INVERSION OF ADJUNCTION IN HIGH CODIMENSION

by

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To the person I admire most, my grandfather Efim Hazan, of blessed memory.

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CHAPTER 1

Introduction

Singularities play a central role in modern algebraic geometry. There is now a large and subtle taxonomy of singularity classes that tends to focus on the properties of the canonical class.

The importance of the canonical class was already clear in classical algebraic geometry. In the classical picture, singularities were investigated through the conditions they impose on adjunction. Specifically, suppose first that A is a smooth variety and $X \subseteq A$ is a smooth hypersurface. Let $z_1, \ldots, z_n$ be local coordinates on A and let X be given as the set $\{f = 0\}$, where f is some polynomial on A. The Poincaré residue is the differential form

$$\omega = \left(\frac{dz_1 \wedge \cdots \wedge dz_{n-1}}{\partial f / \partial z_n} \right)_X . \tag{1.1}$$

Up to sign this form does not depend on the order of the z_i. If X is smooth then ω has no poles and it explicitly realizes the isomorphism

$$(K_A + X)_X \cong K_X.$$

Now suppose that X is singular but A is still smooth. We still have a differential form ω as in (1.1) to restrict to X. Suppose that we have a resolution of singularities $f : X' \to X$ and we consider $f^*\omega$. This differential form *a priori* has coefficients that are only rational functions, so $f^*\omega$ may very well have poles now. If it doesn't then, classically, the singularities were said to impose no conditions on adjunction. For example, the Du Val surfaces are precisely the surfaces whose singularities impose no conditions on adjunction.

Even if the differential form $f^*\omega$ has poles one may consider the ideal of functions $g \in \mathcal{O}_X$ so that $f^*(g \cdot \omega)$ is regular. This construction gives rise to the adjoint ideal. Here we can see how it measures the conditions on singularities imposed by adjunction and the Poincaré residue.

From the modern viewpoint, these considerations amount to a comparison between the canonical bundles of A, X and X'. It is natural to consider the expression

$$K_{X'} - f^*K_X.$$

This divisor is effective precisely when the singularities impose no conditions on adjunction. However, one must first define K_X for singular X and, if the definition does not produce a Cartier divisor, explain how to define the pullback f^*K_X.

To define K_X, let X be a normal variety of dimension n. Let X_{sm} be the open dense subset of the smooth points of X. The canonical class is constructed by taking any divisor $K_{X_{\mathrm{sm}}}$ so that

$$\Omega^n_{X_{\mathrm{sm}}} \cong \mathcal{O}_{X_{\mathrm{sm}}}(K_{X_{\mathrm{sm}}})$$

and taking the divisor on X induced by $K_{X_{\mathrm{sm}}}$ via topological closure in the Zariski topology. This is possible because, since X is normal, the singular locus of X is codimension two or more.

Unfortunately, for a singular variety the construction of K_X does not have to produce a Cartier divisor. We wish to pull back K_X to a resolution of singularities of X and it is not clear how to do this if K_X is not Cartier. A remedy for this problem is the fundamental construction in the theory of singularities of a pair (X, Δ). First, a $\mathbb{Q}$-divisor on X is just a Weil divisor with coefficients in $\mathbb{Q}$ instead of $\mathbb{Z}$. The set of $\mathbb{Q}$-divisors is a $\mathbb{Q}$-vector space in the obvious way. A $\mathbb{Q}$-divisor Δ is $\mathbb{Q}$-Cartier if some multiple $m \cdot \Delta$ has integer coefficients and is Cartier. A pair (X, Δ) is a normal variety X and a $\mathbb{Q}$-divisor Δ so that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. This notion originally arose in the study of boundaries of compactifications of quasi-projective varieties. However, we will want to think of Δ as some kind of error term that corrects some particularly unpleasant aspects of the singularities of X.

With the notion of pairs comes the notion of the singularity of a pair. Note well that, in this context, X can be smooth and (X, Δ) can be very singular. First we need a notion of resolving the singularities of (X, Δ). If $f : Y \to X$ is a morphism then, since $K_X + \Delta$ is $\mathbb{Q}$-Cartier, we can define

$$f^*(K_X + \Delta)$$

in the obvious way. Roughly speaking, a birational morphism f is a log-resolution (of singularities) of (X, Δ) if Y is smooth and the support of $f^*(K_X + \Delta)$, locally analytically near every point $p \in Y$, looks like a union of coordinate hyperplanes in $\mathbb{C}^n$ where n is the dimension of X. Let Δ_Y be the strict transform of Δ in Y. We will review the exact definitions in the conventions in Chapter 2.

Consider the expression

$$R = K_Y + \Delta_Y - f^*(K_X + \Delta).$$

This is a $\mathbb{Q}$-divisor, exceptional for f, that could reasonably be called the relative canonical class of f. If X is smooth and $\Delta = 0$ then all coefficients of R are positive. We do not expect this to happen for general pairs (X, Δ). For pairs, the coefficients of the divisors appearing in R is a generalization of the order of vanishing of Δ at a point. These coefficients measure, in some subtle way, how singular (X, Δ) is. Many singularity classes place bounds on how non-positive the coefficients can be. Two particularly important examples are klt (Kawamata log-terminal) singularities and log-canonical singularities. Klt requires that, for all choices of f, all coefficients of R are > -1 and log-canonical requires that, for all choices of f, all coefficients of R are ≥ -1.

In Section 3.4 we will review the Kawamata-Viehweg vanishing theorem and see why these numbers are important. To see roughly what is happening, suppose X is smooth and Δ is some effective $\mathbb{Q}$-divisor. Consider what happens to the pair $(X, c \cdot \Delta)$ as c ranges over the positive rationals. When c is very small, regardless of what Δ is, the pair $(X, c \cdot \Delta)$ is klt. There is exactly one value of c, called the log-canonical threshold of $(X, c \cdot \Delta)$, where this pair is log-canonical. Once c exceeds this value the pair is not log-canonical anymore.

In this light we see that we want to allow $\mathbb{Q}$-coefficients because we want a notion of smallness for the error and rational numbers can be arbitrarily small. Many singularity classes can be regarded as notions of smallness for Δ. We provide more detail in Chapter 3.

Of course, there are many more classes of singularities than just klt and log-canonical. For example, if $H \subseteq X$ is a reduced, irreducible, divisor in X there is a notion of (X, Δ) being plt (pure log-terminal) near H. We will not define this notion here (see Definition 3.3.4) but this class is very closely related to klt singularities. It can be thought of as allowing (X, Δ) to be log-canonical at the generic point of H but requiring the pair be klt everywhere else. There is a list of singularity classes in Definition 3.3.4 but this list makes no pretense at completeness.

Another class of singularities that will be important to us are Gorenstein and

$\mathbb{Q}$-Gorenstein singularities. A normal variety X is Gorenstein if K_X is Cartier (this is not quite true, Gorenstein requires that the dualizing sheaf ω_X be invertible and this implies that K_X is Cartier) and $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

The next important ingredient is the adjunction formula. This formula imposes non-trivial conditions on the possible singularities of subvarieties. The classical adjunction formula applies to a smooth variety Z contained in a smooth variety X and computes K_Z in terms of $(K_X)_Z$ and the normal bundle of Z in X. So, if we have a presentation of Z in terms of defining equations, we can compute its canonical class and get a lot of information about Z.

There are other ways to present a subvariety. One way that is particularly important is as a locus at which some effective Weil divisor has some fixed order of vanishing or, in more precise language, as a center of log-canonical singularities of a pair. In this situation there is also an adjunction type formula known as the canonical bundle formula of Kodaira and its generalization to higher dimensions. This formula is reviewed in Chapter 5. In Chapter 7 we will give a new adjunction formula for the relative canonical class of certain special resolutions.

We will concentrate on a phenomenon called inversion of adjunction. While the adjunction formula is, in some sense, classical fact, the inversion of adjunction phenomenon is more recent and more subtle. Adjunction imposes conditions on singularities of subvarieties by looking at the singularities of the ambient space. On the other hand, inversion of adjunction goes backwards and predicts the singularities of the ambient space from the singularities of subvarieties. This is surprising since the singularity classes are defined in terms of all subvarieties of X. Thus, it seems initially that singularities of Z seem to only impose conditions on subvarieties of X contained inside Z, leaving out the subvarieties that intersect Z properly.

Specifically, inversion of adjunction is a set of theorems describing how singularity classes behave under restriction to subvarieties. For example, if (X, Δ) is a pair and $H \subseteq X$ is an irreducible divisor, then (X, Δ) is plt near H if and only if (H, Δ_H) is klt. Inversion of adjunction is typically a consequence of an appropriate adjunction formula combined with a vanishing theorem for cohomology. For a review of inversion of adjunction, including a proof of this statement see Chapter 4.

Inversion of adjunction is a crucial part of modern birational geometry. It has played a role in virtually all recent progress in the minimal model program, including the celebrated work of Birkar, Cascini, Hacon and McKernan in [3], Hacon and McKernan in [13] and [14], Siu in [31], and Takayama in [34], among many others. It is thus natural to investigate inversion of adjunction more deeply because it is intrinsically interesting and because we hope to apply it in the future.

The form of inversion of adjunction that we will focus on calculates a subtle and important invariant of singularities called the multiplier ideal. The multiplier ideal is an ideal sheaf on X associated to a pair structure (X, Δ). The deeper the ideal at a given point, the worse the singularity of the pair at that point. In this language, if Z is a subvariety of X, (X, Δ) is a pair and (Z, Δ_Z) is an appropriately chosen pair structure on Z induced by Δ (often Δ_Z is simply the restriction of Δ to Z), inversion of adjunction calculates the multiplier ideal of (Z, Δ_Z) in terms of a multiplier-like adjoint ideal on X that depends only on Δ.

Most known forms of inversion of adjunction apply to irreducible divisors, that is, subvarieties of codimension one. There are some more recent statements, some of which are reviewed in Chapter 4, that apply to certain subvarieties of higher codimension. Our primary contribution is to give some new statements and methods of proof for this higher codimension situation.

In Chapters 5 and 6 we propose to connect these questions to another famous theorem - Kawamata's subadjunction theorem. A similar connection was made in [20], where the author proves an L^2-extension theorem by analytic methods. We prove a similar theorem by algebraic methods in Chapter 8.

Kawamata's subadjunction theorem is an analog of the adjunction formula for so-called exceptional log-canonical centers. A log-canonical pair has a finite collection of subvarieties of X outside which it is klt. These subvarieties are called the log-canonical centers of (X, Δ) (see Definition 5.1.1 for the precise definition). An exceptional log-canonical center is a log-canonical center that is minimal with respect to inclusion and satisfies a technical condition that can always be achieved by perturbing Δ slightly (see Definition 5.1.3). If X is smooth then many subvarieties of X can be exceptional log-canonical centers and *any* subvariety of X, not even necessarily normal, can be written as a generically exceptional log-canonical center of some Δ (see Example 5.1.5). In this case, one can tautologically write

$$\nu^*(K_X + \Delta)_Z \sim_{\mathbb{Q}} K_{Z^n} + \Delta_{Z^n}$$

where $\nu : Z^n \to Z$ is the normalization of Z and Δ_{Z^n} is some sort of non-unique error term.

With this setup, Kawamata's celebrated subadjunction theorem says the following. Suppose A is an ample divisor and $0 < \varepsilon \ll 1$ is a small rational number. Suppose further that (X, Δ) is a pair with $Z \subseteq X$ an exceptional log-canonical center. Then Z is normal and we can choose a Weil $\mathbb{Q}$-divisor Δ_Z on Z so that

$$(K_X + \Delta + \varepsilon A)_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$$

with (Z, Δ_Z) klt. In our context, we can regard this theorem as saying that the error term Δ_Z is small.

In Chapter 6 we will consider the situation of generically exceptional log-canonical centers (recall that if X is smooth then any subvariety $Z \subseteq X$ is such a center for some Δ). In this situation we construct an analogous formula with a specifically constructed Δ_Z that we will call a suitably chosen Kawamata different. We will define an adjoint ideal $\mathrm{adj}_Z(X, \Delta)$, analogous to a multiplier ideal, that measures the failure of Z to be actually exceptional for Δ. By analogy with inversion of adjunction in codimension one, we will prove a theorem that allows us to calculate the multiplier ideal of Δ_{Z^n} in terms of this adjoint ideal on X. Specifically, the statement is as follows.

Theorem (Theorem 6.1.1)**.** *Let $\nu : Z^n \to Z$ be the normalization of Z. Let Δ_{Z^n} be a suitably chosen Kawamata different for Z, as in Definition 5.4.4.*

Recall that $K_{Z^n} + \Delta_{Z^n}$ is $\mathbb{Q}$-Cartier and so we may consider $\mathcal{J}(Z^n, \Delta_{Z^n})$. Then:

1. *$\mathcal{J}(Z^n, \Delta_{Z^n})$ is contained in the conductor ideal of ν.*

2. *The conductor is also an ideal on Z and so $\mathcal{J}(Z^n, \Delta_{Z^n})$ is naturally an ideal on Z. With this identification, we have that*

$$\mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_{Z^n} = \mathcal{J}(Z^n, \Delta_{Z^n}),$$

3. *We have the natural exact sequence*

$$0 \to \mathcal{J}(X, \Delta) \to \mathrm{adj}_Z(X, \Delta) \to \mathcal{J}(Z^n, \Delta_{Z^n}) \to 0.$$

This theorem is a stronger version of Kawamata's theorem. In particular, Kawamata's original subadjunction theorem can be quickly deduced from this. This theorem can be regarded as a form of inversion of adjunction that applies to arbitrary subvarieties $Z \subseteq X$, as well as a description of the mysterious Kawamata different.

Then, in Chapter 7, we consider the theorems of Takagi from [32] and [33]. In these papers, S. Takagi investigates the case of a subvariety $X \subseteq A$ with A smooth and X $\mathbb{Q}$-Gorenstein. Recall that the $\mathbb{Q}$-Gorenstein condition simply means that K_X is $\mathbb{Q}$-Cartier, while Gorenstein is the condition that the dualizing sheaf is invertible, so in particular K_X is Cartier. He also defines an adjoint ideal $\mathrm{adj}_X(A, \Delta)$ and proves that, if X is Gorenstein, then there is a similar inversion of adjunction formula with an error term

$$\mathrm{adj}_X(A, \Delta) \cdot \mathcal{O}_X = \mathcal{J}(X, \Delta_X + \mathbb{V}(J_1))$$

where J_1 is the l.c.i.-defect sheaf (see Section 7.2 for a description of the basic theory of these sheaves). Takagi's proof proceeds by reduction to positive characteristic and the application of tight closure techniques.

Here we present an alternative approach to this theorem that uses only standard characteristic zero methods: resolution of singularities and Kawamata-Viehweg vanishing. We extend the formula to the $\mathbb{Q}$-Gorenstein case and prove that, if X is only $\mathbb{Q}$-Gorenstein with Gorenstein index r (that is, rK_X is Cartier) then

$$\mathrm{adj}_X(A, \Delta) \cdot \mathcal{O}_X = \mathcal{J}\left(X, \Delta_X + \frac{1}{r}\mathbb{V}(J_r)\right).$$

We deduce this from a simple trick with the Leray spectral sequence and an adjunction formula for relative canonical classes of special kinds of embedded resolutions of singularities that appears to be new (see Section 7.4 for a the notion of a strong factorizing resolution):

Theorem (Theorem 7.6.6)**.** *Let A be a smooth variety and let X be a generically smooth equidimensional subscheme. Let $\pi : \overline{A} \to A$ be a factorizing resolution of X inside A and let f be the restriction of π to $\overline{X}$, the strict transform of X along π.*

Write

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_X).$$

Suppose that X is $\mathbb{Q}$-Gorenstein with a Gorenstein index r. Suppose further that f is a log-resolution of $\mathcal{I}_{r,X}$ and J_r. Let D be the divisor defined by

$$J_r \cdot \mathcal{O}_{\overline{X}} = \mathcal{O}_{\overline{X}}(-D).$$

Then

$$K_{\overline{X}/X} - \frac{1}{r}D = (K_{\overline{A}/A} - cR_X)_{|\overline{X}}$$

with equality being equality of $\mathbb{Q}$-divisors on $\overline{X}$.

The proof of this formula consists of writing down carefully chosen differential forms and analyizing their transformation under birational morphisms and restrictions. This produces a formula for determinantal ideals that are similar to the Jacobian ideal, true for arbitrary X, that can then be translated into a formula for relative canonical classes if X is $\mathbb{Q}$-Gorenstein.

Using similar techniques we can also provide a characteristic zero proof of Takagi's subadditivity theorem in our Theorem 7.8.4: if X is $\mathbb{Q}$-Gorenstein and Δ_1, Δ_2 are effective $\mathbb{Q}$-divisors then

$$\mathrm{Jac}_X \cdot \mathcal{J}(X, \Delta_1 + \Delta_2) \subseteq \mathcal{J}(X, \Delta_1) \cdot \mathcal{J}(X, \Delta_2).$$

Finally, in Chapter 8 we investigate a powerful application of inversion of adjunction - the extension theorem for pluri-canonical forms. Specifically, we extend from $Z \subseteq X$ pluri-canonical sections of Cartier divisors of the form $K_X + A + \Delta$ where A is big and nef and (X, Δ) is log-canonical with exceptional log-canonical center Z:

Theorem (Theorem 8.5.6)**.** *Let X be a smooth projective variety and let A and Δ be $\mathbb{Q}$-divisors such that*

1. A is big and nef,

2. (X, Δ) is log-canonical with an exceptional log-canonical center Z,

3. $M = K_X + A + \Delta$ is Cartier.

Then the map

$$H^0(X, mM) \to H^0(Z, mM_Z)$$

is surjective for all $m \geq 1$.

To prove this, and out of independent interest, we investigate a construction due to Christopher Hacon in [15]. This construction is a generalization of the asymptotic multiplier ideal, reviewed in Section 3.5. We begin by working out a version of this idea in the setting of incomplete linear series. We then write down some basic statements about the case of complete linear series in the construction of the Hacon ideal. We obtain an ideal $\mathcal{J}_{-}(X, c \cdot ||M||)$ that we call the restricted multiplier ideal. We prove a vanishing theorem for this ideal and generalize of a theorem of Goodman regarding the detection of nef line bundles by multiplier ideals.

CHAPTER 2

Conventions

- We will work entirely over $\mathbb{C}$.
- We let X denote a normal projective variety over $\mathbb{C}$. $\mathbb{C}(X)$ denotes the function field of X. A variety is an integral separated scheme of finite type over k. We may use the terms *reducible* variety to denote a reduced separated scheme of finite type over k.
- An irreducible divisor $H \subseteq X$ is a reduced, irreducible subvariety of codimension one. It may not be normal and may not be defined by a single equation. A Weil divisor is then a $\mathbb{Z}$-linear combination of irreducible divisors. If the ambient variety X is smooth we may write hypersurface where we mean irreducible divisor.
- A simple normal crossings variety is a possibly reducible variety X, with smooth irreducible components, so that locally analytically at every point of X there exists an isomorphism of X with a subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by unions of intersections of coordinate hyperplanes. A scheme X has simple normal crossings support if X_{red} is a simple normal crossings variety. We say that X has simple normal crossings with Y if $X \cup Y$ has simple normal crossings support. In particular, if X is a subscheme of a smooth variety A, then X has simple normal

crossings support if locally at every point $p \in A$ there exist regular parameters x_i so that the germ at p of the ideal sheaf of X is generated by elements of the form $x_{i_1}^{e_1} \cdots x_{i_s}^{e_s}$.

- If $\pi : X' \to X$ is a birational morphism then we write $\operatorname{exc}(\pi)$ for the set of points of X' at which π is not an isomorphism, endowed with the reduced scheme structure.

- If $\pi : X' \to X$ is a morphism of non-reduced schemes we will say that π is birational if it is an isomorphism on a dense open subset of X'.

- Suppose $\pi : X' \to X$ is a birational morphism. Then π is the blow-up of some ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_X$. Suppose Y is a subscheme of X thats not contained in $\operatorname{exc}(\pi)$. Blowing up the ideal $\mathcal{I} \cdot \mathcal{O}_Y$ gives a birational morphism $f : Y' \to Y$ and Y' is a subscheme of X', called the *strict transform* on Y along π. We will sometimes use *proper transform* as a synonym for strict transform.

  Suppose that Y is a subvariety not contained in $\operatorname{exc}(\pi)$. Then the strict transform of Y is also a subvariety. This subvariety Y' is simply the Zariski closure of $Y \setminus \operatorname{exc}(\pi)$ in X'.

- An embedded resolution of singularities of a generically smooth subscheme X contained in a possibly singular variety A is a birational morphism $\pi : A' \to A$ so that:

  1. A' is smooth and π is an isomorphism at every generic point of X.
  2. The set $\operatorname{exc}(\pi)$ is a divisor with simple normal crossings support.
  3. The strict transform of X in A', denoted X', is smooth and has simple normal crossings with $\operatorname{exc}(\pi)$.

Such a resolution exists whenever $X \not\subseteq A_{\text{sing}}$.

A factorizing resolution of singularities of $X \subseteq A$ as above is a birational morphism $\pi : A' \to A$ that is an embedded resolution of singularities of X in A so that, if X' is the strict transform of X in A', we have that

$$\mathcal{I}_X \cdot \mathcal{O}_{A'} = \mathcal{I}_{X'} \cdot \mathcal{L}$$

with $\mathcal{L}$ a line bundle and the support of $\mathcal{I}_X \cdot \mathcal{O}_{A'}$ is a simple normal crossings variety. If A is smooth these resolutions were shown to exist in [4]. We will show in Lemma 7.4.4 that the case of A singular and $X \not\subseteq A_{\text{sing}}$ follows formally from the smooth case.

Let Z be an $\mathbb{R}_{>0}$-linear combination of subschemes of A with no component of X contained in the support of Z. An embedded resolution of singularities $\pi : A' \to A$ as above is also a log-resolution of Z if $\pi^{-1}Z$ is a divisor with simple normal crossings support and $\operatorname{Supp}(\pi^{-1}Z) \cup \operatorname{exc}(\pi) \cup X'$ is a simple normal crossings variety.

- X is said to be $\mathbb{Q}$-Gorenstein if X is normal and there is some natural number r so that rK_X is a Cartier divisor. Any such r is called a Gorenstein index of X.

- The abbreviation l.c.i. stands for locally complete intersection. We say that a variety X is l.c.i. at a point $p \in X$ if the local ring $\mathcal{O}_{X,p}$ is a locally complete intersection ring. This is equivalent to saying that X is locally a complete intersection for any embedding $X \subseteq A$ with A smooth. Recall that X is locally a complete intersection in some smooth A if and only if it is locally a complete intersection in all smooth A.

- If $X \subseteq A$ is an equidimensional subscheme of a variety A we write $\operatorname{codim}_A(X)$ for the codimension of X in A.

- If $\mathcal{L}$ is a line bundle and $\mathcal{F}$ is a subsheaf of $\mathcal{L}$ then we can write $\mathcal{F} = \mathcal{I} \cdot \mathcal{L}$ for some ideal sheaf $\mathcal{I}$. We will say that $\mathcal{F}$ generates the ideal $\mathcal{I}$.

- If $\mathcal{I}$ is an ideal sheaf, we denote by $\mathbb{V}(\mathcal{I})$ the subscheme defined by $\mathcal{I}$.

- A multi-index of type
$$\binom{n}{m}$$
is an ordered list of integers $(i_1, \ldots, i_m)$ so that $i_s < i_{s+1}$ and $i_s \in [1, n]$ for all s. If I is a multi-index we write
$$dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_m}$$
as short-hand for differential forms.

- If D is a $\mathbb{Q}$-divisor on a smooth variety A and X is a subvariety not contained in the support of D we will write D_X for the intersection of D with X as a $\mathbb{Q}$-divisor on X. If $\mathcal{F}$ is a sheaf on A we will write $\mathcal{F}_X$ for $\mathcal{F} \otimes \mathcal{O}_X$.

- In diagrams of morphisms, the arrow
$$\hookrightarrow$$
denotes a closed immersion.

CHAPTER 3

Singularities of pairs and the multiplier ideal

Before we can proceed with the main body of the exposition and results, we need to recall some facts and fix some notation. That is the purpose of this chapter. Most of the facts here are standard and can be found in [7], [21], [25].

3.1 $\mathbb{Q}$-divisors and pairs

In this section we fix definitions for a basic construction for everything we will do - $\mathbb{Q}$-divisors. Recall that X was a normal projective variety over $\mathbb{C}$.

Definition 3.1.1. A Weil $\mathbb{Q}$-divisor on X is a finite linear combination

$$D := \sum_{i=1}^{n} a_i D_i$$

where n is a natural number, D_i are prime Weil divisors on X and $a_i \in \mathbb{Q}$. The set of all Weil $\mathbb{Q}$-divisors is a $\mathbb{Q}$-vector space in the evident way. If the Weil $\mathbb{Q}$-divisor D is such that $a_i \in \mathbb{Z}$ for all $1 \leq i \leq n$ we will say that D is a Weil divisor on X.

Definition 3.1.2. Let D_1, D_2 be two Weil $\mathbb{Q}$-divisors.

1. We say that D_1 is rationally equivalent to D_2, denoted $D_1 \sim_{\mathbb{Q}} D_2$, if there exists a natural number m so that mD_1 and mD_2 are Weil (integer) divisors and, as Weil divisors, they are rationally equivalent.

2. We similarly say that a Weil $\mathbb{Q}$-divisor D is $\mathbb{Q}$-Cartier if there is a natural number m so that mD is a Weil divisor that is Cartier as a Weil divisor.

$\mathbb{Q}$-Cartier divisors can be pulled back by morphisms as follows.

