MICHIGAN MEMORIAL PHOENIX PROJECT
THE UNIVERSITY OF MICHIGAN

RECENT DEVELOPMENTS IN NEUTRON TRANSPORT THEORY

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CONTENTS

I. The Normal Modes of the Transport Equation

I will describe in these lectures the method of elementary solutions. We will consider neutron transport phenomena under quite severe restrictions. We assume that all neutrons are travelling at the same speed (the "one-group" problem) and that the medium through which they are travelling is homogeneous and isotropic. All essential features of the method appear on consideration of this idealized situation.

Choosing appropriate units we may write the transport equation for the neutron distribution function as

$$\text{(I-1)} \qquad \frac{\partial}{\partial t}\psi(\vec{r},\hat{\Omega},t) + \hat{\Omega}\cdot\vec{\nabla}\psi + \psi = \frac{1}{4\pi}\int f(\hat{\Omega}\cdot\hat{\Omega}')\,\psi(\hat{\Omega}')\,d\hat{\Omega}' + Q(\vec{r},\hat{\Omega},t)\ ,$$

where $f(\hat{\Omega}\cdot\hat{\Omega}')$ is the number of neutrons that enter the system with direction $\hat{\Omega}$ at any point following the absorption of one neutron with direction $\hat{\Omega}'$ at that point.

For the time being, we simplify even further. We assume plane symmetry and steady-state conditions. Then the distribution function $\psi(\vec{r},\hat{\Omega},t)$ will depend only upon x, and $\hat{\Omega}\cdot\hat{X} = \mu$, the cosine of the angle between the neutron velocity and the x-axis. Also, we assume isotropic scattering. The transport equation becomes

$$\text{(I-2)} \qquad \mu\,\frac{\partial\psi(x,\mu)}{\partial x} + \psi = \frac{c}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu' + Q(x,\mu)\ ,$$

where c is the average number of neutrons produced per collision.

We shall look for solutions of the homogeneous (source-free) equation, and then form our general solution as a superposition of these "elementary solutions." In order to accomplish this, it will be necessary to include in the space of admissible solutions not only ordinary functions but also distributions in the sense of Schwarz. Thus, making the ansatz

$$\psi_\nu(x,\mu) = e^{-x/\nu}\,\varphi_\nu(\mu), \tag{I-3}$$

the "functions" $\varphi_\nu(\mu)$ satisfy the integral equation

$$\left(1-\frac{\mu}{\nu}\right)\varphi_\nu(\mu) = \frac{c}{2}\int_{-1}^{1}\varphi_\nu(\mu')\,d\mu'. \tag{I-4}$$

Since this equation is homogeneous in φ_ν, the normalization is arbitrary. For convenience, we shall normalize such that

$$\int_{-1}^{1}\varphi_\nu(\mu)\,d\mu = 1. \tag{I-5}$$

Then (I-4) becomes

$$(\nu-\mu)\,\varphi_\nu(\mu) = \frac{c\nu}{2}, \tag{I-6}$$

with the normalization (I-5) forming a subsidiary condition restricting the allowed values of ν.

There are two classes of solutions.

1. ν does not lie in the interval $[1,1]$ on the real line. For this case

(I-7) $$\varphi_{0\pm}(\mu) = \pm\frac{c}{2}\frac{\nu_0}{\pm\nu_0-\mu} \; ;$$

where $\pm\nu_0$ are the two roots of

(I-8) $$\Lambda(\nu) = 1-\frac{c\nu}{2}\int_{-1}^{1}\frac{d\mu}{\nu-\mu} = 1 - c\nu\tanh^{-1}\frac{1}{\nu} = 0.$$

For $c<1$ ν_0 is real and $\nu_0>1$.

For $c>1$ ν_0 is purely imaginary.

(The case c=1 will be treated separately.)

As is well known, the two roots $\pm\nu_0$ form the entire discrete spectrum for this problem.

2. ν is real and $-1\leq\nu\leq 1$. As before, we write for the solution of (I-6)

(I-9) $$\varphi_\nu(\mu) = \frac{c\nu}{2}\frac{1}{\nu-\mu} \; .$$

Now the expression is purely formal since it contains a singularity. Actually, in any physical problem we shall be working with superpositions of normal modes---expressions of the form

(I-10) $$\psi(\mu) = \int A(\nu)\,\varphi_\nu(\mu)\,d\nu \; ,$$

and for a well-behaved expansion function $A(\nu)$ such integrals are well-defined so long as a prescription is given to specify the path of integration. In general,

$$\int_{-1}^{1} A(\nu)\frac{1}{\nu-\mu}\,d\nu = P\int_{-1}^{1} A(\nu)\frac{1}{\nu-\mu}\,d\nu + CA(\mu) \tag{I-11}$$

(P signifies Cauchy principal value), where C may take on any value depending upon how we choose the path of integration to avoid the pole at $\nu=\mu$. This property of the integral (I-10) can be expressed by writing $\varphi_\nu(\mu)$ in the form

$$\varphi_\nu(\mu) = \frac{c\nu}{2}P\frac{1}{\nu-\mu} + \lambda(\nu)\delta(\mu-\nu). \tag{I-12}$$

The P indicates than when the function φ_ν appears in an integrand, the principal value is to be taken. The number $\lambda(\nu)$ is chosen so that φ_ν satisfies the normalization condition. A solution φ_ν exists for all ν in the continuous range $[-1,1]$ provided we choose

$$\lambda(\nu) = 1 - \frac{c\nu}{2}P\int_{-1}^{1}\frac{d\mu}{\nu-\mu}. \tag{I-13}$$

Consider now (as a generalization) integrals of the form

$$F(z) = \int_{-1}^{1}\frac{f(\mu)\,d\mu}{\mu-z}, \tag{I-14}$$

which are functions of the complex variable z. These "Cauchy integrals" are discussed thoroughly in Muskhelishvili's book.* For our purposes the most useful computational aids are the so-called Plemelj formulae.

$$\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{-1}^{1} \frac{f(\mu)\, d\mu}{\mu - (\nu \pm i\varepsilon)} = \frac{1}{2\pi i} P \int_{-1}^{1} \frac{f(\mu)\, d\mu}{\mu - \nu} \pm \frac{1}{2} f(\nu), \tag{I-15}$$

The proof follows automatically from the definition of the principal value integral and the residue theorem. Furthermore, $F(z)$ is analytic in the complex plane cut from -1 to 1 along the real axis. We introduce the notation

$$F^{\pm}(\nu) = \lim_{\varepsilon \to 0^+} F(\nu \pm i\varepsilon), \tag{I-16}$$

i.e. $F^{\pm}(\nu)$ are the boundary values of the function $F(z)$ as z approaches ν from $\left\{ \begin{matrix} \text{above} \\ \text{below} \end{matrix} \right\}$ the cut. In terms of these boundary values, the sum and difference of the Plemelj formulae yield the useful (and concise) expressions:

$$\begin{aligned} F^{+}(\nu) - F^{-}(\nu) &= f(\nu) \\ \pi i \left(F^{+}(\nu) + F^{-}(\nu) \right) &= P \int_{-1}^{1} \frac{f(\mu)\, d\mu}{\mu - \nu} . \end{aligned} \tag{I-17}$$

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*Muskhelishvili, <u>Singular Integral Equations</u>, Noordhoff, Gröningen, Holland (1953).

In particular, $\lambda(\nu)$ may be written

(I-18) $$\lambda(\nu) = \frac{1}{2}\left(\Lambda^{+}(\nu) + \Lambda^{-}(\nu)\right).$$

The functions φ_ν are orthogonal in the following sense:

(I-19) <u>Theorem 1.</u> $\int_{-1}^{1} \mu\, \varphi_\nu(\mu)\, \varphi_{\nu'}(\mu)\, d\mu = 0$ when $\nu \neq \nu'$.

This orthogonality relation is obtained directly from the integral equation in the usual manner. We multiply (I-4) by $\varphi_{\nu'}(\mu)$ and integrate over μ from -1 to 1.

