Elementary view of fusion implosions

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(Received 9 May 1975; revised 28 July 1975)

The implications of some elementary solutions of the hydrodynamic equations for simple fluids are explored for scaling relations for—and insight into—the dynamics of imploded fusion targets. Because the model is so simple, analytical solutions are available and explicit formulas for such quantities as neutron yield, bremsstrahlung spectra and yield, etc., can be deduced. Where possible, comparison is made with the results of laser-fusion experiments being performed in the KMS Fusion laboratories.

INTRODUCTION

A class of elementary solutions of the hydrodynamic equations for a simple fluid is examined for implications for spherically symmetrically imploded plasmas. The mathematical model of the plasma to be used here is[1]

$$\frac{\partial n}{\partial t}+\nabla\cdot n\mathbf{w}=0, \tag{1a}$$

$$mn\left(\frac{\partial}{\partial t}+\mathbf{w}\cdot\nabla\right)\mathbf{w}+\nabla n\Theta=0, \tag{1b}$$

$$n\left(\frac{\partial}{\partial t}+\mathbf{w}\cdot\nabla\right)\Theta+\tfrac{2}{3}n\Theta\nabla\cdot\mathbf{w}=0, \tag{1c}$$

where n, $\mathbf{w}$, and Θ are the density, velocity, and energy of the fluid, respectively. Since the model neglects viscosity, heat flow, coupling to the external world, and treats the two-component (at least) plasma as a simple fluid, physical interpretation of any solutions obtained should be inferred cautiously. Nevertheless, since simple analytical solutions are readily available, some insight may be gained into the dynamics of implosions, and, more importantly, some potentially useful scaling relations for measured (or measurable) quantities can be easily deduced, for example, neutron yields from imploded D–T plasmas. We reiterate that we do not mean to argue that this mathematical model has any a priori, peculiar relevance to the description of laser-driven implosions. We employ it because analytical solutions are available, and appeal to results obtained for justification (or lack of it) for its use.

Assuming spherical symmetry, that the fluid is entirely contained in a sphere of radius $R(t)$, that $\mathbf{w}=(\mathbf{r}/R)\dot{R}=\mathbf{r}Q$, where $Q=d\ln R/dt$, and that $n(r,t)=\psi(t)h(y)$ and $\Theta(r,t)=\theta(t)f(y)$, where $y\equiv r/R$, we find

$$n=Nh/R^3, \tag{2}$$

$$\theta(t)R^2(t)=\theta_0R_0^{\,2}, \tag{3}$$

$$R^2(t)=R_0^{\,2}[1+(\alpha v_0^{\,2}/R_0^{\,2})t^2], \tag{4}$$

$$v_0^{\,2}=\theta_0/m_I, \tag{5}$$

where

$$\alpha\equiv-4\pi\int_0^1 y\,dy\,\frac{dhf}{dy}\,. \tag{6}$$

Here, N is the total number of particles with density profile $h(y)$, and h and f are fairly arbitrary except that h is subject to the normalization

$$4\pi\int_0^1 y^2\,dy\,h(y)=1. \tag{7}$$

The quantities with the subscript zero are evaluated at $t=0$, the time when compression ceases and expansion begins. Hereafter we make the isothermal assumption and set $f=1$. [This guarantees that the thermal energy in the fluid at any instant is $N\theta(t)$.] Actually, if we impose the reasonable requirement that the sum of the directed energy and the thermal energy be constant, we find that $f=1-y^2$. Similar arguments led to the same conclusion in Ref. 1. Nevertheless, we use the $f=1$ assumption throughout, since the quantitative effect of the parabolic y dependence is insignificant given the oversimplifications of the model.

