rise occurred earlier and the current continued rising at a faster rate (curves c, d). With the voltage exceeding U_0 by only a few volts the current became excessive, being limited mainly by the external resistance. The critical voltage agreed closely with the "turn-over" value obtained on application of a continuous 50-cycle voltage.[9]

No proper explanation seems to have been given so far for the large increase in leakage current observed as the voltage is raised and the contact point reaches high temperature. In the usual rectifier model[1,2] current carriers are taken to flow from the metal into the semiconductor on application of inverse voltage; but the number of carriers available in the metal is not increased by high temperature. The explanation seems to be that by the mechanisms 1 and 2 described above—image force and tunnel effect—the current rises beyond the saturation value postulated in the simple theory. The ensuing power loss raises the local temperature at the contact point until intrinsic conduction sets in. Electrons are thus raised into the conduction band at a fast rate, leaving the same number of holes behind in the valency band. These electron-hole pairs are generated within the barrier or in its neighborhood sufficiently close by to drift towards it within their lifetime (10^{-6} to 10^{-4} second). Although the barrier is practically insuperable for the predominant carriers (electrons in n-type germanium), there is, of course, no barrier to stop the minority carriers (holes). This part of the leakage current is thus expected to rise exponentially with temperature and to become dominant as the intrinsic temperature is approached. The possibility of electrons and holes simultaneously carrying the current has already been discussed,[10] but the origin of the minority carriers, namely, intrinsic generation, seems to have escaped explanation so far.

[9] The current pulse as displayed on the cathode-ray screen remained steady for curves a and b. When the mean power exceeded about 25 mw, the current pulse crept up visibly. The power loss then apparently exceeded the heat dissipation, with the rectifier temperature increasing from one pulse to the next.

[10] J. Bardeen and W. H. Brattain, Phys. Rev. **75**, 1208 (1949).

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Equivalence Theorems for Pseudoscalar Coupling*

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By a unitary transformation a rigorous equivalence theorem is established for the pseudoscalar coupling of pseudoscalar mesons (neutral and charged) to a second-quantized nucleon field. By the transformation the linear pseudoscalar coupling is eliminated in favor of a nonlinear pseudovector coupling term together with other terms. Among these is a term corresponding to a variation of the effective rest mass of the nucleons with position through its dependence on the meson potentials. The question of the connection of the nonlinear pseudovector coupling with heuristic proposals that such a coupling may account for the saturation of nuclear forces and the independence of single nucleon motions in nuclei is briefly discussed. The new representation of the Hamiltonian may have particular value in constructing a strong coupling theory of pseudoscalar coupled meson fields. Some theorems on a class of unitary transformations of which the present transformation is an example are stated and proved in an appendix.

INTRODUCTION

In a recent communication[1] by one of the present authors it was demonstrated that by performing a particular unitary transformation on the Hamiltonian describing the interaction of a nucleon with a neutral pseudoscalar meson field through pseudoscalar coupling a new representation of the Hamiltonian was obtained which gave prominence to some features of this theory which are obscured in the usual representation. The present paper is the first of a series whose purpose is to generalize these results and to investigate, and perhaps exploit, their significance in our understanding of nucleonic properties and nuclear forces.

The unitary transformation referred to above can be regarded as generating a rigorous equivalence theorem, correct to all orders in the coupling constant, connecting the simple pseudoscalar coupling in the original representation with pseudovector coupling in the new representation. The latter is particularly interesting because the pseudovector coupling term has a nonlinear character of a type which one would anticipate would lead to saturating of the nucleon-meson coupling in the presence of large meson potentials. The attractive possibility is therefore suggested that within the framework of the ordinary pseudoscalar meson theory with simple pseudoscalar coupling may lie just the elements of nonlinear behavior responsible for the saturation of nuclear forces and for the relative independence of one-particle motions in nuclei (as evidenced by nuclear shell structure) which have been proposed on a heuristic

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[1] L. L. Foldy, Phys. Rev. **84**, 168 (1951).

basis by several workers.[2] The new representation also has the advantage of exhibiting the Hamiltonian in a form in which nonrelativistic strong-coupling methods can be applied to its investigation; in the conventional representation, the pseudoscalar theory with pseudoscalar coupling has evaded such treatment. However, further discussion of these points will be left for succeeding papers in this series and the present paper will be concerned only with the establishment of rigorous equivalence theorems. We may then justify our attention to the present results simply on the basis that in view of recent experimental evidence favoring the pi-meson as having pseudoscalar character,[3] any new results concerning the properties of pseudoscalar fields may prove to have value in the future.