(I-20) $$\int_{-1}^{1} \varphi_{\nu'}(\mu)\left(1 - \frac{\mu}{\nu}\right)\varphi_\nu(\mu)\, d\mu = \frac{c}{2}\int_{-1}^{1}\varphi_\nu(\mu)\, d\mu \int_{-1}^{1}\varphi_{\nu'}(\mu)\, d\mu$$

The right hand side is invariant under interchange of ν and ν'. So then must be the left hand side. This implies that

(I-21) $$\left(\frac{1}{\nu} - \frac{1}{\nu'}\right)\int \mu\, \varphi_{\nu'}(\mu)\, \varphi_\nu(\mu)\, d\mu = 0,$$

from which Theorem 1 follows immediately. ν or ν' may come from either the discrete or continuous parts of the spectrum. Since we know the functions φ_ν, we may calculate the normalization integrals for the case $\nu = \nu'$. For the discrete modes we have simply

(I-22) $$\int_{-1}^{1} \mu\, \varphi_{0\pm}^{2}(\mu)\, d\mu = N_{0\pm} = \pm\frac{c}{2}\nu_0^{3}\left[\frac{c}{\nu_0^{2}-1} - \frac{1}{\nu_0^{2}}\right].$$

More care is required in dealing with the continuum. Then, the normalization integral is undefined when $\nu = \nu'$. However, we may write

$$\int \mu \varphi_\nu(\mu) \varphi_{\nu'}(\mu) d\mu = N(\nu) \delta(\nu - \nu') ,$$

which may be thought of as an abbreviation for the statement that if we expand a function $\psi(\mu)$ in terms of the φ_ν

(I-23) $$\psi(\mu) = \int_{-1}^{1} A(\nu) \varphi_\nu(\mu) d\nu \quad .$$

The expansion coefficients $A(\nu)$ will be given by

(I-24) $$\int_{-1}^{1} \mu \varphi_\nu(\mu) \psi(\mu) d\mu = N(\nu) A(\nu) .$$

Therefore, using the explicit form of the $\varphi_\nu(\mu)$

$$A(\nu) N(\nu) = \int_{-1}^{1} \mu \varphi_\nu(\mu) \psi(\mu) d\mu = \int_{-1}^{1} \mu \varphi_\nu(\mu) d\mu \int_{-1}^{1} A(\nu') \varphi_{\nu'}(\mu) d\nu'$$

(I-25)

$$= \int_{-1}^{1} \mu \left[\frac{c\nu}{2} P \frac{1}{\nu - \mu} + \lambda(\nu) \delta(\nu - \mu) \right] d\mu \int_{-1}^{1} A(\nu') \left[\frac{c\nu'}{2} P \frac{1}{\nu' - \mu} + \lambda(\nu') \delta(\nu' - \mu) \right] d\nu'$$

$$= \int_{-1}^{1} \mu \left[\frac{c\nu}{2} P \frac{1}{\nu - \mu} + \lambda(\nu) \delta(\nu - \mu) \right] d\mu \left[\frac{c}{2} P \int_{-1}^{1} \frac{\nu' A(\nu') d\nu'}{\nu' - \mu} + \lambda(\mu) A(\mu) \right]$$

$$= \frac{c\nu}{2} P \int_{-1}^{1} \frac{\mu}{\nu - \mu} \left[\frac{c}{2} P \int_{-1}^{1} \frac{\nu' A(\nu') d\nu'}{\nu' - \mu} \right] d\mu \quad +$$

$$+ \frac{c\nu}{2} P\int \frac{\mu}{\nu-\mu} \lambda(\mu) A(\mu)\, d\mu$$

$$+ \frac{c\nu}{2} P\int \frac{\nu'}{\nu'-\nu} \lambda(\nu) A(\nu')\, d\nu'$$

$$+ \nu \lambda^2(\nu) A(\nu) \quad = I_1 + I_2 + I_3 + \nu \lambda^2(\nu) A(\nu) \ .$$

Consider now $\overline{I}_1$ given by

(I-26) $$\overline{I}_1 = \frac{c\nu}{2} P\int_{-1}^{1} \nu' A(\nu') \left[\frac{c}{2} P\int_{-1}^{1} \frac{\mu\, d\mu}{(\nu'-\mu)(\nu-\mu)} \right] d\nu'$$

(i.e. $\overline{I}_1$ is I_1 with the orders of integration interchanged.)

(I-27) $$\overline{I}_1 = \frac{c\nu}{2} P\int_{-1}^{1} \nu' A(\nu') \left[\frac{c}{2} \frac{1}{\nu-\nu'} P\int_{-1}^{1} \left(\frac{\nu'}{\nu'-\mu} - \frac{\nu}{\nu-\mu} \right) d\mu \right] d\nu'$$

$$= \frac{c\nu}{2} P\int_{-1}^{1} \frac{\nu' A(\nu')}{\nu-\nu'} \left[\frac{c\nu'}{2} P\int_{-1}^{1} \frac{d\mu}{\nu'-\mu} - \frac{c\nu}{2} P\int_{-1}^{1} \frac{d\mu}{\nu-\mu} \right] d\nu'$$

$$= \frac{c\nu}{2} P\int_{-1}^{1} \frac{\nu' A(\nu')}{\nu-\nu'} \left(\lambda(\nu) - \lambda(\nu') \right) d\nu' \ .$$

(I-28) Thus $\overline{I}_1 + I_2 + I_3 = 0$.

However, we are dealing with one of those rare instances in which the orders of integration may not be interchanged.

(I-29) $\overline{I}_1 \neq I_1$.

In fact, by the Poincare-Bertrand formula,* we have

(I-30) $$I_1 = \bar{I}_1 + \nu\left(\frac{\pi c \nu}{2}\right)^2 A(\nu) .$$

(I-31) Therefore, $$N(\nu) A(\nu) = \nu\left[\lambda^2(\nu) + \left(\frac{\pi c \nu}{2}\right)^2\right] A(\nu)$$

for any well-behaved A(ν) and hence

(I-32) $$N(\nu) = \nu\left[\lambda^2(\nu) + \left(\frac{\pi c \nu}{2}\right)^2\right] .$$

As an application, we may calculate the Green's function for a uniform, infinite medium for the case c < 1. A unit plane source radiating in the direction μ_0 is located at the origin. Then $\psi(x,\mu)$ satisfies the homogeneous equation except at x = 0 and vanishes as $|x| \to \infty$, so we may write:

(I-33)
$$\begin{aligned}\psi_g(x,\mu'; 0,\mu_0) &= A_{0+}\varphi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)e^{-x/\nu}\varphi_\nu(\mu)\,d\nu \qquad x > 0\\ &= -A_{0-}\varphi_{0-}(\mu)e^{x/\nu_0} - \int_{-1}^0 A(\nu)e^{-x/\nu}\varphi_\nu(\mu)\,d\nu \qquad x < 0\end{aligned}$$

where the A's are to be determined. The source introduces a discontinuity of $\frac{1}{2\pi\mu}\delta(\mu-\mu_0)$ in the distribution function at the origin:

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*Muskhelishvili, p. 56.

$$\text{(I-34)} \qquad A_{0+}\varphi_{0+} + A_{0-}\varphi_{0-} + \int_{-1}^{1} A_\nu \varphi_\nu \, d\nu = \frac{1}{2\pi\mu}\delta(\mu-\mu_0)\,.$$

Thus the A's are just the coefficients in the expansion of $\frac{1}{2\pi\mu}\delta(\mu-\mu_0)$ in terms of the φ_ν, (in the next lecture I shall show that the functions φ_ν form a complete set---thus this expansion is permissible) and may be found by the orthogonality relations and equation (I-24).

The result is

$$\text{(I-35)} \qquad \psi_g(x,\mu\,;\,0,\mu_0) = \frac{1}{2\pi}\left[\frac{\mp\varphi_{0\pm}(\mu_0)\,\varphi_{0\pm}(\mu)\,e^{\mp x/\nu_0}}{N_{0\pm}} + \int_0^1 \frac{\varphi_{\pm\nu}(\mu_0)\,\varphi_{\pm\nu}(\mu)\,e^{\mp x/\nu}\,d\nu}{N(\nu)}\right] \qquad x \gtrless 0.$$

The Green's function for an isotropic plane source may be found by integrating ψ_g over μ_0, and the Green's function for the spatial density may be found by integrating over μ. The result is

$$\text{(I-36)} \qquad \rho^g(x,0) = \frac{1}{2}\left[\frac{e^{-|x|/\nu_0}}{N_{0+}} + \int_0^1 \frac{e^{-|x|/\nu}\,d\nu}{N(\nu)}\right]\,.$$

Note that for large $|x|$ the discrete term dominates.

II. Partial Range Completeness

Last time we made use of the fact that the φ's (discrete + continuum) formed a complete set of functions of μ in the interval $[-1,1]$. Today we shall prove a more general "partial range" completeness theorem which includes this as a special case.

