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RECENT APPLICATIONS OF NEUTRON TRANSPORT THEORY

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BASED ON LECTURES DELIVERED AT
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ERRATA

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I.3.2
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CONTENTS

Reporter's Note: The material contained here will be presented in more detail in a book by K. M. Case and P. F. Zweifel entitled Neutron Transport Theory.

I. INTRODUCTION AND FULL-RANGE APPLICATIONS

I.1 Historical Introduction

The idea of using a set of (singular) normal modes in order to solve the transport equation in a manner analogous to the methods used in other problems of mathematical physics seems to have originated with Davison (1). Van Kampen (2) later applied a similar technique to a problem in plasma oscillations. The collisionless Vlasov equation encountered in that application is quite similar to the equation of neutron transport. Later Case (3), generalizing Van Kampen's technique somewhat, showed its equivalence to the Laplace transform approach used by Landau (4) to study plasmas. Wigner (5) also discussed briefly the general ideas which had already been advanced by Davison. It remained for Case (6), however, to exploit the method fully. In particular, Case proved the extremely important partial-range completeness theorem and showed how the normal mode expansion coefficients of the solution of various problems could be obtained from the solution of a certain class of singular integral equations of a type discussed extensively by Muskhelishvili (7). Case also derived an orthogonality relation for infinite medium problems which permitted the coefficients to be obtained directly.

The great virtue of Case's approach is its analogy with the method of solving "classical" partial differential equations. Thus, the solutions of theoretical transport problems are obtained more directly than by using the rather cumbersome methods previously

$$(\psi_k, \psi_{k'}) = \delta_{kk'} . \tag{I.2.3}$$

The usual method of solution of (I.2.1) is to expand the functions $f(\underline{r})$ and $g(\underline{r})$ as

$$\begin{aligned} f(\underline{r}) &= \sum f_k \psi_k(\underline{r}) \\ g(\underline{r}) &= \sum g_k \psi_k(\underline{r}) \end{aligned} \tag{I.2.4}$$

and use (I.2.3) to obtain

$$f_k = \frac{g_k}{\lambda_k} . \tag{I.2.5}$$

We will apply the same method of attack to a modification of the above problem, namely

$$Q f(\underline{r}, \mu) = 0 , \tag{I.2.6}$$

where $f(\underline{r}\ \mu)$ must now satisfy boundary conditions

$$\lim_{\underline{r} \to \infty} f(\underline{r}, \mu) = 0 \tag{I.2.7}$$

$$f(o, \mu) = f_0(\mu) . \tag{I.2.8}$$

We have introduced explicitly the additional dependence, on the angular coordinate μ, which occurs in transport problems with azimuthal symmetry. In this case we must expand in terms of a set $\psi_k(\underline{r}, \mu)$:

$$f(\underline{r},\mu) = \Sigma' f_k \psi_k(\underline{r},\mu) \tag{I.2.9}$$

where the prime denotes the omission of all $\psi_k(\underline{r},\mu)$ not satisfying (I.2.7). Applying (I.2.8) to (I.2.9) gives

$$f_0(\mu) = \Sigma' f_k \psi_k(0,\mu) \equiv \Sigma' f_k \psi_k(\mu) \tag{I.2.10}$$

and once again the expansion coefficients f_k may be found. Note that the completeness of $\psi_k(\underline{r},\mu)$ is required for $\psi_k(0,\mu)$. Thus, the problem is analogous to that of a partial differential equation in (say) $\underline{r}$ and t with certain initial conditions.

I.3 The Transport Equation

We will consider the one-speed, one-dimensional, time-independent neutron transport equation

$$\mu \frac{\delta \psi(x,\mu,\phi)}{\delta x} + \psi = c\int \psi(x,\mu',\phi') f(\underline{\Omega}\cdot\underline{\Omega}')d\underline{\Omega}' + q(x,\mu,\phi) \tag{I.3.1}$$

where we take units such that $v = \Sigma_t = 1$. Here $\psi(x,\mu,\phi)$ is the neutron angular density as a function of optical position x, polar angle $\cos^{-1}\mu$ between the neutron velocity and the x-axis, and azimuthal angle ϕ . The mean number of secondary neutrons emitted per collision is denoted by c. $q(x,\mu,\phi)$ is the source, and $f(\underline{\Omega}\cdot\underline{\Omega}')$ is the probability that a neutron with velocity vector in the solid angle element $d\underline{\Omega}$ about unit vector $\underline{\Omega}$ has a velocity vector in $d\underline{\Omega}'$ about $\underline{\Omega}'$

after a scattering collision. It is customary to expand

$$f(\underline{\Omega}\cdot\underline{\Omega}') = \sum_{\ell=0}^{N} \frac{2\ell+1}{4\pi}\, b_\ell\, P_\ell\,(\underline{\Omega}\cdot\underline{\Omega}'),\quad b_0 = 1, \tag{I.3.2}$$

where the $P_\ell\,(\underline{\Omega}\cdot\underline{\Omega}')$ are Legendre polynomials. We will assume azimuthal symmetry. Insertion of (I.3.2) into (I.3.1) and use of the spherical harmonics addition theorem gives, after integration over azimuthal angle,

$$\mu\frac{\delta\psi(x,\mu)}{\delta x} + \psi = \frac{c}{2}\sum_{\ell=0}^{N}(2\ell+1)b_\ell P_\ell(\mu)\int_{-1}^{1}\psi(x,\mu')P_\ell(\mu')d\mu' + q(x,\mu). \tag{I.3.3}$$

Except for Part IV, we shall be concerned with isotropic scattering, in which case $b_\ell = \delta_{\ell o}$. Definition of the Boltzmann operator for isotropic scattering as

$$B = \mu\frac{\delta}{\delta x} + 1 - \frac{c}{2}\int_{-1}^{1}d\mu' \tag{I.3.4}$$

gives the equation

$$B\psi(x,\mu) = q(x,\mu). \tag{I.3.5}$$

Since the source may be replaced by a boundary condition,* we shall seek the solution of the homogeneous equation

$$B\psi(x,\mu) = 0 \tag{I.3.6}$$

*Or, in some cases, a "particular solution" must be included, cf. Section II.6. In any case, however, it is the solution of the homogeneous equation which is of prime importance.

which is of the same form as (I.2.6). In order to follow the procedure of Section I.2, we must determine a complete orthogonal set of normal modes of B. The angular density may be expanded as

$$\psi(x,\mu) = \sum a_\nu \psi_\nu(x,\mu) \tag{I.3.7}$$

and the expansion coefficients determined from the orthogonality relations. We will now proceed to determine the eigenfunctions, prove completeness theorems, state orthogonality relations, and apply the method to various problems.

I.4 The Normal Modes

The eigenfunctions of B may be written in the form*

$$\psi_\nu(x,\mu) = e^{-x/\nu}\psi_\nu(\mu) \ . \tag{I.4.1}$$

Substitution of (I.4.1) into (I.3.6) gives

$$(\nu-\mu)\psi_\nu(\mu) = \frac{c\nu}{2}\int_{-1}^{1}\psi_\nu(\mu')\,d\mu' \ . \tag{I.4.2}$$

This is a homogeneous equation so we may normalize in any manner; in particular, we set

$$\int_{-1}^{1}\psi_\nu(\mu)\,d\mu = 1, \tag{I.4.3}$$

*Using translational invariance. That is, since B commutes with all translations, it is an invariant of the translation group which means that its eigenfunctions are the invariant subspaces of the translation group. These are the one-dimensional subspaces $e^{-x/\nu}$.

which is permissible except for the trivial solution $\psi_\nu(\mu) \equiv 0$.

With (I.4.2) and (I.4.3), we have

$$(\nu-\mu)\,\psi_\nu(\mu) = \frac{c\nu}{2}\,, \tag{I.4.4}$$

and the most general solution of (I.4.4) is

$$\psi_\nu(\mu) = \frac{c\nu}{2}\, P\frac{1}{\nu-\mu} + \lambda(\mu)\,\delta(\mu-\nu)\,, \tag{I.4.5}$$

where the symbol P signifies that, in any integrals involving $(\nu-\mu)^{-1}$, the Cauchy Principal value is to be taken. The function $\psi_\nu(\mu)$ is not an ordinary function, but a Schwarzian distribution and has a physical meaning only when it appears inside an integral. Since μ is the cosine of an angle $(-1 \le \mu \le 1)$, there are two ranges of interest of the variable $\nu : \nu \notin [-1, 1]$ and $\nu \in [-1, 1]$.

For $\nu \notin [-1, 1]$, (I.4.5) becomes simply

$$\psi_\nu(\mu) = \frac{\frac{c\nu}{2}}{\nu-\mu} \tag{I.4.6}$$

and (I.4.3) tells us that ν must obey the equation

$$\Lambda(\nu) = 1 - \frac{c\nu}{2}\int_{-1}^{1}\frac{d\mu}{\nu-\mu} = 0\,. \tag{I.4.7}$$

Explicitly,

$$\Lambda(\nu) = 1 - c\nu \tanh^{-1}\frac{1}{\nu}\,. \tag{I.4.8}$$

Note that $\Lambda(\infty) = 1 - c$.

There are two roots, denoted by $\pm\nu_0$, to the symmetric dispersion function $\Lambda(\nu)$. For $c<1$, ν_0 is real and, of course, greater than unity. As $c\to 1$, $\nu_0\to\infty$ (this is actually a double root since the two roots coalesce). For $c>1$, ν_0 is purely imaginary. For a given c, there are only two values of ν for which the homogeneous equation has a solution and the eigenfunctions associated with these discrete eigenvalues are denoted by

$$\psi_{0\pm}(\mu) = \frac{c\nu_0}{2}\,\frac{1}{\nu_0 \mp \mu} \tag{I.4.9}$$

and are the discrete eigenfunctions.

For $\nu\in[-1,1]$, (I.4.5) is the form of the eigenfunction and $\lambda(\nu)$ is determined from (I.4.3) as

$$\lambda(\nu) = 1-\frac{c\nu}{2}P\int_{-1}^{1}\frac{d\mu}{\nu-\mu} = 1-c\nu\tanh^{-1}\nu\,. \tag{I.4.10}$$

The values of $\lambda(\nu)$ may be related to the boundary values of the function $\Lambda(\nu)$ which is analytic in the plane cut from -1 to 1 along the real axis. If we define $\Lambda^{\pm}(\nu)=\lim_{\epsilon\to 0^+}\Lambda(\nu\pm i\epsilon)$, then

$$\Lambda^{\pm}(\nu) = \lambda(\nu)\pm\frac{\pi i c\nu}{2}\,. \tag{I.4.11}$$

The $\lambda(\nu)$ can always be chosen from (I.4.10) for any value of ν between -1 and +1 so there is a continuum of eigenvalues with the associated eigenfunctions (I.4.5). These, plus the discrete modes in (I.4.9), are the normal modes of the transport equation.

I.5 Full-Range Completeness Theorem

Theorem. An "arbitrary"* function $\psi(\mu)$ defined on the full range of $\mu(-1 \leq \mu \leq +1)$ can be expanded in terms of the normal modes of the transport equation in the form:

$$\psi(\mu) = a_{0+} \ \psi_{0+} \ (\mu) + a_{0-} \ \psi_{0-} \ (\mu) + \int_{-1}^{1} A(\nu) \psi_{\nu} (\mu) d\nu \tag{I.5.1}$$

where $a_{0\pm}$ and $A(\nu)$ are the expansion coefficients. This theorem is a special case of the partial-range completeness theorem first proved by Case (6). We attempt first to expand another "arbitrary" function $\psi'(\mu)$ in terms of the continuum modes alone and then determine the necessary restrictions to make the expansion permissible. Consider

$$\psi'(\mu) = \int_{-1}^{1} A(\nu) \psi_{\nu} (\mu) d\nu \tag{I.5.2}$$

and insert (I.4.5) to obtain the singular integral equation

$$\psi'(\mu) = P\int_{-1}^{1} \frac{\frac{c\nu}{2} A(\nu) \, d\nu}{\nu - \mu} + \lambda(\mu) A(\mu) \ . \tag{I.5.3}$$

Using (I.4.11), one obtains

$$\psi'(\mu) = \frac{1}{2} \left(\Lambda^{+} (\mu) + \Lambda^{-} (\mu) \right) A(\mu) + P\int_{-1}^{1} \frac{c\nu}{2} \frac{A(\nu)}{\nu - \mu} d\nu \ . \tag{I.5.4}$$

*The proof is valid if $\psi(\mu)$ is a distribution (29) as well as, of course, an "ordinary" function.

