F-purity of hypersurfaces

by

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Dedication: To my friends and family; I love you all

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Introduction

1 Motivation

Let R be a commutative ring of prime characteristic $p > 0$. The e^{th}-*iterated Frobenius* map $R \xrightarrow{F^e} R$ (defined by $r \mapsto r^{p^e}$) is a ring homomorphism whose image is the subring $R^{p^e} \subseteq R$ consisting of all $(p^e)^{\text{th}}$ powers of elements of R. We call R *F-finite* whenever the Frobenius map is a finite map of rings, and in this thesis we will deal almost exclusively with F-finite rings.

The Frobenius map has been an important tool in commutative algebra since Kunz characterized regular rings as those for which R is flat over R^p [Kun69]. In general, singular rings exhibit pathological behavior with respect to the Frobenius endomorphism, and by imposing conditions on the structure of R as an R^p-module, new classes of singularities can be defined. For example, we say that R is *F-pure* (or *F-split*) if the inclusion $R^p \subseteq R$ splits as a map of R^p-modules [HR76]. This condition is equivalent to the condition that R contain a free R^p-module summand of rank one. Note that a regular ring is F-pure, though the converse need not be true: the ring $\mathbb{F}_p[x, y]/(x^2 + y^2)$ is F-pure for $p \neq 2$, yet always has an isolated singularity at the origin. The notion of F-purity is a critical ingredient in the proof of the well-known Hochster-Roberts Theorem on the Cohen-Macaulay property of rings of invariants [HR74].

Recently, more subtle applications of Frobenius to define singularities has led to new classes of *F-singularities*, many of which are motivated by the theory of *tight closure*. The tight closure of an ideal $I \subseteq R$ is an ideal $I^* \subseteq R$ which contains I and is "tight" in the sense that it is contained in (and is often much smaller than) many of the more common closures of I (e.g., the radical and integral closure of I). The ring R is called *F-regular* if $I^* = I$ for all ideals $I \subseteq R$, and is called *F-rational* if $I^* = I$ for every ideal I generated by a system of parameters. Regular rings are F-regular, and in general, F-regular rings satisfy many nice properties. For example, F-regular rings must be F-pure, *Cohen-Macaulay*, *normal* and must also have *rational singularities* [Smi97a]. For more on tight closure, the reader is referred to the original source [HH90]. For an account of the many applications of tight closure theory, see [Hoc04].

Amazingly, the F-singularities discussed above are closely related to singularity types that appear in the theory of birational geometry for varieties defined over $\mathbb{C}$, and in particular, to those appearing in the so-called *Minimal Model Program* [Fed83, Smi97a, Har98, Smi00, KM98].

It is standard practice in birational geometry to study the singularities of *pairs* $(X, \lambda \bullet Y)$, where X is a variety over $\mathbb{C}$, and $Y \subseteq X$ is a hypersurface in X. Via *integrability conditions* (or alternately, via *resolution of singularities*), one defines the notion of *KLT* and *log canonical* singularities for such pairs [Kol97]. By varying the parameter λ, we may define an important invariant of $Y \subseteq X$. The *log canonical threshold* of $Y \subseteq X$, denoted $\mathbf{lct}(X, Y)$, is defined to be the supremum over all parameters λ such that the pair $(X, \lambda \bullet Y)$ is log canonical. We may replace "log canonical" in the definition of $\mathbf{lct}(X, Y)$ with "KLT" without affecting the value of this invariant. It is a fact that $\mathbf{lct}(X, Y) \in (0, 1]$, with smaller values corresponding to worse singularities. This is readily seen in the case that $X = \mathbb{C}^m$, and $Y = \mathbb{V}(f)$ is the hypersurface in X defined a polynomial $f \in \mathbb{C}[x_1, \cdots, x_m]$, where we have the following concrete description of the log canonical threshold:

$$\mathbf{lct}(\mathbb{C}^m, \mathbb{V}(f)) = \sup \left\{ \lambda \in \mathbb{R}_{\geqslant 0} : \frac{1}{|f|^{2\lambda}} \text{ is locally integrable in } \mathbb{C}^m \right\}.$$

By again using resolution of singularities (or integrability considerations), one may assign to each pair $(X, \lambda \bullet Y)$ an ideal $\mathscr{J}(X, \lambda \bullet Y)$ called the *multiplier ideal of the pair* $(X, \lambda \bullet g)$. The ideal $\mathscr{J}(X, \lambda \bullet Y)$ must be trivial for small values of λ, and get smaller as λ increases. By varying the parameter λ, we may recover the log canonical threshold of (X, Y): $\mathbf{lct}(X, Y) = \sup\{\lambda \geqslant 0 : \mathscr{J}(X, \lambda \bullet Y) \text{ is trivial}\}$. For more on the basic properties of multiplier ideals and their jumping numbers, as well as their role in higher dimensional birational geometry, see (for example) [BL04, Laz09, Laz04].

Motivated by geometric considerations, Hara and Watanabe recently extended the notion of F-purity to pairs of the form $(R, \lambda \bullet f)$, where f is a non-zero, non-unit element of R, and λ is a non-negative real parameter [HW02, Tak04]. For example, we say that the pair $(R, \lambda \bullet f)$ is F-pure if the inclusion $R^{p^e} \cdot f^{\lfloor (p^e - 1)\lambda \rfloor} \subseteq R$ splits as a map of R^{p^e}-modules for all $e \gg 0$. Here, $R^{p^e} \cdot f^N$ denotes the R^{p^e}-submodule of R generated by f^N. This generalizes the definition of F-purity for rings as R is F-pure if and only if the pair $(R, 0 \bullet f)$ is F-pure. This notion, though technical, adds great flexibility to the theory, and allows one to define *F-pure thresholds*. The F-pure threshold of an element $f \in R$, denoted $\mathbf{fpt}(R, f)$, is the supremum over all $\lambda \geqslant 0$ such that the pair $(R, \lambda \bullet f)$ is F-pure.

In the case that f is an element of an F-finite regular local ring $(R, \mathfrak{m})$, the F-pure threshold has the following concrete description:

$$\mathbf{fpt}(R, f) = \sup\left\{ \lambda \in \mathbb{R}_{\geqslant 0} : f^{\lceil p^e \lambda \rceil} \notin \mathfrak{m}^{[p^e]} \text{ for some } e \geqslant 1 \right\},$$

where $\mathfrak{m}^{[p^e]}$ is the ideal of R generated by the image of $\mathfrak{m}$ under $R \xrightarrow{F^e} R$. Note that if $\mathfrak{m} = (g_1, \cdots, g_a)$, then $\mathfrak{m}^{[p^e]} = (g_1^{p^e}, \cdots, g_a^{p^e})$. The definition of the F-pure threshold is analogous to that of the log canonical threshold in complex algebraic geometry.

By again using the Frobenius morphism, one defines a family of ideals $\boldsymbol{\tau}(R, \lambda \bullet f)$ of R indexed by a non-negative real parameter λ. The ideal $\boldsymbol{\tau}(R, \lambda \bullet f)$ is called the *test ideal* of the pair $(R, \lambda \bullet f)$. Test ideals (defined in the context of *tight closure*) were originally introduced in [HH90], and generalized to pairs in [HY03]. Test ideals vary with respect to λ in the same way that multiplier ideals do: they must be trivial for small values of λ, and also get smaller as λ increases. They also may be used to recover the F-pure threshold: $\mathbf{fpt}(R, f) = \sup\{ \lambda \in \mathbb{R}_{\geqslant 0} : \boldsymbol{\tau}(R, \lambda \bullet f) = R \}$. There is a very close relationship between F-purity and log canonical singularities, which we will sketch below in the special case of a hypersurface sitting in $\mathbb{Q}^m \subseteq \mathbb{C}^m$.

Fix $g \in \mathbb{Q}[x_1, \cdots, x_m] = \mathbb{Q}[\underline{\boldsymbol{x}}]$, a non-negative real parameter $\lambda > 0$, and consider the ideal $\mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g) \subseteq \mathbb{Q}[\underline{\boldsymbol{x}}]$. For $p \gg 0$, we may reduce the coefficients of g, as well as those of some fixed set of generators of $\mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g)$ modulo p to obtain a polynomial g_p and ideal $\mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g)_p$ in $\mathbb{F}_p[x_1, \cdots, x_m] = \mathbb{F}_p[\underline{\boldsymbol{x}}]$. The following theorem allows us to compare the ideals $\mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g)_p$ and $\boldsymbol{\tau}(\mathbb{F}_p[\underline{\boldsymbol{x}}], \lambda \bullet g_p)$.

Theorem 1.1. [Smi00, HY03] *For $p \gg 0$, $\boldsymbol{\tau}(\mathbb{F}_p[\underline{\boldsymbol{x}}], \lambda \bullet g_p) \subseteq \mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g)_p$ uniformly in λ. Furthermore, for every λ, there exists a positive integer N_λ such that $\boldsymbol{\tau}(\mathbb{F}_p[\underline{\boldsymbol{x}}], \lambda \bullet g_p) = \mathscr{J}(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet g)_p$ for all $p \geqslant N_\lambda$.*

We stress that the assignment $\lambda \mapsto N_\lambda$ is typically an increasing function of λ. We also have the following relationship between log canonical singularities and F-purity.

Theorem 1.2. [HW02, Tak04] The pair $(\mathbb{Q}[\underline{\boldsymbol{x}}], \lambda \bullet f)$ is log canonical if the pairs $(\mathbb{F}_p[\underline{\boldsymbol{x}}], \lambda \bullet f_p)$ are F-pure for infinitely many $p \gg 0$.

The preceding results imply the following relationship between thresholds:

$$\mathbf{fpt}(f_p) \leqslant \mathbf{lct}(f) \text{ and } \lim_{p \to \infty} \mathbf{fpt}(f_p) = \mathbf{lct}(f). \tag{1.2.1}$$

This behavior is illustrated in the following example.

Example 1.3. If $f = x^2 + y^3 \in \mathbb{Q}[x, y]$, then $\mathbf{lct}(f) = \frac{5}{6}$, and

$$\mathbf{fpt}(f_p) = \begin{cases} 1/2 & p = 2 \\ 2/3 & p = 3 \\ 5/6 & p \equiv 1 \bmod 6 \\ \frac{5}{6} - \frac{1}{6p} & p \equiv 5 \bmod 6 \end{cases}.$$

In this example, it follows from Dirichlet's Theorem on primes in arithmetic progressions that there exist infinitely many primes p such that $\mathbf{fpt}(f_p) = \mathbf{lct}(f)$.

Theorem 1.2 and Example 1.3 motivate the following two conjectures.

Conjecture 1.4. If the pair $(\mathbb{Q}^m, \lambda \bullet \mathbb{V}(f))$ is log canonical, then $(\mathbb{F}_p[\boldsymbol{x}], \lambda \bullet f_p)$ is F-pure for infinitely many p.

Conjecture 1.5. There exist infinitely many primes such that $\mathbf{fpt}(f_p) = \mathbf{lct}(f)$.

It turns out that these conjectures are equivalent (see Theorem 5.19), and their verification represents a long-standing open problem [Fed83, Smi97b, EM06]. They also serve as motivation for much of the work in this thesis.

2 Outline

In Chapter 2, we consider two different notions of singularity. We first consider *log canonical* singularities, which is defined for polynomials over $\mathbb{C}$ via the use of resolution of singularities and L^2-methods. This discussion of log canonical singularities is kept light, as it will only serve as motivation for our main results. We next consider F-purity. Since our main results deal with this singularity type, we include a detailed discussion of F-purity of hypersurfaces, and place emphasis on the special case that R is a regular ring. Chapter 2 concludes with a discussion of the deep connection between log canonical singularities and F-purity. To precisely describe this connection, we provide a careful exposition on the process of reduction to positive characteristic.

In Chapter 4, we examine base p expansions of non-negative real numbers. Though the ideas in this chapter are elementary, they prove to be useful in the study of singularities in positive characteristic.

In Chapter 5, we study F-purity of hypersurfaces in the most general settings. In particular, we begin our study of the *F-pure threshold*. We examine the relationship between different variants of F-purity, and characterize their behavior in terms of the F-pure threshold. For example, in Theorem 5.19, we show that $(R, \mathbf{fpt}(f) \bullet f)$ is F-pure, and is sharply F-pure if and only if $\mathbf{fpt}(f)$ is a rational number with denominator not divisible by the characteristic. In Proposition 5.12, we deduce important restrictions on the set of all F-pure thresholds. Many of the results in Chapter 5 generalize results previously shown to hold in F-finite regular local rings.

In Chapter 6, we associate to polynomial a rational, convex polytope $\boldsymbol{P}$. This polytope is "small" in the sense that it is contained in $[0, 1]^n$ for some $n \geqslant 1$. This polytope will be key to our study of F-purity of polynomials, and in this chapter we develop some of the important properties of $\boldsymbol{P}$.

In Chapter 7, we show that log canonical singularities is "equivalent" to F-purity for most polynomials. Precisely stated, Theorem 7.17 verifies Conjectures 1.4 and 1.5 for polynomials whose associated polytopes satisfy a natural non-degeneracy condition. Theorem 7.18 builds on this to show that a natural generalization of these conjectures holds for a *very general* complex polynomial.

In Chapter 8, we build on the methods of Chapter 7 to study invariants of singularities associated to *diagonal hypersurfaces* over fields of prime characteristic. A diagonal hypersurface is given by a polynomial of the form $x_1^{d_1} + \cdots + x_n^{d_n}$, and we call a diagonal hypersurface *Fermat* whenever $d_1 = d_2 \cdots = d_n = n = d$ for some $d \geqslant 1$. In Theorem 8.1, we present a formula for the F-pure threshold of an arbitrary diagonal hypersurface. Furthermore, we calculate the first non-trivial *test ideal* associated to a diagonal hypersurface. (See Theorem 8.4.) In Theorem 8.6, we examine the existence of, and give formulas for, higher *jumping numbers* of diagonal hypersurfaces.

In Chapter 9, we examine the F-pure threshold of a *binomial* hypersurface. By definition, a binomial is the sum of two distinct monomials. Our main result of this chapter is Algorithm 9.16, an algorithm for computing the F-pure threshold of an arbitrary binomial hypersurface. This algorithm depends on the characteristic of the ambient space and the geometry of the polytope $\boldsymbol{P}$.

CHAPTER 2

Singularities of hypersurfaces

1 Singularities of hypersurfaces defined via L^2 conditions

Let $f \in \mathbb{C}[x_1, \cdots, x_m]$ be a polynomial with complex coefficients, and consider its zero set $Z = \{\, \boldsymbol{z} \in \mathbb{C}^m : f(\boldsymbol{z}) = 0 \,\}$. We also consider its singular set $Z_{\text{sing}} = \left\{ \boldsymbol{z} \in \mathbb{C}^m : \frac{\partial f}{\partial x_1}(\boldsymbol{z}) = \cdots = \frac{\partial f}{\partial x_m}(\boldsymbol{z}) = f(\boldsymbol{z}) = 0 \right\}$. For every non-negative real parameter λ, consider the function $\mathbb{R}^{2m} = \mathbb{C}^m \xrightarrow{\Gamma_\lambda} \mathbb{R}$ defined by $\Gamma_\lambda(x_1, \cdots, x_m) = \frac{1}{|f(\boldsymbol{x})|^\lambda}$. Note that Γ_λ has a pole at every point $\boldsymbol{z} \in Z$, and understanding how "bad" these poles are provides a measure of the singularities of f. Recall that the function Γ_λ is called *locally* L^2 at a point $\boldsymbol{z} \in \mathbb{C}^m$ if $|\Gamma_\lambda(\boldsymbol{x})|^2 = \frac{1}{|f(\boldsymbol{x})|^{2\lambda}}$ is integrable in a neighborhood of $\boldsymbol{z}$. Note that Γ_λ is locally integrable at $\boldsymbol{z}$ for every λ whenever $\boldsymbol{z} \notin Z$.

Definition 2.1. We say that the pair $(\mathbb{C}^m, \lambda \bullet f)$ is *KLT at* $\boldsymbol{z}$ if the function Γ_λ is locally L^2 at $\boldsymbol{z}$. A pair is said to be *KLT* if it is KLT at every point of Z.

Definition 2.2. We say that $(\mathbb{C}^m, \lambda \bullet f)$ is *log canonical* at $\boldsymbol{z}$ if $(\mathbb{C}^m, \varepsilon \bullet f)$ is KLT at $\boldsymbol{z}$ for all $0 \leqslant \varepsilon < \lambda$. A pair is *log canonical* if it is log canonical at every point of Z.

Definition 2.3. We define the *log canonical threshold* of f at $\boldsymbol{z}$, denoted $\mathbf{lct}_{\boldsymbol{z}}(\mathbb{C}^m, f)$, as follows: $\mathbf{lct}_{\boldsymbol{z}}(\mathbb{C}^m, f) = \sup\{\, \lambda \in \mathbb{R}_{\geqslant 0} : (\mathbb{C}^m, \lambda \bullet f) \text{ is log canonical at } \boldsymbol{z} \,\}$. We define the (global) log canonical threshold of f, denoted $\mathbf{lct}(\mathbb{C}^m, f)$, as follows:

$\mathbf{lct}(\mathbb{C}^m, f) = \sup\{\, \lambda \geqslant 0 : (\mathbb{C}^m, \lambda \bullet f)$ is log canonical$\,\}$. We will often write $\mathbf{lct}(f)$ and $\mathbf{lct}_{\boldsymbol{z}}(f)$ instead of $\mathbf{lct}(\mathbb{C}^m, f)$ and $\mathbf{lct}_{\boldsymbol{z}}(\mathbb{C}^m, f)$.

As $(\mathbb{C}^m, \lambda \bullet f)$ being log canonical is a local condition, we have that $\mathbf{lct}(\mathbb{C}^m, f) = \inf\{\, \mathbf{lct}_{\boldsymbol{z}}(\mathbb{C}^m, f) : \boldsymbol{z} \in Z \,\}$. Furthermore, it follows by definition that $\mathbf{lct}_{\boldsymbol{z}}(\mathbb{C}^m, f) = \sup\{\, \lambda \in \mathbb{R}_{\geqslant 0} : (\mathbb{C}^m, \lambda \bullet f)$ is KLT at $\boldsymbol{z}\,\}$, and thus the log canonical threshold may also be called the KLT threshold.

If $f = x_1^{a_1} \cdots x_m^{a_m}$ is a monomial, then $(\mathbb{C}^m, \lambda \bullet f)$ is KLT if the integral

$$\int \frac{dx_1 \cdots dx_m}{|x_1^{a_1} \cdots x_m^{a_m}|^{2\lambda}}$$

exists in a neighborhood of the origin. By Fubini's Theorem, it suffices to show that

$$\int \frac{dx_i}{|x_i|^{2a_i\lambda}} \tag{2.3.1}$$

exists in a neighborhood of the origin for every $1 \leqslant i \leqslant m$, and by referring to polar coordinates, we see that the integral in (2.3.1) exists in neighborhood of the origin if and only if $0 \leqslant \lambda < \frac{1}{a_i}$. It follows that $\mathbf{lct}(\mathbb{C}^m, f) = \min\left\{ \frac{1}{a_1}, \cdots, \frac{1}{a_m} \right\}$. If f is not a monomial, is not at all clear how one might compute $\mathbf{lct}(\mathbb{C}^m, f)$. Fortunately, we can always *monomialize* f via a *log resolution.*

Remark 2.4. Let $\pi : \tilde{X} \to \mathbb{C}^m$ be a holomorphic map of complex varieties. If $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ are coordinates on $\tilde{X}$, we may consider $J_{\mathbb{C}}(\pi)$, the *complex Jacobian* of π on U: if $\pi(\tilde{x_1}, \cdots, \tilde{x_n}) = (g_1(\tilde{\boldsymbol{x}}), \cdots, g_m(\tilde{\boldsymbol{x}}))$ on U, then $J_{\mathbb{C}}(\pi)$ is defined on U as $\left(\frac{\partial g_i}{\partial \tilde{x_j}}\right)(\tilde{\boldsymbol{x}})$. It is clear that this definition defends on the choice of coordinate system. On the other hand, if $(U', \tilde{x}'_1, \cdots, \tilde{x}'_m)$ is another set of local coordinates with $U' \subseteq U$, then the determinant of the Jacobian with respect to the coordinates U and U' differ by a non-zero constant. Thus, the closed set $E_\pi := \left\{ \tilde{\boldsymbol{z}} \in \tilde{X} : \det J_{\mathbb{C}}(\pi)(\tilde{\boldsymbol{z}}) = 0 \right\}$ is well defined, and it is a consequence of the Implicit Function Theorem that $E_\pi = \left\{ \tilde{\boldsymbol{z}} \in \tilde{X} : \pi \text{ is not an isomorphism locally at } \tilde{\boldsymbol{z}} \right\}$.

Theorem 2.5. [Hir64] There exists a smooth complex variety $\tilde{X}$ and a smooth proper map $\pi : \tilde{X} \to \mathbb{C}^m$ of varieties satisfying the following conditions:

1. $\pi(E_\pi) \subseteq Z_{\text{sing}}$.

2. $\tilde{X}$ can be covered by local coordinates $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ such that both $f \circ \pi$ and $\det J_{\mathbb{C}}(\pi)$ are monomials on U. Precisely stated, $f \circ \pi = \alpha \cdot \tilde{x}_1^{a_1} \cdots \tilde{x}_m^{a_m}$ and $\det J_{\mathbb{C}}(\pi) = \beta \cdot \tilde{x}_1^{b_1} \cdots \tilde{x}_m^{b_m}$ for some functions α and β that do not vanish on U.

Remark 2.6. The set of integers appearing as exponents in some local monomialization of $f \circ \pi$ is finite. Similarly, the set integers appearing as an exponent in some local monomialization of $\det J_{\mathbb{C}}(\pi)$ is finite. Indeed, they are the coefficients of the *divisors* $\operatorname{div} \det J_{\mathbb{C}}(\pi)$ and $\operatorname{div} f \circ \pi$ on $\tilde{X}$.

Remark 2.7. There exist local coordinates $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ on $\tilde{X}$ such that $f \circ \pi$ is a non-vanishing multiple of $\tilde{x}_1^{a_1} \cdots \tilde{x}_m^{a_m}$ on U, $\det J_{\mathbb{C}}(\pi)$ is a non-vanishing multiple of $\tilde{x}_1^{b_1} \cdots \tilde{x}_m^{b_m}$ on U, and such that $b_i = 0$ and $a_i \neq 0$ for some i. If not, then $\det J_{\mathbb{C}}(\pi)$ would vanish whenever $f \circ \pi$ vanishes, so that $\pi^{-1}(Z) \subseteq E_\pi$. However, this is impossible as π is an isomorphism at every point in Z not in $\pi(E_\pi)$.

Fix a log resolution $\pi : \tilde{X} \to \mathbb{C}^m$ of f as in Theorem 2.5. By the change of variables formula for integration, we see that the function $\frac{1}{|f|^{2\lambda}}$ is locally integrable in $\mathbb{C}^m$ if and only if the functions

$$\frac{\det J_{\mathbb{R}}(\pi)}{|f \circ \pi|^{2\lambda}} = \left|\frac{\det J_{\mathbb{C}}(\pi)}{(f \circ \pi)^{\lambda}}\right|^2 = \left|\frac{\tilde{x_1}^{b_1} \cdots \tilde{x_m}^{b_m}}{\tilde{x_1}^{a_1\lambda} \cdots \tilde{x_m}^{a_m\lambda}}\right|^2 \tag{2.7.1}$$

are integrable in the local coordinate patches $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ given by Theorem 2.5. In (2.7.1), $J_{\mathbb{R}}(\pi)$ is the *real* Jacobian of π: it follows from the Cauchy-Riemann equations that $\det J_{\mathbb{R}}(\pi) = |\det J_{\mathbb{C}}(\pi)|^2$. By Fubini's Theorem, these functions are integrable on U if and only if each function $|\tilde{x}_i|^{2(b_i-\lambda a_i)}$ is integrable on U, and we may readily verify that this occurs if and only if

$$b_i - a_i\lambda > -1 \text{ for } 1 \leqslant i \leqslant m. \tag{2.7.2}$$

Though Theorem 2.5 guarantees the existence of log resolutions of f, computing such resolutions is often difficult, and it remains highly non-trivial to compute the log canonical threshold of an arbitrary polynomial. However, the use of log resolutions allows one to deduce the following facts regarding log canonical thresholds.

Lemma 2.8. Let $f \in \mathbb{C}[x_1, \cdots, x_m]$ be a non-constant polynomial.

1. $\mathbf{lct}(f) \in (0, 1] \cap \mathbb{Q}$.

2. The pair $(\mathbb{C}^m, \mathbf{lct}(f) \bullet f)$ is log canonical.

Proof. From (2.7.2), it follows that

$$\mathbf{lct}(\mathbb{C}^m, f) = \min\left\{ \frac{b_i + 1}{a_i} : 1 \leqslant i \leqslant m \right\}, \tag{2.8.1}$$

where we range over all coordinates $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ satisfying the condition in Theorem 2.5. This is a minimum, and not an infimum, by Remark 2.6. Furthermore, by Remark 2.7, there exist coordinates $(U, \tilde{x}_1, \cdots, \tilde{x}_m)$ with $b_i = 0$ and $a_i \neq 0$ for some i, which shows that $\mathbf{lct}(f) \leqslant 1$. To see that the pair $(\mathbb{C}^m, \mathbf{lct}(f) \bullet f)$ is log canonical, note that the equations in (2.7.2) are satisfied (by definition) if $0 < \lambda < \mathbf{lct}(f)$. $\square$

We now consider another important invariant of singularities of pairs.

Definition 2.9. We define the *multiplier ideal* of the pair $(\mathbb{C}^m, \lambda \bullet f)$ as follows: $\mathscr{J}(\mathbb{C}^m, \lambda \bullet f) = \left\{ g \in \mathbb{C}[x_1, \cdots, x_m] : \left|\frac{g}{f^\lambda}\right|^2 \text{ is locally integrable on } \mathbb{C}^m \right\}$.

Remark 2.10. Note that $(\mathbb{C}^m, \lambda \bullet f)$ is KLT if and only if $1 \in \mathscr{J}(\mathbb{C}^m, \lambda \bullet f)$, and it follows that $\mathbf{lct}(f) = \sup\{\lambda \in \mathbb{R}_{\geqslant 0} : \mathscr{J}(\mathbb{C}^m, \lambda \bullet f) = \mathbb{C}[x_1, \cdots, x_m]\}$. Furthermore, if $\boldsymbol{z} \in \mathbb{C}^m$ and $\mathfrak{m} \subseteq \mathbb{C}[x_1, \cdots, x_m]$ is the corresponding maximal ideal, we also have that $\mathbf{lct}_{\boldsymbol{z}}(f) = \sup\{\lambda \in \mathbb{R}_{\geqslant 0} : \mathscr{J}(\mathbb{C}^m, \lambda \bullet f)_{\mathfrak{m}} = \mathbb{C}[x_1, \cdots, x_m]_{\mathfrak{m}}\}$. Indeed, this follows from the fact that the ideal $\mathscr{J}(\mathbb{C}^m, \lambda \bullet f)_{\mathfrak{m}}$ is equal to

$$\left\{ g \in \mathbb{C}[x_1, \cdots, x_m]_{\mathfrak{m}} : \left|\frac{g}{f^{\lambda}}\right|^2 \text{ is locally integrable at } \boldsymbol{z} \right\}.$$

Note that $\mathscr{J}(\mathbb{C}^m, \lambda_1 \bullet f) \subseteq \mathscr{J}(\mathbb{C}^m, \lambda_0 \bullet f)$ whenever $\lambda_0 \leqslant \lambda_1$. By definition, $\mathscr{J}(\mathbb{C}^m, 0 \bullet f) = \mathbb{C}[x_1, \cdots, x_m]$, and we have just observed that $\mathscr{J}(\mathbb{C}^m, \varepsilon \bullet f)$ is also trivial for every $0 < \varepsilon < \mathbf{lct}(f)$. Thus, the multiplier ideal is locally constant to the right of $\lambda = 0$, and this behavior is typical: for every $\lambda_0 \geqslant 0$, there exists $\lambda_1 > \lambda_0$ such that

$$\mathscr{J}(\mathbb{C}^m, \varepsilon \bullet f) = \mathscr{J}(\mathbb{C}^m, \lambda_0 \bullet f) \text{ whenever } \lambda_0 \leqslant \varepsilon < \lambda_1. \tag{2.10.1}$$

The statement in (2.10.1) motivates the following definition.

Definition 2.11. We say that $\lambda \geqslant 0$ is an *jumping number* of f if $\lambda = 0$, or if $\lambda > 0$ and $\mathscr{J}(\mathbb{C}^m, \lambda \bullet f)$ is *strictly* contained in $\mathscr{J}(\mathbb{C}^m, (\lambda - \varepsilon) \bullet f)$ for all $0 < \varepsilon < \lambda$.

The following proposition shows that to study the jumping numbers of f, it suffices to study the jumping numbers that are contained in the unit interval.

Proposition 2.12. [Laz04] Every natural number is a jumping number of f, and a positive real number $\lambda \notin \mathbb{N}$ is a jumping number of f if and only if the fractional part of λ, which is contained in $(0, 1)$, is a jumping number of f.

By definition, $\mathbf{lct}(f)$ is the first non-zero jumping number of f, and in this way we may consider the jumping numbers of f to be generalizations of the $\mathbf{lct}(f)$. Though they will not be explicitly considered in this thesis, they will be compared compared to their positive characteristic analogs in Chapter 8.

Remark 2.13. Fix a log resolution $\pi : \tilde{X} \to \mathbb{C}^m$. Then, as in (2.7.1), the condition that $\left|\frac{g}{f^\lambda}\right|^2$ be locally integrable can be translated to an integrability condition on $\tilde{X}$ via π. As before, the advantage in doing so is that f becomes a monomial in the local coordinates of $\tilde{X}$, which allows one to explicitly compute $\mathscr{J}(\mathbb{C}^m, \lambda \bullet f)$ in terms of the exponents a_i and b_i appearing in Theorem 2.5. This way of thinking about multiplier ideals is very powerful, and the statement in (2.10.1) is obvious from this perspective. This characterization also allows one to give a definition of multiplier ideals and log canonical singularities in every setting in which log resolutions exist. This generalization will not be discussed further in this thesis, and the reader is referred to [BL04, EM06] for an introduction, and to [Laz04] for a more rigorous development.

2 Some standard constructions in positive characteristic

Let R be a reduced ring of prime characteristic $p > 0$. For $e \geqslant 1$, let $R \xrightarrow{F^e} R$ denote the e^{th} *iterated Frobenius morphism* defined by $r \mapsto r^{p^e}$. We will use $F^e_* R$ to denote R when considered as an R-algebra via F^e. If $R^{p^e} = \left\{ r^{p^e} : r \in R \right\}$ is the subring consisting of $(p^e)^{\text{th}}$ powers of R, then the R-algebra structure of $F^e_* R$ and the R^{p^e}-algebra structure of R are isomorphic.

For $e \geqslant 1$, let R^{1/p^e} be the set consisting of the formal symbols $\left\{ f^{1/p^e} : f \in R \right\}$. We define a ring structure on R^{1/p^e} by setting $f^{1/p^e} + g^{1/p^e} := (f+g)^{1/p^e}$ and $f^{1/p^e} \cdot g^{1/p^e} := (fg)^{1/p^e}$. If R is a domain, then R^{1/p^e} admits a more concrete description: Let L be a fixed algebraic closure of the fraction field of R, and let f^{1/p^e} denote the unique root of the equation $T^{p^e} - f \in L[T]$ in L. We may then describe

R^{1/p^e} as the subring of L consisting of all $(p^e)^{\text{th}}$-roots of elements of R. For example, if $R = \mathbb{F}_p[x_1, \cdots, x_m]$, then $R^{1/p^e} = \mathbb{F}_p[x_1^{1/p^e}, \cdots, x_m^{1/p^e}]$.

As R is reduced, we have an inclusion $R \subseteq R^{1/p^e}$ given by $r \mapsto (r^{p^e})^{1/p^e}$. This is an inclusion because $\left(r^{p^e}\right)^{1/p^e} = 0 \iff r^{p^e} = 0 \iff r = 0$, and it is a ring map since $r + s \mapsto \left((r+s)^{p^e}\right)^{1/p^e} = \left(r^{p^e} + s^{p^e}\right)^{1/p^e} = \left(r^{p^e}\right)^{1/p^e} + \left(s^{p^e}\right)^{1/p^e}$. We also note that $F^e_* R \cong R^{1/p^e}$ as R-modules via the isomorphism $r \mapsto r^{1/p^e}$. This map and its inverse are often referred to as taking "$(p^e)^{\text{th}}$ roots / raising to $(p^e)^{\text{th}}$ powers." We have an inclusion (of rings) $R^{1/p^d} \subseteq R^{1/p^{e+d}}$ given by $r^{1/p^d} = (r^{p^e})^{1/p^{e+d}}$, and in this way, we identify $(R^{1/p^d})^{1/p^e}$ and $R^{1/p^{e+d}}$ as R^{1/p^d}-modules. Also note that for any multiplicative set of W of R, we may canonically identify $W^{-1}\left(R^{1/p^e}\right)$ and $(W^{-1}R)^{1/p^e}$ as modules over the localization $W^{-1}R$.

We say that R is *F-finite* if $R^{1/p}$ (equivalently, F_*R) is a finitely generated R-module. One can show that R is F-finite if and only if R^{1/p^e} (equivalently, F^e_*R) is a finite R-module for every $e \geqslant 1$, or even for one single $e \geqslant 1$.

Example 2.14. By definition, a field K of characteristic $p > 0$ is F-finite if and only if $[K : K^p] < \infty$, and a finitely generated algebra over K is F-finite if and only if K is F-finite. [BMS08, Example 2.1]

3 Singularities of hypersurfaces defined via the Frobenius map

The Frobenius map has been an important tool in commutative algebra since Kunz characterized regular rings as those for which R^{1/p^e} is flat over R [Kun69]. Recall that that R is said to be *F-pure* (or *F-split*) if the inclusion $R \subseteq R^{1/p^e}$ splits as a map of R-modules for some $e \geqslant 1$. An F-pure ring is necessarily reduced, and R is F-pure if and only if the inclusion $R \subseteq R^{1/p^e}$ splits as a map of R-modules for every $e \geqslant 1$, or even for one single $e \geqslant 1$ [HR76].