The number of neutrons produced in a D–T plasma contracting and expanding according to Eqs. (2)–(6) is (neglecting fuel depletion)

$$N_n^{\,\mathrm{DT}}=\int_{-t_1}^{t_1}dt\int d^3r\,n_{\mathrm{D}}n_{\mathrm{T}}\langle v\sigma_{\mathrm{DT}}\rangle$$
$$=(8\pi N^2\langle v\sigma_{\mathrm{DT}}\rangle_0/R_0^{\,2}v_0)IK, \tag{8}$$

where

$$I\equiv\int_0^{x_1}\frac{dx}{(1+x^2)^{3/2}}G\left(\frac{1}{1+x^2}\right), \tag{9}$$

$$x\equiv\alpha^{1/2}v_0t/R_0, \tag{10}$$

$$\langle v\sigma_{\mathrm{DT}}\rangle\equiv\langle v\sigma_{\mathrm{DT}}\rangle_0G(\theta/\theta_0),\quad G(1)=1, \tag{11}$$

$$K \equiv \alpha^{-1/2} \int_0^1 dy\, y^2 h^2(y), \tag{12}$$

and where we have assumed that $n_D(r,t) = n_T(r,t)$. For peak temperatures around 1 keV, $G(\theta/\theta_0) \approx (\theta/\theta_0)^{13/2}$, and x_1 may be chosen large enough so that I in Eq. (9) is simply a number slightly less than ½. Equation (9) may also be displayed as

$$N_n^{\rm DT} = (128\pi^3/9) n_i^2 c^{2/3} R_i^4 (\langle v\sigma_{\rm DT}\rangle_0 / v_0) IK, \tag{13}$$

where $n_i = n_{Di} = n_{Ti}$ is the initial density of the deuterium (and tritium), $c \equiv (R_i/R_0)^3$ is a compression number, and R_i is the initial radius of the sphere before compression begins. However, the expression in Eq. (13) is more restrictive than the one in Eq. (8) in that the former implies a spatially uniform fuel density. Numerical estimates for comparison with observation are presented later.

The electromagnetic energy in dE about E radiated per second from a homogeneous sphere of radius R is[2]

$$I(E)\,dE = 4\pi R^2 \frac{\mu E^3\, dE}{8\pi^3 \hbar^3 c^2 [\exp(E/\theta) - 1]} \times \{\tfrac{1}{2} + \exp(-k)/k - [1 - \exp(-k)/k^2\}, \tag{14}$$

where

$$k = 2R\epsilon[\exp(E/\theta) - 1]/c, \tag{15}$$

ϵ is the probability per second for photon emission by electrons in the sphere, and μ is the index of refraction of the photons of energy E in the medium in the sphere. The total energy radiated from our contracting and expanding sphere is therefore

$$E_{\rm rad} = \int_{-t_1}^{t_1} dt \int \frac{d^3r}{V_s} \int_0^\infty dE\, I(E, t), \tag{16}$$

where the time dependence of $I(E,t)$ stems from its dependence upon $R(t)$, $\theta(t)$, and $n(t)$ (the latter through ϵ), and V_s is the time-dependent volume of the sphere. The integrals in Eq. (16) probably cannot be performed analytically if I is given by Eq. (14). However, if we assume only bremsstrahlung radiation and that over the important energy range $k << 1$, that is, the sphere is transparent (which also implies $\mu \approx 1$), we find that Eq. (16) can be integrated to yield[3]

$$E_{\rm rad} \simeq 4.9 \times 10^{-30} (Z^3 \theta_0^{1/2} / v_0 R_0^2) N_I^2 K \text{ (joules)}, \tag{17}$$

for θ_0 in keV. It is noteworthy that $(\theta_0^{1/2}/v_0)$ is independent of temperature; hence, the total bremsstrahlung yield (in the transparent limit) depends on temperature only to the extent that the effective Z of the ions (more accurately, an appropriate average value) depends upon temperature. Here, N_I is the total number of ions with effective charge number Z. Because the transparent limit was taken in the integrand of Eq. (16), the result in Eq. (17) is no longer restricted to homogeneous media. Again, numerical estimates based on Eq. (17) are discussed later.