The method described by Muskhelishvili (7) will be used to show that (I.5.4) has a solution i.e. that the expansion (I.5.2) is possible.

Define a function N(z) by

$$N(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{c\nu}{2}\,\frac{A(\nu)d\nu}{\nu - z} \quad . \tag{I.5.5}$$

If a sufficiently well behaved $A(\nu)$ exists, then N(z) has the following properties:

1) N(z) analytic in complex plane cut from -1 to +1;

2) N(z) $\sim$ z^{-1} as $z \to \infty$;

3) $N^{+}(\mu) + N^{-}(\mu) = \frac{1}{\pi i} P\int_{-1}^{1} \frac{c\nu}{2}\,\frac{A(\nu)d\nu}{\nu - \mu}$;

$N^{+}(\mu) - N^{-}(\mu) = \frac{c\mu}{2} A(\mu)$

Equation (I.5.4) may now be reduced to the Hilbert equation

$$\frac{c\mu}{2}\,\psi'(\mu) = \Lambda^{+}(\mu)\,N^{+}(\mu) - \Lambda^{-}(\mu)\,N^{-}(\mu)\ . \tag{I.5.6}$$

Since the function $\Lambda(z)\,N(z)$ is analytic in the complex plane cut from -1 to 1 and is required to vanish like z^{-1} as $z \to \infty$, the Cauchy integral theorem may be applied to prove that

$$N(z) = \frac{1}{2\pi i\,\Lambda(z)}\int_{-1}^{1} \frac{c\mu}{2}\,\frac{\psi'(\mu)d\mu}{\mu - z} \quad . \tag{I.5.7}$$

The function N(z) of (I.5.7) has the behavior required except for simple poles at $\pm\nu_0$ arising from the vanishing of $\Lambda(z)$. In order to remove this difficulty, we require that the numerator also vanish at $\pm\nu_0$, i.e. that

$$\int_{-1}^{1} \frac{c\mu}{2} \frac{\psi'(\mu)d\mu}{\mu \mp \nu_0} = \mp \frac{1}{\nu_0} \int_{-1}^{1} \mu \psi_{0\pm} (\mu) \psi'(\mu) d\mu = 0 \qquad \text{(I.5.8)}$$

which will not be true in general. Remembering that we are presently expanding $\psi'(\mu)$ which is related to $\psi(\mu)$ by

$$\psi(\mu) = \psi'(\mu) + a_{0+} \psi_{0+} (\mu) + a_{0-} \psi_{0-} (\mu) , \qquad \text{(I.5.9)}$$

equation (I.5.8) may be satisfied if

$$\int_{-1}^{1} \mu\psi_{0\pm}(\mu) \psi(\mu)d\mu = a_{0+} \int_{-1}^{1} \mu\psi_{0+}(\mu) \psi_{0\pm}(\mu)d\mu + a_{0-} \int_{-1}^{1} \mu\psi_{0-}(\mu)\psi_{0\pm}(\mu) d\mu , \qquad \text{(I.5.10)}$$

which will be true if the discrete coefficients are defined by

$$a_{0\pm} = \frac{\int_{-1}^{1} \mu \psi_{0\pm}(\mu) \psi(\mu) d\mu}{\int_{-1}^{1} \mu \psi_{0\pm}^{2}(\mu) d\mu} . \qquad \text{(I.5.11)}$$

It will be seen in the next section that the same conditions can be obtained from orthogonality relations.

That we do indeed have a complete set of eigenfunctions may be verified by the use of the boundary values of N(z) given by (I.5.7) to obtain A(μ) and the substitution of this into the r.h.s. of (I.5.2) to regain the l.h.s. of (I.5.2) (30). Recently, the general range completeness theorem has also been proved (31) using the orthogonality relations (28).

I.6 Full-Range Orthogonality Relations

Full-range orthogonality of the eigenfunctions was first observed by Case (6) and may be seen by writing (I.4.2) in terms of ν and ν', multiplying the first equation by $\nu'\psi_{\nu'}(\mu)d\mu$ and the second by $\nu\psi_{\nu}(\mu)d\mu$ and integrating both equations over μ from -1 to +1. Subtraction of the two equations and use of (I.4.3) gives

$$(\nu-\nu')\int_{-1}^{1}\mu\,\psi_{\nu}(\mu)\psi_{\nu'}(\mu)\,d\mu = 0 \,. \tag{I.6.1}$$

The normalization condition for the discrete eigenfunctions is (6, 11)

$$\int_{-1}^{1}\mu\,\psi_{0\pm}^{2}(\mu)\,d\mu = N_{0\pm} = \pm\frac{c\nu_0^3}{2}\left[\frac{c}{\nu_0^2-1}-\frac{1}{\nu_0^2}\right] , \tag{I.6.2}$$

and for the continuum eigenfunctions is (6, 11)

$$\int_{-1}^{1}\mu\,\psi_{\nu}(\mu)\,\psi_{\nu'}(\mu)\,d\mu = N(\nu)\,\delta(\nu-\nu') , \tag{I.6.3}$$

where

$$N(\nu) = \nu\Lambda^{+}(\nu)\,\Lambda^{-}(\nu) = \nu\left[\lambda^2(\nu)+\left(\frac{\pi c\nu}{2}\right)^2\right] = \frac{\nu}{g(c,\nu)} \,. \tag{I.6.4}$$

The function $g(c,\nu)$ is tabulated extensively in Case et al. (32). Equation (I.6.3) is most easily derived using a modified form of the Poincaré-Bertrand formula (7) which is (28)

$$\frac{P}{\nu-\mu}\,\frac{P}{\nu'-\mu} = \frac{1}{\nu-\nu'}\left[P\frac{1}{\nu'-\mu} - P\frac{1}{\nu-\mu}\right] + \pi^2\delta(\nu-\mu)\,\delta(\nu'-\mu). \tag{I.6.5}$$

Equation (I.6.5) is used in the integral term of (I.6.3) containing the double singularity.

I.7 Full-Range Application: Infinite Medium Green's Function

We wish to find the angular density $G(0,\mu_0 \rightarrow x,\mu) \equiv G(x,\mu)$ at any position and direction due to a unit plane source located at the origin and emitting neutrons in the direction μ_0. The angular density satisfies the homogeneous transport equation (I.3.6) except at the origin where the source will be replaced by the boundary condition

$$G(0^+,\mu) - G(0^-,\mu) = \frac{\delta(\mu-\mu_0)}{2\pi\mu} \,. \tag{I.7.1}$$

If $c < 1$, then

$$\lim_{x\rightarrow\pm\infty} G(x,\mu) = 0 \tag{I.7.2}$$

and the diverging modes must not appear; thus we look for a solution of the form*

$$G(x,\mu) = \begin{cases} a_{0+}\,\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\,\psi_\nu(\mu)\,e^{-x/\nu}\,d\nu, & x>0 \\ -a_{0-}\,\psi_{0-}(\mu)e^{-x/\nu_0} - \int_{-1}^0 A(\nu)\,\psi_\nu(\mu)e^{-x/\nu}\,d\nu, & x<0 \,. \end{cases} \tag{I.7.3}$$

*Sign convention is arbitrary and is chosen for convenience.

Applying (I.7.1) gives

$$\frac{\delta(\mu-\mu_0)}{2\pi\mu} = a_{0+}\psi_{0+}(\mu) + a_{0-}\psi_{0-}(\mu) + \int_{-1}^{1} A(\nu)\psi_\nu(\mu)d\nu \ , \qquad \text{(I.7.4)}$$

which is an equation of the form (I.5.1) where the function to be expanded in terms of the normal modes is $\psi(\mu) = \frac{\delta(\mu-\mu_0)}{2\pi\mu}$. Section I.5 verifies the completeness of the expansion and the orthogonality conditions of Section I.6 give the expansion coefficients as

$$a_{0\pm} = \frac{\psi_{0\pm}(\mu_0)}{2\pi N_{0\pm}}$$

$$A(\nu) = \frac{\psi_\nu(\mu_0)}{2\pi N(\nu)} \ . \qquad \text{(I.7.5)}$$

II. HALF-RANGE APPLICATIONS

II.1 The Half-Space Albedo Problem--Need For Another Completeness Theorem

We have seen that a typical "full space" problem involved the expansion of a function $\psi(\mu)$ on the range $-1 \leq \mu \leq 1$ in terms of the normal modes of the homogeneous Boltzmann equation. From the general uniqueness theorem (10), we know that the steady-state angular density for $c < 1$ in volume V is determined by the sources in V and the incoming distribution on the surface of V. For half-space problems with V in the right half-space, this incident distribution is $\psi(0,\mu)$ for $\mu \geq 0$. This leads to a different type of expansion, as we shall see below.

For the half-space albedo problem, we wish to find the angular density at any position and direction due to an incoming distribution on the surface of the form

$$\psi_a(0,\mu) = \delta(\mu-\mu_0)\,, \qquad \mu \geq 0\,. \tag{II.1.1}$$

For $c < 1$,

$$\psi_a(\infty,\mu) = 0\,. \tag{II.1.2}$$

Boundary condition (II.1.2) requires that we exclude all diverging modes of the Boltzmann operator in our expansion. An expression which obeys this condition is

$$\psi_a(x,\mu) = a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu . \qquad \text{(II.1.3)}$$

Using (II.1.1) we seek the solution of

$$\delta(\mu-\mu_0) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)d\nu, \qquad \mu \geq 0 . \qquad \text{(II.1.4)}$$

The boundary condition is imposed only for $\mu \geq 0$ and for this reason is different from the expansion of Section I.5. We will therefore prove a half-range completeness theorem for the eigenfunctions and also examine orthogonality relations which may be used to determine the expansion coefficients. The half-range relations have the form

$$(\nu-\nu')\int_0^1 W(\mu)\psi_\nu(\mu)\psi_{\nu'}(\mu)d\mu = 0 \qquad \text{(II.1.5)}$$

$$\int_0^1 W(\mu)\psi_{0+}^2(\mu)d\mu = K_+ \qquad \text{(II.1.6)}$$

$$\int_0^1 W(\mu)\psi_\nu(\mu)\psi_{\nu'}(\mu)d\mu = H(\nu)\delta(\nu-\nu') , \qquad \text{(II.1.7)}$$

where $W(\mu)$ is the half-range weight function (μ was the full-range weight function). The orthogonality relations and (II.1.4) permit the determination of the expansion coefficients for the half-space albedo problem as

$$a_{0+} = \frac{W(\mu_0)\psi_{0+}(\mu_0)}{K_+}$$

$$A(\nu) = \frac{W(\mu_0)\psi_\nu(\mu_0)}{H(\nu)} \tag{II.1.8}$$

and the problem is solved.

II.2 Half-Range Completeness Theorem

Theorem. An "arbitrary" function $\psi(\mu)$ defined on the half-range $0 \leq \mu \leq 1$* can be expanded in terms of the continuum modes with $0 \leq \nu \leq 1$ plus one of the discrete modes, say $\psi_{0+}(\mu)$, as

$$\psi(\mu) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)d\nu\,, \quad 0 \leq \mu \leq 1\,. \tag{II.2.1}$$

The proof of the theorem**follows along the lines of the full-range theorem and is another specialization of the general-range theorem (6). Here we again attempt to expand not $\psi(\mu)$ but $\psi'(\mu)$ in terms of the continuum modes alone:

$$\psi'(\mu) = \int_0^1 A(\nu)\psi_\nu(\mu)d\nu\,, \quad 0 \leq \mu \leq 1\,. \tag{II.2.2}$$

Following the proof of Section I.5 we are again led to a Hilbert equation of the form

$$\frac{c\mu}{2}\psi'(\mu) = \frac{\Lambda^+(\mu) - \Lambda^-(\mu)}{2\pi i}\psi'(\mu) = \Lambda^+(\mu)N^+(\mu) - \Lambda^-(\mu)N^-(\mu)\,, \quad 0 \leq \mu \leq 1\,, \tag{II.2.3}$$

*There is an analogous theorem for the other half-range $-1 \leq \mu \leq 0$.
**Proved in detail in (11).

where in this case the function N(z) is analytic in the complex plane cut from 0 to 1 rather than from -1 to 1 because the limits of integration on the definition of N(z) are from 0 to 1. Thus the branch cuts of $\Lambda(z)$ and N(z) differ and the right hand side of (II.2.3) is not simply the difference in the boundary values of an analytic function as was previously the case.

