A Recursive Algorithm for Computing Cramer–Rao-Type Bounds on Estimator Covariance

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Abstract—We give a recursive algorithm to calculate submatrices of the Cramer–Rao (CR) matrix bound on the covariance of any unbiased estimator of a vector parameter \( \theta \). Our algorithm computes a sequence of lower bounds that converges monotonically to the CR bound with exponential speed of convergence. The recursive algorithm uses an invertible “splitting matrix” to successively approximate the inverse Fisher information matrix. We present a statistical approach to selecting the splitting matrix based on a “complete-data–incomplete-data” formulation similar to that of the well-known EM parameter estimation algorithm. As a concrete illustration we consider image reconstruction from projections for emission computed tomography.

Index Terms—Multidimensional parameter estimation, estimator covariance bounds, complete–incomplete-data problem, image reconstruction.

I. INTRODUCTION

The Cramer–Rao (CR) bound on estimator covariance is an important tool for predicting fundamental limits on best achievable parameter estimation performance. For a vector parameter \( \theta \in \mathbb{R}^n \), an observation \( Y \), and probability density function (pdf) \( f_Y(y; \theta) \), one seeks a lower bound on the minimum achievable variance of an unbiased estimator \( \hat{\theta} = \hat{\theta}(Y) \) of a scalar parameter \( \theta \), of interest. More generally, if, without loss of generality, the \( p \) parameters \( \theta_1, \ldots, \theta_p \) are of interest, \( p \leq n \), one may want to specify a \( p \times p \) matrix which lower bounds the error variance matrix for unbiased estimators \( \hat{\theta}_1, \ldots, \hat{\theta}_p \). The upper left \( p \times p \) submatrix of the \( n \times n \) inverse Fisher information matrix \( F^{-1} \) provides the CR lower bound for these parameter estimates. Equivalently, the first \( p \) columns of \( F^{-1} \) provide this CR bound. The method of sequential partitioning [1] for computing the upper left \( p \times p \) submatrix of \( F^{-1} \) and Cholesky-based Gaussian elimination techniques [2] for computing the first columns of \( F^{-1} \) are efficient direct methods for obtaining the CR bound, but require \( O(n^3) \) floating point operations. Unfortunately, in many practical cases of interest, e.g., when there are a large number of nuisance parameters, high computation and memory requirements make direct implementation of the CR bound impractical.

In this correspondence we give an iterative algorithm for computing columns of the CR bound which requires only \( O(pn^2) \) floating point operations per iteration. This algorithm falls into the class of “splitting matrix iterations” [2] with the imposition of an additional requirement: the splitting matrix must be chosen to ensure that a valid lower bound results at each iteration of the algorithm. While a purely algebraic approach to specifying a suitable splitting matrix can also be adopted, here we exploit specific properties of Fisher information matrices arising from the statistical model. Specifically, we formulate the parameter estimation problem in a complete-data–incomplete-data setting and apply a version of the “data processing theorem” [3] for Fisher information matrices. This setting is similar to that which underlies the classical formulation of the maximum likelihood expectation maximization (ML-EM) parameter estimation algorithm. The ML-EM algorithm generates a sequence of estimates \( (\hat{\theta}(t))_{t=0}^\infty \) which successively increases the likelihood function and converges to the maximum likelihood estimator. In a similar manner, our algorithm generates a sequence of tighter and tighter lower bounds on estimator covariance which converges to the actual CR matrix bound.

The algorithms given here converge monotonically with exponential rate, where the asymptotic speed of convergence increases as the spectral radius \( \rho(I - F_x^{-1}F_y) \) decreases. Here \( I \) is the \( n \times n \) identity matrix and \( F_x \) and \( F_y \) are the complete- and incomplete-data Fisher information matrices, respectively. Thus when the complete data is only moderately more informative than the incomplete data, \( F_y \) is close to \( F_x \) so that \( \rho(I - F_x^{-1}F_y) \) is close to 0 and the algorithm converges very quickly. To implement the algorithm, one must 1) precompute the first \( p \) columns of \( F_y^{-1} \), and 2) provide a subroutine that can multiply \( F_y^{-1}F_x \) or \( F_y^{-1}F_x[I \otimes (\Theta, \Theta)] \) by a column vector (see (18)). By appropriately choosing the complete-data space, this precomputation can be quite simple, e.g., \( X \) can frequently be chosen to make \( F_x \) sparse or even diagonal. If the complete-data space is chosen intelligently, only a few iterations may be required to produce a bound which closely approximates the CR bound. In this case the proposed algorithm gives an order of magnitude computational savings as compared to conventional methods of computing the CR bound. This allows one to examine small submatrices of the CR bound for estimation problems that would have been intractable by exact methods due to the large dimension of \( F_y \).