We would like to generalize of the notion of F-purity to *pairs*. By definition, a pair consists of the ambient ring R, an element $f \in R$, and a non-negative real parameter λ, and is denoted by $(R, \lambda \bullet f)$.

We say that the inclusion $R \cdot f^{N/p^e} \subseteq R^{1/p^e}$ *splits over* R (or splits as a map of R-modules) if there exists an map $\theta \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $\theta\left(f^{N/p^e}\right) = 1$. Note that if f is a unit, then $R \cdot f^{N/p^e} \subseteq R^{1/p^e}$ splits over R if and only if R is F-pure. For the remainder of this section, R will denote an F-pure ring of characteristic $p > 0$, and f will denote a non-zero, non-unit element of R.

We would like to say that the pair $(R, \lambda \bullet f)$ is F-pure if $R \cdot f^{\lambda} \subseteq R^{1/p^e}$ splits for some (or all) $e \geqslant 1$. Of course, the problem in doing so is that f^{λ} need not represent an element of R^{1/p^e} for *any* $e \geqslant 1$. On way of getting arounds this is to approximate λ by a sequence of rational numbers λ_e with $\lambda_e \in \frac{1}{p^e} \cdot \mathbb{N}$ and $\lim\limits_{e \to \infty} \lambda_e = \lambda$, and then require that some (or all) of the inclusions $R \cdot f^{\lambda_e} \subseteq R^{1/p^e}$ split over R. Of course, there are numerous ways to approximate λ (for example, $\frac{\lceil p^e \lambda \rceil}{p^e}$, $\frac{\lfloor (p^e-1)\lambda \rfloor}{p^e}$, and $\frac{\lceil (p^e-1)\lambda \rceil}{p^e}$ would all work) and each choice may lead to a distinct notion of F-purity for pairs.

Definition 2.15. [Tak04, TW04, Sch08] The pair $(R, \lambda \bullet f)$ is said to be

1. *F-pure* if $R \cdot f^{\lfloor (p^e-1)\lambda \rfloor / p^e} \subseteq R^{1/p^e}$ splits over R for all $e \geqslant 1$,
2. *strongly F-pure* if $R \cdot f^{\lceil p^e \lambda \rceil / p^e} \subseteq R^{1/p^e}$ splits over$R$ for some $e \geqslant 1$, and
3. *sharply F-pure* if $R \cdot f^{\lceil (p^e-1)\lambda \rceil / p^e} \subseteq R^{1/p^e}$ splits over R for some $e \geqslant 1$.

If $\wp \subseteq R$ is a prime ideal, we say that $(R, \lambda \bullet f)$ is (strongly/sharply) F-pure at $\wp$ if the pair $(R_{\wp}, \lambda \bullet f)$ is (strongly/sharply) F-pure.

The following lemma shows that F-purity may be detected locally.

Lemma 2.16. $(R, \lambda \bullet f)$ is (strongly/sharply) F-pure if and only if $(R_\wp, \lambda \bullet f)$ is (strongly/sharply) F-pure at every prime ideal $\wp \subseteq R$.

Proof. We will only prove the statement regarding F-purity. By definition, $(R, \lambda \bullet f)$ is F-pure if the map $\Theta : \operatorname{Hom}_R(R^{1/p^e}, R) \to R$ given by evaluation at $f^{\lfloor (p^e-1)\lambda \rfloor / p^e}$ is surjective for all $e \geqslant 1$. However, Θ is surjective if and only if $\Theta_\wp : \operatorname{Hom}_R(R^{1/p^e}, R)_\wp = \operatorname{Hom}_{R_\wp}\left(R_\wp^{1/p^e}, R_\wp\right) \to R_\wp$ is surjective for every $\wp \in \operatorname{Spec} R$, and under this identification is is easy to verify that $\Theta_\wp$ is also given by evaluation at $f^{\lfloor (p^e-1)\lambda \rfloor / p^e}$. $\square$

Of course, Lemma 2.16 also holds with "prime ideal" replaced by "maximal ideal." We now examine how F-purity for a pair is affected by varying the parameter λ.

Lemma 2.17. If the inclusion $R \cdot f^{N/p^e} \subseteq R^{1/p^e}$ splits as a map of R-modules, so must the inclusion $R \cdot f^{a/p^e} \subseteq R^{1/p^e}$ for all $0 \leqslant a \leqslant N$.

Proof. By hypothesis, there exists a map $\theta \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $\theta\left(f^{N/p^e}\right) = 1$. If we let $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$ denote the composition $R^{1/p^e} \xrightarrow{\cdot f^{\frac{N-a}{p^e}}} R^{1/p^e} \xrightarrow{\theta} R$, then $\phi\left(f^{a/p^e}\right) = 1$. $\square$

Lemma 2.18. Let λ be a positive real number.

1. If $(R, \lambda \bullet f)$ is F-pure, then so is $(R, \varepsilon \bullet f)$ for every $0 \leqslant \varepsilon \leqslant \lambda$.

2. $(R, \lambda \bullet f)$ is not F-pure if $\lambda > 1$.

Proof. As $\lfloor (p^e - 1)\, 0 \rfloor = 0$, the pair $(R, 0 \bullet f)$ is F-pure. As $\lfloor (p^e-1)\,\varepsilon \rfloor \leqslant \lfloor (p^e-1)\lambda \rfloor$, Lemma 2.17 shows that $(R, \varepsilon \bullet f)$ is F-pure whenever $(R, \lambda \bullet f)$ is. For the second point, suppose that $(R, (1+\varepsilon) \bullet f)$ is F-pure for some $\varepsilon > 0$, so that

$$R \cdot f^{\lfloor (p^e-1)(1+\varepsilon) \rfloor / p^e} \subseteq R^{1/p^e} \text{ splits over } R \text{ for every } e \geqslant 1. \tag{2.18.1}$$

Note that $(p^e - 1)(1 + \varepsilon) = p^e - 1 + (p^e - 1) \cdot \varepsilon > p^e$ for $e \gg 0$. By (2.18.1) and Lemma 2.17, we have that $R \cdot f \subseteq R^{1/p^e}$ splits over R for $e \gg 0$. We conclude that there exists a map $\theta \in \operatorname{Hom}_R(R^{1/p^e}, R)$ with $1 = \theta(f) = f \cdot \theta(1)$, which is impossible as f is not a unit. ☐

Definition 2.19. [TW04] We define the F-pure threshold of $f \in R$, denoted $\mathbf{fpt}(R, f)$, as follows: $\mathbf{fpt}(R, f) = \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (R, \lambda \bullet f) \text{ is } F\text{-pure} \}$. Note that $\mathbf{fpt}(R, u) = \infty$ whenever u is a unit in R. We define the F-pure threshold of f at $\wp \in \operatorname{Spec} R$, denoted $\mathbf{fpt}_{\wp}(R, f)$, as follows: $\mathbf{fpt}_{\wp}(R, f) = \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (R, \lambda \bullet f) \text{ is } F\text{-pure at } \wp \}$. When there is possibility for confusion, we write $\mathbf{fpt}(f)$ and $\mathbf{fpt}_{\wp}(f)$ rather than $\mathbf{fpt}(R, f)$ and $\mathbf{fpt}_{\wp}(R, f)$. Note that, by definition, $\mathbf{fpt}_{\wp}(R, f) = \mathbf{fpt}(R_{\wp}, f)$.

Remark 2.20. By Lemma 2.18, we see that $\mathbf{fpt}(R, f) \in [0, 1]$.

Remark 2.21. Though not at all obvious from the definition, $\mathbf{fpt}(f) \in \mathbb{Q}$ whenever R is F-finite and regular [BMS08, Theorem 3.1]. For rationality results in a more general setting, see [KLZ09, BSTZ09].

Proposition 2.22. The F-pure and strongly (respectively, sharply) F-pure threshold agree: $\mathbf{fpt}(R, f) = \sup_{\lambda} \{ (R, \lambda \bullet f) \text{ is strongly (respectively, sharply) } F\text{-pure} \}$.

Proof. The equality of these thresholds is shown in [TW04, Proposition 2.2] and [Sch08, Proposition 5.3]. ☐

Remark 2.23. Let $W \subseteq R$ be any multiplicative set. If $(R, \lambda \bullet f)$ is F-pure, it follows that $(W^{-1}R, \lambda \bullet f)$ is also F-pure, and thus $\mathbf{fpt}(R, f) \leqslant \mathbf{fpt}(W^{-1}R, f)$.

Proposition 2.24. The F-pure threshold may be computed locally. Precisely stated, we have that $\mathbf{fpt}(f) = \inf \{ \mathbf{fpt}_{\wp}(f) : \wp \in \operatorname{Spec} R \}$.

Proof. Note that $\mathbf{fpt}(f) \leqslant \inf\{\,\mathbf{fpt}_\wp(f) : \wp \in \operatorname{Spec} R\,\}$ by Remark 2.23. We may now assume that $\lambda := \inf\{\,\mathbf{fpt}_\wp(f) : \wp \in \operatorname{Spec} R\,\} > 0$. For $0 < \varepsilon < \lambda$, we have that $(R, (\lambda - \varepsilon) \bullet f)$ is F-pure at every $\wp$, so that $(R, (\lambda - \varepsilon) \bullet f)$ is also F-pure by Lemma 2.16. Thus, $\mathbf{fpt}(f) \geqslant \lambda - \varepsilon$, and the claim follows by letting $\varepsilon \to 0$. □

4 F-singularities of hypersurfaces in F-finite regular rings

We will next focus on the case of a smooth ambient ring, and will closely follow [BMS08]. Throughout this section, R will be assumed to be an F-finite regular ring of characteristic $p > 0$, and f will denote a non-zero, non-unit in R. In this setting, we know that R^{1/p^e} (equivalently, F^e_*R) is a finitely generated flat R-module [Kun69].

1 $\left[\frac{1}{p^e}\right]^{\text{th}}$ Frobenius powers

In this subsection, we define (see Definition 2.26) and study the properties of $\left[\frac{1}{p^e}\right]^{\text{th}}$ *Frobenius powers* of principal ideals. These ideals will be used in the next subsection to define the *test ideal* associated to a pair.

For every ideal $I \subseteq R$, let $I^{[p^e]}$ denote the ideal generated by the set $\{\, g^{p^e} : g \in I\,\}$. It is clear that $I^{[p^e]} = IF^e_*R$, and we call $I^{[p^e]}$ the e^{th} *Frobenius power* of I. Note that $\left(I^{[p^e]}\right)^{[p^d]} = I^{[p^{e+d}]}$ by definition. Furthermore, as F^e_*R is a flat R-module, we have that $g^{p^e} \in I^{[p^e]}$ if and only if $g \in I$.

Lemma 2.25. Let $\{\, I_n\,\}_n$ be a family of ideals of R. Then $\bigcap_n {I_n}^{[p^e]} = (\bigcap_n I_n)^{[p^e]}$.

Proof. Let $S := F^e_*R$. As $I^{[p^e]} = IS$, it suffices to show that $\bigcap_n (IS) = (\bigcap_n I)\, S$. Furthermore, it suffices to check this equality after localizing at every prime ideal of S. However, S is a finitely generated flat R-algebra (and hence locally free), in which case it is easy to verify that $\bigcap_n (IS_\wp) = (\bigcap_n I)\, S_\wp$ for all $\wp \in \operatorname{Spec} S$. □

Definition 2.26. [BMS08] We define the $\left[\frac{1}{p^e}\right]^{\text{th}}$ *Frobenius power* of f to be the minimal ideal J, with respect to inclusion, such that $f \in J^{[p^e]}$. We denote the $\left[\frac{1}{p^e}\right]^{\text{th}}$ Frobenius power of f by $(f)^{\left[\frac{1}{p^e}\right]}$. This is well defined by Lemma 2.25.

Lemma 2.27. Let $f \in R$.

1. If $J \subseteq R$ is an ideal, then $(f)^{\left[\frac{1}{p^e}\right]} \subseteq J$ if and only if $f \in J^{[p^e]}$.
2. $(f)^{\left[\frac{1}{p^e}\right]} \subseteq (f)^{\left[\frac{1}{p^{e+d}}\right]^{[p^d]}}$.
3. If $N \geqslant n$ are integers, then $\left(f^N\right)^{\left[\frac{1}{p^e}\right]} \subseteq (f^n)^{\left[\frac{1}{p^e}\right]}$.
4. $\left(f^{p^d}\right)^{\left[\frac{1}{p^{e+d}}\right]} = (f)^{\left[\frac{1}{p^e}\right]}$.

Proof. By definition, we have that $f \in J^{[p^e]} \implies (f)^{\left[\frac{1}{p^e}\right]} \subseteq J$. On the other hand, if $(f)^{\left[\frac{1}{p^e}\right]} \subseteq J$, then $f \in \left((f)^{\left[\frac{1}{p^e}\right]}\right)^{[p^e]} \subseteq J^{[p^e]}$. For the second point, let $J := (f)^{\left[\frac{1}{p^{e+d}}\right]}$. Then $f \in J^{[p^d][p^e]}$, which implies that $(f)^{\left[\frac{1}{p^e}\right]} \subseteq J^{[p^d]}$. The third point follows from the fact that if $f^n \in I^{[p^e]}$, then $f^N \in I^{[p^e]}$, while the last follows from the fact that $f^{p^d} \in I^{[p^{e+d}]} \iff f \in I^{[p^e]}$. □

Lemma 2.28. [BMS08, Lemma 2.7] The formation of $\left[\frac{1}{p^e}\right]^{\text{th}}$ powers commutes with localization: If $W \subseteq R$ is a multiplicative set, then $W^{-1}(f)^{\left[\frac{1}{p^e}\right]} = (f_W)^{\left[\frac{1}{p^e}\right]}$, where f_W denotes the image of f in $W^{-1}R$.

Proposition 2.29. [BMS08, Proposition 2.5] Suppose that R is a finitely-generated free module over the subring $R^{p^e} \subseteq R$. If $\beta_1, \cdots, \beta_N$ is a basis for R over R^{p^e}, and $f = a_1^{p^e}\beta_1 + \cdots + a_n^{p^e}\beta_N$ is the unique representation of f as an R^{p^e}-linear combination of this basis, then $(f)^{\left[\frac{1}{p^e}\right]} = (a_1, \cdots, a_n) \subseteq R$.

Proof. It is obvious that $f \in (a_1, \cdots, a_n)^{[p^e]}$. Next, suppose that $f \in J^{[p^e]}$ for some ideal J. Express $f = \sum_{i=1}^r g_i h_i^{p^e}$ with $h_i \in J$, and express $g_i = \sum_{j=1}^N s_{ij}^{p^e}\beta_j$ as an

R^{p^e}-linear combination of the basis elements $\beta_1, \cdots, \beta_N$. Equating the coefficients of the β_i in the expression $f = \sum_{i=1}^r g_i h_i^{p^e}$ will show that $a_i^{p^e} = \sum_{j=1}^N s_{ij}^{p^e} h_i^{p^e}$, and so we may conclude that $a_i = \sum_{j=1}^N s_{ij} h_i \in J$. □

Corollary 2.30. The ideal $(f)^{\left[\frac{1}{p^e}\right]}$ is the image of the map $\operatorname{Hom}_R(R^{1/p^e}, R) \to R$ given by evaluation at f^{1/p^e}.

Proof. The image of the evaluation map obviously commutes with localization, and Lemma 2.28 shows that the same is true for $(f)^{\left[\frac{1}{p^e}\right]}$, which allows us to assume that R is local. In this case, R is a finitely-generated, flat (and hence free) module over the subring R^{p^e}, and the claim is then an immediate corollary of Proposition 2.29. □

Corollary 2.30 allows us to relate $\left[\frac{1}{p^e}\right]^{\text{th}}$ Frobenius powers with F-purity.

Corollary 2.31. Let $(S, \mathfrak{n})$ be an F-finite regular local ring, and let f be a non-unit of S. Then the inclusion $S \cdot f^{1/p^e} \subseteq S^{1/p^e}$ splits over S if and only if $f \notin \mathfrak{n}^{[p^e]}$.

Proof. By Corollary 2.30, $S \cdot f^{1/p^e} \subseteq S^{1/p^e}$ splits if and only if $(f)^{\left[\frac{1}{p^e}\right]} \nsubseteq \mathfrak{n}$, which by Lemma 2.27 occurs if and only if $f \notin \mathfrak{n}^{[p^e]}$. □

Corollary 2.32. $(R, \lambda \bullet f)$ is F-pure at a maximal ideal $\mathfrak{m}$ if and only if $f^{\lfloor (p^e-1)\lambda \rfloor} \notin \mathfrak{m}^{[p^e]}$ for all $e \geqslant 1$. Similarly, $(R, \lambda \bullet f)$ is strongly F-pure at $\mathfrak{m}$ if and only if $f^{\lceil p^e \lambda \rceil} \notin \mathfrak{m}^{[p^e]}$ for some $e \geqslant 1$, and is sharply F-pure at $\mathfrak{m}$ if and only if $f^{\lceil (p^e-1)\lambda \rceil} \notin \mathfrak{m}^{[p^e]}$ for some $e \geqslant 1$.

Proof. $(R, \lambda \bullet f)$ is F-pure at $\mathfrak{m}$ if and only if the inclusion $R_{\mathfrak{m}} \cdot f^{\lfloor (p^e-1)\lambda \rfloor / p^e} \subseteq R_{\mathfrak{m}}^{1/p^e}$ splits for every $e \geqslant 1$. However, Corollary 2.31 shows that this map splits if and only if $f^{\lfloor (p^e-1)\lambda \rfloor} \notin \mathfrak{m}^{[p^e]} \cdot R_{\mathfrak{m}}$. As $\sqrt{\mathfrak{m}^{[p^e]}} = \mathfrak{m}$, $\mathfrak{m}^{[p^e]}$ is a primary ideal, and hence $\mathfrak{m}^{[p^e]} R^{\mathfrak{m}} \cap R = \mathfrak{m}^{[p^e]}$. From this, we can conclude that $f^{\lfloor (p^e-1)\lambda \rfloor} \notin \mathfrak{m}^{[p^e]} \cdot R_{\mathfrak{m}}$ if and only if $f^{\lfloor (p^e-1)\lambda \rfloor} \notin \mathfrak{m}^{[p^e]}$. □

The next statement follows from Corollary 2.32 and Proposition 2.22.

Corollary 2.33. $\mathbf{fpt}_{\mathfrak{m}}(f) = \sup\left\{ \lambda \in \mathbb{R}_{\geqslant 0} : f^{\lceil p^e \lambda \rceil} \notin \mathfrak{m}^{[p^e]} \text{ for some } e \geqslant 1 \right\}$.

2 Test Ideals and F-jumping numbers

In this subsection, we define and present some of the basic properties of the test ideal corresponding to a pair $(R, \lambda \bullet f)$. In order to define the test ideal, we will need the following lemma.

Lemma 2.34. For every $\lambda \geqslant 0$, we have that $\left(f^{\lceil p^e \lambda \rceil}\right)^{\left[\frac{1}{p^e}\right]} \subseteq \left(f^{\left\lceil p^{e+1} \lambda \right\rceil}\right)^{\left[\frac{1}{p^{e+1}}\right]}$.

Proof. Let $J := \left(f^{\left\lceil p^{e+1} \lambda \right\rceil}\right)^{\left[\frac{1}{p^{e+1}}\right]}$. By Definition 2.26, it suffices to show that $f^{\lceil p^e \lambda \rceil}$ is contained in $J^{[p^e]}$. Note that $p^{e+1}\lambda = p \cdot p^e \lambda \leqslant p \lceil p^e \lambda \rceil$, so that $\lceil p^{e+1} \lambda \rceil \leqslant p \lceil p^e \lambda \rceil$ as well. By definition, we have that $f^{\left\lceil p^{e+1}\lambda \right\rceil} \in J^{[p^{e+1}]}$, and so $\left(f^{\lceil p^e \lambda \rceil}\right)^p \in J^{[p^{e+1}]}$ as well. As R is regular, we may conclude that $f^{\lceil p^e \lambda \rceil} \in J^{[p^e]}$. $\square$

Definition 2.35. [BMS08, Definition 2.9] We define the *test ideal of* $(R, \lambda \bullet f)$ as follows: $\boldsymbol{\tau}(R, \lambda \bullet f) := \bigcup_{e \geqslant 1} \left(f^{\lceil p^e \lambda \rceil}\right)^{\left[\frac{1}{p^e}\right]}$. This is an ideal of R by Lemma 2.34.

Remark 2.36. As R is Noetherian, $\boldsymbol{\tau}(R, \lambda \bullet f) = \left(f^{\lceil p^e \lambda \rceil}\right)^{\left[\frac{1}{p^e}\right]}$ for $e \gg 0$.

Remark 2.37. It follows immediately from Remark 2.36 and Lemma 2.28 that $W^{-1}(\boldsymbol{\tau}(R, \lambda \bullet f)) = \boldsymbol{\tau}(W^{-1}R, \lambda \bullet f)$ for every multiplicative set $W \subseteq R$.

The theory of test ideals is a key component in the theory of *tight closure* [HR76, HH90, HY03]. The following result, which is an immediate corollary of the last point of Lemma 2.27, allows us to determine when this ascending chain stabilizes in an important special case.

Lemma 2.38. [BMS09, Lemma 2.1] If $\lambda \in \frac{1}{p^e} \cdot \mathbb{N}$, then $\boldsymbol{\tau}(R, \lambda \bullet f) = \left(f^{p^e \lambda}\right)^{\left[\frac{1}{p^e}\right]}$.

Proposition 2.39. [BMS08, Proposition 2.11, Corollary 2.16]

1. $\boldsymbol{\tau}(R, \lambda_1 \bullet f) \subseteq \boldsymbol{\tau}(R, \lambda_0 \bullet f)$ if $\lambda_1 \geqslant \lambda_0$.

2. $\boldsymbol{\tau}(R, \lambda \bullet f) = R$ for $0 < \lambda \ll 1$.

3. For every $\lambda \geqslant 0$, there exists $\varepsilon > 0$ so that $\boldsymbol{\tau}(R, \lambda \bullet f) = \boldsymbol{\tau}(R, (\lambda + \varepsilon) \bullet f)$.

Proof. Fix $e \gg 0$ so that $\boldsymbol{\tau}(R, \lambda_1 \bullet f) = \left(f^{\lceil p^e \lambda_1 \rceil}\right)^{\left[\frac{1}{p^e}\right]}$ and $\boldsymbol{\tau}(R, \lambda_0 \bullet f) = \left(f^{\lceil p^e \lambda_0 \rceil}\right)^{\left[\frac{1}{p^e}\right]}$. The assertion that $\boldsymbol{\tau}(R, \lambda_1 \bullet f) \subseteq \boldsymbol{\tau}(R, \lambda_0 \bullet f)$ then follows from Lemma 2.27.

To prove the second point, it suffices to show that $\boldsymbol{\tau}(R, \lambda \bullet f)$ is trivial for some $\lambda > 0$. By Lemma 2.27, the ideals $(f)^{\left[\frac{1}{p^e}\right]}$ form an increasing chain, so we may fix a d such that $(f)^{\left[\frac{1}{p^e}\right]} = (f)^{\left[\frac{1}{p^d}\right]}$ for all $e \geqslant d$. If $(f)^{\left[\frac{1}{p^d}\right]} \neq R$, there exists a maximal ideal $\mathfrak{m}$ with $(f)^{\left[\frac{1}{p^e}\right]} = (f)^{\left[\frac{1}{p^d}\right]} \subseteq \mathfrak{m}$ for all $e \geqslant d$. Lemma 2.27 then shows that $f \in \mathfrak{m}^{[p^e]}$ for all $e \geqslant d$, contradicting the fact that $f \neq 0$. We conclude that $\boldsymbol{\tau}\left(R, \frac{1}{p^d} \bullet f\right) = (f)^{\left[\frac{1}{p^d}\right]} = R$.

For the last point, it suffices to show that $\boldsymbol{\tau}(R, (\lambda + \varepsilon) \bullet f) \supseteq \boldsymbol{\tau}(R, \lambda \bullet f)$ for some $\varepsilon > 0$. Fix d such that $\boldsymbol{\tau}(R, \lambda \bullet f) = \left(f^{\lceil p^d \lambda \rceil}\right)^{\left[\frac{1}{p^d}\right]}$. If $p^d \lambda \notin \mathbb{N}$, there exists $0 < \varepsilon \ll 1$ such that $\lceil p^d \lambda \rceil = \lceil p^d (\lambda + \varepsilon) \rceil$. Then

$$\boldsymbol{\tau}(R, (\lambda + \varepsilon) \bullet f) \supseteq \left(f^{\lceil p^d (\lambda + \varepsilon) \rceil}\right)^{\left[\frac{1}{p^d}\right]} = \left(f^{\lceil p^d \lambda \rceil}\right)^{\left[\frac{1}{p^d}\right]} = \boldsymbol{\tau}(R, \lambda \bullet f).$$

We now assume that $p^d \lambda \in \mathbb{N}$. After possibly increasing d, we may also assume that $\boldsymbol{\tau}\left(R, \frac{1}{p^d} \bullet f\right) = (f)^{\left[\frac{1}{p^d}\right]} = R$. We now show that $\boldsymbol{\tau}(R, \lambda \bullet f) = \left(f^{p^d \lambda}\right)^{\left[\frac{1}{p^d}\right]}$ is contained in $\boldsymbol{\tau}\left(R, \left(\lambda + \frac{1}{p^{2d}}\right) \bullet f\right) = \left(f^{p^{2d} \lambda + 1}\right)^{\left[\frac{1}{p^{2d}}\right]}$. Let $J := \left(f^{p^{2d} \lambda + 1}\right)^{\left[\frac{1}{p^{2d}}\right]}$. By definition, we have that $f^{p^{2d} \lambda + 1} \in J^{[p^{2d}]}$, so that

$$f \in \left(J^{[p^{2d}]} : f^{p^{2d} \lambda}\right) = \left(J^{[p^d]} : f^{p^d \lambda}\right)^{[p^d]}. \tag{2.39.1}$$

However, (2.39.1) and the fact that $(f)^{\left[\frac{1}{p^d}\right]} = R$ imply that $\left(J^{[p^d]} : f^{p^d \lambda}\right) = R$ as well. From this, we conclude that $f^{p^d \lambda} \in J^{[p^d]}$, so that $\left(f^{p^d \lambda}\right)^{\left[\frac{1}{p^d}\right]} \subseteq J$. $\square$

Definition 2.40. We say that $\lambda \geqslant 0$ is an *F-jumping number* of f if $\lambda = 0$, or if $\lambda > 0$ and $\boldsymbol{\tau}(R, \lambda \bullet f)$ is *strictly* contained in $\boldsymbol{\tau}(R, (\lambda - \varepsilon) \bullet f)$ for all $0 < \varepsilon < \lambda$.

The following result, whose proof we omit, shows that the jumping numbers of f are completely determined by those contained in the unit interval.

Proposition 2.41. [BMS08, Proposition 2.25] Every natural number is an F-jumping number of f, and a positive real number $\lambda \notin \mathbb{N}$ is an F-jumping number if and only if the fractional part of λ, which is contained in $(0, 1)$, is an F-jumping number.

We have seen in Proposition 2.39 that $\boldsymbol{\tau}(R, \lambda \bullet f) = R$ for small values of λ, and so the first F-jumping corresponds to the first value at which $\boldsymbol{\tau}(R, \lambda \bullet f)$ is non-trivial. We see below that this first jumping number coincides with $\mathbf{fpt}(R, f)$.

Proposition 2.42. $\mathbf{fpt}(R, f) = \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : \boldsymbol{\tau}(R, \lambda \bullet f) = R \}$

Proof. $\boldsymbol{\tau}(R, \lambda \bullet f) = R$ if and only if $\left(f^{\lceil p^e \lambda \rceil}\right)^{\left[\frac{1}{p^e}\right]} = R$ for some $e \geqslant 1$, which by Corollary 2.30 happens if and only if there exists a map $\theta \in \operatorname{Hom}_R(R^{1/p^e}, R)$ such that $\theta\left(f^{\lceil p^e \lambda \rceil / p^e}\right) = 1$. This is precisely the condition that $(R, \lambda \bullet f)$ be strongly F-pure, and the assertion follows from Proposition 2.22. □

Remark 2.43. As $\mathbf{fpt}_{\mathfrak{m}}(R, f) = \mathbf{fpt}(R_{\mathfrak{m}}, f)$, it follows from Proposition 2.42 and Remark 2.37 that $\mathbf{fpt}_{\mathfrak{m}}(f) = \sup \{ \lambda \geqslant 0 : \boldsymbol{\tau}(R, \lambda \bullet f)_{\mathfrak{m}} = \boldsymbol{\tau}(R_{\mathfrak{m}}, \lambda \bullet f) = R_{\mathfrak{m}} \}$.

CHAPTER 3

Some connections

Suppose that $f \in \mathbb{Q}[x_1, \cdots, x_m]$. By reducing each of the coefficients of f modulo $p \gg 0$, we obtain a *family of models* $f_p \in \mathbb{F}_p[x_1, \cdots, x_m]$. In general, given a polynomial $f \in \mathbb{C}[x_1, \cdots, x_m]$, we may produce a family of positive characteristic models of f via the method of *reduction to positive characteristic.*

We begin with an example. Consider the polynomial $f = 5x + \pi w y^2 + e\sqrt{3}z^{11}$ in $\mathbb{C}[x, y, z, w]$. If we let $A := \mathbb{Z}[\pi, e, \sqrt{3}]_{\pi \cdot e \cdot \sqrt{3}} \subseteq \mathbb{C}$, then A is a finitely-generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, and $f \in A[x, y, z, w]$. Note that from $A[x, y, z, w]$, we can recover $\mathbb{C}[x, y, z, w]$ via base change: $\mathbb{C} \otimes_A A[x, y, z, w] = \mathbb{C}[x, y, z, w]$. If $\mu \subseteq A$ is a maximal ideal, then A/μ is a finite field by Corollary 3.2, which also shows that all but finitely many p appear in the set $\{ \operatorname{char} A/\mu : \mu \subseteq A \text{ is maximal} \}$. By construction, each coefficient of f is a unit in A, and hence will have non-zero image under the map $A \mapsto A/\mu$. Thus, if f_μ denotes the image of f under the map $A[x, y, z, w] \to A/\mu \otimes_A A[x, y, z, w] = (A/\mu)\,[x, y, z, w]$, f_μ is a polynomial over a finite field of positive characteristic whose supporting monomials are the same as the supporting monomials of f. Furthermore, by varying μ we obtain models f_μ over fields of all but finitely many characteristics.

We will discuss generalizations of this process in what follows. Rather than focus on polynomial rings, we will discuss the process of creating positive characteristic models for for any element (and ideal) in a finitely generated $\mathbb{C}$-algebra.

1 Preliminaries

In this section, we gather a collection of results related to the method of reduction to positive characteristic. These results are consequences of the following variant of Noether Normalization.

Lemma 3.1 (Noether Normalization for Domains)**.** Let A be a finitely-generated algebra over a domain D with $D \subseteq A$. There exists $0 \neq N \in D$ such that the extension $D_N \subseteq A_N$ can be factored as $D_N \subseteq D_N[z_1, \cdots, z_d] \subseteq A_N$, where $z_1, \cdots, z_d \in A_N$ are algebraically independent over D_N, and $D_N[z_1, \cdots, z_d] \subseteq A_N$ is module-finite.

If $L = \operatorname{Frac} D$, and R is the localization of A at the non-zero elements of D, then by Noether Normalization for fields, the inclusion $L \subseteq R$ can be factored as a purely transcendental extension followed by a module-finite extension. As only finitely many "denominators" are involved at this process, one can localize at a single element of D and preserve this factorization. For a proof of Lemma 3.1 that is independent of Noether Normalization for fields, we refer the reader to [Hoc].

Corollary 3.2. If $\mathbb{Z} \subseteq A$ is a finitely‘generated $\mathbb{Z}$-algebra, then

1. every maximal ideal of A contains a prime p, and
2. all but finitely many primes p are contained in a maximal ideal of A.
3. Furthermore, A/μ is a finite field for every maximal ideal $\mu \subseteq A$.

Proof. Let $\mu \subseteq A$ be a maximal ideal, and suppose that $\mu \cap \mathbb{Z} = 0$. It follows that A/μ is also a finitely generated $\mathbb{Z}$-algebra, so by Lemma 3.1, there exists an integer $N \neq 0$ such that $A/\mu = (A/\mu)_N$ is module finite over a polynomial ring with coefficients in $\mathbb{Z}_N$. This implies that $0 = \dim A/\mu \geqslant \dim \mathbb{Z}_N = 1$, a contradiction.

For the second point, let $0 \neq N \in \mathbb{Z}$ and $z_1, \cdots, z_d \in A_N$ be as in Lemma 3.1, so that the extension $\mathbb{Z}_N \subseteq A_N$ factors as $\mathbb{Z}_N \subseteq \mathbb{Z}_N[z_1, \cdots, z_d] \subseteq A_N$. Every p not dividing N generates a prime ideal in both $\mathbb{Z}_N$ and $\mathbb{Z}_N[z_1, \cdots, z_d]$, and so by the Lying Over Theorem, there exists a prime (and hence maximal) ideal of A not containing N, and lying over p.

For the last point, we have that A/μ is finitely generated algebra over $\mathbb{F}_p$ for some p, and thus is module finite over a polynomial ring $\mathbb{F}_p[z_1, \cdots, x_d]$ by Lemma 3.1. As A/μ is a field, dimension considerations force that $d = 0$, and thus A/μ is a finite extension of $\mathbb{F}_p$. □

Lemma 3.3. Let $A \subseteq B$ be an inclusion of finitely-generated $\mathbb{Z}$-algebras. If $\mu_B \subseteq B$ is maximal, then so is $\mu_A := \mu \cap A \subseteq A$.