For this reason, we introduce a new function X(z) which is analytic in the plane cut from 0 to 1 and obeys the "ratio condition"

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Lambda^+(\mu)}{\Lambda^-(\mu)} \qquad , 0 \leq \mu \leq 1 \; . \tag{II.2.4}$$

We will also require that X(z) be bounded at infinity and non-zero in the finite complex plane cut from 0 to 1. Using (II.2.4) in (II.2.3) we obtain

$$N^+(\mu)X^+(\mu) - N^-(\mu)X^-(\mu) = \gamma(\mu)\psi'(\mu), \; 0 \leq \mu \leq 1 \tag{II.2.5}$$

where we have defined

$$\gamma(\mu) \equiv \frac{1}{2\pi i}\left[X^+(\mu) - X^-(\mu)\right] = \frac{c\mu}{2}\,\frac{X^-(\mu)}{\Lambda^-(\mu)}, \; 0 \leq \mu \leq 1 . \tag{II.2.6}$$

The Cauchy integral theorem may again be used to prove that

$$N(z) = \frac{1}{2\pi i\, X(z)} \int_0^1 \frac{\gamma(\mu)\psi'(\mu)\,d\mu}{\mu - z} \; , \tag{II.2.7}$$

provided the function N(z) has the properties ascribed to it.

A function $X_o(z)$ which obeys the ratio condition and is analytic in the cut plane is

$$X_0(z) = \exp \frac{1}{2\pi i} \int_0^1 \frac{\left[\ln \Lambda^+(\mu) - \ln \Lambda^-(\mu)\right] d\mu}{\mu - z}, \tag{II.2.8}$$

which can be reduced to

$$X_0(z) = \exp \frac{1}{\pi} \int_0^1 \frac{\theta(\mu)\, d\mu}{\mu - z} \tag{II.2.9}$$

with the definition

$$\theta(\mu) = \arg \Lambda^+(\mu). \tag{II.2.10}$$

Near the endpoint $z = 1$ of the cut, $X_o(z)$ goes to zero as $(1 - z)$. Thus, (II.2.7) gives $N(z)$ a pole at $z = 1$ which is not permissible (since from the definition of $N(z)$, $z = 1$ is a branch point, not a pole). However, the function

$$X(z) = \frac{X_0(z)}{1 - z} \tag{II.2.11}$$

also obeys the condition (II.2.4) and leads to an acceptable $N(z)$.

Now that the proper $X(z)$ function is determined for the half-range case under consideration, we see from (II.2.7) that for large z,

$$N(z) \sim \frac{-z}{2\pi i} \int_0^1 \frac{\gamma(\mu)\psi'(\mu)d\mu}{\mu - z} = \tag{II.2.12}$$

$$\frac{1}{2\pi i} \int_0^1 \gamma(\mu)\psi'(\mu) \left[1 + \frac{\mu}{z} + \frac{\mu^2}{z^2} + \cdots\right] d\mu.$$

The function N(z) will not vanish like z^{-1} for large z (cf. Section I.5) unless the condition

$$\int_0^1 \gamma(\mu)\,\psi'(\mu)\,d\mu = 0 \tag{II.2.13}$$

is introduced. Again this condition can be met since we are trying to expand $\psi(\mu)$ and not $\psi'(\mu)$. For

$$\psi(\mu) \equiv \psi'(\mu) + a_{0+}\psi_{0+}(\mu), \tag{II.2.14}$$

(II.2.13) can be met if we choose

$$a_{0+} = \frac{\int_0^1 \gamma(\mu)\,\psi(\mu)\,d\mu}{\int_0^1 \psi_{0+}(\mu)\,\gamma(\mu)\,d\mu} . \tag{II.2.15}$$

The continuum expansion coefficient is found from

$$A(\nu) = \frac{2}{c\nu}\left[N^+(\nu) - N^-(\nu)\right] \tag{II.2.16}$$

as before and again the completeness of the expansion may be verified by substituting the value of $A(\nu)$ into (II.2.2).

II.3 Properties of X(z)

It has been shown (11) that X(z) obeys the following identities:

Identity A

$$X(z) = \int_0^1 \frac{\gamma(\mu)\,d\mu}{\mu - z} \tag{II.3.1}$$

Identity B

$$X(z)X(-z) = \frac{\Lambda(z)}{(\nu_0^2 - z^2)(1-c)} \tag{II.3.2}$$

Identity C

$$X(z) = \frac{c}{2(1-c)} \int_{-1}^{0} \frac{\mu d\mu}{(\nu_0^2 - \mu^2) X(\mu)(\mu + z)} \, . \tag{II.3.3}$$

Equation (II.3.3) is a non-linear, non-singular integral equation which permits numerical determination of X(z). Identity C follows from Identities A and B with the observation that (II.3.2) can be specialized to give

$$X(0) = \frac{1}{\nu_0(1-c)^{1/2}} \tag{II.3.4}$$

and

$$\frac{X(\mu)}{\Lambda^-(\mu)} = \frac{2}{c\mu}\, \gamma(\mu) = \frac{1}{(\nu_0^2 - \mu^2)(1-c)X(-\mu)} \, , \; 0 \le \mu \le 1 \, . \tag{II.3.5}$$

Equation (II.3.5) means that all answers can be expressed in terms of $X(-\mu)$ so that $X^{\pm}(\mu)$ and $\Lambda^{\pm}(\mu)$ need not be tabulated. These values of $X(-\mu)$ have been tabulated (<u>15</u>) along with Chandrasekhar's H-function (<u>8</u>) and Davison's h^+-function (<u>9</u>), which are related to the X-function by (<u>15</u>)

$$X(-z) = \frac{1}{(\nu_0 + z)(1-c)^{1/2} H(z)} \tag{II.3.6}$$

$$h^+\left(\frac{1}{\nu}\right) = \nu_0(1-c)^{1/2} \, (1+\nu) X(-\nu) \, . \tag{II.3.7}$$

Another form of Identity C which gives a better starting point for the iterative evaluation of X(z) is (13)

$$X(z) = \frac{1}{\nu_0(1-c)^{1/2}} - \frac{cz}{2(1-c)} \int_{-1}^{0} \frac{d\mu}{(\nu_0^2-\mu^2)X(\mu)(\mu+z)} \ . \tag{II.3.8}$$

The most satisfactory scheme (13) of evaluating X(z) is given by

$$\Omega(z) = 1 - \frac{zc\nu_0^2}{2} \int_{-1}^{0} \frac{\left[1-\mu^2 X^2(0)\right] d\mu}{(\nu_0^2+\mu^2)\Omega(\mu)(\mu+z)} \ , \tag{II.3.9}$$

where $\Omega(z)$ is defined by

$$\Omega(z) = \left(\frac{1}{X(0)} - z\right) X(z) \ . \tag{II.3.10}$$

II.4 Half-Range Orthogonality Relations

The half-range relations are a special case of the general-range relations and are (28)

$$\int_0^1 W(\mu)\psi_{0+}(\mu)\psi_\nu(\mu)\,d\mu = 0 \tag{II.4.1}$$

$$\int_0^1 W(\mu)\psi_\nu(\mu)\psi_{\nu'}(\mu)\,d\mu = \frac{W(\nu)N(\nu)}{\nu}\,\delta(\nu-\nu') \ , \tag{II.4.2}$$

where

$$W(\mu) = (\nu_0-\mu)\gamma(\mu) \tag{II.4.3}$$

and

$$N(\nu) = \nu \Lambda^+(\nu) \Lambda^-(\nu) \, . \qquad \text{(I.6.4)}$$

Other important formulae are (28)

$$\int_0^1 W(\mu) \psi_{0-}(\mu) \psi_\nu(\mu) d\mu = c\nu_0 \nu X(-\nu_0) \psi_{0-}(\nu) \qquad \text{(II.4.4)}$$

$$\int_0^1 W(\mu) \psi_{0\pm}(\mu) \psi_{0+}(\mu) d\mu = \mp \left(\frac{c\nu_0}{2}\right)^2 X(\pm\nu_0) \, . \qquad \text{(II.4.5)}$$

Since we have now proved completeness of the half-range expansion and have the orthogonality relations, the solution of the half-space albedo problem of Section II.1 is complete. For the angular density, we have

$$\psi_a(x,\mu) = \frac{-2\gamma(\mu_0)}{c\nu_0 X(\nu_0)} \psi_{0+}(\mu) e^{-x/\nu_0} + (\nu_0 - \mu_0)\gamma(\mu_0) \int_0^1 \frac{\nu \psi_\nu(\mu_0) \psi_\nu(\mu) e^{-x/\nu} d\nu}{N(\nu)\gamma(\nu)(\nu_0 - \nu)} \, , \qquad \text{(II.4.6)}$$

where the first term on the right hand side is the asymptotic term (since $|\nu_0| > |\nu|$ for $0 < c \leq 1$) and the second term is the transient term (also known as the branch cut integral) which vanishes more rapidly for large x than the asymptotic term.

II.5 Two Other Half-Space Applications

a) Milne Problem. This is the solution for the neutron distribution in a source-free half-space with zero incident flux at the surface. A source at infinity provides neutrons for the system. The boundary conditions are *

$$\psi_m(x,\mu) \to \psi_{0-}(x,\mu) \text{ as } x \to \infty \tag{II.5.1}$$

$$\psi_m(0,\mu) = 0, \quad \mu \geq 0. \tag{II.5.2}$$

The solution of the Milne problem is the linear combination of the half-range normal modes plus the $\psi_{0-}(x,\mu)$. Therefore, (II.5.1) is satisfied by

$$\psi_m(x,\mu) = \psi_{0-}(\mu)e^{x/\nu_0} + a_{0+}\,\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu \tag{II.5.3}$$

and application of (II.5.2) gives

$$-\psi_{0-}(\mu) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\,\psi_\nu(\mu)\,d\nu\,, \quad \mu \geq 0\,. \tag{II.5.4}$$

The orthogonality relations of Section II.4 can be used to obtain

$$a_{0+} = \frac{X(-\nu_0)}{X(\nu_0)} \tag{II.5.5}$$

$$A(\nu) = \frac{-c\nu_0\nu^2 X(-\nu_0)\psi_{0-}(\nu)}{(\nu_0-\nu)\gamma(\nu)N(\nu)}\,. \tag{II.5.6}$$

The above solution can be used to give the Milne problem

*From Section I.7 we know that the asymptotic distribution far away from a plane source varies as e^{-x/ν_0} . Therefore, the angular density rises as e^{x/ν_0} when approaching the source at infinity.

extrapolation distance, z_o, which is the point where the asymptotic neutron density vanishes. As always, the asymptotic contribution to the angular density comes from the discrete modes. Using the normalization condition (I.4.3),

$$\frac{\rho_{as}(x)}{2\pi} = \frac{\int_{-1}^{1} \psi_{as}(x,\mu)}{2\pi} = e^{x/\nu_0} + a_{0+} e^{-x/\nu_0} , \tag{II.5.7}$$

so

$$\frac{\rho_{as}(-z_0)}{2\pi} = 0 = e^{-z_0/\nu_0} + a_{0+} e^{z_0/\nu_0} . \tag{II.5.8}$$