Definition 3.1.3. Let $f : X \to Y$ be a morphism of complex projective varieties and let D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. Let m be such that mD is a Cartier divisor and define

$$f^*(D) = \frac{1}{m} f^*(mD).$$

This is only a $\mathbb{Q}$-divisor in general. It is straightforward to check that this definition does not depend on the choice of m.

We can allow $\mathbb{R}$-coefficients, and this is often done, but a divisor with real coefficients cannot have its denominator cleared and our subsequent definitions become more subtle. See Section 1.3.B in [24] for the details.

We now proceed to the definition of pairs. First, we need to define the canonical class.

Definition 3.1.4. Let X a normal quasi-projective variety of dimension d. There is a canonical Weil divisor class K_X defined as follows. On the dense open smooth locus $X_{\mathrm{sm}} \subseteq X$, there is a canonical line bundle

$$\mathcal{O}_{X_{\mathrm{sm}}}(K_{X_{\mathrm{sm}}}) := \bigwedge^d \Omega_{X_{\mathrm{sm}}}.$$

Write

$$K_{X_{\mathrm{sm}}} = \sum_{i=1}^{s} a_i D_i.$$

We let K_X to be the closure of this divisor in X:

$$K_X = \sum_{i=1}^{s} a_i \overline{D_i}$$

where $\overline{D_i}$ is the topological closure of D_i in X with the induced reduced scheme structure.

Unfortunately, there is no reason why K_X has to be Cartier or even $\mathbb{Q}$-Cartier. In particular, if $f : Y \to X$ is a morphism, there is no obvious way to define f^*K_X.

There is a standard way to fix this problem.

Definition 3.1.5. Let X be a normal projective variety over $\mathbb{C}$ and let Δ be a Weil $\mathbb{Q}$-divisor on X. We say that (X, Δ) is a pair if $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Of course, without some restrictions on Δ we can write any Weil $\mathbb{Q}$-divisor this way, so we really need a notion of smallness for Δ. First we need to discuss an important subclass of divisors that will play a central role in our work.

3.2 Big line bundles

Here we recall the definitions and properties of big divisors.

Definition 3.2.1. Let X be a projective variety of dimension n and let $\mathcal{L}$ be a line bundle on X. We say that $\mathcal{L}$ is *big* if

$$\limsup_{m\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes m})}{m^n} > 0.$$

We define the *volume* of $\mathcal{L}$ to be

$$\mathrm{vol}_X(\mathcal{L}) = \limsup_{m\to\infty} \frac{h^0(X, \mathcal{L}^{\otimes m})}{m^n/n!}.$$

The Riemann-Roch formula immediately implies that, if $\mathcal{L}$ is ample, then

$$\mathrm{vol}_X(\mathcal{L}) = (\mathcal{L})^n,$$

where $(\mathcal{L})^n$ is the top self-intersection number of $\mathcal{L}$.

We summarize some of the basic facts about big line bundles. See Section 2.2 in [25] for the proofs, as well as [26] for an interesting new point of view on the entire theory.

Proposition 3.2.2. *Let X be a smooth projective variety of dimension n and let $\mathcal{L}$ be a line bundle on X. Then (all citations are from [24])*

1. *The* $\limsup$ *in the definition is always finite and is always a limit (Remark 2.2.50).*

2. *The property of being big depends only on the numerical class of $\mathcal{L}$ (Corollary 2.2.8).*

3. *All classes of the form $H + C$, H an ample $\mathbb{Q}$-divisor and C an effective $\mathbb{Q}$-divisor, are big. Conversely, every big $\mathbb{Q}$-divisor M can be written as*

$$M \sim_{\mathbb{Q}} H + C$$

   where H is ample and C is effective, H and C are $\mathbb{Q}$-divisors (Corollary 2.2.7).

4. *The numerical classes of big line bundles form an open pointed (not containing a line) cone in $N^1(X) \otimes \mathbb{Q}$, denoted* $\mathrm{Big}(X)$*. The closure of the cone of big classes is called the cone of pseudoeffective classes, denoted* $\overline{\mathrm{Big}(X)}$*.* A pseudoeffective class is not necessarily even numerically equivalent to an effective class *(Section 2.2.B).*

5. *The function vol : $N^1(X) \to \mathbb{R}$ is continuous on* $\overline{\mathrm{Big}(X)}$ *and is zero on the boundary of this cone (Corollary 2.2.45).*

6. *If $\pi : Y \to X$ is a birational morphism and $\mathcal{L}$ is a big line bundle then $\pi^*\mathcal{L}$ is also big (this is an immediate consequence of the projection formula).*

For every big line bundle there are associated three important subsets of X, see [7] for more information.

Definition 3.2.3. Let X be a smooth complex projective variety and let $\mathcal{L} = \mathcal{O}_X(D)$ be a big line bundle on X.

1. We define the *augmented base locus*

$$\mathbb{B}_+(D) = \bigcap_{\substack{D \equiv_{\text{num}} H+E \\ H,E \ \mathbb{Q}\text{-divisors} \\ H \text{ ample, } E \text{ effective}}} E.$$

   Note that, if $Z \subseteq X$ is a subvariety with $Z \not\subseteq \mathbb{B}_+(D)$ then $\mathcal{L}_Z$ is big.

2. We define the *stable base locus*

$$\mathbb{B}(\mathcal{L}) = \bigcup_{m \geq 1} \mathbf{B}(mD)$$

   where $\mathbf{B}(mD)$ denotes the base locus of mD.

3. We define the *restricted base locus*

$$\mathbb{B}_-(D) = \bigcup_{A \text{ ample } \mathbb{Q}\text{-divisor}} \mathbb{B}(D+A).$$

It is easy to see that $\mathbb{B}_+(D)$ and $\mathbb{B}_-(D)$ depend only on the numerical class of D while $\mathbb{B}(D)$ does not. We furthermore have the following easy sequence of inclusions

$$\mathbb{B}_+(D) \subseteq \mathbb{B}(D) \subseteq \mathbb{B}_-(D).$$

The sets $\mathbb{B}_+(D)$ and $\mathbb{B}(D)$ are subvarieties while $\mathbb{B}_-(D)$ is *a priori* a countable union of subvarieties. As of this writing it is not known whether $\mathbb{B}_-(D)$ is always a variety.

3.3 Singularities of pairs and multiplier ideals

In this section we recall the definitions of discrepancy and the associated singularity classes.

Definition 3.3.1. A divisor over X is a divisorial rank 1 discrete valuation $\nu : \mathbb{C}(X) \to \mathbb{Z}$ on the function field of X. By a theorem of Zariski (Lemma 2.45 in [21]) all such valuations are realized as follows - there exists a birational morphism $\pi : Y \to X$ and a prime divisor $E \subseteq Y$ so that $\nu(g) = \mathrm{ord}_E(\pi^* g)$ for all $g \in C(X)$. We say that π extracts E. We will often simply say that E is a divisor over X.

Remark 3.3.2. Note that E need not be exceptional for π in this definition.

Definition 3.3.3. Let (X, Δ) be a pair. Let E be a divisor over X and let $\pi : Y \to X$ be a log-resolution of Δ that extracts E. We define the discrepancy of (X, Δ) along E as

$$a(E; X, \Delta) := \mathrm{ord}_E \left(\lceil K_Y - \pi^*(K_X + \Delta) \rceil \right).$$

We define the *total discrepancy* of (X, Δ) at a (not necessarily closed) point η as

$$\mathrm{totaldiscrep}(\eta; X, \Delta) := \inf_{E, \eta \in \mathrm{center}(E)} a(E; X, \Delta).$$

If η is the generic point of X we simply write $\mathrm{totaldiscrep}(X, \Delta)$. We also want to define the total discrepancy of ideals. Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_s \subseteq \mathcal{O}_X$ be a finite collection of ideals and let $c_i \in \mathbb{Q}_{>0}$. Let $\pi : Y \to X$ be a log-resolution of (X, Δ) and of the $\mathfrak{a}_i$. Write $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$. We define

$$a\left(E; (X, \Delta), \mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_s^{c_s}\right) := \mathrm{ord}_E \left(K_Y - \pi^*(K_X + \Delta) - \sum c_i F_i \right).$$

If Z_i is a finite collection of subschemes of X and $a_i \in \mathbb{Q}_{>0}$ then we define

$$a\left(E; (X, \Delta), \sum_i a_i Z_i\right) := a\left(E; (X, \Delta), \prod \mathcal{I}_{Z_i}^{c_i}\right).$$

Finally, we define

$$\mathrm{totaldiscrep}\left(\eta; (X, \Delta), \sum_i a_i Z_i\right) := \inf_{E, \eta \in \mathrm{center}(E)} a\left(E; (X, \Delta), \sum_i a_i Z_i\right).$$

We will use the following three classes of singularities.

Definition 3.3.4. Let (X, Δ) be a pair and let $H \subseteq X$ be an irreducible divisor. We say that (X, Δ) is

1. klt if, for every divisor E over X, $a(E; X, \Delta) > -1$,
2. plt if $a(E; X, \Delta) > -1$ for every *exceptional* divisor over X,
3. plt along H if $a(E; X, \Delta + H) > -1$ for every divisor over X with center different from H,
4. log-canonical if $a(E; X, \Delta) \geq -1$ for every divisor over X.

We also recall the standard definitions of multiplier and adjoint ideals, see Section 9.2 in [25] for an excellent expanded discussion. First, the following lemma follows immediately from the definition.

Lemma 3.3.5. *Let $\pi : Y \to X$ be a birational morphism with Y smooth and let*

$$D = \sum a_i D_i$$

be a Weil divisor on Y. Let U be an open subset of X. Then we have the following description:

$$\begin{aligned} H^0(U, \pi_* \mathcal{O}_Y(D)) &= \left\{ f \in \mathbb{C}(X) \mid \pi^*(f) \in H^0(\pi^{-1}(U), \mathcal{O}_Y(D)) \right\} \\ &= \left\{ f \in \mathbb{C}(X) \mid ord_{D_i} \pi^*(f) \geq -a_i \right\}. \end{aligned}$$

Here we recall the definition of multiplier ideals. We will review their basic properties in the next section.

Definition 3.3.6. Let (X, Δ) be a pair and let $\pi : Y \to X$ be a log-resolution of Δ. We define

$$\mathcal{J}(X, \Delta) := \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta) \rceil).$$

We also want a definition for multiplier ideals of ideals, so let $\mathfrak{a}_1, \ldots, \mathfrak{a}_s \subseteq \mathcal{O}_X$ be a finite collection of ideals and let $c_i \in \mathbb{Q}_{>0}$ (or even in $\mathbb{R}_{>0}$). Let $\pi : Y \to X$ be a log-resolution of (X, Δ) and of the $\mathfrak{a}_i$. Write $\mathfrak{a}_i \cdot \mathcal{O}_Y = \mathcal{O}_Y(-F_i)$. We define

$$\mathcal{J}((X,\Delta); \mathfrak{a}_1^{c_1} \cdots \mathfrak{a}_s^{c_s}) := \pi_* \mathcal{O}_Y \left(\left\lceil K_Y - \pi^*(K_X + \Delta) - \sum c_i F_i \right\rceil \right).$$

If Z_i is a finite collection of subschemes of X and $a_i \in \mathbb{Q}_{>0}$ then we define

$$\mathcal{J}\left((X, \Delta); \sum_i a_i Z_i \right) := \mathcal{J}\left((X,\Delta); \prod_i \mathcal{I}_{Z_i}^{a_i} \right).$$

The following is straightforward to check from Lemma 3.3.5.

Theorem 3.3.7. *Let (X, Δ) be a pair and let $\pi : Y \to X$ be a log-resolution of Δ. Then*

$$\mathcal{J}(X, \Delta) = \mathcal{O}_X$$

if and only if (X, Δ) is klt and

$$\mathcal{J}(X, (1-\varepsilon)\Delta) = \mathcal{O}_X \text{ for all } \varepsilon \ll 1$$

if and only if (X, Δ) is log-canonical.

We will also discuss the adjoint ideal along an irreducible divisor.

Definition 3.3.8. Let (X, Δ) be a pair and let $H \subseteq X$ be a (reduced) irreducible divisor. Let $\pi : Y \to X$ be a log-resolution of $\Delta + H$. Let $H' \subseteq Y$ be the strict transform of H along π. We define

$$\mathrm{adj}_H(X, \Delta) := \pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H) + H' \rceil).$$

The straightforward analog of the previous theorem for adjoint ideals is the following.

Theorem 3.3.9. *Let* (X, Δ) *be a pair, let* $H \subseteq X$ *be an irreducible divisor and let* $\pi : Y \to X$ *be a log-resolution of* $\Delta + H$*. Then*

$$\mathrm{adj}_H(X, \Delta) = \mathcal{O}_X$$

if and only if (X, Δ) *is plt along* H*.*

3.4 Basic properties of multiplier and adjoint ideals

In this section we recall some basic facts about multiplier and adjoint ideals. The following lemma follows immediately from Lemma 3.3.5.

Lemma 3.4.1. *Let* (X, Δ) *be a pair and let* $\pi : Y \to X$ *be a log-resolution of this pair. Let* $U \subseteq X$ *be an open set and let* $f \in \mathcal{O}_X(U)$*. Then* $f \in \Gamma(U, \mathcal{J}(X, \Delta))$ *if and only if*

$$\mathrm{div}(\pi^* f) \leq K_Y - \pi^*(K_X + \Delta).$$

We begin with the following theorem. Due to its central importance and our later use of the proof, we provide a proof.

Theorem 3.4.2. *The definitions of* $\mathcal{J}(X, \Delta)$ *and* $\mathrm{adj}_H(X, \Delta)$ *are independent of the choice of log-resolution* $\pi : Y \to X$*.*

Proof. We will prove this for $\mathcal{J}(X, \Delta)$. The statement for $\mathrm{adj}_H(X, \Delta)$ is completely analogous and in any event we will prove a stronger version in Proposition 5.1.8.

So, suppose that $\pi_1 : Y_1 \to X$ and $\pi_2 : Y_2 \to X$ are two log-resolutions of (X, Δ). There exists a birational morphism $\pi : W \to X$ that factors through π_1 and π_2, that is, there is a diagram

$$\begin{tikzcd} & W \arrow[dl] \arrow[dr] & \\ Y_1 \arrow[dr, "\pi_1"'] & & Y_2 \arrow[dl, "\pi_2"] \\ & X & \end{tikzcd}$$

and so that π is a log-resolution of (X, Δ). If we can show that

$$\pi_{1,*}\mathcal{O}_{Y_1}(\lceil K_{Y_1} - \pi_1^*(K_X + \Delta)\rceil) = \pi_*\mathcal{O}_W(\lceil K_W - \pi^*(K_X + \Delta)\rceil)$$

then it follows by symmetry that

$$\pi_{1,*}\mathcal{O}_{Y_1}(\lceil K_{Y_1} - \pi_1^*(K_X + \Delta)\rceil) = \pi_{2,*}\mathcal{O}_{Y_2}(\lceil K_{Y_2} - \pi_2^*(K_X + \Delta)\rceil)$$

as required.

We are thus reduced to the following setup. Suppose we are given two maps $f : W \to Y$ and $g : Y \to X$ so that $\pi = f \circ g$ and g are log-resolutions of (X, Δ). Then we need to show that

$$\pi_*\mathcal{O}_W(\lceil K_W - \pi^*(K_X + \Delta)\rceil) = g_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta)\rceil).$$

We claim that it is enough to show that

$$\lceil K_W - \pi^*(K_X + \Delta)\rceil = f^*(\lceil K_Y - g^*(K_X + \Delta)\rceil) + B \tag{3.1}$$

with B an effective exceptional divisor. This is because the projection formula says that

$$\begin{aligned}\pi_*\mathcal{O}_W(\lceil K_{W/X} - \pi^*\Delta\rceil) &= \pi_*\mathcal{O}_W(f^*(\lceil K_{Y/X} - g^*\Delta\rceil) + B) \\ &= g_*(\mathcal{O}_Y(\lceil K_{Y/X} - g^*\Delta\rceil) \otimes f_*\mathcal{O}_W(B)) \\ &= g_*(\mathcal{O}_Y(\lceil K_{Y/X} - g^*\Delta\rceil)\end{aligned}$$

by Lemma 3.4.1.

To see (3.1), we first write

$$\lceil K_W - \pi^*(K_X + \Delta)\rceil = K_{W/Y} - \lfloor -f^*(K_Y - \pi^*(K_X + \Delta))\rfloor.$$

By adding the Cartier divisor $f^*\lfloor -(K_Y - \pi^*(K_X + \Delta))\rfloor$ to both sides of (3.1) we can assume that

$$\lfloor -(K_Y - \pi^*(K_X + \Delta))\rfloor = 0$$

and that $-(K_Y - \pi^*(K_X + \Delta))$ is an effective $\mathbb{Q}$-divisor. We are reduced to proving that, in this special case,

$$K_{W/Y} - \lfloor -f^*(K_Y - \pi^*(K_X + \Delta))\rfloor \geq 0.$$

But then part (3) of Corollary 2.31 in [21] applies to our situation and says that

$$a(E; X, -(K_Y - \pi^*(K_X + \Delta))) > -1$$

for all divisors E over X. After recalling the definition of the discrepancy we see that this is exactly the statement we are looking for. □

One of the crucial facts about the multiplier ideal formalism is the Kawamata-Viehweg vanishing theorem and its multiplier ideal version, known as the Nadel vanishing theorem. We will use the following statement of Kawamata-Viehweg vanishing (see Theorem 9.1.18 in [25] for the proof).

Theorem 3.4.3 (The Kawamata-Viehweg vanishing theorem)**.** *Let Y be a smooth complex projective variety and let $\mathcal{O}_Y(D)$ be a line bundle on Y. Suppose that we can write*

$$D \equiv_{\text{num}} K_Y + A + \Delta$$

with A a big and nef $\mathbb{Q}$-divisor and Δ a simple normal crossings $\mathbb{Q}$-divisor with $\lfloor \Delta \rfloor = 0$. Then

$$H^i(Y, \mathcal{O}_Y(D)) = 0 \text{ for all } i > 0.$$

One may think of Δ as some kind of error term. The following two theorems follow directly from the Kawamata-Viehweg vanishing theorem.

Theorem 3.4.4 (Local vanishing)**.** *Let $f : Y \to X$ be a proper birational morphism with X, Y projective varieties and Y smooth. Let D be a $\mathbb{Q}$-divisor that has simple normal crossings support and is numerically equivalent to $K_{Y'} + f^*D'$ where D' is any $\mathbb{Q}$-divisor on X. Then*

$$R^i f_* \mathcal{O}_{Y'}(\lceil D \rceil) = 0$$

for all $i > 0$.

Proof. An argument using the Leray spectral sequence and Serre vanishing shows that, if H is ample on X, then

$$H^i(Y', \mathcal{O}_{Y'}(\lceil D \rceil + f^*(mH))) = 0$$

for all $i > 0$ and $m \gg 0$. But $D' + mH$ is ample for all $m \gg 0$. Then

$$\lceil D \rceil + f^*(mH) \equiv_{\text{num}} K_{Y'} + f^*(D' + mH) + \Delta$$

where Δ is a simple normal crossings $\mathbb{Q}$-divisor with $\lfloor \Delta \rfloor = 0$. Since $D' + mH$ is ample and f is birational, $f^*(D' + mH)$ is big and nef. The required vanishing now follows from the Kawamata-Viehweg vanishing theorem. □

Theorem 3.4.5 (Nadel vanishing)**.** *Let (X, Δ) be a pair and let D be a Cartier divisor on X so that*

$$D \equiv_{\text{num}} K_X + \Delta + A$$

with A a big and nef $\mathbb{Q}$-divisor. Then

$$H^i(X, \mathcal{O}_X(D) \otimes \mathcal{J}(X, \Delta)) = 0$$

for all $i > 0$.

Proof. Let $f : Y \to X$ be a log-resolution of (X, Δ). By the projection formula,

$$\mathcal{O}_X(D) \otimes \mathcal{J}(X, \Delta) = f_* \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta - D) \rceil).$$

By the local vanishing theorem 3.4.4,

$$R^i f_* \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta - D) \rceil) = 0$$

for all $i > 0$. By the Leray spectral sequence,

$$H^i(X, \mathcal{O}_X(D) \otimes \mathcal{J}(X, \Delta)) = H^i(Y, \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta - D) \rceil)).$$

But the hypotheses imply that

$$K_X + \Delta - D \equiv_{\text{num}} -A,$$

and so

$$\lceil K_Y - f^*(K_X + \Delta - D) \rceil \equiv_{\text{num}} K_Y + f^*A + \Delta$$

where $\Delta = \{K_Y - f^*(K_X + \Delta - D)\}$ is a simple normal crossings divisor with $\lfloor \Delta \rfloor = 0$. Since f is a birational morphism, f^*A is again big and nef. It follows from the Kawamata-Viehweg vanishing theorem 3.4.3 that

$$H^i(Y, \mathcal{O}_Y(\lceil K_Y - f^*(K_X + \Delta - D) \rceil)) = 0$$

for all $i > 0$, as required. $\square$

3.5 Asymptotic multiplier ideals

Asymptotic multiplier ideals will play a central role in the extension theorem of the last chapter, so we recall them. See Chapter 11 of [25] for an excellent explanation of this material. First we will need the following setup.

Definition 3.5.1. Let X be a projective variety. A *graded system of ideals* is a sequence of ideals $\mathfrak{a}_i \subseteq \mathcal{O}_X$, $i \in \mathbb{N}$, with $\mathfrak{a}_i \cdot \mathfrak{a}_j \subseteq \mathfrak{a}_{i+j}$. Let $\mathcal{L}$ be a line bundle. A graded linear series W_l, $l \in \mathbb{N}$, is a sequence of subspaces

$$W_l \subseteq H^0(X, \mathcal{L}^{\otimes l})$$

so that

$$|W_i| + |W_j| \subseteq |W_{i+j}|.$$

Note that the ideals $\mathfrak{a}_l := \mathfrak{b}(W_l)$, where W_l is a graded linear series, are a graded system of ideals.

Now let (X, Δ) be a pair. To define the asymptotic multiplier ideal, let $\mathfrak{a}_i$ be a graded system of ideals and let c be a positive real number. We can then consider the ideals

$$\mathcal{J}_i = \mathcal{J}((X, \Delta); \mathfrak{a}_i^{c/i}).$$

Lemma 3.5.2. *For every $k \in \mathbb{N}$ we have the inclusion*

$$\mathcal{J}_i \subseteq \mathcal{J}_{ki}.$$

Proof. Fix a $k \in \mathbb{N}$. Let $\pi : Y \to X$ be a log-resolution of (X, Δ), $\mathfrak{a}_i$ and $\mathfrak{a}_{ki}$ simultaneously. Let $\mathcal{O}_Y(-F_i) = \mathfrak{a}_i \cdot \mathcal{O}_Y$ and similarly for F_{ki}. Since $\mathfrak{a}_i^k \subseteq \mathfrak{a}_{ki}$ we must have

$$kF_i \geq F_{ki}.$$

It follows that

$$\begin{aligned}
\mathcal{J}_i &= \mathcal{J}((X,\Delta);\mathfrak{a}^{c/i}) \\
&= \pi_*\mathcal{O}_Y\left(K_Y - \left\lfloor \pi^*(K_X+\Delta) + \frac{c}{i}F_i \right\rfloor\right) \\
&\subseteq \pi_*\mathcal{O}_Y\left(K_Y - \left\lfloor \pi^*(K_X+\Delta) + \frac{c}{ki}F_{ki} \right\rfloor\right) \\
&= \mathcal{J}((X,\Delta);\mathfrak{a}^{c/ki}) = \mathcal{J}_{ki}.
\end{aligned}$$

□

This lemma is the crucial point in the construction of the asymptotic multiplier ideal.

Theorem 3.5.3. *The family of ideals $\mathcal{J}_i$ has a unique maximal element that we will call the asymptotic multiplier ideal of the graded system of ideals $\mathfrak{a}_\bullet$ and denote*

$$\mathcal{J}((X,\Delta);c\cdot\mathfrak{a}_\bullet).$$

This ideal is computed by the ideals $\mathcal{J}_k$ for k divisible by some (potentially large) integer k' that depends on the ideals $\mathfrak{a}_i$.

Proof. It follows immediately from the previous lemma and the ascending chain condition that there is a maximal element $\mathcal{J}_p$. Suppose $\mathcal{J}_q$ is also maximal. By the previous lemma they must then both be equal to $\mathcal{J}_{pq}$. This proves that the maximal element is unique. □

Finally, we define the asymptotic multiplier ideal of a graded linear series.

Definition 3.5.4. If W_l is a graded linear series and $c \in \mathbb{Q}_{>0}$ then we set

$$\mathcal{J}((X,\Delta);c\cdot||W_\bullet||) = \mathcal{J}((X,\Delta);c\cdot\mathfrak{a}_\bullet)$$

where $\mathfrak{a}_i = \mathfrak{b}(|W_i|)$ are the base loci of the W_l. If W_l is the complete linear series $|lM|$ for all (or just sufficiently divisible) l then we write

$$\mathcal{J}((X, \Delta); c \cdot ||M||)$$

for the resulting asymptotic multiplier ideal.

CHAPTER 4

Inversion of adjunction - (some of) the story so far

In this chapter we will briefly survey some of the highlights of what is known about inversion of adjunction, especially those theorems closest to our subsequent results. This material is purely expository.

Inversion of adjunction is a central tool in the theory of high-dimensional projective varieties. Since Mori's foundational work it became apparent that, on the one hand, singularities are a fact of life for the classification of varieties of dimension more than three, and on the other hand, that not all singularities are equally bad. There is now a subtle taxonomy of singularities and we recalled some of these classes of singularities in the previous chapter (see Definition 3.3.4 for a very incomplete list).