Previous derivations of equivalence theorems[4] have been concerned with obtaining a pseudoscalar coupling which is equivalent to a given pseudovector coupling. In the present treatment the emphasis is reversed, pseudoscalar couplings being reduced (to within certain other coupling terms) to an equivalent pseudovector coupling. The virtue of this procedure is that it makes easier the transition to an appropriate nonrelativistic Hamiltonian for the study of nonrelativistic problems. In the present paper the results of the earlier communication referred to above, which dealt with the case of a single nucleon interacting with a neutral pseudoscalar field, have been generalized by dealing with nucleons in the second-quantized representation and by considering four types of meson fields: two types of neutral fields, the charged field, and the symmetrical field. The work has also been carried out only in the Schrödinger representation though there appears to exist no difficulty in generalizing the results to the interaction representation. The execution of the transformations is effected in the following section. Use is made there of several theorems on unitary transformations which may have wider application than in the establishment of the present results. Statements and proofs of these theorems are given in an appendix.

RIGOROUS EQUIVALENCE THEOREMS

In this section we shall actually carry out the type of canonical transformation referred to in the introduction in order to derive rigorous equivalence theorems for pseudoscalar coupling. We do not attempt to derive the theorem in its most general form but content ourselves with four special cases: (1) the charge-symmetrical neutral theory in which the coupling constants of the neutron and proton to the neutral meson field have the same sign, (2) the charge-antisymmetrical neutral theory in which the coupling constants have opposite sign, (3) the charged theory, and (4) the symmetrical theory. The last is, of course, a combination of (2) and (3) with equal coupling constants for neutral and charged mesons. Complete generality has not been attempted in that an arbitrary mixture of all meson fields has not been considered, nor has the inclusion of pseudovector coupling in the original Hamiltonian. The interaction of the particles with electromagnetic fields has also been omitted. The methods employed can all be generalized to take care of these omissions, but it was not felt worthwhile to consider a more general case without some specific application of it in view. In order to avoid duplication of effort and to save space we shall introduce a notation which allows all of these cases to be treated together.

Our notation is as follows: We employ φ_0 to represent the potential of the charge-symmetrical neutral meson field, φ_1 and φ_2 to represent the potentials of the charged meson field, and φ_3 to represent the potential of the charge-antisymmetrical neutral meson field; π_0, π_1, π_2, and π_3 represent respective canonically conjugate momenta.[5] We employ τ_1, τ_2, and τ_3 to represent the components of the isotopic spin vector of the nucleon field, and where convenient we also use τ_0 to represent the unit isotopic spin matrix. To unify the treatment of the four different cases we introduce the quantities

$$\Phi=\tau_i\varphi_i,\quad \Pi=\tau_i\pi_i,\quad \phi=(\varphi_i\varphi_i)^{\frac{1}{2}}.$$

Here and in what follows a repeated Latin subscript is to be summed over the following values:

0 (only) in the charge-symmetrical neutral theory;
3 (only) in the charge-antisymmetrical neutral theory;
1, 2 in the charged theory;
1, 2, and 3 in the symmetrical theory.

The following relation then holds in all cases:

$$\Phi^2=\phi^2.$$

We may now write the Hamiltonian in all cases in the form

$$\mathcal{H}=\mathcal{H}_n+\mathcal{H}_\mu+\mathcal{H}_i,$$

where

$$\mathcal{H}_n=\int\psi^*(\beta M+\boldsymbol{\alpha}\cdot\mathbf{p})\psi d\mathbf{x}$$

$$\mathcal{H}_\mu=\tfrac{1}{2}\int[\pi_i\pi_i+\nabla\varphi_i\cdot\nabla\varphi_i+\mu^2\varphi_i\varphi_i]d\mathbf{x}$$

$$\mathcal{H}_i=if\int\psi^*\beta\gamma^5\Phi\psi d\mathbf{x}.$$

Here ψ^* and ψ represent the wave functions of the

[2] L. I. Schiff, Phys. Rev. **80**, 137 (1950); **83**, 239 (1951); **84**, 1, 10 (1951). See also: W. Heisenberg, Z. Naturforsch. **5a**, 251 (1950); R. Finkelstein and M. Ruderman, Phys. Rev. **81**, 655 (1951); F. Bloch, Phys. Rev. **83**, 1062 (1951).