This reduces to

$$z_0 = \frac{-\nu_0}{2} \ln(-a_{0+}) = \frac{-\nu_0}{2} \ln\left[\frac{-X(-\nu_0)}{X(\nu_0)}\right] . \tag{II.5.9}$$

b) Green's Function. The half-space Green's function is the solution for the angular density in a half-space with a plane source somewhere in the interior, x_o, and zero incoming flux at the surface. The boundary conditions are

$$\lim_{x \to \infty} \Psi_g(x,\mu) = 0 \tag{II.5.10}$$

$$\Psi_g(x_0^+,\mu) - \Psi_g(x_0^-,\mu) = \frac{\delta(\mu-\mu_0)}{2\pi\mu} \tag{II.5.11}$$

$$\Psi_g(o,\mu) = 0, \quad \mu \geq 0 . \tag{II.5.12}$$

The infinite-medium Green's function of Section I.7 satisfies the transport equation and (II.5.10) and (II.5.11) so the half-space Green's function is taken as a linear combination of the normal modes which vanish at infinity and $G(x_0,\mu_0 \to x,\mu) \equiv G(x,\mu)$:

$$\Psi_g(x,\mu) = G(x,\mu) + a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu. \tag{II.5.13}$$

The expansion coefficients are then selected to satisfy (II.5.12) and may be obtained by applying the orthogonality relations to

$$-G(o,\mu) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)d\nu, \quad \mu \geq 0 . \tag{II.5.14}$$

II.6 <u>Some Half-Space Simplifications</u>

a) Angular Density. The most general expression for the angular density which appears in various half-space applications is

$$\Psi(x,\mu) = f(x,\mu) + a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu , \tag{II.6.1}$$

where $f(x,\mu)$ is the particular solution for a problem and is given by, for example,

$$
f(x,\mu) = \begin{cases} 0 & \text{albedo} \\ \psi_{0-}(\mu)e^{x/\nu_0} & \text{Milne} \\ G(x_0,\mu_0 \to x,\mu) & \text{Green's function} \\ \dfrac{q}{1-c} & \text{constant isotropic source.} \end{cases} \tag{II.6.2}
$$

The "expansion function" $\psi(\mu)$ for a given problem is defined by

$$
\psi(\mu) = \Psi(0,\mu) - f(0,\mu) = a_{0+}\,\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)\,d\nu \tag{II.6.3}
$$

and the expansion coefficients are

$$
a_{0+} = \frac{-1}{\nu_0 X(\nu_0)(1-c)} \int_0^1 \frac{\mu\psi(\mu)\,d\mu}{X(-\mu)(\nu_0^2-\mu^2)} \tag{II.6.4}
$$

$$
A(\nu) = \frac{(\nu_0+\nu)X(-\nu)}{N(\nu)} \left[\frac{c\nu}{2} P\int_0^1 \frac{\mu\psi(\mu)\,d\mu}{(\nu_0+\mu)X(-\mu)(\nu-\mu)} + \frac{\nu\psi(\nu)\lambda(\nu)}{(\nu_0+\nu)X(-\nu)} \right] \tag{II.6.5}
$$

where (II.3.5) has been substituted for $\gamma(\mu)$ and $\gamma(\nu)$. If numerical evaluation of the principal value integral is necessary, it may always be performed by use of the identity

$$
P\int_\alpha^\beta \frac{f(z)\,dz}{z-z_0} = \int_\alpha^\beta \frac{f(z)-f(z_0)}{z-z_0}\,dz + f(z_0)\ln\left(\frac{\beta-z_0}{z_0-\alpha}\right). \tag{II.6.6}
$$

b) Density. The neutron density is defined by

$$
\frac{\rho(x)}{2\pi} = \int_{-1}^1 \Psi(x,\mu)\,d\mu = \int_{-1}^1 f(x,\mu)\,d\mu + a_{0+}e^{-x/\nu_0} + \int_0^1 A(\nu)e^{-x/\nu}\,d\nu \tag{II.6.7}
$$

where the normalization condition (I.4.3) has been used.

c) Current. The neutron current is defined by

$$\frac{j(x)}{2\pi}=\int_{-1}^{1}\mu\Psi(x,\mu)\,d\mu=\int_{-1}^{1}\mu f(x,\mu)\,d\mu+(1-c)\left[a_{0+}\nu_0 e^{-x/\nu_0}+\int_0^1 \nu A(\nu)e^{-x/\nu}\,d\nu\right] \tag{II.6.8}$$

where (I.4.3) and partial fractions have been used.

d) Emergent Angular Density. The outgoing angular density from the surface of the half-space may be simplified. The emergent distribution is

$$\Psi(0,\mu)=f(0,\mu)+a_{0+}\psi_{0+}(\mu)+\int_0^1 A(\nu)\psi_\nu(\mu)\,d\nu,\quad \mu\le 0 \tag{II.6.9}$$

and by use of a_{0+} and $A(\nu)$ from a), one can obtain

$$\Psi(0,\mu)=f(0,\mu)+\frac{1}{X(+\mu)}\left[\int_0^1\gamma(\mu')\psi(\mu')\left(\frac{1}{\mu'-\mu}-\frac{1}{\nu_0-\mu}\right)d\mu'\right] \tag{II.6.10}$$
$$\mu\le 0,$$

where the following identity has been used:

Identity D.

$$\int_0^1\frac{\nu\psi_\nu(\mu)\psi_\nu(\mu')\,d\nu}{(\nu_0-\nu)\gamma(\nu)N(\nu)}=\frac{1}{\nu_0-\mu}\left[\frac{1}{\nu-\mu'}\,\frac{1}{X(\nu_0)}-\frac{1}{\mu-\mu'}\,\frac{1}{X(\mu)}\right],\ \mu<0,\mu'>0. \tag{II.6.11}$$

As an example, the emergent angular density for the albedo problem with $\Psi_a(0,\mu)=\delta(\mu-\mu_0),\mu\ge 0$ & $f_a(0,\mu)=0$ $\mu\ge 0$ reduces to the form

$$\Psi_a(0,\mu)=\frac{\frac{c\mu_0}{2}}{\mu_0-\mu}\;\frac{1}{(\nu_0+\mu_0)X(-\mu_0)(1-c)(\nu_0-\mu)X(\mu)},\quad \mu\le 0 \tag{II.6.12}$$

and the emergent angular density for the Milne problem with $\psi(\mu) = -\psi_{0-}(\mu)$ simplifies to

$$\Psi_m(o,\mu) = \frac{c\nu_0^2 X(-\nu_0)}{X(\mu)(\nu_0^2-\mu^2)} \ . \tag{II.6.13}$$

Using (II.6.13) and Identity C, one can also show that (11)

$$\frac{\rho_m(0)}{2\pi} = 2\,\frac{X(-\nu_0)}{X(0)} \tag{II.6.14}$$

$$\frac{j_m(0)}{2\pi} = \frac{-2X(-\nu_0)}{X^2(0)} \ . \tag{II.6.15}$$

II.7 Two Slab Problems

Slab problems lead, in general, to Fredholm integral equations for the expansion coefficients rather than closed form expressions which were obtained in the half-space problems. In general, the Fredholm equations do not have simple solutions in closed form.

a) Albedo Problem. Assuming the slab boundaries to be at x = 0 and x = d, we wish to solve the transport equation for $c<1$ subject to the boundary conditions

$$\psi(o,\mu) = \delta(\mu-\mu_0) \ , \quad \mu \geq 0 \tag{II.7.1}$$

$$\psi(d,\mu) = 0 \ , \quad \mu \leq 0 \ . \tag{II.7.2}$$

The angular density is expanded as

$$\psi(x,\mu) = a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + a_{0-}\psi_{0-}(\mu)e^{x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu + \int_0^1 A(-\nu)\psi_{-\nu}(\mu)e^{x/\nu}d\nu\,, \tag{II.7.3}$$

and the application of the boundary conditions to (II.7.3) gives (17)

$$\begin{aligned}\psi_\pm(\mu) &= \delta(\mu-\mu_0) - b_\pm\psi_{0+}(\mu) \mp \int_0^1 B_\pm(\nu)e^{-d/\nu}\psi_{-\nu}(\mu)d\nu \\ &= \int_0^1 B_\pm(\nu)\psi_\nu(\mu)d\nu\,, \quad \mu \geq 0\,,\end{aligned} \tag{II.7.4}$$

where

$$b_\pm = a_{0+} \pm a_{0-}e^{d/\nu_0} \tag{II.7.5}$$

$$B_\pm(\nu) = A(\nu) \pm A(-\nu)e^{d/\nu}. \tag{II.7.6}$$

Equations (II.7.4) are two half-range expansions of the type we have been dealing with and can be reduced, by means of the orthogonality relations of Section II.4 and partial fractions, to (17)

$$b_\pm = \frac{\frac{2}{c}\gamma(\mu_0) \mp \int_0^1 \nu B_\pm(\nu)e^{-d/\nu}X(-\nu)d\nu}{\nu_0 X(-\nu_0)\left[e^{2z_0/\nu_0} \pm e^{-d/\nu_0}\right]} \tag{II.7.7}$$

$$B_{\pm}(\nu) = \frac{\nu(\nu_0^2-\nu^2)(1-c)X(-\nu)}{N(\nu)}\left[\frac{2\gamma(\mu_0)\psi_\nu(\mu_0)}{c\nu}\right.$$

$$-b_{\pm}\psi_{0+}(\nu)X(\nu_0) \mp b_{\pm}e^{-d/\nu_0}\psi_{0-}(\nu)X(-\nu_0)$$

$$\left.\mp\int_0^1 B_{\pm}(\mu)e^{-d/\mu}\psi_{-\mu}(\nu)X(-\mu)\,d\mu\right], \tag{II.7.8}$$

where z_o is the Milne problem extrapolation distance of Section II.5. Approximations to the above equations give approximate values for the expansion coefficients $a_{0\pm}$ and $A(\pm\nu), \nu \geq 0$. For example, for large d, the integral terms with $e^{-d/\nu}$ vanish more rapidly than terms proportional to e^{-d/ν_0} and the expansion coefficients become (17)

$$a_{0+}^{(0)} = -e^{(2z_0+2d)/\nu_0}\,a_{0-}^{(0)} = \frac{\gamma(\mu_0)e^{d/\nu_0}}{c\nu_0 X(-\nu_0)\sinh\left[(2z_0+d)/\nu_0\right]} \tag{II.7.9}$$

$$A^{(0)}(\nu) = \frac{\nu(\nu_0^2-\nu^2)(1-c)X(-\nu)}{N(\nu)}\left[\frac{2\gamma(\mu_0)\psi_\nu(\mu_0)}{c\nu}\right. \tag{II.7.10}$$

$$\left.-a_{0+}^{(0)}\psi_{0+}(\nu)X(\nu_0) - a_{0-}^{(0)}\psi_{0-}(\nu)X(-\nu_0)\right]$$

$$A^{(0)}(-\nu) = -e^{-d/\nu}\,\frac{\nu(\nu_0^2-\nu^2)(1-c)X(-\nu)}{N(\nu)}\left[a_{0-}^{(0)}e^{d/\nu_0}\psi_{0+}(\nu)X(\nu_0)\right. \tag{II.7.11}$$

$$\left.+a_{0+}^{(0)}e^{-d/\nu_0}\psi_{0-}(\nu)X(-\nu_0)\right].$$

b) Critical Problem. If the slab is critical, then by symmetry $\psi(x,\mu)=\psi(d-x,-\mu)$ and there are no incident neutrons at the surface. Setting $\mu_0=0$ and imposing the symmetry condition gives

$$a_{0+} = a_{0-} e^{d/\nu_0} \tag{II.7.12}$$

$$A(\nu) = A(-\nu) e^{d/\nu} \tag{II.7.13}$$

and therefore

$$b_- = B_-(\nu) = 0 \tag{II.7.14}$$

and (II.7.7) and (II.7.8) become the equations for the critical slab problem (16, 18):

$$a_{0+} \nu_0 X(-\nu_0) \left[e^{2z_0/\nu_0} + e^{-d/\nu_0} \right] = -\int_0^1 \nu A(\nu) e^{-d/\nu} X(-\nu) d\nu \tag{II.7.15}$$

$$A(\nu) = -\frac{\nu(\nu_0^2 - \nu^2)(1-c) X(-\nu)}{N(\nu)} \left[a_{0+} \left\{ \psi_{0+}(\nu) X(\nu_0) + e^{-d/\nu_0} \psi_{0-}(\nu) X(-\nu_0) \right\} + \int_0^1 A(\mu) e^{-d/\mu} \psi_{-\mu}(\nu) X(-\mu) d\mu \right]. \tag{II.7.16}$$

For the diffusion approximation (where there is no continuum contribution and $A(\nu) \equiv 0$), (II.7.15) is the critical condition

$$e^{2z_0/\nu_0} + e^{-d/\nu_0} = 0 \tag{II.7.17}$$

and a solution to (II.7.17) exists for

$$\cos\left[(2z_0+d)/2\nu_0\right] = 0 \quad . \qquad \text{(II.7.18)}$$

Thus,

$$d = \pi|\nu_0| - 2z_0 \; . \qquad \text{(II.7.19)}$$

III. TWO HALF-SPACE APPLICATIONS

III.1 Introduction And Normal Modes

For the two half-space problems, we consider two adjacent semi-infinite media of different properties with an interface at x = 0. Denote all functions in the right and left half-spaces by subscripts 1 and 2, respectively.

