The double integral could also be thought of as a positive definite function with a negative time derivative, namely as a Liapunov function. Whatever the interpretation, the important result of that discussion is that we have related the nonlinear stability of the system to linear properties which can be computed for analysis and experiment. They also can be measured experimentally for verification and comparison of theory.

We also discussed a technique for the derivation of conditions of boundedness of solutions of nonlinear ordinary differential equations based on the local stability properties of equilibrium states at infinity. This technique has been used extensively by us for problems in which the nonlinearities are of the rational polynomial type. The essential point here is that if the equilibrium states at infinity are locally unstable, then the solutions are bounded.

IV.2 AN EXAMPLE

As an example of the application of the method of using the stability properties of the equilibrium states at infinity to derive conditions for boundedness, let us consider a simple

xenon problem. The normalized equations which represent this problem are

$$\frac{d\emptyset}{dt} = \omega_o (1 - X)\emptyset \tag{20}$$

$$\frac{dX}{dt} = \lambda_x (\alpha\emptyset + \beta I - X - \gamma\emptyset X) \tag{21}$$

$$\frac{dI}{dt} = \lambda_i (\emptyset - I) , \tag{22}$$

where $\emptyset$ is the flux, X is the xenon concentration, I is the iodine concentration, λ_x and λ_i are decay constants for the appropriate radioactive species.

$$\omega_o = \delta_o / \tau_e \quad ; \quad y = y_x + y_I$$

$$\alpha = y_x / (y - c\delta_o) ; \; \beta = y_I (y - c\delta_o) ; \; \gamma = c\delta_o / (y - c\delta_o) .$$

δ_o is the reactivity added, τ_e is the equivalent lifetime for neutrons (i.e. it is dominated by the delayed neutrons), y_x is the xenon yield, y_I is the iodine yield and c is a constant which converts the xenon concentration into reactivity effects.

This problem was first studied by Jack Chernick.[5] Since the exact yields of xenon and iodine in a reactor are not known,

5. J. Chernick, NSE 8, 233 (1960).

Chernick considered the entire range of possible yields in order to study the stability of the reactor.

We now want to analyze this problem by the method of equilibrium states at infinity. By this method we will derive both necessary and sufficient conditions for boundedness.

In order to put the equilibrium states at infinity in evidence, we replace $\emptyset$ by $\emptyset/z$, X by X/z and I by I/z and then let z approach zero. The details of this procedure will be omitted. However, for our purposes it suffices to say that after the change of variables and elimination of the time between the resulting equations we find that:

$$\left\| \begin{array}{cccc} d\emptyset & dX & dI & dz \\ \emptyset & X & I & z \\ \omega_o(z-X)\emptyset & \lambda_x(\alpha\emptyset z + \beta I z - Xz - \gamma\emptyset X) & \lambda_i(\emptyset - I)z & 0 \end{array} \right\| = 0$$

where $\| \quad \|$ means that any 3x3 determinant in this matrix is zero or, said differently, the rank of the matrix is two. It is evident that any 3x3 determinant in the above example yields a combination of some two of the original three equations from which time has been eliminated.

The equilibrium states at infinity are found by letting $z \longrightarrow 0$. Thus we obtain

$$\left| \begin{array}{ccc} d\emptyset & dX & dI \\ \emptyset & X & I \\ -\omega_o\emptyset X & -\lambda_x\gamma\emptyset X & 0 \end{array} \right| = 0 \qquad (23)$$