Theorem 2. Let α, β be two real numbers such that $-1 \leq \alpha < \beta \leq 1$. Then the set of functions $[\varphi^{\alpha\beta}]$ is complete for all functions defined in the interval $\alpha \leq \mu \leq \beta$ where:

a) if $\alpha \neq -1$ and $\beta \neq 1$ $[\varphi^{\alpha\beta}]$ contains the φ_ν $\alpha \leq \nu \leq \beta$;

b) otherwise $[\varphi^{\alpha\beta}]$ contains the φ_ν $\alpha \leq \nu \leq \beta$ as well as φ_{0-} if $\alpha = -1$ and φ_{0+} if $\beta = 1$.

(Of course, in the case treated yesterday, $\alpha = -1$, $\beta = 1$ and all the φ's were required.)

Let us consider a function $\psi'(\mu)$ $\alpha \leq \mu \leq \beta$ and try to expand it in terms of the φ_ν $\alpha \leq \nu \leq \beta$.

$$\text{(II-1)} \qquad \psi'(\mu) = \int_\alpha^\beta A(\nu)\,\varphi_\nu(\mu)\,d\nu = \lambda(\mu)A(\mu) + \frac{c}{2}P\int_\alpha^\beta \frac{\nu A(\nu)\,d\nu}{\nu - \mu}$$

This is a singular integral equation for the function $A(\mu)$. To investigate completeness we would only need to show that this integral equation possesses a solution. In the course of the proof, we will actually construct the $A(\nu)$ by solving the integral equation directly. This constructive aspect of the proof is important since only in the special case $\alpha = -1$, $\beta = 1$ is there an orthogonality relation to determine the expansion coefficients $A(\nu)$.

[Reporter's Note: The complete proof, as given in the lecture, has been published elsewhere (K. M. Case, Annals of Physics, Jan. 1960). The essential features of the proof may be seen by considering the special case $\alpha = 0$, $\beta = 1$ (half-range completeness) which is treated here. Actually, the half-range and full-range cases provide the only results of physical interest. Moreover, the full-range case is degenerate in the sense that the central feature of the proof, the construction of the function $X(z)$ (see equation (II-11)), is accomplished by inspection. In fact, $X_0(z) = \Lambda(z)$.]

When $\alpha = 0$, $\beta = 1$ we have

(II-2) $$\psi'(\mu) = \lambda(\mu) A(\mu) + \frac{c}{2} P\int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - \mu} \qquad 0 \leq \mu \leq 1 .$$

Let us assume that a well-behaved solution $A(\nu)$ exists. Then the function $N(z)$, defined by

(II-3) $$N(z) = \frac{1}{2\pi i} \frac{c}{2} \int_0^1 \frac{\nu A(\nu)\, d\nu}{\nu - z} ,$$

possesses the following properties:

1) $N(z)$ is analytic in the complex plane cut from 0 to 1;

2) $N(z)$ vanishes at least as fast as $\frac{1}{z}$ as $z \to \infty$;

3) $N^+(\mu)$ and $N^-(\mu)$ are defined for all μ except perhaps $\mu = 1$, and in the neighborhood of $z = 1$, $N(z)$ is bounded by $\frac{1}{|z-1|^\gamma}$ where $\gamma < 1$.

Property (2) follows from

$$\text{(II-4)} \qquad \int_0^1 \frac{\nu A(\nu)\,d\nu}{\nu - z} = -\frac{1}{z}\int_0^1 \frac{\nu A(\nu)\,d\nu}{1-\nu/z} = -\frac{1}{z}\int_0^1 \nu A(\nu)\left[1 + \frac{\nu}{z} + \frac{\nu^2}{z^2} + \cdots\right]d\nu$$

Thus for large z

$$\text{(II-5)} \qquad \int_0^1 \frac{\nu A(\nu)}{\nu - z}\,d\nu \cong -\frac{1}{z}\int_0^1 \nu A(\nu)\,d\nu \,.$$

Note that $\int_0^1 \frac{\nu A(\nu)\,d\nu}{\nu - z}$ will vanish as fast as $\frac{1}{z^n}$ as $|z| \to \infty$ if and only if

$$\text{(II-6)} \qquad \int_0^1 \nu^m A(\nu)\,d\nu = 0 \,, \qquad m = 0, 1, 2, \cdots, n-1 \,.$$

Property (3) is discussed in Muskhelishvili, Chapter IV.

The boundary values $N^{\pm}$ satisfy

$$\text{(II-7)} \qquad \begin{aligned} N^+(\mu) - N^-(\mu) &= \frac{c}{2}\mu A(\mu) \\ \pi i\left[N^+(\mu) + N^-(\mu)\right] &= \frac{c}{2} P\int_0^1 \frac{\nu A(\nu)\,d\nu}{\nu - \mu} \,. \end{aligned}$$

The integral equation may now be written in terms of the boundary values of $N(z)$ and $\Lambda(z)$

$$\text{(II-8)} \qquad \frac{c}{2}\mu\,\Psi'(\mu) = \Lambda^+(\mu)\,N^+(\mu) - \Lambda^-(\mu)\,N^-(\mu) \,.$$

We shall solve this equation for $N(z)$. Then $A(\nu)$ will be given by equation (II-7). Of course, it will be necessary later to verify that the function $N(z)$ so obtained possesses the properties which we already have ascribed to it. We have

(II-9) $$G(\mu)\, N^+(\mu) - N^-(\mu) = \frac{c}{2}\, \frac{\mu\, \psi'(\mu)}{\Lambda^-(\mu)} ,$$

(II-10) where $G(\mu) = \dfrac{\Lambda^+}{\Lambda^-} = \dfrac{\Lambda^+}{(\Lambda^+)^*} = e^{2i\Theta(\mu)}$ and $\Theta(\mu) = \arg \Lambda^+(\mu)$.

We solve now by constructing a function $X(z)$ which is analytic in the complex plane cut from 0 to 1 (from α to β in the general case), non-vanishing along with its boundary values $X^{\pm}$ in the entire plane, and whose boundary values satisfy the "ratio condition:"

(II-11) $$\frac{X^+(\mu)}{X^-(\mu)} = G(\mu)$$

Taking the logarithm of each side, the ratio condition becomes

(II-12) $$\ln X^+ - \ln X^- = 2i\,\Theta(\mu).$$

Then, by the Plemelj formulae (I-17), the function $X_0(z)$ given by

(II-13) $$X_0(z) = e^{\Gamma(z)}$$

(II-14) where $$\Gamma(z) = \frac{1}{2\pi i}\int_0^1 \frac{2i\,\Theta(\mu)\,d\mu}{\mu - z}$$

will satisfy the ratio condition. $X_0(z)$ is also analytic in the cut plane.

Now consider the behavior of $X_0(z)$ in the neighborhood of the cut. For ν on the cut (not an end point) the boundary values $\Gamma^{\pm}(\nu)$ are well defined and hence $X_0^{\pm}(\nu)$ will not vanish. Moreover, since $\Lambda^+(o) = \Lambda^-(o)$, $\Theta(\mu) \rightarrow o$ as $\mu \rightarrow o$, so $\Gamma(o)$ also exists and hence $X(o) \neq o$. However, it may be shown that $\Theta(\mu) \rightarrow \pi$ as $\mu \rightarrow 1$, so in the neighborhood of $z = 1$

(II-15) $$\Gamma(z) \approx \frac{\Theta(1)}{\pi}\int^1 \frac{d\mu}{\mu - z} = \frac{\Theta(1)}{\pi}\ln(1-z) = \ln(1-z)$$

(II-16) and $$X_0(z) = e^{\Gamma(z)} \approx 1 - z$$

Therefore, an appropriate (non-vanishing) $X(z)$ is given by

(II-17) $$X(z) = \frac{X_0(z)}{1 - z} .$$

Now we may write

(II-18) $$X^+(\mu)\,N^+(\mu) - X^-(\mu)\,N^-(\mu) = \Upsilon(\mu)\,\psi'(\mu)$$

where we have defined

$$\text{(II-19)} \qquad \gamma(\mu) = \frac{c}{2}\mu \frac{X^-(\mu)}{\Lambda^-(\mu)} \quad ,$$

and if we consider the function F(z) defined by

$$\text{(II-20)} \qquad F(z) = X(z)N(z) - \frac{1}{2\pi i}\int_0^1 \frac{\gamma(\mu')\psi'(\mu')\,d\mu'}{\mu' - z}$$

we see that F(z) is analytic in the cut plane and has been constructed so as to have no discontinuity across the cut, so F(z) is analytic in the entire plane. Moreover, F(z) vanishes as $|z|\to\infty$ Hence,