[3] See, for example, R. E. Marshak, Revs. Modern Phys. **23**, 137 (1951).

[4] E. C. Nelson, Phys. Rev. **60**, 830 (1941); F. J. Dyson, Phys. Rev. **73**, 929 (1948); K. M. Case, Phys. Rev. **76**, 1 (1949).

[5] The connection between φ_1, φ_2, π_1, π_2 and the usual variables φ, φ^*, π, π^* employed to represent the charged meson field is given by the following equations:

$$\varphi_1=(\varphi^*+\varphi)/\sqrt{2},\quad \varphi_2=i(\varphi^*-\varphi)/\sqrt{2},$$
$$\pi_1=(\pi+\pi^*)/\sqrt{2},\quad \pi_2=i(\pi-\pi^*)/\sqrt{2}.$$

proton-neutron field in the usual eight-component representation, M represents the nucleon mass, μ the meson mass, g is the coupling constant (in rationalized units), $\mathbf{p}$ is the operator $-i\nabla$, and units are employed in which $\hbar$ and c are unity. Quantities which are functions of position such as $\psi^*(\mathbf{x})$ and $\pi(\mathbf{x})$ will be primed when they are to be evaluated at a primed coordinate, *viz.*, $\psi^{*\prime}=\psi^*(\mathbf{x}')$, $\pi'=\pi(\mathbf{x}')$, etc.

We now wish to perform a unitary transformation on the Hamiltonian:

$$H=e^{iS}\mathcal{H}e^{-iS},$$

with the provision that the generator of the transformation S be so determined that the pseudoscalar coupling term is eliminated from the Hamiltonian in favor of a pseudovector coupling term. We assume that S can be written in the form

$$S=\int\psi^* s\psi d\mathbf{x},\quad s=\gamma^5(\Phi/\phi)w,$$

where $w'=w[\phi(x')]$ is a function of ϕ' only.

The actual execution of the unitary transformation is facilitated by the use of several general theorems on unitary transformations of this type whose statement and proof are given in the Appendix. By the use of Theorem III of the Appendix we may write

$$\begin{aligned}
e^{iS}&\int\psi^*(\beta M+if\beta\gamma^5\Phi)\psi d\mathbf{x}e^{-iS}\\
&=\int\psi^* e^{is}(\beta M+if\beta\gamma^5\Phi)e^{-is}\psi d\mathbf{x}\\
&=\int\psi^*\left(\cos w+i\gamma^5\frac{\Phi}{\phi}\sin w\right)(\beta M+if\beta\gamma^5\Phi)\\
&\qquad\times\left(\cos w-i\gamma^5\frac{\Phi}{\phi}\sin w\right)\psi d\mathbf{x}\\
&=\int\psi^*\left[\beta M\left(\cos 2w+\frac{f\phi}{M}\sin 2w\right)\right.\\
&\qquad\left.+if\beta\gamma^5\Phi\left(\cos 2w-\frac{M}{f\phi}\sin 2w\right)\right]\psi d\mathbf{x},
\end{aligned}$$