From a technical standpoint, many proofs in high-dimensional geometry proceed by induction on the dimension, even including Hironaka's fundamental theorem on the resolution of singularities. However, unlike resolution of singularities, we generally do not expect many of the more subtle theorems, such as the minimal model program, to hold in the arbitrarily singular case. As such, induction on the dimension is presented with two complications. The first, not too difficult to resolve, is that for a pair (X, Δ) we need to be able to bound the singularities of (Z, Δ_Z) where

$Z \subseteq X$ is a subvariety and Δ_Z is an appropriately chosen restriction of Δ to Z. The second is that we need to be able to lift data from the irreducible divisor to the larger variety. For example, suppose we know that the singularities of (Z, Δ_Z) are bounded in some way. Do we obtain a bound for the singularities of (X, Δ) near Z?

Inversion of adjunction was originally a statement to this effect when (X, Δ) is a pair and Z is an irreducible divisor with $Z \not\subseteq \mathrm{Supp}(\Delta)$. Here we take Δ_Z to be simply the component-wise restriction of Δ to Z. In this situation, inversion of adjunction says that (Z, Δ_Z) is klt if and only if (X, Δ) is plt near Z. Note the intrinsic geometric appeal of this statement - modulo some subtle distinctions between klt and plt, log-terminal singularities do not ever see transversality problems, in constrant with the generic situation in Bertini's theorem. It is interesting to generalize this theorem, both intrinsically and for several technical applications.

4.1 Inversion of adjunction for klt pairs

In this section we revisit the proof of the original statement of inversion of adjunction. This was originally a proven by Shokurov in dimension 3 in [30] and extended to all dimensions in [1], sections 17.6 - 17.7. In the same papers, the authors conjecture a much stronger statement about minimal log-discrepancies that is still open as of the time of writing. We will give the statement in less than full generality to make the exposition more transparent.

We begin with the following interesting lemma. This is essentially a packaging of local vanishing and, as we will see, can often be circumvented by the use of multiplier ideals. It is, however, of intrinsic geometric interest. We again reduce the generality of the statement to ease exposition.

Lemma 4.1.1 (Connectedness lemma)**.** *Let X be normal, Y smooth, and let f :*

$Y \to X$ be a proper birational morphism. Let

$$D = \sum a_i D_i$$

be a simple normal crossings $\mathbb{Q}$-divisor so that f_*D is effective and

$$K_Y + D \equiv_{\text{num}} f^*B$$

for some divisor B on X. Write

$$A = \sum_{i:a_i<1} a_i D_i, \text{ and } F = \sum_{i:a_i\geq 1} a_i F_i.$$

Then the support of F is connected in a neighborhood of any fiber of f.

Proof. This proof is taken from the proof of Theorem 5.48 in [21]. Notice that

$$\lceil -A \rceil - \lfloor F \rfloor \equiv_{\text{num}} K_Y - f^*B + \{A\} + \{F\}.$$

By the local vanishing theorem 3.4.4, we have that

$$R^1 f_* \mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor) = 0.$$

We therefore can push down the exact sequence

$$0 \to \mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor) \to \mathcal{O}_Y(\lceil -A \rceil) \to \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \to 0$$

to get that the map of sheaves

$$f_* \mathcal{O}_Y(\lceil -A \rceil) \to f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)$$

is surjective.

Let D_i be a component of the support of A. Since g_*D is effective, either D_i is f-exceptional or $a_i > 0$. In particular, $\lceil -A \rceil$ is f-exceptional and effective. It follows that $f_* \mathcal{O}_Y(\lceil -A \rceil) = \mathcal{O}_X$ and we get an exact sequence

$$\mathcal{O}_X \to f_* \mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \to 0.$$

This easily implies the conclusion. Indeed, suppose that for some $x \in X$ we can write $\lfloor F \rfloor = F_1 + F_2$ with F_1 and F_2 having disjoint support near $f^{-1}(x)$. But then

$$f_*\mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_x = f_*\mathcal{O}_{\lfloor F_1 \rfloor}(\lceil -A \rceil)_x \oplus f_*\mathcal{O}_{\lfloor F_2 \rfloor}(\lceil -A \rceil)_x$$

with both summands non-zero, and $\mathcal{O}_{X,x}$, which is a local integral domain, cannot surject onto this. □

Theorem 4.1.2. *Let (X, Δ) be a pair and let $H \subseteq X$ be a normal irreducible divisor that is not a component of the support of Δ. Suppose that H is Cartier, that is, $\mathcal{O}_X(H)$ is an invertible sheaf. Then the pair (X, Δ) is plt near H if and only if (H, Δ_H) is klt.*

Proof. This proof is taken from [21], Theorem 5.50. Since H is Cartier, $(X, \Delta + H)$ is a pair. Let $f : Y \to X$ be a log-resolution of $(X, \Delta + H)$. Write

$$-D = K_Y - f^*(K_X + \Delta + H),$$

note carefully the minus sign.

Let H' be the strict transform of H and let A and F as in the lemma above, applied with our choice of D. Write $F = H' + F'$. Combining the fact that $D - H' = A + F'$ and the adjunction formula we obtain

$$K_{H'} = f^*(K_H + \Delta_H) - (A + F')_{H'},$$

even as divisors (this can be checked with the Poincaré residue, see [19]). In other words,

$$K_{H'} - f^*(K_H + \Delta_H) = -(A + F')_{H'}.$$

If we unwind all the definitions and use the fact that everything in sight is simple normal crossings, we see that

1. $(X, \Delta + H)$ is plt near H if and only if $F' \cap f^{-1}(H) = \emptyset$, and

2. (H, Δ_H) is klt if and only if $F' \cap H' = \emptyset$.

It is now obvious that $(X, \Delta + H)$ plt implies that (H, Δ_H) is klt. Note that we did not need to apply any vanishing theorems to deduce this. Conversely, suppose that (H, Δ_H) is klt. By the connectedness lemma directly above, for every $x \in H$ we have that

$$(H' \cup F') \cap f^{-1}(x)$$

is connected. But $F' \cap H' = \emptyset$, so $F' \cap f^{-1}(x) = \emptyset$. This holds for every $x \in H$, so $F' \cap f^{-1}(H) = \emptyset$ and we are done. □

4.2 The adjoint ideal and the restriction theorem for multiplier ideals

Recall that the multiplier ideal $\mathcal{J}(X, \Delta)$ measures the failure of this pair to be klt and the adjoint ideal $\mathrm{adj}_H(X, \Delta)$ measures the failure of the pair to be plt near H. Therefore, one can expect there to be a version of inversion of adjunction involving these ideals, and indeed there is one.

Theorem 4.2.1 (The restriction theorem)**.** *Let (X, Δ) be a pair and let $H \subseteq X$ be an irreducible divisor. Then*

$$\mathrm{adj}_H(X, \Delta) \cdot \mathcal{O}_H = \mathcal{J}(H, \Delta_H)$$

and we have an exact sequence describing the kernel as

$$0 \to \mathcal{J}(X, \Delta + H) \to \mathrm{adj}_H(X, \Delta) \to \mathcal{J}(H, \Delta_H) \to 0.$$

Proof. Let $\pi : Y \to X$ be a log-resolution of $(X, \Delta + H)$, let H' be the strict transform of H in Y and consider the short exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H)\rceil) \\ &\to \mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H) + H'\rceil) \\ &\to \mathcal{O}_{H'}(\lceil K_Y - \pi^*(K_X + \Delta + H) + H'\rceil)_{H'} \to 0. \end{aligned}$$

On the one hand,

$$\lceil K_Y - \pi^*(K_X + \Delta + H) + H'\rceil_{H'} = \lceil (K_Y - \pi^*(K_X + \Delta + H) + H')_{H'}\rceil$$

because everything in sight has simple normal crossings support. But the adjunction formula says that

$$(K_Y - \pi^*(K_X + \Delta + H) + H')_{H'} = K_{H'} - \pi^*(K_H + \Delta_H),$$

As before, one can show that thisholds at the level of divisors using the Poincaré residue (see [19]) or the adjunction formula we shall prove in Theorem 7.6.6.

On the other hand, the local vanishing theorem 3.4.4 says that

$$R^1\pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H)\rceil) = 0.$$

It follows that the sequence above pushes down to the exact sequence

$$\begin{aligned} 0 &\to \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H)\rceil) \\ &\to \pi_*\mathcal{O}_Y(\lceil K_Y - \pi^*(K_X + \Delta + H) + H'\rceil) \\ &\to \pi_*\mathcal{O}_{H'}(\lceil K_Y - \pi^*(K_X + \Delta + H) + H'\rceil)_{H'} \to 0 \end{aligned}$$

and, combined with our adjunction calculation, this is the exact sequence in the statement of the theorem. $\square$

The explicit description of the kernel as a multiplier ideal is very powerful when combined with Nadel vanishing since it gives a method for lifting sections of adjoint bundles $\mathcal{O}_X(K_X + A)$ with A sufficiently positive from H back to X. This is particularly useful for proving extension theorems for pluricanonical forms. We will use this now standard idea in Theorem 8.5.6 to generalize Siu's famous extension theorem to the high codimension situation.

4.3 A sampling of other forms of inversion of adjunction

Theorem 4.2.1 is the starting point of a rich and developing theory. In this section we will review, without proof, some of the more recent work on inversion of adjunction.

The notions of klt and plt can be interpreted in terms of the total discrepancy. There is a statement of inversion of adjunction, due to Kawakita in [17] and independently Ein and Mustaţă in [8], for the total discrepancy.

Theorem 4.3.1 (Kawakita, Ein-Mustaţă)**.** *Let A be a smooth variety and let $X \subseteq A$ be a closed normal subvariety of codimension c. Let Δ be a $\mathbb{Q}$-divisor on A with $X \not\subseteq \operatorname{Supp}(\Delta)$. Let $W \subseteq X$ be a proper closed subset. Suppose r is a Gorenstein index of X and J_r is the l.c.i.-defect ideal of X (see Section 7.2). Then*

$$\operatorname{totaldiscrep}\left(W; X, \Delta_X + \frac{1}{r}\mathbb{V}(J_r)\right) = \operatorname{totaldiscrep}(W; A, cX + \Delta).$$

The proof uses motivic integration on spaces of arcs. We also have inversion of adjunction of log-canonical singularities.

Theorem 4.3.2 (Kawakita)**.** *Let $(X, S + B)$ be a pair so that S is a reduced irreducible divisor with $S \not\subseteq \operatorname{Supp}(B)$. Let S^ν be the normalization of S and let B^ν be the different of B on S^ν defined by adjunction:*

$$\nu^*(K_X + S + B)_S = K_{S^\nu} + B^\nu.$$

Then $(X, S + B)$ is log-canonical near S if and only if (S^ν, B^ν) is log-canonical.

The work of Takagi from [33] and [32], reviewed in Section 7.3, is another direction of generalization for inversion of adjunction that we will explore in detail. In a superficial sense it is similar to the theorem of Kawakita and Ein-Mustaţă, although the methods and conclusions are different.

CHAPTER 5

Kawamata subadjunction

We now change gears slightly and review the subadjunction theorem of Kawamata. One can regard this theorem as a generalization of the adjunction formula. To understand the analogy, suppose X is smooth and $\Delta = H$ where H is a smooth hypersurface. We then have the formula

$$(K_X + \Delta)_H \sim_{\mathbb{Q}} K_H,$$

(we even have that the two sides are linearly equivalent but we will not focus on that). This Δ is log-canonical and klt outside H. In a sense that we will make precise in the first section of this chapter, H is center of log-canonical singularities for this Δ.

In precise language, H is an exceptional log-canonical center of Δ (see Definition 5.1.3). In general, these centers do not have to be hypersurfaces; many $Z \subseteq X$ can be exceptional log-canonical centers of some Δ; and at least if X is smooth, *every* $Z \subseteq X$ is a generically exceptional log-canonical center of some Δ (see Example 5.1.5 for the proof). In this case, one can tautologically write

$$(K_X + \Delta)_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$$

where Δ_Z is some sort of non-unique error term.

One can ask if there are some choices of Δ_Z that are better than others. Since Z is a generically exceptional log-canonical center, it comes with a morphism $f : E \to Z$, unique up to birational equivalence. This f is not birational in general but it is projective with connected fibers and E can be chosen to be smooth. It is easy to construct a pair structure (E, R) so that, if F is a general fiber of f then (F, R_F) is klt and log-Calabi-Yau. There is some expectation that Δ_Z can be explicitly described as the pullback of an ample divisor by a morphism that this structure induces from Z to an appropriate moduli space of log-Calabi-Yau varieties. Unfortunately, as of the time of writing, the construction of reasonable moduli spaces of such varieties is a major unsolved problem.

Fortunately, all is not lost. Where the moduli space does not exist we can sometimes substitute the induced morphism to the moduli space with a variation of Hodge structure. A deep fact is that this Δ_Z can be described in an explicit Hogde-theoretic way through such a variation of Hodge structure. Although we do not review the Hodge theory, we do review (and slightly generalize) this construction in Section 5.3.

This construction recovers enough information for Kawamata to prove in [18] the following celebrated result. Suppose A is an ample divisor and $0 < \varepsilon \ll 1$ is a small rational number. Suppose further that (X, Δ) is a pair with $Z \subseteq X$ an exceptional log-canonical center of Δ. Then Z is normal and we can choose a Weil $\mathbb{Q}$-divisor Δ_Z on Z so that

$$(K_X + \Delta + \varepsilon A)_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$$

with Δ_Z klt. In our context, we can regard this theorem as saying that the error term Δ_Z is small in an appropriate sense.

In this chapter we state the Hodge-theoretic result and slightly generalize the construction of Δ_Z to the case where Z is only a generically exceptional log-canonical center. We are moving in the direction of generalizing Kawamata's theorem to calculate $\mathcal{J}(Z, \Delta_Z)$ in this much more general situation.

Almost all of the material in this section is expository. The material that is not expository is only a minor variation on existing results and will be indicated as such.

5.1 Log-canonical centers

We begin by recalling the definition of log-canonical centers. We also review exceptional log-canonical centers and define our notion of a generically exceptional log-canonical center. We then define a variant of the adjoint ideal that detects how exceptional a general pair (X, Δ) is near a generically exceptional log-canonical center Z. Generically exceptional log-canonical centers and our new adjoint ideal will play a central role in our statement of inversion of subadjunction.

Definition 5.1.1. Let (X, Δ) be a log-canonical pair. A subvariety $Z \subseteq X$ is called a log-canonical center if there exists a log-resolution $\pi : Y \to X$ of Δ and a divisor $E \subseteq Y$ with $a(E; X, \Delta) = -1$ so that $\pi(E) = Z$.

The following standard theorem is a crucial part of the theory.

Theorem 5.1.2. *Let (X, Δ) be a log-canonical pair. There exists a log-resolution $\pi : Y \to X$ of Δ so that all log-canonical centers of Δ are realized by π, in other words, for every subvariety $Z \subseteq X$ that is a log-canonical center of Δ there is a divisor $E \subseteq Y$ so that $Z = \pi(E)$.*

Proof. The proof of Corollary 2.31 in [21] shows that the number of log-canonical centers is finite. Once this is known the theorem is obvious. $\square$

For our purposes it will be important to define the following special cases of log-canonical centers.

Definition 5.1.3. Let (X, Δ) be a log-canonical pair and let $\pi : Y \to X$ be a log-resolution of Δ realizing all log-canonical centers of Δ. Let $Z \subseteq X$ be a log-canonical center of Δ.

1. Z is a minimal log-canonical center if Z is a minimal element of the set of log-canonical centers of Δ with respect to inclusion.

2. Z is an exceptional log-canonical center if Z is minimal and the divisor $E \subseteq Y$ with $\pi(E) = Z$ and $a(E; X, \Delta) = -1$ is unique.

3. Z is generically an exceptional log-canonical center if there is a dense open subset $U \subseteq X$ containing the generic point of Z so that Z_U is an exceptional log-canonical center of (U, Δ_U). In other words, the divisor $E \subseteq Y$ with $\pi(E) = Z$ and $a(E; X, \Delta) = -1$ is unique but Z may not be a minimal log-canonical center.

Example 5.1.4. Suppose that X is smooth and Δ is a reduced simple normal crossings Weil divisor, say

$$\Delta = \sum E_i.$$

Then (X, Δ) is log-canonical. The log-canonical centers of Δ are simply intersections of the E_i. The minimal centers are the intersections of maximal subcollections E_{i_α} so that

$$Z = \bigcap_{i_\alpha} E_{i_\alpha}$$

is not empty. Every intersection of any subcollection of the E_i is generically exceptional. All minimal centers are exceptional.

Example 5.1.5. Let $X = \mathbb{C}^2$ and let Δ be the cusp, that is, the image of the morphism $\mathbb{C} \to \mathbb{C}^2$ that sends $t \mapsto (t^2, t^3)$. Then $\Delta' = c \cdot \Delta$ is log-canonical when

$c = 5/6$. The origin is an exceptional log-canonical center of (X, Δ'). The cusp itself is a generically exceptional log-canonical center of (X, Δ) even though (X, Δ) is not log-canonical.

Example 5.1.6. Let X be a smooth quasi-projective variety and let $Z \subseteq X$ be an arbitrary subvariety of X. We claim that there exists a Δ on X so that Z is a generically exceptional log-canonical center of Δ. Indeed, the problem is local on X so we may assume that Z is smooth. Let A be a divisor so ample that $\mathcal{O}_X(A) \otimes \mathcal{I}_Z$ is globally generated. Then we can choose divisors

$$H_1, \ldots, H_s \in |\mathcal{O}_X(A) \otimes \mathcal{I}_Z|$$

so that $H_1 \cap \cdots \cap H_s = Z$ and the $\mathbb{Q}$-divisor

$$\Delta = c \cdot (H_1 + \cdots + H_s)$$

is simple normal crossings outside Z, where $c \in \mathbb{Q}_{>0}$ is to be determined. Take a log-resolution $f : Y \to X$ of (X, Δ). Consider

$$K_{Y/X} - f^*\Delta = K_{Y/X} - cf^*(H_1 + \cdots H_s) = \sum_i b_i E_i.$$

Since Δ was simple normal crossings outside Z we can choose a c with $0 < c \leq 1$ so that Δ is log-canonical outside Z, all b_i with E_i dominating Z are ≤ -1 and some b_i with E_i dominating Z is equal to -1. Then the locus where Δ is not log-canonical does not contain Z and we may remove it from X.

We have shown that there exists an open subset $U \subseteq X$ and a pair structure (X, Δ) so that $Z \cap U$ as a minimal log-canonical center of Δ_U. We can then apply the tie-breaking procedure as in, for example, Proposition 8.7.1 in [5], to get an (X, Δ') with $Z \cap U$ an exceptional log-canonical center of Δ'_U, that is, Z is a generically exceptional log-canonical center of (X, Δ').

Just as in the codimension one case, there is a natural adjoint ideal that measures the failure of a log-canonical center to be exceptional. The adjoint ideal we define here seems to be new.

Definition 5.1.7. Let (X, Δ) be a pair and let $Z \subseteq X$ be a generically exceptional log-canonical center of Δ. Let $g : X' \to X$ be a log-resolution Δ. Define

$$\mathrm{adj}_Z(X, \Delta) = g_* \mathcal{O}_{X'} \left(\lceil K_{X'/X} - g^*\Delta \rceil + E \right)$$

where E is the unique divisor dominating Z with discrepancy -1.

Proposition 5.1.8. *This ideal does not depend on the choice of g.*

Proof. Consider a sequence of birational maps

$$\begin{tikzcd} X'' \arrow[r, "f"] \arrow[dr, "\pi"'] & X' \arrow[d, "g"] \\ & X \end{tikzcd}$$

with g a log-resolution of Δ and f a log-resolution of $g^*(\Delta) + \mathrm{Exc}(g)$. Note that π is a log-resolution Δ. As usual, it is enough to show that g and π compute the same ideal.

Now, let

$$B = \lceil K_{X'/X} - g^*\Delta \rceil + E.$$

As in Theorem 3.4.2, it is enough to show that, if E^π and E^g are the exceptional divisors of discrepancy -1 dominating W on X'' and X' respectively, then

$$\lceil K_{X''/X} - \pi^*\Delta \rceil + E^\pi = f^* \left(\lceil K_{X'/X} - g^*\Delta \rceil + E^g \right) + B \tag{5.1}$$

with B effective and f-exceptional. But this follows from already established facts. Indeed, the proof of 3.4.2 shows verbatim the statement that

$$\lceil K_{X''/X} - \pi^*\Delta \rceil = f^* \lceil K_{X'/X} - g^*\Delta \rceil + B' \tag{5.2}$$

with B' effective and f-exceptional. On the other hand, by definition of discrepancy we have that

$$\operatorname{ord}_{E^\pi}\left(\lceil K_{X''/X} - \pi^*\Delta \rceil\right) = \operatorname{ord}_{E^g}\left(\lceil K_{X'/X} - g^*\Delta \rceil\right) = -1.$$

It follows that

$$\operatorname{ord}_E\left(\lceil K_{X''/X} - \pi^*\Delta \rceil + E^\pi\right) = \operatorname{ord}_E\left(\lceil K_{X'/X} - g^*\Delta \rceil + E^g\right) = 0. \tag{5.3}$$

Thus, (5.2) shows that (5.1) holds for all divisors except E and (5.3) shows that (5.1) holds for E. This gives (5.1). □

The importance of these definitions will become clear shortly when we discuss their roles in generalizing the adjunction formula. First however we need to take a detour into some Hodge theory.

5.2 The Kodaira canonical bundle formula - motivation

We begin with Fujita's version of Kodaira's original formula. This is explained in the excellent article of Kollár in [5], Section 8.2. Let $f : S \to C$ be a *relatively minimal* elliptic surface. The basic associated invariants to this morphism are the induced moduli morphism $j : C \to \overline{\mathcal{M}_1} \cong \mathbb{P}^1$ and the set of points $\mathcal{S}$ of C that have singular fibers E_x, $x \in \mathcal{S}$. In the original formula of Kodaira we needed to know the monodromy of f around the singular fibers E_x but it is now recognized that the only necessary number is the log-canonical threshold $c(E_x)$ of the pair (S, E_x).

All these data determine the formula

$$K_S \sim_{\mathbb{Q}} f^*\left(K_C + \frac{1}{12}j^*\mathcal{O}_{\mathbb{P}^1}(1) + \sum_{x\in\mathcal{S}}(1 - c(E_x))[x]\right).$$

The important features of the formula are

$$K_S \sim_{\mathbb{Q}} f^*(K_C + J + B)$$

where J is the pull-back of some ample divisor on the moduli space parameterizing the fibers via a moduli map, and B is some $\mathbb{Q}$-divisor that depends only on easily computable information about the singularities of the singular fibers.

There are many obstacles to generalizing this formula to the log-Calabi-Yau situation. First of all, there are no known reasonable moduli spaces of log-Calabi-Yau varieties. We hope to deal with this by replacing the moduli space with a variation of Hodge structure. Second, the fiber space is not minimal and cannot be made minimal without introducing additional singularities. Let's deal with the second issue now. First suppose that we still have an elliptic surface $f : S \to C$ which may no longer be minimal. Let $\pi : S \to S'$ be the blow-down morphism to the minimal surface and let $f' : S' \to C$ be the resulting Iitaka fibration. Then $K_S = \pi^* K_{S'} + E$ for some exceptional divisor E and we can apply the original canonical bundle formula to get that

$$K_S - E \sim_{\mathbb{Q}} f^* \left(K_C + \frac{1}{12} j^* \mathcal{O}_{\mathbb{P}^1}(1) + \sum_{x \in \mathcal{S}} (1 - c(E_x))[x] \right).$$

This seems bad, as we have no easy way to compute E purely in terms of the given morphism $f : S \to C$.

So, we adjust our expectations and again focus only on the important features. We do know that the support of E consists of singular fibers and so we can try to absorb E into B and write

$$K_S + R_S \sim_{\mathbb{Q}} f^*(K_C + J + B)$$

where R_S is some divisor supported on the singular fibers. We lose uniqueness here, because if we change the coefficient of $[x]$ from $1 - c(E_x)$ to $1 - c(E_x) + d$ we can compensate for this by adding dE_x to R_S. This is not all bad, it just means that, while R_S and B are not geometrically meaningful, the geometric content is hidden in $R_S - f^*B$.

Given that we've given up on making R_S and B meaningful by themselves we need to decide what we will take for B. Our choice will be compensated for by R_S. Write

$$B = \sum_{x \in \mathcal{S}} a(x)[x]$$

for some coefficients $a(x) \in \mathbb{Q}$ that are to be determined. One choice is to just take $a(x) = 0$. This removes the singular fibers from B and puts them into R. While this will work, in our setup B is what we control and R is the unknown, so we would prefer to keep some information about the singular fibers in B. The next obvious choice is $a(x) = 1$. This works well, and generalizes well.

We can now write out exactly what the balancing act between R_S and f^*B amounts to in theorem form. We get the following statement.

Theorem 5.2.1. *Let $f : S \to C$ be an elliptic surface that is not minimal. Let $\mathcal{S} \subseteq C$ be the set of points that have singular fibers. There is then a unique $\mathbb{Q}$-divisor R on S so that we can write*

$$K_S + R \sim_{\mathbb{Q}} f^*(K_C + J + B)$$

where:

1. *J is the (ample)* moduli part*, obtained by pull-back from $\mathcal{M}_1$ of an ample divisor by the moduli map,*

2. *$B = \sum_{x \in \mathcal{S}} [x]$,*

3. *$\operatorname{Supp}(R) \subseteq f^{-1}(\mathcal{S})$,*

4. *(S, R) is log-canonical (so that R is not somehow too big), and*

5. *for every $x \in \mathcal{S}$ there is a point $q \in f^{-1}(x)$ so that (S, R) is not klt at q (so that R is not somehow too small).*

5.3 The higher-dimensional Kodaira canonical bundle formula

Theorem 5.2.1 generalizes well to higher dimensions and to the situation of a log-canonical pair. The main question is what the replacement of the moduli map by a variation of Hodge structure actually gives. What do we expect to get? Suppose we had a moduli space. Then we expect to write $J = j^*A$ where j is the moduli map that the fiber space induces and A is an ample divisor. In the case of an elliptic surface we had that the base of the fiber space was a curve and the moduli space was also a curve. If the fiber space is not isotrivial we get that j is a finite morphism and J is therefore actually ample. In general we do not expect this dimensional coincidence and so we expect J to be only semi-ample (which includes the isotrivial case), at least if the fiber space is nice enough to give us a morphism to the moduli space and not only a rational map. This conjecture is known as the adjunction conjecture of Kawamata and Shokurov and it is wide open at the time of writing.