(II-21) $F(z) = 0$ (Liouville's theorem)

$$\text{(II-22)} \qquad \text{and} \quad 2\pi i N(z) = \frac{1}{X(z)}\int_0^1 \frac{\gamma(\mu')\psi'(\mu')\,d\mu'}{\mu' - z} \quad .$$

Does N(z) possess the properties which it must have? It is analytic in the cut plane (remember that $X(z)$ has been constructed so as not to vanish in the cut plane) and well behaved in the neighborhood of the cut. However, since

$$\text{(II-23)} \qquad \frac{1}{X(z)} \sim z \qquad \text{as } z \to \infty ,$$

$N(z)$ will vanish $\sim \frac{1}{z}$ at ∞ as required only if

(II-24) $$\int_0^1 \frac{\gamma(\mu)\,\psi'(\mu)\,d\mu}{\mu - z} \sim \frac{1}{z^2} \qquad as \quad z \rightarrow \infty \quad ,$$

and this in turn will be true if (see equation (II-6))

(II-25) $$\int_0^1 \gamma(\mu')\,\psi'(\mu')\,d\mu' = 0 \ .$$

In other words, if $\psi'(\mu)$ satisfies equation (II-25), the integral equation (II-2) possesses a solution. Thus, any function $\psi'(\mu)$ satisfying (II-25) may be expanded:

(II-26) $$\psi'(\mu) = \int_0^1 A(\nu)\,\varphi_\nu(\mu)\,d\nu$$

with $A(\nu)$ given by (II-7).

Therefore, we may expand a completely arbitrary function $\psi(\mu)$ in the form

(II-28) $$\psi(\mu) = a_{0+}\,\varphi_{0+}(\mu) + \int_0^1 A(\nu)\,\varphi_\nu(\mu)\,d\nu ,$$

where a_{0+} is chosen so that the function

(II-29) $$\psi'(\mu) = \psi(\mu) - a_{0+}\,\varphi_{0+}(\mu)$$

satisfies (II-25).

This condition becomes

(II-30) $$\int_0^1 \gamma(\mu)\left[\psi(\mu) - a_{0+}\,\varphi_{0+}(\mu)\right]d\mu = 0$$

(II-31) or $$a_{0+} = \frac{\int_0^1 \gamma(\mu)\,\psi(\mu)\,d\mu}{\int_0^1 \gamma(\mu)\,\varphi_{0+}(\mu)\,d\mu} = \frac{\int_0^1 \gamma(\mu)\,\psi(\mu)\,d\mu}{M_{0+}}$$

where we have defined

(II-32) $$M_{0+} = \int_0^1 \gamma(\mu)\,\varphi_{0+}(\mu)\,d\mu$$

To recapitulate, we have proved the completeness theorem (for the case $\alpha = 0$, $\beta = 1$) by explicitly obtaining the coefficients $A(\nu)$, a_{0+} in the expansion of an arbitrary function $\psi(\nu)$. The results may be summarized for future reference:

(II-7) $$A(\nu) = \frac{2}{c\nu}\left[N^+(\nu) - N^-(\nu)\right]$$

(II-22) $$N(z) = \frac{1}{2\pi i}\,\frac{1}{X(z)}\int_0^1 \frac{\gamma(\mu')\,\psi'(\mu')\,d\mu'}{\mu' - z}$$

(II-17) $$X(z) = \frac{X_0(z)}{1 - z}$$

(II-13) $$X_0(z) = e^{\Gamma(z)}$$

(II-14) $$\Gamma(z) = \frac{1}{2\pi i}\int_0^1 \frac{2i\,\Theta(\mu)\,d\mu}{\mu - z}$$

(II-10) $$\Theta(\mu) = \arg \Lambda^+(\mu)$$

(II-19) $$\gamma(\mu) = \frac{c\mu}{2}\,\frac{X^-(\mu)}{\Lambda^-(\mu)}$$

(II-29) $$\psi'(\mu) = \psi(\mu) - a_{0+}\varphi_{0+}(\mu)$$

(II-31) $$a_{0+} = \frac{1}{M_{0+}}\int_0^1 \gamma(\mu)\,\psi(\mu)\,d\mu$$

(II-32) $$M_{0+} = \int_0^1 \gamma(\mu)\,\varphi_{0+}(\mu)\,d\mu$$

III. Half Space Problems

We now apply the results of the first two lectures to half space problems, by writing the solution to a given problem as a superposition of elementary solutions and applying the appropriate boundary conditions to obtain the expansion coefficients. Recall the notation

$$\psi_\nu = e^{-x/\nu} \varphi_\nu(\mu) \qquad \psi_{0\pm} = e^{\mp x/\nu_0} \varphi_{0\pm}(\mu) . \tag{III-1}$$

As a simple example, we solve the albedo problem. A beam of neutrons moving at an angle $\theta_0 = \cos^{-1}\mu_0$ with the x-axis is incident on a half space $x > o$. We wish to know $\Psi_a(x, \mu)$, the distribution of neutrons within the half space. Ψ_a will satisfy the homogeneous equation for $x > o$ with boundary conditions:

$$\text{a) } \Psi_a \to 0 \quad \text{as } x \to \infty \text{ (we take } c < 1); \tag{III-2}$$

$$\text{b) } \Psi_a(0, \mu) = \delta(\mu - \mu_0) \quad \text{for all } \mu \geq 0.$$

Boundary condition a) suggests we exclude the ψ's that grow as $x \to \infty$, so

$$\Psi_a(x, \mu) = a_{0+} \psi_{0+}(x, \mu) + \int_0^1 A(\nu) \psi_\nu(x, \mu)\, d\nu . \tag{III-3}$$

Boundary condition b) requires

$$\delta(\mu - \mu_0) = a_{0+} \varphi_{0+}(\mu) + \int_0^1 A(\nu) \varphi_\nu(\mu)\, d\nu \qquad \mu \geq 0. \tag{III-4}$$

Thus we are led to a half-range expansion of the function

$$\psi(\mu) = \delta(\mu - \mu_0) \qquad \mu \geqslant 0 . \tag{III-5}$$

Using eq. (II-31) we obtain

$$a_{0+} = \frac{\gamma(\mu_0)}{M_{0+}} . \tag{III-6}$$

We could also exhibit explicit expressions for the A (ν), but the discrete term again dominates in the asymptotic region (cf. eq. (I-36)).

We may solve the classical Milne problem: given a half space with no incident neutrons, we want to find $\Psi_0(x, \mu)$, the distribution of neutrons within the medium provided that Ψ_0 is bounded by e^x as $x \to \infty$. The boundary conditions are

$$\text{(III-7)} \quad \text{a)} \quad \Psi_0(x,\mu) - \psi_{0-}(x,\mu) \to 0 \quad as \quad x \to \infty ,$$

$$\text{b)} \quad \Psi_0(0,\mu) = 0 \qquad \mu \geqslant 0 .$$

Condition a) is satisfied by writing

$$\Psi_0(x,\mu) = \psi_{0-}(x,\mu) + a_{0+}\psi_{0+}(x,\mu) + \int_0^1 A(\nu)\psi_\nu(x,\mu)\,d\nu \tag{III-8}$$

and condition b) yields

$$0 = \varphi_{0-}(\mu) + a_{0+}\varphi_{0+}(\mu) + \int_0^1 A_\nu(\mu)\varphi_\nu(\mu)\,d\nu \qquad \mu \geqslant 0 , \tag{III-9}$$

which tells us that the A(ν) and a_{0_+} are the coefficients in the half-range expansion of $-\varphi_{0_-}$.