The time-integrated bremsstrahlung energy spectrum is also of interest. Again we content ourselves with the transparent limit of Eq. (14) to obtain, for the power radiated per cm³ per unit energy at t,

$$\frac{I(E,t)}{V_s} = \frac{(2/\pi)^{1/2}}{6\pi} \frac{Z^3 e^6 n_I^2}{\hbar m^{3/2} c^3} \frac{\exp(-z) K_0(z)}{\theta^{1/2}}, \tag{18}$$

where $z = E/2\theta$. Integrating over space and time, we find

$$P(E) = \int_{-t_1}^{t_1} dt \int \frac{d^3r}{V_s} I(E,t) = \frac{2}{3}\left(\frac{2}{\pi}\right)^{1/2} \frac{Z^3 e^6 N_I^2 K}{m^{3/2} \hbar c^3} \frac{\exp(-z_0) Q(z_0)}{v_0 \theta_0^{1/2} R_0^2}, \tag{19}$$

where we have defined

$$Q(z_0) \equiv \int_{-x_1}^{x_1} \frac{dx \exp(-z_0 x^2)}{(1+x^2)} K_0[z_0(1+x^2)]. \tag{20}$$

Because the function $[\exp(-z_0x^2)]/(1+x^2)$ is so sharply peaked at the origin, the quantity Q can probably be adequately approximated as

$$Q(z_0) \simeq K_0(z_0) \int_{-\infty}^{\infty} \frac{dx \exp(-z_0 x^2)}{(1+x^2)} \simeq \left(\frac{\pi}{z_0+1}\right)^{1/2} K_0(z_0). \tag{21}$$

Thus the implosion-integrated spectrum in Eq. (19) is not markedly different in its energy dependence ($z_0 = E/2\theta_0$) from that of Eq. (18).

Another potentially interesting quantity is the number of D–T neutrons produced during the compression of a purely deuterium-fueled target,[4] or, perhaps more conveniently, the ratio $R_{\rm DD}^{\rm DT} \approx N_n^{\rm DT}/N_n^{\rm DD}$. Neglecting the burn-up of either the deuterium or the tritium, but taking due account of the fact that the only tritium present is due to D–D reactions, this ratio may be displayed as

$$R_{\rm DD}^{\rm DT} = \Big[\int_{-t_1}^{t_1} dt \int d^3r \langle v\sigma_{\rm DT}\rangle n_{\rm D}(t) \times \int_{-t_1}^{t} dt' \langle v\sigma_{\rm DD}\rangle' n_{\rm D}^2(t') \Big] \times \Big[\int_{-t_1}^{t_1} dt \int d^3r \langle v\sigma_{\rm DD}\rangle n_{\rm D}^2(t) \Big]^{-1}. \tag{22}$$

Recalling Eqs. (2)–(5), we find that we may reduce Eq. (22) to the expression

$$R_{DD}{}^{DT} = N_D \langle v\sigma_{DT}\rangle_0 L\eta / v_0 R_0{}^2 K, \qquad (23)$$

where Eqs. (11) and (12) have been resorted to and three new quantities have been introduced, namely,

$$\langle v\sigma_{DD}\rangle \equiv \langle v\sigma_{DD}\rangle_0 F(\theta/\theta_0), \quad F(1) = 1, \qquad (24)$$

$$L \equiv \alpha^{-1} \int_0^1 y^2\, dy\, h^3(y), \qquad (25)$$

and

$$\eta \equiv \left[\int_{-x_1}^{x_1} dx\, G\left(\frac{1}{1+x^2}\right) \int_{-x_1}^{x} \frac{dx'}{(1+x'^2)^3} F\left(\frac{1}{1+x'^2}\right)\right] \times \left[\int_{-x_1}^{x_1} \frac{dx}{(1+x^2)^{3/2}} F\left(\frac{1}{1+x^2}\right)\right]^{-1}, \qquad (26)$$

where, as before, $x = (\alpha^{1/2} v_0/R_0)t$. Here we have assumed the same density profile $h(y)$, for both deuterons and tritons, but it is otherwise arbitrary except for the normalization of Eq. (7). For θ_0 of the order of 1 keV, the functions G and F may be roughly approximated by

$$G(\theta/\theta_0) \simeq (\theta/\theta_0)^{13/2}, \quad F(\theta/\theta_0) \simeq (\theta/\theta_0)^{11/2}, \qquad (27)$$

in which case it is estimated that $\eta \approx 1/3$.