With the variation of Hodge structure approach it turns out that we get only that J is nef, not semi-ample. We are now finally ready to give the full statement of the theorem we will base all our calculations on.

Theorem 5.3.1 (Theorem 8.5.1 in [5])**.** *Let E and W be smooth projective varieties and let $f : E \to W$ be a dominant morphism. Let F be a general fiber of f. Let R be a $\mathbb{Q}$-divisor on E and let B be a reduced divisor on W so that:*

1. $K_E + R \sim_{\mathbb{Q}} f^*(\textit{some divisor on } W)$*,*

2. $\kappa(F, K_F + R_F) = 0$*,*

3. $f : E \to W$*,* R *and* B *satisfy the* standard normal crossings assumptions*:*

(a) E and W are smooth (as assumed in our statement),

(b) $R + f^*B$ and B are simple normal crossings,

(c) f is smooth over $W \setminus B$, and

(d) if F' is a fiber of f over a point $p \in W \setminus B$ then $R_{F'}$ is klt.

Then we can write

$$K_E + R \sim_{\mathbb{Q}} f^*(K_W + J(E/W, R) + B_R)$$

where:

- *$J(E/W, R)$ is a divisor, defined only up to linear equivalence. It is the so-called* moduli part*. It depends only on (F, R_F) and on W. Under our standard normal crossings assumptions it is nef.*

- *B_R is a $\mathbb{Q}$-divisor that is uniquely defined once we fix the divisor B as follows: it is the unique $\mathbb{Q}$-divisor for which there is a codimension ≥ 2 subset $S \subset W$ such that*

  1. *$(E \setminus f^{-1}(S), R + f^*(B - B_R))$ is log-canonical, and*
  2. *every irreducible component of B is dominated by a log-canonical center of $(E, R + f^*(B - B_R))$.*

Condition 1 essentially determines R, condition 2 is an analog of the log-Calabi-Yau condition and condition 3 is essentially the semi-stable reduction that is expected to turn the moduli map into a morphism. Of the conditions on B, condition 1 says that $R - f^*B_R$ is not too big and condition 2 says that $R - f^*B_R$ is not too small.

5.4 The Kawamata different and the subadjunction theorem

We now come to one of the main ideas of our point of view on the subadjunction theorem. Consider the classical adjunction formula: if X is a smooth projective variety and H is a smooth hypersurface then

$$(K_X + H)_H \sim K_H.$$

The directly analogous higher codimension formula involves the normal bundle and so needs information about defining equations, and if H is singular then it is not immediately clear what analog of this formula we want. We can compute dualizing sheaves in this manner, but we are often interested in the canonical class.

What does this have to do with log-canonical centers? Let $\Delta = H$. It is immediate that H is an exceptional log-canonical center of Δ and so

$$(K_X + \Delta)_H \sim K_H.$$

What then if Z is a normal exceptional log-canonical center[1] of some Δ?

Example 5.4.1. Consider the twisted cubic $C \subseteq \mathbb{P}^3$. It is the intersection of three degree 2 hypersurfaces H_1', H_2', H_3'. The intersection $H_1' \cap H_2'$ is equal to $C \cup L$ with L a line that intersects C at a point. Blowing up this point writes an image of $\mathbb{P}^1$ as a component of the intersection of two smoooth hypersurfaces H_1, H_2 in a threefold X. Let $\Delta = H_1 + H_2$; this Δ is log-canonical. A direct calculation shows that $(K_{\mathbb{P}^3} + \Delta)_C$ has degree 2, while C is abstractly isomorphic to $\mathbb{P}^1$, so $(K_{\mathbb{P}^3} + \Delta)_C \neq K_C$. Note however that the difference between the two has positive degree, suggesting a positivity result.

To try to recover a formula, we can tautologically write

$$(K_X + \Delta)_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z$$

for lots of choices of Δ_Z. The Hodge theoretic result of the previous section can be used to show that this Δ_Z does indeed have something to do with adjunction.

[1] We will see later that these are always normal.

Unfortunately, due to $J(E/W, R)$ only being nef and not semi-ample we need to add some ampleness to the situation by perturbing Δ.

We emphasize that the content of the following theorem is the explicit construction of Δ_{Z^n}. The construction is slightly more general than Kawamata's in [18] in order to accomodate log-canonical centers that are only generically exceptional.

Theorem 5.4.2. *Let X be a smooth projective variety, let H be an ample divisor on X and let $\epsilon > 0$. Let Δ be a $\mathbb{Q}$-divisor on X and suppose that Z is a generically exceptional log-canonical center of (X, Δ). Let $\nu : Z^n \to Z$ be the normalization of Z. With the above setup there exists an explicitly constructed $\mathbb{Q}$-divisor Δ_{Z^n} on Z^n that we will call the* Kawamata different*, so that*

1. *$K_{Z^n} + \Delta_{Z^n}$ is $\mathbb{Q}$-Cartier,*
2. *$\nu^*(K_X + \Delta + \epsilon H)_Z \sim_{\mathbb{Q}} K_{Z^n} + \Delta_{Z^n}$.*

Before giving the proof, we remark that the theorem of Kawamata applies when Z is an exceptional log-canonical center of Δ. He shows that in this case Z is normal and Δ_Z is klt. We will not prove this statement here, although it is a corollary of our Theorem 6.1.1 (see Corollary 6.2.3).

Proof of Theorem 5.4.2. We begin with the following claim.

Claim 5.4.3. *There exists a log-resolution $g : Y' \to X$ of Δ with the following properties. Let E be the unique divisor on Y' of discrepancy -1 lying over Z and let $g_E : E \to Z$ be the restriction of g. Then we can arrange for g to factor as in the* *following diagram*

(5.4)
$$\begin{tikzcd}
E \arrow[d, "f_E"] \arrow[dd, bend right, "g_E"'] & \subseteq & Y' \arrow[d, "f"'] \arrow[dd, bend left, "g"] \\
W \arrow[d, "\pi_E"] & \subseteq & Y \arrow[d, "\pi"'] \\
Z & \subseteq & X
\end{tikzcd}$$

so that, if we write

$$K_{Y'} + E + \Delta' \sim_{\mathbb{Q}} g^*(K_X + \Delta)$$

with $g_*\Delta' = \Delta$ *and let* $R = \Delta'_E$ *(this divisor has simple normal crossings support), then* W *carries a divisor* B *that, together with* R *and* f*, satisfies the* standard normal crossings assumptions *of Theorem 5.3.1.*

First, we will finish the proof of Theorem 5.4.2 assuming the truth of the claim. Note that, since Z is generically an exceptional log-canonical center of Δ, R is klt on a generic fiber of g_E.

Using Theorem 5.3.1, we obtain a divisor B_R, supported on B so that

$$K_E + R \sim_{\mathbb{Q}} f_E^*(K_W + J(E/W, R) + B_R).$$

Since H is ample and $J(E/W, R)$ is nef, the sum $J(E/W, R) + \epsilon\pi_E^* H$ is big and nef and so is $\mathbb{Q}$-equivalent to some effective divisor J_ϵ. But recall that $K_E + R \sim_{\mathbb{Q}} g_E^*(K_X + \Delta)_Z$. It follows that

$$K_W + J_\epsilon + B_R \sim_{\mathbb{Q}} \pi_E^*(K_X + \Delta + \epsilon H)_Z.$$

Now, π_E must factor through the normalization $\nu : Z^n \to Z$; write $\pi_E = \nu \circ h$ for this factorization. Pushing the above formula forward along h yields

$$\nu^*(K_X + \Delta + \epsilon H)_Z \sim_{\mathbb{Q}} K_{Z^n} + h_*(J_\epsilon + B_R).$$

Set $\Delta_{Z^n} = h_*(J_\epsilon + B_R)$. $\square$

Proof of Claim. For clarity, we will construct the required resolution in several steps.

Step 1: Begin with any log-resolution of Δ, say $g : X_1 \to X$, and let R and E as in the statement of the claim. Take any reduced divisor B_0 so that $\mathrm{Supp}(B_0)$ contains Sing (Z), and so that the locus of points at which g is not smooth or R and F_p are not simple normal crossings, where F_p is the fiber of $g_E : E \to Z$ over p.

Step 2: Take a smooth blow-up $\pi : Y \to X$ that does not blow up the generic point of Z and that induces a birational morphism $\pi_E : W \to Z$ with W smooth and $B = \pi_E^* B_0$ simple normal crossings. Note that, currently, π is related to g only because B depends on g. We have so far the following diagram

$$\begin{tikzcd} W \arrow[r, hook] \arrow[dr, "\pi_E"'] & Y \arrow[dr, "\pi"] & & X_1 \arrow[dl, "g"'] \\ & Z \arrow[r, hook] & X & \end{tikzcd}$$

Step 3: In the above diagram we can identify two opportunities for a fiber product:

$$\begin{tikzcd} & E \arrow[d] & & X_1 \arrow[d] \\ W \arrow[r] & Z & Y \arrow[r] & X \end{tikzcd}$$

Let $E' \subseteq E \times_Z W$ be the component of the fiber product dominating W and let $X' \subseteq X_1 \times_X Y$ be the component of the fiber product containing E'. Note that the blow-up $X' \to Y$ is an isomorphism outside B so that outside B we have that E' is isomorphic to E and X' is isomorphic to X_1. In particular, X' is a log-resolution of Δ outside B.

Step 4: We have the following diagram

$$\begin{tikzcd} E' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ W \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

with $X' \to X$ isomorphic to $X_1 \to X$ outside B. Complete the diagram to a new diagram

$$\begin{tikzcd} E \arrow[r, hook] \arrow[d] & Y' \arrow[d] \\ E' \arrow[r, hook] \arrow[d] & X' \arrow[d] \\ W \arrow[r, hook] \arrow[d] & Y \arrow[d] \\ Z \arrow[r, hook] & X \end{tikzcd}$$

where the morphism $g : Y' \to X' \to X$ has the following properties. Let $k : Y' \to X'$ be the factoring morphism. We require that Y' be a log-resolution of Δ and that, if we replace E by its strict transform under k, E becomes smooth. Let $R' = k^*R$, write

$$K_{Y'} + E + \Delta' \sim_{\mathbb{Q}} g^*(K_X + \Delta)$$

with $g_*\Delta' = \Delta$ and replace R with $R = \Delta'_E$. The next property for which we ask is that the exceptional set of k should have simple normal crossings. Note that then R differs from R' by divisors exceptional for k, so $R + f^*B$ is simple normal crossings.

As all blow-up centers of k had centers on X that are contained in B, these choices construct Diagram 5.4 with the given initial choice of $g : X_1 \to X$ and B. $\square$

We finally introduce a condition on Δ_{Z^n} that should be thought of as saying that Δ_{Z^n} is sufficiently generic.

Definition 5.4.4. Notation as in the previous theorem. We say that a Kawamata different (Z^n, Δ_{Z^n}) is *suitably chosen* if, in addition to the requirements in Claim 5.4.3, the following are satisfied.

- The map π_E is sufficiently high - the Rees valuations of $\mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_{Z^n}$ are extracted by π_E,

- J_ε is general - $\lfloor J_\epsilon + B_R \rfloor = \lfloor B_R \rfloor$,
- B is sufficiently large - the components of B include the Rees valuations of $\mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_{Z^n}$,

Remark 5.4.5. To achieve this we make our choices in Claim 5.4.3 as follows. We select B large enough to contain the support of $\mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_Z$ and we select $\pi : Y \to X$ to factor through the blow-up of $\mathrm{adj}_Z(X, \Delta)$, this makes π_E sufficiently high and B sufficiently lage. To make J_ε sufficiently general, we use the fact that J_ε is big and nef to choose it to be of the form $H + \varepsilon C$, where H is a general ample divisor, C is effective and ε is sufficiently small. Note that all this may *a priori* change B_R and Δ_{Z^n}.

CHAPTER 6

Inversion of subadjunction

We now arrive at the first of our main results. The material in this chapter is adapted from [10].

Let X be a smooth projective variety, let Δ be a $\mathbb{Q}$-divisor and let Z be a generically exceptional log-canonical center of Δ. As we discussed earlier, the statement of Kawamata's subadjunction theorem can be regarded as a kind of adjunction formula:

$$\nu^*(K_X + \Delta + \varepsilon H)_Z \sim_{\mathbb{Q}} K_Z + \Delta_Z,$$

where Δ_Z can be viewed as an error term. In this light, Kawamata's theorem says that when the singularities of Δ near Z are as nice as possible, that is, Z is an exceptional log-canonical center of Δ), then the singularities of Δ_Z are nice, that is, klt. In this chapter we prove one of our main theorems. We generalize Kawamata's statement to compute $\mathcal{J}(Z^n, \Delta_{Z^n})$ in terms of $\mathrm{adj}_Z(X, \Delta)$.

While our result immediately implies Kawamata's theorem, it should not be regarded as a new proof of it. Instead it is a strictly stronger statement that is deduced via similar methods by doing more work.

6.1 The statement

We will first state the theorem, then recall and prove a few lemmas, and then finally give the proof of the theorem.

Theorem 6.1.1 (Inversion of subadjunction)**.** *Let X be a smooth projective variety and let Δ be a $\mathbb{Q}$-divisor on X. Suppose that $Z \subseteq X$ is a generically exceptional log-canonical center of (X, Δ). Let*

$$\nu : Z^n \to Z$$

be the normalization of Z. Let Δ_{Z^n} be a suitably chosen Kawamata different for Z, as in Definition 5.4.4.

As $K_{Z^n} + \Delta_{Z^n}$ is $\mathbb{Q}$-Cartier, we may consider the multiplier ideal $\mathcal{J}(Z^n, \Delta_{Z^n}) \subseteq \mathcal{O}_{Z^n}$. Then:

1. *$\mathcal{J}(Z^n, \Delta_{Z^n})$ is contained in the conductor ideal of ν.*
2. *The conductor is also an ideal on Z and so $\mathcal{J}(Z^n, \Delta_{Z^n})$ can naturally be viewed as an ideal on Z. With this identification, we have that*

$$\operatorname{adj}_Z(X, \Delta) \cdot \mathcal{O}_{Z^n} = \mathcal{J}(Z^n, \Delta_{Z^n}),$$

3. *We have an exact sequence*

$$0 \to \mathcal{J}(X, \Delta) \to \operatorname{adj}_Z(X, \Delta) \to \mathcal{J}(Z^n, \Delta_{Z^n}) \to 0$$

of sheaves on Z.

Remark 6.1.2. In the statement of Theorem 6.1.1, the short exact sequence has the form

$$0 \to \mathcal{J}(X, \Delta) \to \operatorname{adj}_Z(X, \Delta) \to \mathcal{J}(Z^n, \Delta_{Z^n}) \to 0.$$

On the other hand, in codimension one the statement of Theorem 4.2.1 contains the short exact sequence

$$0 \to \mathcal{J}(X, \Delta + H) \to \operatorname{adj}_H(X, \Delta) \to \mathcal{J}(H, \Delta_H) \to 0.$$

The reader may be worried that we have $\mathcal{J}(X, \Delta)$ on the one hand and $\mathcal{J}(X, \Delta+H)$ on the other. To deduce Theorem 4.2.1 from Theorem 6.1.1, suppose we are given H and Δ as in Theorem 4.2.1. We set Δ in Theorem 6.1.1 to be $\Delta + H$.

The following lemma will play a key role in our discussion.

Lemma 6.1.3. *Let $f : Y' \to X$ be a proper birational morphism between projective varieties and let Z be a subvariety of X. Let $E \subseteq Y'$ be an irreducible divisor lying over Z. Let $f_E : E \to Z$ be the restriction of f and let D be a Cartier divisor on Y' with $E \not\subseteq \operatorname{Supp}(D)$. Suppose that the natural map of sheaves*

$$f_* \mathcal{O}_{Y'}(D) \to f_{E,*} \mathcal{O}_E(D_E)$$

induced by restriction of sections is surjective. Let U be an open subset of Z. Then we can describe the sheaf $f_{E,*}\mathcal{O}_E(D_E)$ by the rule

$$\Gamma(U, f_{E,*}\mathcal{O}_E(D_E)) = \left\{ p \in \mathbb{C}(Z) \mid f_E^*(p) \in \Gamma(f_E^{-1}(U), \mathcal{O}_E(D_E)) \right\}.$$

In other words, every rational function in the set $\Gamma(f_E^{-1}(U), \mathcal{O}_E(D_E))$ is a pull-back of a rational function from Z.

Proof of Lemma. Let

$$\mathcal{S}(U) = \left\{ p \in \mathbb{C}(Z) \mid f_E^*(p) \in \Gamma(f_E^{-1}(U), \mathcal{O}_E(D_E)) \right\}.$$

It is easy to see that this assignment, together with the obvious restriction maps, defines a sheaf $\mathcal{S}$ on Z (even an $\mathcal{O}_Z$-module). On the other hand, since $E \not\subseteq \operatorname{Supp}(D)$, we can define $f_*\mathcal{O}_{Y'}(D) \cdot \mathcal{O}_Z$ in the usual way since all rational functions in $\mathcal{O}_{Y'}(D)$ are regular at the generic point of E and obtain $\mathcal{S}$ more intrinsically.

By the definition of $\mathcal{S}$, there is a natural map of sheaves

$$\varphi : \mathcal{S} \to f_{E,*}\mathcal{O}_E(D_E)$$

given by $p \mapsto f_E^*(p)$. We wish to show that φ is an isomorphism. It is injective since E dominates Z. Since both source and target are sheaves, if φ is surjective as a map of sheaves then $\varphi_U : \Gamma(U, \mathcal{S}) \to \Gamma(U, f_{E,*}\mathcal{O}_E(D_E))$ is an isomorphism for every open subset U of Z.

Notice however that we can factor φ as follows. Let U be an open subset of Z, let $p \in \Gamma(U, \mathcal{S})$ and let p' be any rational function on X so that $p'_Z = g$. Then $f^*(p')_E = \varphi_U(p)$. But, since $E \not\subseteq \mathrm{Supp}(D)$, the map

$$f_*\mathcal{O}_{Y'}(D) \to f_{E,*}\mathcal{O}_E(D_E)$$

is nothing more than the map that, for an open subset V of X takes a rational function $p' \in \Gamma(V, f_*\mathcal{O}_{Y'}(D))$ and maps it to $f^*(p')_E$. By hypothesis, this map is surjective as a map of sheaves. But this map clearly factors through φ and so φ is also surjective, as required. $\square$

Remark 6.1.4. Note that the conclusion of the lemma is equivalent to the statement that the natural "base change" map

$$f_*\mathcal{O}_{Y'}(D) \cdot \mathcal{O}_Z \to f_{E,*}\mathcal{O}_E(D_E)$$

is an isomorphism. In fact, this is how the proof of the lemma proceeds. We will make use of this equivalent formulation.

We now turn to the proof of Theorem 6.1.1.

Proof of Theorem 6.1.1. To make the proof more clear, we will proceed in several steps. As shorthand, set

$$\mathfrak{b} = \mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_Z.$$

Step 1: We show that there is a natural exact sequence

$$0 \to \mathcal{J}(X, \Delta) \to \mathrm{adj}_Z(X, \Delta) \to \mathfrak{b} \to 0,$$

that $\mathfrak{b}$ is contained in the conductor of ν, that it is integrally closed on Z^n, and we describe its local sections. We accomplish this by combining local vanishing and Lemma 6.1.3 in the following manner. First, we construct the diagram of morphisms

$$\begin{tikzcd}
E \arrow[dd, "g_E"'] \arrow[dr, "h_E"] & & \subseteq & & Y' \arrow[dl, "h"'] \arrow[dd, "g"] \\
& Z^n \arrow[dl, "\nu_E"] & \subseteq & X_0 \arrow[dr, "\nu"'] & \\
Z & & \subseteq & & X
\end{tikzcd}$$

using the following steps:

- Step 1: Let $\nu_E : Z^n \to Z$ is the normalization map. It is proper and birational and therefore it is given by the blowing up of some ideal sheaf $\mathcal{I}$ on Z. Lift $\mathcal{I}$ in an arbitrary manner to an ideal sheaf on X and blow up this ideal sheaf to obtain X_0 and $\nu : X_0 \to X$. Thus, X_0 is reduced but possibly not normal.

- Step 2: Complete ν to a log-resolution $g : Y' \to X$ as in Definition 5.1.7. Let E be the unique divisor lying over Z with discrepancy -1 and let $g_E : E \to Z$ be the restriction of g to E.

- Step 3: With these choices, g_E factors through ν_E and g factors through ν. Let the factorizations be $g_E = \nu_E \circ h_E$ and $g = \nu \circ h$, here h_E is the restriction of h and ν_E is the restriction of ν.

Now, let

$$D = K_{Y'/X} - g^*\Delta + E.$$

Consider the short exact sequence

$$0 \to \mathcal{O}_{Y'}(\lceil D - E \rceil) \to \mathcal{O}_{Y'}(\lceil D \rceil) \to \mathcal{O}_{Y'}(\lceil D \rceil)_E \to 0. \tag{6.1}$$

of sheaves on Y'. Because of our assumption that Z is an exceptional log-canonical center of Δ near the generic point of Z, E cannot be in the support of D. As we are in a simple normal crossings situation, we have

$$\mathcal{O}_{Y'}(\lceil D \rceil)_E = \mathcal{O}_E(\lceil D_E \rceil).$$

By the local vanishing theorem 3.4.4 applied to (6.1) and the morphism g, we get the short exact sequence

$$0 \to \mathcal{J}(X, \Delta) \to \mathrm{adj}_Z(X, \Delta) \to g_{E,*}\mathcal{O}_E(\lceil D_E \rceil) \to 0.$$

Then Lemma 6.1.3 says that the natural map

$$\mathfrak{b} := \mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_Z \to g_{E,*}\mathcal{O}_E(\lceil D_E \rceil)$$

is an isomorphism.

We can also apply local vanishing to (6.1) and the morphism h. We obtain that

$$R^1 h_* \mathcal{O}_{Y'}(\lceil D - E \rceil) = 0.$$

Lemma 6.1.3 again says that the natural map

$$h_*\mathcal{O}_{Y'}(\lceil D \rceil) \cdot \mathcal{O}_{Z^n} \to h_{E,*}\mathcal{O}_E(\lceil D_E \rceil)$$

is an isomorphism. In particular, the sheaf $h_{E,*}\mathcal{O}_E(\lceil D_E \rceil)$ is naturally a subsheaf of the function field of Z. But we have just seen that

$$g_{E,*}\mathcal{O}_E(\lceil D_E \rceil) = \nu_{E,*}(h_{E,*}\mathcal{O}_E(\lceil D_E \rceil))$$

is an ideal of $\mathcal{O}_Z$ (in a compatible sense) and so $\mathfrak{b}$ is contained in the conductor of ν.

Note that this describes the local sections of $\mathfrak{b}$ as follows. Let $p \in \Gamma(U, \mathcal{O}_Z)$ be a regular function on an open set U. Then $g_E^*(p) \in \Gamma(g^{-1}(U), \mathcal{O}_E(\lceil D_E \rceil))$ if and only if $p \in \Gamma(U, \mathfrak{b})$. In particular, since the membership criteria for $\mathfrak{b}$ are clearly given by valuations and $\mathfrak{b}$ is an ideal subsheaf of the sheaf of integrally closed rings $\mathcal{O}_{Z^n}$, $\mathfrak{b}$ is integrally closed.

We also emphasize that, since $\mathfrak{b} = \mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_Z$, the ideal $\mathfrak{b}$ does not depend on any choices of log-resolutions or Kawamata boundaries.

Step 2: Next we make use of the fact, just proven, that $\mathfrak{b}$ is integrally closed in order to make our choice of log-resolution and other parameters for the rest of the proof. Let R_i be the finite set of divisors over Z that compute membership in the integrally closed ideal $\mathfrak{b}$, that is, the Rees valuations of $\mathfrak{b}$. As in Claim 5.4.3, our diagram of morphisms will be as follows:

$$\begin{tikzcd}
E \arrow[r, hook] \arrow[d, "f_E"] \arrow[dd, bend right=60, "g_E"'] & Y' \arrow[d, "f"'] \arrow[dd, bend left=60, "g"] \\
W \arrow[r, hook] \arrow[d, "\pi_E"] & Y \arrow[d, "\pi"'] \\
Z \arrow[r, hook] & X
\end{tikzcd}$$

where:

- $g : Y' \to X$ is a log-resolution of Δ as in Definition 5.1.7, E is the unique divisor of discrepancy -1 lying over Z, and $g_E : E \to Z$ is the restriction of g to E,

- $\pi_E : W \to Z$ is a proper birational morphism with simple normal crossings exceptional divisor, chosen so that it extracts the R_i. We choose $\pi : Y \to X$ to be a proper birational morphism that induces $\pi_E : W \to Z$ by restriction.

- As observed after Definition 5.4.4, we may additionally choose g and π_E in such way as to have a reduced divisor B on W with the properties that:

- B satisfies the *standard normal crossings assumptions* of Theorem 5.3.1. Denote by B_R the divisor constructed from B in Theorem 5.3.1.
- $R_i \subseteq \operatorname{Supp}(B)$,

- Again, $\pi_E : W \to Z$ factors through the normalization $\nu : Z^n \to Z$ and we write $\pi_E = \nu \circ h$ for the factorization.