Again the asymptotic behavior is given by the discrete term, with

(III-10) $$a_{0_+} = \frac{-\int_0^1 \gamma(\mu)\,\varphi_{0_-}(\mu)\,d\mu}{\int_0^1 \gamma(\mu)\,\varphi_{0_+}(\mu)\,d\mu} = -\frac{M_{0_-}}{M_{0_+}} .$$

This immediately suggests a generalization. We seek functions $\Psi_\nu(x,\mu)$ $(-1 \leq \nu \leq 0)$ satisfying the homogeneous equation and the following boundary conditions (instead of (III-7)):

(III-11)

a) $$\Psi_\nu(0,\mu) = 0 \qquad \mu \geq 0 ,$$

b) $$\Psi_\nu(x,\mu) - \psi_\nu(x,\mu) \longrightarrow 0 \quad \text{as} \quad x \longrightarrow \infty .$$

As in the Milne problem we get

(III-12) $$\Psi_\nu(x,\mu) = \psi_\nu(x,\mu) + a_{0_+}\psi_{0_+}(x,\mu) + \int_0^1 A(\nu')\,\psi_{\nu'}(x,\mu)\,d\nu'$$

with the dominant coefficient given by

(III-13) $$a_{0_+} = \frac{-\int_0^1 \gamma(\mu)\,\varphi_\nu(\mu)\,d\mu}{\int_0^1 \gamma(\mu)\,\varphi_{0_+}(\mu)\,d\mu} .$$

Using these Ψ_ν we may construct the Green's function for the half space. It must satisfy:

(III-14) $$\mu\frac{\partial \Psi_g(x,\mu)}{\partial x} + \Psi_g(x,\mu) = \frac{c}{2}\int_{-1}^{1}\Psi_g(x,\mu')\,d\mu' + \frac{\delta(x-x_0)\,\delta(\mu-\mu_0)}{2\pi}, \qquad \begin{matrix} x > 0 \\ x_0 > 0 \end{matrix}$$

with boundary conditions

$$\Psi_g(0,\mu) = 0 \qquad \mu > 0$$
$$\lim_{x\to\infty} \Psi_g(x,\mu) = 0 \quad . \tag{III-15}$$

The solution is

$$\Psi_g(x,\mu;x_0,\mu_0) = \psi_g^{inf}(x,\mu;x_0,\mu_0) - \frac{1}{2\pi}\left\{\frac{\varphi_{0-}(\mu_0)}{N_{0-}}\left[\Psi_0(x,\mu) - \psi_{0-}(x,\mu)\right]e^{-x_0/\nu_0}\right.$$
$$\left. + \frac{1}{2\pi}\int_{-1}^{0}\frac{\varphi_\nu(\mu_0)}{N(\nu)}\left[\Psi_\nu(x,\mu) - \psi_\nu(x,\mu)\right]e^{-x_0/\nu}\,d\nu \right. \tag{III-16}$$

(ψ_g^{inf} is the infinite medium Green's function.)

as may be verified by noting that:

a) Ψ_g satisfied the source-free equation for $x \neq x_0$ (since it is a superposition of normal modes in the regions $x < x_0$, $x > x_0$;

b) Ψ_g possesses the correct source discontinuity at $x = x_0$ $\mu = \mu_0$ (by virtue of the inclusion of ψ_g^{inf});

c) Ψ_g vanishes as $x \to \infty$ (since ψ_g^{inf} vanishes along with the terms in brackets);

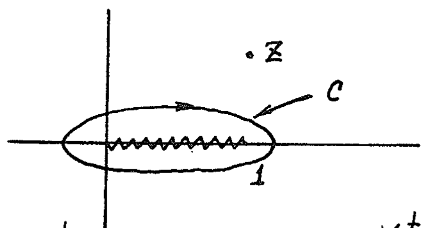
d) Ψ_g represents a solution with zero incoming flux, since the functions Ψ_0 and Ψ_ν are constructed to have zero incoming flux, and hence

$$\Psi_g(0,\mu;x_0,\mu_0) = \psi_g^{inf}(0,\mu;x_0,\mu_0) - \frac{1}{2\pi}\frac{\varphi_{0-}(\mu_0)\,\varphi_{0-}(\mu)}{N_{0-}}e^{-x_0/\nu_0}$$
$$+ \frac{1}{2\pi}\int_{-1}^{0}\frac{\varphi_\nu(\mu_0)\,\varphi_\nu(\mu)}{N(\nu)}e^{-x_0/\nu}\,d\nu \qquad \mu > 0$$

$= 0$ by virtue of (I-35).

We may simplify our expressions for the classical Milne problem by examining some of the properties of the function X(z). Since X(z) is analytic in the complex plane cut from 0 to 1 and vanishes $\sim \frac{1}{z}$ at ∞ , we have, by applying Cauchy's theorem to the region bounded by a large circle and the countour $\mathcal{C}$:

$$X(z) = \frac{1}{2\pi i}\int_{C} \frac{X(z')\,dz'}{z'-z} = \frac{1}{2\pi i}\int_0^1 \frac{X^+(\mu') - X^-(\mu')}{\mu' - z}\,d\mu' \tag{III-18}$$



the contour C,

but

$$X^+ - X^- = X^-\left(\frac{X^+}{X^-} - 1\right) = X^-\left(\frac{\Lambda^+}{\Lambda^-} - 1\right) \tag{III-19}$$

$$= \frac{X^-}{\Lambda^-}\left(\Lambda^+ - \Lambda^-\right) = 2\pi i\,\gamma(\mu)\ ,$$

where we have used equations (II-11) (II-19) and the relation

$$\Lambda^+(\mu) - \Lambda^-(\mu) = 2\pi i\,\frac{c\mu}{2}\ , \tag{III-20}$$

which follows directly from the Plemlj formulae (I-17). This yields the identity:

$$X(z) = \int_0^1 \frac{\gamma(\mu')\,d\mu'}{\mu' - z}\ . \tag{III-21}$$

Applications of the Identity (III-21) to the Milne problem result.

1. The asymptotic behavior of Ψ_0 is determined by the coefficient a_{0+} which may be written:

$$a_{0+} = \frac{-\int_0^1 \gamma(\mu)\varphi_{0-}(\mu)\,d\mu}{\int_0^1 \gamma(\mu)\varphi_{0+}(\mu)\,d\mu} = \frac{-\int \frac{c}{2}\frac{\nu_0}{-\nu_0-\mu}\gamma(\mu)\,d\mu}{\int \frac{c}{2}\frac{\nu_0}{\nu_0-\mu}\gamma(\mu)\,d\mu} = \frac{X(-\nu_0)}{X(\nu_0)} \tag{III-22}$$

(III-23)

2. Since $\psi'(\mu) = -\varphi_{0-}(\mu) - a_{0+}\varphi_{0+}(\mu)$ we may write a reasonably simple expression for the function N(z) which determines the coefficients A(ν).

Using the explicit forms of the functions φ_{0+} and φ_{0-} we have

$$2\pi i\,N(z) = \frac{1}{X(z)}\int_0^1 \frac{\gamma(\mu')}{\mu'-z}\left[\frac{-c\nu_0}{2}\frac{1}{\nu_0+\mu'} - \frac{a_{0+}c\nu_0}{2}\frac{1}{\nu_0-\mu'}\right]d\mu' , \tag{III-24}$$

which becomes, after separating the integrand into partial fractions and applying (III-22) and the identity (III-21),

$$2\pi i\,N(z) = \frac{-c\nu_0}{2}\left[\frac{1}{\nu_0+z} + \frac{a_{0+}}{\nu_0-z}\right] - \frac{c\nu_0^2\,X(-\nu_0)}{X(z)\,(z^2-\nu_0^2)} . \tag{III-25}$$

3. The expression for the emerging angular density is simplified since the appropriate φ_ν's are not singular for $\mu < 0$

$$\Psi_0(0,\mu) = \varphi_{0-}(\mu) + a_{0+}\varphi_{0+}(\mu) + \int_0^1 \frac{c}{2}\frac{\nu A(\nu)\,d\nu}{\nu-\mu} \qquad \mu < 0 \tag{III-26}$$

$$= \varphi_{0-}(\mu) + a_{0+}\varphi_{0+}(\mu) + 2\pi i\,N(\mu)$$

which, in view of Eq. (III-25) becomes simply

$$\Psi_0(0,\mu) = \frac{c\nu_0^2 X(-\nu_0)}{X(\mu)(\nu_0^2-\mu^2)} \qquad \text{(III-27)}$$

Another Identity

$$X(z)X(-z) = \frac{\Lambda(z)}{(\nu_0^2-z^2)(1-c)} \qquad \text{(III-28)}$$

Proof: consider F(z) defined by

$$F(z) = \frac{\Lambda(z)}{X(z)X(-z)(\nu_0^2-z^2)(1-c)} \qquad \text{(III-29)}$$

X(z) has been constructed to be analytic and non-vanishing in the complex plane cut from 0 to 1. So X(z)X(-z) is analytic and non-vanishing in the complex plane cut from -1 to 1. Since $\Lambda(\nu_0) = \Lambda(-\nu_0) = 0$, $\dfrac{\Lambda(z)}{\nu_0^2-z^2}$ is also analytic in the cut plane. Therefore F(z) is analytic in the cut plane. But the ratio condition

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} \qquad \text{(II-11) (II-10)}$$

insures that F(z) will be continuous across the cut. Thus F(z) is analytic in the entire complex plane and is determined entirely by its behavior as $z \to \infty$.