Proof. By Corollary 3.2, we know that B/μ_B is a finite field, and the inclusion $A/\mu_A \subseteq B/\mu_B$ then shows that A/μ_A must also be a finite field. □

Corollary 3.4. Let A be a domain, $A \subseteq B$ be an inclusion of finitely-generated A-algebras, and let $\pi : \operatorname{Spec} B \to \operatorname{Spec} A$ be the induced map of schemes. Then the (inverse) image of a dense set under π is also dense. Furthermore, the image under π of a non-empty open set contains a non-empty open subset of A.

Proof. We will first show that if $W \subset \operatorname{Spec} B$ is dense in $\operatorname{Spec} B$, then $\pi(W)$ is dense in $\operatorname{Spec} A$. By means of contradiction, suppose that $\pi(W)$ is *not* dense in $\operatorname{Spec} A$, so that $\pi(W) \cap U = \varnothing$ for some non-empty open set $U \subseteq \operatorname{Spec} A$. It follows that

$$W \cap \pi^{-1}(U) \subseteq \pi^{-1}(\pi(W)) \cap \pi^{-1}(U) = \varnothing. \tag{3.4.1}$$

However, π, being induced by the inclusion $A \subseteq B$, is dominant, and so $\pi^{-1}(U) \neq \varnothing$. As W is dense in $\operatorname{Spec} B$, we must have that $\pi^{-1}(U) \cap W \neq \varnothing$, contradicting (3.4.1).

We will now show that if $\Lambda \subseteq \operatorname{Spec} A$ is dense, then $\pi^{-1}(\Lambda)$ is dense in $\operatorname{Spec} B$. It suffices to show that $\operatorname{Spec} B_f \cap \pi^{-1}(\Lambda) \neq \varnothing$ for every non-zero element $f \in B$. By assumption, B is a finitely-generated A-algebra, and thus so is $B_f = B[T]/(1 - Tf)$. By Lemma 3.1, there exists $N \in A$ non-zero such that $A_N \subseteq B_{fN}$ factors as

$$A_N \subseteq A_N[z_1, \cdots, z_d] \subseteq B_{fN}, \tag{3.4.2}$$

where $z_1, \cdots, z_d$ are elements of B_{fN} that are algebraically independent over A_N, and the second inclusion in (3.4.2) is module finite. By the Lying Over Theorem, it follows that the map $\operatorname{Spec} B_{fN} \to \operatorname{Spec} A_N$ induced by (3.4.2) is surjective. As Λ is dense in $\operatorname{Spec} A$, we have that $\Lambda \cap \operatorname{Spec} A_N \neq \varnothing$. The previous surjection shows that $\operatorname{Spec} B_{fN} \cap \pi^{-1}(\Lambda) \neq \varnothing$, and thus that $\operatorname{Spec} B_f \cap \pi^{-1}(\Lambda) \neq \varnothing$.

Let $U \subseteq \operatorname{Spec} B$ be an open set, and suppose that U contains the basic open set $\varnothing \neq \operatorname{Spec} B_f$. We have seen in the preceding paragraph that there exists $N \neq 0$ in A such that $\operatorname{Spec} A_N = \pi(\operatorname{Spec} B_{fN}) \subseteq \pi(\operatorname{Spec} B_f) \subseteq \pi(U)$. $\square$

The following result shows an explicit way to reduce a polynomial over $\mathbb{C}$ to positive characteristic, and will be important in Chapter 7.

Corollary 3.5. Let $f \in \mathbb{C}[x_1, \cdots, x_s]$ with $f(\mathbf{0}) = 0$. Then there exists a finitely-generated $\mathbb{Z}$-subalgebra $A \subseteq \mathbb{C}$ with $f \in A[x_1, \cdots, x_s]$ satisfying the following property: for $p \gg 0$, $\pi^{-1}(p) \neq \varnothing$ and $\operatorname{Support}(f_{\mu_p}) = \operatorname{Support}(f)$ for every maximal ideal $\mu_p \in \pi^{-1}(p)$. Here $\pi : \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ denotes the map induced by $\mathbb{Z} \subseteq A$.

Proof. Let $f = u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n}$ for some coefficients $u_i \in \mathbb{C}^*$, and let $B = \mathbb{Z}[u_1^{\pm 1}, \cdots, u_m^{\pm 1}]$; note that B is a finitely-generated $\mathbb{Z}$-algebra. Choose an integer N so the inclusion $\mathbb{Z}_N \subseteq B_N$ factors as in Lemma 3.1. We may take $A := B_N$. $\square$

2 On reduction to positive characteristic

If $\mathbb{Z} \subseteq A \subseteq \mathbb{C}$ is a finitely-generated $\mathbb{Z}$-algebra and $g \in \mathbb{C}[x_1, \cdots, x_s]$, we say that g is *defined over* A if $g \in A[x_1, \cdots, x_s] \subseteq \mathbb{C}[x_1, \cdots, x_s]$. Similarly, we say that an ideal $J \subseteq \mathbb{C}[x_1, \cdots, x_s]$ is *defined over* A if some fixed set of generators of J are defined over A. This condition is equivalent to the existence of an ideal $J_A \subseteq A[x_1, \cdots, x_s]$ with the property that $J_A \cdot \mathbb{C}[x_1, \cdots, x_s] = J$. For example, $J = (\pi x^2, y + \sqrt{2}z)$ and $f = \ln 5 \cdot x + y \in S$ are both defined over $\mathbb{Z}[\pi, \sqrt{2}, \ln 5]$. In fact, if J and f are defined over A, then they are defined over any finitely-generated $\mathbb{Z}$-algebra $B \subseteq \mathbb{C}$ with $A \subseteq B$. If Σ is any *finite* set of ideals (or elements) of S, there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ such that every member of Σ is defined over A.

Let $S = \mathbb{C}[x_1, \cdots, x_s]/I$ be a finitely-generated $\mathbb{C}$-algebra. Suppose that I is defined over A, and fix an ideal $I_A \subseteq A[x_1, \cdots, x_s]$ that expands to I. We say that S is *defined over* A, and set $S_A := A[x_1, \cdots, x_s]/I_A$. We now point out some of the subtleties of these definitions and constructions.

Remark 3.6. First, note that if I is defined over A, then I does not canonically determine an ideal of $A[x_1, \cdots, x_s]$. Indeed, if I_A and I'_A are both ideals of $A[x_1, \cdots, x_s]$ that expand to I, then I_A and I'_A *need not be equal* as ideals of $A[x_1, \cdots, x_s]$. For example, choose complex numbers α and β that are pairwise algebraically independent over $\mathbb{Q}$ (i.e., π and e is very likely an example of such a pair). Let $A = \mathbb{Z}[\alpha, \beta]$, and consider the ideals $I_A = (x, \alpha \cdot y)$ and $I'_A = (x, \beta \cdot y)$ in $A[x, y]$. Note that $I_A \neq I'_A$ as ideals of $A[x, y]$, though $I_A \cdot \mathbb{C}[x, y] = I'_A \cdot \mathbb{C}[x, y] = (x, y)$. In this specific case, we see that though $I_A \neq I'_A$, we still have that $A[x, y]/I_A \equiv A[x, y]/I'_A$. This suggests that distinct choices for I_A might yield isomorphic quotients.

However, $S_A = A[x_1, \cdots, x_m]/I_A$ may depend on the choice of I_A. For example, let $A = \mathbb{Z}[\pi]$, and consider the ideals $I_A = (x)$ and $I'_A = (\pi x)$ of $A[x]$. Then $I_A \cdot \mathbb{C}[x] = I'_A \cdot \mathbb{C}[x] = (x)$, yet $A[x]/I_A \not\equiv A[x]/I'_A$ as rings, for

$$A[x]/I_A = A \equiv \mathbb{Z}[t] \not\equiv \mathbb{Z}[t,x]/(tx) \equiv A[x]/I'_A,$$

where t is an indeterminate over $\mathbb{Z}$. That S_A depends on the choice of A is clear.

Many of the ambiguities described above can be resolved by expanding our "coefficient" base A. Indeed, if $I_A \subseteq A[x_1, \cdots, x_s]$ and $I_B \subseteq B[x_1, \cdots, x_s]$ are ideals whose expansions to $\mathbb{C}[x_1, \cdots, x_s]$ are equal, then there exists a finitely generated $\mathbb{Z}$-algebra C containing both A and B such that the expansions of I_A and I_B are equal in $C[x_1, \cdots, x_s]$. Indeed, one need only attach the finitely many coefficients involved in expressing generators of I_A in terms of those for I_B over $\mathbb{C}$, and vice versa.

Another unifying point is that, regardless of the choice of A or I_A, we can always recover A from S_A, as $\mathbb{C} \otimes_A S_A = S$. This justifies the point of view that S_A, though not canonically determined, is a good approximation of S.

Consider the map $\operatorname{Spec} S_A \to \operatorname{Spec} A$ induced by the map $A \to S_A$. The fiber over the generic point $(0) \in \operatorname{Spec} A$ is the variety corresponding to $\operatorname{Frac} A \otimes_A S_A$, while the fiber over a closed point (i.e., maximal ideal) $\mu \subseteq A$ is the variety corresponding to $A/\mu \otimes_A S_A$. By Corollary 3.2, this is a variety over the finite field A/μ.

By extending the scalars of the closed fiber from $\operatorname{Frac} A$ to $\mathbb{C}$, we obtain the variety corresponding to S. This follows from the fact that $\mathbb{C} \otimes_A (\operatorname{Frac} A \otimes_A S_A) = \mathbb{C} \otimes_A S_A = S$. As the extension $\operatorname{Frac} A \subseteq \mathbb{C}$ is nice (i.e., faithfully flat), we see that much of the information carried by S is carried by the fiber of $\operatorname{Spec} S_A \to \operatorname{Spec} A$ over the generic point of $\operatorname{Spec} A$, regardless of whatever choices were made. On the other hand, it is a general principle that properties of the generic fiber are closely related to those of the closed fibers. However, each closed fiber is a variety over a finite field, and can be studied via the Frobenius map on the ambient ring.

Suppose that we want to study a finitely-generated $\mathbb{C}$-algebra S. If S is defined over A, we may then hope to use properties satisfied by the fibers of $\operatorname{Spec} S_A \to \operatorname{Spec} A$ over "most" closed points of $\operatorname{Spec} A$ to say something about S. In doing so, we would like to know that this is independent of the choices being made.

3 On F-purity for pairs reduced from $\mathbb{C}$

The aim of this section is to examine the F-purity property for pairs that are reduced from positive characteristic. Let f denote a polynomial with non-zero image in $S := \mathbb{C}[x_1, \cdots, x_s]/I$. Suppose that $A \subseteq \mathbb{C}$ is a finitely-generated $\mathbb{Z}$-algebra such that f and I are both defined over A. Fix an ideal $I_A \subseteq A[x_1, \cdots, x_m]$ with $I_A \cdot \mathbb{C}[x_1, \cdots, x_m] = I$, and let $S_A = A[x_1, \cdots, x_m]/I_A$ be as defined above, so that $\mathbb{C} \otimes_A S_A = S$. For every maximal ideal $\mu_A \subseteq A$, f_{μ_A} will denote the image of f under the map $A[x_1, \cdots, x_s] \to S_A \to S_A(\mu_A) := A/\mu \otimes_A S_A$. Let $B \subseteq \mathbb{C}$ be another finitely-generated $\mathbb{Z}$-algebra with f and I both defined over B, and let $I_B \subseteq B[x_1, \cdots, x_m], S_B = B[x_1, \cdots, x_m]/I_B$, and $f_{\mu_B} \in S_B(\mu_B)$ be as above.

Though the construction of S_A depends on the choice of A and I_A, we will show that the issue of whether or not the positive characteristic pairs $(S_A(\mu_A), \lambda \bullet f_{\mu_A})$ are F-pure for "most" maximal ideals $\mu_A \subseteq A$ is independent of these choices. Here, "most" maximal ideals will mean all maximal ideals in some dense (or dense open) subset of $\operatorname{Spec} A$. In doing so, we will rely on the following result.

Lemma 3.7. Let $K \subseteq L$ be perfect fields of characteristic $p > 0$. Let R be a K-algebra, and consider the inclusion $R \subseteq T := L \otimes_K R$. Then there exists a map $\theta_R \in \operatorname{Hom}_R\left(R^{1/p^e}, R\right)$ with $\theta_R(g^{1/p^e}) = 1$ if and only if there exists a map $\theta_T \in \operatorname{Hom}_T\left(T^{1/p^e}, T\right)$ with $\theta_T\left(g^{1/p^e}\right) = 1$.

Proof. As T is a free R-module (and hence faithfully flat), the map of R-modules

$$\operatorname{Hom}_R\left(R^{1/p^e}, R\right) \to R$$

given by evaluation at g^{1/p^e} is surjective if and only if the map of T-modules

$$\operatorname{Hom}_T\left(T \otimes_R R^{1/p^e}, T\right) \to T \tag{3.7.1}$$

given by evaluation at $1 \otimes g^{1/p^e}$ is surjective. We identify T^{1/p^e} and $T \otimes_R R^{1/p^e}$ as T-algebras via the canonical isomorphisms

$$\begin{aligned} T \otimes_R R^{1/p^e} = (L \otimes_K R) \otimes_R R^{1/p^e} = L \otimes_K R^{1/p^e} &\equiv L^{1/p^e} \otimes_{K^{1/p^e}} R^{1/p^e} \\ &\equiv (L \otimes_K R)^{1/p^e} \equiv T^{1/p^e}. \end{aligned}$$

Here, we have crucially used that L and K are perfect. It is easy to check that the isomorphism $T \otimes_R R^{1/p^e} \equiv T^{1/p^e}$ induces an isomorphism between the map $\operatorname{Hom}_T\left(T^{1/p^e}, T\right) \to T$ given by evaluation at $g^{1/p^e} \in R^{1/p^e} \subseteq T^{1/p^e}$ with the map appearing in (3.7.1). $\square$

Lemma 3.8. Suppose that $A \subseteq B$, and that $I_B \subseteq B[x_1, \cdots, x_m]$ is equal to the expansion of $I_A \subseteq A[x_1, \cdots, x_m]$ to $B[x_1, \cdots, x_m]$. Let μ_B be a maximal ideal of $\operatorname{Spec} B$, and let $m_A := \mu_B \cap A$ denote the corresponding maximal ideal of A. Then $(S_A(\mu_A), \lambda \bullet f_{\mu_A})$ is F-pure if and only if $(S_B(\mu_B), \lambda \bullet f_{\mu_B})$ is F-pure. In particular, $\mathbf{fpt}(f_{\mu_A}) = \mathbf{fpt}(f_{\mu_B})$.

Proof. Let $p := \operatorname{char} A/\mu_A = \operatorname{char} B/\mu_B$. Note that $A/\mu_A \subseteq B/\mu_B$ is an extension of finite (hence, perfect) fields. That $I_A \cdot B[x_1, \cdots, x_m] = I_B$ implies that $S_B \equiv B \otimes_A S_A$, and furthermore we have

(3.8.1)

$$\begin{aligned} S_B(\mu_B) = S_B \otimes_B B/\mu_B \equiv (S_A \otimes_A B) \otimes_B B/\mu_B \equiv & S_A \otimes_A B/\mu_B \\ \equiv & S_A \otimes_A \left(A/\mu_A \otimes_{A/\mu_A} B/\mu_B\right) \\ \equiv & S_A(\mu_A) \otimes_{A/\mu_A} B/\mu_B \end{aligned}$$

Under this identification, it is easy to see that $f_{\mu_A} \mapsto f_{\mu_B}$ under the inclusion $S_A(\mu_A) \subseteq S_A(\mu_A) \otimes_{A/\mu_A} B/\mu_B \equiv S_B(\mu_B)$. Finally, it follows from Lemma 3.7 that there exists a map in $\operatorname{Hom}_{S_A(\mu_A)}\left(S_A(\mu_A)^{1/p^e}, S_A(\mu_A)\right)$ sending f_A^{N/p^e} to 1 if and only if there exists a map in $\operatorname{Hom}_{S_B(\mu_B)}\left(S_B(\mu_B)^{1/p^e}, S_B(\mu_B)\right)$ sending f_B^{N/p^e} to 1, and the claim follows. $\square$

Corollary 3.9. The pair $(S_A(\mu_A), \lambda \bullet f_{\mu_A})$ is F-pure for all maximal ideals μ_A in some dense (respectively, dense open) subset of $\operatorname{Spec} S_A$ if and only if the pair $(S_B(\mu_B), \lambda \bullet f_{\mu_B})$ is F-pure for all maximal ideals μ_B in some dense (respectively, dense open) subset of $\operatorname{Spec} S_B$.

Proof. We will first assume that $A \subseteq B$ and that $I_B = I_A \cdot B[x_1, \cdots, x_m]$, so that Lemma 3.8 applies. Suppose there exists a dense (respectively, dense open) set $W_A \subseteq \operatorname{Spec} A$ such that $(S_A(\mu_A), \lambda \bullet f_{\mu_A})$ is F-pure for every maximal ideal $\mu_A \in W_A$, and set $W_B := \pi^{-1}(W_A)$. By Corollary 3.4, W_B is a dense (respectively, dense open) subset of $\operatorname{Spec} B$. Let μ_B denote a maximal ideal in W_B, and let $\mu_A = \mu_B \cap A$ be the image of μ_B under π. It follows from Lemma 3.8 that $(S_B(\mu_B), \lambda \bullet f_{\mu_B})$ is F-pure.

Instead, suppose there exists a dense (respectively, dense open) set $W_B \subseteq \operatorname{Spec} B$ such that $(S_B(\mu_B), \lambda \bullet f_{\mu_B})$ is F-pure for every maximal ideal $\mu_B \in W_B$. Let W_A denote any dense (or dense open) subset of $\pi(W_B)$ (such a set exists by Corollary 3.4), and let μ_A denote a maximal ideal in W_A. By definition, we may choose a maximal ideal μ_B in B such that $\mu_A = \pi(\mu_B) = \mu_B \cap A$. Once again, Lemma 3.8 shows that $(S_A(\mu_A), \lambda \bullet f_{\mu_A})$ must also be F-pure.

We now address the general case. As $I_A \cdot \mathbb{C}[x_1, \cdots, x_s] = I_B \cdot \mathbb{C}[x_1, \cdots, x_s]$, there is a finitely-generated $\mathbb{Z}$-algebra $D \subseteq \mathbb{C}$ such that $A, B \subseteq D$, and $I_A \cdot D[x_1, \cdots, x_s] = I_B \cdot D[x_1, \cdots, x_s] := I_D$. By construction, our initial argument applies to the inclusions $A \subseteq D$ and $B \subseteq D$, and thus the claim comparing A and B follows. $\square$

4 A connection between singularities

In this subsection, $f \in S = \mathbb{C}[x_1, \cdots, x_m]$ will denote a polynomial with $f(\mathbf{0}) = 0$. If $A \subseteq \mathbb{C}$ is a finitely-generated $\mathbb{Z}$-algebra, then S_A will denote the polynomial ring $A[x_1, \cdots, x_m] \subseteq S$. If $\mu \subseteq A$ is a maximal ideal, we will use $S_A(\mu)$ to denote the polynomial ring $A/\mu \otimes_A S_A = (A/\mu)[x_1, \cdots, x_m]$. By Corollary 3.2, $S_A(\mu)$ is a polynomial ring over a finite field. By abuse of notation, we will use $\mathfrak{m}$ to denote the ideal generated by the variables in the polynomial rings S, S_A, and $S_A(\mu)$.

Definition 3.10. We say that $(\mathbb{C}^m, \lambda \bullet f)$ has *dense F-pure type* at $\mathbf{0}$ if for every finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ with $f \in S_A$, the positive characteristic pairs $(S_A(\mu), \lambda \bullet f_\mu)$ are F-pure at $\mathfrak{m}$ for all maximal ideals μ in some dense (though not neccesarily open) subset of $\operatorname{Spec} A$.

This definition of dense F-pure type can easily be extended to study the singularities of f at any point $\boldsymbol{z}$ with $f(\boldsymbol{z}) = 0$. One way to do so is as follows: Fix an automorphism $S \xrightarrow{\phi} S$ such that $\boldsymbol{z} \mapsto \mathbf{0}$ under the induced isomorphism $\mathbb{C}^m \to \mathbb{C}^m$, and let $g = \phi(f)$. Note that $g(\mathbf{0}) = 0$. Then $(\mathbb{C}^m, \lambda \bullet f)$ has dense F-pure type at $\boldsymbol{z}$ if $(\mathbb{C}^m, \lambda \bullet g)$ has dense F-pure type at $\mathbf{0}$.

Remark 3.11. If follows from Corollary 3.9 that it suffices to check the condition in Definition 3.10 for one single A with $f \in S_A$.

The following theorem allows us to compare log canonical singularities with F-pure singularities.

Theorem 3.12. [HW02, Tak04] If the pair $(\mathbb{C}^m, \lambda \bullet f)$ has dense F-pure type at $\mathbf{0}$, then it is also log canonical at $\mathbf{0}$.

The following result further illustrates the close relationship between singularities in characteristic zero and those in characteristic $p > 0$.

Theorem 3.13. [Smi00, HY03] There exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ with $f \in S_A$, and ideals $\mathscr{H}^\lambda \subseteq (S_A)_\mathfrak{m}$ for every $\lambda \geqslant 0$ with $\mathscr{H}^\lambda \cdot S_\mathfrak{m} = \mathscr{J}\left(\mathbb{C}^m, \lambda \bullet f\right)_\mathfrak{m}$ satisfying the following conditions:

1. There exists a dense open set $U \subseteq \operatorname{Spec} A$ such for every maximal ideal $\mu \in U$ and for every $\lambda \geqslant 0$, we have $\boldsymbol{\tau}\left(S_A(\mu), \lambda \bullet f_\mu\right)_\mathfrak{m} \subseteq \mathscr{H}^\lambda \cdot S_A(\mu)_\mathfrak{m}$.

2. For every $\lambda \geqslant 0$, there exists a dense open set $U_\lambda \subseteq \operatorname{Spec} A$ such that for every maximal ideal $\mu \in U_\lambda$, we have $\boldsymbol{\tau}\left(S_A(\mu), \lambda \bullet f_\mu\right)_\mathfrak{m} = \mathscr{H}^\lambda \cdot S_A(\mu)_\mathfrak{m}$.

We emphasize that the set $U \subseteq \operatorname{Spec} A$ does not depend on λ, while the sets U_λ do, and typically get smaller as λ increases. As a corollary of Theorem 3.13, we obtain the following relationship between thresholds.

Theorem 3.14. For every $A \subseteq \mathbb{C}$ with $f \in S_A$, the following hold:

1. There exists a dense open set $W \subseteq \operatorname{Spec} A$ such that $\mathbf{fpt}_\mathfrak{m}(f_\mu) \leqslant \mathbf{lct}_\mathbf{0}(f)$ for every maximal ideal $\mu \in U$.

2. For every $0 < \lambda < \mathbf{lct}_\mathbf{0}(f)$, there exists a dense open set $W_\lambda \subseteq \operatorname{Spec} A$ such that $\lambda \leqslant \mathbf{fpt}_\mathfrak{m}(f_\mu) \leqslant \mathbf{lct}_\mathbf{0}(f)$ for every maximal ideal $\mu \in W_\lambda$.

We again emphasize that the set U_λ varies with λ, and typically shrinks as λ increases.

Proof. Corollary 3.9 can be used to show that the conclusions of the theorem hold if and only if they hold for a specific A with $f \in S_A$. Let A be as in Theorem 3.13. For the first point, let $W = U$ be as in the first statement in Theorem 3.13. For $\mu \in U$, the inclusion $\boldsymbol{\tau}\left(S_A(\mu), f_\mu \bullet \lambda\right)_{\mathfrak{m}} \subseteq \mathscr{H}^\lambda \cdot S_A(\mu)$ and Remark 2.43 imply that

(3.14.1)

$$\mathbf{fpt}_{\mathfrak{m}}(f_\mu) = \sup\{\lambda : \boldsymbol{\tau}\left(S_A(\mu), f_\mu \bullet \lambda\right)_{\mathfrak{m}} = S_A(\mu)_{\mathfrak{m}}\} \leqslant \sup\{\lambda : \mathscr{H}^\lambda \cdot S_A(\mu)_{\mathfrak{m}} = S_A(\mu)_{\mathfrak{m}}\}.$$

However, as $(S_A)_{\mathfrak{m}} \to S_{\mathfrak{m}}$ and $(S_A)_{\mathfrak{m}} \to S_A(\mu)_{\mathfrak{m}}$ are local maps of local rings, we have that $\mathscr{H}^\lambda \cdot S_A(\mu)_{\mathfrak{m}}$ is trivial $\iff$ $\mathscr{H}^\lambda$ is trivial $\iff$ $\mathscr{H}^\lambda \cdot S_{\mathfrak{m}} = \mathscr{J}\left(\mathbb{C}^m, \lambda \bullet f\right)_{\mathfrak{m}}$ is trivial. This observation, combined with (3.14.1) and Remark 2.10 show that $\mathbf{fpt}_{\mathfrak{m}}(f_\mu) \leqslant \mathbf{lct}_{\mathbf{0}}(f)$.

For the second point, let $0 < \lambda < \mathbf{lct}_{\mathbf{0}}(f)$, let $W_\lambda = U_\lambda \cap U$, where U and U_λ are as in Theorem 3.13, and fix a maximal ideal $\mu \in W_\lambda$. That $\mathbf{fpt}_{\mathfrak{m}}(f_\mu) \leqslant \mathbf{lct}_{\mathbf{0}}(f)$ follows from the preceding paragraph. By our choice of λ, Remark 2.10 shows that $\mathscr{J}\left(\mathbb{C}^m, \lambda \bullet f\right)_{\mathfrak{m}}$ is trivial. This shows that $\mathscr{H}^\lambda \cdot S_{\mathfrak{m}} = \mathscr{J}\left(\mathbb{C}^m, \lambda \bullet f\right)_{\mathfrak{m}}$ is also trivial, which we have already seen holds if and only if $\mathscr{H}^\lambda \cdot S_A(\mu)_{\mathfrak{m}} = \boldsymbol{\tau}\left(S_A(\mu), \lambda \bullet f_\mu\right)_{\mathfrak{m}}$ is trivial as well. It then follows from Remark 2.43 that $\lambda \leqslant \mathbf{fpt}_{\mathfrak{m}}(f_\mu)$. ☐

Conjecture 3.15. For every $A \subseteq \mathbb{C}$ with $f \in S_A$, there exists a dense set U in $\operatorname{Spec} A$ such that $\mathbf{fpt}(f_\mu) = \mathbf{lct}(f)$ for all maximal ideals $\mu \in U$.

Whenever f has rational coefficients, this conjecture is equivalent to the statement that $\mathbf{fpt}_{\mathfrak{m}}(f_p) = \mathbf{lct}_{\mathbf{0}}(f)$ for infinitely many $p \gg 0$. For $f \in \mathbb{Q}[x_1, \cdots, x_m]$, the set of all primes such that $\mathbf{fpt}_{\mathfrak{m}}(f_p) = \mathbf{lct}_{\mathbf{0}}(f)$ also appears to encode subtle arithmetic information of f, as illustrated by the following example.

Example 3.16. Let $f \in \mathbb{Q}[x, y, z]$ be a homogeneous polynomial of degree 3 with isolated singuarity at $\mathbf{0}$, so that f defines an elliptic curve $E \subseteq \mathbb{P}^2$. Then, $\mathbf{lct}_{\mathbf{0}}(f) = 1$, and $\mathbf{fpt}_{\mathfrak{m}}(f_p) = 1$ if and only if $E_p = \mathbb{V}(f_p)$ is not *supersingular*. It is known that there are infinitely many primes such that E_p is not supersingular, though the collection of such primes may be small (i.e., have density zero in the set of all primes) [Ser72]. It is also not possible that $\mathbf{fpt}_{\mathfrak{m}}(f_p) = \mathbf{lct}_{\mathbf{0}}(f)$ for all $p \gg 0$, as it is known that there are infinitely many primes for which E_p *is* supersingular [Elk87]. See [MTW05, Example 4.6] for more details on this example.

CHAPTER 4

On base p expansions

Much of this thesis is dedicated to studying the properties of F-pure thresholds, and by Remark 2.20, we know that these invariants are always contained in the unit interval. In this section, we introduce some notation regarding base p (or p-adic) expansions of numbers contained in the unit interval, and derive some easy results. Though elementary, the idea of studying a number via its base p expansion will be useful when applied to F-pure thresholds.

Definition 4.1. Let $\alpha \in (0,1]$. A *non-terminating base p expansion of α* is an expression $\alpha = \sum_{i \geqslant 1} \frac{a_i}{p^i}$, with $0 \leqslant a_i \leqslant p-1$, such that $\forall\, N > 0, \exists\, e \geqslant N$ with $a_e \neq 0$. The number a_e is called the e^{th} *digit* of the non-terminating base p expansion of α.

Remark 4.2. Though base p expansions are not unique in general, we note that every $\alpha \in (0,1]$ possesses a unique non-terminating base p expansion.

Example 4.3. The non-terminating base p expansion of $\frac{1}{p}$ is $\frac{1}{p} = \frac{0}{p} + \sum_{e \geqslant 2} \frac{p-1}{p^e}$.

Example 4.4. Let $\alpha = \frac{a}{b}$ be a rational number in $(0,1]$, and fix a prime $p \equiv 1 \bmod b$, so that $p = b\omega + 1$ for some $\omega \geqslant 1$. Dividing both sides by p shows that $1 = \frac{b\omega}{p} + \frac{1}{p}$, and multiplying both sides by $\alpha = \frac{a}{b}$ shows that

$$\alpha = \frac{a\omega}{p} + \frac{1}{p} \cdot \alpha. \tag{4.4.1}$$

As $a \leqslant b$, we also have that $a\omega \leqslant b\omega = p - 1$. This, along with (4.4.1), shows that the non-terminating base p expansion of α is constant, and is given by $\alpha = \sum_{e \geqslant 1} \frac{a\omega}{p^e}$.

We refer the reader to [HW08, Chapter 9] for the standard algorithm on computing base p expansions.

Definition 4.5. Let $\alpha \in (0, 1]$, and let p be a prime number, and fix $e \geqslant 1$.

1. $0 \leqslant c_e(\alpha) \leqslant p - 1$ will denote the e^{th} digit in the non-terminating base p expansion of α. By convention, $c_\alpha(0) = 0$.

2. For $e \geqslant 1$, we define the e^{th} *truncation of the non-terminating base p expansion of α* by $\langle \alpha \rangle_e := \frac{c_1(\alpha)}{p} + \cdots + \frac{c_e(\alpha)}{p^e}$. By convention, $\langle 0 \rangle_e = 0$.

3. For $e \geqslant 0$, we define the e^{th} *tail of the non-terminating base p expansion of α* by $[\![\alpha]\!]_e := \sum_{d \geqslant e+1} \frac{c_d(\alpha)}{p^d}$. By convention, $[\![0]\!]_e = 0$.

4. For $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_n) \in [0, 1]^n$, we set $\langle \boldsymbol{\alpha} \rangle_e := (\langle \alpha_1 \rangle_e, \cdots, \langle \alpha_n \rangle_e)$.

Lemma 4.6. Let $\alpha \in (0, 1]$. Then the following hold:

1. $\langle \alpha \rangle_e \in \frac{1}{p^e} \cdot \mathbb{N}$.

2. $0 < [\![\alpha]\!]_e \leqslant \frac{1}{p^e}$, with equality if and only if $\alpha \in \frac{1}{p^e} \cdot \mathbb{N}$.

3. $\alpha = \langle \alpha \rangle_e + [\![\alpha]\!]_e$.

4. $\langle \alpha \rangle_e < \alpha$ and $[\![\alpha]\!]_e > 0$ for all e.

5. If $\alpha \notin \frac{1}{p^e} \cdot \mathbb{N}$, then $\langle \alpha \rangle_e = \frac{\lfloor p^e \alpha \rfloor}{p^e}$.

6. $\lceil p^e \alpha \rceil = p^e \langle \alpha \rangle_e + 1$.

7. $p^e \langle \alpha \rangle_e - 1 \leqslant \lfloor (p^e - 1)\alpha \rfloor \leqslant p^e \langle \alpha \rangle_e$.

8. $p^e \langle\alpha\rangle_e \leqslant \lceil (p^e-1)\alpha \rceil \leqslant p^e \langle\alpha\rangle_e + 1$.

9. If $\beta \in (0,1] \cap \frac{1}{p^e} \cdot \mathbb{N}$ and $\alpha > \beta$, then $\langle\alpha\rangle_e \geqslant \beta$.

Proof. Points 1 - 4 follow by definition. For 5, note that $\lfloor p^e\alpha \rfloor = \lfloor p^e \langle\alpha\rangle_e + p^e [\![\alpha]\!]_e \rfloor = p^e \langle\alpha\rangle_e + \lfloor p^e [\![\alpha]\!]_e \rfloor$. As $\alpha \notin \frac{1}{p^e} \cdot \mathbb{N}$, we have that $0 < [\![\alpha]\!]_e < \frac{1}{p^e}$, and so $\lfloor p^e [\![\alpha]\!]_e \rfloor = 0$. For 6, we have that $p^e\alpha = p^e \langle\alpha\rangle_e + p^e [\![\alpha]\!]_e$, with $0 < p^e [\![\alpha]\!]_e \leqslant 1$ by 2. Thus, we see that $\lceil p^e\alpha \rceil = p^e \langle\alpha\rangle_e + 1$. As $|p^e [\![\alpha]\!]_e - \alpha| < 1$, points 7 and 8 follow from rounding the equation $(p^e-1)\alpha = p^e\alpha - \alpha = p^e \langle\alpha\rangle_e + p^e [\![\alpha]\!]_e - \alpha$. We now prove the last point. By 2, we have that $\langle\alpha\rangle_e + \frac{1}{p^e} \geqslant \alpha > \beta$, and multiplying by p^e shows that $p^e \langle\alpha\rangle_e + 1 > p^e\beta$. As both sides of this inequality are integers, we can conclude that $p^e \langle\alpha\rangle_e \geqslant p^e\beta$. ∎

Lemma 4.7. If $(p^d-1) \cdot \alpha \in \mathbb{N}$, then $\langle\alpha\rangle_{ed+d} = \langle\alpha\rangle_{ed} + \frac{1}{p^{ed}} \cdot \langle\alpha\rangle_d$ for all $e \geqslant 1$.