APPLICATIONS AND DISCUSSION

In order to obtain numerical results, we must evaluate certain integrals involving the density profiles of the particles participating in the contraction and expansion of the sphere of fluid. Initially, we assume a flat profile which is satisfactorily realized by modeling $h(y)$ by the formula

$$h(y) = A(1 - y^n) \qquad (28)$$

with n large. The constant A is determined by Eq. (7). With this choice for the density profile, it is found that $K = 1.1 \times 10^{-2}$.

Under these circumstances we may use Eq. (13) to estimate the neutron yield from an imploded D–T pellet. Evaluating $\langle v\sigma_{DT}\rangle_0/v_0$ at 0.5 keV, and making use of the fact that this function varies approximately like $\theta_0{}^6$ in this temperature range, we find

$$N_n{}^{DT} \simeq 4.6 \times 10^{-30} n_i{}^2 R_i{}^4 c^{2/3} (2\theta_0)^6. \qquad (29)$$

Some measurements of neutron yield have been made[5] for which $n_i = 9 \times 10^{20}$ atoms/cm³, $R_i = 3 \times 10^{-3}$ cm, $c = 10^3$, and $\theta_0 = 0.5$ keV. From Eq. (29) we find

$$N_n{}^{DT} = 3.0 \times 10^4 \text{ neutrons}, \qquad (30)$$

which is in excellent agreement with the observed yields. The temperature at peak compression is perhaps the least accurately known quantity needed for the yield calculation. If it were estimated to be 0.73 keV instead of 0.5 keV, the estimated yield would be 3.0×10^5—yields that have been observed several times. We remark that, if we had calculated Eq. (29) using $f = 1 - y^2$ instead of $f = 1$, the numerical factor in this equation would have been reduced by about one order of magnitude.

To estimate the bremsstrahlung energy radiated by these pellets, we may rewrite Eq. (17) in the form

$$E_{rad} \simeq 8.6 \times 10^{-29} (Z^3 \theta_0{}^{1/2} n_i{}^2 c^{2/3} R_i{}^4 K / v_0) \text{ (joules)}. \qquad (31)$$

Using the same values as before plus the facts that $Z = 1$ and $v_0 = 1.9 \times 10^7$ cm/sec, we find

$$E_{rad} \simeq 2.3 \times 10^{-4} \text{ J}. \qquad (32)$$

Suppose, however, there should be some $Z > 1$ contaminant uniformly mixed with the imploding fuel. If fn_i is the density of the contaminant, then Eq. (32) would be modified to read

$$E_{rad} \simeq 2.3 \times 10^{-4} (1 + fZ + fZ^2 + f^2 Z^3). \qquad (33)$$

In such an event, radiation losses would severely compete with fuel heating during the implosion.

It is worth noting here that, with the pellet parameters presently under discussion, the time required for contraction from 30 to 3 μm is, according to Eq. (4), 90 psec. The rate of neutron production is governed by the integrand of the integral displayed in Eq. (9). If again we approximate $G(\theta/\theta_0) \approx (\theta/\theta_0)^{13/2}$, the rate of neutron production is given by the function $(1 + x^2)^{-8}$. [Recall that $x^2 = (\alpha v_0{}^2/R_0{}^2)t^2$.] This is a very sharply peaked function about the origin, falling to half its value in 2.7 psec and to one-tenth in only 5.2 psec. Thus, according to this model, neutrons are produced only during a small fraction of the total time of contraction and expansion.

In Eq. (23) a formula was presented for the estimate of the ratio of the number of D–T neutrons to the number of D–D neutrons expected from the implosion of a purely deuterium-fueled target. In the present case of an essentially flat density profile, that formula may be rewritten as

$$R_{DD}{}^{DT} = (4\pi/3)(L\eta/K) n_i c^{2/3} \times R_i (\langle v\sigma_{DT}\rangle_0/v_0)_{1/2} (2\theta_0)^6, \qquad (34)$$

where the notation $(\langle v\sigma_{DT}\rangle_0/v_0)_{1/2}$ implies evaluation of the quantity at 0.5 keV, that is, $(\langle v\sigma_{DT}\rangle_0/v_0)_{1/2} = 1.9 \times 10^{-30}$ cm². Thus, by recalling that $\eta \approx 1/3$, $K = 1.1 \times 10^{-2}$, and evaluating $L = 1.5 \times 10^{-3}$, Eq. (34) reduces to

$$R_{DD}{}^{DT} = 3.6 \times 10^{-31} n_i c^{2/3} R_i (2\theta_0)^6. \qquad (35)$$