The paper concludes with an implementation of the recursive algorithm for bounding the minimum achievable error of reconstruction for a small region of interest (ROI) in an image reconstruction problem arising in emission computed tomography. By using the complete data specified for the standard EM algorithm for PET reconstruction [4], [5], \( F_x \) is shown to be diagonal and the implementation of the recursive CR bound algorithm is very simple. As in the ML-EM PET reconstruction algorithm, the rate of convergence of the iterative CR bound algorithm depends on the image intensity and the tomographic system response matrix. Furthermore, due to the sparseness of the tomographic system response matrix, the computation of each column of the CR bound matrix recursion requires only
A. Background and General Assumptions

Let \( \Theta \) be an open subset of the real line \( \mathbb{R} \). Define \( \hat{\theta} = [\theta_1, \cdots, \theta_p]^\top \) a real, nonrandom parameter vector residing in \( \Theta = \Theta_1 \times \cdots \times \Theta_p \). Let \( \{P_{\theta}\}_{\theta \in \Theta} \) be a family of probability measures for a certain random variable \( Y \) taking values in a set \( \mathcal{Y} \). Assume that for each \( \theta \in \Theta, P_{\theta} \) is absolutely integrable with respect to a dominating measure \( \mu \), so that for each \( \theta \) there exists a density function \( f(Y; \theta) = dP_{\theta}(y)/d\mu \) for \( Y \). When \( \int |f(y; \theta)| d\mu < \infty \), we define the expectation \( E_{\theta}[Y] = \int y f(y; \theta) d\mu \).

The family of densities \( \{f_{\theta}(y; \theta)\}_{\theta \in \Theta} \) is said to be a regular family [6] if \( \Theta \) is an open subset of \( \mathbb{R}^n \) and 1) \( f_{\theta}(y; \theta) \) is a continuous function on \( \Theta \) for \( \mu \)-almost all \( y \); 2) \( \log f(Y; \theta) \) is mean-square differentiable in \( \theta \); and 3) \( \log f(Y; \theta) \) is mean-square continuous in \( \theta \). These three conditions guarantee that the nonnegative definite \( n \times n \) Fisher information matrix \( F_\theta(\theta) \) exists and is finite:

\[
F_{\theta}(\theta) = E_{\theta} [\nabla_\theta \log f(Y; \theta) \nabla_\theta \log f(Y; \theta)] = E_{\theta} \left[ \begin{bmatrix} \frac{\partial^2}{\partial \theta_i \partial \theta_j} & \frac{\partial^2}{\partial \theta_i \partial \theta_k} & \cdots & \frac{\partial^2}{\partial \theta_i \partial \theta_p} \end{bmatrix} \right],
\]

where \( \nabla_\theta = \left( \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \cdots, \frac{\partial}{\partial \theta_p} \right) \) is the (row) gradient operator.

Finally we recall convergence results for linear recursions of the form

\[
e^{i+1} = A e^i, \quad i = 1, 2, \ldots,
\]

where \( e^i \) is a vector and \( A \) is a matrix. Let \( \rho(A) \) denote the spectral radius, i.e., the maximum magnitude of the eigenvalues, of \( A \). If \( \rho(A) < 1 \), then \( e^i \) converges exponentially to zero and the asymptotic rate of convergence increases as the root convergence factor \( \rho(A) \) decreases [7].

B. The CR Lower Bound

Let \( \hat{\theta} = \hat{\theta}(Y) \) be an unbiased estimator of \( \theta \in \Theta \), and assume that the densities \( f_{\theta}(y; \theta) \in \Theta \) are a regular family. Additionally assume that the Fisher information \( F_{\hat{\theta}} \) is positive definite. Then the covariance matrix of \( \hat{\theta} \) satisfies the matrix CR lower bound [6]:

\[
\text{cov}_{\theta} (\hat{\theta}) \geq B (\hat{\theta}) = F_{\hat{\theta}}^{-1}(\hat{\theta}).
\]

We refer to the above as the unbiased CR bound.

Assume that among the \( n \) unknown quantities \( \theta = [\theta_1, \cdots, \theta_n]^\top \), only a small number \( p \ll n \) of parameters \( \theta^* = [\theta_{i_1}, \cdots, \theta_{i_p}]^\top \), are of interest, the remaining \( n - p \) parameters being considered "nuisance parameters." Partition the Fisher information matrix \( F_{\theta} \) as

\[
F_{\theta} = \begin{bmatrix} F_{11} & F_{12}^T \\ F_{12} & F_{22} \end{bmatrix},
\]

where \( F_{11} \) is the \( (n - p) \times (n - p) \) Fisher information matrix for the parameters \( \theta^* \) of interest, \( F_{22} \) is the \( (n - p) \times p \) information coupling matrix, \( \text{cov}_{\theta}(\hat{\theta}^*) \geq 2 F_{11}^{-1} \), of the parameters of interest is simply the \( p \times p \) submatrix in the upper left-hand corner of \( F_{11}^{-1} \):

\[
\text{cov}_{\theta}(\hat{\theta}^*) \geq 2 F_{11}^{-1} = 2 F_{11}^{-1}.
\]