Proof. Left to the reader. ∎

Definition 4.8. If $\alpha \in [0,1]$, let $\overline{\langle\alpha\rangle}_e = \sum_{n\geqslant 0} \frac{\langle\alpha\rangle_e}{p^{en}} = \langle\alpha\rangle_e \cdot \sum_{n\geqslant 0} \frac{1}{p^{en}} = \langle\alpha\rangle_e \cdot \frac{p^e}{p^e-1}$. We observe that $\overline{\langle\alpha\rangle}_e$ is the rational number whose base p expansion is obtained by "repeating" the first e digits of the non-terminating base p expansion of α.

Lemma 4.9. If $\alpha \in [0,1]$, then the following hold:

1. $\left\langle \overline{\langle\alpha\rangle}_e \right\rangle_e = \langle\alpha\rangle_e$.

2. $p^e \langle\alpha\rangle_e = (p^e-1)\overline{\langle\alpha\rangle}_e$.

Proof. Left to the reader. ∎

Lemma 4.10. Let $\alpha \in [0,1]$. For $e \geqslant 1$, the following conditions are equivalent:

1. $\lfloor (p^e-1)\alpha \rfloor = p^e \langle\alpha\rangle_e$.

2. $\alpha \leqslant p^e \llbracket \alpha \rrbracket_e$.

3. $\alpha \geqslant \overline{\langle \alpha \rangle_e}$.

Proof. We may assume that $\alpha > 0$. By Lemma 4.6, $\alpha = \langle \alpha \rangle_e + \llbracket \alpha \rrbracket_e$, and so

$$(p^e - 1)\alpha = p^e \langle \alpha \rangle_e + p^e \llbracket \alpha \rrbracket_e - \alpha. \tag{4.10.1}$$

From this, we gather that $\lfloor (p^e - 1)\alpha \rfloor = p^e \langle \alpha \rangle_e + \lfloor p^e \llbracket \alpha \rrbracket_e - \alpha \rfloor$, which shows that (1) holds if and only if (2) holds. By Lemma 4.9, $p^e \langle \alpha \rangle_e = (p^e - 1)\overline{\langle \alpha \rangle_e}$. Substituting this into (4.10.1) and gathering the $(p^e - 1)$ terms yields the equation

$$(p^e - 1)\left(\alpha - \overline{\langle \alpha \rangle_e}\right) = p^e \llbracket \alpha \rrbracket_e - \alpha. \tag{4.10.2}$$

Thus, (4.10.2) shows that (2) holds if and only if (3) holds. □

Definition 4.11. Let $(\alpha_1, \cdots, \alpha_n) \in [0,1]^n$, and let p be a prime number.

1. We say that the non-terminating base p expansions of $\alpha_1, \cdots, \alpha_n$ *add without carrying* if $c_e(\alpha_1) + \cdots + c_e(\alpha_n) \leqslant p - 1$ for every $e \geqslant 1$.

2. For $\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$, we say that the base p expansions of $k_1, \cdots, k_n$ *add without carrying* if the obvious analogous condition holds.

Remark 4.12. The non-terminating base p expansions of $\alpha_1, \cdots, \alpha_n$ add without carrying if and only if the base p expansions of the integers $p^e \langle \alpha_1 \rangle_e, \cdots, p^e \langle \alpha_n \rangle_e$ add without carrying for all $e \geqslant 1$.

Remark 4.13. If the non-terminating base p expansions of $\alpha_1, \cdots, \alpha_n$ add without carrying and $\alpha := \alpha_1 + \cdots + \alpha_n$, then $c_e(\alpha) = c_e(\alpha_1) + \cdots + c_e(\alpha_n)$ for all $e \geqslant 1$.

The following classical results will play a key role in this thesis.

Lemma 4.14 ([Dic02, Luc78])**.** Let $\boldsymbol{k} = (k_1, \cdots k_n) \in \mathbb{N}^n$, and set $N := |\boldsymbol{k}| = \sum k_i$. Then $\binom{N}{\boldsymbol{k}} := \frac{N!}{k_1! \cdots k_m!} \not\equiv 0 \mod p$ if and only if the base p expansions of the entries of $\boldsymbol{k}$ add without carrying.

Theorem 4.15 (Dirichlet)**.** For any collection $\alpha_1, \cdots, \alpha_n$ of rational numbers, there exist infinitely many primes p such that $(p-1) \cdot \alpha_i \in \mathbb{N}$ for $1 \leqslant i \leqslant n$.

Lemma 4.16. Let $(\alpha_1, \cdots, \alpha_n) \in \mathbb{Q}^n \cap [0,1]^n$.

1. If $\alpha_1 + \cdots + \alpha_n \leqslant 1$, then there exist infinitely many primes p such that the non-terminating base p expansions of $\alpha_1, \cdots, \alpha_n$ add without carrying.

2. If $\alpha_1 + \cdots + \alpha_n > 1$, then there exist infinitely many primes p such that $c_1(\alpha_1) + \cdots + c_1(\alpha_n) \geqslant p$.

Proof. By Theorem 4.15, there exist infinitely many primes p congruent to 1 modulo the denominator of each α_i. As in Example 4.4, we have that the non-terminating base p expansion of each α_i is constant. In the case that $\alpha_1 + \cdots + \alpha_n \leqslant 1$, one must have that $\sum_i c_e(\alpha_i) \leqslant p - 1$, and so (1) follows. Similarly, if $\alpha_1 + \cdots + \alpha_n > 1$, then one must have that $\sum_i c_e(\alpha_i) \geqslant p$, and so (2) follows. ☐

The following lemma will be very useful in Chapter 9 when computing the F-pure threshold of binomial hypersurfaces.

Lemma 4.17. Let $(\alpha, \beta) \in [0,1]^2$. If $c_{e+1}(\alpha) + c_{e+1}(\beta) \geqslant p$, then

$$\langle \alpha \rangle_e + \langle \beta \rangle_e + \frac{1}{p^e} = \langle \alpha + \beta \rangle_e .$$

Proof. If both α and β are in $\frac{1}{p^e} \cdot \mathbb{N}$, then so is $\alpha + \beta$. By Lemma 4.6, we have that $\langle \alpha \rangle_e + \frac{1}{p^e} + \langle \beta \rangle_e + \frac{1}{p^e} = \alpha + \beta = \langle \alpha + \beta \rangle_e + \frac{1}{p^e}$, and the result follows. We may now assume that $\alpha \notin \frac{1}{p^e} \cdot \mathbb{N}$, so that $[\![\alpha]\!]_e < \frac{1}{p^e}$. By our choice of e, we also have that $[\![\alpha]\!]_e + [\![\beta]\!]_e > \frac{c_e(\alpha)}{p^{e+1}} + \frac{c_{e+1}(\beta)}{p^{e+1}} \geqslant \frac{1}{p^e}$. Thus, $\frac{1}{p^e} < [\![\alpha]\!]_e + [\![\beta]\!]_e < \frac{2}{p^e}$ and $\alpha + \beta = \langle\alpha\rangle_e + \langle\beta\rangle_e + \frac{1}{p^e} + \left([\![\alpha]\!]_e + [\![\beta]\!]_e - \frac{1}{p^e}\right)$, which shows that

$$\alpha + \beta = \langle\alpha\rangle_e + \langle\beta\rangle_e + \frac{1}{p^e} + \delta \text{ for some } 0 < \delta < \frac{1}{p^e}. \tag{4.17.1}$$

From (4.17.1), we see that $\alpha + \beta \notin \frac{1}{p^e} \cdot \mathbb{N}$ and $\lfloor p^e(\alpha+\beta)\rfloor = p^e\langle\alpha\rangle_e + p^e\langle\beta\rangle_e + 1$. The result then follows from point 5 from Lemma 4.6. $\square$

CHAPTER 5

F-pure thresholds of hypersurfaces

The main results of this chapter clarify the relationship between the various types of F-purity appearing in the literature, namely F-purity and (sharp/strong) F-purity. Theorem 5.19 tells us that (strong) F-purity is always a stronger condition than sharp F-purity, which is (possibly) stronger than the condition of F-purity. We also see that F-purity and sharp F-purity are the same if and only if $\mathbf{fpt}(R, f)$ is a rational number whose denominator is not divisible by p, and that $(R, \mathbf{fpt}(f) \bullet f)$ is always F-pure. In Proposition 5.12, we derive some interesting restrictions on the set of all F-pure thresholds in a fixed characteristic. We emphasize that these results are valid assuming only that R is F-pure, and thus generalize facts which are known to hold whenever R is an F-finite, (complete) regular local ring.

1 Truncations of the F-pure threshold

Throughout this chapter, R will denote an F-pure ring of characteristic $p > 0$, and f will denote a non-unit in R. In this section, we will prove Key Lemma 5.2 (which shows that the truncations of the F-pure threshold encode important "splitting data"), and deduce some important consequences. In doing so, we will use the fact that an R-linear map $\theta : R^{1/p^e} \to R$ gives rise, in a natural way, to an

R^{1/p^ℓ}-linear map $\theta^{1/p^\ell} : R^{1/p^{e+\ell}} \to R^{1/p^\ell}$ defined by $\theta^{1/p^\ell}(r^{1/p^{e+\ell}}) = \theta(r^{1/p^e})^{1/p^\ell}$. We call θ^{1/p^ℓ} the $(p^\ell)^{th}$*-root* of the map θ.

Lemma 5.1. Let $\alpha \in [0,1] \cap \frac{1}{p^d} \cdot \mathbb{N}$. If the inclusion $R \cdot f^{\lceil p^e \alpha \rceil / p^e} \subseteq R^{1/p^e}$ splits as a map of R-modules for some $e \geqslant 1$, so must the inclusion $R \cdot f^\alpha \subseteq R^{1/p^d}$.

Proof. Suppose $R \cdot f^{\lceil p^e \alpha \rceil / p^e} \subseteq R^{1/p^e}$ splits, so that $\theta(f^{\lceil p^e \alpha \rceil / p^e}) = 1$ for some map $\theta \in \operatorname{Hom}_R(R^{1/p^e}, R)$. If $e \geqslant d$, then $\alpha \in \frac{1}{p^d} \cdot \mathbb{N} \subseteq \frac{1}{p^e} \cdot \mathbb{N}$, and so $f^\alpha \in R^{1/p^d} \subseteq R^{1/p^e}$. It is then clear that $f^{\lceil p^e \alpha \rceil / p^e} = f^\alpha$ maps to 1 under the composition $R^{1/p^d} \subseteq R^{1/p^e} \xrightarrow{\theta_e} R$.

Instead, suppose that $d > e$, so that $R^{1/p^e} \subseteq R^{1/p^d}$. Note that $p^e \alpha \leqslant \lceil p^e \alpha \rceil$, so $\alpha \leqslant \lceil p^e \alpha \rceil / p^e$. By Lemma 2.17, it suffices to show that there exists a map $R^{1/p^d} \to R$ sending $f^{\lceil p^e \alpha \rceil / p^e}$ to 1. As R is F-pure, there exists an R^{1/p^e}-linear map $\phi : R^{1/p^d} \to R^{1/p^e}$ with $\phi(1) = 1$. If σ denotes the composition $R^{1/p^d} \xrightarrow{\phi} R^{1/p^e} \xrightarrow{\theta_e} R$, then $\sigma \left(f^{\lceil p^e \alpha \rceil / p^e} \right) = 1$. $\square$

Key Lemma 5.2. Let R be an F-pure ring of characteristic $p > 0$, and fix $d \geqslant 1$. Then $p^d \langle \mathbf{fpt}(f) \rangle_d = \max \left\{ a \in \mathbb{N} : R \cdot f^{a/p^d} \subseteq R^{1/p^d} \text{ splits as a map of } R\text{-modules} \right\}$.

Proof. Set $\lambda := \mathbf{fpt}(f)$ and $\nu_f(p^d) := \max \left\{ a : R \cdot f^{a/p^d} \subseteq R^{1/p^d} \text{ splits} \right\}$. Note that $0 \leqslant \nu_f(p^d) \leqslant p^d - 1$. If $\lambda = 0$, then $\nu_f(p^d) = 0$, and we will now assume that $\lambda \in (0,1]$. By Lemma 4.6, $\langle \lambda \rangle_d < \lambda$, it follows from Proposition 2.22 that $(R, \langle \lambda \rangle_d \bullet f)$ is strongly F-pure, and hence there exists $e \geqslant 1$ such that $R \cdot f^{\lceil p^e \langle \lambda \rangle_d \rceil / p^e} \subseteq R^{1/p^e}$ splits. By Lemma 5.1, we may assume that $e = d$, and it follows that $p^d \langle \lambda \rangle_d \leqslant \nu_f(p^d)$.

To prove equality, we must show that the inclusion $R \cdot f^{\langle \lambda \rangle_d + \frac{1}{p^d}} \subseteq R^{1/p^d}$ *never* splits. By Lemma 4.6, $p^d \langle \lambda \rangle_d + 1 = \lceil p^d \lambda \rceil$, so it suffices to show that $\theta \left(f^{\lceil p^d \lambda \rceil / p^d} \right) \neq 1$ for every $\theta \in \operatorname{Hom}_R(R^{1/p^d}, R)$. If $\lambda \notin \frac{1}{p^d} \cdot \mathbb{N}$, it follows that $\lceil p^d \lambda \rceil = \lceil p^d (\lambda + \varepsilon) \rceil$ for $0 < \varepsilon \ll 1$. By Definition 2.19, it follows that $\theta \left(f^{\lceil p^d \lambda \rceil / p^d} \right) = \theta \left(f^{\lceil p^d (\lambda + \varepsilon) \rceil / p^d} \right) \neq 1$ for every $\theta \in \operatorname{Hom}_R(R^{1/p^d}, R)$.

Now, suppose that $\lambda \in \frac{1}{p^d} \cdot \mathbb{N}$, so that $\lceil p^d \lambda \rceil / p^d = \lambda$. By way of contradiction, suppose that $\theta\left(f^{\lambda}\right) = 1$ for some $\theta \in \operatorname{Hom}_R(R^{1/p^d}, R)$. As $0 \neq \lambda$, it follows that $\lambda \geqslant \frac{1}{p^d}$. By Lemma 2.17, there exists an R-linear map $R^{1/p^d} \to R$ sending f^{1/p^d} to 1. Taking $(p^d)^{\text{th}}$ roots of this map produces an R^{1/p^d}-linear map $\phi : R^{1/p^{2d}} \to R^{1/p^d}$ with the property that $\phi(f^{1/p^{2d}}) = 1$. Under the composition $R^{1/p^{2d}} \xrightarrow{\phi} R^{1/p^d} \xrightarrow{\theta} R$, it follows from the R^{1/p^d}-linearity of ϕ that

$$f^{\lambda + \frac{1}{p^{2d}}} = f^{\lambda} \cdot f^{1/p^{2d}} \mapsto f^{\lambda} \cdot \phi(f^{1/p^{2d}}) = f^{\lambda} \cdot 1 \mapsto \theta(f^{\lambda}) = 1.$$

We see that $R \cdot f^{\lambda + \frac{1}{p^{2d}}} \subseteq R^{1/p^{2d}}$ splits, contradicting the definition of $\lambda = \mathbf{fpt}(f)$. $\square$

Remark 5.3. It follows from Key Lemma 5.2 and Lemma 2.17 that $R \cdot f^{N/p^e} \subseteq R^{1/p^e}$ splits as a map of R-modules if and only if $\frac{N}{p^e} \leqslant \langle \mathbf{fpt}(f) \rangle_e$.

Proposition 5.4. Let $\alpha \in [0, 1]$ with $(p^d - 1) \cdot \alpha \in \mathbb{N}$. If $R \cdot f^{\langle \alpha \rangle_d} \subseteq R^{1/p^d}$ splits as a map of R-modules, then so does $R \cdot f^{\langle \alpha \rangle_{ed}} \subseteq R^{1/p^{ed}}$ for every $e \geqslant 1$.

Proof. We induce on e, the base case being our hypothesis. Suppose $R \cdot f^{\langle \alpha \rangle_d} \subseteq R^{1/p^d}$ and $R \cdot f^{\langle \alpha \rangle_{ed}} \subseteq R^{1/p^{ed}}$ split as maps of R-modules, so that there exists

1. an R-linear map $R^{1/p^d} \to R$ with $f^{\langle \alpha \rangle_d} \mapsto 1$, and
2. an R-linear map $\theta : R^{1/p^{ed}} \to R$ with $\theta(f^{\langle \alpha \rangle_{ed}}) = 1$.

We now show that $R \cdot f^{\langle \alpha \rangle_{ed+d}} \subseteq R^{1/p^{ed+d}}$ splits as a map of R-modules. By taking $(p^{ed})^{\text{th}}$-roots of the map in 1, we obtain

(3) an $R^{1/p^{ed}}$-linear map $\phi : R^{1/p^{ed+d}} \to R^{1/p^{ed}}$ with $\phi\left(f^{\frac{\langle \alpha \rangle_d}{p^{ed}}}\right) = 1$.

By Lemma 4.7, we have that $\langle \alpha \rangle_{ed+d} = \langle \alpha \rangle_{ed} + \frac{\langle \alpha \rangle_d}{p^{ed}}$ for all $e \geqslant 1$, and it follows that

$$f^{\langle \alpha \rangle_{ed+d}} = f^{\frac{\langle \alpha \rangle_d}{p^{ed}}} \cdot f^{\langle \alpha \rangle_{ed}}. \tag{5.4.1}$$

Under the composition $R^{1/p^{ed+d}} \xrightarrow{\phi} R^{1/p^{ed}} \xrightarrow{\theta} R$, it follows from (5.4.1), and the $R^{1/p^{ed}}$-linearity of ϕ, that $f^{\langle\alpha\rangle_{ed+d}} = f^{\frac{\langle\alpha\rangle_d}{p^{ed}}} \cdot f^{\langle\alpha\rangle_{ed}} \mapsto f^{\langle\alpha\rangle_{ed}} \cdot \phi\left(f^{\frac{\langle\alpha\rangle_d}{p^{ed}}}\right) = f^{\langle\alpha\rangle_{ed}} \cdot 1 \mapsto \theta(f^{\langle\alpha\rangle_{ed}}) = 1$. □

Corollary 5.5. Let $\alpha \in [0,1]$ with $(p^d - 1) \cdot \alpha \in \mathbb{N}$. If $R \cdot f^{\langle\alpha\rangle_d} \subseteq R^{1/p^d}$ splits, then $\alpha \leqslant \mathbf{fpt}(f)$.

Proof. By Proposition 5.4, $R \cdot f^{\langle\alpha\rangle_{ed}} \subseteq R^{1/p^{ed}}$ splits for all $e \geqslant 1$. By Key Lemma 4.6, we have that $\langle\alpha\rangle_{ed} \leqslant \langle\mathbf{fpt}(f)\rangle_{ed}$ for every $e \geqslant 1$, and taking the limit as $e \to \infty$ gives the desired inequality. □

We arrive at the following generalization of [MTW05, Proposition 2.16].

Corollary 5.6. $\mathbf{fpt}(f) = 1$ if and only if $R \cdot f^{(p-1)/p} \subseteq R^{1/p}$ splits over R.

Proof. If $\mathbf{fpt}(f) = 1$, then $\langle\mathbf{fpt}(f)\rangle_1 = \frac{p-1}{p}$, and it follows from Key Lemma 5.2 that $R \cdot f^{(p-1)/p} \subseteq R^{1/p}$ splits. On the other hand, if $R \cdot f^{(p-1)/p} \subseteq R^{1/p}$ splits, then Corollary 5.5 shows that $\mathbf{fpt}(f) \geqslant 1$. The claim then follows, as $\mathbf{fpt}(f) \leqslant 1$ by Remark 2.20. □

Corollary 5.7. If $(R, \mathfrak{m})$ is an F-finite regular local ring, then $R/(f)$ is F-pure if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$ if and only if $\mathbf{fpt}(f) = 1$.

Proof. The first assertion is just Feder's Criteria [Fed83, Proposition 2.1]. One the other hand, Corollary 2.31 shows that $f^{p-1} \notin \mathfrak{m}^{[p]}$ if and only if $R \cdot f^{(p-1)/p} \subseteq R^{1/p}$ splits, which by Corollary 5.6 holds if and only if $\mathbf{fpt}(f) = 1$. □

2 The set of all F-pure thresholds

Definition 5.8. $\mathbf{FPT}_p$ will denote the set of all characteristic p F-pure thresholds:

$\mathbf{FPT}_p := \{\, \mathbf{fpt}(R, f) : 0 \neq f, \text{ a non-unit in an } F\text{-pure ring } R \text{ of characteristic } p \,\}.$

Remark 5.9. We stress that the ring R is allowed to vary in Definition 5.8. By Remark 2.20, we have that $\mathbf{FPT}_p \subseteq [0,1]$.

Recall from Definition 4.8 that if $\lambda \in [0,1]$, then $\overline{\langle \lambda \rangle}_e = \frac{p^e-1}{p^e} \cdot \langle \lambda \rangle_e \in [0,1]$ is the rational number whose base p expansion is obtained by repeating the first e digits of the non-terminating base p expansion of λ.

Proposition 5.10. For any $\lambda \in \mathbf{FPT}_p$, and for any $e \geqslant 1$, we have that $\overline{\langle \lambda \rangle}_e \leqslant \lambda$.

Proof. There exists an F-pure ring R and an element $f \in R$ such that $\lambda = \mathbf{fpt}(R, f)$. Set $\alpha := \overline{\langle \lambda \rangle}_e$. By Lemma 4.9, it follows that

1. $\langle \alpha \rangle_e = \left\langle \overline{\langle \lambda \rangle}_e \right\rangle_e = \langle \lambda \rangle_e$, and
2. $(p^e - 1) \cdot \alpha = p^e \langle \lambda \rangle_e \in \mathbb{N}$.

By Key Lemma 5.2 and 1, the inclusion $R \cdot f^{\langle \alpha \rangle_e} = R \cdot f^{\langle \lambda \rangle_e} \subseteq R^{1/p^e}$ splits as a map of R-modules. Then, Corollary 5.5 and 2 imply that $\lambda \geqslant \alpha$. $\square$

Corollary 5.11. For any $e \geqslant 1$ and $\lambda \in \mathbf{FPT}_p$, the following hold:

1. $\lfloor (p^e - 1)\lambda \rfloor = p^e \langle \lambda \rangle_e$.
2. $\lambda \leqslant p^e \llbracket \lambda \rrbracket_e$.
3. $\lambda \geqslant \overline{\langle \lambda \rangle}_e$.

Proof. The three assertions are equivalent by Lemma 4.10, and the third point follows from Proposition 5.10. $\square$

Corollary 5.11 places severe restrictions on the set $\mathbf{FPT}_p$.

Proposition 5.12. Fix $e \geqslant 1$. For any $\beta \in [0,1] \cap \frac{1}{p^e} \cdot \mathbb{N}$ we have that

$$\mathbf{FPT}_p \cap \left(\beta, \frac{p^e}{p^e - 1} \cdot \beta \right) = \varnothing.$$

Proof. Let $\lambda \in \mathbf{FPT}_p$. If $\lambda > \beta$, then $\langle\lambda\rangle_e \geqslant \beta$ by Lemma 4.6. Combining this with Corollary 5.11 and Definition 4.8 shows that $\lambda \geqslant \overline{\langle\lambda\rangle}_e = \frac{p^e}{p^e-1} \cdot \langle\lambda\rangle_e \geqslant \frac{p^e}{p^e-1} \cdot \beta$. □

Remark 5.13. Proposition 5.12 is a generalization of [BMS09, Proposition 4.3], in which it is assumed that R is an F-finite regular ring.

Example 5.14. Proposition 5.12 states that for every $e \geqslant 1$, there exist $p^e - 1$ disjoint open subintervals of $[0, 1]$ that do not intersect $\mathbf{FPT}_p$. Figure 5.14.1 shows the intervals corresponding to $e = 1, 2$ and 3 that cannot intersect $\mathbf{FPT}_2$.

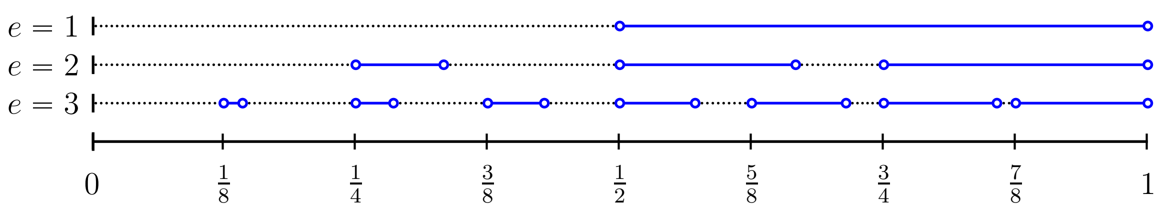


Figure 5.14.1: Three intervals that do not intersect $\mathbf{FPT}_2$.

Remark 5.15. As illustrated by Example 5.14, many of the intervals from Proposition 5.12 overlap. However, as $\sum_{\beta \in [0,1] \cap \frac{1}{p^e} \cdot \mathbb{N}} \text{length}\left(\beta, \frac{p^e}{p^e-1} \cdot \beta\right) = \frac{1}{2}$, Proposition 5.14 shows that for every $e \geqslant 1$, there is a set of Lebesgue measure $\frac{1}{2}$ that does not intersect $\mathbf{FPT}_p$. This was first observed in [BMS09].

Remark 5.16. Proposition 5.12 may be used to show that $\mathbf{FPT}_p$ is a set of Lebesgue measure zero. This is not surprising, as very often $\mathbf{FPT}_p$ is contained in $[0, 1] \cap \mathbb{Q}$ [BMS08, BMS09, KLZ09, BSTZ09]. We stress, however, that in the generality dealt with in this chapter, the issue of whether $\mathbf{FPT}_p \subseteq \mathbb{Q}$ is open.

3 Purity at the F-pure threshold

In this section, we prove Theorem 5.19. We now recall the various singularity types and invariants associated to pairs $(R, \lambda \bullet f)$ via the Frobenius morphism.

Definition 5.17. The pair $(R, \lambda \bullet f)$ is said to be

1. *F-pure* if $R \cdot f^{\lfloor (p^e-1)\lambda \rfloor / p^e} \subseteq R^{1/p^e}$ splits over R for all $e \geqslant 1$,

2. *strongly F-pure* if $R \cdot f^{\lceil p^e \lambda \rceil / p^e} \subseteq R^{1/p^e}$ splits over R for some $e \geqslant 1$,

3. *sharply F-pure* if $R \cdot f^{\lceil (p^e-1)\lambda \rceil / p^e} \subseteq R^{1/p^e}$ splits over R for some $e \geqslant 1$.

The F-pure threshold f is defined as $\mathbf{fpt}(f) = \sup_\lambda \{\, (R, \lambda \bullet f) \text{ is } F\text{-pure}\,\}$, and we may define the (strongly/sharply) F-pure thresholds similarly. By Proposition 2.22, all of these thresholds agree, and consequently, F-purity, strong F-purity, and sharp F-purity are equivalent conditions on pairs $(R, \lambda \bullet f)$ with $0 \leqslant \lambda < \mathbf{fpt}(f)$. The following example shows that these conditions need not be equivalent at the parameter $\mathbf{fpt}(f)$.

Example 5.18. Consider the pair $\left(\mathbb{F}_p[[x]], \frac{1}{p} \bullet x^p\right)$, and let $\mathfrak{m} = (x) \subseteq \mathbb{F}_p[[x]]$. It is easy to see that $\mathbf{fpt}(x^p) = \frac{1}{p}$. Note that $\left\lfloor (p^e-1)\frac{1}{p} \right\rfloor = \left\lfloor p^{e-1} - \frac{1}{p} \right\rfloor = p^{e-1} - 1$, and so $(x^p)^{\left\lfloor (p^e-1)\frac{1}{p} \right\rfloor} = x^{p^e - p} \notin \mathfrak{m}^{[p^e]}$. Similarly, $\left\lceil (p^e-1)\frac{1}{p} \right\rceil = p^{e-1}$, and consequently $(x^p)^{\left\lceil (p^e-1)\cdot\frac{1}{p} \right\rceil} = x^{p^e} \in \mathfrak{m}^{[p^e]}$ for every $e \geqslant 1$. Corollary 2.32 allows us to conclude that $\left(\mathbb{F}_p[[x]], \frac{1}{p} \bullet x^p\right)$ is F-pure, but *not* sharply F-pure.

In the main result of this chapter, we see that the structure of the base p expansion of $\mathbf{fpt}(f)$ completely determines whether these are equivalent conditions in general.

Theorem 5.19. The pair $(R, \mathbf{fpt}(f) \bullet f)$ is F-pure, not strongly F-pure, and sharply F-pure $\iff (p^e - 1) \cdot \mathbf{fpt}(f) \in \mathbb{N}$ for some $e \geqslant 1$.

Remark 5.20. The first assertion above generalizes [Har06, Proposition 2.6], in which it is assumed that R is a complete, F-finite regular local ring. The third is a generalization of [Sch08, Corollary 5.4 and Remark 5.5], in which it is assumed that R is an F-finite regular local ring.

Remark 5.21. The condition that $(p^e - 1) \cdot \mathbf{fpt}(f) \in \mathbb{N}$ for some $e \geqslant 1$ is equivalent to the condition that $\mathbf{fpt}(f) \in \mathbb{Q}$, and that the denominator of $\mathbf{fpt}(f)$ is not divisible by p. We will notice in Chapters 8 and 9 that the denominator of $\mathbf{fpt}(f)$ is often a *power* of p, and more often is divisible by p. Thus, there are many instances in which F-purity is not equal to sharp F-purity.

Proof of Theorem 5.19. By Corollary 5.11, $\lfloor (p^e - 1)\mathbf{fpt}(f) \rfloor = p^e \langle \mathbf{fpt}(f) \rangle_e$, and so Key Lemma 5.2 implies that the inclusion $R \cdot f^{\lfloor (p^e-1)\mathbf{fpt}(f) \rfloor / p^e} \subseteq R^{1/p^e}$ splits as a map of R-modules. We see that $(R, \mathbf{fpt}(f) \bullet f)$ is F-pure.

By Lemma 4.6, $\lceil p^e \mathbf{fpt}(f) \rceil = p^e \langle \mathbf{fpt}(f) \rangle_e + 1$, and so Key Lemma 5.2 shows that the inclusion $R \cdot f^{\lceil p^e \mathbf{fpt}(f) \rceil / p^e} \subseteq R^{1/p^e}$ *never* splits over R. We see that $(R, \mathbf{fpt}(f) \bullet f)$ is not strongly F-pure.

By Definition 5.17, $(R, \mathbf{fpt}(f) \bullet f)$ is sharply F-pure if and only if $R \cdot f^{\lceil (p^e-1)\mathbf{fpt}(f) \rceil / p^e}$ splits off from R^{1/p^e} over R for some $e \geqslant 1$, and Remark 5.3 show that this inclusion splits if and only if

$$\lceil (p^e - 1)\mathbf{fpt}(f) \rceil \leqslant p^e \langle \mathbf{fpt}(f) \rangle_e . \tag{5.21.1}$$

However, Lemma 4.6 shows that (5.21.1) holds if and only if

$$\lceil (p^e - 1)\mathbf{fpt}(f) \rceil = p^e \langle \mathbf{fpt}(f) \rangle_e = \lfloor (p^e - 1)\mathbf{fpt}(f) \rfloor , \tag{5.21.2}$$

where the last equality follows from Corollary 5.11. Finally, we observe that (5.21.2) holds if and only if $(p^e - 1) \cdot \mathbf{fpt}(f) \in \mathbb{N}$. $\square$

CHAPTER 6

Splitting polytopes

In this chapter, we associate to any collection of n distinct monomials a rational polytope $\boldsymbol{P} \subseteq [0,1]^n$, and derive some basic properties of $\boldsymbol{P}$. The polytopes $\boldsymbol{P}$ have appeared previously in [MTW05, ST09, LM10, BMS06], and the geometry of $\boldsymbol{P}$ will be used crucially in Chapter 7, 8, and 9 in giving bounds for and explicitly computing F-pure threshold of certain special classes of polynomials.

Throughout this chapter, let $R := K[x_1, \cdots, x_m]$ denote a polynomial ring over a field K of arbitrary characteristic, and $\boldsymbol{C} := \{ \boldsymbol{x}^{\boldsymbol{a}_1}, \cdots, \boldsymbol{x}^{\boldsymbol{a}_n} \}$ will denote a collection of distinct monomials in R. Furthermore, if $\boldsymbol{s} = (s_1, \cdots, s_n) \in \mathbb{R}^n$, we will use $|\boldsymbol{s}|$ to denote the coordinate sum $s_1 + \cdots + s_n$. We stress that $|\cdot|$ is *not* the usual Euclidean norm on $\mathbb{R}^n$.

Definition 6.1. We call the matrix $\boldsymbol{M} = \boldsymbol{M}_{\boldsymbol{C}} = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_n) \in M_{m \times n}(\mathbb{N})$ the *splitting matrix associated to* $\boldsymbol{C}$.

Remark 6.2. Definition 6.1 is motivated by the following fact: For every n-tuple of integers $\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$, we have that $(\boldsymbol{x}^{\boldsymbol{a}_1})^{k_1} \cdots (\boldsymbol{x}^{\boldsymbol{a}_n})^{k_n} = x_1^{[\boldsymbol{M}\boldsymbol{k}]_1} \cdots x_m^{[\boldsymbol{M}\boldsymbol{k}]_m} = \boldsymbol{x}^{\boldsymbol{M}\boldsymbol{k}}$, where $[\boldsymbol{M}\boldsymbol{k}]_i$ denotes the i^{th} entry of $\boldsymbol{M}\boldsymbol{k} \in \mathbb{N}^m$. It follows from this that $(u_1\boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n\boldsymbol{x}^{\boldsymbol{a}_n})^N = \sum_{|\boldsymbol{k}|=N} \binom{N}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} \cdot \boldsymbol{x}^{\boldsymbol{M}\boldsymbol{k}}$, where the u_i are elements of K, and $\binom{N}{\boldsymbol{k}} = N!/(k_1! \cdots k_n!)$.