According to (I-8)

$$\Lambda(z) \xrightarrow[|z|\to\infty]{} 1-C \qquad \text{(III-30)}$$

and since $\Gamma(z)$ vanishes as $|z| \to \infty$

(III-31) $$X_0(z) \xrightarrow[|z|\to\infty]{} 1 \qquad ;$$

(III-32) thus $$X(z)\,X(-z)\,(\nu_0^2 - z^2) = X_0(z)\,X_0(-z)\,\frac{\nu_0^2 - z^2}{1 - z^2} \xrightarrow[|z|\to\infty]{} 1$$

(III-33) thus $$F(z) \xrightarrow[|z|\to\infty]{} 1$$

(III-34) and $$F(z) \equiv 1$$ (Liouville's theorem).

Q.E.D.

<u>Applications of the Identity (III-29)</u>

1. If we apply the identity (III-29) to equation (III-18), we obtain the (non-linear) integral equation satisfied by $X(z)$

(III-35) $$X(z) = \frac{c}{2(1-c)} \int_{-1}^{0} \frac{\mu'\, d\mu'}{(\nu_0^2 - \mu'^2)\, X(\mu')\, (\mu' + z)} \quad .$$

2. This integral equation enables us to write a simple expression for the emerging neutron density

(III-36) $$\rho_0(0) = \int_{-1}^{1} \Psi_0(0,\mu')\, d\mu' = \int_{-1}^{0} \frac{c\,\nu_0^2\, X(-\nu_0)\, d\mu'}{X(\mu')\,(\nu_0^2 - \mu'^2)}$$

$$= \nu_0^2\, 2\,(1-c)\, X(0)\, X(-\nu_0)$$

(III-37) but $[X(0)]^2 = \dfrac{\Lambda(0)}{\nu_0^2(1-c)} = \dfrac{1}{\nu_0^2(1-c)}$,

(III-38) thus $\rho_0(0) = 2\nu_0\sqrt{1-c}\; X(-\nu_0)$.

3. Similarly for the emerging neutron current:

(III-39) $j_0(0) = \int_{-1}^{0} \mu' \Psi_0(0,\mu')\,d\mu' = c\nu_0^2 X(-\nu_0) \int_{-1}^{0} \dfrac{\mu'\,d\mu'}{X(\mu')(\nu_0^2-\mu'^2)}$,

(III-40) but $\displaystyle\int_{-1}^{0} \frac{\mu'\,d\mu'}{X(\mu')(\nu_0^2-\mu'^2)} = \lim_{z\to\infty} \int_{-1}^{0} \frac{\mu'\,d\mu'}{X(\mu')(\nu_0^2-\mu'^2)(1+\frac{\mu'}{z})}$

$$= \frac{2(1-c)}{c} \lim_{z\to\infty} z X(z) = -\frac{2(1-c)}{c} \lim_{z\to\infty} X_0(z)$$

$$= \frac{-2(1-c)}{c} ,$$

(III-41) thus $j_0(0) = -2(1-c)\nu_0^2 X(-\nu_0)$.

4. Finally, the emerging neutron angular density at grazing angle is simply

(III-42) $\Psi_0(0,0) = c\nu_0\sqrt{1-c}\; X(-\nu_0)$.

The Extrapolated End·Point

For large values of x, the distribution function is approximately given by

(III-43) $$\Psi_0(x,\mu) \approx \psi_{0_-}(x,\mu) + a_{0_+}\psi_{0_+}(x,\mu)$$

(III-44) or $$\rho_{\text{asymptotic}} = e^{x/\nu_0} + a_{0_+} e^{-x/\nu_0} .$$

In terms of the "extrapolated end·point" z_0, defined by

(III-45) $$a_{0_+} = -e^{-2z_0/\nu_0} ,$$

$\rho_{\text{asymptotic}}$ becomes:

(III-46) $$\rho_{\text{asymptotic}} = 2e^{-z_0/\nu_0} \sinh\left[\frac{x+z_0}{\nu_0}\right] .$$

(III-47) Writing $$z_0 = \frac{\nu_0}{2}\left\{-\ln a_{0_+} + \pi i\right\} ,$$

we obtain (after much algebra)

(III-48) $$z_0 = \frac{c}{2}\int_0^1 \frac{1}{\Lambda_+(\mu')\Lambda_-(\mu')}\left[1 + \frac{c\mu'^2}{1-\mu'^2}\right]\nu_0 \tanh^{-1}\frac{\mu'}{\nu_0}\,d\mu' ,$$

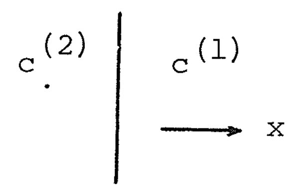
which goes over to the more familiar form when we recall that

(III-49) $$\Lambda^+(\mu)\Lambda^-(\mu) = (1 - c\mu\tanh^{-1}\frac{1}{\mu})^2 + \frac{c^2\pi^2\mu^2}{4} .$$

IV. Applications and Extensions

In this lecture, I will sketch some applications and extensions of the method.

Consider first the Milne problem for two adjacent half spaces characterized by $c^{(1)}$ and $c^{(2)}$.



The distribution function satisfies

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \begin{cases} \frac{c^{(1)}}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu' & x > 0 \\ \frac{c^{(2)}}{2} \int_{-1}^{1} \psi(x,\mu')\, d\mu' & x < 0 \end{cases} \quad . \tag{IV-1}$$

We can construct modes $\psi_\nu^{(1)}(x,\mu)$ and $\psi_{0\pm}^{(1)}(x,\mu)$ appropriate to medium 1, and $\psi_\nu^{(2)}(x,\mu)$ and $\psi_{0\pm}^{(2)}(x,\mu)$ appropriate to medium 2. The boundary conditions for the Milne problem are:

a) $$\lim_{x\to\infty} \left(\Psi_0(x,\mu) - \psi_{0-}^{(1)}(x,\mu) \right) = 0 \quad ;$$

(IV-2) b) $$\lim_{x\to -\infty} \Psi_0(x,\mu) = 0 \quad ;$$

c) $$\Psi_0(0^+,\mu) = \Psi_0(0^-,\mu) \quad .$$

These imply:

$$\text{a)}\quad \Psi_0(x,\mu) = \psi_{0-}^{(1)}(x,\mu) + a_{0+}\psi_{0+}^{(1)}(x,\mu) + \int_0^1 A(\nu)\,\psi_\nu^{(1)}(x,\mu)\,d\nu \qquad x>0\ ;$$

$$\text{(IV-3)}\qquad \text{b)}\quad \Psi_0(x,\mu) = -a_{0-}\psi_{0-}^{(2)}(x,\mu) - \int_{-1}^0 A(\nu)\,\psi_\nu^{(2)}(x,\mu)\,d\nu \qquad x<0\ ;$$

$$\text{c)}\quad -a_{0-}\varphi_{0-}^{(2)}(\mu) - \int_{-1}^0 A(\nu)\,\varphi_\nu^{(2)}(\mu)\,d\nu = \varphi_{0-}^{(1)}(\mu) + a_{0+}\varphi_{0+}^{(1)}(\mu) + \int_0^1 A(\nu)\,\varphi_\nu^{(1)}(\mu)\,d\nu\,,$$

which gives

$$\text{(IV-4)}\qquad \psi'(\mu) = \int_{-1}^0 A(\nu)\,\varphi_\nu^{(2)}(\mu)\,d\nu + \int_0^1 A(\nu)\,\varphi_\nu^{(1)}(\mu)\,d\nu$$

$$\text{(IV-5)}\qquad \text{where}\quad \psi'(\mu) \equiv -\varphi_{0-}^{(1)}(\mu) - a_{0+}\varphi_{0+}^{(1)}(\mu) - a_{0-}\varphi_{0-}^{(2)}(\mu)\ .$$