C. A Recursive CR Bound Algorithm

The basic idea of the algorithm is to replace the difficult inversion of \( F_{\theta} \) with an easily inverted matrix \( F \). To simplify notation, we drop the dependence on \( \theta \). Let \( F \) be an \( n \times n \) matrix. Assume that \( F_{\theta} \) is positive definite and that \( F = 2, F - F_{\theta} \) is nonnegative definite. It follows that \( F \) is positive definite, so let \( F^{1/2} \) be the positive definite matrix-square-root factor of \( F \). Then,

\[
1 - F^{-1/2}(F - F_{\theta}) F^{-1/2} = F^{-1/2} F_{\theta} F^{-1/2} > 0,
\]

and

\[
F^{-1/2}(F - F_{\theta}) F^{-1/2} \geq 0.
\]

Hence \( 0 \leq I - F^{-1/2} F_{\theta} F^{-1/2} < I \), so that all of the eigenvalues of \( I - F^{-1/2} F_{\theta} F^{-1/2} \) are nonnegative and strictly less than 1. Since \( I - F^{-1/2} F_{\theta} F^{-1/2} \) is similar to \( I - F^{-1} \), it follows that the eigenvalues of \( I - F^{-1} \) lie in \([0, 1)\) [8, Corollary 1.3.4]. Thus, applying the matrix form of the geometric series [8, Corollary 5.6.16]:

\[
B = [F_{\theta}]^{-1} = [F - (F - F_{\theta})]^{-1}
= [I - (F - F_{\theta})^{-1}]^{-1}
= \sum_{k=0}^{\infty} [I - F^{-1} F_{\theta}]^{k} F^{-1}.
\]

This infinite series expression for the unbiased \( n \times n \) CR bound \( B \) is the basis for the matrix recursion given in the following theorem.

Theorem 1: Assume that \( F_{\theta} \) is positive definite and \( F \geq F_{\theta} \). When initialized with the \( n \times n \) matrix of zeros \( B^{(0)} = 0 \), the
following recursion yields a sequence of matrix lower bounds $B^{(k)} = B^{(k)}(\hat{g})$ on the $n \times n$ covariance of unbiased estimators $\hat{g}$ of $g$. This sequence asymptotically converges to the $n \times n$ unbiased CR bound $F_{\gamma}$ with root convergence factor $\rho(A)$.

**Recursive Algorithm:** For $k = 0, 1, 2, \ldots$,

$$B^{(k+1)} = A \cdot B^{(k)} + F^{-1},$$

where $A = I - F^{-1}F_{\gamma}$ has eigenvalues in [0,1]. Furthermore, the convergence is monotone in the sense that $B^{(k)} \leq B^{(k+1)} \leq B = F_{\gamma}$, for $k = 0, 1, 2, \ldots$.

**Proof:** Since all eigenvalues of $I - F^{-1}F_{\gamma}$ are in the range [0,1], we obviously have $\rho(I - F^{-1}F_{\gamma}) < 1$. Now consider

$$B^{(k+1)} - F_{\gamma}^{-1} = (I - F^{-1}F_{\gamma})B^{(k)} + F^{-1} - F_{\gamma}^{-1}$$

$$= (I - F^{-1}F_{\gamma})(B^{(k)} - F_{\gamma}^{-1}).$$

(9)

Since the eigenvalues of $I - F^{-1}F_{\gamma}$ are in [0,1], this establishes that $B^{(k+1)} \rightarrow F_{\gamma}^{-1}$ as $k \rightarrow \infty$ with root convergence factor $\rho(I - F^{-1}F_{\gamma})$. Similarly,

$$B^{(k+1)} - B^{(k)} = (I - F^{-1}F_{\gamma})(B^{(k)} - B^{(k-1)}), \quad k = 1, 2, \ldots,$$

with initial condition $B^{(0)} = F_{\gamma}^{-1}$. By induction we have

$$B^{(k+1)} - B^{(k)} = F_{\gamma}^{-1/2}(I - F^{-1/2}F_{\gamma}F^{-1/2})F_{\gamma}^{-1/2},$$

which is nonnegative definite for all $k \geq 0$. Hence the convergence is monotone.

By right multiplying each side of the equality (8) by the matrix $\mathcal{S} = [e_1, \ldots, e_p]$, where $e_j$ is the $j$th unit vector in $\mathbb{R}^n$, we obtain a recursion for the first $p$ columns $B^{(k)}e_j$ of $B^{(k)}$. Furthermore, the first $p$ rows $\mathcal{S}^TB^{(k)}e_j$ correspond to the upper left-hand corner $p \times p$ submatrix of $B^{(k)}$ and, since $\mathcal{S}^TB^{(k+1)}e_j = \mathcal{S}^TB^{(k)}e_j$ is nonnegative definite, by Theorem 1, $\mathcal{S}^TB^{(k)}e_j$ converges monotonically to $\mathcal{S}^TF_{\gamma}^{-1}e_j$. Thus we have the following corollary to Theorem 1.