Definition 6.3. We define the *splitting polytope associated to $\boldsymbol{C}$* as follows:

$$\boldsymbol{P} = \boldsymbol{P_C} = \left\{ \boldsymbol{s} \in \mathbb{R}^n_{\geqslant 0} : [\boldsymbol{Ms}]_i \leqslant 1 \ \forall \ i = 1, \cdots, m \right\},$$

where $[\boldsymbol{Ms}]_i$ denotes the i^{th} entry of the element $\boldsymbol{Ms} \in \mathbb{R}^m$. By Definition 6.3, we have that $\boldsymbol{P} \subseteq [0,1]^n$.

Definition 6.4. A point $\boldsymbol{\eta} \in \boldsymbol{P}$ is called *maximal* if $|\boldsymbol{\eta}| = \max\{ |\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P} \}$, and $\boldsymbol{P}_{\max}$ will denote the face consisting of all maximal points of $\boldsymbol{P}$.

Definition 6.5. We say that $\boldsymbol{P}$ *contains a unique maximal point* if ${}^{\#}\boldsymbol{P}_{\max} = 1$.

Lemma 6.6. If $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$, then $\boldsymbol{\eta}$ must be a vertex of $\boldsymbol{P}$. Furthermore, if the set of vertices of $\boldsymbol{P}$ contains a unique element $\boldsymbol{\eta}$ with maximal coordinate sum, then $\boldsymbol{\eta}$ must also be the unique maximal point of $\boldsymbol{P}$.

Proof. If $\boldsymbol{P}_{\max} = \{ |\boldsymbol{\eta}| \}$, it follows from the fact that $\boldsymbol{P}_{\max}$ is a face of $\boldsymbol{P}$ that $\boldsymbol{\eta}$ must be a vertex of $\boldsymbol{P}$ [Bar02]. On the other hand, suppose that $\boldsymbol{\eta}$ is the unique vertex of $\boldsymbol{P}$ with maximal coordinate sum, so that $\boldsymbol{\eta} \in \boldsymbol{P}_{\max}$. If $\boldsymbol{\eta}$ is not the unique maximal point of $\boldsymbol{P}$, then the convex polytope $\boldsymbol{P}_{\max}$ contains infinitely many points, and so must contain a vertex $\boldsymbol{\nu} \neq \boldsymbol{\eta}$. As vertices of $\boldsymbol{P}_{\max}$ are vertices of $\boldsymbol{P}$, this contradicts the uniqueness of $\boldsymbol{\eta}$. $\square$

Example 6.7. Let $\boldsymbol{C} = \{x^a, y^b, x^c y^c\}$. Then, $\boldsymbol{M} = \begin{pmatrix} a & 0 & c \\ 0 & b & c \end{pmatrix}$, and consequently $\boldsymbol{P} = \left\{ \boldsymbol{s} \in \mathbb{R}^3_{\geqslant 0} : \begin{array}{c} as_1 + cs_3 \leqslant 1 \\ bs_2 + cs_3 \leqslant 1 \end{array} \right\}$. Note that $\boldsymbol{P}$ is the convex hull of the points $\boldsymbol{0}, v_1 = \left(\frac{1}{a}, 0, 0\right), v_2 = \left(0, \frac{1}{b}, 0\right), v_3 = \left(\frac{1}{a}, \frac{1}{b}, 0\right)$, and $v_4 = \left(0, 0, \frac{1}{c}\right)$. See Figure 6.7 for a picture of $\boldsymbol{P}$. By Lemma 6.6, ${}^{\#}\boldsymbol{P} = 1$ if and only if $|v_3| = \frac{1}{a} + \frac{1}{b} \neq \frac{1}{c} = |v_4|$, in which case $\boldsymbol{P}_{\max} = \{ v_3 \}$ or $\{ v_4 \}$. In the case that $\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$, $\boldsymbol{P}_{\max}$ is equal to the edge determined by the vertices v_3 and v_4.

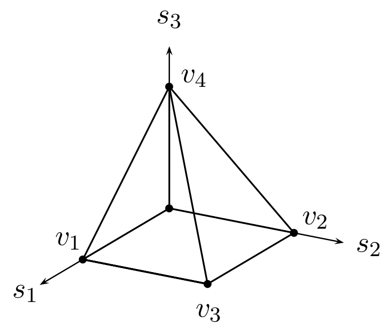


Figure 6.7.1: The rational polytope $\boldsymbol{P} \subseteq [0,1]^3$ associated to $\{x^a, y^b, x^c y^c\}$.

Example 6.8. If $\boldsymbol{C} = \left\{ x_1^{a_1} \cdots x_m^{a_m}, x_1^{b_1} \cdots x_m^{b_m} \right\}$, $\boldsymbol{P} \subseteq [0,1]^2$ contains a unique maximal point if $a_i \neq b_i$ for all $1 \leqslant i \leqslant m$. (see Corollary 9.5.)

Example 6.9. If $\boldsymbol{C} = \left\{ x_1^{d_1}, \cdots, x_n^{d_n} \right\}$, then $\boldsymbol{P} = \left\{ \boldsymbol{s} \in \mathbb{R}^n_{\geqslant 0} : s_i \leqslant \frac{1}{d_i} \text{ for } 1 \leqslant i \leqslant n \right\}$, and $\boldsymbol{P}_{\max} = \left\{ \left(\frac{1}{d_1}, \cdots, \frac{1}{d_n} \right) \right\}$.

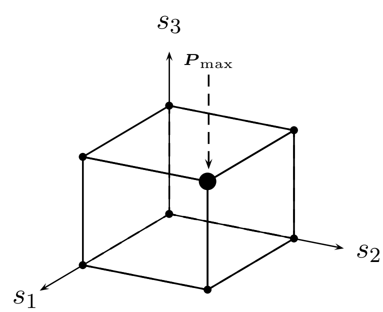


Figure 6.9.1: The rational polytope $\boldsymbol{P} \subseteq [0,1]^3$ associated to $\left\{ x_1^{d_1}, x_2^{d_2}, x_3^{d_3} \right\}$.

Remark 6.10. It follows from the proof of Lemma 6.6 that if $\boldsymbol{P}$ contains a unique maximal point $\boldsymbol{\eta}$, then $\boldsymbol{\eta}$ must be a vertex of $\boldsymbol{P}$. As $\boldsymbol{P}$ is defined by hyperplanes with coefficients in $\mathbb{N}$, it follows that $\boldsymbol{\eta}$ must have rational coordinates. In fact, if $H = \{ \boldsymbol{s} \in \mathbb{R}^n : L(\boldsymbol{s}) \leqslant 1 \}$ is a halfspace defined by a linear equation L with rational coefficients, then every vertex of the polytope $\boldsymbol{P} \cap H$ also has rational coordinates.

Lemma 6.11. Fix $e \geqslant 1$. If $\boldsymbol{s}$ is in the boundary of $\boldsymbol{P}$, then $\langle \boldsymbol{s} \rangle_e = (\langle s_1 \rangle_e, \cdots, \langle s_n \rangle_e)$ is in the interior of $\boldsymbol{P}$.

Proof. As the defining inequalities of $\boldsymbol{P}$ have coefficients in $\mathbb{N}$, the assertion follows from the fact that $\langle s_i \rangle_e < s_i$. $\square$

Recall from Remark 6.2 that $(u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n})^N = \sum_{|\boldsymbol{k}|=N} \binom{N}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} \boldsymbol{x}^{\boldsymbol{M}\boldsymbol{k}}$. We will soon be interested in knowing, after gathering of terms, what the coefficient of a given monomial in this expression is. Specifically, we are interested in knowing when there exist indices $\boldsymbol{k} \neq \boldsymbol{k}'$ in $\mathbb{N}^n$ such that $|\boldsymbol{k}| = |\boldsymbol{k}'| = N$ and $\boldsymbol{M}\boldsymbol{k} = \boldsymbol{M}\boldsymbol{k}'$. The following lemma allows us to address this issue whenever the geometry of $\boldsymbol{P}$ is nice.

Lemma 6.12. Suppose that $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$.

1. If $|\boldsymbol{s}| = |\langle \boldsymbol{\eta} \rangle_e|$ and $\boldsymbol{M}\boldsymbol{s} = \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e$ for some $\boldsymbol{s} \in \frac{1}{p^e} \cdot \mathbb{N}^n$, then $\boldsymbol{s} = \langle \boldsymbol{\eta} \rangle_e$.

2. If $|\boldsymbol{s}| = |\boldsymbol{\nu}|$, $\boldsymbol{M}\boldsymbol{s} = \boldsymbol{M}\boldsymbol{\nu}$, and $\langle \boldsymbol{\eta} \rangle_e - \boldsymbol{\nu} \in \frac{1}{p^e} \cdot \mathbb{N}^n$ for some $\boldsymbol{\nu}, \boldsymbol{s} \in \frac{1}{p^e} \cdot \mathbb{N}^n$, then $\boldsymbol{s} = \boldsymbol{\nu}$.

Proof. For the first statement, let $\boldsymbol{\eta}' := \boldsymbol{s} + \boldsymbol{\eta} - \langle \boldsymbol{\eta} \rangle_e$. By definition, $\boldsymbol{\eta}' \in \mathbb{R}^n_{\geqslant 0}$. By assumption, we also have $\boldsymbol{M}\boldsymbol{\eta}' = \boldsymbol{M}\boldsymbol{s} + \boldsymbol{M}\boldsymbol{\eta} - \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e = \boldsymbol{M}\boldsymbol{\eta}$, and $|\boldsymbol{\eta}'| = |\boldsymbol{s}| + |\boldsymbol{\eta}| - |\langle \boldsymbol{\eta} \rangle_e| = |\boldsymbol{\eta}|$, which shows that $\boldsymbol{\eta}'$ is a maximal point of $\boldsymbol{P}$. Thus $\boldsymbol{\eta}' = \boldsymbol{\eta}$, and $\boldsymbol{s} = \langle \boldsymbol{\eta} \rangle_e$. For the second statement, let $\boldsymbol{s}' := \boldsymbol{s} + \langle \boldsymbol{\eta} \rangle_e - \boldsymbol{\nu} \in \frac{1}{p^e} \cdot \mathbb{N}^n$. By hypothesis, $\boldsymbol{s}' \in \frac{1}{p^e} \cdot \mathbb{N}^n$, $|\boldsymbol{s}'| = |\langle \boldsymbol{\eta} \rangle_e|$, and $\boldsymbol{M}\boldsymbol{s}' = \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e$. The first statement, applied to $\boldsymbol{s}'$, shows that $\boldsymbol{s}' = \langle \boldsymbol{\eta} \rangle_e$, and thus $\boldsymbol{s} = \boldsymbol{\nu}$. $\square$

Corollary 6.13. Suppose that $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$.

1. The coefficient of $\boldsymbol{x}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e}$ in $(u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n})^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$ is $\binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \boldsymbol{u}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e}$.

2. Let $\boldsymbol{\nu} \in \frac{1}{p^e} \cdot \mathbb{N}^n$ be an index such that $\langle \boldsymbol{\eta} \rangle_e - \boldsymbol{\nu} \in \frac{1}{p^e} \cdot \mathbb{N}^n$. Then the coefficient of $\boldsymbol{x}^{p^e \boldsymbol{M} \boldsymbol{\nu}}$ in $(u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n})^{p^e |\boldsymbol{\nu}|}$ is $\binom{p^e |\boldsymbol{\nu}|}{p^e \boldsymbol{\nu}} \boldsymbol{u}^{p^e \boldsymbol{M} \boldsymbol{\nu}}$.

CHAPTER 7

F-purity and log canonical singularities

The main result of this chapter, Theorem 7.6, gives bounds for, and in an important special case, allows us to explicitly compute the F-pure threshold of a polynomial f whenever the polytope $\boldsymbol{P}$ associated to the supporting monomials of f contains a unique maximal point. These bounds will be crucial when computing the F-pure threshold of diagonal and binomial hypersurfaces in Chapters 8 and 9.

Let f denote a polynomial over the complex numbers. In Theorem 7.17, we invoke Theorem 7.6 to show that Conjecture 3.15, which states that log canonical singularities is equivalent to dense F-pure type, holds for f whenever the associated polytope $\boldsymbol{P}$ has a unique maximal point. Finally, in Theorem 7.18, we apply these methods to show that Conjecture 3.15 holds for *very general* complex polynomials.

1 Some background on F-purity of monomial ideals

Let $R = K[x_1, \cdots, x_m]$ denote a polynomial ring over a field of characteristic $p > 0$ with $[K : K^p] < \infty$ K. By Example 2.14, R is an F-finite regular ring. We will let $\mathfrak{m} \subseteq R$ denote the ideal generated by the variables, and f will denote a non-zero polynomial in $\mathfrak{m}$.

In Chapter 2, we introduced the notion of F-purity for pairs of the form $(R, \lambda \bullet f)$, and to this point we have dealt exclusively with pairs of this form. However, the notion of (strong/sharp) F-purity can be extended to pairs of the form $(R, \mathfrak{a} \bullet f)$, where $\mathfrak{a}$ is any ideal of R [Tak04]. For example, we say that the pair $(R, \lambda \bullet f)$ is *F-pure at $\mathfrak{m}$* if $\mathfrak{a}^{\lfloor (p^e-1)\lambda \rfloor} \nsubseteq \mathfrak{m}^{[p^e]}$ for all $e \gg 0$, and is *strongly F-pure at $\mathfrak{m}$* if $\mathfrak{a}^{\lceil p^e \lambda \rceil} \nsubseteq \mathfrak{m}^{[p^e]}$ for some $e \geqslant 1$ (c.f, Corollary 2.32). We also define the *F-pure threshold of $\mathfrak{a}$ at $\mathfrak{m}$* as in the principal case: $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) := \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (R, \lambda \bullet \mathfrak{a}) \text{ is } F\text{-pure at } \mathfrak{m} \}$. As in Proposition 2.22, we have that $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (R, \lambda \bullet \mathfrak{a}) \text{ is } F\text{-strongly pure at } \mathfrak{m} \}$. Below, we gather these facts to give our working definition for $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a})$ for this thesis.

Definition 7.1. Let $\mathfrak{a} \subseteq \mathfrak{m}$ be an ideal of R. Then

1. $\mathbf{fpt}_{\mathfrak{m}}(f) = \sup \left\{ \lambda \in \mathbb{R} : f^{\lceil p^e \lambda \rceil} \notin \mathfrak{m}^{[p^e]} \text{ for some } e \geqslant 1 \right\}$, and
2. $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \sup \left\{ \lambda \in \mathbb{R} : \mathfrak{a}^{\lceil p^e \lambda \rceil} \nsubseteq \mathfrak{m}^{[p^e]} \text{ for some } e \geqslant 1 \right\}$.

The first point in Definition 7.1 above is a restatement of Corollary 2.33. Our goal in this Chapter is to compare the value of $\mathbf{fpt}_{\mathfrak{m}}(f)$ with that of $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a})$, where here $\mathfrak{a}$ is the monomial ideal generated by the *supporting monomials* of f.

Definition 7.2. If $f = \sum_{\boldsymbol{a}} \boldsymbol{u_a} \boldsymbol{x^a} \in R$, $\operatorname{Support}(f) := \{ \boldsymbol{x^a} : \boldsymbol{u_a} \neq 0 \}$. We call $\operatorname{Support}(f)$ the set of *supporting monomials* of f.

Notation 7.3. For the rest of this section, we will assume that $\operatorname{Support}(f) = \{ \boldsymbol{x}^{\boldsymbol{a}_1}, \cdots, \boldsymbol{x}^{\boldsymbol{a}_n} \}$. We will let $\mathfrak{a} = (\operatorname{Support}(f)) \subseteq R$ denote the monomial ideal generated by $\operatorname{Support}(f)$. Following the notation of Chapter 6, $\boldsymbol{M} = (\boldsymbol{a}_1 \cdots \boldsymbol{a}_n)$ will denote the splitting matrix associated to $\operatorname{Support}(f)$, and $\boldsymbol{P}$ will denote the polytope associated to $\operatorname{Support}(f)$. Recall that $P \subseteq [0,1]^n$.

We now gather some facts regarding F-pure thresholds, and relate $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a})$ to the geometry of $\boldsymbol{P}$.

Proposition 7.4.

1. $\mathbf{fpt}_{\mathfrak{m}}(f) \in \mathbb{Q} \cap (0,1]$.

2. $(R, \lambda \bullet f)$ is F-pure at $\mathbf{0}$ if and only if $0 \leqslant \lambda \leqslant \mathbf{fpt}_{\mathfrak{m}}(f)$.

3. $\mathbf{fpt}_{\mathfrak{m}}(f) \leqslant \min\{\,1, \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a})\,\}$.

4. $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \max\{\,|\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P}\,\}$.

Proof. The first point follows from Remarks 2.20 and 2.21, while the second is a restatement of Theorem 5.19. The third point follows from the fact that $f \in \mathfrak{a}$.

It remains to prove the last point. If $\mathfrak{a}^{\lceil cp^e \rceil} \not\subseteq \mathfrak{m}^{[p^e]}$, then at least one of the generators of $\mathfrak{a}^{\lceil cp^e \rceil}$ is not in $\mathfrak{m}^{[p^e]}$, and so there exists $\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$ such that $k_1 + \cdots + k_n = \lceil cp^e \rceil$ and $(\boldsymbol{x}^{\boldsymbol{a}_1})^{k_1} \cdots (\boldsymbol{x}^{\boldsymbol{a}_n})^{k_n} = \boldsymbol{x}^{\boldsymbol{M}\boldsymbol{k}} \notin \mathfrak{m}^{[p^e]}$, i.e., every entry of $\boldsymbol{M}\boldsymbol{k}$ is less than p^e. Thus, $\frac{1}{p^e} \cdot k \in \boldsymbol{P}$, and consequently,

$$c \leqslant \frac{\lceil cp^e \rceil}{p^e} = \frac{|\boldsymbol{k}|}{p^e} = \left| \frac{1}{p^e} \cdot \boldsymbol{k} \right| \leqslant \max\{\,|\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P}\,\}\,. \tag{7.4.1}$$

It follows from Definition 7.1 and (7.4.1) that $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) \leqslant \max\{\,|\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P}\,\}$.

Next, choose $\boldsymbol{\eta} \in \boldsymbol{P}_{\max}$, and fix $e \geqslant 1$. By Lemma 6.11, every entry of $\boldsymbol{M}\,\langle \boldsymbol{\eta} \rangle_e$ is less than 1. Thus, $(\boldsymbol{x}^{\boldsymbol{a}_1})^{p^e \langle \eta_1 \rangle_e} \cdots (\boldsymbol{x}^{\boldsymbol{a}_n})^{p^e \langle \eta_n \rangle_e} = \boldsymbol{x}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e}$ is contained $\mathfrak{a}^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$, but not in $\mathfrak{m}^{[p^e]}$. By Definition 7.1, we have that $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) \geqslant |\,\langle \boldsymbol{\eta} \rangle_e\,|$. As $\lim_{e \to \infty} \langle \boldsymbol{\eta} \rangle_e = \boldsymbol{\eta}$, this shows that $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) \geqslant |\boldsymbol{\eta}| = \max\{|\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P}\}$. $\square$

Remark 7.5. The inequality in the third point of Proposition 7.4 above may be strict. Indeed, it is easily verified from Definition 7.1 that $\mathbf{fpt}_{\mathfrak{m}}(x^p + y^p) = \frac{1}{p}$, while $\mathbf{fpt}_{\mathfrak{m}}(x^p, y^p) = \frac{2}{p}$.

2 F-pure thresholds for polynomials with good support

Recall that if $\eta \in [0,1]$, then $c_e(\eta)$ denotes the e^{th} digit in the non-terminating base p expansion of η. (See Chapter 4.)

Theorem 7.6. Suppose that $\boldsymbol{P} \subseteq [0,1]^n$ contains a unique maximal point $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_n)$, and let $L = \sup \{ N : c_e(\eta_1) + \cdots + c_e(\eta_n) \leqslant p-1 \text{ for all } 0 \leqslant e \leqslant N \}$.

1. If $L = \infty$, then $\mathbf{fpt}_{\mathfrak{m}}(f) = \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}) = \eta_1 + \cdots + \eta_n$.

2. If $L < \infty$, then $\langle \eta_1 \rangle_L + \cdots + \langle \eta_n \rangle_L + \frac{1}{p^L} \leqslant \mathbf{fpt}_{\mathfrak{m}}(f)$.

The following lemmas will be used often in the proof of Theorem 7.6.

Lemma 7.7. For every $e \geqslant 1$, we have that $p^e \langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e = \max \left\{ a \in \mathbb{N} : f^a \notin \mathfrak{m}^{[p^e]} \right\}$.

Proof. This is a restatement of [MTW05, Proposition 1.9], and a special case of Lemma 5.2 □

Lemma 7.8. Let $\alpha \in [0,1]$ be a rational number such that $(p^e - 1) \cdot \alpha \in \mathbb{N}$ for some $e \geqslant 1$. If $f^{(p^e-1)\alpha} \notin \mathfrak{m}^{[p^e]}$, then $\alpha \leqslant \mathbf{fpt}_{\mathfrak{m}}(f)$.

Proof. This is an immediate consequence of Corollary 5.5 and Corollary 2.31. □

Proof of Theorem 7.6. We may write $f = u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n}$, where each u_i is non-zero. We first assume that the non-terminating base p expansions of $\eta_1, \cdots, \eta_n$ add without carrying. By Proposition 7.4, it suffices to show that $\mathbf{fpt}_{\mathfrak{m}}(f) \geqslant |\boldsymbol{\eta}|$. If $\langle \boldsymbol{\eta} \rangle_e = (\langle \eta_1 \rangle_e, \cdots, \langle \eta_n \rangle_e)$, then Corollary 6.13 shows that

$$\binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \boldsymbol{u}^{p^e \langle \boldsymbol{\eta} \rangle_e} \boldsymbol{x}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e}$$

appears as a summand of $f^{p^e |\langle \boldsymbol{\eta} \rangle_e|}$.

1. By definition, each $u_i \in K^*$, so $\boldsymbol{u}^{p^e \langle \boldsymbol{\eta} \rangle_e} \neq 0$.

2. By Remark 4.12, the integers $p^e \langle \eta_1 \rangle_e, \cdots, p^e \langle \eta_n \rangle_e$ add without carrying for all $e \geqslant 1$. This, combined with Lemma 4.14, shows that

$$\binom{p^e |\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \not\equiv 0 \bmod p.$$

3. By Lemma 6.11, every entry of $\boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e$ is less than 1, and consequentially every entry of $p^e \cdot \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e$ is less than p^e.

The first two points above show that $\boldsymbol{x}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e} \in \operatorname{Support}(f^{p^e |\langle \boldsymbol{\eta} \rangle_e|})$, while the third point shows that $\boldsymbol{x}^{p^e \boldsymbol{M} \langle \boldsymbol{\eta} \rangle_e} \notin \mathfrak{m}^{[p^e]}$. Thus, $f^{p^e |\langle \boldsymbol{\eta} \rangle_e|} \notin \mathfrak{m}^{[p^e]}$, and Lemma 7.7 allows us to conclude that $\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e \geqslant |\langle \boldsymbol{\eta} \rangle_e| = \langle \eta_1 \rangle_e + \cdots + \langle \eta_n \rangle_e$. Taking the limit as $e \to \infty$ shows that $\mathbf{fpt}_{\mathfrak{m}}(f) \geqslant \eta_1 + \cdots + \eta_n = \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a})$, while the opposite inequality holds by Proposition 7.4.

We now assume that the non-terminating base p expansions of $\eta_1, \cdots, \eta_n$ add *with* carrying. Let $c_e(t_i)$ and L be as in the statement of the Theorem. Note that

4. $0 \leqslant L < \infty$ (as $c_0(t_i) = 0$ by convention), and that

5. $c_{L+1}(\eta_1) + \cdots + c_{L+1}(\eta_n) \geqslant p$.

By replacing each $c_{L+1}(t_i)$ with an integer less than or equal to $c_{L+1}(t_i)$, we see that there exist integers $\delta_1, \cdots, \delta_n$ such that $\delta_1 + \cdots + \delta_n = p - 1$ and $0 \leqslant \delta_i \leqslant c_{L+1}(t_i)$, with the second inequality being strict for at least one index. We will assume without loss of generality that $\delta_1 < c_{L+1}(\eta_1)$. For $e \geqslant L + 2$, set

$$\boldsymbol{\nu}(e) = \langle \boldsymbol{\eta} \rangle_L + \left(\frac{\delta_1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e}, \frac{\delta_2}{p^{L+1}}, \cdots, \frac{\delta_n}{p^{L+1}} \right). \tag{7.8.1}$$

The following summarizes the important properties of $\boldsymbol{\nu}(e) = (\boldsymbol{\nu}_1(e), \cdots, \boldsymbol{\nu}_n(e))$.

6. $\boldsymbol{\nu}(e) \in \frac{1}{p^e} \cdot \mathbb{N}^n$.

7. As $\delta_i \leqslant c_{L+1}(t_i)$ for $2 \leqslant i \leqslant n$, it follows that $\boldsymbol{\nu}_i(e) \leqslant \langle t_i \rangle_{L+1}$, while the fact that $\delta_1 < c_{L+1}(\eta_1)$ shows that $\boldsymbol{\nu}_1(e) < \langle \eta_1 \rangle_{L+1}$. Thus, we have that $\mathbf{0} \neq \langle \boldsymbol{\eta} \rangle_e - \boldsymbol{\nu}(e) \in \frac{1}{p^e} \cdot \mathbb{N}^n$ for $e \geqslant L+2$.

8. As $\delta_1 + \cdots + \delta_n = p-1$, it follows from the definition of L that the base p expansions of the integers $p^e \nu_1(e), \cdots, p^e \nu_n(e)$ add without carrying.

9. Finally, we have that

$$\begin{aligned} |\boldsymbol{\nu}(e)| &= |\langle \boldsymbol{\eta} \rangle_L| + \frac{1}{p^{L+1}} \cdot \left(\sum_{i=1}^{n} \delta_i \right) + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e} \\ &= |\langle \boldsymbol{\eta} \rangle_L| + \frac{p-1}{p^{L+1}} + \frac{p-1}{p^{L+2}} + \cdots + \frac{p-1}{p^e}. \end{aligned}$$

Points 6 and 7 above, along with Remark 6.2 and Corollary 6.13, show that, after gathering terms, the monomial

$$\binom{p^e |\boldsymbol{\nu}(e)|}{p^e \boldsymbol{\nu}(e)} \boldsymbol{u}^{p^e \boldsymbol{\nu}(e)} \boldsymbol{x}^{p^e \boldsymbol{M} \boldsymbol{\nu}(e)}$$

appears as a summand of $f^{p^e |\boldsymbol{\nu}(e)|}$. Point 8 above, along with Lemma 4.14, gives that $\binom{p^e |\boldsymbol{\nu}(e)|}{p^e \boldsymbol{\nu}(e)} \neq 0 \bmod p$. We can conclude from this that $\boldsymbol{x}^{p^e \boldsymbol{M} \boldsymbol{\nu}(e)} \in \operatorname{Support}(f^{p^e |\boldsymbol{\nu}(e)|})$ and $\boldsymbol{x}^{p^e \boldsymbol{\nu}(e)} \notin \mathfrak{m}^{[p^e]}$. Thus, $f^{p^e |\boldsymbol{\nu}(e)|} \notin \mathfrak{m}^{[p^e]}$, and by Lemma 7.7, we have that

$$\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e \geqslant |\boldsymbol{\nu}(e)| = |\langle \boldsymbol{\eta} \rangle_L| + \frac{p-1}{p^{L+1}} + \cdots + \frac{p-1}{p^e}, \tag{7.8.2}$$

where the last inequality follows from point 9 above. Finally, letting $e \to \infty$ in (7.8.2) gives that $\mathbf{fpt}_{\mathfrak{m}}(f) \geqslant |\langle \boldsymbol{\eta} \rangle_L| + \frac{1}{p^L}$. $\square$

3 Log canonical singularities and dense F-pure type

In this section, we prove Conjecture 3.15, which states that log canonical singularities is equivalent to dense F-pure type, for a large class of polynomials over the complex numbers. Let S denote a polynomial ring over $\mathbb{C}$, and let $f \in S$ be a polynomial with $f(\mathbf{0}) = 0$. Recall that the pair $(S, \lambda \bullet f)$ is said to be *KLT* at $\mathbf{0}$ if the function $\frac{1}{|f|^{2\lambda}}$ is locally integrable at $\mathbf{0}$, and is said to be *log canonical* at $\mathbf{0}$ if it is KLT for every $0 \leqslant \varepsilon < \lambda$. As with singularities defined via Frobenius, these singularity types can be extended to pairs of the form $(S, \lambda \bullet \mathfrak{a})$, where now $\mathfrak{a}$ is allowed to be any ideal of S [Kol97]. For example, if $\mathfrak{a} = (f_1, \cdots, f_a)$ is a set of generators for f, we say that $(S, \lambda \bullet \mathfrak{a})$ is KLT at $\mathbf{0}$ if the function $\left(\frac{1}{\sum_{i=1}^{a} |f_i|^2}\right)^{\lambda}$ is integrable in some neighborhood of the origin. By again referring to a log resolution, this can be shown to be independent of the set of generators. We say that $(S, \lambda \bullet \mathfrak{a})$ is log canonical at $\mathbf{0}$ if it is KLT at $\mathbf{0}$ for every $0 \leqslant \varepsilon < \lambda$. We may also define the log canonical threshold of $\mathfrak{a}$ at $\mathbf{0}$, which we denote $\mathbf{lct_0}(\mathfrak{a})$, to be the common value of $\sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (S, a \bullet \lambda) \text{ is KLT at } \mathbf{0} \} = \sup \{ \lambda \in \mathbb{R}_{\geqslant 0} : (S, a \bullet \lambda) \text{ is log canonical at } \mathbf{0} \}$ [BL04, EM06, Laz09].

Below, we gather some important facts regarding log canonical thresholds.

Proposition 7.9. If $\mathfrak{a}$ is the monomial ideal of S generated by Support(f), then

1. $\mathbf{lct_0}(f) \in \mathbb{Q} \cap (0, 1]$,
2. $(S, \lambda \bullet f)$ is log canonical at $\mathbf{0}$ if and only if $0 \leqslant \lambda \leqslant \mathbf{lct_0}(f)$, and
3. $\mathbf{lct_0}(f) \leqslant \min \{ 1, \mathbf{lct_0}(\mathfrak{a}) \}$.

Proof. The first two points above are a restatement of Lemma 2.8, while the third point follows from the fact that $f \in \mathfrak{a}$. $\square$

Remark 7.10. By Proposition 7.4, each of the points in Proposition 7.9 holds whenever "log canonical at $\mathbf{0}$" is replaced by "F-pure at $\mathfrak{m}$" and "$\mathbf{lct_0}$" is replaced by "$\mathbf{fpt_m}$."

The following well-known lemma shows that the log canonical threshold and the F-pure threshold of a monomial ideal agree in all characteristics.

Lemma 7.11. Let K be a finite field of characteristic $p > 0$ with $[K : K^p] < \infty$, and let $\boldsymbol{C}$ be a collection of monomials in the variables $x_1, \cdots, x_m$. By abuse of notation, let $\mathfrak{a}$ denote the monomial ideal generated by $\boldsymbol{C}$ in both $\mathbb{C}[x_1, \cdots, x_m]$ and $K[x_1, \cdots, x_m]$. Then, $\mathbf{lct}_{\mathbf{0}}(\mathfrak{a}_{\mathbb{C}}) = \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_K)$.

Proof. Let $\boldsymbol{N} \subseteq \mathbb{R}^m_{\geqslant 0}$ denote the Newton polytope associated to $\mathfrak{a}$, and let $\boldsymbol{P}$ denote the polytope associated to the set $\boldsymbol{C}$. Then

$$\mathbf{lct}_{\mathbf{0}}(\mathfrak{a}) = \max\{\, \lambda \in \mathbb{R}_{>0} : \mathbf{1} \in \lambda \cdot \boldsymbol{N} \,\} = \max\{\, |\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P} \,\} = \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}), \tag{7.11.1}$$

where $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^m$. The first equality in (7.11.1) follows from [How01, Example 5], the second by setting $w = \mathbf{1}$ in [BMS06, Lemma 4.3], and the last by Proposition 7.4. ◻

Notation 7.12. For the rest of this chapter, f will denote a non-zero element of $S := \mathbb{C}[x_1, \cdots, x_m] = \mathbb{C}[\underline{\boldsymbol{x}}]$ with $f(\mathbf{0}) = 0$. Furthermore, we set $\operatorname{Support}(f) = \{\, \boldsymbol{x}^{\boldsymbol{a}_1}, \cdots, \boldsymbol{x}^{\boldsymbol{a}_n} \,\}$, and $\mathfrak{a} = (\operatorname{Support}(f)) \subseteq S$. If A is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, S_A will denote the subring $A[x_1, \cdots, x_m] = A[\underline{\boldsymbol{x}}] \subseteq S$. Note that $\mathbb{C} \otimes_A S_A = S$. For a maximal ideal $\mu \subseteq A$, $S_A(\mu)$ will denote the polynomial ring $S_A \otimes_A A/\mu = S_A/\mu S_A = (A/\mu)\,[\underline{\boldsymbol{x}}]$. By Corollary 3.2, $\operatorname{char} A/\mu > 0$. For $g \in S_A$, g_μ will denote the image of g in $S_A(\mu)$. Similarly, $\mathfrak{a}_\mu$ will denote the ideal in $S_A(\mu)$ generated by $\operatorname{Support}(f_\mu)$. Finally, the symbol $\mathfrak{m}$ will always be used to denote the ideal generated by the variables $x_1, \cdots, x_m$ in the polynomial rings S, S_A, and $S_A(\mu)$.