The solution to the transport equation in each half-space is expanded as

$$\psi(x,\mu) = \begin{cases} f_1(x,\mu) + a_{0+}\psi_{01+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A(\nu)\psi_{\nu 1}(\mu)e^{-x/\nu}d\nu, & x>0 \\ -f_2(x,\mu) - a_{0-}\psi_{02-}(\mu)e^{x/\nu_{02}} - \int_{-1}^0 A(\nu)\psi_{\nu 2}(\mu)e^{-x/\nu}d\nu, & x<0 \end{cases} \quad \text{(III.1.1)}$$

where $f_i(x,\mu)$ is the particular solution. We have excluded all diverging normal modes since they appear only in the Milne problem, in which case they may be included as particular solutions.

All two half-space problems must satisfy the interface condition

$$\psi(0^+,\mu) - \psi(0^-,\mu) = \psi(\mu) \quad \text{(III.1.2)}$$

which means that

$$\psi(\mu) - f_1(0,\mu) - f_2(0,\mu) = a_{0+}\psi_{01+}(\mu) + a_{0-}\psi_{02-}(\mu) + \int_0^1 A(\nu)\psi_{\nu 1}(\mu)d\nu$$

$$+ \int_{-1}^0 A(\nu)\psi_{\nu 2}(\mu)\, d\nu \,. \qquad \text{(III.1.3)}$$

This can be rewritten as

$$\psi'(\mu) = \psi(\mu) - f_1(0,\mu) - f_2(0,\mu) - a_{0+}\psi_{01+}(\mu) - a_{0-}\psi_{02-}(\mu)$$
$$= \int_0^1 A(\nu)\psi_{\nu 1}(\mu)d\nu + \int_{-1}^0 A(\nu)\psi_{\nu 2}(\mu)d\nu \,. \qquad \text{(III.1.4)}$$

The continuum normal modes of the homogeneous transport equation, $\psi_{\nu i}(\mu)$, for i = 1,2, have the same form as (I.4.5) except the mean number of secondaries is c_i. Likewise, the roots $\pm\nu_{0i}$ are the roots of the dispersion functions $\Lambda_i(\nu)$ which are functions of c_i. The normal modes $\psi_{0i\pm}(\mu)$ have the form of (I.4.11) with the proper values of c_i and ν_{0i} . This suggests the following notation (11, 14, 15):

$$c(\nu) = \begin{cases} c_1 & \nu > 0 \\ c_2 & \nu < 0 \end{cases} \qquad \text{(III.1.5)}$$

$$L(\nu) = \begin{cases} \Lambda_1(\nu) & \nu > 0 \\ \Lambda_2(\nu) & \nu < 0 \end{cases} \qquad \text{(III.1.6)}$$

$$\ell(\nu) = 1 - \frac{\nu c(\nu)}{2} P\int_{-1}^1 \frac{d\mu}{\nu - \mu} \qquad \text{(III.1.7)}$$

$$L^{\pm}(\nu) = \ell(\nu) \pm \frac{\pi i \nu c(\nu)}{2} \qquad \text{(III.1.8)}$$

$$\Phi_\nu(\mu) = \frac{\nu c(\nu)}{2} \frac{P}{\nu-\mu} + \ell(\nu)\delta(\nu-\mu) . \tag{III.1.9}$$

We can now write (III.1.4) as

$$\psi'(\mu) = \int_{-1}^{1} A(\nu)\,\Phi_\nu(\mu)\,d\nu \tag{III.1.10}$$

and we have a modified form of the full-range expansion which was discussed in Part I.

III.2 Completeness Theorem

The two half-space completeness theorem closely follows the proof of Section II.2. Defining

$$\eta(z) = \frac{1}{2\pi i}\int_{-1}^{1} \frac{\nu c(\nu)}{2} \frac{A(\nu)\,d\nu}{\nu - z} \tag{III.2.1}$$

reduces (III.1.10) to

$$L^+(\mu)\eta^+(\mu) - L^-(\mu)\eta^-(\mu) = \frac{\mu c(\mu)}{2} \psi'(\mu) . \tag{III.2.2}$$

The function L(z) is not continuous so another function must be introduced. A function X(z) is needed which is analytic in the complex plane cut from -1 to 1, which is non-vanishing along with its boundary values in the entire finite plane, and whose boundary values satisfy the ratio condition

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{L^+(\mu)}{L^-(\mu)} = \begin{cases} \dfrac{\Lambda_1^+}{\Lambda_1^-} & , \mu > 0 \\[2ex] \dfrac{\Lambda_2^+}{\Lambda_2^-} & , \mu < 0 . \end{cases} \tag{III.2.3}$$

By inspection (11), such a function is

$$X(z) = X_1(z) X_2(-z)\,, \tag{III.2.4}$$

where $X_i(z)$ is the half-space function of Section II.2 corresponding to c_i (which is continuous for $-z$). With the use of (III.2.3), equation (III.2.2) becomes

$$X^+(\mu)\,\eta^+(\mu) - X^-(\mu)\,\eta^-(\mu) = \Gamma(\mu)\,\psi'(\mu)\,, \tag{III.2.5}$$

where

$$\Gamma(\mu) = \frac{\mu c(\mu)}{2}\;\frac{X^-(\mu)}{L^-(\mu)}\;,\quad -1 \le \mu \le 1\,. \tag{III.2.6}$$

The solution of (III.2.5) is

$$\eta(z) = \frac{1}{2\pi i\, X(z)} \int_{-1}^{1} \frac{\Gamma(\mu)\,\psi'(\mu)\,d\mu}{\mu - z} \tag{III.2.7}$$

where the conditions

$$\int_{-1}^{1} \mu^n\, \Gamma(\mu)\,\psi'(\mu)\,d\mu = 0\,,\quad n = 0, 1\,, \tag{III.2.8}$$

must be imposed for $\eta(z)$ to have the proper behavior at infinity. The two discrete modes $\psi_{01+}(\mu)$ and $\psi_{02-}(\mu)$ are available in (III.1.4) to satisfy the above two conditions. This completes the proof of completeness.

The function $X(z)$ satisfies the following identities (28)

$$z^n X(z) + \delta_{n2} = \int_{-1}^{1} \mu^n \Gamma(\mu) \frac{d\mu}{\mu - z} \quad , \quad n = 0, 1, 2 \tag{III.2.9}$$

which may be written as

$$z^n X(z) + \delta_{n2} = \frac{c_2}{2(1-c_2)} \int_{-1}^{0} \frac{\mu^{n+1} X_1(\mu) d\mu}{X_2(\mu)(\nu_{02}^2 - \mu^2)(\mu - z)} \tag{III.2.10}$$

$$+ \frac{c_1}{2(1-c_1)} \int_{0}^{1} \frac{\mu^{n+1} X_2(-\mu) d\mu}{X_1(-\mu)(\nu_{01}^2 - \mu^2)(\mu - z)} \quad , \quad n = 0, 1, 2.$$

III.3 Orthogonality Relations

The orthogonality relations for problems in two adjacent half-spaces with weight function

$$W(\mu) = (\nu_{01} - \mu)(\nu_{02} + \mu) \Gamma(\mu) \tag{III.3.1}$$

are (28)

$$\int_{-1}^{1} \Phi_\nu(\mu) \Phi_{\nu'}(\mu) W(\mu) d\mu = W(\nu) L^+(\nu) L^-(\nu) \delta(\nu - \nu') \tag{III.3.2}$$

$$\int_{-1}^{1} \psi_{01+}(\mu) \Phi_\nu(\mu) W(\mu) d\mu = 0 \tag{III.3.3}$$

$$\int_{-1}^{1} \psi_{02-}(\mu) \Phi_\nu(\mu) W(\mu) d\mu = 0 \tag{III.3.4}$$

$$\int_{-1}^{1} \psi_{01+}(\mu) \psi_{02-}(\mu) W(\mu) d\mu = 0 \tag{III.3.5}$$

$$\int_{-1}^{1} \psi_{01-}(\mu) \Phi_\nu(\mu) W(\mu) d\mu = \frac{1}{2} c_1 \nu c(\nu) \nu_{01}^2 \frac{\nu_{02} - \nu_{01}}{\nu_{01} + \nu} X(-\nu_{01}) \tag{III.3.6}$$

$$\int_{-1}^{1} \psi_{01\pm}(\mu)\psi_{01+}(\mu)W(\mu)d\mu = -\left(\frac{c_1\nu_{01}}{2}\right)^2(\nu_{01} \pm \nu_{02})X(\pm\nu_{01}) \qquad \text{(III.3.7)}$$

$$\int_{-1}^{1} \psi_{02\pm}(\mu)\psi_{02-}(\mu)W(\mu)d\mu = \left(\frac{c_2\nu_{02}}{2}\right)^2(\nu_{02} \mp \nu_{01})X(\pm\nu_{02}) \qquad \text{(III.3.8)}$$

$$\int_{-1}^{1} \psi_{01-}(\mu)\psi_{02-}(\mu)W(\mu)d\mu = \frac{1}{2}\, c_1 c_2 \nu_{01}^2 \nu_{02} X(-\nu_{01})\,. \qquad \text{(III.3.9)}$$

The orthogonality relations of Sections I.6 and II.4 may be obtained from these results. Upon letting $c_1 = c_2 = c$,

$$X(z) = X_1(z)\,X_1(-z) = \frac{\Lambda_1(z)}{(\nu_{01}^2 - z^2)(1-c_1)}\;; \qquad \text{(III.3.10)}$$

so

$$W(\mu) = \frac{c}{2(1-c)}\,\mu\,, \qquad \text{(III.3.11)}$$

which agrees with Section I.6. For $c_2 = 0$, $\nu_{02} = 1$ and

$$X_2(z) = \frac{1}{1-z} \quad ;\ \text{so} \quad X(z) = \frac{X_1(z)}{1+z} \qquad \text{(III.3.12)}$$

and

$$W(\mu) = (\nu_{01}-\mu)(1+\mu)\Gamma(\mu) = \begin{cases} (\nu_{01}-\mu)\gamma(\mu) & 0 \le \mu \le 1 \\ 0 & -1 \le \mu \le 0 \end{cases}, \qquad \text{(III.3.13)}$$

which agrees with Section II.4.

Likewise, letting $c_2 = 0$ in the solutions of the two half-space problems of the following section gives the half-space results of the corresponding problems.