Note that these conditions say that the resulting Kawamata different is suitably chosen in the sense of Definition 5.4.4. We define E_i to be the components of B.

Next, we adopt the notation from the proof of Kawamata's subadjunction theorem in Theorem 5.4.2. Notice that in fact

$$-(J_\varepsilon + B_R) = K_W - \pi_E^*(K_{Z^n} + \Delta_{Z^n})$$

as $\mathbb{Q}$-divisors. Indeed, their non-exceptional parts are equal by definition and it follows that the exceptional parts are $\mathbb{Q}$-equivalent, hence equal. In particular,

$$h_* \mathcal{O}_W(\lceil -B_R \rceil) = \mathcal{J}(Z^n, \Delta_{Z^n}).$$

Step 3: We finally compare $\mathfrak{b}$ and $\mathcal{J}(Z^n, \Delta_{Z^n})$. For each index i, let F_i^α be the divisors on E that dominate E_i (the indices α runs through depend on i). Note that we do *not* claim that $\operatorname{ord}_{F_i^\alpha}(f^* B_R) = \operatorname{ord}_{F_i^\alpha}(R)$ for all i and α!

To make the comparison, recall from the definition of B_R that

(a) For any irreducible divisor G on W, $(E, R + f_E^*(B - B_R))$ is log-canonical in a neighborhood of the generic point of every component of $f_E^* G$ that dominates G,

(b) every component of B is dominated by a log-canonical center of $(E, R + f_E^*(B - B_R))$.

Since $R + f_E^*(B - B_R)$ is a simple normal crossings divisor by assumption, our choice of B from step 2 and condition (a) say that

$$\mathrm{ord}_{F_i^\alpha}(R - f_E^* B_R) \le 1 - \mathrm{ord}_{F_i^\alpha}(f_E^* B) \le 0$$

for all i and α. This says that $\mathcal{J}(Z^n, \Delta_{Z^n}) = h_* \mathcal{O}_W(\lceil -B_R \rceil) \subseteq \mathfrak{b}$.

For the reverse inequality, notice that condition (b) says that for every i there is an α so that

$$\mathrm{ord}_{F_i^\alpha}(-R + f_E^*(B_R - B)) = -1.$$

So suppose that (locally) there were to exist an element $p \in \mathfrak{b} \setminus \mathcal{J}(Z^n, \Delta_{Z^n})$. Then, on the one hand, we have

$$\mathrm{ord}_{F_i^\alpha}(g_E^* p) \ge \mathrm{ord}_{F_i^\alpha}(R)$$

for all i and α. On the other hand, there must exist an index i with

$$\mathrm{ord}_{E_i}(\pi_E^* p) < -\lceil -\mathrm{ord}_{E_i}(B_R) \rceil = \lfloor \mathrm{ord}_{E_i}(B_R) \rfloor.$$

Since the left hand side is an integer, this inequality is satisfied if and only if

$$\mathrm{ord}_{E_i}(\pi_E^* p) \le \mathrm{ord}_{E_i}(B_R) - 1.$$

Pulling back we obtain, for this i and all F_i^α,

$$\mathrm{ord}_{F_i^\alpha}(g_E^* p) \le \mathrm{ord}_{F_i^\alpha}(f_E^*(B_R - B)).$$

Putting the two inequalities together we see that, we must have

$$\mathrm{ord}_{F_i^\alpha}(R) \le \mathrm{ord}_{F_i^\alpha}(g_E^* p) \le \mathrm{ord}_{F_i^\alpha}(f_E^*(B_R - B)).$$

Then, for this i and all F_i^α, $\mathrm{ord}_{F_i^\alpha}(-R + f_E^*(B_R - B)) \ge 0$, a contradiction. $\square$

6.2 Corollaries

This theorem has a number of immediate corollaries, including Kawamata's subadjunction statement as well as a naive version of inversion of subadjunction.

Corollary 6.2.1. *All suitably chosen Kawamata differents are effective. All suitably chosen Kawamata differents have the same multiplier ideal.*

Proof. By the theorem, $\mathcal{J}(Z^n, \Delta_{Z^n})$ is always an ideal. This is equivalent to the assertion that Δ_{Z^n} is effective. Also, $\mathcal{J}(Z^n, \Delta_{Z^n}) = \mathrm{adj}_Z(X, \Delta) \cdot \mathcal{O}_{Z^n}$ and this latter ideal does not depend on the choice of Δ_{Z^n}. □

Example 6.2.2. To illustrate this phenomenon we present an example that can be found in [2]. Let $C \subseteq \mathbb{P}^2$ be the curve defined by the equation $x^2 z - y^3 = 0$. The normalization of this curve is a $\mathbb{P}^1$. Direct computation shows that

$$(K_{\mathbb{P}^2} + C)_C = K_{C^\nu} + 2p$$

with $p \in \mathbb{P}^1$ a point. C is, of course, a generically exceptional log-canonical center of $\Delta = C$.

The twisted cubic $C \subseteq \mathbb{P}^3$ is another example. There are two quadrics $H_1, H_2 \subseteq \mathbb{P}^2$ with $H_1 \cap H_2 = C \cup L$ with L a line at infinity. Setting $\Delta = H_1 + H_2$ we can check by direct computation that C is a generically exceptional log-canonical center of Δ, although it is not minimal - the point $C \cap L$ is also a log-canonical center. We can easily check that $(K_{\mathbb{P}^3} + \Delta)_C$ is ample while C is, of course, Fano, so the difference is ample and, in particular, effective.

Corollary 6.2.3 (Kawamata subadjunction)**.** *If Δ is log-canonical and Z is an exceptional log-canonical center of Δ, then Z is normal and the suitably chosen Kawamata different is effective and klt.*

Proof. If Δ is log-canonical and Z is a minimal center then

$$\mathrm{adj}_Z(X,\Delta) = \mathcal{O}_X.$$

It follows from Theorem 6.1.1 that $\mathcal{J}(Z^n, \Delta_{Z^n}) = \mathcal{O}_{Z^n}$. But the theorem also tells us that $\mathcal{J}(Z^n, \Delta_{Z^n})$ is contained in the conductor of ν. This conductor is therefore the unit ideal, that is, Z is normal. Furthermore, the formula $\mathcal{J}(Z, \Delta_Z) = \mathcal{O}_Z$ immediately implies that Δ_Z is effective and klt. □

Corollary 6.2.4 (Naive inversion of subadjunction)**.** *Suppose Z is an exceptional log-canonical center of Δ in a neighborhood of the generic point of Z. Then a suitably chosen Kawamata different is klt on Z^n if and only if Δ is log-canonical and Z is a minimal log-canonical center of Δ.*

Proof. Since $\mathrm{adj}_Z(X,\Delta)\cdot\mathcal{O}_{Z^n} = \mathcal{J}(Z^n, \Delta_{Z^n})$, the equivalence follows from checking when each side of this equation can be equal to $\mathcal{O}_{Z^n}$. □

Corollary 6.2.5 (Kawamata-Viehweg vanishing for $\mathrm{adj}_E(X,\Delta)$)**.** *Suppose that Z is normal and A is a big and nef $\mathbb{Q}$-divisor with A_Z again big (in particular, if $Z \not\subseteq \mathbb{B}_+(A)$). Suppose that L is a Cartier divisor with $A \equiv_{\mathrm{num}} L - \Delta$. Then*

$$H^i(X, \mathcal{O}_X(K_X + L) \otimes \mathrm{adj}_Z(X,\Delta)) = 0$$

for all $i > 0$.

Proof. This follows immediately from Kawamata-Viehweg vanishing applied to the long exact sequence in cohomology that we get from the short exact sequence in Theorem 6.1.1. □

CHAPTER 7

The special case of a $\mathbb{Q}$-Gorenstein center

In the previous chapter we gave a statement of inversion of adjunction for arbitrary subvarieties. One may expect that for special subvarieties or special Δ there may be more precise statements. In this chapter we will discuss inversion of adjunction when restricting to a high codimension subvariety with $\mathbb{Q}$-Gorenstein singularities (see Definition 7.1.2). Much of our work here is inspired by earlier work of S. Takagi in [33] and [32]. Takagi uses reduction to positive characteristic and tight closure methods, our work here can be regarded as a characteristic zero proof of his theorems. We do obtain slightly stronger results here than he did.

Our main new contribution is an adjunction formula for relative canonical classes in what are known as strong factorizing resolutions. We define strong factorizing resolutions in Definition 7.4.3, the proof of their existence is in [4]. Our formula computes the relative canonical class of an appropriate embedded resolution of a $\mathbb{Q}$-Gorenstein subvariety in terms of the relative canonical class of the ambient variety and the l.c.i.-defect sheaf (see 7.2) of the subvariety. This formula, combined with a simple Leray spectral sequence trick, give our proofs of Takagi's theorems.

The material in this chapter is adapted from [11].

7.1 Gorenstein and $\mathbb{Q}$-Gorenstein singularities

Here we briefly review the definitions of Gorenstein and $\mathbb{Q}$-Gorenstein singularities. First we review the classical notion of a Gorenstein singularity (see [16], Section V.9). We do not aim for maximum generality.

Definition 7.1.1. Let X be a projective variety. We say that X is *Gorenstein* if the dualizing complex of X is quasi-isomorphic to an invertible sheaf. In particular, not only the canonical sheaf but even the dualizing sheaf of X is invertible.

We will not use this notion very much. Instead we will use the following notion, which is also often called Gorenstein.

Definition 7.1.2. Let X be a normal projective variety. We say that X is *quasi-Gorenstein* if K_X is Cartier. We say that X is $\mathbb{Q}$-Gorenstein if K_X is $\mathbb{Q}$-Cartier.

As we already implied, one can show that Gorenstein implies quasi-Gorenstein but not vice versa. We give a few examples.

1. Any hypersurface in a smooth variety is Gorenstein.

2. Almost all affine toric varieties that are $\mathbb{Q}$-Gorenstein are not Gorenstein. Write $X = \mathrm{Spec}\mathbb{C}[\sigma^\vee \cap M]$ and let σ have primitive generators $a_1, \ldots, a_s$. Then X is $\mathbb{Q}$-Gorenstein if and only if there is a $u \in M$ so that $u(a_i) = -r \in \mathbb{Z}$ for all $1 \leq i \leq s$. If $r \neq 1$ then X is not Gorenstein (see Exercise 31 in [28]).

7.2 The Jacobian ideal and the l.c.i.-defect ideal

Here we briefly review the relevant notions of l.c.i.-defect sheaves. For proofs see the extremely clear appendix of [8]. There is a good explanation of this material in [17] as well. We adopt the notation of [8] to emphasize the dependence of the l.c.i.-defect sheaves on the Gorenstein index.

Recall that any irreducible scheme X has an associated Jacobian ideal sheaf Jac_X, defined locally as follows. Suppose that $X \subseteq \mathbf{A}^{n+c}$ is affine of codimension c, defined by an ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbf{A}^{n+c}}$. Let $x \in X$ and suppose that $\mathcal{I}$ is generated near x by $f_1, \ldots, f_s$. We can form the matrix of partials

$$\frac{\partial f_i}{\partial z_j}$$

and the Jacobian ideal is generated by the $c \times c$-minors of this matrix. This ideal does not depend on the choice of generators f_i or on the choice of local coordinates z_j (see [9], Section 20.2).

Suppose that X is $\mathbb{Q}$-Gorenstein with a Gorenstein index r. We get a map

$$(\Omega_X^n)^{\otimes r} \to \mathcal{O}_X(rK_X)$$

given by restricting a section on the left to $X \setminus X_{\text{sing}}$ and extending this restricted section to a section of $\mathcal{O}_X(rK_X)$, which is possible since X is normal and $\mathcal{O}_X(rK_X)$ is an invertible sheaf. Since $\mathcal{O}_X(rK_X)$ is an invertible sheaf there must be an ideal sheaf $\mathcal{I}_{r,X}$ so that the image of this map is $\mathcal{I}_{r,X} \cdot \mathcal{O}_X(rK_X)$. Then it is true that $(\mathrm{Jac}_X)^r \subseteq \mathcal{I}_{r,X}$ with equality if and only if X is locally a complete intersection. In general there is an ideal sheaf J_r so that

$$\overline{J_r \cdot \mathcal{I}_{r,X}} = \overline{\mathrm{Jac}_X^r},$$

where $\overline{\mathcal{I}}$ indicates the integral closure of the ideal $\mathcal{I}$. The ideal sheaf J_r is called the r-th l.c.i.-defect sheaf of X. Its support is precisely the set of points where $\mathcal{O}_{X,x}$ is not an l.c.i. local ring.

We should also mention the fact that $(J_r)^s \subseteq J_{rs}$ and $\overline{(J_r)^s} = \overline{J_{rs}}$. This ensures that the conclusions of Theorem 7.7.1 do not depend on the arbitrary choice of Gorenstein index.

7.3 A brief summary of the work of Takagi in the $\mathbb{Q}$-Gorenstein case

In [33], Takagi defines an adjoint ideal that turns out to be analogous to our adjoint ideal from Definition 5.1.7. Specifically, his definition is as follows.

Definition 7.3.1. [The Takagi adjoint ideal] Let A be a smooth complex variety and let Δ be an effective $\mathbb{Q}$-divisor on A. Let X be a reduced closed subscheme of pure codimension c so that no component of X is contained in the support of Δ. Let $f : A' \to A$ be the blow-up of X and let $E_1, \ldots, E_s$ be the resulting divisors dominating a component of X. Let $g : \overline{A} \to A$ be a log-resolution of $(A', f^*\Delta + f^{-1}X)$ so that the strict transform of $E_1 \cup \ldots \cup E_s$ is smooth. Set $\pi = f \circ g$. Define

$$\operatorname{adj}_X(A, \Delta) = \pi_* \mathcal{O}_{\overline{A}} \left(K_{\overline{A}/A} - \lfloor \pi^* \Delta \rfloor - c\pi^{-1}(X) + \sum_{j=1}^{s} E_j \right).$$

Takagi proves the following proposition, which gives a sense of what singularities the ideal $\operatorname{adj}_X(A, \Delta)$ detects.

Proposition 7.3.2. *Over an open set $U \subseteq X$, the local sections of $\operatorname{adj}_X(A, \Delta)$ consist of those $f \in \mathcal{O}_X(U)$ that satisfy the inequality*

$$ord_E(f) + a(E; A, cX + Y) > 0$$

for all divisors E over A with center intersecting U and contained in $X_{\mathrm{sing}} \cup \mathrm{Supp}(\Delta)$.

With this setup, Takagi's main theorem is the following.

Theorem 7.3.3. *If X is a normal, Gorenstein, closed subvariety of codimension c that is not contained in $\mathrm{Supp}(\Delta)$ then*

$$\mathcal{J}(X, \Delta_X + \mathbb{V}(J_1)) = \operatorname{adj}_X(A, \Delta) \cdot \mathcal{O}_X,$$

where the multiplier ideal of an $\mathbb{R}_{>0}$-linear combination of subschemes is defined as in Definition 3.3.6.

To prove this theorem, Takagi proceeds by reducing the problem to positive characteristic, where the adjoint ideal becomes an appropriate modification of the test ideal from tight closure theory. The problem is then reduced to a problem in tight closure theory.

In the earlier paper [32], Takagi also proves the following generalization of the subadditivity formula for multiplier ideals. The proof proceeds via tight closure techniques as well.

Theorem 7.3.4. *Let X be a $\mathbb{Q}$-Gorenstein complex variety. Let Jac_X be the Jacobian ideal of X and let D_1 and D_2 be two $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Then*

$$\mathrm{Jac}_X \cdot \mathcal{J}(X, D_1 + D_2) \subseteq \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2).$$

In the case of X smooth, this theorem is deduced from the restriction theorem by considering the diagonal $\Delta \subseteq X \times X$ and computing

$$\mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2) \cdot \mathcal{O}_\Delta = p_1^{-1}\mathcal{J}(X, D_1) \cdot p_2^{-1}\mathcal{J}(X, D_2).$$

Our proof of Takagi's subadditivity theorem will proceed along similar lines.

7.4 Strong factorizing resolutions

In our approach to Takagi's theorems we make crucial use of so-called strong factorizing resolutions of Bravo and Villamayor in [4]. Here we summarize the main results of their work. First we define precisely the meaning of a simple normal crossings variety.

Definition 7.4.1. A simple normal crossings variety is a possibly reducible variety X, with smooth irreducible components, so that locally analytically at every point of X there exists an isomorphism of X with a subvariety of $\mathbb{A}^n_{\mathbb{C}}$ defined by unions of intersections of coordinate hyperplanes. A scheme X has simple normal crossings support if X_{red} is a simple normal crossings variety. We say that X has simple normal crossings with Y if $X \cup Y$ has simple normal crossings support.

In particular, if X is a subscheme of a smooth variety A then X has simple normal crossings support if locally at every point $p \in A$ there exist regular parameters x_i so that the germ at p of the ideal sheaf of X is generated by elements of the form $x_{i_1}^{e_1} \cdots x_{i_s}^{e_s}$.

These varieties are, in a precise sense, the smoothest varieties that can be achieved through embedded resolutions of singularities. The following definition makes this precise.

Definition 7.4.2. An embedded resolution of singularities of a generically smooth subscheme X contained in a possibly singular variety A is a birational morphism $\pi : A' \to A$ so that:

1. A' is smooth and π is an isomorphism at every generic point of X.
2. The set $\operatorname{exc}(\pi)$ is a divisor with simple normal crossings support.
3. The strict transform of X in A', denoted X', is smooth and has simple normal crossings with $\operatorname{exc}(\pi)$.

It is a standard fact in the theory of resolutions of singularities (see e.g. the excellent exposition in [22]) that such a resolution exists whenever $X \not\subseteq A_{\text{sing}}$. Next we state the work of Bravo and Villamayor.

Definition 7.4.3. A factorizing resolution of singularities of $X \subseteq A$ as above is a birational morphism $\pi : A' \to A$ that is an embedded resolution of singularities of X in A so that, if X' is the strict transform of X in A', we have that

$$\mathcal{I}_X \cdot \mathcal{O}_{A'} = \mathcal{I}_{X'} \cdot \mathcal{L}$$

with $\mathcal{L}$ a line bundle, and such that moreover support of $\mathcal{I}_X \cdot \mathcal{O}_{A'}$ is a simple normal crossings variety.

Let Δ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with no component of X contained in $\mathrm{Supp}(\Delta)$. An embedded resolution of singularities $\pi : A' \to A$ as above is also a log-resolution of Δ if $\pi^*\Delta$ is a divisor with simple normal crossings support and $\mathrm{Supp}(\pi^*\Delta) \cup \mathrm{exc}(\pi) \cup X'$ is a simple normal crossings variety.

If $Z_1, \ldots, Z_s$ are subschemes with ideals $\mathcal{I}_{Z_i}$, we similarly define an embedded resolution $\pi : A' \to A$ that is also a log-resolution of the Z_i to be an embedded resolution as above so that

$$\mathcal{I}_{Z_1} \cdots \mathcal{I}_{Z_s} \cdot \mathcal{O}_A$$

is the ideal sheaf of a divisor with simple normal crossings support. Call this divisor F. We require that $F \cup \mathrm{exc}(\pi) \cup X'$ be a simple normal crossings variety. A resolution of a finite linear combination

$$\sum a_i Z_i$$

of subschemes of A is just a resolution of the Z_i.

If A is smooth these resolutions were shown to exist in [4]. We prove in the next lemma that the existence of these resolutions in the case that A is singular and $X \not\subseteq A_{\mathrm{sing}}$ follows formally from the smooth case.

Lemma 7.4.4. *Let $X \subseteq A$ be a generically smooth subscheme of a not necessarily smooth variety A. Let $\pi_1 : A' \to A$ be a birational morphism from a smooth variety A' that is an isomorphism at the generic points of the components of X. Let X' be the strict transform of X in A' and let E be a divisor on A' with simple normal crossing support so that no component of X' is contained in E. Then there exist* *morphisms*

$$\begin{array}{ccc} \overline{X} & \hookrightarrow & \overline{A} \\ \downarrow & & \downarrow \pi_2 \\ X' & \hookrightarrow & A' \\ \downarrow & & \downarrow \pi_1 \\ X & \hookrightarrow & A \end{array}$$

where $\overline{X}$ *is the strict transform of* X *in* $\overline{A}$ *so that* $\pi := \pi_1 \circ \pi_2$ *is a factorizing resolution of* X *inside* A *and* $\overline{X} \cup \operatorname{exc}(\pi) \cup \operatorname{Supp}(\pi_2^* E)$ *is a simple normal crossings variety.*

Proof. We perform the following procedure. Take a factorizing resolution of

$$(\pi_1^{-1}(X))_{\mathrm{red}}.$$

Note that the strict transforms of all irreducible components of $(\pi_1^{-1}(X))_{\mathrm{red}}$ are smooth and disjoint. Blow up the supports of the strict transforms of all irreducible components of $(\pi_1^{-1}(X))_{\mathrm{red}}$ other than the strict transforms of the components of X. Let $\pi^\circ : A'' \to A'$ be the resulting morphism, let $\pi'' : A'' \to A$ be its composition with π_1 and consider the subscheme X'' of A'' defined by the ideal sheaf

$$\mathcal{I}_{X''} := (\mathcal{I}_X \cdot \mathcal{O}_{A''}) \cdot \mathcal{O}_{A''}(-(\pi^\circ)^* E).$$

This is a scheme supported on the strict transform of X and a union of divisors on a smooth variety, but it may have some embedded primes.

Since A'' is smooth the divisorial components of X'' are locally principal. The embedded primes of X'' are supported either on the strict transform of X, which is generically reduced, or on one of these divisorial components. Write the divisorial part of X'' as

$$\sum a_i D_i$$

with $a_i > 0$ and let

$$\mathcal{L} = \mathcal{O}_{A''}\left(\sum (a_i - 1) D_i\right).$$

The subscheme Y of A'' defined by the ideal $\mathcal{I}_Y = \mathcal{I}_{X''} \cdot \mathcal{L}$ is generically reduced. We conclude by taking a factorizing resolution $\overline{A}$ of Y, which is now possible since it is generically smooth, and noticing that the expansion of $\mathcal{I}_Y$ and $\mathcal{I}_{X''}$ to this resolution must differ by the pull-back of $\mathcal{L}$. By definition of X'' it follows that

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = \mathcal{I}_{\overline{X}} \cdot \mathcal{M}$$

for a line bundle $\mathcal{M}$ and the simple normal crossings hypothesis in the lemma is also satisfied. □

Corollary 7.4.5. *If no component of X is contained in A_{sing} then a factorizing resolution of X always exists. Furthermore we can choose this resolution to be a log-resolution π of any $\mathbb{R}_{>0}$-linear combination Z of subschemes of A not containing any component of X in its support.*

Proof. Let $\pi' : A' \to A$ be any birational morphism with A' smooth that is an isomorphism at the generic points of X. Take a log-resolution $\pi'' : A'' \to A'$ of $(\pi')^{-1}Z + \text{exc}(\pi)$. Let $\pi_1 = \pi' \circ \pi''$ and let

$$E = (\pi'')^{-1}(\text{Supp}(\pi')^{-1}Z + \text{exc}(\pi)).$$

We apply the previous lemma to this π_1 and E to conclude. □

7.5 A different definition of the Takagi adjoint ideal

In this section we give a new way to compute the Takagi adjoint ideal. This new way is analogous to the more familiar definition from [25].

Definition 7.5.1. Let X be a generically smooth equidimensional subscheme of a variety A. Let $c = \operatorname{codim}_A(X)$ and let Δ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor with no component of X contained in $\operatorname{Supp}(\Delta) \cup A_{\text{sing}}$. Finally suppose that A is $\mathbb{Q}$-Gorenstein. Let $\pi : \overline{A} \to A$ be a log-resolution of Δ that is also a factorizing resolution of X in the sense of Corollary 7.4.5. Let $\overline{X}$ be the strict transform of X in $\overline{A}$. Write

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_X).$$

We define

$$\operatorname{adj}'_X(A, \Delta) := \pi_* \mathcal{O}_{\overline{A}}(\lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X).$$

First we prove that our definition always computes the Takagi adjoint ideal.

Proposition 7.5.2. *Keep the notation of the preceding definition. Then*

$$\operatorname{adj}_X(A, \Delta) = \operatorname{adj}'_X(A, \Delta).$$

Proof. Let π be a factorizing resolution as in Definition 7.5.1 and let A'' be the blow-up of $\overline{A}$ along $\overline{X}$. Let $\pi'' : A'' \to \overline{A}$ be the blow-up morphism. Let π' be the composition $A'' \to A$. Notice that due to the simple normal crossings hypotheses the composition $A'' \to A$ satisfies the conditions of Definition 7.3.1. Let E be the (reduced) union of the exceptional divisors lying above the generic points of the irreducible components of X. Since π'' is a blow-up of smooth centers transverse to the exceptional locus of π we compute:

$$\begin{aligned}(\pi'')^* \mathcal{O}_{\overline{A}}(\lceil K_{\overline{A}/A} - \pi^*\Delta - cR_X \rceil) &= \mathcal{O}_{A''}(\lceil K_{A''/A} - (\pi')^*\Delta - cR_X \rceil - (c-1)E) \\ &= \mathcal{O}_{A''}(\lceil K_{A''/A} - (\pi') * \Delta - c(\pi')^{-1}(X) + E \rceil).\end{aligned}$$

By the universal property of blow-ups, π' must factor through the blow-up of X in A. But then the divisor we just arrived at computes Takagi's adjoint. We conclude by the projection formula. $\square$

From now on we will conflate the two notations, that is, we will write $\operatorname{adj}'_X(A,\Delta)$ as $\operatorname{adj}_X(A,\Delta)$. We have observed that Takagi has already shown that $\operatorname{adj}_X(A,\Delta)$ does not depend on the choice of resolution and so our adjoint does not either.