We recall the definition of *dense F-pure type*, and its relationship with log canonical singularities.

Definition 7.13. The pair $(S, \lambda \bullet f)$ is of *dense F-pure type at* $\mathfrak{m}$ if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ with $f \in S_A$, and a dense subset $W \subseteq \operatorname{Spec} A$ such that $(S_A(\mu), \lambda \bullet f_\mu)$ is F-pure at $\mathfrak{m}$ for all maximal ideals $\mu \subseteq W$.

Theorem 7.14. [HW02, Tak04] If $(S, \lambda \bullet f)$ is of dense F-pure type at $\mathfrak{m}$, then $(S, \lambda \bullet f)$ is log canonical at $\mathbf{0}$.

The converse to Theorem 7.14 is Conjecture 3.15, and verifying this conjecture remains a long-standing open problem [Fed83, Smi97b, EM06].

Definition 7.15. If Conjecture 3.15 holds, we say that *log canonical singularities at* $\mathbf{0}$ *implies dense F-pure type at* $\mathfrak{m}$ for f.

Below, we give a slightly different characterization of dense F-pure type.

Lemma 7.16. Log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for f if there exists a finitely generated $\mathbb{Z}$-algebra $A \subseteq \mathbb{C}$ with $\mathbb{Z} \subseteq A$ and $f \in S_A$, and an infinite set of primes $\mathbf{\Lambda}$ satisfying the following property: For every $p \in \mathbf{\Lambda}$, there exists a non-empty set $W_p \subseteq \pi^{-1}(p)$, dense in the fiber $\pi^{-1}(p)$, such that for every maximal ideal $\mu_p \in W_p$, $\operatorname{Support}(f_{\mu_p}) = \operatorname{Support}(f)$, and $\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) = \min\left\{1, \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_{\mu_p})\right\}$. Here, $\pi : \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ denotes the map induced $\mathbb{Z} \subseteq A$.

Proof. As W_p is dense in $\pi^{-1}(p)$, it follows that $\overline{W_p} = \pi^{-1}(p)$, and thus

$$\overline{\bigcup_{p \in \mathbf{\Lambda}} W_p} \supseteq \bigcup_{p \in \mathbf{\Lambda}} \overline{W_p} = \bigcup_{p \in \mathbf{\Lambda}} \pi^{-1}(p) = \pi^{-1}(\mathbf{\Lambda}). \tag{7.16.1}$$

By Lemma 3.4, $\pi^{-1}(\mathbf{\Lambda})$ is dense in $\operatorname{Spec} A$, and so (7.16.1) shows that $\bigcup_{p \in \mathbf{\Lambda}} W_p$ is also dense in $\operatorname{Spec} A$. For $p \in \mathbf{\Lambda}$ and $\mu_p \in \pi^{-1}(p)$, it follows from Theorem 3.14 that

$$\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) \leqslant \mathbf{lct}_{\mathbf{0}}(f) \text{ for } p \gg 0. \tag{7.16.2}$$

As ${}^{\#}\mathbf{\Lambda} = \infty$, we may replace $\mathbf{\Lambda}$ with a slightly smaller set (7.16.1) remains true after this replacement), and thus assume that (8.10.1) holds for all $p \in \mathbf{\Lambda}$ and $\mu_p \in \pi^{-1}(p)$. Note that

$$\text{(7.16.3)} \quad \mathbf{lct_0}(f) \leqslant \min\{1, \mathbf{lct_0}(\mathfrak{a})\} = \min\left\{1, \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_{\mu_p})\right\} = \mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) \leqslant \mathbf{lct_0}(f).$$

Indeed, the first inequality in (7.16.3) above holds by Proposition 7.9, the second by Lemma 7.11, the third by assumption, and the last by (8.10.1).

We know from Proposition 7.9 that $(S, \lambda \bullet f)$ is log canonical at $\mathbf{0}$ if and only if $0 \leqslant \lambda \leqslant \mathbf{lct_0}(f)$. If $\mu \subseteq A$ is a maximal ideal, we know that the pair $(S_A(\mu), \lambda \bullet f_\mu)$ is F-pure at $\mathfrak{m}$ if and only if $0 \leqslant \lambda \leqslant \mathbf{fpt}_{\mathfrak{m}}(f_\mu)$ by Proposition 7.4. Examining Definitions 7.15 and 7.13, we see that to demonstrate that log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for f, it suffices to show that $\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) = \mathbf{lct_0}(f)$ for every $p \in \mathbf{\Lambda}$ and maximal ideal $\mu_p \in W_p$. However, this is precisely the content of (7.16.3). ☐

Theorem 7.17. *If $\boldsymbol{P}$ contains a unique maximal point, then log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for f.*

Proof. Let A be a finitely-generated $\mathbb{Z}$-algebra satisfying the conclusion of Corollary 3.5, and let $\pi : \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ denote the map induced by the inclusion $\mathbb{Z} \subseteq A$. Let μ_p denote an arbitrary element of $\pi^{-1}(p)$. As $\operatorname{Support}(f) = \operatorname{Support}(f_{\mu_p})$ for $p \gg 0$, $\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p})$ can be computed via $\boldsymbol{P}$, as in Theorem 7.6. Let $\boldsymbol{\eta}$ be the unique maximal point of $\boldsymbol{P}$.

If $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_{\mu_p}) = |\boldsymbol{\eta}| \leqslant 1$, let $\mathbf{\Lambda}$ consist of all primes $p \gg 0$ such that the non-terminating base p expansions of $\eta_1, \cdots, \eta_n$ add without carrying. By Lemma 4.16, ${}^{\#}\mathbf{\Lambda} = \infty$, and it follows from Theorem 7.6 that $\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) = \mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_{\mu_p})$ for all $p \in \mathbf{\Lambda}$.

If $\mathbf{fpt}_{\mathfrak{m}}(\mathfrak{a}_{\mu_p}) = |\boldsymbol{\eta}| > 1$, instead let $\boldsymbol{\Lambda}$ denote the set of all primes $p \gg 0$ such that

$$c_1(\eta_1) + \cdots + c_1(\eta_n) \geqslant p. \tag{7.17.1}$$

By Lemma 4.16, ${}^{\#}\boldsymbol{\Lambda} = \infty$. Theorem 7.6 and (7.17.1) above show that $\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) \geqslant 1$, and Proposition 7.9 shows that equality must hold. If we set $W_p := \pi^{-1}(p)$, we see that $A, \boldsymbol{\Lambda}$, and W_p satisfy the hypotheses of Lemma 7.16, and so we are done. $\square$

Theorem 7.18. Suppose that the coefficients of the supporting monomials of f are algebraically independent over $\mathbb{Q}$. Then log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for f.

Remark 7.19. Let $\boldsymbol{Z}$ denote a countable union of Zariski-closed subsets of $(\mathbb{C}^*)^n$, and let g be a polynomial over $\mathbb{C}$ such that $g(\mathbf{0}) = 0$ and ${}^{\#}\operatorname{Support}(g) = n$. If $u_1, \cdots, u_n$ denote the coefficients of the supporting monomials of g, we say that g is *very general* (with respect to $\boldsymbol{Z}$) if $(u_1, \cdots, u_n) \notin \boldsymbol{Z}$. Note that the coefficients of the supporting monomials of g are algebraically independent over $\mathbb{Q}$ if and only if g is very general with respect to $\boldsymbol{Z}_{\mathbb{Q}}$, where

$$\boldsymbol{Z}_{\mathbb{Q}} := \left[\bigcup_{h \in \mathbb{Q}[\eta_1, \cdots, \eta_n]} \mathbb{V}(h) \right] \bigcap \, (\mathbb{C}^*)^n . \tag{7.19.1}$$

In (7.19.1) the t_i denote the coordinates of $(\mathbb{C}^*)^n$. In this language, Theorem 7.18 shows that log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for a *very general complex polynomial.*

Proof of Theorem 7.18. Let $\operatorname{Support}(f) = \{ \boldsymbol{x}^{\boldsymbol{a}_1}, \cdots, \boldsymbol{x}^{\boldsymbol{a}_n} \}$, and let $\boldsymbol{P} \subseteq [0,1]^n$ denote the polytope associated to $\operatorname{Support}(f)$. Set $\alpha = \min\{ 1, \max\{ |\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P} \} \}$, and fix a point $\boldsymbol{\eta} \in \boldsymbol{P}$ with $|\boldsymbol{\eta}| = \alpha$. By Remark 6.10 , we may assume that $\boldsymbol{\eta} \in \mathbb{Q}^n$. If $\boldsymbol{\Lambda}$ denotes the set of primes p such that $(p-1) \cdot \boldsymbol{\eta} \in \mathbb{N}^n$, then ${}^{\#}\boldsymbol{\Lambda} = \infty$ by Theorem 4.15. We may write $f = u_1 \boldsymbol{x}^{\boldsymbol{a}_1} + \cdots + u_n \boldsymbol{x}^{\boldsymbol{a}_n}$, where the u_i are elements of $\mathbb{C}^*$.

If $A := \mathbb{Z}[u_1, \cdots, u_n]_{\prod u_i} \subseteq \mathbb{C}$, then $f \in S_A$, and $\operatorname{Support}(f_\mu) = \operatorname{Support}(f)$ for all maximal ideals $\mu \subseteq A$. Consequently, the F-pure threshold of the ideal generated by $\operatorname{Support}(f_\mu)$ in $S_A(\mu)$ is equal to $\max\{\, |\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P} \,\}$. Fix a prime $p \in \boldsymbol{\Lambda}$.

Remark 6.2 shows that the monomial $\boldsymbol{x}^{(p-1)\boldsymbol{M\eta}}$ appears in $f^{(p-1)\alpha}$ with coefficient

$$0 \neq \Theta_{\boldsymbol{\eta},p}(\boldsymbol{u}) = \sum_{\substack{|\boldsymbol{k}|=(p-1)\alpha \\ \boldsymbol{Mk}=(p-1)\cdot\boldsymbol{M\eta}}} \binom{(p-1)\cdot\alpha}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} \in \mathbb{Z}[u_1, \cdots, u_n] \subseteq A. \tag{7.19.2}$$

As $\alpha \leqslant 1$, we have that $\binom{(p-1)\cdot\alpha}{\boldsymbol{k}} \neq 0 \bmod p$ for each $\boldsymbol{k}$ in (8.13.3). By hypothesis, $\mathbb{Z}[\underline{\boldsymbol{u}}]$ is a polynomial ring, and it follows that $\Theta_{\boldsymbol{\eta},p}(\boldsymbol{u})$ induces a non-zero element of the polynomial ring $\mathbb{Z}/p\mathbb{Z}[\underline{\boldsymbol{u}}] \subseteq A/pA$. Let $\pi : \operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}$ be the map induced by the inclusion $\mathbb{Z} \subseteq A$. If we define $W_p := D(\Theta_{\boldsymbol{\eta},p}(\boldsymbol{u})) \cap \pi^{-1}(p)$, we have just shown that $W_p \neq \varnothing$, and so W_p is a dense (open) subset of the fiber $\pi^{-1}(p)$.

Let μ_p denote an arbitrary maximal ideal in W_p. By construction, $\Theta_{\boldsymbol{\eta},p}(\boldsymbol{u})$ has non-zero image in A/μ_p, and so (8.13.3) shows that $\boldsymbol{x}^{(p-1)\boldsymbol{M\eta}} \in \operatorname{Support}\left(\left(f_{\mu_p}\right)^{(p-1)\alpha}\right)$. As $\boldsymbol{\eta} \in \boldsymbol{P}$, every entry of $(p-1)M\boldsymbol{\eta}$ is less than p, and so $\boldsymbol{x}^{(p-1)\boldsymbol{M\eta}} \notin \mathfrak{m}^{[p]}$. Thus,

$$\left(f_{\mu_p}\right)^{(p-1)\alpha} \notin \mathfrak{m}^{[p]} \subseteq (A/\mu_p)\,[\underline{\boldsymbol{x}}] = S_A(\mu_p). \tag{7.19.3}$$

By (7.19.3), Lemma 7.8, applied to $f_{\mu_p} \in S_A(\mu_p)$, shows that

$$\mathbf{fpt}_{\mathfrak{m}}(f_{\mu_p}) \geqslant \alpha = \min\{\, 1, \max\{\, |\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P} \,\}\,\}, \tag{7.19.4}$$

and Proposition 7.4 shows that equality must hold in (7.19.4). We have just shown that $A, \boldsymbol{\Lambda}$, and $W_p \subseteq \pi^{-1}(p)$ satisfy the hypotheses of Lemma 7.16. We conclude that log canonical singularities at $\mathbf{0}$ implies dense F-pure type at $\mathfrak{m}$ for f. $\square$

Remark 7.20. An important difference between Theorem 7.17 and Theorem 7.18 is that $W_p = \pi^{-1}(p)$ in the former, while we only know that $W_p \subseteq \pi^{-1}(p)$ in the latter. It would be interesting to investigate under what conditions F-pure thresholds remain constant over the fibers of certain distinguished primes in $\operatorname{Spec} \mathbb{Z}$.

CHAPTER 8

F-singularities of diagonal hypersurfaces

In this chapter we compute various F-invariants of *diagonal hypersurfaces.* A diagonal hypersurface is a polynomial of the form $u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$, and a diagonal hypersurface $u_1 x_1^d + \cdots + u_n x_d^d$ is called a *Fermat*.

1 A detailed discussion of the main results

Throughout, $R = K[x_1, \cdots, x_n]$ will denote a polynomial ring over an F-finite field K of characteristic $p > 0$, and $\mathfrak{m} = (x_1, \cdots, x_n)$ will denote the ideal generated by the variables of R.

1 The F-pure threshold of a diagonal hypersurface

Our main result, Theorem 8.1 below, gives a formula for the F-pure threshold of any diagonal hypersurface as a function of the characteristic p. Recall that $c_e(\alpha)$ denotes the e^{th} digit in the non-terminating base p expansion of α. (See Chapter 4.)

Theorem 8.1. Let $(d_1, \cdots, d_n) \in \mathbb{N}^n$, and let $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$.
If $L := \sup \left\{ N : c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p - 1 \text{ for all } 0 \leqslant e \leqslant N \right\}$, then

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{d_1} + \cdots + \frac{1}{d_n} & \text{if } L = \infty \\ \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L} & \text{if } L < \infty \end{cases}$$

Formulas for the F-pure threshold of the hypersurfaces x^2+y^3 and x^2+y^7 are given in [MTW05, Example 4.3/4.4]. At first glance, these formulas appear to be quite different from those appearing in Theorem 8.1 above. Below, we give an example of how Theorem 8.1 may be used to recover the formulas already in the literature.

Example 8.2. We adopt decimal notation for base p expansions. For example, if a and b are integers with $0 \leqslant a, b \leqslant p-1$, then the expression $.\overline{a\ b}\,(\text{base}\,p)$ will denote the unique number α with the property that $c_e(\alpha) = a$ for e odd and $c_e(\alpha) = b$ for e even. Let $f = u_1x^2 + u_2y^3$. We now use Theorem 8.1 to compute $\mathbf{fpt}_{\mathfrak{m}}(f)$ as a function of the congruency class of p modulo 6. If $p = 3$, then

$$\frac{1}{2} = .\overline{1}\ (\text{base}\,3) \text{ and } \frac{1}{3} = .1 = .0\ \overline{2}\ (\text{base}\,3). \tag{8.2.1}$$

Carrying is required to add the expansions in (8.2.1), and the first carry occurs at the second spot. We conclude that $\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{1}{2} \right\rangle_1 + \left\langle \frac{1}{3} \right\rangle_1 + \frac{1}{3} = 0 + \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$. Similarly, one can show that $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{2}$ if $p = 2$. If $p = 6\omega+1$ for some $\omega \geqslant 1$, then

$$\frac{1}{2} = .\overline{3\omega}\ (\text{base}\,p) \text{ and } \frac{1}{3} = .\overline{2\omega}\ (\text{base}\,p). \tag{8.2.2}$$

We notice that the expansions in (8.2.2) add without carrying, and consequently $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Finally, if $p = 6\omega + 5$ for some $\omega \geqslant 0$, then

$$\frac{1}{2} = .\overline{3\omega+2}\ (\text{base}\,p) \text{ and } \frac{1}{3} = .\overline{2\omega+1 \quad 4\omega+3}\ (\text{base}\,p). \tag{8.2.3}$$

In adding the expansions in (8.2.3), the first carry occurs at the second spot, and so

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{1}{2} \right\rangle_1 + \left\langle \frac{1}{3} \right\rangle_1 + \frac{1}{p} = \frac{3\omega+2}{p} + \frac{2\omega+1}{p} + \frac{1}{p} = \frac{5\omega+4}{p}. \tag{8.2.4}$$

The reader may verify that $\frac{5\omega+4}{p} + \frac{1}{6p} = \frac{5}{6}$, and so (8.2.4) becomes $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{5}{6} - \frac{1}{6p}$.

As a special case, we recover the following formula from [MTW05, Example 4.3]:

$$\mathbf{fpt}_{\mathfrak{m}}(x^2+y^3) = \begin{cases} 1/2 & p=2 \\ 2/3 & p=3 \\ 5/6 & p\equiv 1 \bmod 6 \\ \frac{5}{6}-\frac{1}{6p} & p \equiv 5 \bmod 6 \end{cases}.$$

By using methods similar to those in the Example 8.2, we are able to give a formula for the F-pure threshold of the degree d Fermat hypersurface in terms of the lease positive residue of p modulo d.

Corollary 8.3. Let $f = u_1x_1^d + \cdots + u_dx_d^d \in K[x_1, \cdots, x_d]$. Then

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{p^\ell} & p^\ell \leqslant d < p^{\ell+1} \text{ for some } \ell \geqslant 1 \\ 1-\frac{a-1}{p} & 0 < d < p \text{ and } p \equiv a \bmod d \text{ with } 1 \leqslant a < d \end{cases}$$

2 Test ideals and higher jumping numbers

By Proposition 2.42, $\boldsymbol{\tau}(R, \lambda \bullet f) = R$ for $0 \leqslant \lambda < \mathbf{fpt}(f)$, while necessarily we have that $\boldsymbol{\tau}(R, \mathbf{fpt}(f) \bullet f) \neq R$. We now focus on understanding $\boldsymbol{\tau}(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f)$ for diagonal hypersurfaces f. By Lemma 8.10, the ideals $\boldsymbol{\tau}(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f)$ and $\boldsymbol{\tau}(R, \mathbf{fpt}(f) \bullet f)$ are equal whenever p does not divide any of the exponents in f, and so Theorem 8.4 below often allows to to compute the first non-trivial test ideal.

Theorem 8.4. Let $f = u_1x_1^{d_1} + \cdots + u_nx_n^{d_n}$. Then

$$\boldsymbol{\tau}(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f) = \begin{cases} (f) & \mathbf{fpt}_{\mathfrak{m}}(f) = 1 \\ \mathfrak{m} & \mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{d_1} + \cdots + \frac{1}{d_n} \\ \mathfrak{m} & \mathbf{fpt}_{\mathfrak{m}}(f) < \min\left\{1, \sum \frac{1}{d_i}\right\} \text{ and } p > \max\{d_1, \cdots, d_n\} \end{cases}$$

Remark 8.5. Note that $\boldsymbol{\tau}(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f)$ need not equal $\mathfrak{m}$ if $\mathbf{fpt}_{\mathfrak{m}}(f)$ is less than $\min\left\{1, \frac{1}{d_1} + \cdots + \frac{1}{d_n}\right\}$ and p is less than or equal to some exponent of f. For examples of this pathological behavior, see [MY09, Proposition 4.2].

Our final result computes higher jumping numbers for the degree d Fermat hypersurface. By Proposition 2.41, it suffices to only consider those jumping numbers contained in $(0, 1]$. As we have seen in Corollary 8.3, the F-pure threshold of such a hypersurface depends strongly on the congruence class of p modulo d. In Theorem 8.6 below, we will see that the existence (and value) of a second jumping number strictly between $\mathbf{fpt}(f)$ and 1 also depends strongly on information encoded by the congruence class of p modulo d.

Theorem 8.6. Let $f = u_1 x_1^d + \cdots + u_d x_d^d$. Suppose that $p > d$ and write $p = d \cdot \omega + a$ for some $\omega \geqslant 1$ and $1 \leqslant a < d$. Let $\sigma_p := \omega + \lceil 2a/d \rceil$.

1. If $a = 1$, then $\mathbf{fpt}(f) = 1$ is the only F-jumping number of f in $(0, 1]$.

We now assume that $a \geqslant 2$.

2. If $p < a(d-1)$, then $\mathbf{fpt}(f) < \frac{p-a+\sigma_p}{p} \leqslant 1$ are F-jumping numbers of f in $(0, 1]$.

3. If $p > a(d-1)$, then $\mathbf{fpt}(f) < 1$ are the only F-jumping numbers of f in $(0, 1]$.

Remark 8.7. As a is strictly less than d, the condition that $p > a(d-1)$ in the last point above is automatically satisfied whenever p is larger than $(d-1)^2$. Thus, Theorem 8.6 says that if $p > (d-1)^2$, then $\mathbf{fpt}(f)$ and 1 are the only jumping numbers of f in $(0, 1]$.

Example 8.8. Suppose that $d = 4$, and $p = 7$. Then $\omega = 1$, $a = 3$, and $p < a(d-1)$. Note that $\sigma_p = \omega + \lceil 2a/d \rceil = 1 + \lceil 6/4 \rceil = 3 = a$, and thus $\frac{p-a+\sigma_p}{p} = 1$. In this case, Theorem 8.6 provides no new information.

Example 8.9. Let $d = 6$ and $p = 11$, so that $\omega = 1, a = 5$, and $p < a(d-1)$. We have that $\sigma_p = \omega + \left\lceil \frac{2a}{d} \right\rceil = 1 + \left\lceil \frac{10}{6} \right\rceil = 3$, and Theorem 8.6 shows that

$$\mathbf{fpt}(f) = \frac{p-a+1}{p} = \frac{7}{11}, \frac{p-a+\sigma_p}{p} = \frac{9}{11}, \text{ and } 1$$

are F-jumping numbers of f contained in $(0,1]$. In this case, the reader may verify that these are *all* of the F-jumping numbers of f in $(0,1]$

2 F-pure thresholds of diagonal hypersurfaces

Throughout this chapter, $R = K[x_1, \cdots, x_n]$ will denote a polynomial ring over an F-finite field K of characteristic $p > 0$, and $\mathfrak{m} = (x_1, \cdots, x_n)$ will denote the ideal generated by the variables of R. $R^{p^e} = \left\{ r^{p^e} : r \in R \right\} = K^{p^e}[x_1^{p^e}, \cdots, x_n^{p^e}]$ will denote the subring of R consisting of $(p^e)^{\text{th}}$ powers of elements of R. As $[K : K^p] < \infty$, we have that $[K : K^{p^e}] < \infty$ for all $e \geqslant 1$. If Σ^e is a finite basis for K over K^{p^e}, the reader may verify that

$$\left\{ \sigma \cdot \mu : \mu \text{ monomial }, \mu \notin \mathfrak{m}^{[p^e]}, \sigma \in \Sigma^e \right\} \tag{8.9.1}$$

is a free basis for R as an R^{p^e}-module. If K is perfect, so that $K^{p^e} = K$, the basis in (8.9.1) is the unique free basis for R as an R^{p^e}-module consisting of monomials.

The following lemma says that to compute the F-pure threshold of a diagonal polynomial f, it often suffices to compute the F-pure threshold of f at $\mathfrak{m}$.

Lemma 8.10. Let $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$ be a diagonal polynomial in R. If $p \nmid d_i$ for every $1 \leqslant i \leqslant n$, then $\mathbf{fpt}(f) = \mathbf{fpt}_{\mathfrak{m}}(f)$.

Proof. It follows from the Jacobian criterion for regularity that $R_\wp / f$ is regular (and hence F-pure) for every $\wp \neq \mathfrak{m}$. It follows that $\mathbf{fpt}_\wp(f) = 1$ by Corollary 5.7, and the claim then follows from Proposition 2.24. □

We will now prove Theorem 8.1, and in doing so, we will rely heavily on (the methods of) Theorem 7.6.

Theorem 8.1. Let $(d_1, \cdots, d_n) \in \mathbb{N}^n$, and let $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$. If $L := \sup \left\{ N : c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p-1 \text{ for all } 0 \leqslant e \leqslant N \right\}$, then

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{d_1} + \cdots + \frac{1}{d_n} & \text{if } L = \infty \\ \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L} & \text{if } L < \infty \end{cases}$$

Proof. Let $\boldsymbol{P}$ denote the rational polytope $\left\{ \boldsymbol{s} : 0 \leqslant s_i \leqslant \frac{1}{d_i} \right\} \subseteq [0,1]^n$. Then $\boldsymbol{P}$ is also the polytope associated to Support(f), and $\left(\frac{1}{d_1}, \cdots, \frac{1}{d_n}\right)$ is the unique maximal point of $\boldsymbol{P}$. (See Chapter 6.) It follows from Theorem 7.6 that $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{d_1} + \cdots + \frac{1}{d_n}$ whenever the non-terminating base p expansions of $\frac{1}{d_1}, \cdots, \frac{1}{d_n}$ add without carrying.

Suppose now that $L < \infty$. It follows from Theorem 7.6 that

$$\mathbf{fpt}_{\mathfrak{m}}(f) \geqslant \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L}. \tag{8.10.1}$$

By way of contradiction, suppose the inequality in (8.10.1) is strict. By Lemma 4.6, we have that $\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_L \geqslant \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + \left\langle \frac{1}{d_n} \right\rangle_L + \frac{1}{p^L}$, and Lemma 7.7 implies that

$$f^{p^L \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + p^L \left\langle \frac{1}{d_n} \right\rangle_L + 1} = \sum_{\boldsymbol{k}} \binom{|\boldsymbol{k}|}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} x_1^{d_1 k_1} \cdots x_n^{d_n k_n} \notin \mathfrak{m}^{[p^L]}. \tag{8.10.2}$$

It follows from (8.10.2) that there exists an index $\boldsymbol{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$ with

$$|\boldsymbol{k}| = p^L \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + p^L \left\langle \frac{1}{d_n} \right\rangle_L + 1 \tag{8.10.3}$$

such that $d_i k_i < p^L$ for all i. Restated, $\frac{1}{d_i} > \frac{1}{p^L} \cdot k_i$, and applying Lemma 4.6 to these inequalities yields the inequalities $\left\langle \frac{1}{d_i} \right\rangle_L \geqslant \frac{1}{p^L} \cdot k_i$. These inequalities imply the bounds $k_i \leqslant p^L \left\langle \frac{1}{d_i} \right\rangle_L$ for all i, and summing these yields

$$|\boldsymbol{k}| = k_1 + \cdots + k_n \leqslant p^L \left\langle \frac{1}{d_1} \right\rangle_L + \cdots + p^L \left\langle \frac{1}{d_n} \right\rangle_L,$$

which contradicts (8.10.3). Thus, equality holds in (8.10.1), and we are done. $\square$

Corollary 8.3. Let $f = u_1 x_1^d + \cdots + u_d x_d^d \in K[x_1, \cdots, x_d]$. Then

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \begin{cases} \frac{1}{p^\ell} & \text{if } p^\ell \leqslant d < p^{\ell+1} \text{ for some } \ell \geqslant 1 \\ 1 - \frac{a-1}{p} & \text{if } 0 < d < p \text{ and } p \equiv a \bmod d \text{ with } 1 \leqslant a < d \end{cases}$$

Proof. If $p^\ell \leqslant d < p^{\ell+1}$ for some $\ell \geqslant 1$, then $\frac{1}{p^{\ell+1}} < \frac{1}{d} \leqslant \frac{1}{p^\ell}$. Consequently, the non-terminating base p expansion of $\frac{1}{d}$ is of the form $\frac{1}{d} = \sum_{e \geqslant \ell+1} \frac{\omega_e}{p^e}$, with $\omega_{\ell+1} \neq 0$. Thus, $c_e\left(\frac{1}{d}\right) = 0$ for $1 \leqslant e \leqslant \ell$, $c_{\ell+1}\left(\frac{1}{d}\right) = \omega_{\ell+1} \neq 0$, and adding d copies of $c_{\ell+1}\left(\frac{1}{d}\right)$ yields $c_{\ell+1}\left(\frac{1}{d}\right) + \cdots + c_{\ell+1}\left(\frac{1}{d}\right) = d\omega_{\ell+1} \geqslant d \geqslant p^\ell \geqslant p$. In the notation of Theorem 8.1, we have that $L = \ell$, and as $\left\langle \frac{1}{d} \right\rangle_\ell = 0$, we have $\mathbf{fpt}_{\mathfrak{m}}(f) = d \cdot \left\langle \frac{1}{d} \right\rangle_\ell + \frac{1}{p^\ell} = \frac{1}{p^\ell}$.

For the remainder, we will assume that $p > d$. Fix an integer $\omega \geqslant 1$ such that $p = d \cdot \omega + a$ with $1 \leqslant a < d$. From this equation, we see that $\frac{1}{d} = \frac{\omega}{p} + \frac{a}{d} \cdot \frac{1}{p}$, from which we may conclude that $c_1\left(\frac{1}{d}\right) = \omega$ and $c_{e+1}\left(\frac{1}{d}\right) = c_e\left(\frac{a}{d}\right)$ for $e \geqslant 1$. If $a = 1$, it follows that $c_e\left(\frac{1}{d}\right) = \omega$ for all $e \geqslant 1$. As $d \cdot c_e\left(\frac{1}{d}\right) = d \cdot \omega = p - 1$ for all $e \geqslant 1$, it follows that the non-terminating base p expansions of d copies of $\frac{1}{d}$ add without carrying, so that $\mathbf{fpt}_{\mathfrak{m}}(f) = 1$ by Theorem 8.1.

Next, suppose that $a \geqslant 2$. Adding d copies of $c_1\left(\frac{1}{d}\right)$ gives $c_1\left(\frac{1}{d}\right) + \cdots + c_1\left(\frac{1}{d}\right) = d \cdot \omega = p - a \leqslant p - 1$, while $d \cdot c_2\left(\frac{1}{d}\right) > p$ by Lemma 8.19. This shows that, in adding the expansions of d copies of $\frac{1}{d}$, the first carry occurs at the second digit, and Theorem 8.1 implies that $\mathbf{fpt}_{\mathfrak{m}}(f) = d \cdot \left\langle \frac{1}{d} \right\rangle_1 + \frac{1}{p} = \frac{d \cdot \omega}{p} + \frac{1}{p} = \frac{p-a+1}{p}$. ∎

3 Test ideals of diagonal hypersurfaces

1 Proof of Theorem 8.4

Having computed $\mathbf{fpt}_{\mathfrak{m}}(f)$ for diagonal hypersurfaces, we now change our focus to computing the corresponding test ideals at these parameters. We will follow the previous notation. In particular, $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n}$ and $\boldsymbol{\eta} = \left(\frac{1}{d_1}, \cdots, \frac{1}{d_n}\right) \in \mathbb{Q}^n$.

We begin with Proposition 8.11 , a technical result that will be key to the proof of Theorem 8.4. We will prove Proposition 8.11 in the next subsection.

Proposition 8.11. Suppose that $d_i \leqslant p^e$, and that $d_i \neq p^r$ for any $1 \leqslant r \leqslant e-1$. Furthermore, assume that $c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p-2$ and $\binom{|p^e\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \neq 0 \bmod p$. Then $x_i \in \left(f^{p^e\left\langle \frac{1}{d_1}\right\rangle_e + \cdots + p^e\left\langle \frac{1}{d_n}\right\rangle_e + 1}\right)^{\left[\frac{1}{p^e}\right]}$.

Theorem 8.4. Let $f = u_1 x_1^{d_1} + \cdots + u_n x_n^{d_n} \in R := K[x_1, \cdots, x_n]$ be a diagonal hypersurface over a perfect field of characteristic $p > 0$. Then

$$\boldsymbol{\tau}\left(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f\right) = \begin{cases} (f) & \mathbf{fpt}_{\mathfrak{m}}(f) = 1 \\ \mathfrak{m} & \mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{d_1} + \cdots + \frac{1}{d_n} \\ \mathfrak{m} & \mathbf{fpt}_{\mathfrak{m}}(f) < \min\left\{1, \sum \frac{1}{d_i}\right\} \text{ and } p > \max\left\{d_1, \cdots, d_n\right\}. \end{cases}$$

Proof. It follows immediately from Definition 2.26 and Lemma 2.38 that $(f) = (f^p)^{\left[\frac{1}{p}\right]} = \boldsymbol{\tau}(R, 1 \bullet f)$. We will now assume that $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{d_1} + \cdots + \frac{1}{d_n} < 1$.

Let $\boldsymbol{\eta} := \left(\frac{1}{d_1}, \cdots, \frac{1}{d_n}\right) \in \mathbb{Q}^n$. In this case, it follows from Theorem 8.1 that the non-terminating base p expansions of the entries of $\boldsymbol{\eta}$ add without carrying, so that

1. $d_i \neq p^r$ for all $r \geqslant 1$ and $1 \leqslant i \leqslant n$ (for, otherwise, carrying would be necessary),

2. and $\binom{|p^e\langle \boldsymbol{\eta} \rangle_e|}{p^e \langle \boldsymbol{\eta} \rangle_e} \neq 0 \bmod p$ for all $e \geqslant 1$ (by Lemma 4.14).

As $\mathbf{fpt}_{\mathfrak{m}}(f) < 1$, $c_e\left(\mathbf{fpt}_{\mathfrak{m}}(f)\right) \leqslant p-1$ for some $e \geqslant 1$. Choose $e \gg 0$ such that

3. $d_i < p^e$ for $1 \leqslant i \leqslant n$,

4. $c_e\left(\mathbf{fpt}_{\mathfrak{m}}(f)\right) = c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p-2$ (see Remark 4.13), and

5. $\boldsymbol{\tau}\left(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f\right) = \left(f^{\lceil p^e \mathbf{fpt}_{\mathfrak{m}}(f)\rceil}\right)^{\left[\frac{1}{p^e}\right]} = \left(f^{p^e\left\langle \frac{1}{d_1}\right\rangle_e + \cdots + p^e\left\langle \frac{1}{d_n}\right\rangle_e + 1}\right)^{\left[\frac{1}{p^e}\right]}$.