III.4 Three Applications

a) Milne Problem. For two adjacent, source-free half-spaces with $c_1 < 1$ and $c_2 < 1$ and with a source at $+\infty$, we have (14,15)

$$\mu \frac{\delta\psi(x,\mu)}{\delta x} + \psi(x,\mu) = \begin{cases} \frac{c_1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' & x>0 \\ \frac{c_2}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' & x<0 \end{cases} \quad \text{(III.4.1)}$$

where

$$\lim_{x\to+\infty}\psi(x,\mu)=\psi_{01-}(x,\mu) \text{ (the slowest diverging mode)} \quad \text{(III.4.2)}$$

$$\lim_{x\to-\infty}\psi(x,\mu)=0 \quad \text{(III.4.3)}$$

$$\psi(0^+,\mu)=\psi(0^-,\mu). \text{ (continuity at the interface)} \quad \text{(III.4.4)}$$

The angular density, from (III.1.1), is

$$\psi(x,\mu) = \begin{cases} \psi_{01-}(\mu)e^{x/\nu_{01}} + a_{0+}\psi_{01+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A(\nu)\psi_{\nu 1}(\mu)e^{-x/\nu}d\nu, & x>0 \\ -a_{0-}\psi_{02-}(\mu)e^{x/\nu_{02}} - \int_{-1}^{0} A(\nu)\psi_{\nu 2}(\mu)e^{-x/\nu}d\nu, & x<0. \end{cases} \quad \text{(III.4.5)}$$

These equations reduce to (III.1.10) where

$$\psi'(\mu) = -\psi_{01-}(\mu) - a_{0+}\psi_{01+}(\mu) - a_{0-}\psi_{02-}(\mu) = \int_{-1}^{1} A(\nu)\Phi_\nu(\mu)d\nu \quad \text{(III.4.6)}$$

and the orthogonality relations yield

$$a_{0+} = \frac{X(-\nu_{01})(\nu_{02}-\nu_{01})}{X(\nu_{01})(\nu_{02}+\nu_{01})} \tag{III.4.7}$$

$$a_{0-} = -\frac{X(-\nu_{01})2c_1\nu_{01}^2}{X(-\nu_{02})c_2\nu_{02}(\nu_{02}+\nu_{01})} \tag{III.4.8}$$

$$A(\nu) = \begin{cases} \dfrac{-\nu c_1(1-c_1)\nu_{01}^2(\nu_{02}-\nu_{01})X(-\nu_{01})X_1(-\nu)}{(\nu_{02}+\nu)N_1(\nu)X_2(-\nu)} & \nu > 0 \\[2ex] \dfrac{-\nu c_1(1-c_2)\nu_{01}^2(\nu_{02}-\nu_{01})X(-\nu_{01})X_2(\nu)}{(\nu_{01}^2-\nu^2)N_2(\nu)X_1(\nu)} & \nu < 0 \end{cases} \tag{III.4.9}$$

and the solution for $\psi(x,\mu)$ is now complete since the expansion coefficients are known.

b) Constant Source. For a uniform isotropic source in the right-hand half-space,

$$\mu\frac{\delta\psi(x,\mu)}{\delta x} + \psi(x,\mu) = \begin{cases} \dfrac{c_1}{2}\displaystyle\int_{-1}^{1}\psi(x,\mu')d\mu' + q_0 & x > 0 \\[2ex] \dfrac{c_2}{2}\displaystyle\int_{-1}^{1}\psi(x,\mu')d\mu' & x < 0\,. \end{cases} \tag{III.4.10}$$

The boundary conditions for $c_1 < 1$ and $c_2 < 1$ are

$$\lim_{x\to+\infty}\psi(x,\mu) = \frac{q_0}{1-c_1} \tag{III.4.11}$$

$$\lim_{x\to-\infty}\psi(x,\mu) = 0 \tag{III.4.12}$$

$$\psi(o^+,\mu) = \psi(o^-,\mu) \tag{III.4.13}$$

and the angular density is

$$\psi(x,\mu) = \begin{cases} \frac{q_0}{1-c} + a_{0+}\psi_{01+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A(\nu)\psi_{\nu 1}(\mu)e^{-x/\nu}d\nu, & X>0 \\ -a_{0-}\psi_{02-}(\mu)e^{x/\nu_{02}} - \int_{-1}^0 A(\nu)\psi_{\nu 2}(\mu)e^{-x/\nu}d\nu, & X<0. \end{cases} \tag{III.4.14}$$

The expansion coefficients (14,15) are obtained by application of the orthogonality relations to

$$\frac{-q_0}{1-c} - a_{0+}\psi_{01+}(\mu) - a_{0-}\psi_{02-}(\mu) = \int_{-1}^{1} A(\nu)\,\Phi_\nu(\mu)\,d\nu\,. \tag{III.4.15}$$

c) Green's Function. To find the Green's function for two adjacent half-spaces with a source plane arbitrarily at x_o in the right half-space, one considers the equation

$$\mu\frac{\delta\psi(x,\mu)}{\delta x} + \psi(x,\mu) = \begin{cases} \frac{c_1}{2}\int_{-1}^{1}\psi(x,\mu')d\mu' + \frac{\delta(x-x_0)\delta(\mu-\mu_0)}{2\pi}, & X>0 \\ \frac{c_2}{2}\int_{-1}^{1}\psi(x,\mu')d\mu', & X<0 \end{cases} \tag{III.4.16}$$

with the boundary conditions

$$\lim_{X\to\pm\infty}\psi(x,\mu) = 0 \tag{III.4.17}$$

$$\psi(o^+,\mu) = \psi(o^-,\mu)\,. \tag{III.4.18}$$

The angular density which satisfies (III.4.16) and (III.4.17) is

$$\psi(x,\mu) = \begin{cases} G_1(x_0,\mu_0 \to x,\mu) + a_{0+}\psi_{01+}(\mu)e^{-x/\nu_{01}} + \int_0^1 A(\nu)\psi_{\nu 1}(\mu)e^{-x/\nu}d\nu, & x>0 \\ -a_{0-}\psi_{02-}(\mu)e^{x/\nu_{02}} - \int_{-1}^0 A(\nu)\psi_{\nu 2}(\mu)e^{-x/\nu}d\nu, & x<0 \end{cases} \qquad \text{(III.4.19)}$$

where G_1 is the infinite medium Green's function of Section I.7 with $c = c_1$. Equations (III.4.18) and (III.4.19) combine to give

$$-G_1(x_0,\mu_0 \to 0,\mu) - a_{0+}\psi_{01+}(\mu) - a_{0-}\psi_{02-}(\mu) = \int_{-1}^1 A(\nu)\Phi_\nu(\mu)d\nu \qquad \text{(III.4.20)}$$

and solutions are again available (14,15).

IV. ANISOTROPIC SCATTERING APPLICATIONS

IV.1 General Anisotropic Scattering

We now consider the transport equation

$$\mu \frac{\delta\psi(x,\mu)}{\delta x} + \psi(x,\mu) = \frac{c}{2}\sum_{\ell=0}^{N}(2\ell+1)b_\ell P_\ell(\mu)\int_{-1}^{1}\psi(x,\mu')P_\ell(\mu')d\mu' + q(x,\mu), \qquad (I.3.3)$$

with $b_0 = 1$. Considering the homogeneous form of (I.3.3) and using (I.4.1) to separate the variables gives

$$(\nu-\mu)\,\psi_\nu(\mu) = \frac{c\nu}{2}\sum_{\ell=0}^{N}(2\ell+1)b_\ell P_\ell(\mu)\int_{-1}^{1}\psi_\nu(\mu')P_\ell(\mu')d\mu' . \qquad (IV.1.1)$$

Multiplying this equation by $P_k(\mu)d\mu$, integrating over μ from - 1 to 1, and using the orthogonality and recursion relations for Legendre polynomials gives

$$\nu(1-cb_k)\psi_{\nu k} - \frac{k+1}{2k+1}\psi_{\nu,k+1} - \frac{k}{2k+1}\psi_{\nu,k-1} = 0 \qquad (IV.1.2)$$

where $\psi_{\nu k}$ is defined as

$$\psi_{\nu k} = \int_{-1}^{1}\psi_\nu(\mu)P_k(\mu)d\mu . \qquad (IV.1.3)$$

We normalize the solution to

$$\psi_{\nu 0} = \int_{-1}^{1}\psi_\nu(\mu)d\mu = 1 \qquad (IV.1.4)$$

to conform with (I.4.3) and obtain

$$\psi_{\nu 1} = \nu(1-c)$$

$$\psi_{\nu 2} = \frac{3\nu^2}{2}(1-cb_1)(1-c) - \frac{1}{2} \tag{IV.1.5}$$

and so on.

The most general solution of (IV.1.1) is

$$\psi_\nu(\mu) = \frac{c\nu}{2} P \frac{M(\mu,\nu)}{\nu-\mu} + \lambda(\mu)\delta(\mu-\nu) , \tag{IV.1.6}$$

where $M(\mu,\nu)$ is a known function, defined as

$$M(\mu,\nu) = \sum_{\ell=0}^{N} (2\ell+1) b_\ell P_\ell(\mu) \psi_{\nu\ell} . \tag{IV.1.7}$$

The normalization condition (IV.1.4) gives the dispersion function $\Lambda(\nu)$ whose zeros define the eigenvalues ν not on the real line between -1 and 1:

$$\Lambda(\nu) = 1 - \frac{c\nu}{2} \sum_{\ell=0}^{N} (2\ell+1) b_\ell \psi_{\nu\ell} \int_{-1}^{1} \frac{P_\ell(\mu)d\mu}{\nu-\mu} = 0 . \tag{IV.1.8}$$

This may be written as

$$\Lambda(\nu) = 1 - c\nu \sum_{\ell=0}^{N} (2\ell+1) b_\ell \psi_{\nu\ell} Q_\ell(\nu) , \tag{IV.1.9}$$

where $Q_\ell(\nu)$ are the Legendre polynomials of the second kind (33). The roots of $\Lambda(\nu)=0$ are the discrete eigenvalues and are investigated by Mika (12). For ν on the real line from -1 to 1, equation (IV.1.4) determines $\lambda(\nu)$, which is related to the boundary values of $\Lambda(z)$ by

$$\Lambda\pm(\nu) = \lambda(\nu) \pm \frac{\pi i c \nu M(\nu,\nu)}{2} . \qquad \text{(IV.1.10)}$$

Another method of determination of the discrete eigenvalues is also available for anisotropic scattering. Multiplying (IV.1.6) by $P_k(\mu)d\mu$ and integrating over μ gives

$$\psi_{\nu k}(\mu) = \frac{c\nu}{2} \sum_{\ell=0}^{N} (2\ell+1) b_\ell \psi_{\nu\ell} \int_{-1}^{1} \frac{P_k(\mu)P_\ell(\mu)d\mu}{\nu-\mu} , \qquad \text{(IV.1.11)}$$

which may be rewritten in the form

$$\sum_{\ell=0}^{N} \left\{ (2\ell+1) c\nu^2 b_\ell A_{\ell k}(\tfrac{1}{\nu}) - \delta_{\ell k} \right\} \psi_{\nu\ell} = 0 . \qquad \text{(IV.1.12)}$$

Here, $A_{\ell k}(\frac{1}{\nu})$ is defined as (34)

$$A_{\ell k}(\tfrac{1}{\nu}) = \frac{1}{2\nu} \int_{-1}^{1} \frac{P_\ell(\mu) P_k(\mu) d\mu}{\nu-\mu} , \qquad \text{(IV.1.13)}$$

with

$$A_{\ell k}(\tfrac{1}{\nu}) = \nu Q_\ell(\nu) P_k(\nu) \quad , \quad \ell \geq k . \qquad \text{(IV.1.14)}$$

The system of equations (IV.1.12) can have a non-trivial solution only if

$$\det \left[(2\ell+1) c\nu^2 b_\ell A_{\ell k}(\tfrac{1}{\nu}) - \delta_{\ell k} \right] = 0 , \qquad \text{(IV.1.15)}$$

and this yields the discrete eigenvalues. From this we see that for c = 1, $\nu = \infty$ is always a (double) root since

$$\lim_{\nu\to\infty} \nu^2 A_{\ell k}(\tfrac{1}{\nu}) = \lim_{\nu\to\infty} \int_{-1}^{1} \frac{P_\ell(\mu)P_k(\mu)d\mu}{1-\mu/\nu} = \frac{2}{2\ell+1} \delta_{\ell k} \qquad \text{(IV.1.16)}$$

and (IV.1.15) becomes, in the limit as $\nu \to \infty$,

$$\det \left[b_\ell - 1 \right] \delta_{\ell k} = 0 . \qquad \text{(IV.1.17)}$$

Equation (IV.1.17) is the identity 0 = 0 since $b_o = 1$. The root is a double root since $\Lambda(\nu) = \Lambda(-\nu)$.