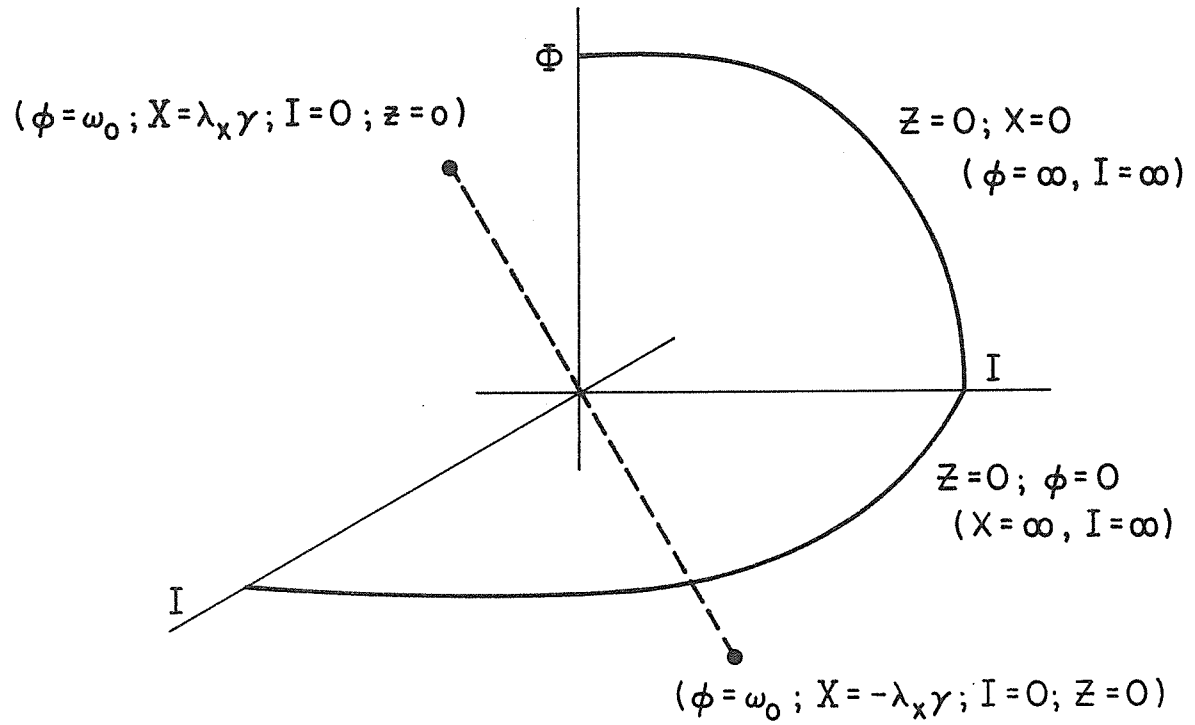
Equation (23) has the following solutions

(a) $z = 0 \quad ; \quad \emptyset = 0$

(b) $z = 0 \quad ; \quad X = 0$

(c) $z = 0 \quad ; \quad \emptyset = \pm \omega_o \quad ; \quad X = \pm \lambda_x \gamma \quad ; \quad I = 0.$

The first two are lines at infinity while the last consists of two points. A sketch of the possible singularities at infinity is shown below:

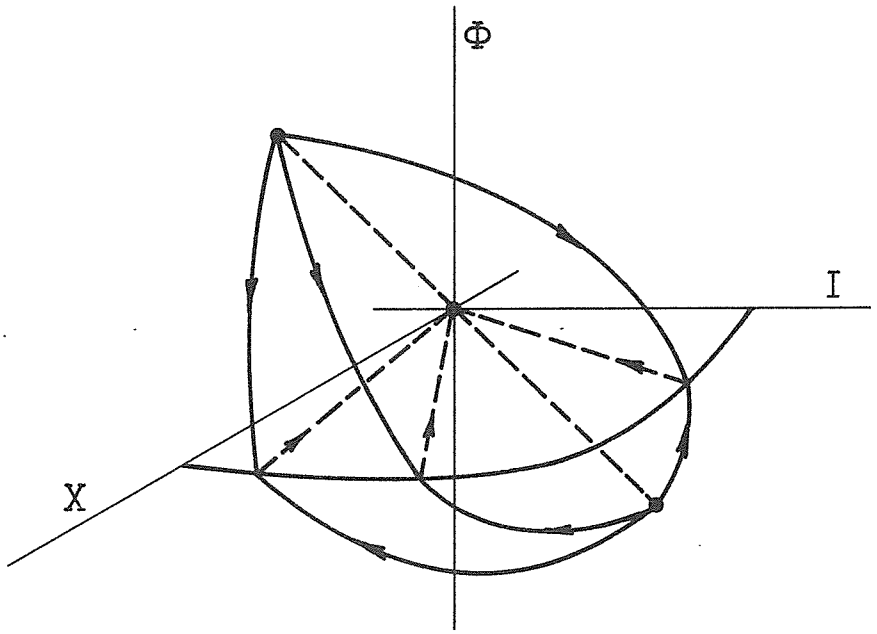


In order to examine the types of stability at the two distinct points and along the lines $X = 0$ and $\emptyset = 0$, we must make a projective transformation to bring the points from infinity to a finite region of space and we must make the transformation in such a manner that the eigenvalues at the equilibrium states are not altered. After making this transformation

we then linearize the system around the projected equilibrium points and determine whether they are stable or unstable.

If we do all this we find that the three eigenvalues at each of the two distinct points ($\phi = \pm\omega_0; X = \pm\lambda_x\gamma; I = 0; Z = 0$) are equal to $\gamma\lambda_x$. Both of these equilibrium states are thus unstable and trajectories can only originate from these two points.

To consider the results along the line $\emptyset = 0$; $z = 0$, we first note that the hyper-surface $z = 0$ is an integral hyper-surface of the system, and in addition that $\emptyset = 0$ is also an integral surface of the system. On $z = 0$ we established that the two distinct points are unstable and thus that trajectories must emanate from them. In other words the trajectories will tend to end on the line $\emptyset = 0$; $z = 0$ as shown in the sketch below.



On the other integral surface, $\phi = 0$, the trajectories are described by the two equations

$$\frac{dX}{dt} = \lambda_x X + \lambda_x \beta I$$

$$\frac{dI}{dt} = -\lambda_I I .$$

This is a linear system of equations and it has one stable equilibrium state at the origin (the eigenvalues are $-\lambda_x$ and $-\lambda_I$). Therefore, the trajectories on the integral surface, $\phi=0$, must approach the origin as straight lines and appear as originating from the line $\phi=0$; $z=0$. Thus, the intersection of the two integral surfaces, $\phi=0$ and $z=0$, which is the line of interest, is effectively a saddle line and consequently an unstable line.

Finally we have the line $X=0$, $z=0$. For this case there are certain conditions on the coefficients that must be considered. When these conditions are satisfied,no trajectories end on this line. When these conditions are not satisfied, trajectories do end on the line $X = 0$; $z = 0$. Consequently these conditions are the necessary and sufficient conditions for stability or instability of the system. We will not write these conditions here explicitly.

Although the analysis of problems of this kind may be involved, the results of this type of analysis are gratifying since one obtains necessary as well as sufficient conditions for boundedness by examining local stability of equilibrium states at infinity. We feel that the method has a lot of merit

and should be further exploited.

IV.3 APPLICABILITY TO FAST REACTORS

So far no explicit mention of the dynamics of fast reactors has been made even though this presumably is a conference on Fast Reactors. The reason for the omission is intentional because we feel that there is no difference either in the form of the equations or the analytical procedures which are used in fast or thermal reactors. The only difference lies in the way we calculate, if we can, the coefficients which enter the kinetics equations. These calculations may at times become very involved, particularly when we try to account for reactivity phenomena arising either from structural charges such as bowing effects or from phase changes such as boiling etc.

In this regard, a special problem which requires specific techniques for the calculation of reactivity etc. and which is peculiar to fast reactors is the problem of core meltdown and reassembly into a supercritical mass. Some techniques have been developed but will not be discussed here.