In point 5, we have used that $p^e\left\langle \frac{1}{d_1}\right\rangle_e + \cdots + p^e\left\langle \frac{1}{d_n}\right\rangle_e = p^e\left\langle \mathbf{fpt}_{\mathfrak{m}}(f)\right\rangle_e$ (as the entries of $\boldsymbol{\eta}$ add without carrying) and that $\lceil p^e \mathbf{fpt}_{\mathfrak{m}}(f)\rceil = p^e\left\langle \mathbf{fpt}_{\mathfrak{m}}(f)\right\rangle_e + 1$. (See Lemma 4.6.) Points $1-5$ above allow us to apply Proposition 8.11, which says that

$$(x_1, \cdots, x_n) \subseteq \left(f^{p^e\left\langle \frac{1}{d_1}\right\rangle_e + \cdots + p^e\left\langle \frac{1}{d_n}\right\rangle_e + 1}\right)^{\left[\frac{1}{p^e}\right]} = \boldsymbol{\tau}\left(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f\right).$$

For the remaining case, we assume that $\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{1}{d_1}\right\rangle_L + \cdots + \left\langle \frac{1}{d_n}\right\rangle_L + \frac{1}{p^L} < 1$, where $L = \max\{e : c_1\left(\frac{1}{d_e}\right) + \cdots + c_n\left(\frac{1}{d_e}\right) \leqslant p-1\}$. By Remark 4.13, we have that $c_e(\mathbf{fpt}_{\mathfrak{m}}(f)) = c_1\left(\frac{1}{d_e}\right) + \cdots + c_n\left(\frac{1}{d_e}\right)$ for $1 \leqslant e \leqslant L$, and that $c_e\left(\mathbf{fpt}_{\mathfrak{m}}(f)\right) = p-1$ for $e \geqslant L+1$. As $\mathbf{fpt}_{\mathfrak{m}}(f) < 1$, we must have that $c_e\left(\mathbf{fpt}_{\mathfrak{m}}(f)\right) \leqslant p-2$ for some $1 \leqslant e \leqslant L$.

6. Choose $1 \leqslant \ell \leqslant L$ so that $c_\ell(\mathbf{fpt}_{\mathfrak{m}}(f)) = c_1\left(\frac{1}{d_\ell}\right) + \cdots + c_n\left(\frac{1}{d_\ell}\right) \leqslant p-2$ and $c_e\left(\mathbf{fpt}_{\mathfrak{m}}(f)\right) = p-1$ for $e \geqslant \ell+1$.

7. By our choice of ℓ, we have that $\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{1}{d_1}\right\rangle_\ell + \cdots + \left\langle \frac{1}{d_n}\right\rangle_\ell + \frac{1}{p^\ell}$.

8. We also see that $\binom{\left|p^\ell\langle\boldsymbol{\eta}\rangle_\ell\right|}{p^\ell\langle\boldsymbol{\eta}\rangle_\ell} \not\equiv 0 \bmod p$ by Lemma 4.14.

9. As $p > \max\{d_1, \cdots, d_n\}$, each d_i is strictly less than any power of p.

As before, points $6-9$ and Proposition 8.11 allow us to conclude that

$$(x_1, \cdots, x_n) \subseteq \left(f^{p^\ell\left\langle \frac{1}{d_1}\right\rangle_\ell + \cdots + p^\ell\left\langle \frac{1}{d_n}\right\rangle_\ell + 1}\right)^{\left[\frac{1}{p^\ell}\right]} = \boldsymbol{\tau}\left(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f\right).$$

□

2 Some supporting lemmas

In this section, we prove Proposition 8.11, and we do so via a series of lemmas.

Lemma 8.12. Let d and e be positive integers. Then $d\left(1 + p^e\left\langle \frac{1}{d}\right\rangle_e\right) - p^e \in \mathbb{N}$. Furthermore, if $d \leqslant p^e$, then $d\left(1 + p^e\left\langle \frac{1}{d}\right\rangle_e\right) - p^e \leqslant p^e - 1$.

Proof. It is clear that $d\left(1+p^e\left\langle\frac{1}{d}\right\rangle_e\right)-p^e\in\mathbb{Z}$. By definition, $\frac{1}{d}=\left\langle\frac{1}{d}\right\rangle_e+\left[\!\left[\frac{1}{d}\right]\!\right]_e$, and consequently $d\left(1+p^e\left\langle\frac{1}{d}\right\rangle_e\right)-p^e=d\left(1+p^e\left(\left\langle\frac{1}{d}\right\rangle_e-\frac{1}{d}\right)\right)=d\left(1-p^e\left[\!\left[\frac{1}{d}\right]\!\right]_e\right)$. Finally, by Lemma 4.6, we have that $0<\left[\!\left[\frac{1}{d}\right]\!\right]_e\leqslant\frac{1}{p^e}$, so $0\leqslant d\left(1-p^e\left[\!\left[\frac{1}{d}\right]\!\right]_e\right)<d$. ☐

Lemma 8.13. Fix $e\geqslant 1$, let i be an integer with $1\leqslant i\leqslant n$, and let v_i denote the element of $\mathbb{N}^n$ with 1 in the i^{th} spot and zeroes elsewhere. Suppose that $d_i\leqslant p^e$, $d_i\neq p^r$ for any $1\leqslant r\leqslant e-1$, and $\binom{|p^e\langle\boldsymbol{\eta}\rangle_e|+1}{p^e\langle\boldsymbol{\eta}\rangle_e+\mathrm{v}_i}\not\equiv 0 \bmod p$. Let

$$\mu_i := x_i^{d_i\left(1+p^e\left\langle\frac{1}{d_i}\right\rangle_e\right)-p^e}\, x_1^{d_1p^e\left\langle\frac{1}{d_1}\right\rangle_e}\cdots\underbrace{x_i^{d_ip^e\left\langle\frac{1}{d_i}\right\rangle_e}}_{\text{omitted}}\cdots x_n^{d_np^e\left\langle\frac{1}{d_n}\right\rangle_e}.$$

By Lemma 8.12,

$$\beta_i := \binom{|p^e\langle\boldsymbol{\eta}\rangle_e|+1}{p^e\langle\boldsymbol{\eta}\rangle_e+\mathrm{v}_i}\boldsymbol{u}^{p^e\langle\boldsymbol{\eta}\rangle_e+\mathrm{v}_i}\cdot\mu_i$$

is part of a free basis for R over R^{p^e} as described in (8.9.1). When $f^{p^e\left\langle\frac{1}{d_1}\right\rangle_e+\cdots+p^e\left\langle\frac{1}{d_n}\right\rangle_e+1}$ is written as an R^{p^e}-linear combination of this basis, the coefficient of β_i is equal to $x_i^{p^e}$.

Proof. After relabeling the variables, we may assume that $i=1$. We have that

$$f^{p^e\left\langle\frac{1}{d_1}\right\rangle_e+\cdots+p^e\left\langle\frac{1}{d_n}\right\rangle_e+1}=\sum_{\boldsymbol{k}}\binom{|\boldsymbol{k}|}{\boldsymbol{k}}\boldsymbol{u}^{\boldsymbol{k}}x_1^{d_1k_1}\cdots x_n^{d_nk_n}, \tag{8.13.1}$$

where the sum ranges over all $\boldsymbol{k}\in\mathbb{N}^n$ such that $|\boldsymbol{k}|=p^e\left\langle\frac{1}{d_1}\right\rangle_e+\cdots+p^e\left\langle\frac{1}{d_n}\right\rangle_e+1$. Note that the monomial given by the index $\boldsymbol{k}=p^e\langle\boldsymbol{\eta}\rangle_e+\mathrm{v}_1$ is

$$x_1^{d_1\left(p^e\left\langle\frac{1}{d_1}\right\rangle_e+1\right)}x_2^{d_2p^e\left\langle\frac{1}{d_2}\right\rangle_e}\cdots x_n^{d_np^e\left\langle\frac{1}{d_n}\right\rangle_e}=x_1^{p^e}\cdot\mu_1. \tag{8.13.2}$$

Monomials in (8.13.1) that correspond to distinct indices are themselves distinct. Thus, it suffices to show that the monomial given by $p^e\cdot\langle\boldsymbol{\eta}\rangle_e+\mathrm{v}_1$ is the *only* monomial in (8.13.1) that is an R^{p^e}-multiple of μ_1. However, a monomial in (8.13.1) corresponding to the index $\boldsymbol{k}=(\kappa_1,\cdots,\kappa_n)$ is an R^{p^e}-multiple of μ_1 whenever

$$x_1^{d_1\kappa_1}\cdots x_n^{d_n\kappa_n}=x_1^{a_1p^e}\cdots x_n^{a_np^e}\cdot\mu_1 \tag{8.13.3}$$

for some $(a_1, \cdots, a_n) \in \mathbb{N}^n$. Equating the exponents in (8.13.3) shows that

$$d_1\kappa_1 = p^e(a_1 - 1) + d_1\left(1 + p^e\left\langle\frac{1}{d}\right\rangle_e\right) \text{ and } d_i\kappa_i = p^e\left(a_i + d_i\left\langle\frac{1}{d_i}\right\rangle_e\right) \text{ for } i \geqslant 2.$$

Dividing each equation by the appropriate d_i and adding the resulting equalities shows that $|\boldsymbol{k}| = p^e \cdot \left(\frac{a_1-1}{d_1} + \sum_{i\geqslant 2}\frac{a_i}{d_i}\right) + p^e\left\langle\frac{1}{d_1}\right\rangle_e + \cdots + p^e\left\langle\frac{1}{d_n}\right\rangle_e + 1$, and comparing this with the equality $|\boldsymbol{k}| = p^e\left\langle\frac{1}{d_1}\right\rangle_e + \cdots + \left\langle\frac{1}{d_n}\right\rangle_e + 1$ shows that $\frac{a_1-1}{d_1} + \sum_{i\geqslant 2}\frac{a_i}{d_i} = 0$. As $(a_1, \cdots, a_n) \in \mathbb{N}^n$, we conclude that $a_1 = 1$ and $a_i = 0$ for $i \geqslant 2$. Substituting these values into (8.13.3) shows that

$$x_1^{d_1 k_1} \cdots x_n^{d_n k_n} = x_1^{p^e} \cdot \mu_1, \tag{8.13.4}$$

and comparing (8.13.4) with (8.13.2) show that $\boldsymbol{k} = p^e\langle\boldsymbol{\eta}\rangle_e + \mathrm{v}_1$. $\square$

Lemma 8.14. If $c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p - 2$ and $\binom{|p^e\langle\boldsymbol{\eta}\rangle_e|}{p^e\langle\boldsymbol{\eta}\rangle_e} \not\equiv 0 \bmod p$, then

$$\binom{|p^e\langle\boldsymbol{\eta}\rangle_e| + 1}{p^e\langle\boldsymbol{\eta}\rangle_e + \mathrm{v}_i} \not\equiv 0 \bmod p \text{ for } 1 \leqslant i \leqslant n.$$

Proof. The assumption that $c_e\left(\frac{1}{d_i}\right) \leqslant c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) \leqslant p - 2$ implies that $p^e\left\langle\frac{1}{d_i}\right\rangle_e + 1 \equiv c_e\left(\frac{1}{d_i}\right) + 1 \not\equiv 0 \bmod p$ for $1 \leqslant i \leqslant n$. Similarly, we also have that $|p^e\langle\boldsymbol{\eta}\rangle_e| + 1 \equiv c_e\left(\frac{1}{d_1}\right) + \cdots + c_e\left(\frac{1}{d_n}\right) + 1 \not\equiv 0 \bmod p$.

The result then follows by reducing the equality

$$\binom{|p^e\langle\boldsymbol{\eta}\rangle_e| + 1}{p^e\langle\boldsymbol{\eta}\rangle_e + \mathrm{v}_i} = \binom{|p^e\langle\boldsymbol{\eta}\rangle_e|}{p^e\langle\boldsymbol{\eta}\rangle_e} \cdot \frac{|p^e\langle\boldsymbol{\eta}\rangle_e| + 1}{p^e\left\langle\frac{1}{d_i}\right\rangle_e + 1}$$

(over $\mathbb{Q}$) modulo the prime p. $\square$

Proof of Proposition 8.11. This is an immediate consequence of Proposition 2.29 and Lemmas 8.13 and 8.14. $\square$

4 On higher jumping numbers of Fermat hypersurfaces

Throughout this section, $f := u_1 x_1^d + \cdots + u_d x_d^d$ will now denote the degree d Fermat hypersurface in R. Our goal is to prove Theorem 8.6, which is concerned with the higher jumping numbers of f. Note that, by Proposition 2.41, it suffices to only consider those jumping numbers contained in $(0,1]$. Theorem 8.6 shows that whenever p is greater than d, yet still "small", there exist exotic jumping numbers. Theorem 8.6 also shows that for $p \gg 0$, the only jumping numbers in $(0,1]$ are the expected ones (namely, $\mathbf{fpt}(f)$ and 1).

Notation 8.15. In what follows, we will always assume that $p > d$. Accordingly, we fix integers $\omega \geqslant 1$ and $1 \leqslant a < d$ such that $p = d \cdot \omega + a$.

Theorem 8.6. Let $\sigma_p := \omega + \lceil 2a/d \rceil \in \mathbb{N}$.

1. If $a = 1$, then $\mathbf{fpt}(f) = 1$ is the only F-jumping number of f in $(0,1]$.

We now assume that $a \geqslant 2$.

2. If $p < a(d-1)$, then $\mathbf{fpt}(f) < \frac{p-a+\sigma_p}{p} \leqslant 1$ are F-jumping numbers of f in $(0,1]$.
3. If $p > a(d-1)$, then $\mathbf{fpt}(f) < 1$ are the only F-jumping numbers of f in $(0,1]$.

The proof of Theorem 8.6 will depend heavily on the following two lemmas, whose proofs we postpone until the next subsection.

Lemma 8.16. Suppose that $a \geqslant 2$, and $p < a(d-1)$. Then

1. $\boldsymbol{\tau}\left(R, \left(\frac{p-a+\sigma_p-1}{p} + \frac{p-1}{p^2} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right) = \mathfrak{m}$ for all $e \geqslant 1$.
2. $\boldsymbol{\tau}\left(R, \left(\frac{p-a+\sigma_p}{p}\right) \bullet f\right) \neq \mathfrak{m}$.

Lemma 8.17. If $a \geqslant 2$ and $p > a(d-1)$, then $\boldsymbol{\tau}\left(R, \left(\frac{p-1}{p} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right) = \mathfrak{m}$ for all $e \geqslant 1$.

Proof of Theorem 8.6. By Corollary 8.3, $\mathbf{fpt}_{\mathfrak{m}}(f) = 1$ whenever $a = 1$. As the F-pure threshold is the smallest jumping number, the first assertion of Theorem 8.6 follows. We now assume that $a \geqslant 2$.

As $p > d$, it follows from Theorem 8.4 that $\boldsymbol{\tau}\left(R, \mathbf{fpt}_{\mathfrak{m}}(f) \bullet f\right) = \mathfrak{m}$. Lemma 8.16 shows that if $p < a(d-1)$, then $\boldsymbol{\tau}\left(R, \lambda \bullet f\right)$ continues being equal to $\mathfrak{m}$ for all λ with $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{p-a+1}{p} \leqslant \lambda < \frac{p-a+\sigma_p}{p}$, and that $\boldsymbol{\tau}\left(R, \frac{p-a+\sigma_p}{p} \bullet f\right) \neq \mathfrak{m}$. It follows from Definition 2.40 that $\frac{p-a+\sigma_p}{p}$ is an F-jumping number of f. The assertion that $\frac{p-a+\sigma_p}{p} \leqslant 1$ follows from Lemma 8.20.

Similarly Lemma 8.17 shows that $\boldsymbol{\tau}\left(R, \lambda \bullet f\right) = \mathfrak{m}$ for every $\lambda \in (\mathbf{fpt}_{\mathfrak{m}}(f), 1)$. As we have already seen that $\boldsymbol{\tau}\left(R, 1 \bullet f\right) = (f) \neq \mathfrak{m}$, this shows that $\mathbf{fpt}_{\mathfrak{m}}(f) < 1$ are the only F-jumping numbers of f contained in $(0, 1]$. □

1 More supporting lemmas

We now prove the two technical lemmas cited in the proof of Theorem 8.6.

Remark 8.18. From the equation $p = d \cdot \omega + a$, we have that $\frac{1}{d} = \frac{\omega}{p} + \frac{a}{d} \cdot \frac{1}{p}$, from which we may conclude that $c_1\left(\frac{1}{d}\right) = \omega$ while $c_{e+1}\left(\frac{1}{d}\right) = c_e\left(\frac{a}{d}\right)$ for $e \geqslant 1$.

Lemma 8.19. If $a \geqslant 2$, then $(d-1) \cdot c_2\left(\frac{1}{d}\right) \geqslant p + 1$.

Proof. Let $\omega_1 := c_1\left(\frac{a}{d}\right)$. By Remark 8.18, it suffices to show that $(d-1) \cdot \omega_1 > p$. We may write $\frac{a}{d} = \frac{\omega_1}{p} + \left[\!\left[\frac{a}{d}\right]\!\right]_1$, from which it follows that

$$(8.19.1) \qquad \frac{(d-1) \cdot a}{d} = \frac{(d-1) \cdot \omega_1}{p} + (d-1) \cdot \left[\!\left[\frac{a}{d}\right]\!\right]_1 \leqslant \frac{(d-1) \cdot \omega_1}{p} + \frac{d-1}{p}.$$

First, suppose that $a \geqslant 3$, so that $\frac{(d-1) \cdot a}{d} = a - \frac{a}{d} > a - 1 \geqslant 2$. Substituting this into

(8.19.1) shows that

$$2 < \frac{(d-1)\cdot\omega_1}{p} + \frac{d-1}{p} < \frac{(d-1)\cdot\omega_1}{p} + \frac{p-1}{p} = \frac{(d-1)\cdot\omega_1 - 1}{p} + 1.$$

From this, we may conclude that $1 < \frac{(d-1)\cdot\omega_1 - 1}{p}$, so that $(d-1)\cdot\omega_1 > p+1$.

We now deal with the case that $a = 2$. By way of contradiction, suppose that $(d-1)\cdot\omega_1 \leqslant p$. As p is prime, the equality $(d-1)\cdot\omega_1 = p$ implies that $d-1 = p$ or $\omega_1 = p$. The later is impossible as ω_1 is a digit in a base p expansion, while the former is impossible as $p > d$. Thus, we may assume that $(d-1)\cdot\omega_1 \leqslant p-1$, and substituting this into (8.19.1) shows that

$$\frac{2d-2}{d} \leqslant \frac{p-1}{p} + \frac{d-1}{p} = 1 + \frac{d-2}{p}.$$

Consequently, $\frac{d-2}{d} \leqslant \frac{d-2}{p}$, so $\frac{1}{d} \leqslant \frac{1}{p}$, which implies that $p \leqslant d$, a contradiction. ☐

Lemma 8.20. Let $\sigma_p := \omega + \left\lceil \frac{2a}{d} \right\rceil$.

1. $p < d(\omega + \sigma_p - 1) < 2p$.
2. If $p < a(d-1)$, then $\sigma_p \leqslant a$.
3. If $p > a(d-1)$, then $p < d(\omega + a - 1) < 2p$.

Proof. For 1, substituting the identity $d\omega = p - a$ into $d(\omega + \sigma_p - 1)$ yields

$$d(\omega + \sigma_p - 1) = 2p - 2a + d\left\lceil \frac{2a}{d} \right\rceil - d. \tag{8.20.1}$$

Substituting the inequalities $\frac{2a}{d} \leqslant \left\lceil \frac{2a}{d} \right\rceil < \frac{2a}{d} + 1$ into (8.20.1) shows that

$$2p - d \leqslant d(\omega + \sigma_p - 1) < 2p.$$

For 2, we have that $d\omega + a = p < a(d-1)$ by assumption. It follows that

$$d\omega + a < a(d-1) \implies \omega + \frac{2a}{d} < a \implies \sigma_p = \left\lceil \omega + \frac{2a}{d} \right\rceil \leqslant a.$$

The third point follows along the same lines, and is left to the reader. ☐

Lemma 8.21. Let μ, μ_1, μ_2 be monomials with $\mu_1, \mu_2 \in \operatorname{Support}(f^N)$ and $\mu \notin \mathfrak{m}^{[p^e]}$. Suppose that $d \geqslant 2$ is not a power of p. If $\mu_1 = x_i^{p^e} \cdot \mu$ for some i and μ_2 is another R^{p^e}-multiple of μ, then $\mu_1 = \mu_2$.

Proof. By assumption, we have that

$$\mu_1 = x_i^{p^e} \cdot \mu \text{ and } \mu_2 = \boldsymbol{x}^{p^e \boldsymbol{a}} \cdot \mu \text{ for some } \boldsymbol{a} \in \mathbb{N}^n. \tag{8.21.1}$$

The assumption that all $d_1 = \cdots = d_n = d$ implies that f^N is homogeneous, so that $\deg \mu_1 = \deg \mu_2$. Applying this to (8.21.1) shows that $|\boldsymbol{a}| = 1$, so that $\mu_2 = x_j^{p^e} \cdot \mu$ for some j. If $i \neq j$, then $\deg_{x_i} \mu_1 = p^e + \deg_{x_i} \mu$, while $\deg_{x_i} \mu_2 = \deg_{x_i} \mu$, and so

$$\deg_{x_i} \mu_1 - \deg_{x_i} \mu_2 = p^e. \tag{8.21.2}$$

We have already observed that the left-hand side of (8.21.2) is divisible by d, contradicting the fact that d is not a power of p. Thus, $i = j$, and so $\mu_1 = \mu_2$. ☐

Lemma 8.16. Suppose that $a \geqslant 2$ and $p < a(d-1)$. Then

1. $\boldsymbol{\tau}\left(R, \left(\frac{p-a+\sigma_p-1}{p} + \frac{p-1}{p^2} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right) = \mathfrak{m}$ for all $e \geqslant 1$, and
2. $\boldsymbol{\tau}\left(R, \left(\frac{p-a+\sigma_p}{p}\right) \bullet f\right) \neq \mathfrak{m}$.

Proof. By Lemma 8.19, we know what $(d-1) \cdot c_2\left(\frac{1}{d}\right) \geqslant p+1$. Thus, there exist non-negative integers $\delta_1, \cdots, \delta_{d-1}$ such that $\sum \delta_i = p-1$ and $\delta_i \leqslant c_2\left(\frac{1}{d}\right)$ for $1 \leqslant i \leqslant d-1$, with the preceeding inequality being strict for at least one index i (which we are free to choose). In what follows, we will assume that $\delta_{d-1} < c_2\left(\frac{1}{d}\right)$. Fix $e \geqslant 1$, and let

(8.21.3)
$$\boldsymbol{s} := \left(\frac{\omega}{p} + \frac{\delta_1}{p^2}, \cdots, \frac{\omega}{p} + \frac{\delta_{d-2}}{p^2}, \frac{\omega}{p} + \frac{\delta_{d-1}}{p^2} + \frac{p-1}{p^3} + \cdots + \frac{p-1}{p^e}, \frac{\omega + \sigma_p - 1}{p}\right).$$

Note that $\boldsymbol{s} \in \frac{1}{p^e} \cdot \mathbb{N}$. Furthermore, we claim that the first $d-1$ entries of $\boldsymbol{s}$ are less than or equal to $\left\langle \frac{1}{d} \right\rangle_2 = \frac{\omega}{p} + \frac{c_2\left(\frac{1}{d}\right)}{p^2}$. Indeed, this follows from the fact that $\delta_i \leqslant c_2\left(\frac{1}{d}\right)$ for $1 \leqslant i \leqslant d-1$, with the inequality being strict for $i = d-1$. Let $\boldsymbol{k} := p^e \cdot \boldsymbol{s} \in \mathbb{N}$. We summarize some important properties of $\boldsymbol{k}$ below.

1. By the preceding remarks, the first $d-1$ entries of $\boldsymbol{k}$ are less than or equal to $p^e \left\langle \frac{1}{d} \right\rangle_e$, and it follows that the first $d-1$ entries of $d \cdot \boldsymbol{k}$ are less than or equal to $p^e - 1$.

2. By Lemma 8.20, the last entry of $d \cdot \boldsymbol{k}$ is *strictly* between p^e and $2p^e$.

3. $$\begin{aligned} |\boldsymbol{k}| &= p^e \cdot \left(\frac{d\omega + \sigma_p - 1}{p} + \sum \frac{\delta_i}{p^2} + \frac{p-1}{p^3} + \cdots + \frac{p-1}{p^e} \right) \\ &= p^e \cdot \left(\frac{p - a + \sigma_p - 1}{p} + \frac{p-1}{p^2} + \frac{p-1}{p^3} \cdots + \frac{p-1}{p^e} \right). \end{aligned}$$

4. By construction, the integers $k_1, \cdots, k_d$ add without carrying. By Lemma 4.14, we conclude that $\binom{|\boldsymbol{k}|}{\boldsymbol{k}} \neq 0 \bmod p$.

As there is no gathering of monomials when expanding $f^{|\boldsymbol{k}|}$ using the multinomial theorem, the last point above shows that $\boldsymbol{x}^{d \cdot \boldsymbol{k}} \in \operatorname{Support}(f^{|\boldsymbol{k}|})$, while the first two points show that

$$\boldsymbol{x}^{d \cdot \boldsymbol{k}} = x_d^{p^e} \cdot \mu, \tag{8.21.4}$$

where μ is some monomial with $\mu \notin \mathfrak{m}^{[p^e]}$. By point (4) above, the element $\beta := \binom{|\boldsymbol{k}|}{\boldsymbol{k}} u^{\boldsymbol{k}} \cdot \mu$ is part of a free basis for R over R^{p^e} as described in (8.9.1), and Lemma 8.21 (combined with (8.21.4)) show that, when writing $f^{|\boldsymbol{k}|}$ in terms of this basis, the coefficient β is given by $x_d^{p^e}$. We see that

$$x_d \in \left(f^{|\boldsymbol{k}|}\right)^{\left[\frac{1}{p^e}\right]} = \boldsymbol{\tau}\left(R, \left(\frac{p - a + \sigma_p - 1}{p} + \frac{p-1}{p^2} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right),$$

where the last equality holds by Lemma 2.38. As this argument is symmetric in the variables, the first claim follows.

It remains to show that $\boldsymbol{\tau}\left(R, \frac{p-a+\sigma_p}{p} \bullet f\right) \neq \mathfrak{m}$. By way of contradiction, suppose that $x_1 \in \boldsymbol{\tau}\left(R, \frac{p-a+\sigma_p}{p} \bullet f\right) = (f^{p-a+\sigma_p})^{\left[\frac{1}{p}\right]}$. It follows that there exists $\boldsymbol{k} \in \mathbb{N}^d$ with $|\boldsymbol{k}| = p - a + \sigma_p$ such that $\boldsymbol{x}^{d \cdot \boldsymbol{k}} \in \operatorname{Support}(f^{p-a+\sigma_p})$, $p \leqslant dk_1 < 2p$, and $dk_i \leqslant p-1$ for $2 \leqslant i \leqslant d$. Note that

$$dk_i \leqslant p - 1 = d\omega + a - 1 \implies k_i \leqslant \omega + \frac{a-1}{d} \implies k_i \leqslant \omega, \tag{8.21.5}$$

where here we have used that $k_i \in \mathbb{N}$ and that $1 \leqslant a < d$. Thus, from the conclusion of (8.21.5) we obtain the stronger bound $dk_i \leqslant dm_1 = p - a$ for $2 \leqslant i \leqslant d$. It follows that

$$d \cdot (p - a + \sigma_p) = d \cdot |\boldsymbol{k}| \leqslant dk_1 + (d-1)(p-a),$$

from which we can conclude that

$$p - a + d\sigma_p \leqslant dk_1 < 2p. \tag{8.21.6}$$

However, substituting the definition of σ_p into (8.21.6) shows that

$$\begin{aligned} p - a + d\sigma_p = p - a + d \cdot \left(\omega + \left\lceil \frac{2a}{d} \right\rceil\right) &= p - a + d\omega + d\left\lceil \frac{2a}{d} \right\rceil \\ &= p - a + p - a + d\left\lceil \frac{2a}{d} \right\rceil \\ &\geqslant 2p - 2a + 2a = 2p. \end{aligned}$$

This contradicts (8.21.6), and it follows that x_1 is not contained in $\boldsymbol{\tau}\left(R, \frac{p-a+\sigma_p}{p} \bullet f\right)$. Hence, $\boldsymbol{\tau}\left(R, \frac{p-a+\sigma_p}{p} \bullet f\right) \neq \mathfrak{m}$. ☐

Lemma 8.17. If $a \geqslant 2$ and $p > a(d-1)$, then $\boldsymbol{\tau}\left(R, \left(\frac{p-1}{p} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right) = \mathfrak{m}$ for all $e \geqslant 1$.

Proof. As in the proof of Lemma 8.16, Lemma 8.19 guarantees that there exists non-negative integers $\delta_1, \cdots, \delta_{d-1}$ such that $\sum_{i=1}^{d-1} \delta_i = p-1$ and $\delta_i \leqslant \omega_2$ for $1 \leqslant i \leqslant d-1$, with at least one inequality being strict. Again, we will assume that $\delta_{d-1} < \omega_2$. Fix $e \geqslant 1$, and let

(8.21.7)
$$\boldsymbol{s} := \left(\frac{\omega_1}{p} + \frac{\delta_1}{p^2}, \cdots, \frac{\omega_1}{p} + \frac{\delta_{d-2}}{p^2}, \frac{\omega_1}{p} + \frac{\delta_{d-1}}{p^2} + \frac{p-1}{p^3} + \cdots + \frac{p-1}{p^e}, \frac{\omega_1 + a - 1}{p}\right).$$

Note that $\boldsymbol{s} \in \frac{1}{p^e} \cdot \mathbb{N}$. As in the proof of Lemma 8.16, we have that the first $d-1$ entries of $\boldsymbol{s}$ are less than or equal to $\left\langle \frac{1}{p} \right\rangle_e$. Let $\boldsymbol{k} := p^e \cdot \boldsymbol{s} \in \mathbb{N}$.

1. The first $d-1$ entries of $d \cdot \boldsymbol{k}$ are less than or equal to $p^e - 1$.

2. By Lemma 8.20, the last entry of $d \cdot \boldsymbol{k}$ is *strictly* between p^e and $2p^e$

3. $$\begin{aligned} |\boldsymbol{k}| &= p^e \cdot \left(\frac{dm_1 + a - 1}{p} + \sum_{i=1}^{d-1} \frac{\delta_i}{p^2} + \frac{p-1}{p^3} + \cdots + \frac{p-1}{p^e}\right) \\ &= p^e \cdot \left(\frac{p-1}{p} + \frac{p-1}{p^2} + \frac{p-1}{p^3} \cdots + \frac{p-1}{p^e}\right) \end{aligned}$$

4. The integers $k_1, \cdots, k_d$ add without carrying, so $\binom{|\boldsymbol{k}|}{\boldsymbol{k}} \neq 0 \bmod p$ by Lemma 4.14.

As in the proof of Lemma 8.16, these points imply that

$$x_d \in \left(f^{|\boldsymbol{k}|}\right)^{\left[\frac{1}{p^e}\right]} = \boldsymbol{\tau}\left(R, \left(\frac{p - a + \sigma_p - 1}{p} + \frac{p-1}{p^2} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right).$$

By the symmetry of the argument, we conclude that $\mathfrak{m} \subseteq \boldsymbol{\tau}\left(R, \left(\frac{p-1}{p} + \cdots + \frac{p-1}{p^e}\right) \bullet f\right)$.

$\square$

CHAPTER 9

F-pure thresholds of binomial hypersurfaces

This chapter is dedicated to the computation of F-pure threshold for *binomial* hypersurfaces. Recall that a binomial, by definition, a K-linear combination of two distinct monomials. For *most* binomials f, Theorem 9.14 provides a formula for $\mathbf{fpt}_{\mathfrak{m}}(f)$ in terms of the the characteristic and certain quantities determined by the geometry of $\boldsymbol{P}$. This formula is also the key step in Algorithm 9.16, an algorithm for computing the F-pure threshold of an *arbitrary* binomial hypersurface.

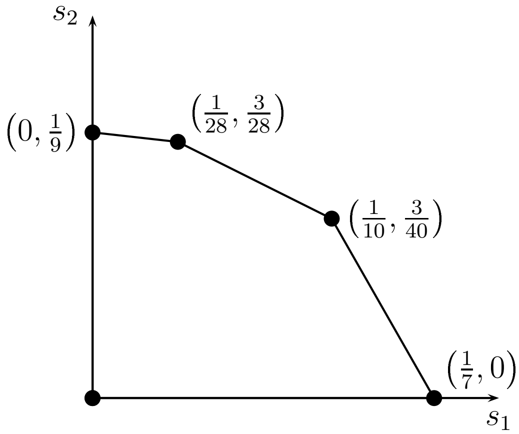
1 The polytope $\boldsymbol{P}$ associated to a binomial

Given any binomial f, we may use the methods of Chapter 6 to construct the polytope $\boldsymbol{P}$ associated to Support(f). As f is a binomial, we have that ${}^{\#}\,\text{Support}(f) = 2$, and thus $\boldsymbol{P} \subseteq [0,1]^2$. In this case, the polytope $\boldsymbol{P}$ is more easily studied. For the convenience of the reader, we restate Definition 6.3 in this simplified setting. We will use "$*$" to denote the standard *inner product* (or *dot product*) on $\mathbb{R}^2$: $\boldsymbol{s} * \boldsymbol{\sigma} = s_1\sigma_1 + s_1\sigma_2$.