The completeness theorem for the expansion of the angular density for general anisotropic scattering was proved by Mika (12). From now on, however, we will restrict our attention to the case of linearly anisotropic scattering where N = 1. For $c < 1$ and $N = 1$, there are only two discrete eigenvalues $\pm \nu_0$ (12) which have corresponding eigenfunctions

$$\psi_{0\pm}(\mu) = \frac{c\nu_0}{2} \, \frac{M(\mu, \pm\nu_0)}{\nu_0 \mp \mu} , \qquad \text{(IV.1.18)}$$

where

$$M(\mu, \pm\nu_0) = 1 \pm 3b_1(1-c)\mu\nu_0 . \qquad \text{(IV.1.19)}$$

IV.2 Orthogonality Relations For Linearly Anisotropic Scattering

a) Full-Range Case. The orthogonality relations are (12)

$$\int_{-1}^{1} \mu \psi_\nu(\mu) \psi_{\nu'}(\mu) d\mu = S(\nu) \delta(\nu - \nu') \qquad \text{(IV.2.1)}$$

$$\int_{-1}^{1} \mu \psi_{0\pm}(\mu) \psi_\nu(\mu) d\mu = 0 \qquad \text{(IV.2.2)}$$

$$\int_{-1}^{1} \mu \psi_{0+}(\mu) \psi_{0-}(\mu) d\mu = 0 \qquad \text{(IV.2.3)}$$

$$\int_{-1}^{1} \mu \psi_{0\pm}(\mu) \psi_{0\pm}(\mu) d\mu = S_{0\pm} \quad , \tag{IV.2.4}$$

where

$$S(\nu) = \nu \Lambda^{+}(\nu) \Lambda^{-}(\nu) = \nu \left\{ \lambda^{2}(\nu) + \left[\frac{\pi c \nu M(\nu,\nu)}{2} \right]^{2} \right\} \tag{IV.2.5}$$

and where $S_{0\pm}$ is known (12).

b) Half-Range Case. The weight function depends upon b_1 and c and has the form

$$W(\mu) = (\nu_0 - \mu) \gamma(\mu) = (\nu_0 - \mu) \frac{c\mu}{2} \frac{X^{-}(\mu)}{\Lambda^{-}(\mu)} , \tag{IV.2.6}$$

but the normal modes are no longer orthogonal (28). Indeed, it is necessary to define a constant B which is also a function of b_1 and c such that

$$B = \frac{3b_1(1-c)(\nu_0 - \bar{\nu})}{M(\bar{\nu}, \nu_0)} , \tag{IV.2.7}$$

where

$$\bar{\nu} = \frac{\int_0^1 \mu \gamma(\mu) d\mu}{\int_0^1 \gamma(\mu) d\mu} . \tag{IV.2.8}$$

The eigenfunctions obey the following bi-orthogonality relations (28, 35):

$$\int_0^1 \psi_{\nu}(\mu) \left[\psi_{\nu'}(\mu) + B \frac{c\nu'}{2} \right] W(\mu) d\mu = \frac{W(\nu) S(\nu)}{\nu} \delta(\nu - \nu') \tag{IV.2.9}$$

$$\int_0^1 \psi_{0+}(\mu) \left[\psi_{\nu'}(\mu) + B \frac{c\nu'}{2} \right] W(\mu) d\mu = 0 \tag{IV.2.10}$$

$$\int_0^1 \psi_{0-}(\mu)\left[\psi_{\nu'}(\mu)+B\frac{c\nu'}{2}\right]W(\mu)d\mu = c\nu_0\nu X(-\nu_0)\psi_{0-}(\nu)\,\frac{M(\nu_0,\nu_0)\,M(\bar{\nu},-\nu)}{M(\nu_0,-\nu)\,M(\bar{\nu},\nu_0)} \tag{IV.2.11}$$

$$\int_0^1 \psi_{\nu}(\mu)\left[\psi_{0+}(\mu)+B\frac{c\nu_0}{2}\right]W(\mu)d\mu = 0 \tag{IV.2.12}$$

$$\int_0^1 \psi_{0\pm}(\mu)\left[\psi_{0+}(\mu)+B\frac{c\nu_0}{2}\right]W(\mu)\,d\mu = \mp\left(\frac{c\nu_0}{2}\right)^2 X(\pm\nu_0)\,\frac{M(\nu_0,\nu_0)\,M(\bar{\nu},\pm\nu_0)}{M(\bar{\nu},\nu_0)}\ . \tag{IV.2.13}$$

These relations reduce to the pure orthogonal forms obtained for isotropic scattering if $b_1 = 0$ and may be used in exactly the same manner.

IV.3 Three Problems With Linearly Anisotropic Scattering

a) Infinite-Medium Green's Function. The angular density due to a source plane at the origin is expanded as (12)

$$G(o,\mu_0\to x,\mu) = \begin{cases} a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0}+\displaystyle\int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu, & x\geq 0 \\ -a_{0-}\psi_{0-}(\mu)e^{x/\nu_0}-\displaystyle\int_{-1}^0 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu, & x\leq 0\ . \end{cases} \tag{IV.3.1}$$

Following the analysis of Section I.7, we obtain

$$a_{0\pm} = \frac{\psi_{0\pm}(\mu_0)}{2\pi S_{0\pm}} \tag{IV.3.2}$$

$$A(\nu) = \frac{\psi_\nu(\mu_0)}{2\pi S(\nu)}\ . \tag{IV.3.3}$$

b) Half-Space Milne Problem. Following the analysis of Section II.5 and the completeness theorem (12), the angular density is expanded as

$$\psi_m(x,\mu) = \psi_{0-}(\mu)e^{x/\nu_0} + a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu \quad \text{(IV.3.4)}$$

and the bi-orthogonality relations are applied to

$$-\psi_{0-}(\mu) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)d\nu\ , \quad \mu \geq 0\ . \quad \text{(IV.3.5)}$$

The expansion coefficients are

$$a_{0+} = \frac{X(-\nu_0)M(\bar{\nu},-\nu_0)}{X(\nu_0)M(\bar{\nu},\nu_0)} \quad \text{(IV.3.6)}$$

$$A(\nu) = -\frac{c\nu_0\nu^2 X(-\nu_0)\psi_{0-}(\nu)M(\nu_0,\nu_0)M(\bar{\nu},-\nu)}{(\nu_0-\nu)\gamma(\nu)S(\nu)M(\nu_0,-\nu)M(\bar{\nu},\nu_0)} \quad \text{(IV.3.7)}$$

and were obtained by Shure and Natelson (13). Following the analysis of Section II.5, the Milne problem extrapolation distance, z_0, is now given by

$$z_0 = \frac{-\nu_0}{2}\ln(-a_{0+}) = \frac{-\nu_0}{2}\ln\left[-\frac{X(-\nu_0)M(\bar{\nu},-\nu_0)}{X(\nu_0)M(\bar{\nu},-\nu_0)}\right]\ . \quad \text{(IV.3.8)}$$

c) Half-Space Albedo Problem. We have

$$\psi_a(x,\mu) = a_{0+}\psi_{0+}(\mu)e^{-x/\nu_0} + \int_0^1 A(\nu)\psi_\nu(\mu)e^{-x/\nu}d\nu \quad \text{(IV.3.9)}$$

and

$$\delta(\mu-\mu_0) = a_{0+}\psi_{0+}(\mu) + \int_0^1 A(\nu)\psi_\nu(\mu)d\nu\ , \quad \mu \geq 0 \quad \text{(IV.3.10)}$$

and find (35)

$$a_{0+} = \frac{-2\gamma(\mu_0)}{c\nu_0 X(\nu_0)} \; \frac{M(\bar{\nu},\mu_0)}{M(\bar{\nu},\nu_0)} \tag{IV.3.11}$$

$$A(\nu) = \frac{(\nu_0-\mu_0)\,\gamma(\mu_0)\,\nu}{(\nu_0-\nu)\,\gamma(\nu)\,S(\nu)} \left[\psi_\nu(\mu_0) + B\,\frac{c\nu}{2} \right] \; . \tag{IV.3.12}$$

V. TIME-DEPENDENT APPLICATIONS

V.1 Eigenvalues And Eigenfunctions

The homogeneous time-dependent transport equation for plane geometry and isotropic scattering is

$$\frac{\delta\psi(x,\mu,t)}{\delta t} + \mu\,\frac{\delta\psi(x,\mu,t)}{\delta x} + \psi(x,\mu,t) = \frac{c}{2}\int_{-1}^{1}\psi(x,\mu',t)\,d\mu' . \qquad (V.1.1)$$

For an absorbing medium ($c < 1$), the angular distribution denoted by $\psi^{(c)}(x,\mu,t)$ can be related to the angular distribution for a non-absorbing medium by the substitution (20)

$$\psi^{(c)}(x',\mu,t') = c e^{-(1-c)t'}\,\psi^{(1)}(cx',\mu,ct') , \qquad (V.1.2)$$

so it is sufficient to study time-dependent problems with $c = 1$. For convenience, define

$$\psi^{(1)}(cx',\mu,ct') = \psi(x,\mu,t) . \qquad (V.1.3)$$

Following the approach of Bowden (19), multiply (V.1.1) by $e^{(1-s)t}dt$ and integrate from 0 to ∞ to obtain

$$\mu\,\frac{\delta\psi_s(x,\mu)}{\delta x} + s\psi_s(x,\mu) = \frac{1}{2}\int_{-1}^{1}\psi_s(x,\mu')\,d\mu' , \qquad (V.1.4)$$

where the Laplace transform of $\psi(x,\mu,t)$ is defined as

$$\psi_s(x,\mu) = \int_0^{\infty}\psi(x,\mu,t)\,e^{(1-s)t}\,dt \qquad (V.1.5)$$

and the integral converges for $\mathrm{Re}(s) > 1$. The time-dependent solution is obtained from $\psi_s(x,\mu)$ by the inverse transformation

$$\psi(x,\mu,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \psi_s(x,\mu)e^{-(1-s)t}ds \ , \tag{V.1.6}$$

where $\gamma > 1$ is to the right of all singularities of the integrand.

Equation (V.1.4) is the same in appearance as the time-independent equation (I.3.6) already studied if s^{-1} is replaced by c and xs is replaced by x. The time-independent results of Section I.4 can be used to give (20)

$$\psi_{\nu s}(x,\mu) = e^{-sx/\nu}\psi_{\nu s}(\mu) \tag{V.1.7}$$

$$\int_{-1}^{1}\psi_{\nu s}(\mu)d\mu = 1 \tag{V.1.8}$$

$$\psi_{\nu s}(\mu) = \frac{\nu}{2s} P \frac{1}{\nu-\mu} + \lambda_s(\nu)\,\delta(\nu-\mu) \tag{V.1.9}$$

$$\lambda_s(\nu) = 1 - \frac{\nu}{s}\tanh^{-1}\nu \tag{V.1.10}$$

$$\Lambda_s(z) = 1 - \frac{z}{s}\tanh^{-1}\frac{1}{z} \tag{V.1.11}$$

$$\Lambda_s\left[\pm\nu_0(s)\right] = 0 \tag{V.1.12}$$

$$\Lambda_s^{\pm}(\nu) = \lambda_s(\nu) \pm \frac{\pi i \nu}{2s} \ , \quad -1<\nu<1 \tag{V.1.13}$$

$$\psi_{s\pm}(\mu) = \frac{\nu_0}{2S} \frac{1}{\nu_0 \mp \mu} \quad . \tag{V.1.14}$$

For the time-independent case, s = 1 and $\Lambda_1(\nu)$ has two zeros. The time-dependent problem under consideration differs since s may be complex. Again the roots are paired and denoted as $\pm\nu_0$ and may be observed from the conformal mapping of $s = \nu_0 \tanh^{-1} \frac{1}{\nu_0}$ of (say) the right half-plane (20):