We will now prove a formula that may seem technical at first, but that packages the application of the local vanishing theorem that we will use to deduce our restriction theorems.

Lemma 7.5.3. *Keep the notation of Definition 7.5.1. Let*

$$D := \lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X,$$

so that we have the usual short exact sequence

$$0 \to \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D) \to \mathcal{O}_{\overline{A}}(D) \to \mathcal{O}_{\overline{X}}(D_{\overline{X}}) \to 0. \tag{7.1}$$

Then

$$R^i\pi_*(\mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)) = 0$$

for all $i > 0$. In particular, if f is the restriction of π to $\overline{X}$,

$$\operatorname{adj}_X(A,\Delta) \cdot \mathcal{O}_X = f_*(\mathcal{O}_{\overline{A}}(\lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X) \cdot \mathcal{O}_{\overline{X}}).$$

In other words we may restrict first then push forward. Furthermore,

$$\pi_*\mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D) = \mathcal{J}(A, cX + \Delta).$$

Proof. We calculate as follows. Let $\pi'' : A'' \to \overline{A}$ be the blow-up of $\overline{X}$ with reduced exceptional divisor E. Let π' be the composition $A'' \to A$. Then

$$\begin{aligned}(\mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{A''}) \cdot (\pi'')^*\mathcal{O}_{\overline{A}}(\lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X) &= \mathcal{O}_{A''}((\pi'')^*(\lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X) - E) \\ &= \mathcal{O}_{A''}(\lceil K_{A''/A} - (\pi')^*\Delta - c(\pi')^{-1}X \rceil).\end{aligned}$$

This has vanishing higher direct images by local vanishing (see Theorem 3.4.4). Furthermore,

$$\pi_* \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D) = \pi'_* \mathcal{O}_{A''}(\lceil K_{A''/A} - (\pi')^*\Delta - c(\pi')^{-1}X \rceil) = \mathcal{J}(A, cX + \Delta).$$

In turn, local vanishing for π'' implies that

$$R^i \pi''_* (\mathcal{O}_{A''}(\lceil K_{A''/A} - (\pi')^*\Delta - c(\pi')^{-1}X \rceil)) = 0$$

for all $i > 0$. We conclude by the following lemma. $\square$

Lemma 7.5.4. *Suppose we are given a diagram of proper morphisms*

$$\begin{tikzcd} & X'' \arrow[dl, "h"'] \arrow[dd, "f"] \\ X' \arrow[dr, "g"'] & \\ & X \end{tikzcd}$$

and a coherent sheaf $\mathcal{F}$ on X''. Suppose that $R^j h_* \mathcal{F}$ and $R^j f_* \mathcal{F}$ vanish for all $j > 0$. Then $R^j g_*(h_* \mathcal{F}) = 0$ for all $j > 0$.

Proof. This is an easy consequence of the Leray spectral sequence. Indeed, the Leray spectral sequence has the form

$$R^i g_*(R^j h_* \mathcal{F}) \Rightarrow R^{i+j} f_* \mathcal{F}.$$

Since $R^j h_* \mathcal{F} = 0$ for all $j > 0$ the spectral sequence degenerates at the E_2 sheet. Now the assumption that $R^j f_* \mathcal{F} = 0$ for all $j > 0$ immediately implies the conclusion. $\square$

7.6 A high-codimension adjunction formula for relative canonical classes

We now discuss the adjunction formula that we will use in our proof of Takagi's restriction theorem. We begin with a formula for Jacobian ideals. First we introduce some notation.

Definition 7.6.1. Let A be a matrix of elements of a commutative ring R. We denote by $[A]_n$ the ideal of R generated by the $n \times n$-minors of A. This definition extends to the situation of a map $\varphi : \mathcal{F} \to \mathcal{G}$ of locally free sheaves to give $[\varphi]_n$.

Let $f : Y \to X$ be a morphism of possibly reducible varieties of pure dimension with Y smooth and $\dim(X) = \dim(Y) = n$. Consider the natural map $f^*\Omega^n_X \to \Omega^n_Y$. The image of this map is, by definition, given by $\mathrm{Jac}_f \cdot \Omega^n_Y$. If X is also smooth then, in local coordinates, Jac_f is just $[df]_n$.

Our next goal is to prove a general lemma regarding Jacobian ideals that can be viewed as a kind of chain rule. It will be useful in the current generality in our investigation of the subadditivity theorem. First, we make a few definitions.

Setup 7.6.2. *Here we indicate assumptions that will be in force later.*

(a) Denote by A a smooth variety of dimension N and X an equidimensional possibly reducible subvariety of dimension n and codimension c. Let $\mathfrak{a}$ to be an ideal sheaf on A contained in the ideal sheaf of X. We denote by $\pi : A' \to A$ a birational morphism with A' smooth that is furthermore an isomorphism at every generic point of X. Denote by X' the strict transform of X along π. We assume that X' is smooth. Let $f : X' \to X$ be the restriction of π. The diagram of morphisms is as follows.

$$\begin{tikzcd} X' \arrow[r, hook] \arrow[d, "f"'] & A' \arrow[d, "\pi"] \\ X \arrow[r, hook] & A \end{tikzcd}$$

(b) Let $p \in A'$ and let the germ of $\mathfrak{a}$ at $\pi(p)$ be generated by $(h_1, \ldots, h_m)$. Let $w_1, \ldots, w_N$ be local coordinates of A' at p and let $z_1, \ldots, z_N$ be local coordinates of A at $\pi(p)$. We suppose finally that $w_1, \ldots, w_n$ restrict to local coordinates on X' at p and that all other w_j restrict to zero on X'.

(c) To distinguish the two constructions, in the case where π is a factorizing resolution for X we will write $\overline{A}$ instead of A' and $\overline{X}$ instead of X'.

With these choices we have the following formula.

Lemma 7.6.3. *Keep the notation of Setup 7.6.2. As germs at p,*

$$\mathrm{Jac}_f \cdot \left(\left[\frac{\partial(h_i \circ \pi)}{\partial w_j} \right]_c \right) \cdot \mathcal{O}_{X'} = \left(\mathrm{Jac}_\pi \cdot \left[\frac{\partial h_i}{\partial z_j} \right]_c \cdot \mathcal{O}_{A'} \right) \cdot \mathcal{O}_{X'}.$$

Here π need not be factorizing for X.

Proof. If $m < c$ the statement is trivial so let I be a multi-index of type $\binom{N}{n}$ and let J be a multi-index of type $\binom{m}{c}$. Consider the form

$$\omega_{I,J} = d(z_{i_1} \circ \pi) \wedge \cdots \wedge d(z_{i_n} \circ \pi) \wedge d(h_{j_1} \circ \pi) \wedge \cdots \wedge d(h_{j_c} \circ \pi).$$

The form $\omega_{I,J}$ is an element of the module $(\Omega^N_{A'})_p$. Let $\mathfrak{b}$ be the ideal generated by the $\omega_{I,J}$ for all choices of I and J. On the one hand,

$$\begin{aligned} \omega_{I,J} &= \pi^*(dz_{i_1} \wedge \cdots \wedge dz_{i_n} \wedge dh_{j_1} \wedge \cdots \wedge dh_{j_c}) \\ &= \pm \pi^* \left(m_{I^c,J} \cdot (dz_1 \wedge \cdots \wedge dz_N) \right) \\ &= \pm \mathrm{Jac}_\pi \cdot (m_{I^c,J} \circ \pi) \cdot (dw_1 \wedge \cdots \wedge dw_N). \end{aligned}$$

where $m_{I^c,J}$ is the minor of the matrix of partials $\frac{\partial h_i}{\partial z_j}$ corresponding to the rows $(1, \ldots, N) \setminus I$ and columns J. It follows that

$$\mathfrak{b} \cdot \mathcal{O}_{X'} = \left(\mathrm{Jac}_\pi \cdot \left[\frac{\partial h_i}{\partial z_j} \right]_c \cdot \mathcal{O}_{A'} \right) \cdot \mathcal{O}_{X'}.$$

Now observe that for any i we have

$$d(h_i \circ \pi)_{X'} = d((h_i \circ \pi)_{X'}) = 0$$

since the h_i vanish on X. On the other hand,

$$d(h_i \circ \pi)_{X'} = \sum_{j=1}^{N} \left(\frac{\partial(h_i \circ \pi)}{\partial w_j} \right)_{X'} d(w_{j,X'}).$$

We choose $w_{1,X'}, \ldots, w_{n,X'}$ to be local coordinates on X' at p, so the $d(w_{j,X'})$ are linearly independent for $1 \leq j \leq n$ while the rest are zero. It follows that we must have

$$\left(\frac{\partial(h_i \circ \pi)}{\partial w_j}\right)_{X'} = 0$$

for $1 \leq j \leq n$.

Now, if we define $\pi_i = z_i \circ \pi$, then

$$\omega_{I,J} = \left(\sum_{S \text{ type } \binom{N}{n}} \left(\frac{\partial \pi_{i_1}}{\partial w_{s_1}} \cdots \frac{\partial \pi_{i_n}}{\partial w_{s_n}} \right) \cdot dw_{s_1} \wedge \cdots \wedge dw_{s_n} \right) \wedge$$
$$\left(\sum_{T \text{ type } \binom{N}{c}} \left(\frac{\partial(h_{j_1} \circ \pi)}{\partial w_{t_1}} \cdots \frac{\partial(h_{j_c} \circ \pi)}{\partial w_{t_c}} \right) \cdot dw_{t_1} \wedge \cdots \wedge dw_{t_c} \right).$$

By our calculation of the derivatives of $h_i \circ \pi$, the terms

$$\left(\frac{\partial(h_{j_1} \circ \pi)}{\partial w_{t_1}} \cdots \frac{\partial(h_{j_c} \circ \pi)}{\partial w_{t_c}} \right)_{X'}$$

are non-zero only if $T = (n+1, \ldots, N)$. It follows that

$$\mathfrak{b} \cdot \mathcal{O}_{X'} = \left([d\pi]_{n,(1,\ldots,n)} \cdot \left[\frac{\partial(h_i \circ \pi)}{\partial w_j} \right]_c \right) \cdot \mathcal{O}_{X'}$$

where $[d\pi]_{n,(1,\ldots,n)}$ is the ideal of $n \times n$-minors of $d\pi$ with the choice of columns (here the columns give the variables that we differentiate with respect to) equal to $(1, \ldots, n)$. But, since $w_1, \ldots, w_n$ were chosen to restrict to the local coordinates of X' and $w_{n+1}, \ldots, w_N$ were chosen to restrict to zero on X' it is immediate that

$$([d\pi]_{n,(1,\ldots,n)}) \cdot \mathcal{O}_{X'} = \mathrm{Jac}_f.$$

$\square$

The following lemma is essentially the adjunction formula that we will use to deduce Takagi's restriction theorem.

Lemma 7.6.4. *Keep the notation of Setup 7.6.2. Suppose furthermore that π is a factorizing resolution*[1] *of X. Write*

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_X).$$

Then

$$(\mathrm{Jac}_\pi \cdot \mathcal{O}_{\overline{X}}) \cdot (\mathrm{Jac}_X \cdot \mathcal{O}_{\overline{X}}) = \mathrm{Jac}_f \cdot (\mathcal{O}_{\overline{A}}(-cR_X)) \cdot \mathcal{O}_{\overline{X}}.$$

Proof. We apply the previous lemma. Choose $\mathfrak{a}$ to be the ideal of X. Suppose that, at p, $I = (h_1, \ldots, h_m)$. We get that

$$\mathrm{Jac}_f \cdot \left(\left[\frac{\partial (h_i \circ \pi)}{\partial w_j} \right]_c \right) \cdot \mathcal{O}_{\overline{X}} = \left(\mathrm{Jac}_\pi \cdot \left[\frac{\partial h_i}{\partial z_j} \right]_c \cdot \mathcal{O}_{\overline{A}} \right) \cdot \mathcal{O}_{\overline{X}}.$$

By definition,

$$\left(\left[\frac{\partial h_i}{\partial z_j} \right]_c \right) \cdot \mathcal{O}_X = \mathrm{Jac}_X.$$

Now, as germs at p we may write

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = (h_1 \circ \pi, \ldots, h_m \circ \pi) = (g\overline{h}_1, \ldots, g\overline{h}_m)$$

with g a local generator of $\mathcal{O}_{\overline{A}}(-R_X)$. But

$$\frac{\partial (h_i \circ \pi)}{\partial w_j} = \frac{\partial (g\overline{h}_i)}{\partial w_j} = g\frac{\partial (\overline{h}_i)}{\partial w_j} + \overline{h_i}\frac{\partial g}{\partial w_j}.$$

Since $\overline{h}_i$ restricts to zero on $\overline{X}$, we see that

$$\left(\left[\frac{\partial (h_i \circ \pi)}{\partial w_j} \right]_c \right) \cdot \mathcal{O}_{\overline{X}} = g^c \cdot \mathrm{Jac}_{\overline{X}} = (g^c)$$

since $\overline{X}$ is smooth. □

The next step is to interpret the various Jacobian ideals that appear in this formula in terms of relative canonical classes.

[1] Recall that, in Setup 7.6.2, we agreed to use $\overline{A}$ and $\overline{X}$ in the notation for factorizing resolutions.

Lemma 7.6.5. *Let X be a $\mathbb{Q}$-Gorenstein possibly reducible variety and let $f : \overline{X} \to X$ be a birational morphism with $\overline{X}$ smooth. Let r be a Gorenstein index of X. Then*

$$\mathrm{Jac}_f^r = (\mathcal{I}_{r,X} \cdot \mathcal{O}_{\overline{X}}) \cdot \mathcal{O}_{\overline{X}}(-rK_{\overline{X}/X}).$$

Proof. Write

$$K_{\overline{X}} + K^- = f^*K_X + K^+$$

with K^+, K^- effective. We get a map

$$f^*\mathcal{O}_X(rK_X) \otimes \mathcal{O}_{\overline{X}}(-rK^-) \to \mathcal{O}_{\overline{X}}(r(K_{\overline{X}})).$$

The image of this map is given by $I \cdot \mathcal{O}_{\overline{X}}(rK_{\overline{X}})$ where I is the ideal $\mathcal{O}_{\overline{X}}(-rK^+)$. Next we have a commutative diagram

$$\begin{tikzcd}
f^*(\Omega_X^n)^{\otimes r} \otimes \mathcal{O}_{\overline{X}}(-rK^-) \arrow[r] \arrow[dr] & \mathcal{O}_{\overline{X}}(rK_{\overline{X}}) \\
& f^*\mathcal{O}_{\overline{X}}(rK_X) \otimes \mathcal{O}_{\overline{X}}(-rK^-). \arrow[u]
\end{tikzcd}$$

By computing the images of these maps we see that

$$\mathrm{Jac}_f^r \cdot \mathcal{O}_{\overline{X}}(-rK^-) = (\mathcal{I}_{r,X} \cdot \mathcal{O}_{\overline{X}}) \cdot \mathcal{O}_{\overline{X}}(-rK^+).$$

The required statement follows by rearranging this equation. ∎

We can now finally prove our adjunction formula.

Theorem 7.6.6. *Let A be a smooth variety and let X be a generically smooth equidimensional subscheme. Let $\pi : \overline{A} \to A$ be a factorizing resolution of X inside A and let f be the restriction of π to $\overline{X}$, the strict transform of X along π. Write*

$$\mathcal{I}_X \cdot \mathcal{O}_{\overline{A}} = \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(-R_X).$$

Suppose that X is $\mathbb{Q}$-Gorenstein with a Gorenstein index r. Suppose further that f is a log-resolution of $\mathcal{I}_{r,X}$ and J_r. Let D be the divisor defined by

$$J_r \cdot \mathcal{O}_{\overline{X}} = \mathcal{O}_{\overline{X}}(-D).$$

Then

$$K_{\overline{X}/X} - \frac{1}{r}D = (K_{\overline{A}/A} - cR_X)_{\overline{X}}$$

with equality being equality of $\mathbb{Q}$-divisors on $\overline{X}$.

Proof. This follows easily from the previous two lemmas and the definition of J_r. □

Note that the necessary π, that is, a log-resolution of $\mathcal{I}_{r,X}$ and J_r can always be found by Lemma 7.4.4. Lastly we record the following easy fact that will be useful for the subadditivity theorem.

Lemma 7.6.7. *Let $f_1 : A \to B$ and $f_2 : B \to C$ be birational morphisms and let $f = f_2 \circ f_1$. Suppose that A and B are smooth. Then*

$$\mathrm{Jac}_f = (\mathrm{Jac}_{f_2} \cdot \mathcal{O}_A) \cdot \mathrm{Jac}_{f_1}$$

Proof. Let n be the dimension of the varieties involved. Consider the composition of natural maps

$$f^*\Omega^n_C \to f_2^*\Omega^n_B \to \Omega^n_A.$$

The composition is df and the formula follows easily since Ω^n_B and Ω^n_A are line bundles and the second morphism is just multiplication by a generator of Jac_{f_1}. □

7.7 Takagi's restriction theorem in the $\mathbb{Q}$-Gorenstein case

The tools developed so far enable us to give a quick proof of a stronger form of the restriction theorem given by Takagi in his paper [33].

Theorem 7.7.1. *Let $X \subseteq A$ be a $\mathbb{Q}$-Gorenstein (in particular, reduced) equidimensional subscheme of an ambient smooth variety with a Gorenstein index r and codimension c. Let Δ be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor not containing any component of X in its support. Then there exists a short exact sequence*

$$0 \to \mathcal{J}(A, cX + \Delta) \to \mathrm{adj}_X(A, \Delta) \to \mathcal{J}\left(X, \frac{1}{r}\mathbb{V}(J_r) + \Delta_X\right) \to 0$$

with the first map given by inclusion and the last map given by restriction to X.

Proof. Let $\pi : \overline{A} \to A$ be a factorizing resolution as in Definition 7.5.1 and in Theorem 7.6.6. Lemma 7.5.3 gives the short exact sequence

$$0 \to \mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D) \to \mathcal{O}_{\overline{A}}(D) \to \mathcal{O}_{\overline{X}}(D_{\overline{X}}) \to 0$$

where $D := \lceil K_{\overline{A}/A} - \pi^*\Delta \rceil - cR_X$. The same lemma states that

$$R^i \pi_*(\mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)) = 0$$

for all $i > 0$. We have already seen in Lemma 7.5.3 that

$$\pi_*(\mathcal{I}_{\overline{X}} \cdot \mathcal{O}_{\overline{A}}(D)) = \mathcal{J}(A, cX + \Delta)$$

and

$$\pi_* \mathcal{O}_{\overline{A}}(D) = \mathrm{adj}_X(A, \Delta)$$

by definition. We must only check that

$$f_* \mathcal{O}_{\overline{X}}(D_{\overline{X}}) = \mathcal{J}\left(X, \frac{1}{r}\mathbb{V}(J_r) + \Delta_X\right).$$

This follows immediately from the following expresssion

$$\mathcal{O}_{\overline{X}}(D_{\overline{X}}) = \mathcal{O}_{\overline{X}}\left(\left\lceil K_{\overline{X}/X} - \frac{1}{r}\pi^{-1}(J_r) - \pi^*\Delta_X) \right\rceil\right) \tag{7.2}$$

which itself follows immediately from the formula of Theorem 7.6.6. □

The following form of our restriction theorem answers a question of Takagi's in [33], Remark 3.2, (3).

Corollary 7.7.2. *In the situation of the theorem we have the formulas*

$$\mathrm{adj}_X(A,\Delta)\cdot\mathcal{O}_X = \mathcal{J}\left(X, \frac{1}{r}\mathbb{V}(J_r)+\Delta_X\right)$$

and

$$\mathcal{J}\left(X, \frac{1}{r}\mathbb{V}(J_r)+\Delta_X\right) \subseteq \mathcal{J}(A,\Delta)\cdot\mathcal{O}_X.$$

Proof. The first expression follows immediately from Theorem 7.7.1 while the second follows from the easy observation that $\mathrm{adj}_X(A,\Delta)\subseteq\mathcal{J}(A,\Delta)$ by definition. $\square$

Corollary 7.7.3. *Keep the notation of Definition 7.5.1. The adjoint ideal satisfies local vanishing, that is,*

$$R^i\pi_*\mathcal{O}_{\overline{A}}(\lceil K_{\overline{A}/A}-\pi^*\Delta\rceil - cR_X) = 0$$

for all $i>0$.

Proof. This follows from the long exact sequence for $R^i\pi_*$ that arises from pushing forward

$$0\to\mathcal{I}_{\overline{X}}\cdot\mathcal{O}_{\overline{A}}(D)\to\mathcal{O}_{\overline{A}}(D)\to\mathcal{O}_{\overline{X}}(D_{\overline{X}})\to 0$$

in Lemma 7.5.3. In this lemma we have already seen that the term on the left has vanishing higher direct images. On the other hand, the expression for $\mathcal{O}_{\overline{X}}(D_{\overline{X}})$ in (7.2) shows that this sheaf has vanishing higher direct images by the local vanishing theorem again (see Theorem 3.4.4). $\square$

7.8 A characteristic zero proof of Takagi's subadditivity theorem

We present a proof of Takagi's subadditivity theorem that uses only standard algebro-geometric characteristic zero techniques: resolution of singularities and the vanishing theorem of Kawamata-Viehweg. Our approach will be similar to the approach of [6]. Specifically, we will make use of the following observation.

Lemma 7.8.1. *Let X_1, X_2 be $\mathbb{Q}$-Gorenstein varieties and let D_1 and D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X_1 and X_2, respectively. Let p_1 and p_2 be the projections from $X_1 \times X_2$ to X_1 and X_2 respectively. Then*

$$\mathcal{J}(X_1 \times X_2, p_1^* D_1 + p_2^* D_2) = p_1^{-1}\mathcal{J}(X_1, D_1) \cdot p_2^{-1}\mathcal{J}(X_2, D_2).$$

Proof. The proof in [25], Proposition 9.5.22, goes through without the requirement that X_1 and X_2 be smooth. $\square$

From now on let X be a $\mathbb{Q}$-Gorenstein variety of dimension n and Gorenstein index r and let $\Delta \subseteq X \times X$ be the diagonal. Let $g : X' \to X$ be a proper birational morphism from a smooth variety X' and let $\rho : X' \times X' \to X \times X$ be the product morphism. Let Δ' be the strict transform of Δ in $X' \times X'$. Notice that Δ' is the diagonal of $X' \times X'$ and the induced morphism $\Delta' \to \Delta$ is just g.

The obstruction to the proof of Theorem 7.8.4 is the restriction theorem: it requires a smooth ambient space. We will show that, in our very special situation of the diagonal in $X \times X$, an appropriate restriction theorem holds. To this end we need to uncover the analog of the adjunction formula. The following lemma becomes precisely the required analog once we use our earlier work to translate the Jacobian ideals into relative canonical classes.

Lemma 7.8.2. *Let*

$$\begin{tikzcd}
\overline{\Delta} \arrow[r, hook] \arrow[d, "f"] \arrow[dd, bend right=60, "h"'] & A' \arrow[d, "\pi"] \arrow[dd, bend left=60, "\sigma"] \\
\Delta' \arrow[r, hook] \arrow[d, "g"] & X' \times X' \arrow[d, "\rho"] \\
\Delta \arrow[r, hook] & X \times X
\end{tikzcd}$$

be a factorizing resolution of Δ *given by Lemma 7.4.4. Let*

$$\mathcal{I}_{\Delta} \cdot \mathcal{O}_{A'} = \mathcal{I}_{\overline{\Delta}} \cdot \mathcal{O}_{A'}(-R_{\Delta}).$$

Then

$$(\mathrm{Jac}_{\pi} \cdot \mathcal{O}_{\overline{\Delta}}) \cdot (\mathrm{Jac}_{g} \cdot \mathcal{O}_{\overline{\Delta}})^2 \subseteq \mathrm{Jac}_{h} \cdot (\mathcal{O}_{A'}(-nR_{\Delta})) \cdot \mathcal{O}_{\overline{\Delta}}.$$

Proof. To simplify and unify notation, let $A = X' \times X'$ and $B = X \times X$ and fix a point $p \in A'$. Let w_j be coordinates on A' and let z_j be coordinates on A at $\pi(p)$ as in Setup 7.6.2. Let $\mathrm{pr}_1, \mathrm{pr}_2 : X \times X \to X$ be the two projections. Let $s_{1,j}$ be generators on X of the maximal ideal of the local ring at $\mathrm{pr}_1(\sigma(p))$, where $1 \leq j \leq M$ and similarly let $s_{2,j}$ be generators on X of the maximal ideal of the local ring at $\mathrm{pr}_2(\sigma(p))$. Finally, let $x_i = \mathrm{pr}_1^*(s_{1,i})$ and $y_i = \mathrm{pr}_2^*(s_{2,i})$. Then

$$(\mathcal{I}_{\Delta})_{\sigma(p)} = (g_1, \ldots, g_M)$$

where $g_i = x_i - y_i$. By Lemma 7.6.3 applied with the above choices, $h_i = g_i \circ \rho$, $X = \Delta'$ and $X' = \overline{\Delta}$ we have that

$$\mathrm{Jac}_f \cdot \left(\left[\frac{\partial (g_i \circ \sigma)}{\partial w_j} \right]_n \right) \cdot \mathcal{O}_{\overline{\Delta}} = \left(\mathrm{Jac}_{\pi} \cdot \left[\frac{\partial (g_i \circ \rho)}{\partial z_j} \right]_n \cdot \mathcal{O}_{A'} \right) \cdot \mathcal{O}_{\overline{\Delta}}. \tag{7.3}$$

Suppose that we can show that on Δ' we have

$$(\mathrm{Jac}_g)^2 \subseteq \mathrm{Jac}_g \cdot \left(\left[\frac{\partial (g_i \circ \rho)}{\partial z_j} \right]_n \cdot \mathcal{O}_{\Delta'} \right). \tag{7.4}$$

Then, after multiplying both sides of (7.3) by Jac_g and using Lemma 7.6.7, we obtain first of all that

$$(\mathrm{Jac}_{\pi} \cdot \mathcal{O}_{\overline{\Delta}}) \cdot (\mathrm{Jac}_{g} \cdot \mathcal{O}_{\overline{\Delta}})^2 \subseteq \mathrm{Jac}_{h} \cdot \left(\left[\frac{\partial (g_i \circ \sigma)}{\partial w_j} \right]_n \cdot \mathcal{O}_{\overline{\Delta}} \right).$$

As in the proof of Lemma 7.6.4, write next

$$(\mathcal{I}_{\Delta} \cdot \mathcal{O}_{A'})_p = (g_1 \circ \sigma, \ldots, g_M \circ \sigma) = (r\overline{g}_1, \ldots, r\overline{g}_M)$$

where r is a local generator of the sheaf $\mathcal{O}_{A'}(-R_{\overline{\Delta}})$. We now have that

$$\left(\frac{\partial(g_i \circ \sigma)}{\partial w_j}\right)_{\overline{\Delta}} = \left(r\frac{\partial \overline{g}_i}{\partial w_j} + \overline{g}_i \frac{\partial r}{\partial w_j}\right)_{\overline{\Delta}} = \left(r\frac{\partial \overline{g_i}}{\partial w_j}\right)_{\overline{\Delta}}$$

and so

$$\left(\left[\frac{\partial(g_i \circ \sigma)}{\partial w_j}\right]_n \cdot \mathcal{O}_{\overline{\Delta}}\right) = r^n \cdot \mathrm{Jac}_{\overline{\Delta}} = (r^n)$$

since $\overline{\Delta}$ is smooth. But this concludes the proof, assuming (7.4).