Thus, it is required to expand $\psi'(\mu)$ in terms of the $\varphi_\nu^{(1)}(\mu)$ $[0<\nu\leq 1]$ and the $\varphi_\nu^{(2)}(\mu)$ $[-1\leq\nu<0]$. The construction of this expansion proceeds along the same lines as the completeness theorem of lecture II. We introduce

$$\text{(IV-6)}\qquad \lambda(\mu) = \begin{cases} \lambda^{(1)}(\mu) & \mu>0 \\ \lambda^{(2)}(\mu) & \mu<0 \end{cases} \qquad \Lambda^{\pm}(\mu) = \begin{cases} \Lambda_{(1)}^{\pm}(\mu) & \mu>0 \\ \Lambda_{(2)}^{\pm}(\mu) & \mu<0 \end{cases}$$

$$C(\mu) = \begin{cases} C^{(1)} & \mu>0 \\ C^{(2)} & \mu<0 \end{cases} \qquad G(\mu) = \begin{cases} G^{(1)}(\mu) & \mu>0 \\ G^{(2)}(\mu) & \mu<0 \end{cases}\ .$$

The integral equation satisfied by the expansion coefficient $A(\nu)$ is

$$\psi'(\mu) = \lambda(\mu)A(\mu) + \frac{1}{2}P\int_{-1}^{1}\frac{c(\mu)\nu A(\nu)d\nu}{\nu-\mu} \quad . \tag{IV-7}$$

As before, the crucial step is the construction of a function $X(z)$ which is analytic in the complex plane cut from -1 to 1, non-vanishing along with its boundary values $X^{\pm}(\mu)$ in the entire plane, and whose boundary values satisfy the ratio condition

$$\frac{X^{+}(\mu)}{X^{-}(\mu)} = G(\mu) \quad . \tag{IV-8}$$

In this case, however, the result may be written down immediately:

$$X(z) = X^{(1)}(z)\,X^{(2)}(-z) \quad , \tag{IV-9}$$

where $X^{(1)}(z)$ and $X^{(2)}(z)$ are the X-functions for a single half space with the appropriate value of c. This is clear since $X^{(1)}(z)$ satisfies the ratio condition along the right hand half of the cut (where $X^{(2)}(-z)$ is continuous) and $X^{(2)}(-z)$ satisfies the ratio condition along the left hand half of the cut.

The function $\psi'(\mu)$ can be expanded if and only if it satisfies: (see equation (II-25))

$$\int_{-1}^{1}\mu'^{\ell}\,\gamma(\mu')\,\psi'(\mu')\,d\mu' = 0 \qquad \ell = 0, 1 \tag{IV-10}$$

(IV-11) where $$\gamma(\mu) = \frac{c(\mu)}{2}\,\mu\,\frac{X^-(\mu)}{\Lambda^-(\mu)} \quad .$$

These conditions determine a_{0_+} and a_{0_-} .

The results are

(IV-12) $$\mu\,\frac{c(\mu)\,A(\mu)}{2} = N^+(\mu) - N^-(\mu)$$

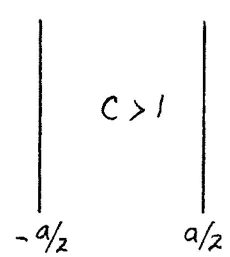
(IV-13) where $$N(z) = \frac{1}{2\pi i}\,\frac{1}{X(z)}\int_{-1}^{1}\frac{c(\mu')\,\mu'\,\psi'(\mu')\,X^-(\mu')\,d\mu'}{\Lambda^-(\mu')\,(\mu'-z)}$$

(IV-14) and $$X(z) = X^{(1)}(z)\,X^{(2)}(-z) = \frac{1}{1-z^2}\,e^{\frac{1}{2\pi i}\int_{-1}^{1}\frac{\ln G(\mu')\,d\mu'}{\mu'-z}} \quad .$$

Other problems with this geometry may be treated similarly. In fact, the most complicated problems which yield answers in closed form are those involving two half spaces, each with arbitrary (anisotropic) scattering law.

One of the simplest "not exactly soluble" problems is that of slab geometry. Here we can obtain an explicit criticality condition and arrive at useful approximations.

(IV-15)



We seek to know the critical size of the slab: that value of a for which the source free equation

$$\mu\,\frac{\partial\psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{c}{2}\int_{-1}^{1}\psi(x,\mu')\,d\mu'$$

together with the boundary conditions for no incident neutrons

$$\text{(IV-16)} \qquad \begin{aligned} \psi\left(\tfrac{a}{2}, \mu\right) &= 0 \qquad & \mu < 0 \\ \psi\left(-\tfrac{a}{2}, \mu\right) &= 0 \qquad & \mu > 0 \end{aligned}$$

possess a non-trivial solution.

Thus we write

$$\text{(IV-17)} \qquad \psi(x,\mu) = \psi_{0_+}(x,\mu) + \psi_{0_-}(x,\mu) + \int_0^1 A(\nu)\left\{\psi_\nu(x,\mu) + \psi_{-\nu}(x,\mu)\right\} d\nu ,$$

where we have utilized the reflection symmetry to set

$$\text{(IV-18)} \qquad \psi(x,\mu) = \psi(-x,-\mu)$$

and chosen $a_{0_+} = 1$.

The boundary conditions imply:

$$\text{(IV-19)} \qquad 0 = \varphi_{0_+}(\mu)e^{a/2\nu_0} + \varphi_{0_-}(\mu)e^{-a/2\nu_0} + \int_0^1 A(\nu)e^{-a/2\nu}\varphi_{-\nu}(\mu)\,d\nu + \int_0^1 A(\nu)e^{a/2\nu}\varphi_\nu(\mu)\,d\nu, \qquad \mu > 0,$$

giving

$$\text{(IV-20)} \qquad -\varphi_{0_+}(\mu)e^{a/2\nu_0} - \varphi_{0_-}(\mu)e^{-a/2\nu_0} - \int_0^1 B(\nu)e^{-a/\nu}\varphi_{-\nu}(\mu)\,d\nu$$

$$= \int_0^1 B(\nu)\,\varphi_\nu(\mu)\,d\nu \qquad \mu > 0$$

(IV-21) where $$B(\nu) = A(\nu)\, e^{a/2\nu} .$$

Suppose we regard the left hand side of equation (IV-20) as a known function of μ for $\mu > 0$. Then equation (IV-20) says that the unknown $B(\nu)$ is the expansion function for the left hand side in terms of the $\varphi_\nu(\mu)$ $[0 \leq \nu \leq 1]$. The expansion will be possible only if the left hand side satisfies equation (II-25). Using equation (III-21) and the explicit forms of the φ's, this condition becomes:

(IV-22) $$\nu_0 X(\nu_0)\, e^{a/2\nu_0} - \nu_0 X(-\nu_0)\, e^{-a/2\nu_0} - \int_0^1 \nu B(\nu)\, e^{-a/\nu} X(\nu)\, d\nu = 0 .$$

As a first approximation, valid for $a \gg 1$, we may neglect the integral term altogether (recall that ν_0 is imaginary in this case) to give

(IV-23) $$\frac{X(\nu_0)}{X(-\nu_0)} = e^{-a/\nu_0}$$

(IV-24) or $$a = \frac{\pi}{2} |\nu_0| - z_0$$ where we have used (III-22) and (III-45)

Further approximations require some knowledge of the function $B(\nu)$. To obtain this, we must carry out the expansion implied by (IV-20). Using (II-24) and (III-19) we obtain

(IV-25) $$\mu B(\mu) = \frac{1}{2\pi i}\left\{\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)}\right\}\left\{\frac{-\nu_0 X(\nu_0)\, e^{a/2\nu_0}}{\mu - \nu_0} + \frac{\nu_0 X(-\nu_0)\, e^{-a/2\nu_0}}{\mu + \nu_0} + \int_0^1 \frac{\nu B(\nu)\, e^{-a/\nu} X(-\nu)\, d\nu}{\mu + \nu}\right\} .$$

This is a Fredholm integral equation for $B(\nu)$. As before, the first approximation is obtained by neglecting the integral term. Successive approximations follow by iteration.

Let me conclude by giving some idea of how the method may be generalized to the case of anisotropic scattering. First we deal with a non-absorbing medium with linear anisotropic scattering, i.e. a medium for which c = 1 and for which the scattering function $f(\hat{\Omega}\cdot\hat{\Omega}')$ (see equation (I-1)) is of the form

(IV-26) $$f(\hat{\Omega}\cdot\hat{\Omega}') = 1 + C_1 \hat{\Omega}\cdot\hat{\Omega}' .$$

This serves as an introduction to the more general anisotropic scattering law, and also illustrates the special complication that arises when c = 1.