**Corollary 1:** Assume that $F_{\gamma}$ is positive definite and $F_{\gamma} \geq F$, and let $\mathcal{S} = [e_1, \ldots, e_p]$ be the $n \times p$ elementary matrix whose columns are the first $p$ unit vectors in $\mathbb{R}^n$. When initialized with the $n \times p$ matrix of zeros $\mathcal{S}^TB^{(0)} = 0$, the top $p \times p$ block $\mathcal{S}^TB^{(k)}$ of $\mathcal{S}^TB^{(k)}$ in the following recursive algorithm yields a sequence of lower bounds on the covariance of any unbiased estimator of $\theta^T = [\theta_1, \ldots, \theta_q]^T$ which asymptotically converges to the $p \times p$ CR bound $\mathcal{S}^TF_{\gamma}\mathcal{S}$ with root convergence factor $\rho(A)$.

**Recursive Algorithm:** For $k = 0, 1, 2, \ldots$,

$$\mathcal{S}^TB^{(k+1)} = A \cdot \mathcal{S}^TB^{(k)} + \mathcal{S}^TF_{\gamma}^{-1},$$

(10)

where $A = I - F^{-1}F_{\gamma}$ has eigenvalues in [0,1] and $\mathcal{S}^TF_{\gamma}^{-1} = F^{-1}\mathcal{S}^TF_{\gamma}$ is the $n \times p$ matrix consisting of the first $p$ columns of $F^{-1}$. Furthermore, the convergence is monotone in the sense that $\mathcal{S}^TB^{(k)} \leq \mathcal{S}^TB^{(k+1)} \leq \mathcal{S}^TF_{\gamma}^{-1}$, for $k = 0, 1, 2, \ldots$.

Given $F^{-1}$ and $A$ the $n \times n$ times $n \times p$ matrix multiplication $A \cdot B^{(k)}$ requires only $O(np^2)$ floating point operations.

**D. Discussion**

We make the following comments on the recursive algorithms of Theorem 1 and Corollary 1.

1) In order that the algorithm (10) for computing columns of $F_{\gamma}^{-1}$ have significant computational advantages relative to the direct sequential partitioning and Cholesky-based methods discussed in Section II.B, the precomputation of the matrix inverse $F^{-1}$ must be simple, and the iterations must converge reasonably quickly. By choosing $F$ that is sparse or diagonal, the computation of $F^{-1}$ requires only $O(n^2)$ floating point operations. If in addition $F$ can be chosen such that $\rho(I - F^{-1}F_{\gamma})$ is small, then the algorithm (10) will converge to within a small fraction of the corresponding column of $F_{\gamma}^{-1}$ with only a few iterations and thus will be an order of magnitude less costly than direct methods requiring $O(n^3)$ operations.

2) From the relation $I = F_{\gamma}^{-1}F_{\gamma}^{-1} - F_{\gamma}^{-1}F_{\gamma}^{-1}$, obtained in a similar manner as (9) of the proof of Theorem 1, we obtain the following recursion for the normalized difference $\Delta B^{(k)} = F_{\gamma}^{-1}B^{(k)} - B^{(k)}$ between $B^{(k)}$ and its asymptotic limit $F_{\gamma}^{-1}$:

$$\Delta B^{(k+1)} = A'\Delta B^{(k)}, \quad k = 1, 2, \ldots,$$

with $\Delta B^{(0)} = \mathcal{S}^TF_{\gamma}^{-1}$. This recursion can be implemented in parallel with (10) to monitor the progress of the iterative CR bound algorithm towards its limit.

3) For $p = 1$, the iteration of Corollary 1 is related to the "matrix splitting" method [2] for iteratively approximating the solution $g$ to a linear equation $Cg = d$. In this method, a decomposition $C = F + N$ is found for the nonsingular matrix $C$ such that $F$ is nonsingular and $\rho(F^{-1}N) < 1$. Once this decomposition is found, the algorithm below produces a sequence of vectors $\mathcal{S}g^{(k)}$ which converges to the solution $g = C^{-1}d$ as $k \rightarrow \infty$:

$$\mathcal{S}g^{(k+1)} = F^{-1}N\mathcal{S}g^{(k)} + F^{-1}d.$$
approach to choosing the matrix \( F \); \( F \) is chosen to be the Fisher information matrix of the complete data that is intrinsic to a related EM parameter estimation algorithm. This approach guarantees that \( F \geq F_y \) due to the Fisher information version of the data processing inequality.