It remains to show (7.4). In fact, it is clearly enough to show that

$$\mathrm{Jac}_g \subseteq \left[\frac{\partial(g_i \circ \rho)}{\partial z_j}\right]_n \cdot \mathcal{O}_{\Delta'}. \tag{7.5}$$

For this we choose the z_j as follows: let $\mathrm{pr}_1', \mathrm{pr}_2' : X' \times X' \to X'$ be the two projections and let $s_{1,1}', \ldots, s_{1,n}'$ be local coordinates on X' at $\mathrm{pr}_1'(\pi(p))$, $s_{2,1}', \ldots, s_{2,n}'$ local coordinates on X' at $\mathrm{pr}_2'(\pi(p))$. Let $x_i' = (\mathrm{pr}_1')^*(s_{1,i}')$ and $y_i' = (\mathrm{pr}_2')^*(s_{2,i}')$. Set $z_j = x_j'$ for $1 \leq j \leq n$ and $z_j = y_{j-n}'$ for $n+1 \leq j \leq 2n$.

Notice that, since $y_i \circ \rho$ does not depend on x_j' we have that

$$\frac{\partial(y_i \circ \rho)}{\partial x_j'} = 0.$$

It follows that, in these coordinates, the matrix of partials

$$\frac{\partial(g_i \circ \rho)}{\partial z_j} = \frac{\partial(x_i \circ \rho)}{\partial z_j} - \frac{\partial(y_i \circ \rho)}{\partial z_j}$$

is block diagonal with two blocks,

$$\frac{\partial(x_i \circ \rho)}{\partial x_j'}, \ -\frac{\partial(y_i \circ \rho)}{\partial y_j'}.$$

It is furthermore clear that the ideal of $n \times n$ - minors of each of these two blocks is Jac_g. But this in particular proves (7.5), as required. □

Corollary 7.8.3. *Keep the notation of the lemma. Suppose that h furthermore log-resolves $\mathcal{I}_{r,X}$ and J_r. Let F be the divisor defined by*

$$\mathcal{I}_{r,\Delta} \cdot \mathcal{O}_{\overline{\Delta}} = \mathcal{O}_{\overline{\Delta}}(-F).$$

Then the following inequality is holds:

$$K_{\overline{\Delta}/\Delta} - \frac{1}{r}F \leq (K_{A'/B} - nR_{\Delta})_{\overline{\Delta}}.$$

Proof. It follows from Lemma 7.6.5 that

$$\mathrm{Jac}_h^r = (\mathcal{I}_{r,X} \cdot \mathcal{O}_{\overline{\Delta}}) \cdot \mathcal{O}_{\overline{\Delta}}(-rK_{\overline{\Delta}/\Delta}),$$

and similarly for Jac_g^r. Combining this with the inequality of ideals

$$(\mathrm{Jac}_\pi \cdot \mathcal{O}_{\overline{\Delta}}) \cdot (\mathrm{Jac}_g \cdot \mathcal{O}_{\overline{\Delta}})^2 \subseteq \mathrm{Jac}_h \cdot (\mathcal{O}_A(-nR_\Delta)) \cdot \mathcal{O}_{\overline{\Delta}}$$

we get the following inequality of divisors:

$$-(K_{A'/A})_{\overline{\Delta}} - \frac{2}{r}F - 2f^*K_{\Delta'/\Delta} \leq -\frac{1}{r}F - K_{\overline{\Delta}/\Delta} - (nR_\Delta)_{\overline{\Delta}}.$$

Since the morphism $\rho : A \to B$ is the product $g \times g : X' \times X' \to X \times X$, we have that

$$2K_{\Delta'/\Delta} = (K_{A/B})_{\Delta'}.$$

The inequality simplifies to

$$-(K_{A'/A})_{\overline{\Delta}} - \frac{1}{r}F - f^*(K_{A/B})_{\Delta'} \leq -K_{\overline{\Delta}/\Delta} - (nR_\Delta)_{\overline{\Delta}}.$$

The corollary now follows easily. □

We can now finally prove our version of Takagi's subadditivity theorem.

Theorem 7.8.4. *Let X be a $\mathbb{Q}$-Gorenstein variety and let D_1, D_2 be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Then*

$$\overline{\mathrm{Jac}_X} \cdot \mathcal{J}(X, D_1 + D_2) \subseteq \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2).$$

Proof. It is enough to show that

$$\overline{\mathrm{Jac}_X} \cdot \mathcal{J}(X, D_1 + D_2) \subseteq \mathrm{adj}_\Delta(X \times X, p_1^* D_1 + p_2^* D_2) \cdot \mathcal{O}_\Delta$$

where $p_1, p_2 : X \times X \to X$ are the two projections, since we have the easy inequality

$$\mathrm{adj}_\Delta(X \times X, p_1^* D_1 + p_2^* D_2) \subseteq \mathcal{J}(X \times X, p_1^* D_1 + p_2^* D_2)$$

and we will conclude by applying Lemma 7.8.1. We let $\sigma : A' \to X \times X$ be the log-resolution as in Lemma 7.8.2 and we choose $g : X' \to X$ to also log-resolve Jac_X, $\mathcal{I}_{r,X}$, J_r and D_1 and D_2, and we assume finally that σ is a log-resolution of all of these that is also a factorizing resolution for Δ. Then, with the notation of the preceding corollary, we obtain that

$$K_{\overline{\Delta}/\Delta} - \frac{1}{r} F \leq (K_{A'/B} - nR_\Delta)_{\overline{\Delta}}. \tag{7.6}$$

Let G be the divisor defined by

$$\mathrm{Jac}_X \cdot \mathcal{O}_{\overline{\Delta}} = \mathcal{O}_{\overline{\Delta}}(-G).$$

Since $\mathcal{I}_{r,\Delta}$ is an ideal that contains Jac_X^r we finally obtain the inequality

$$K_{\overline{\Delta}/\Delta} - G \leq (K_{A'/B} - nR_\Delta)_{\overline{\Delta}}. \tag{7.7}$$

But by Lemma 7.5.3 we have

$$\mathrm{adj}_\Delta(X \times X, p_1^* D_1 + p_2^* D_2) \cdot \mathcal{O}_\Delta = h_*(\mathcal{O}_{A'}(\lceil K_{A'/B} - \sigma^{-1}(p_1^* D_1 + p_2^* D_2) - nR_\Delta \rceil)_{\overline{\Delta}}).$$

Putting this together with our inequality we are done. □

Remark 7.8.5. In fact, the proof also shows that

$$\langle \mathcal{I}_{r,X} \rangle^{1/r} \cdot \mathcal{J}(X, D_1 + D_2) \subseteq \mathcal{J}(X, D_1) \cdot \mathcal{J}(X, D_2)$$

in the sense of Kawakita's $\mathbb{Q}$-ideals and his partial ordering on them (see [17]). Indeed, to obtain this we simply have to skip the estimate in inequality 7.7 and use the inequality in 7.6 on the sheaf

$$h_*(\mathcal{O}_{A'}(\lceil K_{A'/B} - \sigma^{-1}(p_1^* D_1 + p_2^* D_2) - nR_\Delta \rceil)_{\overline{\Delta}}).$$

CHAPTER 8

Hacon linear series and the extension theorem

In this chapter we will switch gears slightly and investigate a construction due to Christopher Hacon in [15]. This construction is a generalization of the asymptotic multiplier ideal. We begin by working out a version of this idea in the setting of incomplete linear series. This leads us to consider sequences $\overline{W}_l^k$ of linear series whose section ring is $\mathbb{N}^2$-graded, we call these bigraded linear series. There is a special subsequence of $\overline{W}_l^k$, which we call the associated vertical subseries W_l^k, that is a graded linear series in k for each fixed l. We write down a condition that we call the Hacon inequality that ensures that the asymptotic multiplier ideals $\mathcal{J}(X, c\cdot|W_l^\bullet|)$ stabilize.

We then write down some basic statements about the case of complete linear series in the construction of the Hacon ideal. We obtain an ideal $\mathcal{J}_-(X, c \cdot ||M||)$ that we call the restricted multiplier ideal. We prove a vanishing theorem for this ideal and a generalization of a theorem of Goodman regarding the detection of nef line bundles by multiplier ideals. A similar exploration was independently undertaken in [27].

We finally use this formalism to generalize the famous extension theorem of Siu to the high-codimension situation. Specifically, we extend from $Z \subseteq X$ pluri-canonical sections of Cartier divisors of the form $K_X + A + \Delta$ where A is big and nef and

(X, Δ) is log-canonical with exceptional log-canonical center Z.

8.1 Hacon linear series

In this section, we adopt the construction of Hacon in [15] to the case of incomplete linear series. We are led to consider sequences of linear series $\overline{W}_l^k$ that are $\mathbb{N}^2$-graded and we write down an important subsequence that we call the associated vertical subseries. We conclude by adapting the argument of Hacon to write down a condition that ensures that the multiplier ideals $\mathcal{J}(X, c \cdot |W_l^k|)$ stabilize. We call this condition the Hacon inequality.

Definition 8.1.1. Let X be a projective variety. Let M and H be two divisors on X. A sequence of linear series

$$\overline{W}_l^k \subseteq H^0(X, \mathcal{O}_X(lM + kH))$$

is called a bigraded linear series if

1. For all l there is a sufficiently divisible $k := k(l)$ so that $\overline{W}_l^k \neq \{0\}$.
2. $\overline{W}_0^0 = \mathbb{C}$.
3. The inequality
$$|\overline{W}_{l_1}^{k_1}| + |\overline{W}_{l_2}^{k_2}| \subseteq |\overline{W}_{l_1+l_2}^{k_1+k_2}|$$
is satisfied.

Bigraded linear series are very general objects. We propose the following, much more special, notion. We propose to name it after Hacon who seems to have noticed first their importance.

Definition 8.1.2. A bigraded linear series $\overline{W}_l^k$ is called a Hacon linear series (in k with respect to V) if M is pseudoeffective and there is a base-point free linear series

$V \subseteq |H|$ so that

$$|\overline{W}_l^k| + |V| \subseteq |\overline{W}_l^{k+1}|$$

for all k and l.

We start what a few easy examples.

1. Let H be very ample and M pseudo-effective. We can set

$$\overline{W}_l^k = |lM + kH|.$$

This is a Hacon linear series in k with respect to $|H|$. We will call this the complete Hacon linear series. We will study this example in more detail in the next section.

2. More generally, let $f : X \to Y$ be a morphism and let H be base-point free on Y. Then

$$\overline{W}_l^k = |lM + kf^*H|$$

is a Hacon linear series in k with respect to $|f^*H|$. This was the Hacon linear series used in [15]. Similar ideas were used in [23] with f the Albanese morphism.

3. Let $|W_l| \subseteq |lM|$ be a graded linear series, let V be a base-point free linear series and let

$$|\overline{W}_l^k| = |W_l| + k|V|.$$

From our point of view, this Hacon linear series reduces our theory to the theory of the usual asymptotic multiplier ideal.

The importance of Hacon linear series stems from the following.

Definition 8.1.3. Let $\overline{W}_l^k$ be a Hacon linear series. The associated vertical subseries is the sequence of linear series given by

$$W_l^k := \overline{W}_{lk}^k.$$

The reason for the name, and the intended use, stems from the observation that, if $D \in |W_l^k|$,

$$\frac{1}{lk} D \sim_{\mathbb{Q}} M + \frac{1}{l} H.$$

Lemma 8.1.4. *The associated vertical subseries is graded in k and satisfies the Hacon inclusion:*

$$l|W_{l+1}^k| + k|V| \subseteq |W_l^{(l+1)k}|. \tag{8.1}$$

Proof. Both of these follow easily from the definitions. Indeed,

$$|W_l^{k_1}| + |W_l^{k_2}| = |\overline{W}_{lk_1}^{k_1}| + |\overline{W}_{lk_2}^{k_2}| \subseteq |\overline{W}_{l(k_1+k_2)}^{k_1+k_2}| = |W_l^{k_1+k_2}|$$

and

$$l|W_{l+1}^k| + k|V| = l|\overline{W}_{(l+1)k}^k| + k|V| \subseteq |\overline{W}_{l(l+1)k}^{lk}| + k|V| \subseteq |\overline{W}_{l(l+1)k}^{(l+1)k}| = |W_l^{(l+1)k}|.$$

□

Let $c \in \mathbb{R}^+$. It follows that we get a sequence of ideals

$$\mathcal{J}_l^k = \mathcal{J}\left(X, \frac{c}{kl} \cdot |W_l^k|\right)$$

and

$$\mathcal{J}_l^\bullet = \mathcal{J}\left(X, \frac{c}{l} \cdot ||W_l^\bullet||\right).$$

An essential point, noticed by Hacon, is that these ideals stabilize.

Proposition 8.1.5. *We have that $\mathcal{J}_{l+1}^\bullet \subseteq \mathcal{J}_l^\bullet$. Furthermore, the sequence of ideals $\mathcal{J}_l^\bullet$ becomes stationary for all $l \gg 0$.*

Proof. We claim that $\mathcal{J}_{l+1}^k \subseteq \mathcal{J}_l^{(l+1)k}$. To see this we calculate on a common log-resolution $\pi : X' \to X$ on the base loci of W_{l+1}^k and $W_l^{(l+1)k}$. Let F_{l+1}^k the effective divisor on X' so that $\mathcal{O}_{X'}(-F^k_{l+1})$ is the expansion of the base locus ideal of W^k_{l+1} to X', and similarly for $F_l^{(l+1)k}$. Taking the base divisors in the Hacon inequality (8.1) and dividing by $(l+1)lk$ gives

$$\frac{1}{(l+1)k}F^k_{l+1} \geq \frac{1}{(l+1)lk}F_l^{(l+1)k}. \tag{8.2}$$

This proves our claim that

$$\mathcal{J}^k_{l+1} = \mathcal{J}\left(X, \frac{c}{k(l+1)} \cdot |W^k_{l+1}|\right) \subseteq \mathcal{J}\left(X, \frac{c}{kl(l+1)} \cdot |W_l^{(l+1)k}|\right) = \mathcal{J}_l^{(l+1)k}$$

and, in particular, that

$$\mathcal{J}^\bullet_{l+1} = \mathcal{J}\left(X, \frac{c}{l+1} \cdot |W^\bullet_{l+1}|\right) \subseteq \mathcal{J}\left(X, \frac{c}{l} \cdot |W^\bullet_l|\right) = \mathcal{J}^\bullet_l.$$

The next step is to show that this sequence actually stabilizes. This is a consequence of uniform global generation. By Castelnuovo-Mumford regularity, we may choose a sufficiently ample divisor G so that the sheaves

$$\mathcal{O}_X(M+G) \otimes \mathcal{J}_l$$

are all globally generated. Indeed,

$$\mathcal{J}_l = \mathcal{J}\left(X, \frac{1}{kl} \cdot |W^k_l|\right)$$

and

$$\frac{1}{kl}(k(lM+H)) \sim_{\mathbb{Q}} M + \frac{1}{k}H$$

and so it suffices to choose G so that $G - \frac{1}{k}H$ is sufficiently ample. Hence we have that

$$H^0(X, \mathcal{O}_X(M+G) \otimes \mathcal{J}_{l+1}) \subseteq H^0(X, \mathcal{O}_X(M+G) \otimes \mathcal{J}_l)$$

and we have equality if and only if $\mathcal{J}_{l+1} = \mathcal{J}_l$. Since the spaces in question are finite-dimensional, this must eventually happen for some l. $\square$

This was essentially first noticed by Hacon in [15]. Notice that the same argument works for adjoint ideals along a divisor.

Definition 8.1.6. We denote the limit ideals by

$$\mathrm{adj}_D(X, c\cdot ||W^\bullet_\bullet||),\ \mathcal{J}(X, c\cdot ||W^\bullet_\bullet||).$$

Any log-resolution of W^k_l with l large enough and k divisible enough is said to compute the ideal $\mathcal{J}(X, c\cdot ||W^\bullet_\bullet||)$.

Similarly we may consider mixed multiplier ideals of the form

$$\mathcal{J}((X,\Delta); c\cdot ||W^\bullet_\bullet||),\ \mathcal{J}((X,\mathfrak{d}^b); c\cdot ||W^\bullet_\bullet||)$$

where Δ is a $\mathbb{Q}$-divisor, $\mathfrak{d}$ is an ideal and $b \in \mathbb{R}^+$, for which the construction also goes through *mutatis mutandis*. Essentially any permutation of these definitions is possible, as long as the ideal in question satisfies some analog of the Nadel vanishing theorem.

Remark 8.1.7. Note that, in Example (3), $\mathcal{J}(X, c\cdot ||W^\bullet_\bullet||) = \mathcal{J}(X, c\cdot ||W_\bullet||)$, the usual asymptotic multiplier ideal.

8.2 The uniformity lemma

We know that the Hacon ideal exists but we don't yet know anything about the way in which the ideals $\mathcal{J}(X, c\cdot |W^k_l|)$ stabilize. In particular, we don't even know yet that there are finitely many of them. In this section we prove a technical lemma that says that the convergence of these ideals to $\mathcal{J}(X, ||W^\bullet_\bullet||)$ is somehow "uniform." We then provide two corollaries of this lemma that will be useful when applying the Hacon ideal formalism later in the chapter.

Lemma 8.2.1 (Uniformity lemma). *Let W_l^k be the associated vertical subseries of a Hacon linear series, let Δ be an effective $\mathbb{Q}$-divisor and let $c, d \in \mathbb{R}^+$. There exists a proper birational morphism $\pi : X' \to X$ that computes $\mathcal{J}((X, d\cdot\Delta); c\cdot ||W_\bullet^\bullet||)$ and a finite set of divisors E_α on X' so that the following condition for membership holds. If (locally) $f \in \mathcal{O}_X$ then $f \in \mathcal{J}((X, d\cdot\Delta); c\cdot||W_\bullet^\bullet||)$ if and only if, for all large l and divisible k,*

$$ord_{E_\alpha}(\pi^* f) \leq ord_{E_\alpha}\left(K_{X'/X} - \left\lfloor \pi^*(d\cdot\Delta) + \frac{c}{kl}F_l^k \right\rfloor\right) \tag{8.3}$$

for each E_α. Here F_l^k is the divisorial part of $\mathfrak{b}(W_l^k)\cdot\mathcal{O}_{X'}$. Furthermore, the finite set of the E_α depends only on the support of Δ and the base locus of W_l^k for large l and divisible k.

Proof. Let l' and k' be natural numbers so that

$$\mathcal{J}((X, d\cdot\Delta), c\cdot||W_\bullet^\bullet||) = \mathcal{J}\left((X, d\cdot\Delta), \frac{c}{lk}|W_l^k|\right)$$

for all $l \geq l'$, $k'|k$.

Now, take log-resolutions $\pi^{(l,k)} : X_l^k \to X$ of W_l^k and Δ and let them factor through $\pi^{(l',k')}$ via morphisms

$$\pi^{(l,k)\to(l',k')} : X_l^k \to X_{l'}^{k'}.$$

Let

$$\mathcal{O}_{X_l^k}(-F_l^k) = \mathfrak{b}(W_l^k)\cdot\mathcal{O}_{X_l^k}.$$

We will write simply π when the superscripts are clear from the context.

Consider the divisors

$$T_l^k := -\pi_*^{(l,k)\to(l',k')}\left\lfloor \pi^*(d\cdot\Delta) + \frac{c}{lk}F_l^k \right\rfloor.$$

Since W_l^k is graded in k, for every l there is a $K(l)$ so that the divisors $T_l^{mK(l)}$ are constant for all $m \geq 1$. Indeed, for k sufficiently divisible, the T_l^{mk} are increasing in m, negative and integral and so must eventually stabilize.

Next, starting with l' and k', take for each l a $k(l)$ divisible enough to compute the asymptotic multiplier ideal

$$\mathcal{J}\left((X, d \cdot \Delta), \frac{c}{lk(l)}|W_l^{k(l)}|\right) = \mathcal{J}\left((X, d \cdot \Delta), \frac{c}{l} \cdot ||W_l^{\bullet}||\right)$$

and to be divisible by $K(l)$ and $K(l-1)$. We set

$$D_l := \pi_*^{(l,k(l)) \to (l',k')} \left(K_{X_l^{k(l)}/X} - \left\lfloor \pi^*(d \cdot \Delta) + \frac{c}{lk(l)} F_l^{k(l)} \right\rfloor\right).$$

By our choice of $k(l)$ we have that $lk(l)$ and $lk(l-1)$ are divisible by $K(l-1)$. Combining this with inequality (8.2) we get that

$$\begin{aligned}
D_l &\leq \pi_*^{(l-1,k(l)) \to (l',k')} \left(K_{X_{l-1}^{k(l)}/X} - \left\lfloor \pi^*(d \cdot \Delta) + \frac{c}{(l-1)lk(l)} F_{l-1}^{lk(l)} \right\rfloor\right) \\
&= \pi_*^{(l-1,k(l-1)) \to (l',k')} \left(K_{X_{l-1}^{k(l-1)}/X} - \left\lfloor \pi^*(d \cdot \Delta) + \frac{c}{(l-1)lk(l-1)} F_{l-1}^{lk(l-1)} \right\rfloor\right) \\
&= D_{l-1}.
\end{aligned}$$

Since the divisorial push-forward drops some conditions for membership, we have that

$$\begin{aligned}
\mathcal{J}((X, d \cdot \Delta), c \cdot ||W_{\bullet}^{\bullet}||) &= \pi_*^{(l,k(l))} \mathcal{O}_{X_l^{k(l)}} \left(K_{X_l^{k(l)}/X} - \left\lfloor \pi^*(d \cdot \Delta) + \frac{c}{lk(l)} F_l^{k(l)} \right\rfloor\right) \\
&\subseteq \pi_*^{(l',k')} \mathcal{O}_{X_{l'}^{k'}}(D_l) \subseteq \pi_*^{(l',k')} \mathcal{O}_{X_{l'}^{k'}}(D_{l'}) \\
&= \mathcal{J}((X, d \cdot \Delta), c \cdot ||W_{\bullet}^{\bullet}||).
\end{aligned}$$

It follows that we have equalities throughout. It remains to choose the E_α to be the components of the support of $K_{X_{l'}^{k'}/X} + \pi^*\Delta + F_{l'}^{k'}$. $\square$

The following lemmas follow easily from the uniformity lemma.

Lemma 8.2.2. *Let W_l^k be the associated vertical subseries of a Hacon linear series and let $c \in \mathbb{R}^+$. Let C be an effective $\mathbb{Q}$-divisor. There exists an $\varepsilon > 0$ so that*

$$\mathcal{J}((X, \varepsilon C); c \cdot ||W_\bullet^\bullet||) = \mathcal{J}(X, c \cdot ||W_\bullet^\bullet||).$$

Proof. Let

$$\mathcal{J}^\varepsilon = \mathcal{J}((X, \varepsilon C); ||W_\bullet^\bullet||).$$

It follows easily from Proposition 8.1.5 that, if $\varepsilon_2 \leq \varepsilon_1$ then $\mathcal{J}^{\varepsilon_1} \subseteq \mathcal{J}^{\varepsilon_2}$. By the Noetherian property these ideals must stabilize as $\varepsilon \to 0$. In particular, there are only finitely many ideals involved as long as ε is bounded above. Let $\mathcal{J}$ be the limit ideal.