The homogeneous transport equation for such a medium is

(IV-27) $$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1} \psi(x,\mu')\, d\mu' + \frac{C_1 \mu}{2}\int_{-1}^{1} \mu' \psi(x,\mu')\, d\mu' .$$

If we integrate this over μ and x we obtain

(IV-28) $$\int \mu' \psi(x,\mu')\, d\mu' = const. = J .$$

Hence the equation is

$$\mu \frac{\partial \psi(x,\mu)}{\partial x} + \psi(x,\mu) = \frac{1}{2}\int_{-1}^{1} \psi(x,\mu')\, d\mu' + \frac{C_1 \mu}{2} J \ . \tag{IV-29}$$

This is formally identical to the problem of a medium with isotropic scattering (and c = 1), but with a spatially uniform source of strength $\frac{C_1 \mu}{2} J$. Thus we may solve the isotropic scattering problem assuming the parameter J is known and then determine J by the consistency condition (IV-28).

Therefore, consider the elementary solutions for the isotropic scattering case with c = 1. The continuum modes are the same as before (they are given in equations (I-7)-(I-9)).

The discrete case is somewhat different since as $c \longrightarrow 1$ the two roots of $\Lambda(\nu) = 0$ coelesce at $\nu_0 = \infty$. We may write our discrete solutions formally as before,

$$\psi_0(x,\mu) = \frac{1}{2} \frac{\nu_0}{\nu_0 - \mu} e^{-x/\nu_0} \ , \tag{IV-30}$$

where ν_0 is a root of $\Lambda(\nu_0) = 0$. Two linearly independent solutions of the transport equation are:

$$\psi_1(x,\mu) = \lim_{\nu_0 \to \infty} \left[\frac{1}{2} \frac{\nu_0}{\nu_0 - \mu} e^{-x/\nu_0} \right] = \frac{1}{2} \ ; \tag{IV-31}$$

$$\psi_2(x,\mu) = \lim_{\nu_0 \to \infty} \left[\nu_0^2 \frac{\partial}{\partial \nu_0} \left(\frac{1}{2} \frac{\nu_0}{\nu_0 - \mu} e^{-x/\nu_0} \right) \right] = \frac{1}{2}(x - \mu) \ , \tag{IV-32}$$

as may be verified.

We may now solve the Milne problem for the linear anisotropic medium. A particular solution of equation (IV-29) is

$$\psi_0(x,\mu) = \frac{c_1 \mu}{2} J \ . \tag{IV-33}$$

Then

$$\Psi_0(x,\mu) = b_1 \psi_1(x,\mu) + b_2 \psi_2(x,\mu) + \int_{-1}^{1} B(\nu)\psi_\nu(x,\mu)\, d\nu + \frac{c_1\mu}{2} J \tag{IV-34}$$

where the ψ's are normal modes for the isotropic scattering medium. The boundary conditions are:

$$\text{(IV-35)} \quad \begin{aligned} &\text{a)} \quad \lim_{x \to \infty} \Psi_0(x,\mu) - \psi_2(x,\mu) < \infty \ ; \\ &\text{b)} \quad \Psi_0(0,\mu) = 0 \qquad\qquad \mu > 0 \ . \end{aligned}$$

Condition (a) implies

$$\Psi_0(x,\mu) = \psi_2(x,\mu) + b_1\psi_1(x,\mu) + \int_0^1 B(\nu)\psi_\nu(x,\mu)\, d\nu + \frac{c_1\mu}{2} J \ , \tag{IV-36}$$

and condition (b) gives

$$\left[\frac{1 - c_1 J}{2}\right]\mu - \frac{b_1}{2} = \int_0^1 B(\nu)\varphi_\nu(\mu)\, d\nu \qquad \mu > 0 \tag{IV-37}$$

or, with the abbreviation

$$A(\nu) = \frac{2B(\nu)}{1 - c_1 J} \qquad a_1 = \frac{b_1}{1 - c_1 J} \quad , \tag{IV-38}$$

we get

$$\mu - a_1 = \int_0^1 A(\nu)\, \varphi_\nu(\mu)\, d\nu \qquad \mu > 0 \quad . \tag{IV-39}$$

This is of the familiar form. The left hand side is to be expanded in terms of the $\varphi_\nu(\mu)$ $\nu \geqslant 0$, with the number a_1 chosen so that equation (II-25) is satisfied. A(ν) will be given by equation (II-33) et seq.

Thus

$$\Psi_0(x,\mu) = \frac{c_1 \mu J}{2} + \frac{x - \mu}{2} + \frac{1 - c_1 J}{2} \left[a_1 + \int_0^1 A(\nu)\, \psi_\nu(x,\mu)\, d\nu \right] \tag{IV-40}$$

the condition $\int_{-1}^{1} \mu' \Psi(x,\mu')\, d\mu' = J$ may be applied (note that equation (I-4) implies that $\int_{-1}^{1} \mu' \varphi_\nu(\mu')\, d\mu' = 0$ when c = 1) to give

$$J = \frac{c_1 J - 1}{2} \int_{-1}^{1} \mu'^2\, d\mu' = \frac{c_1 J - 1}{3} \tag{IV-41}$$

or

$$1 - c_1 J = -3J = \frac{3}{3 - c_1} \quad . \tag{IV-42}$$

The final result is

(IV-43) $$\Psi_0(x,\mu) = \frac{1}{2}\left[x - \frac{3}{3-c_1}\mu\right] + \frac{3}{2(3-c_1)}\left[a_1 + \int_0^1 A(\nu)\psi_\nu(x,\mu)\,d\nu\right] .$$

The term in the first brackets represents a discrete solution of the transport equation (IV-27). The other discrete solution is represented by the constant term. This is true for an arbitrary scattering law. As Mika has pointed out, whenever c = 1 there will be two roots of $\Lambda(\nu) = 0$ at $\nu_0 = \infty$, with corresponding normal modes

(IV-44)
$$\psi_1(x,\mu) = A$$
$$\psi_2(x,\mu) = B\left(x - \frac{3}{3-c_1}\mu\right),$$ A, B constants,

where c_1 is the coefficient of $P_1(\hat{\Omega}\cdot\hat{\Omega}')$ in the expansion of the scattering function $f(\hat{\Omega}\cdot\hat{\Omega}')$ in Legendre polynomials.

Mika has treated in detail the extension to the case of arbitrary scattering law.

(IV-45) $$f(\hat{\Omega}\cdot\hat{\Omega}') = \sum_{\ell=1}^{N} c_\ell\, P_\ell(\hat{\Omega}\cdot\hat{\Omega}')$$

One may still find elementary solutions of the form

(IV-46) $$\psi_\nu(x,\mu) = e^{-x/\nu}\varphi_\nu(\mu)$$

with the resulting integral equation

(IV-47) $$\left(1-\frac{\mu}{\nu}\right)\varphi_\nu(\mu) = \int f(\hat{\Omega}\cdot\hat{\Omega}')\,\varphi_\nu(\mu')\,d\hat{\Omega}' .$$

Thus the φ's will be orthogonal in the sense that

(IV-48) $$\int \mu'\,\varphi_\nu(\mu')\,\varphi_{\nu'}(\mu')\,d\mu' = 0 \qquad \text{for} \quad \nu \neq \nu' .$$

The function $\Lambda(\nu)$ is more complicated than in the isotropic scattering case. There will still be a continuum $[-1 \leq \nu \leq 1]$ with eigenfunctions similar in form to those given by equation (I-7). However, the number of discrete modes is not two, in general. Mika has shown that when $f(\hat{\Omega}\cdot\hat{\Omega}')$ is a polynomial of the N^{th} degree, there may be from 2 to 2N + 2 discrete roots of $\Lambda(\nu) = 0$, the number of roots depending on the coefficients c . Also, some of the roots may be degenerate. Thus if $\Lambda(\nu_0) = 0$ has an m-fold root, there will be m eigenfunctions given by

(IV-49) $$\psi_{\nu_0}^{(i)} = \left(\frac{\partial}{\partial\nu}\right)^i \psi(x,\nu)\Big|_{\nu=\nu_0} \qquad i = 0, 1, \cdots, m-1 .$$

Finally, some of the roots may be imbedded in the continuum, i.e. it may happen that $\Lambda(\nu_0) = 0$ for some real ν_0 between -1 and 1. The associated non-singular normal modes must also be taken into account.

Mika has shown that the totality of eigenfunctions form a complete set, when $f(\hat{\Omega}\cdot\hat{\Omega})$ is a polynomial of arbitrary order.