and the employment of the same theorem gives us

$$\begin{aligned}
e^{iS}&\int\psi^*\boldsymbol{\alpha}\cdot\mathbf{p}\psi d\mathbf{x}e^{-iS}=\int\psi^* e^{is}\boldsymbol{\alpha}\cdot\mathbf{p}e^{-is}\psi d\mathbf{x}\\
&=\int\psi^*\left(\cos w+i\gamma^5\frac{\Phi}{\phi}\sin w\right)\boldsymbol{\alpha}\cdot\mathbf{p}\\
&\qquad\times\left(\cos w-i\gamma^5\frac{\Phi}{\phi}\sin w\right)\psi d\mathbf{x}\\
&=\int\psi^*\left[\boldsymbol{\alpha}\cdot\mathbf{p}-\frac{\Phi}{\phi}\frac{dw}{d\phi}\boldsymbol{\sigma}\cdot\nabla\phi-\frac{1}{2}\sin 2w\boldsymbol{\sigma}\cdot\nabla\left(\frac{\Phi}{\phi}\right)\right.\\
&\qquad\left.-i\sin^2 w\frac{\Phi}{\phi}\boldsymbol{\alpha}\cdot\nabla\left(\frac{\Phi}{\phi}\right)\right]\psi d\mathbf{x}\\
&=\int\psi^*\left[\boldsymbol{\alpha}\cdot\mathbf{p}-\frac{dw}{d\phi}\boldsymbol{\sigma}\cdot\nabla\Phi-\left\{\frac{1}{2}\sin 2w-\phi\frac{dw}{d\phi}\right\}\boldsymbol{\sigma}\cdot\nabla\left(\frac{\Phi}{\phi}\right)\right.\\
&\qquad\left.-\frac{i}{2}\sin^2 w\left\{\frac{\Phi(\boldsymbol{\alpha}\cdot\nabla\Phi)-(\boldsymbol{\alpha}\cdot\nabla\Phi)\Phi}{\phi^2}\right\}\right]\psi d\mathbf{x}.
\end{aligned}$$

We also have

$$e^{iS}\tfrac{1}{2}\int[\nabla\varphi_i\cdot\nabla\varphi_i+\mu^2\varphi_i\varphi_i]d\mathbf{x}e^{-iS}=\tfrac{1}{2}\int[\nabla\varphi_i\cdot\nabla\varphi_i+\mu^2\varphi_i\varphi_i]d\mathbf{x},$$

so that the only remaining term to be evaluated is $e^{iS}\frac{1}{2}\int\pi_i\pi_i d\mathbf{x}e^{-iS}$. To evaluate this term we write it as

$$\tfrac{1}{2}\int e^{iS}\pi_i e^{-iS}e^{iS}\pi_i e^{-iS}d\mathbf{x},$$

and then make use of Theorem I of the Appendix to write

$$\begin{aligned}
e^{iS}\pi_i e^{-iS}&=\pi_i+i\int\psi^{*\prime}[s',\pi_i]\psi' d\mathbf{x}'\\
&=\pi_i-\psi^*\gamma^5\frac{d\Phi}{d\varphi_i}\frac{dw}{d\phi}\\
&\quad-\psi^*\pi^5\left[\frac{1}{2}\sin 2w-\phi\frac{dw}{d\phi}\right]\frac{d}{d\varphi_i}\left(\frac{\Phi}{\phi}\right)\psi\\
&\quad-\frac{i}{2}\psi^*\frac{\sin^2 w}{\phi^2}\left[\Phi\frac{d\Phi}{d\varphi_i}-\frac{d\Phi}{d\varphi_i}\Phi\right]\psi.
\end{aligned}$$

On examining these results we note that the pseudoscalar coupling term can be eliminated by the choice

$$w(\phi)=\tfrac{1}{2}\tan^{-1}(f\phi/M).$$

Combining all of the above results, carrying out some simplifications, and introducing the abbreviation

$$M^*=M(1+f^2\phi^2/M^2)^{\frac{1}{2}},$$

the transformed Hamiltonian can be written as

$$H=e^{iS}\mathcal{H}e^{-iS}=\mathcal{H}_n+\mathcal{H}_\mu+H_1+H_2+H_3+H_4+H_5,$$

where $\mathcal{H}_n$ and $\mathcal{H}_\mu$ have the same meanings as above and

$$H_1=\int\psi^*(M^*-M)\psi d\mathbf{x},$$

$$H_2=-\frac{f}{4}\int\psi^*\left[\frac{M}{(M^*)^2}(\boldsymbol{\sigma}\cdot\nabla\Phi+\gamma^5\Pi)+(\boldsymbol{\sigma}\cdot\nabla\Phi+\gamma^5\Pi)\frac{M}{(M^*)^2}\right]\psi d\mathbf{x},$$

$$H_3=-\frac{f}{4}\int\psi^*\left[\frac{M^*-M}{\phi^2(M^*)^2}(\phi^2\tau_i-\Phi\varphi_i)(\boldsymbol{\sigma}\cdot\nabla\varphi_i+\gamma^5\pi_i)+(\boldsymbol{\sigma}\cdot\nabla\varphi_i+\gamma^5\pi_i)(\phi^2\tau_i-\Phi\varphi_i)\frac{M^*-M}{\phi^2(M^*)^2}\right]\psi d\mathbf{x},$$