A. Incomplete-Data Formulation

Many estimation problems can be conveniently formulated as an incomplete-complete-data problem. The setup is the following. Imagine that there exists a different set of measurements \( X \) taking values in a set \( \mathcal{X} \) whose probability density \( f_x(x; \theta) \) is also a function of \( \theta \). Further assume that this hypothetical set of measurements \( X \) is larger and more informative as compared to \( Y \) in the sense that the conditional distribution of \( Y \) given \( X \) is functionally independent of \( \theta \). Note that our definition contains as a special case the definition of incomplete-complete data is equivalent to defining \( Y \) as a random variable which has a deterministic functional transformation \( Y = h(X) \), where \( h: \mathcal{X} \rightarrow \mathcal{Y} \) is many-to-one.

1) The EM Algorithm

For an initial point \( \theta^{(0)} \), the EM algorithm produces a sequence of estimates \( \theta^{(k)} \) by alternating between computing an estimate \( Q(Y; \hat{\theta}^{(k)}) \) of the complete-data log-likelihood function \( f_y(y; \theta) \), called the expectation (E) step, and finding the maximum of \( Q(Y; \hat{\theta}^{(k)}) \) over \( \theta \), called the maximization (M) step [10]:

**EM Algorithm:** For \( k = 0, 1, 2, \ldots \), do:

\[
\text{(E)} \quad Q(Y; \hat{\theta}^{(k)}) = E_{Y|X}(\log f_X(X; \theta)) \quad \text{and} \quad \hat{\theta}^{(k+1)} = \arg\max_{\theta} Q(Y; \theta^{(k)})
\]

It can be shown [11] that the sequence \( \hat{\theta}^{(k)} \) monotonically increases likelihood in the sense that \( f_y(Y; \hat{\theta}^{(k+1)}) \geq f_y(Y, \hat{\theta}^{(k)}) \), for all \( k \). Furthermore, if the likelihood function is strictly concave on \( \theta \), \( \hat{\theta}^{(k)} \) converges to the maximum likelihood estimate.

2) A Data Processing Theorem

Assume that a complete-data set \( X \) has been specified. For regular probability densities \( f_x(x; \theta), f_y(y; \theta), f_{X|Y}(x|y; \theta) \), we define the associated Fisher information matrices \( F_x(\theta) = -E_{X|Y}(\nabla^2 \log f_x(X; \theta)) \), \( F_y(\theta) = -E_{X|Y}(\nabla^2 \log f_y(Y; \theta)) \), \( F_{X|Y}(\theta) = -E_{X|Y}(\nabla^2 \log f_{X|Y}(X|Y; \theta)) \), respectively. The following gives a decomposition for \( F_y(\theta) \) in terms of \( F_x(\theta) \) and \( F_{X|Y}(\theta) \).

**Lemma 1:** Let \( X \) and \( Y \) be random variables which have a joint probability density \( f_{X,Y}(x,y; \theta) \) relative to some product measure \( \mu_X \times \mu_Y \). Assume that \( X \) is more informative than \( Y \) in the sense that the conditional distribution of \( Y \) given \( X \) is functionally independent of \( \theta \). Assume also that \( f_x(x; \theta^{(k)}) \) is a regular family of densities with mixed partials \( \partial^2 \) \( f_x(x; \theta) \) which are continuous in \( \theta \) and absolutely integrable in \( x \). Then \( f_y(y; \theta^{(k)}) \) is a regular family of densities with continuous and absolutely integrable mixed partials, the above-defined Fisher information matrices \( F_x(\theta), F_y(\theta), \) and \( F_{X|Y}(\theta) \) exist, are finite, and

\[
F_y(\theta) = F_x(\theta) - F_{X|Y}(\theta).
\]

**Proof of Lemma 1:** Since \( X, Y \) has the density \( f_{X,Y}(x,y; \theta) \) with respect to the measure \( \mu_X \times \mu_Y \), there exists versions \( f_{X,Y}(x,y; \theta) \) and \( f_{X,Y}(x,y; \theta) \) of the conditional densities. Furthermore, by assumption, \( f_{X,Y}(x,y; \theta) = f_{X,Y}(x,y; \theta) \) does not depend on \( \theta \). Since \( f_x(x; \theta) = f_x(x; \theta) \) does not depend on \( \theta \), it is straightforward to show that the family \( f_x(x; \theta) \) inherits the regularity properties of the family \( f_{X,Y}(x,y; \theta) \). Now for any \( y \) such that \( f_y(y; \theta) > 0 \), we have from Bayes’ rule,

\[
f_{X|Y}(y; \theta) = \frac{f_{X,Y}(x,y; \theta)}{f_y(Y; \theta)}.
\]