For $n_i = 9 \times 10^{20}$ cm^{-3}, $c = 10^3$, $R_i = 3 \times 10^{-3}$ cm, and $\theta_0 = 0.5$ keV, we find

$$R_{DD}{}^{DT} = 9.7 \times 10^{-11}. \quad (36)$$

The extreme smallness of this ratio is due firstly to the low peak temperature, and secondly to rather low initial density. However, most importantly the smallness of this ratio is due to the assumption that the tritons are in thermal equilibrium with the deuterons. If we had assumed that the tritons were all at an energy of about 50 keV, the ratio would be increased by a factor of the order of 10^7. Evidently, this ratio can provide a useful diagnostic only if the mean energy of the tritons can be rather accurately estimated (or an actual energy spectrum calculated) for a given implosion.

These implosions imply only a small amount of thermal energy in the plasma at peak compression. This thermal energy is

$$E_T = \tfrac{3}{2}(N_D + N_T + N_e)\theta_0$$
$$= 6N\theta_0, \quad (37)$$

where $N = N_D = N_T = 0.5N_e$. In these small targets, $N = 1.0 \times 10^{14}$ so that, for $\theta_0 = 0.5$ keV, $E_T = 48$ mJ.

Finally, we reiterate that this simple model neglects viscosity, heat flow, laser–plasma coupling, and effects due to charge separation among other things. To account for some or all of these phenomena, more physically realistic mathematical models must be employed and numerical analysis resorted to. In order to appreciate the greater complexity that the more realistic treatment implies, the reader is referred, for example, to the discussions of this subject by Nuckolls *et al.*[6] and Brueckner and Jorna.[7] It is our opinion that the main value of the analysis based upon this simple model is that it leads to analytic results which, in turn, lead to scaling relations which appear to be useful.

APPENDIX

The analysis required to proceed from the model equations [Eqs. (1a)–(1c)] to the solutions displayed in Eq. (2)–(6) is somewhat involved. Furthermore, our approach is mildly unconventional.[1] Thus, we here append an outline of this analysis as an aid to the interested (or skeptical) reader.

Given the model, it is then assumed that

$$\mathbf{W}(\mathbf{r}, t) = \mathbf{r}(\dot{R}/R)$$
$$= \mathbf{r}Q(t)$$
$$= \mathbf{y}\dot{R}(t), \quad (A.1)$$

where we have defined $Q \equiv \dot{R}/R = d \ln R/dt$ and $\mathbf{y} \equiv \mathbf{r}/R(t)$. Here, $R(t)$ is the time-dependent radius of the sphere of fluid. It is then further assumed that the other dependent variables n and $\oplus$ are separable functions of the new independent variables y and t; that is,

$$n(r, t) = n(y, t) = h(y)\psi(t), \quad (A.2)$$

and

$$\oplus(r, t) = \oplus(y, t) = f(y)\theta(t). \quad (A.3)$$

It then follows that

$$\frac{\partial}{\partial t} + \mathbf{W} \cdot \nabla \rightarrow \frac{\partial}{\partial t} \quad (A.4)$$

and Eqs. (1a)–(1c) reduce to

$$\frac{\partial n}{\partial t} + 3nQ = 0, \quad (A.5)$$

$$\hat{\mathbf{r}}\left[mny\frac{d^2R}{dt^2} + \frac{1}{R}\frac{\partial n\oplus}{\partial y}\right] = 0. \quad (A.6)$$

If one now makes use of the separability assumptions [Eqs. (A.2) and (A.3)], the solutions given in the text follow straightforwardly.

It is interesting to note that, if α defined in Eq. (6) is negative, our solutions are the same as those principally studied by Kidder in Ref. 1.

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