$$H_4=-\frac{i}{8}\int\psi^*\left[\frac{M^*-M}{\phi^2M^*}(\Phi\tau_i-\tau_i\Phi)(\boldsymbol{\alpha}\cdot\nabla\varphi_i+\pi_i)+(\boldsymbol{\alpha}\cdot\nabla\varphi_i+\pi_i)(\Phi\tau_i-\tau_i\Phi)\frac{M^*-M}{\phi^2M^*}\right]\psi d\mathbf{x},$$

$$H_5=\frac{1}{8}\int\left\{\psi^*\left[\gamma^5\frac{fM\tau_i}{(M^*)^2}+\frac{f\gamma^5(M^*-M)}{\phi^2(M^*)^2}(\phi^2\tau_i-\Phi\varphi_i)+\frac{iM^*-M}{2\phi^2M^*}(\Phi\tau_i-\tau_i\Phi)\right]\psi\right\}^2 d\mathbf{x}.$$

We now discuss briefly the nature of the various interaction terms. H_1 represents a nonlinear spin-independent coupling of the meson field to the nucleons; its (meson) vacuum expectation value can be directly interpreted as a contribution to the renormalization of the nucleon mass. H_2 represents a simple pseudovector coupling term but with a nonlinear coefficient of a type which one would expect to lead to saturation effects in strong meson potentials. H_3 is a somewhat more complicated nonlinear pseudovector coupling term which vanishes in the two neutral meson theories. It is probably associated with the correlations existing between the emission of successive charged mesons resulting from the change in the isotopic spin coordinate of the nucleons. H_4 which also vanishes in the two neutral meson theories, represents a nonlinear vector coupling of the meson field to the nucleons of a rather unfamiliar character. H_5 represents a "contact" or "direct" interaction term; the second and third terms in the bracket in H_5 vanish for the two neutral meson theories.

It will be noted that when $\mathcal{H}_n$ and H_1 are combined they form a Hamiltonian corresponding to nucleons whose effective rest mass M^* is a function of position through its dependence on the meson potential: $M^*=(M^2+f^2\phi^2)^{\frac{1}{2}}$. In a succeeding paper some indications will be pointed out that this position dependence of the mass leads to such strong repulsion between nucleons at separations of the order of the meson Compton wavelength or smaller as possibly to preclude the existence of any bound states between two nucleons.

A final point which is worth mentioning is that the present calculations (as well as the earlier calculation of Dyson[4]) demonstrate that the distinction between a meson-nucleon coupling which is linear in the meson potential and one which is nonlinear is largely artificial in that a linear coupling in one representation of the Hamiltonian may be strongly nonlinear in another representation.

APPENDIX. SOME THEOREMS ON UNITARY TRANSFORMATIONS

Notation:

Let $\psi_\alpha^*(\mathbf{x})$ and $\psi_\alpha(\mathbf{x})$ with α running through the values $1, 2, \cdots, n$ be a set of operators which are functions of position and which obey the Jordan-Pauli anticommutation relations:

$$[\psi_\alpha^*(\mathbf{x}),\psi_\beta(\mathbf{x}')]_+=\delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{x}'),$$
$$[\psi_\alpha^*(\mathbf{x}),\psi_\beta^*(\mathbf{x}')]_+=[\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{x}')]_+=0.$$

The quantities $s_{\alpha\beta}(\mathbf{x})$ represent the matrix elements of an $n\times n$ Hermitian matrix $s(\mathbf{x})$. The matrix elements of the matrix s are assumed to commute with one another and with ψ^* and ψ. The quantity S is defined by

$$S=\int\psi_\alpha^*(\mathbf{x})s_{\alpha\beta}(\mathbf{x})\psi_\beta(\mathbf{x})d\mathbf{x},$$

where repeated indices are summed from 1 to n in accordance with the usual summation convention. The $\alpha\beta$ element of a function $f(s)$ of the matrix s will be written $\{f(s)\}_{\alpha\beta}$. For example,

$$\{e^{is}\}_{\alpha\beta}=\delta_{\alpha\beta}+is_{\alpha\beta}-s_{\alpha\gamma}s_{\gamma\beta}/2!-is_{\alpha\gamma}s_{\gamma\delta}s_{\delta\beta}/3!+\cdots.$$

The quantity $Q_{\alpha\beta}(\mathbf{x},\partial/\partial\mathbf{x})$ is the $\alpha\beta$ element of an $n\times n$ matrix which is not necessarily Hermitian. The matrix elements of Q are assumed to commute with $\psi^*(\mathbf{x}')$ and $\psi(\mathbf{x}')$, but because of the presence of the differentiation operators, they do not commute with $\psi^*(\mathbf{x})$ and $\psi(\mathbf{x})$; they do not necessarily commute with the matrix elements of s.