Note that \( f_{X,Y}(x,y; \theta) > 0 \) implies that \( f_y(y; \theta) > 0 \). Hence, we can use (14) to express

\[
\log f_{X|Y}(y; \theta) = \log f_{X,Y}(x,y; \theta) - \log f_y(y; \theta) + \log f_{X,Y}(x,y; \theta),
\]

whenever \( f_{X,Y}(x,y; \theta) > 0 \). From this relation it is seen that \( f_{X,Y}(x,y; \theta) \) inherits the regularity properties of the \( X \) and \( Y \) densities. Therefore, since the set \( \{x, y: f_{X,Y}(x,y; \theta) > 0\} \) has probability 1, we obtain from (15):

\[
E_{X|Y}(-\nabla^2 \log f_{X|Y}(X|Y; \theta)) = E_{X|Y}(-\nabla^2 \log f_{X,Y}(X|Y; \theta)) - E_{X|Y}(-\nabla^2 \log f_{X,Y}(X|Y; \theta)).
\]

This establishes the lemma.

Since the Fisher information matrix \( F_{X|Y} \) is nonnegative definite, an important consequence of the decomposition of Lemma 1 is the inequality

\[
F_x(\theta) \geq F_y(\theta)
\]

The inequality (16) is a Fisher matrix version of the “data processing theorem” of information theory [3], which asserts that any irreversible processing of data \( X \) entails a loss in information in the resulting data \( Y \).

B. Remarks

1) The inequality (16) on \( F_X \) is precisely the condition required of the splitting matrix \( F \) by the recursive CR bound algorithm (10). Furthermore, in many applications of the EM algorithm, the complete-data space is chosen such that the dependence of \( X \) on \( \theta \) is “uncoupled,” so that \( F_x \) is diagonal or very sparse. Since many of the problems in which \( F_y \) is difficult to invert are problems for which the EM algorithm has been applied, the Fisher information \( F_{X|Y} \) of the corresponding complete-data space is thus a natural choice for \( F \).

2) If the incomplete-data Fisher matrix \( F_y \) is available, the matrix \( A \) in the recursion (8) can be precomputed as

\[
A = I - F_x^{-1} F_y.
\]

On the other hand, if the Fisher matrix \( F_y \) is not available, the matrix \( A \) in the recursion (8) can be computed directly from \( Q(Y, \theta) = H(X; Y; \theta) \) arising from the E step of the EM parameter estimation algorithm (12). Note that, under the assumption that exchange of order of differentiation and expectation is justified:

\[
F_{X|Y}(\theta) = E_{X|Y}[ -\nabla^2 \log f_{X|Y}(X|Y; \theta)] - E_{X|Y}[ -\nabla^2 H(\theta; \theta)],
\]

where \( H(\theta; \theta) = E_{X|Y}[ \log f_{X|Y}(X|Y; \theta)] \). We can make use of an identity [10, Lemma 2]: \( \nabla^2 H(\theta; \theta) = -\nabla^2 \theta^T H(\theta; \theta) \). Furthermore, \( \nabla^2 H(\theta; \theta) = \nabla^2 Q(\theta; \theta) \).
gives the identity \( F_{X|Y}(\theta) = E_{\theta}[\nabla^1 Q(\theta; \theta)] \); yielding an alternative expression to (17) for \( A \):
\[
A = F_y^{-1} E_{\theta}[\nabla^1 Q(\theta; \theta)].
\]  
(18)

3) The form \( \rho(I - F^{-1}F_y) \) for the rate of convergence of the algorithms (8) and (10) implies that when \( F = F_y \), for rapid convergence the complete-data space \( \mathcal{X} \) should be chosen such that \( X \) is not significantly more informative than \( Y \) relative to the parameter \( \theta \).