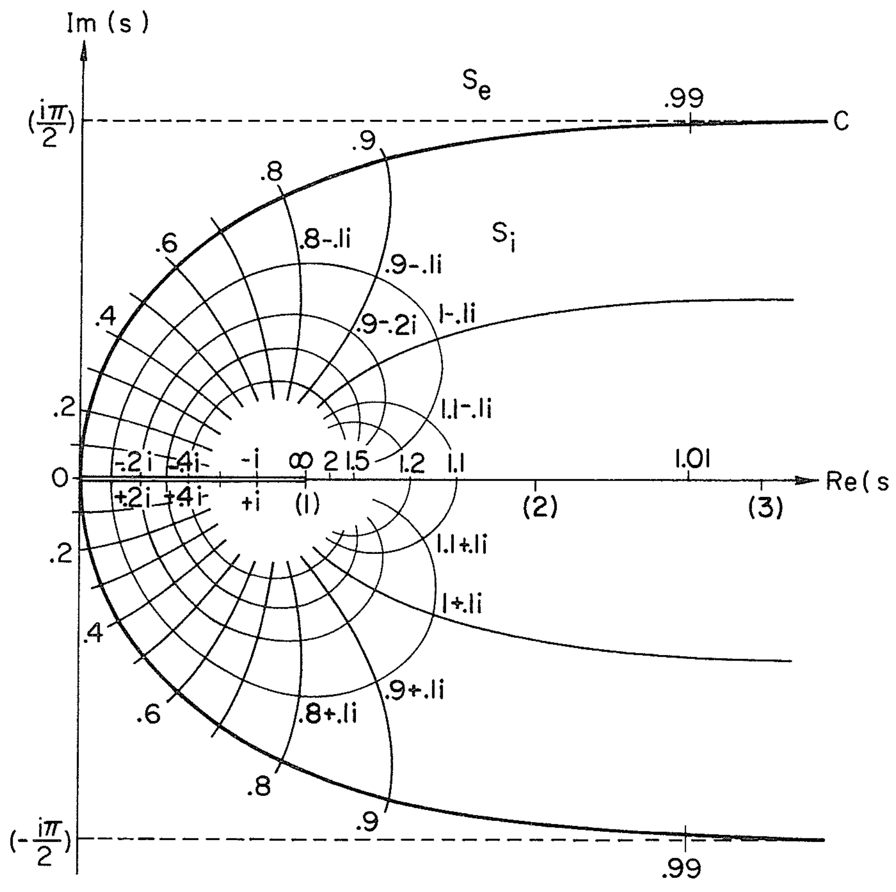

FIGURE 1

Here, C is the conformal map of the branch cut 0 to 1 in the ν plane (which arises because of the factor $\tanh^{-1}\frac{1}{\nu}$). There is a branch cut from 0 to 1 in the s-plane because of the branch point of $\nu_0(s)$ at s = 1. In region S_e there are no discrete eigenvalues and in region S_i there are two values. By choosing ν_0 to be the particular root which is positive for s > 1, then $Re(\nu_0) \geq 0$ in the entire cut region S_i.

V.2 Completeness Theorems

a) Full-Range. An "arbitrary" function $\psi(\mu)$ defined for $-1 \leq \mu \leq 1$ may be expanded in terms of the continuum modes $\psi_{\nu s}(\mu)$ for any s where, if $s \in S_i$, the set of expansion coefficients must also include the discrete modes $\psi_{s\pm}(\mu)$.

For $s \in S_i$, the proof follows that of Section I.5. For $s \in S_e$, the function $\Lambda_s(z)$ has no zeros and the discrete modes are not needed. The full-range theorem again is adequate for the treatment of infinite-medium problems.

b) Half-Range. An "arbitrary" function $\psi(\mu)$ defined for $0 \leq \mu \leq 1$ (or $-1 \leq \mu \leq 0$) may be expanded in terms of the continuum modes $\psi_{\nu s}(\mu)$, $0 \leq \nu \leq 1$ (or $-1 \leq \nu \leq 0$), for $s \in S_e$ and in terms of $\psi_{\nu s}(\mu)$ and one discrete mode $\psi_{s+}(\mu)$ (or $\psi_{s-}(\mu)$) for $s \in S_i$.

For $s \in S_i$, the proof is analogous to that of Section II.2. For $s \in S_e$, the function $X_{0s}(z)$ behaves like a constant near the endpoint z = 1 so the discrete eigenfunction is not needed.

Notice that we have an infinite number of eigenfunctions since we expand $\psi_s(x,\mu)$ for each value of s. We must therefore study the behavior of $\psi_s(x,\mu)$ as a function of s in any particular problem in order to find the time-dependent angular density from (V.1.6).

V.3 Two Applications

a) Half-Space Albedo Problem. The transformation of the boundary conditions

$$\psi_a(o,\mu,t) = \delta(\mu-\mu_0)\delta(t), \quad \mu \geq 0 \tag{V.3.1}$$

$$\lim_{x\to\infty} \psi_a(x,\mu,t) = 0 \tag{V.3.2}$$

gives

$$\psi_{as}(o,\mu) = \delta(\mu-\mu_0), \quad \mu \geq 0 \tag{V.3.3}$$

$$\lim_{x\to\infty} \psi_{as}(x,\mu) = 0 . \tag{V.3.4}$$

Use of the completeness theorem shows that we expand the transform of the angular density as

$$\psi_{as}(x,\mu) = \begin{cases} a_{s+}\psi_{s+}(\mu)e^{-x/\nu_0} + \int_0^1 A_s(\nu)\psi_{\nu s}(\mu)e^{-x/\nu}d\nu, & s\in S_i \\ \int_0^1 A_s(\nu)\psi_{\nu s}(\mu)e^{-x/\nu}d\nu & s\in S_e \end{cases} \tag{V.3.5}$$

and solve the equations

$$\delta(\mu-\mu_0) = \begin{cases} a_{s+}\psi_{s+}(\mu) + \int_0^1 A_s(\nu)\psi_{\nu s}(\mu)d\nu, & \mu \geq 0, \quad s \in S_i \\ \int_0^1 A_s(\nu)\psi_{\nu s}(\mu)d\nu & , \mu \geq 0, \quad s \in S_e \end{cases} \qquad \text{(V.3.6)}$$

for the expansion coefficients. The value of $\psi_{as}(x,\mu)$ for $s \in S_e$ is given by (20)

$$\psi_{as}(x,\mu) = \frac{1}{(1+\mu_0)X_s(-\mu_0)} \int_0^1 \frac{(1+\nu)X_s(-\nu)\psi_{\nu s}(\mu_0)\psi_{\nu s}(\mu)e^{-sx/\nu}d\nu}{\nu\Lambda_s^+(\nu)\Lambda_s^-(\nu)} \qquad \text{(V.3.7)}$$

where

$$X_s(z) = \frac{1}{1-z} \exp \frac{1}{2\pi i} \int_0^1 \ln \frac{\Lambda_s^+(\nu)}{\Lambda_s^-(\nu)} \frac{d\nu}{\nu - z} \quad . \qquad \text{(V.3.8)}$$

The function $\psi_{as}(x,\mu)$ is analytic in the cut region S_i and in the region S_e. It is also continuous and analytic across contour C (20) so that $\psi_{as}(x,\mu)$ is regular in the entire right-hand half-plane of s except for the cut from $0 \leq s \leq 1$. The shifted contour for the inverse transformation of $\psi_{as}(x,\mu)$ thus appears as in Figure 2.

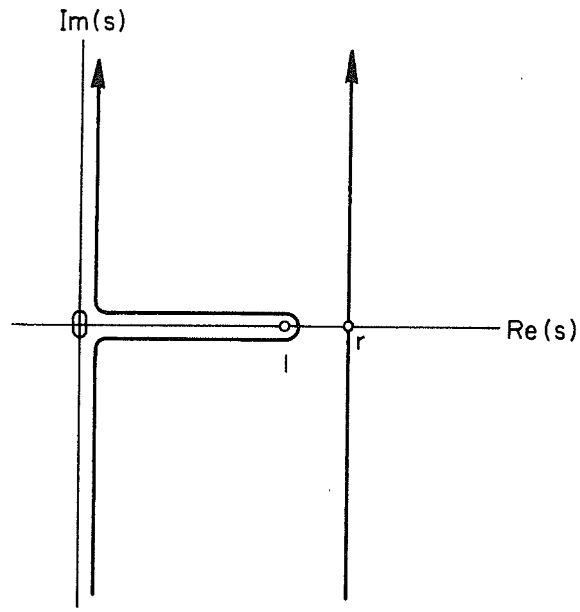

FIGURE 2

The inverse transformation is therefore

$$2\pi i\, \psi_a(x,\mu,t) = \int_{-i\infty}^{i\infty} \psi_{as}(x,\mu) e^{-(1-S)t} dS - \int_0^1 \left[\psi_{as}^{+}(x,\mu) - \psi_{as}^{-}(x,\mu) \right] e^{-(1-S)t} dS \qquad (V.3.9)$$

which can be reduced to the final form (20)

$$\psi_a(x,\mu,t) = \frac{2}{\pi}\int_0^1 (1-S)\,|\nu_0(S)|\,\psi_{ms}(0,-\mu_0)\psi_{ms}(x,\mu)e^{-(1-S)t}\,dS$$
$$+\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\psi_{as}(x,\mu)e^{-(1-S)t}\,dS\,, \tag{V.3.10}$$

where $\psi_{as}(x,\mu)$ is given in (V.3.7) and where $\psi_{ms}(x,\mu)$ is the transform of the angular density for the steady-state Milne problem normalized to unit outgoing current.* The explicit expressions for $\psi_{ms}(x,\mu)$ and $\psi_{ms}(0,-\mu_0)$ are (20)

$$\psi_{ms}(x,\mu) = \frac{s}{2(s-1)\nu_0^2}\left[\frac{1}{X_s(\nu_0)}\psi_{s+}(\mu)e^{-sx/\nu_0} + \frac{1}{X_s(-\nu_0)}\psi_{s-}(\mu)e^{sx/\nu_0}\right]$$
$$-\frac{1}{2s}\int_0^1 \frac{X_s(-\nu)\psi_{\nu s}(\mu)e^{-sx/\nu}\,d\nu}{\Lambda_s^+(\nu)\Lambda_s^-(\nu)} \tag{V.3.11}$$

$$\psi_{ms}(0,-\mu_0) = \left[2(s-1)(\nu_0^2-\mu_0^2)X_s(-\mu_0)\right]^{-1} . \tag{V.3.12}$$

For x = 0, the path of integration may be deformed into the left half-plane because $e^{-sx/\nu}\to 1$. The emergent angular density from a half-space pulsed by an incident beam at t = 0 is thus given by

$$\psi_a(0,-\mu,t) = \frac{2}{\pi}\int_0^1 (1-s)|\nu_0(s)|\,\psi_{ms}(0,-\mu_0)\psi_{ms}(0,-\mu)e^{-(1-s)t}\,dS\,, \quad \mu \geq 0. \tag{V.3.13}$$

*The solution of Section II.5 is normalized to unit outgoing density.

The initial value of the emergent angular density can be obtained from (V.3.13) as (20)

$$\psi_a(0,-\mu,0) = \left[2(\mu+\mu_0)\right]^{-1} \quad , \quad \mu \geq 0 \,, \tag{V.3.14}$$

and is due to neutrons which are scattered only once.

b) Slab Albedo Problem. The slab problem (19) closely resembles the half-space problem, but there is no branch point at s = 1 in the complex s-plane since both discrete eigenfunctions $\psi_{s\pm}(\mu)e^{-sx/\nu_0}$ are involved in the expansion of $\psi_s(x,\mu)$. Therefore, there is no branch cut along the real axis $0 \leq s \leq 1$. Instead, there are a finite number of poles (36) at certain values of $s = s_j$, j = 1 to N, in the interval $0 < s < 1$ (19). These poles correspond to the solutions of the slab critical problem and fill up the interval as the slab thickness is increased. Equation (V.3.10) is now replaced by (19)

$$\psi_a(x,\mu,t) = \sum_{j=1}^{N} \psi_{ms_j}(0,-\mu_0)\psi_{ms_j}(x,\mu)e^{-(1-s_j)t} + \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} \psi_{as}(x,\mu)e^{-(1-s)t}\,dS \,, \tag{V.3.15}$$

where the $\psi_{ms_j}(x,\mu)$ are normalized to (36)

$$\int_0^d dx \int_{-1}^{1} \psi_{ms_j}(x,\mu)\psi_{ms_j}(x,-\mu)\,d\mu = 1 \,. \tag{V.3.16}$$

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