Where no confusion can arise, subscripts will be dropped from matrix products, *viz.*,

$$\psi_\alpha^*(\mathbf{x}')\{e^{is(\mathbf{x}')}\}_{\alpha\beta}Q_{\beta\gamma}(\mathbf{x})\{e^{-is(\mathbf{x}')}\}_{\gamma\delta}\psi_\delta(\mathbf{x}')\equiv\psi^*(\mathbf{x}')e^{is(\mathbf{x}')}Q(\mathbf{x})e^{-is(\mathbf{x}')}\psi(\mathbf{x}').$$

Position coordinates of which operators are functions will be suppressed by writing ψ_α^* for $\psi_\alpha^*(\mathbf{x})$ and $Q_{\alpha\beta}'$ for $Q_{\alpha\beta}(\mathbf{x}')$, for example.

The following readily established theorem will be used extensively in the proofs given below:

$$\begin{aligned}[\psi^*A\psi,\psi^{*\prime}B'\psi']&=[\psi_\alpha^*A_{\alpha\beta}\psi_\beta,\psi_\gamma^{*\prime}B_{\gamma\delta}'\psi_\delta']\\&=\psi_\gamma^{*\prime}\psi_\alpha^*[A_{\alpha\beta}B_{\gamma\delta}'-B_{\gamma\delta}'A_{\alpha\beta}]\psi_\beta\psi_\delta'\\&\quad+\psi_\alpha^*[A_{\alpha\beta}B_{\beta\gamma}'-B_{\alpha\beta}'A_{\beta\gamma}]\psi_\gamma\delta(\mathbf{x}-\mathbf{x}')\\&=\psi_\gamma^{*\prime}\psi_\alpha^*[A_{\alpha\beta},B_{\gamma\delta}']\psi_\beta\psi_\delta'\\&\quad+\psi_\alpha^*[A,B']_{\alpha\gamma}\psi_\gamma\delta(\mathbf{x}-\mathbf{x}').\end{aligned}\tag{A-1}$$

Here A and B' are two $n\times n$ matrices.

Theorem I:

$$e^{iS}\psi_\alpha e^{-iS}=\{e^{-is}\}_{\alpha\beta}\psi_\beta=\{e^{is}\psi\}_\alpha,$$

or

$$e^{iS}\psi e^{-iS}=e^{-is}\psi.$$

Proof: Define

$$\psi_\alpha{}^\xi=e^{i\xi S}\psi_\alpha e^{-i\xi S}.$$

Then

$$\psi_\alpha{}^{\xi+d\xi}=\psi_\alpha{}^\xi+(\partial\psi_\alpha{}^\xi/\partial\xi)d\xi=e^{id\xi S}\psi_\alpha{}^\xi e^{-id\xi S}$$
$$=\psi_\alpha{}^\xi+id\xi[S,\psi_\alpha{}^\xi],$$

so

$$\partial\psi_\alpha{}^\xi/\partial\xi=i[S,\psi_\alpha{}^\xi]. \tag{A-2}$$

We will now show that

$$\psi_\alpha{}^\xi=\{e^{-i\xi s}\}_{\alpha\beta}\psi_\beta \tag{A-3}$$

is a solution of Eq. (A-2).