4) The matrix recursion of Theorem 1 can be related to the following Frobenius normalization method for inverting a sparse matrix \( C \):
\[
B^{(k+1)} = (I - \alpha C)B^{(k)} + \alpha I,
\]  
(19)
where \( \alpha = 1/\|[C]_2 \) is the inverse of the Frobenius norm of \( C \). When initialized with \( B^{(0)} = I \), the above algorithm converges to \( C^{-1} \) as \( k \to \infty \). For the case that \( C \) is the Fisher matrix \( F \), the matrix recursion (19) can be interpreted as a special case of the algorithm of Theorem 1 for a particular choice of complete data \( X \). Specifically, let the complete data be defined as the concatenation \( X = [Y^T, S^T]^T \) of the incomplete data \( Y \) and a hypothetical data set \( S = [S_1, \cdots, S_n]^T \) defined by the following
\[
S = \xi(\theta) + W,
\]  
(20)
where \( W = [W_1, \cdots, W_n]^T \) are i.i.d. standard Gaussian random variables independent of \( Y \), and \( \xi = [\xi_1, \cdots, \xi_n]^T \) is a vector function of \( \theta \). It is readily verified that the Fisher matrix \( F \) based on observing \( S \) is of the form \( F_S = \sum_{j=1}^{n} \nabla^2 \xi_j C(\theta) \xi_j \). Now since \( S \) and \( Y \) are independent, \( F_S = F_y + F_F \), so that if we could choose \( \xi(\theta) \) such that \( F_F = \|F_y\| - I - F_y \), the recursion of Theorem 1 would be equivalent to (19) with \( F_F = C, F_F^{-1} = \alpha I \), \( A = I - \alpha F_y \). In particular, for the special case that \( F_F \) is functionally independent of \( \theta \) we can take \( m \) equal to \( n \) and take the hypothetical data \( S = [S_1, \cdots, S_n]^T \) as the \( n \)-dimensional linear Gaussian model:
\[
S_j = \xi_j \theta^T + W_j, \quad j = 1, \cdots, n,
\]
where
\[
\xi_j = \left[ (\|F_y\|_2 - \sqrt{\lambda_j}) \right] Y_j, \quad j = 1, \cdots, n,
\]
and \( \{\xi_1, \cdots, \xi_n\} \) are the eigenvectors and \( \{\lambda_1, \cdots, \lambda_n\} \) are the eigenvalues of \( F_y \). With this definition of \( S, F_F = \sum_{j=1}^{n} \lambda_j Y_j^2 \) is simply the eigendecomposition of the matrix \( \|F_y\| - I - F_y \), so that \( F_F = \|F_y\| - I - F_y \) as required.

IV. APPLICATION TO ECT IMAGE RECONSTRUCTION

We consider the case of positron emission tomography (PET), where a set of \( m \) detectors is placed around an object to measure positions of emitted gamma rays. The mathematical formulation of PET is as follows [12]. Over a specified time interval, a number \( N_b \) of gamma rays are randomly emitted from pixels \( b, b = 1, \cdots, n \), and a number \( Y_d \) of these gamma rays are detected at detectors \( d, d = 1, \cdots, m \). The average number of emissions in pixels \( 1, \cdots, n \) is an unknown vector \( \theta = [\theta_1, \cdots, \theta_n]^T \), called the object intensity. It is assumed that the \( N_b \)'s are independent Poisson random variables with rates \( \theta_0, b = 1, \cdots, n \), and the \( Y_d \)'s are independent Poisson distributed with rates \( \mu_y = \sum_{j=1}^{m} P_{db} \theta_0 \), where \( P_{db} \) is the transition probability corresponding to emitter location \( b \) and detector location \( d \). For simplicity, we assume that \( \mu_y > 0, \forall d \). The objective is to estimate a subset \( \theta_1, \cdots, \theta_n \);
\[ p < n, \]  
of the object intensities within a \( p \)-pixel region of interest (ROI). In this section we develop the recursive CR bound for this estimation problem.

The log-likelihood function for \( \theta \) based on \( Y = [Y_1, \cdots, Y_m]^T \) is simply
\[
\ln f_Y(Y; \theta) = \ln \prod_{k=1}^{m} \frac{\mu_y^{Y_k}}{Y_k!} e^{-\mu_y}
\]  
(21)
\[
= - \sum_{d=1}^{m} \mu_y + \sum_{d=1}^{m} Y_d \ln \mu_y + \text{constant}. \quad \text{(22)}
\]
From this, the Hessian matrix with respect to \( \theta \) is simply calculated and, using the fact that \( E_y[Y_k] = \mu_y \), the \( n \times n \) Fisher information matrix \( F_y \) is obtained:
\[
F_y = \sum_{d=1}^{m} \frac{1}{\mu_y} P_{dy} P_{dy}^T,
\]  
(23)
\[
= \left( \sum_{d=1}^{m} \sum_{d'=1}^{m} P_{dy} P_{dy}^T \right)_{i,j=1,\cdots,n},
\]
where \( P_{dy} = [P_{dy}, \cdots, P_{dy}] \) is the \( d \)th row of the \( m \times m \) system matrix \( (F_y^{-1}) \). If \( m \geq n \), and the linear span of \( (P_{dy})_{1} \) on \( \mathbb{R}^n \), then \( F_y \) is invertible and the CR bound exists. However, even for an imaging system of moderate resolution, e.g., a 256 x 256 pixel plane, direct computation of the \( p \times p \) ROI submatrix \( E_y^T F_y^{-1} E_y \) is impractical.

The standard choice of complete data for estimation of \( \theta \) via the EM algorithm is the set \( \{N_b, Y_b \}_{b=1}^{n} \), where \( N_b \) denotes the number of emissions in pixel \( b \) which are detected at detector \( d \) [4], [5]. \( N_{ab} \) is independent Poisson random variables with intensity \( E_y^T (F_y^{-1}) E_y \). We can use in Theorem 1 to obtain a monotonically convergent CR bound recursion.