Let $\pi : X' \to X$ be the birational morphism from Lemma 8.2.1 applied to the ideal $\mathcal{J}$. We get that, for all $0 < \varepsilon \ll 1$, we have that if (locally) $f \in \mathcal{O}_X$ then $f \in \mathcal{J}((X, \varepsilon C); ||W_\bullet^\bullet||)$ if and only if

$$\operatorname{ord}_{E_\alpha}(\pi^* f) \leq \operatorname{ord}_{E_\alpha}\left(K_{X'/X} - \left\lfloor \pi^*(\varepsilon \cdot \Delta) + \frac{c}{kl} F_l^k \right\rfloor\right)$$

for all E_α in some finite set $\mathcal{D}$. We may assume that $\pi : X' \to X$ is also the morphism obtained from Lemma 8.2.1 applied to $\mathcal{J}(X, c \cdot ||W_\bullet^\bullet||)$, that is, $\mathcal{D}$ is large enough so that the following condition holds. If (locally) $f \in \mathcal{O}_X$ then $f \in \mathcal{J}(X, c \cdot ||W_\bullet^\bullet||)$ if and only if, for all large l' and divisible k,

$$\operatorname{ord}_{E_\alpha}(\pi^* f) \leq \operatorname{ord}_{E_\alpha}\left(K_{X'/X} - \left\lfloor \frac{c}{kl} F_l^k \right\rfloor\right)$$

for all E_α in $\mathcal{D}$.

But, for ε sufficiently small,

$$\left\lfloor \pi^*(\varepsilon \cdot \Delta) + \frac{c}{kl} F_l^k \right\rfloor = \left\lfloor \frac{c}{kl} F_l^k \right\rfloor.$$

This proves the lemma. □

Lemma 8.2.3. *Let W_l^k and V_l^k be the associated vertical subseries of two Hacon linear series and let $c \in \mathbb{R}^+$. Suppose that there exists an ideal $\mathfrak{d}$ and, for every l a $k(l)$ so that for all l and all k divisible by $k(l)$ we have*

$$\mathfrak{b}(W_l^k) \cdot \mathfrak{d}^k \subseteq \mathfrak{b}(V_l^k).$$

Then

$$\mathcal{J}(X, c \cdot ||W_\bullet^\bullet||) \subseteq \mathcal{J}(X, c \cdot ||V_\bullet^\bullet||).$$

Proof. Without loss of generality we may make $k(l)$ even more divisible and so we may assume that, for a fixed l and for all k divisible by $k(l)$, we have

$$\mathcal{J}\left(X, (\mathfrak{b}(W_l^k) \cdot (\mathfrak{d}^k))^{c/lk}\right) = \mathcal{J}\left(X, \mathfrak{b}(W_l^k)^{c/lk} \cdot \mathfrak{d}^{c/l}\right) = \mathcal{J}\left(X, \mathfrak{b}(W_l^\bullet)^{c/l} \cdot \mathfrak{d}^{c/l}\right).$$

By Proposition 8.1.5, we have that for large l

$$\mathcal{J}\left(X, \mathfrak{b}(W_l^\bullet)^{c/l} \cdot \mathfrak{d}^{c/l}\right) = \mathcal{J}\left((X, \mathfrak{d}^{c/l}), c \cdot ||W_\bullet^\bullet||\right)$$

By the previous lemma, we have that for l sufficiently large

$$\mathcal{J}\left((X, \mathfrak{d}^{c/l}), c \cdot ||W_\bullet^\bullet||\right) = \mathcal{J}(X, c \cdot ||W_\bullet^\bullet||).$$

On the other, hand, by assumption we have

$$\mathcal{J}\left(X, (\mathfrak{b}(W_l^k) \cdot (\mathfrak{d}^k))^{c/lk}\right) \subseteq \mathcal{J}\left(X, \mathfrak{b}(V_l^k)^{c/lk}\right) = \mathcal{J}(X, c \cdot ||V_\bullet^\bullet||),$$

as required. $\square$

8.3 The restricted multiplier ideal

An obvious choice of $\overline{W}_l^k$ is to take the complete linear series

$$\overline{W}_l^k = |lM + kH|.$$

Since in general we very much *do not* have that $|lM + kH| = |lM| + |kH|$ for even sufficiently divisible l and k, we expect this construction to be non-trivial. We propose to call the resulting ideal the restricted multiplier ideal, by analogy with the restricted base locus in [7]. Here we develop the basic facts about these ideals: they do not depend on the choice of H, they satisfy a slightly stronger analog of Nadel vanishing, and they satisfy $\mathcal{J}_{-}(X, c \cdot ||(m+1)M||) \subseteq \mathcal{J}_{-}(X, c \cdot ||mM||)$.

Definition 8.3.1. Let $c \in \mathbb{R}_{\geq 0}$, let M be a pseudo-effective divisor and let H be an ample divisor. Let $\overline{W}_l^k$ be the linear series

$$\overline{W}_l^k = |lM + kH|.$$

Then $\overline{W}_l^k$ is a bigraded linear series and, since,

$$|\overline{W}_l^k| + |H| \subseteq |\overline{W}_l^{k+1}|$$

it is also a Hacon linear series. We obtain an ideal

$$\mathcal{J}^H(X, c \cdot ||M||) := \mathcal{J}(X, c \cdot ||W_{\bullet}^{\bullet}||)$$

that we will call the restricted multiplier ideal of M with respect to H. It is equal, for sufficiently large l and k and sufficiently small $\varepsilon > 0$, to the ideal

$$\mathcal{J}\left(X, \frac{c}{kl}|k(lM + H)|\right) = \mathcal{J}(X, ||M + \varepsilon H||).$$

Note that we need not take k to be divisible since $lM + H$ is big and therefore has exponent one. From now on we will write l and k for sufficiently large integers without further comment.

We begin with the following basic proposition.

Proposition 8.3.2. *Let H and H' be two ample divisors and let M be a pseudoeffective divisor. Then*

$$\mathcal{J}^{H'}(X, c \cdot ||M||) = \mathcal{J}^{H}(X, c \cdot ||M||).$$

In light of this proposition, we make the following definition.

Definition 8.3.3. By analogy with the definition of $\mathbb{B}_{-}(M)$ in [7], let

$$\mathcal{J}_{-}(X, c \cdot ||M||)$$

be $\mathcal{J}^{H}(X, c \cdot ||M||)$ for any ample divisor H.

Proof of Proposition 8.3.2. It is enough to show that

$$\mathcal{J}^{H}(X, c \cdot ||M||) \subseteq \mathcal{J}^{H'}(X, c \cdot ||M||).$$

Pick an $m \gg 0$ so that $mH - H'$ is base-point free. Then

$$|klM + kH'| + |kmH - kH'| \subseteq |klM + kmH|$$

for all k. Since $|kmH - kH'|$ is a base-point free linear series we obtain that

$$\mathcal{J}\left(X, c \cdot \frac{1}{l}||lM + H'||\right) \subseteq \mathcal{J}\left(X, c \cdot \frac{1}{l}||lM + mH||\right)$$

and therefore

$$\mathcal{J}^{H'}(X, c \cdot ||M||) \subseteq \mathcal{J}^{mH}(X, c \cdot ||M||).$$

It is therefore enough to show that

$$\mathcal{J}^{mH}(X, c \cdot ||M||) = \mathcal{J}^{H}(X, c \cdot ||M||).$$

But we can write

$$\begin{aligned}\mathcal{J}^{mH}(X, c\cdot ||M||) &= \mathcal{J}\left(X, c\cdot \frac{1}{lmk}|k(lmM+mH)|\right)\\ &= \mathcal{J}\left(X, c\cdot \frac{1}{l(mk)}|mk(lM+H)|\right)\\ &= \mathcal{J}\left(X, c\cdot \frac{1}{l}||lM+H||\right)\\ &= \mathcal{J}^{H}(X, c\cdot ||M||),\end{aligned}$$

concluding the proof. □

First we record the version of Nadel vanishing that holds for the restricted multiplier ideal.

Theorem 8.3.4 (Nadel vanishing)**.** *Let L and M be divisors so that M is pseudo-effective and let $c \in \mathbb{R}^+$. Suppose that $L - cM \equiv_{\text{num}} K_X + A$ with A big and nef. Then*

$$H^i(X, \mathcal{O}_X(L) \otimes \mathcal{J}_-(X, c\cdot ||M||)) = 0$$

for all $i > 0$.

Proof. Since A is big we can write $A \sim_{\mathbb{Q}} H + C$ with H ample and C effective $\mathbb{Q}$-divisors. Use Lemma 8.2.2 to choose $\varepsilon' > 0$ small enough so that

$$\mathcal{J}_-((X, \varepsilon' C), c\cdot ||M||) = \mathcal{J}_-(X, c\cdot ||M||).$$

Next choose $\varepsilon > 0$ small enough so that

$$\mathcal{J}_-((X, \varepsilon' C), c\cdot ||M||) = \mathcal{J}\left((X, \varepsilon' C), c\cdot \left|\left|M + \frac{\varepsilon}{c}H\right|\right|\right)$$

and $\varepsilon' > \varepsilon$. We compute

$$\begin{aligned} L - cM - \varepsilon H &\equiv_{\text{num}} K_X + A - \varepsilon H \\ &\equiv_{\text{num}} K_X + (1-\varepsilon')A + \varepsilon'(H + C) - \varepsilon H \\ &= K_X + (1-\varepsilon')A + (\varepsilon' - \varepsilon)H + \varepsilon' C \\ &= K_X + \text{ample} + \varepsilon' C. \end{aligned}$$

It follows from the usual Nadel vanishing now that

$$\begin{aligned} H^i(X, \mathcal{O}_X(L) \otimes \mathcal{J}_-(X, c \cdot ||M||)) &= H^i\left(X, \mathcal{O}_X(L) \otimes \mathcal{J}\left((X, \varepsilon' C), c \cdot \left|\left|M + \frac{\varepsilon}{c} H\right|\right|\right)\right) \\ &= 0, \end{aligned}$$

as required. $\square$

We note the following easy property of this ideal:

Proposition 8.3.5. *The usual equality*

$$\mathcal{J}_-(X, c \cdot ||mM||) = \mathcal{J}_-(X, cm \cdot ||M||)$$

holds true. Hence, the restricted asymptotic multiplier ideal satisfies the usual inequality

$$\mathcal{J}_-(X, c \cdot ||(m+1)M||) \subseteq \mathcal{J}_-(X, c \cdot ||mM||).$$

Proof. By the preceding proposition we have that

$$\begin{aligned} \mathcal{J}^H(X, c \cdot ||mM||) &= \mathcal{J}^{mH}(X, c \cdot ||mM||) \\ &= \mathcal{J}\left(X, c \cdot \frac{1}{l}||lmM + mH||\right) \\ &= \mathcal{J}\left(X, cm \cdot \frac{1}{l}||lM + H||\right) \\ &= \mathcal{J}^H(X, cm \cdot ||M||). \end{aligned}$$

For the second statement, it is easy to see that

$$\mathcal{J}_{-}(X, c \cdot ||M||) \subseteq \mathcal{J}_{-}(X, d \cdot ||M||)$$

for $d \leq c$. But then we are done by the first statement. $\square$

The following related statement can be useful when comparing asymptotic ideal constructions.

Proposition 8.3.6. *Let M be a pseudo-effective divisor and let H be very ample. Then*

$$\mathcal{J}(X, ||(l+1)M + H||) \subseteq \mathcal{J}(X, ||lM + H||).$$

Proof. We compute

$$\begin{aligned}
\mathcal{J}(X, ||(l+1)M + H||) &= \mathcal{J}\left(X, (l+1) \cdot \left|\left|M + \frac{1}{l+1}H\right|\right|\right) \\
&\subseteq \mathcal{J}\left(X, l \cdot \left|\left|M + \frac{1}{l+1}H\right|\right|\right) \\
&= \mathcal{J}\left(X, \frac{l}{(l+1)l} \cdot ||l(l+1)M + lH||\right) \\
&\subseteq \mathcal{J}\left(X, \frac{l}{(l+1)l} \cdot ||l(l+1)M + (l+1)H||\right) \\
&= \mathcal{J}(X, ||lM + H||)
\end{aligned}$$

where the penultimate step follows from the fact that H is base-point free. $\square$

8.4 A generalization of a theorem of Goodman

We now take a diversion into an application of the restricted multiplier ideal. Recall that a big line bundle M is nef if and only if $\mathcal{J}(X, ||mM||) = \mathcal{O}_X$ for all m (see [25], Proposition 11.2.18). See also *Ibid.*, Example 11.2.19 and Remark 11.2.20 for the related theorems of Goodman in [12] and Russo in [29]. The restricted multiplier ideal and its vanishing theorem allow us to write down a natural generalization of this fact.

Theorem 8.4.1. *A line bundle M is nef if and only if it is pseudo-effective and $\mathcal{J}_{-}(X, ||mM||) = \mathcal{O}_X$ for all m.*

Proof. The proof is essentially the same as for Proposition 11.2.18 in [25], we reproduce it for completeness. A nef line bundle is pseudo-effective and, since $lM + H$ is ample, we have

$$\mathcal{J}_{-}(X, ||mM||) = \mathcal{J}\left(X, \frac{1}{l}||lmM + H||\right) = \mathcal{O}_X$$

for all m. Conversely, by the standard Castelnuovo-Mumford regularity argument we may choose an ample divisor G so that

$$\mathcal{O}_X(mM + G) \otimes \mathcal{J}_{-}(X, ||mM||)$$

is globally generated. But $\mathcal{J}_{-}(X, ||mM||) = \mathcal{O}_X$ for all m and so $\mathcal{O}_X(mM + G)$ is globally generated. This easily implies that M is nef. Indeed, let C be any curve on X. Since $\mathcal{O}_X(mM + G)$ is globally generated, $(mM + G.C) \geq 0$ and so

$$(M.C) = \lim_{m \to \infty} \frac{1}{m}(mM + G.C) \geq 0.$$

$\square$

8.5 An extension theorem for pluricanonical forms from centers of high codimension

The extension theorem is an important application of the multiplier ideal formalism. Hacon linear series and the restricted multiplier ideal are particularly well-suited for extension arguments. Here we prove an extension theorem for pluri-canonical sections from exceptional log-canonical centers, similar [20], using algebraic methods. To do this we first have to develop some basic formalism for dealing with asymptotic versions of the adjoint ideal from Definition 5.1.7.

For the rest of this section let X denote a smooth projective variety and Δ a log-canonical $\mathbb{Q}$-divisor with an exceptional log-canonical center Z. Let $\mathfrak{a}_\bullet$ be a graded system of ideals.

Definition 8.5.1. Let $\mathfrak{a}$ be an ideal and $c \in \mathbb{R}_+$. We define $\mathrm{adj}_Z((X,\Delta),\mathfrak{a}^c)$, in the notation of Definition 5.1.7,

$$\mathrm{adj}_Z((X,\Delta),\mathfrak{a}^c) = g_*\mathcal{O}_{X'}\left(\lceil K_{X'/X} - g^*\Delta - cF\rceil + E\right)$$

where $g : X' \to X$ is also a log-resolution of $\mathfrak{a}$ and $\mathcal{O}_{X'}(-F) = \mathfrak{a}\cdot\mathcal{O}_{X'}$. We define $\mathrm{adj}_Z((X,\Delta), c\cdot||\mathfrak{a}_\bullet||)$ to be the unique maximal element of the set of ideals

$$\mathrm{adj}_Z\left((X,\Delta),\frac{c}{l}\cdot\mathfrak{a}^l\right).$$

Let M be a big Cartier divisor. The ideal $\mathrm{adj}_Z((X,\Delta);c\cdot||M||)$ is defined in the evident way - we take $\mathfrak{a}_l$ to be the base locus of $|lM|$.

Lemma 8.5.2. *We have*

$$\mathfrak{a}_1 \subseteq \mathrm{adj}_Z((X,\Delta),\mathfrak{a}_\bullet).$$

Proof. Take $g : X' \to X$ as in Definition 8.5.1. We may assume that g is also a log-resolution of $\mathfrak{a}_1$. Write $\mathcal{O}_{X'}(-F_k) = \mathfrak{a}_k\cdot\mathcal{O}_{X'}$ for k either a fixed large and divisible number or $k = 1$. The definition of a graded system of ideals and the hypothesis imply that

$$-F_1 \leq -\frac{1}{k}F_k.$$

We obtain

$$\begin{aligned}\mathfrak{a}_1 &\subseteq g_*\mathcal{O}_{X'}(\lceil K_{X'/X} - g^*\Delta + E_g\rceil - F_1)\\ &= g_*\mathcal{O}_{X'}(\lceil K_{X'/X} - g^*\Delta + E_g - F_1\rceil)\\ &\subseteq g_*\mathcal{O}_{X'}\left(\left\lceil K_{X'/X} - g^*\Delta + E_g - \frac{1}{k}F_k\right\rceil\right)\\ &= \mathrm{adj}_Z((X,\Delta),\mathfrak{a}_\bullet),\end{aligned}$$

as desired. ☐

Remark 8.5.3. Let M be a big Cartier divisor. By Corollary 6.2.5, the ideal

$$(\mathrm{adj}_Z)_-((X,\Delta);||M||)$$

exists.

Definition 8.5.4. By Theorem 6.1.1 we have a short exact sequence (this defines $\mathfrak{b}^X_\Delta(\mathfrak{a}_\bullet)$)

$$0 \to \mathcal{J}((X,\Delta);\mathfrak{a}_\bullet) \to \mathrm{adj}_Z((X,\Delta),\mathfrak{a}_\bullet) \to \mathfrak{b}^X_\Delta(\mathfrak{a}_\bullet) \to 0 \tag{8.4}$$

with $\mathfrak{b}^X_\Delta(\mathfrak{a}_\bullet)$ a multiplier ideal. Let M be a big Cartier divisor. If $Z \not\subseteq \mathbb{B}_-(M)$, the short exact sequence (8.4) defines a short exact sequence

$$0 \to \mathcal{J}_-((X,\Delta);||M||) \to (\mathrm{adj}_Z)_-((X,\Delta);||M||) \to (\mathfrak{b}^X_\Delta)_-(||M||) \to 0. \tag{8.5}$$

Lemma 8.5.5. *We have*

$$\mathfrak{a}_1 \cdot \mathcal{O}_Z \subseteq \mathfrak{b}^X_\Delta(\mathfrak{a}_\bullet).$$

More generally, if $\mathfrak{b}_i$ is a graded system of ideals and W^k_l is a Hacon linear series so that $\mathfrak{b}_{lk} \subseteq \mathfrak{b}(W^k_l)$ then

$$\mathfrak{b}_1 \cdot \mathcal{O}_Z \subseteq \mathfrak{b}^X_\Delta(||W^\bullet_\bullet||).$$

Proof. This follows immediately from Lemma 8.5.2. ☐

We are now ready for the main theorem of the chapter.

Theorem 8.5.6. *Let X be a smooth projective variety and let A and Δ be $\mathbb{Q}$-divisors such that*

1. *A is big and nef,*

2. *(X,Δ) is log-canonical with an exceptional log-canonical center Z,*

3. $M = K_X + A + \Delta$ *is Cartier.*

Then the map

$$H^0(X, mM) \to H^0(Z, mM_Z)$$

is surjective for all $m \geq 1$*.*

Proof. If M_Z is not $\mathbb{Q}$-effective there is nothing to prove. Fix a very ample divisor H which we will take sufficiently ample later. We have the following two lemmas.

Lemma 8.5.7 (Basic Lifting)**.** *Let* σ *be a section in* $H^0(Z, ((m+1)M + H)_Z)$*. Suppose that* σ *vanishes along* $\mathfrak{b}^X_\Delta(||mM + H||)$*. Then there exists a section* $\overline{\sigma} \in H^0(X, (m+1)M + H)$ *that restricts to* σ*.*

Proof. Twist (8.4) by $\mathcal{O}_X((m+1)M + H)$ to get

$$\begin{aligned} 0 \to \mathcal{O}_X((m+1)M + H) \otimes \mathcal{J}((X, \Delta); ||mM||) \\ \to \mathcal{O}_X((m+1)M + H) \otimes \mathrm{adj}_Z((X, \Delta), ||mM||) \\ \to \mathcal{O}_Z((m+1)M_Z + H_Z) \otimes \mathfrak{b}^X_\Delta(||mM||) \to 0. \end{aligned}$$

Note that

$$(m+1)M + H - (\Delta + mM) = K_X + A + \Delta + H - \Delta = K_X + \text{ample}.$$

We conclude by Kawamata-Viehweg vanishing. □

Lemma 8.5.8. *Let* $\mathfrak{a}$ *be an ideal, let* $\mathcal{L}$ *be a line bundle and suppose that* $\mathcal{L} \otimes \mathfrak{a}$ *is globally generated. Suppose that* $\mathfrak{b}$ *is another ideal and that*

$$H^0(X, \mathcal{L} \otimes \mathfrak{a}) \subseteq H^0(X, \mathcal{L} \otimes \mathfrak{b}).$$

Then $\mathfrak{a} \subseteq \mathfrak{b}$*.*

Proof. This follows immediately from unwinding the definition of a globally generated line bundle. □

First make H ample enough so that $M+H$ is base-point free. Next, pick any $0<\varepsilon<1/2$ and let Δ_Z be the $\mathbb{Q}$-divisor given by the equation

$$(K_X+\Delta+\varepsilon H)\sim_{\mathbb{Q}} K_Z+\Delta_Z.$$

Make H so ample (without changing ε) to make Δ_Z ample. We can then assume that (Z,Δ_Z) is klt. Since $M+H$ is base-point free we have

$$\mathcal{J}((Z,\Delta_Z);||M_Z||)\subseteq \mathfrak{b}(|M+H|_Z).$$

We also have

$$\begin{aligned}(m+1)M_Z+H_Z-\Delta_Z-mM_Z &\sim_{\mathbb{Q}} M_Z+H_Z-(K_X+\Delta+\varepsilon H)_Z\\ &= A_Z+(1-\varepsilon)H_Z.\end{aligned}$$

Choose H so ample that

$$A_Z+(1-2\varepsilon)H_Z-(K_X+\Delta)_Z\sim_{\mathbb{Q}}(\dim(Z)+1)B$$

with B ample. Rearranging this equation we see that

$$A_Z+(1-\varepsilon)H_Z\sim_{\mathbb{Q}} K_Z+\Delta_Z+(\dim(Z)+1)B.$$

With these choices we also have, by Castelnuovo-Mumford regularity, that

$$\mathcal{O}_Z(((m+1)M+H)_Z)\otimes\mathcal{J}((Z,\Delta_Z);||mM_Z||)$$

are globally generated for all $m\geq 1$.

We will first prove by induction on m that

$$\mathcal{J}((Z,\Delta_Z);||mM_Z||)\subseteq \mathfrak{b}(|mM+H|_Z)$$

for all m and this choice of H. We already chose H so that the base case $m = 1$ holds. So suppose that the statement is true for a given m. By Lemma 8.5.5,

$$\mathfrak{b}(|mM + H|_Z) \subseteq \mathfrak{b}^X_\Delta(||mM + H||).$$

Now, all this gives us the following inclusions. First, the induction hypothesis says that

$$H^0(Z,\mathcal{O}_Z((m+1)M + H)_Z \otimes \mathcal{J}((Z,\Delta_Z); ||mM_Z||)) \subseteq$$
$$H^0(Z,\mathcal{O}_Z((m+1)M + H)_Z \otimes \mathfrak{b}^X_\Delta(||mM + H||)).$$

Next, Basic Lifting gives

$$H^0(Z,\mathcal{O}_Z((m+1)M + H)_Z \otimes \mathfrak{b}^X_\Delta(||mM + H||)) \subseteq$$
$$\mathrm{Im}(H^0(X,\mathcal{O}_X((m+1)M + H)) \to H^0(Z,\mathcal{O}_Z((m+1)M + H)_Z)).$$

By our global generation assumption on H and Lemma 8.5.8, it follows that

$$\mathcal{J}((Z,\Delta_Z); ||mM_Z||) \subseteq \mathfrak{b}(|(m+1)M + H|_Z).$$

The standard inequality

$$\mathcal{J}((Z,\Delta_Z); ||(m+1)M_Z||) \subseteq \mathcal{J}((Z,\Delta_Z); ||mM_Z||)$$

now implies that

$$\mathcal{J}((Z,\Delta_Z); ||(m+1)M_Z||) \subseteq \mathfrak{b}(|(m+1)M + H|_Z),$$

concluding the induction. Note that, in particular, if M_Z is $\mathbb{Q}$-effective then $Z \not\subseteq \mathbb{B}_-(M)$.

Fix an integer $l \geq 1$. It follows from the fact that (Z,Δ_Z) is klt and the above that

$$\mathfrak{b}(|mlM_Z|) \subseteq \mathcal{J}((Z,\Delta_Z); ||mlM_Z||) \subseteq \mathfrak{b}(|mlM + H|_Z)$$

for all m. Since H is base-point free we can even write

$$\mathfrak{b}(|mklM_Z|) \subseteq \mathfrak{b}(|mklM + kH|_Z)$$

for all $k \geq 1$. By Lemma 8.5.5 we get that

$$\mathfrak{b}(|lM_Z|) \subseteq (\mathfrak{b}^X_\Delta)_-(||lM||) \subseteq (\mathfrak{b}^X_\Delta)_-(||(l-1)M||).$$

We claim that this is enough. Indeed, this inclusion of the base locus shows that the natural inclusion

$$H^0(Z, \mathcal{O}_Z(lM_Z) \otimes (\mathfrak{b}^X_\Delta)_-(||(l-1)M||)) \subseteq H^0(Z, \mathcal{O}_Z(lM_Z))$$

is an equality. Since $Z \not\subseteq \mathbb{B}_-(M)$ there is an exact sequence

$$0 \to \mathcal{J}_-((X,\Delta); ||(l-1)M||) \to (\mathrm{adj}_Z)_-((X,\Delta); ||(l-1)M||) \to$$
$$\to (\mathfrak{b}^X_\Delta)_-(||(l-1)M||) \to 0.$$

Twist this exact sequence by $\mathcal{O}_X(lM)$. To conclude we simply apply Nadel vanishing to the sheaf on the left-hand side in the form of Proposition 8.3.4. $\square$

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