We have on differentiating (A-3)

$$\partial\psi_\alpha{}^\xi/\partial\xi=-i\{e^{-i\xi s}\}_{\alpha\beta}s_{\beta\gamma}\psi_\gamma,$$

and on evaluating the commutator

$$[S,\psi_\alpha{}^\xi]=\int[\psi_\mu{}^{*\prime}s_{\mu\nu}{}'\psi_\nu{}'\{e^{-i\xi s}\}_{\alpha\beta}\psi_\beta-\{e^{-i\xi s}\}_{\alpha\beta}\psi_\beta\psi_\mu{}^{*\prime}s_{\mu\nu}{}'\psi_\nu{}']d\mathbf{x}'$$
$$=-\int\{e^{-i\xi s}\}_{\alpha\beta}\delta_{\beta\mu}\delta(\mathbf{x}-\mathbf{x}')s_{\mu\nu}{}'\psi_\nu{}'d\mathbf{x}'$$
$$=-\{e^{-\xi s}\}_{\alpha\beta}s_{\beta\gamma}\psi_\gamma.$$

Hence we see that the equation is satisfied. Since $\psi_\alpha{}^0=\psi_\alpha$, we obtain on setting $\xi=1$:

$$(\psi_\alpha{}^\xi)_{\xi=1}=e^{iS}\psi_\alpha e^{-iS}=\{e^{-is}\}_{\alpha\beta}\psi_\beta.$$

Corollary:

$$e^{iS}\psi^*e^{-iS}=\psi^*e^{is}.$$

Theorem II:

If

$$[s_{\alpha\beta}{}'',[s_{\gamma\delta}{}',Q_{\mu\nu}]]=0,$$

then

$$e^{iS}Q_{\mu\nu}e^{-iS}=Q_{\mu\nu}+\int\psi_\alpha{}^{*\prime}[\{e^{is'}\}_{\alpha\beta}Q_{\mu\nu}\{e^{-is'}\}_{\beta\gamma}-Q_{\mu\nu}\delta_{\alpha\gamma}]\psi_\gamma{}'d\mathbf{x}'$$
$$=Q_{\mu\nu}+i\int\psi_\alpha{}^{*\prime}[s_{\alpha\beta}{}',Q_{\mu\nu}]\psi_\beta{}'d\mathbf{x}'.$$

Proof: Define

$$Q_{\mu\nu}{}^\xi=e^{i\xi S}Q_{\mu\nu}e^{-i\xi S},$$

whence, as in the proof of Theorem I, one finds

$$\partial Q_{\mu\nu}{}^\xi/\partial\xi=i[S,Q_{\mu\nu}{}^\xi].$$

Then making use of the hypothesis together with (A-1) one can readily show that

$$Q_{\mu\nu}{}^\xi=Q_{\mu\nu}+i\xi\int\psi_\alpha{}^{*\prime}[s_{\alpha\beta}{}',Q_{\mu\nu}]\psi_\beta{}'d\mathbf{x}'$$

is a solution of the above equation and by the same reasoning as in Theorem I, one establishes the present theorem. Note that $Q_{\mu\nu}$ is not included in the matrix product occurring under the integral sign. This theorem also holds if $Q_{\mu\nu}$ is not a matrix element of an $n\times n$ matrix but still satisfies the hypothesis. Corollary: If the matrix element $Q_{\mu\nu}$ commutes with all matrix elements of s, then

$$e^{iS}Q_{\mu\nu}e^{-iS}=Q_{\mu\nu}.$$

Theorem III:

If all matrix elements of Q commute with all matrix elements of s then

$$e^{iS}\int\psi_\mu{}^*Q_{\mu\nu}\psi_\nu d\mathbf{x}e^{-iS}\equiv e^{iS}\int\psi^*Q\psi d\mathbf{x}e^{-iS}$$
$$=\int\psi^*e^{is}Qe^{-is}\psi d\mathbf{x}.$$

Proof: We have

$$e^{iS}\int\psi_\mu{}^*Q_{\mu\nu}\psi_\nu d\mathbf{x}e^{-iS}$$
$$=\int e^{iS}\psi_\mu{}^*e^{-iS}e^{iS}Q_{\mu\nu}e^{-iS}e^{iS}\psi_\nu e^{-iS}d\mathbf{x},$$

and using Theorem I and the corollary to Theorem II, we obtain

$$e^{iS}\int\psi^*Q\psi d\mathbf{x}e^{-iS}=\int\psi_\mu{}^*\{e^{is}\}_{\mu\alpha}Q_{\alpha\beta}\{e^{-is}\}_{\beta\nu}\psi_\nu d\mathbf{x}$$
$$\equiv\int\psi^*e^{is}Qe^{-is}\psi d\mathbf{x}.$$

This theorem could also be established by the methods employed in the proof of Theorems I and II.