The log-likelihood function associated with the complete-data set \( X = \{N_{ab}, Y_b \}_{b=1}^{n} \) is of similar form to (22):
\[
\ln f_X(X; \theta) = - \sum_{d=1}^{m} \sum_{b=1}^{n} P_{db} \theta_b + \sum_{d=1}^{m} \sum_{b=1}^{n} N_{ab} \ln \theta_b + \text{constants}.
\]

The Hessian \( V_\theta^2 \ln f_X(X; \theta) \) is easily calculated, and, assuming \( \theta_b > 0, \forall b \), the Fisher information matrix \( F_X \) is obtained as
\[
F_X = \text{diag} \left( \sum_{d=1}^{m} P_{db} \theta_b \right)
\]  
(24)
where diag*(a_{ab}) denotes a diagonal \( n \times n \) matrix with \( a_{ab} \)'s indexed successively along the diagonal.

Using the results (24) and (23) above, we obtain
\[
A = I - F_X^{-1} F_y
\]  
(25)
In many SPECT and PET tomographic geometries, the $m \times n$ ($m \geq n$) system response $((P_{ij}))$ is a sparse matrix, i.e., its number of nonzero elements is only $O(n)$ as compared to $O(n^2)$ for the non-sparse case. Note, however, that even when the system response matrix is sparse, the matrix $A$ (25) is not generally sparse, and it would appear that the recursive algorithm (10) of Corollary 1 requires $O(n^2)$ memory storage to store the $n \times n$ matrix $A$. In the present case, however, we only require $O(n)$ memory storage since it is seen that, using (25) in (10), the recursion collapses into a set of $p$ vector recursions which only require storing the $n$ parameters of the vector $\phi$, the $np$ entries of $g^{(k)}$, and the $O(n)$ nonzero entries of the sparse matrix $((P_{ij}))$. Because of this feature, we have been able to implement this recursive CR bound on relatively large image reconstruction problems [13].

The rate of convergence of the recursive CR bound algorithm is determined by the maximum eigenvalue $p(A)$ of $A$ specified by (25). For a fixed system matrix $((P_{ij}))$, the magnitude of this eigenvalue will depend on the image intensity $\theta$. Assume for simplicity that with probability 1 any emitted gamma ray is detected at some detector, i.e., $\sum_{j=1}^{n} P_{i,j} = 1$ for all $b$. Since $\text{trace}(A) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_i$ are the eigenvalues of $A$, using (25) it is seen that the maximum eigenvalue $p(A)$ must satisfy

$$
\frac{1}{n} \text{trace}(A) = 1 - \frac{1}{n} \sum_{j=1}^{n} \frac{1}{P_{i,j}^{(k)}} \theta_j^2 \leq p(A) < 1.
$$

A consequence of the inequality $\left(\sum_{j=1}^{n} P_{i,j} \theta_j^2\right)^2 \leq \sum_{i=1}^{n} \theta_i^2 \sum_{j=1}^{n} P_{i,j}^2 \theta_j$ is

$$
\frac{1}{n} \text{trace}(A) = 1 - \frac{1}{n}.
$$

where equality occurs if $P_{i,j}$ is independent of $i$. On the other hand, as the intensity $\phi$ concentrates an increasing proportion $1 - \epsilon$ of its mass on a single pixel $k_{\epsilon}$, $\phi_{k_{\epsilon}}$, we obtain $(1/n) \text{trace}(A) = 1 - 1/n + O(\epsilon)$. Thus for this case we have, from (26), $1 - 1/n + O(\epsilon) \leq p(A) < 1$. Since the number of pixels $n$ is typically very large, this implies that the asymptotic convergence rate of the recursive algorithm will suffer for image intensities which approach that of an ideal point source, at least for this particular choice of splitting matrix $F_X$.

V. CONCLUSION AND FUTURE WORK

We have given a recursive algorithm which can be used to compute submatrices of the CR lower bound $F_X^{-1}$ on unbiased multidimensional parameter estimation error covariance. The algorithm successively approximates the inverse Fisher information matrix $F_X^{-1}$ via a monotonically convergent splitting matrix iteration. We have also given a statistical methodology for selecting an appropriate splitting matrix $F$ which involves application of a data processing theorem to a complete-data-incomplete-data formulation of the estimation problem. We are developing analogous recursive algorithms to compute matrix CR-type bounds for constrained and biased estimation, such as those developed in [14], [15].

REFERENCES


Bounds on Achievable Convergence Rates of Parameter Estimators via Universal Coding

Neri Merhav

Abstract—Lower bounds on achievable convergence rates of parameter estimators towards the true parameter are derived via universal coding considerations. It is shown that for a parametric class of finite-alphabet information sources, if there exists a universal lossless code whose redundancy decays sufficiently rapidly, then it induces a limitation on the fastest achievable convergence rate of any parameter estimator, at any value of the true parameter, with a possible exception of a vanishingly small subset of parameter values. A specific choice of a universal

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