Definition 9.1. Let $\boldsymbol{x}^{\boldsymbol{\nu}} \neq \boldsymbol{x}^{\boldsymbol{\omega}}$ denote distinct monomials in $K[x_1, \cdots, x_m]$ such that every variable appears in either $\boldsymbol{x}^{\boldsymbol{\nu}}$ or $\boldsymbol{x}^{\boldsymbol{\omega}}$. We define the *polytope associated to* $\{\, \boldsymbol{x}^{\boldsymbol{\nu}}, \boldsymbol{x}^{\boldsymbol{\omega}} \,\}$ as follows: $\boldsymbol{P} = \left\{\, \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : (\nu_i, \omega_i) * \boldsymbol{s} \leqslant 1 \text{ for all } i = 1, \cdots, m \,\right\}$.

Remark 9.2. As $\boldsymbol{\nu}, \boldsymbol{\omega} \in \mathbb{N}^m$, we have that $\boldsymbol{P} \subseteq [0,1]^2$.

Example 9.3. In Figure 9.3, we consider the polytope $\boldsymbol{P}$ for $\boldsymbol{\nu} = (1,4,7)$ and $\boldsymbol{\omega} = (9,8,4)$. Observe that adding the condition $2s_1 + 3s_2 \leqslant 1$ would not change $\boldsymbol{P}$, which motivates the Definition 9.4 below.



$$\boldsymbol{P} = \left\{ (s_1, s_2) \in \mathbb{R}^2_{\geqslant 0} : \begin{array}{r} s_1 + 9s_2 \leqslant 1 \\ 4s_1 + 8s_2 \leqslant 1 \\ 7s_1 + 4s_2 \leqslant 1 \end{array} \right\}$$

Figure 9.3.1: The rational polytope $\boldsymbol{P} \subseteq [0,1]^2$ associated to $\{ xy^4z^7, x^9y^8z^4 \}$.

Definition 9.4. We say that (ν_ℓ, ω_ℓ) is *active* on $\boldsymbol{P}$ if $\boldsymbol{P} \cap \{ \boldsymbol{s} : (\nu_i, \omega_i) * \boldsymbol{s} = 1 \} \neq \varnothing$. It follows that $\boldsymbol{P} = \{ \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : (\nu_i, \omega_i) * \boldsymbol{s} \leqslant 1 \text{ for all active } (\nu_i, \omega_i) \}$. We say that $\{\boldsymbol{x}^{\boldsymbol{\nu}}, \boldsymbol{x}^{\boldsymbol{\omega}}\}$ *minimally define* $\boldsymbol{P}$ if (ν_i, ω_i) is active on $\boldsymbol{P}$ for all i.

Recall that a point $\boldsymbol{\eta} \in \boldsymbol{P}$ is called a *maximal point of* $\boldsymbol{P}$ if $|\boldsymbol{\eta}| = \max\{|\boldsymbol{s}| : \boldsymbol{s} \in \boldsymbol{P}\}$.

Corollary 9.5. $\boldsymbol{P}$ has a unique maximal point $\iff \nu_i \neq \omega_i$ for all active (ν_i, ω_i).

Proof. By Lemma 6.6, it suffices to show that the set of vertices of $\boldsymbol{P}$ contains a unique with maximal coordinate sum if and only if $\nu_i \neq \omega_i$ for all active (ν_i, ω_i). However, it is fairly straightforward to verify that there exist two distinct vertices with maximal coordinate sum if and only if some bounding line segment of $\boldsymbol{P}$ has slope equal to -1. As the equations for the bounding line segments of $\boldsymbol{P}$ are given by the active (ν_i, ω_i), the claim follows. $\square$

Corollary 9.5 shows that $\boldsymbol{P}$ will have a unique maximal point for most choices of $\boldsymbol{\nu}$ and $\boldsymbol{\omega}$. Furthermore, if $\boldsymbol{P}$ does not have a unique maximal point, we may eliminate some variables from the monomials $\boldsymbol{x}^{\boldsymbol{\nu}}$ and $\boldsymbol{x}^{\boldsymbol{\omega}}$ to obtain monomials whose associated polytope does have a unique maximal point, as shown in the following remark.

Remark 9.6. Let g be a binomial over K with $\mathrm{Support}(g) = \{\boldsymbol{x}^{\boldsymbol{a}}, \boldsymbol{x}^{\boldsymbol{b}}\}$ for some $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}^m$. Let $\mu = \prod_{a_i = b_i} x_i^{a_i}$. Then $g = \mu \cdot h$ for some binomial h satisfying the condition that no variable appears with the same exponent in both of its supporting monomials. By Corollary 9.5, the polytope associated to $\mathrm{Support}(g)$ must contain a unique maximal point.

Definition 9.7. Suppose that $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} = (\eta_1, \eta_2) \in \boldsymbol{P}$. We define the subpolytopes ${}^{\star}\boldsymbol{P}$, $\boldsymbol{R}$, and $\boldsymbol{P}_{\star}$ of $\boldsymbol{P}$ as follows:

1. ${}^{\star}\boldsymbol{P} := \boldsymbol{P} \cap \{\, \boldsymbol{s} \in \mathbb{R}^2 : s_2 \geqslant \eta_2 \,\}$.
2. $\boldsymbol{R} := \{\, \boldsymbol{s} \in \mathbb{R}^2 : s_1 \leqslant \eta_1 \text{ and } s_2 \leqslant \eta_2 \,\}$.
3. $\boldsymbol{P}_{\star} := \boldsymbol{P} \cap \{\, \boldsymbol{s} \in \mathbb{R}^2 : s_1 \geqslant \eta_1 \,\}$.

Note that this gives a decomposition $\boldsymbol{P} = {}^{\star}\boldsymbol{P} \cup \boldsymbol{R} \cup \boldsymbol{P}_{\star}$.

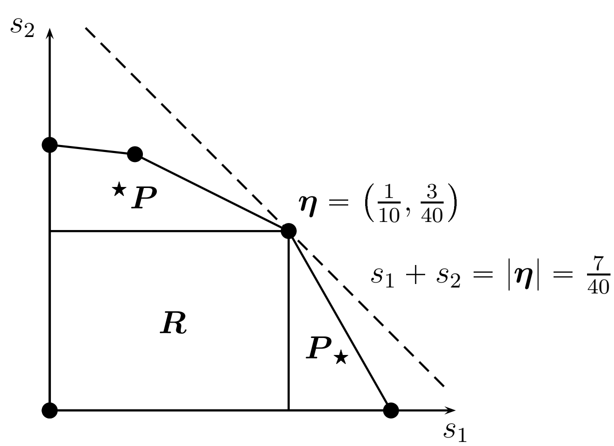


Figure 9.7.1: The decomposition of $\boldsymbol{P}$ from Example 9.3.

Figure 9.7.1 illustrates some important properties of the polytope decomposition $\boldsymbol{P} = {}^{\star}\boldsymbol{P} \cup \boldsymbol{R} \cup \boldsymbol{P}_{\star}$ which we now summarize. As $\boldsymbol{P}$ is defined by equations with non-negative integer coefficients, the bounding line segments of $\boldsymbol{P}$ that are not on the axes must have negative slope. Furthermore, the convexity of $\boldsymbol{P}$ shows that the slope of these bounding line segments must strictly decrease as s_1 increases.

If $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$, then the first time a line of the form $s_1 + s_2 = \alpha$ intersects $\boldsymbol{P}$ occurs when $\alpha = |\boldsymbol{\eta}|$, and this intersection consists precisely of the point $\boldsymbol{\eta}$. It follows that none of the bounding line segments have slope equal to negative one (for such a line segment would consist entirely of maximal points). Thus, the bounding line segments of $\boldsymbol{P}$ not on the axes have negative slope, and none of these slopes are equal to negative one. Furthermore, it is easy to verify that the line segments with slopes greater than negative one give the non-trivial bounding line segments for ${}^{\star}\boldsymbol{P}$. Similarly, the segments with slopes less than negative one give the non-trivial bounding line segments for $\boldsymbol{P}_{\star}$. We record these observations in Lemma 9.8 below.

Lemma 9.8. Suppose that $\boldsymbol{P}$ has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$. Then:

1. ${}^{\star}\boldsymbol{P} := \left\{ \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : s_2 \geqslant \eta_2 \text{ and } (\nu_i, \omega_i) * \boldsymbol{s} \leqslant 1 \text{ for active } (\nu_i, \omega_i) \text{ with } \nu_i < \omega_i \right\}$.
2. $\boldsymbol{P}_{\star} := \left\{ \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : s_1 \geqslant \eta_1 \text{ and } (\nu_i, \omega_i) * \boldsymbol{s} \leqslant 1 \text{ for active } (\nu_i, \omega_i) \text{ with } \nu_i > \omega_i \right\}$.

2 Some lemmas on the computation of $\mathbf{fpt}_{\mathfrak{m}}(f)$

In this subsection, we derive some lemmas which will simplify our computation of the F-pure threshold of a binomial ideal.

Lemma 9.9. Let $f = u_1 \boldsymbol{x}^{\boldsymbol{\nu}} + u_2 \boldsymbol{x}^{\boldsymbol{\omega}}$ be a binomial over K. Then there is no gathering of monomial terms in the expansion of f^N given by the binomial theorem: if $\boldsymbol{k}$ and

$\boldsymbol{\kappa} \in \mathbb{N}^2$ with $|\boldsymbol{k}| = |\boldsymbol{\kappa}| = N$, then $k_1\boldsymbol{\nu} + k_2\boldsymbol{\omega} = \kappa_1\boldsymbol{\nu} + \kappa_2\boldsymbol{\omega}$ if and only if $\boldsymbol{k} = \boldsymbol{\kappa}$. In particular, if $\boldsymbol{k} \in \mathbb{N}^2$ with $|\boldsymbol{k}| = N$, then $\boldsymbol{x}^{k_1\boldsymbol{\nu}+k_2\boldsymbol{\omega}} \in \operatorname{Support}(f^N)$ if and only if $\binom{N}{\boldsymbol{k}} \neq 0 \bmod p$.

Proof. By hypothesis, we have that $\boldsymbol{k}-\boldsymbol{\kappa}$ is in the kernel of the matrix $M = \begin{pmatrix} \boldsymbol{\nu} & \boldsymbol{\omega} \\ 1 & 1 \end{pmatrix}$. However, $\ker M \neq 0 \iff \boldsymbol{\nu} = \boldsymbol{\omega}$, which is impossible as f is a binomial over K. $\square$

Lemma 9.10. Let $f \in K[x_1, \cdots, x_m]$ be a binomial with $\operatorname{Support}(f) = \{\boldsymbol{x}^{\boldsymbol{\nu}}, \boldsymbol{x}^{\boldsymbol{\omega}}\}$, and let $\boldsymbol{P}$ denote the polytope associated to $\operatorname{Support}(f)$. Reorder the variables and fix $\ell \geqslant 1$ so that (ν_i, ω_i) is active on $\boldsymbol{P}$ precisely when $1 \leqslant i \leqslant \ell$. Let $g \in K[y_1, \cdots, y_\ell]$ denote the image of f under the map $K[x_1, \cdots, x_m] \to K[y_1, \cdots, y_\ell]$ given by $x_i \mapsto y_i$ for $1 \leqslant i \leqslant \ell$ and $x_i \mapsto 1$ otherwise. Then $f^N \in (x_1, \cdots, x_m)^{[p^e]}$ if and only if $g^N \in (y_1, \cdots, y_\ell)^{[p^e]}$.

Proof. Write $f = u_1\boldsymbol{x}^{\boldsymbol{\nu}} + u_2\boldsymbol{x}^{\boldsymbol{\omega}}$ for some u_1 and $u_2 \in K^*$. Let $\boldsymbol{\nu'} = (\nu_1, \cdots, \nu_\ell)$ and $\boldsymbol{\omega'} = (\omega_1, \cdots, \omega_\ell)$, so that $g = u_1\boldsymbol{y}^{\boldsymbol{\nu'}} + u_2\boldsymbol{y}^{\boldsymbol{\omega'}}$. We may write

$$f^N = \sum_{|\boldsymbol{k}|=N} \binom{N}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} \boldsymbol{x}^{k_1\boldsymbol{\nu}+k_2\boldsymbol{\omega}} \text{ and } g^N = \sum_{|\boldsymbol{k}|=N} \binom{N}{\boldsymbol{k}} \boldsymbol{u}^{\boldsymbol{k}} \boldsymbol{y}^{k_1\boldsymbol{\nu'}+k_2\boldsymbol{\omega'}}. \tag{9.10.1}$$

By Lemma 9.9, (9.10.1) gives the unique expression for f^N and g^N as a K-linear combination of distinct monomials in their respective polynomial rings. Thus, $f^N \in (x_1, \cdots, x_m)^{[p^e]}$ if and only if there exists $\boldsymbol{k} \in \mathbb{N}^2$ with $\binom{N}{\boldsymbol{k}} \neq 0 \bmod p$ and $\frac{1}{p^e-1} \cdot \boldsymbol{k} * (\nu_i, \omega_i) \leqslant 1$ for all $1 \leqslant i \leqslant m$. Similarly, $g^N \in (y_1, \cdots, y_\ell)^{[p^e]}$ if and only if there exists $\boldsymbol{k} \in \mathbb{N}^2$ with $\binom{N}{\boldsymbol{k}} \neq 0 \bmod p$ and $\frac{1}{p^e-1} \cdot \boldsymbol{k} * (\nu_i, \omega_i) \leqslant 1$ for all $1 \leqslant i \leqslant \ell$. However, if $\boldsymbol{s} := \frac{1}{p^e-1} \cdot \boldsymbol{k}$, then $\boldsymbol{s} * (\nu_i, \omega_i) \leqslant 1$ for all $1 \leqslant i \leqslant m$ if and only if $\boldsymbol{s} \in \boldsymbol{P}$, which (by our choice of ℓ) holds if and only if $\boldsymbol{s} * (\nu_i, \omega_i) \leqslant 1$ for all $1 \leqslant i \leqslant \ell$. $\square$

Corollary 9.11. Let $f \in K[x_1, \cdots, x_m]$ and $g \in K[y_1, \cdots, y_\ell]$ be as in Lemma 9.10, and let $\mathfrak{m} = (x_1, \cdots, x_m)$ and $\mathfrak{n} = (y_1, \cdots, y_\ell)$. Then, the polytopes associated to Support(f) and Support(g) are equal and is minimally defined by Support(g). Furthermore, $\mathbf{fpt}_{\mathfrak{m}}(f) = \mathbf{fpt}_{\mathfrak{n}}(g)$.

As $\boldsymbol{P} \subseteq [0,1]^2$, it follows that the maximal points of $\boldsymbol{P}$ should have small coordinate sum. Whenever $\boldsymbol{P}$ contains a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$ and is minimally defined by Support(f), the following lemma will allow us to focus on the typical case in which $|\boldsymbol{\eta}| \leqslant 1$ when computing the F-pure threshold of f.

Lemma 9.12. Suppose $\boldsymbol{P}$ is minimally defined by Support$(f) = \{ \boldsymbol{x}^{\boldsymbol{\nu}}, \boldsymbol{x}^{\boldsymbol{\omega}} \}$ and has a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$. If $|\boldsymbol{\eta}| > 1$, then after renaming variables, we have that $f = y + z^n$ for some $n \geqslant 1$.

Proof. As $P \subseteq [0,1]^2$, it follows that both η_1 and η_2 are non-zero. Thus, there exist bounding line segments of ${}^{\star}\boldsymbol{P}$ and $\boldsymbol{P}_{\star}$ that contain $\boldsymbol{\eta}$, so we may fix i such that $\nu_i < \omega_i$ and $j \neq i$ with $\nu_j > \omega_j$ such that

$$(\nu_i, \omega_i) * \boldsymbol{\eta} = (\nu_j, \omega_j) * \boldsymbol{\eta} = 1. \tag{9.12.1}$$

As $\nu_i < \omega_i$, it follows that $\nu_i < \nu_i \cdot |\boldsymbol{\eta}| = (\nu_i, \nu_i) * \boldsymbol{\eta} < (\nu_i, \omega_i) * \boldsymbol{\eta} = 1$, and as $\nu_i \in \mathbb{N}$, it follows that $\nu_i = 0$. Similarly, the equation $(\nu_j, \omega_j) * \boldsymbol{\eta} = 1$ implies that $\omega_j = 0$, and substituting the values $\nu_i = \omega_j = 0$ into (9.12.1) shows that $\boldsymbol{\eta} = \left(\frac{1}{\nu_j}, \frac{1}{\omega_i} \right)$. We also see that the bounding line segments of ${}^{\star}\boldsymbol{P}$ and $\boldsymbol{P}_{\star}$ that intersect $\boldsymbol{\eta}$ are given by $(\nu_j, 0)$ (which corresponds to a horizontal line) and $(0, \omega_i)$ (which corresponds to a vertical line). This shows that $\boldsymbol{P} = \left\{ \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : s_1 \leqslant \frac{1}{\nu_j} \text{ and } s_2 \leqslant \frac{1}{\omega_i} \right\}$, and as we are assuming that Support(f) minimally defines $\boldsymbol{P}$, we have that $f = x_j^{\nu_j} + x_i^{\omega_i}$. Finally, the equality $\frac{1}{\nu_j} + \frac{1}{\omega_i} = |\boldsymbol{\eta}| > 1$ shows that either ν_j or ω_i must equal 1. $\square$

Lemma 9.13. Let K be an F-finite field, and let $\mathfrak{m}$ and $\mathfrak{n}$ denote the homogeneous maximal ideals of the polynomial rings $K[\underline{\boldsymbol{x}}]$ and $K[\underline{\boldsymbol{y}}]$, respectively. If $f \in \mathfrak{m}$ and

$g \in \mathfrak{n}$, then $\mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) = \min\{\mathbf{fpt}_{\mathfrak{m}}(f), \mathbf{fpt}_{\mathfrak{n}}(g)\}$, where $\mathfrak{m}+\mathfrak{n}$ is the homogeneous maximal ideal of the polynomial ring $K[\underline{\boldsymbol{x}}, \underline{\boldsymbol{y}}]$.

Proof. Suppose $\lambda_1 := \mathbf{fpt}_{\mathfrak{m}}(f) \leqslant \lambda_2 := \mathbf{fpt}_{\mathfrak{n}}(g)$. We now show that $\mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) = \lambda_1$. By Lemma 4.6, $\langle \lambda_1 \rangle_e \leqslant \langle \lambda_2 \rangle_e$ for all e, and so Lemma 7.7 implies that $f^{p^e \langle \lambda_1 \rangle_e} \notin \mathfrak{m}^{[p^e]}$ and $g^{p^e \langle \lambda_2 \rangle_e} \notin \mathfrak{n}^{[p^e]}$. Fix monomials $\mu_1 \in \operatorname{Support}(f^{p^e \langle \lambda_1 \rangle_e})$ and $\mu_2 \in \operatorname{Support}(g^{p^e \langle \lambda_2 \rangle_e})$ such that $\mu_1 \notin \mathfrak{m}^{[p^e]}$ and $\mu_2 \notin \mathfrak{n}^{[p^e]}$. As μ_1 and μ_2 are in different sets of variables, $\mu_1\mu_2 \in \operatorname{Support}((fg)^{p^e \langle \lambda_1 \rangle_e})$ but not in $\mathfrak{m}^{[p^e]} + \mathfrak{n}^{[p^e]} = (\mathfrak{m}+\mathfrak{n})^{[p^e]}$, and so $(fg)^{p^e \langle \lambda_1 \rangle_e} \notin (\mathfrak{m}+\mathfrak{n})^{[p^e]}$. By Lemma 7.7, $f^{p^e \langle \lambda_1 \rangle_e + 1} \in \mathfrak{m}^{[p^e]}$, and it follows that $(fg)^{p^e \langle \lambda_1 \rangle_e + 1} \in \mathfrak{m}^{[p^e]} \subseteq (\mathfrak{m}+\mathfrak{n})^{[p^e]}$. Thus, $p^e \langle \lambda_1 \rangle_e = \max\{a : (fg)^a \notin (\mathfrak{m}+\mathfrak{n})^{[p^e]}\} = p^e \langle \mathbf{fpt}_{\mathfrak{m}+\mathfrak{n}}(fg) \rangle_e$ (see Lemma 7.7) for all $e \geqslant 1$. $\square$

3 The F-pure threshold of a binomial hypersurface

Theorem 9.14. Suppose $\boldsymbol{P}$ is minimally defined by $\operatorname{Support}(f)$ with unique maximal point $\boldsymbol{\eta}$ satisfying $|\boldsymbol{\eta}| \leqslant 1$. Let $L = \sup\{N : c_e(\eta_1) + c_e(\eta_2) \leqslant p-1 \; \forall \; 0 \leqslant e \leqslant N\}$.

1. If $L = \infty$, then $\mathbf{fpt}_{\mathfrak{m}}(f) = |\boldsymbol{\eta}|$.

For the remainder, we will assume that $L < \infty$ and both η_1 and η_2 are non-zero. Let $\ell := \max\{e \leqslant L : c_e(\eta_1) + c_e(\eta_2) \leqslant p-2\}$. (We will see that $1 \leqslant \ell \leqslant L$.)

2. If neither $\langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0\right)$ nor $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)$ is contained in the interior of $\boldsymbol{P}$, then $\mathbf{fpt}_{\mathfrak{m}}(f) = \langle |\boldsymbol{\eta}| \rangle_L$.

3. Otherwise, let $\varepsilon = \max\left\{ r : \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, r\right) \text{ or } \langle \boldsymbol{\eta} \rangle_\ell + \left(r, \frac{1}{p^\ell}\right) \text{ is in } \boldsymbol{P} \right\}$. Then
   (a) $0 < \varepsilon \leqslant [\![|\boldsymbol{\eta}|]\!]_L$, with $\varepsilon = [\![|\boldsymbol{\eta}|]\!]_L \iff$ either η_1 or η_2 is in $\frac{1}{p^\ell} \cdot \mathbb{N}$, and
   (b) $\mathbf{fpt}_{\mathfrak{m}}(f) = \langle |\boldsymbol{\eta}| \rangle_L + \varepsilon \leqslant |\boldsymbol{\eta}|$, with $\mathbf{fpt}_{\mathfrak{m}}(f) = |\boldsymbol{\eta}| \iff$ either η_1 or η_2 is in $\frac{1}{p^\ell} \cdot \mathbb{N}$.

1 An example

We present an example which we hope clarifies the statement of Theorem 9.14.

Example 9.15. Let $f = u_1 x^7 y^2 + u_2 x^5 y^6 \in K[x, y]$ with $u_1, u_2 \in K^*$, so that

$$P = \left\{ (s_1, s_2) \in \mathbb{R}^2_{\geqslant 0} : \begin{array}{c} 2s_1 + 6s_2 \leqslant 1 \\ 7s_1 + 5s_2 \leqslant 1 \end{array} \right\}.$$

Note that $\boldsymbol{P}$ is minimally defined by Support(f) and contains a unique maximal point $\boldsymbol{\eta} = \left(\frac{1}{32}, \frac{5}{32}\right)$ (see Figures 9.15.1 and 9.15.2), so that $\mathbf{fpt}_{\mathfrak{m}}(f)$ is computable via Theorem 9.14.

Theorem 9.14 states that for primes p with the property that the non-terminating base p expansions of $\frac{1}{32}$ and $\frac{5}{32}$ add without carrying, then $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{1}{32} + \frac{5}{32} = \frac{3}{16}$. As in Example 4.4, if $p \equiv 1 \bmod 32$, (for example, if $p = 97, 193, 257, 353$ or 449) then the expansions of $1/32$ and $5/32$ are constant, and thus add without carrying. However, there exist primes p such that $\mathbf{fpt}_{\mathfrak{m}}(f) = \frac{3}{16}$ with $p \not\equiv 1 \bmod 32$. For example, if $p = 47$, then $\frac{1}{32} = .\,\overline{1\ 22}$ (base 47) and $\frac{5}{32} = .\,\overline{7\ 16}$ (base 47).

We will now compute $\mathbf{fpt}_{\mathfrak{m}}(f)$ when $p = 43$. As

(9.15.1)
$$\frac{1}{32} = .\,\overline{1\ 14\ 33\ 25\ 22\ 36\ 12\ 4} \text{ (base 43) and } \frac{5}{32} = .\,\overline{6\ 30\ 38\ 41\ 28\ 9\ 17\ 20} \text{ (base 43)},$$

we see that carrying occurs at the second spot, and that $\ell = L = 1$. By (9.15.1), we see that $\langle \boldsymbol{\eta} \rangle_1 = \left(\frac{1}{43}, \frac{6}{43}\right)$, and Figure 9.15.1 shows that

$$\left\{ \langle \boldsymbol{\eta} \rangle_1 + \left(\frac{1}{43}, 0\right), \langle \boldsymbol{\eta} \rangle_1 + \left(0, \frac{1}{43}\right) \right\} \cap \boldsymbol{P} = \varnothing.$$

By Theorem 9.14, we conclude that

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \langle |\boldsymbol{\eta}| \rangle_1 = \left\langle \frac{3}{16} \right\rangle_1 = \frac{8}{43} \text{ when } p = 43.$$

We now compute $\mathbf{fpt}_{\mathfrak{m}}(f)$ when $p = 37$. As

(9.15.2)
$$\frac{1}{32} = .\overline{1\ 5\ 28\ 33\ 19\ 24\ 10\ 15}\ (\text{base}\, 37) \text{ and } \frac{5}{32} = .\overline{5\ 28\ 33\ 19\ 24\ 10\ 15\ 1}\ (\text{base}\, 37),$$

we see that the first carry occurs at the third spot, and that $\ell = L = 2$. We also see from (9.15.2) that $\langle \boldsymbol{\eta} \rangle_2 = \left(\frac{1}{37} + \frac{5}{37^2}, \frac{5}{37} + \frac{28}{37^2}\right)$, and Figure 9.15.2 shows that

$$\left\{ \langle \boldsymbol{\eta} \rangle_2 + \left(\frac{1}{37^2}, 0\right), \langle \boldsymbol{\eta} \rangle_2 + \left(0, \frac{1}{37^2}\right) \right\} \subseteq \text{Interior}\, \boldsymbol{P}.$$

From Figure 9.15.2, we also see that $\varepsilon = \max\left\{ r : \langle \boldsymbol{\eta} \rangle_2 + \left(\frac{1}{37^2}, r\right) \in \boldsymbol{P} \right\}$. Moreover, Figure 9.15.2 shows that the point $\langle \boldsymbol{\eta} \rangle_2 + \left(\frac{1}{37^2}, \varepsilon\right)$ lies on the hyperplane $7s_1 + 5s_2 = 1$, and an easy calculation shows that $\varepsilon = \frac{3}{6845} = .0\ 0\ \overline{22\ 7\ 14\ 29}\ (\text{base}\, 37)$. Thus, Theorem 9.14 shows that

$$\mathbf{fpt}_{\mathfrak{m}}(f) = \langle |\boldsymbol{\eta}| \rangle_2 + \varepsilon = \left\langle \frac{3}{16} \right\rangle_2 + \varepsilon = .6\ 34\ \overline{22\ 7\ 14\ 29}\ (\text{base}\, 37) \text{ when } p = 37.$$

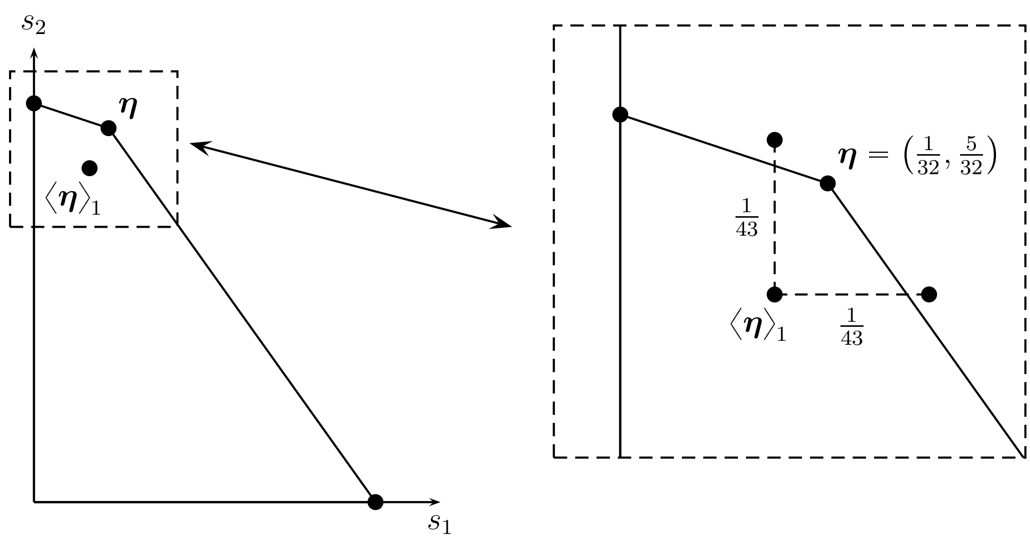


Figure 9.15.1: $p = 43$, $\ell = L = 1$, $\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{3}{16} \right\rangle_1 = .8$ (base 43).

2 An algorithm

We now show how Theorem 9.14 may be used to construct an algorithm for computing the F-pure threshold at $\mathfrak{m}$ of *any* binomial over K.

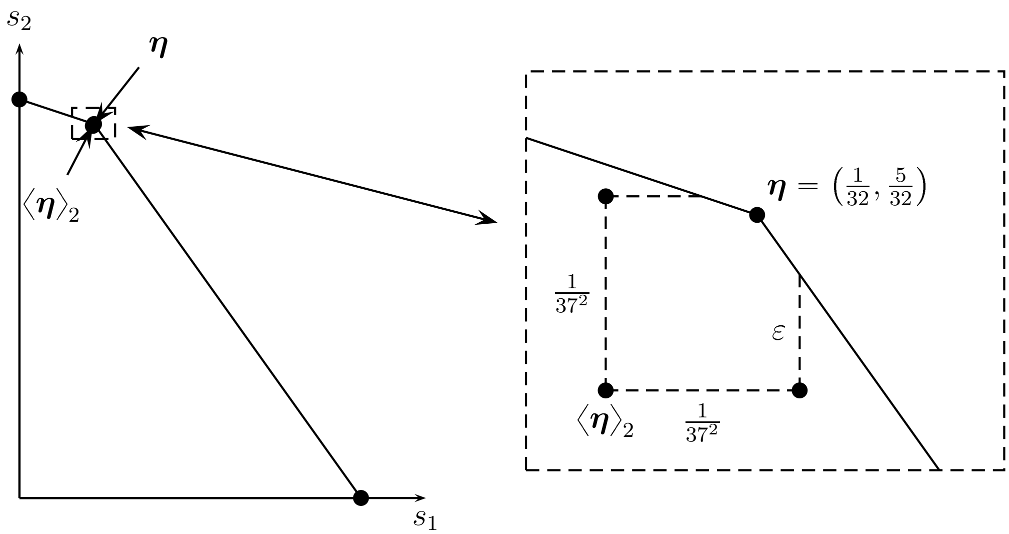


Figure 9.15.2: $p = 37$, $\ell = L = 2$, $\mathbf{fpt}_{\mathfrak{m}}(f) = \left\langle \frac{3}{16} \right\rangle_2 + \varepsilon = .6\ 34\ \overline{22\ 7\ 14\ 29}$ (base 37).

Algorithm 9.16. Let $g \in K[x_1, \cdots, x_m]$ be a binomial.

Step 1: Factor g as $g = \mu \cdot h$ for some monomial μ and binomial h so that no variable appearing in μ appears in h, and so that no variable appears with the same exponent in both supporting monomials of h. As in Remark 9.6, the polytope $\boldsymbol{P}$ associated to Support(h) will contain a unique maximal point $\boldsymbol{\eta} \in \boldsymbol{P}$.

Step 2: Reorder the variables so that $\mu = x_1^{a_1} \cdots x_d^{a_d}$. By Lemma 9.13,

$$\mathbf{fpt}_{\mathfrak{m}}(g) = \mathbf{fpt}_{\mathfrak{m}}(\mu \cdot h) = \min \left\{ \mathbf{fpt}_{(x_1, \cdots, x_d)}(\mu), \mathbf{fpt}_{(x_{d+1}, \cdots, x_m)}(h) \right\},$$

and it is an easy exercise to verify that $\mathbf{fpt}_{(x_1, \cdots, x_d)}(\mu) = \min \left\{ \frac{1}{a_1}, \cdots, \frac{1}{a_d} \right\}$.

Step 3: We must now compute the F-pure threshold of h. As in Corollary 9.11, eliminate inactive variables from h to obtain a polynomial f with the property that Support(f) minimally defines $\boldsymbol{P}$.

Step 4: If $|\boldsymbol{\eta}| > 1$, it follows from Lemma 9.12 that $f = y + z^n$ for some $n \geqslant 1$. In this case, it is an easy consequence of Theorem 8.1 that $\mathbf{fpt}_{(y,z)}(f) = 1$.

Step 5: If $|\boldsymbol{\eta}| \leqslant 1$, we may compute the F-pure threshold of f using Theorem 9.14.

3 Proof of Theorem 9.14

This subsection is dedicated to the proof of Theorem 9.14. We will rely heavily on the following two technical lemmas whose proofs will be postponed until the following subsection.

Lemma 9.17. Let $\ell \leqslant L < \infty$ be as in Theorem 9.14. Then

1. $1 \leqslant \ell \leqslant L$ and
2. $\langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 + \eta_2 \rangle_L$.
3. Furthermore,

$$f^{p^L \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^L]} \iff f^{p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1} \notin \mathfrak{m}^{[p^\ell]}$$
$$\iff \left\{ \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0 \right), \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell} \right) \right\} \cap \text{Interior}\, \boldsymbol{P} \neq \varnothing.$$

Lemma 9.18. If $\varepsilon := \sup \left\{ r : \left\{ \langle \boldsymbol{\eta} \rangle_\ell + \left(r, \frac{1}{p^\ell} \right), \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, r \right) \right\} \cap \boldsymbol{P} \neq \varnothing \right\} > 0$, then

1. $0 < \varepsilon \leqslant [\![|\boldsymbol{\eta}|]\!]_L$, with $\varepsilon = [\![|\boldsymbol{\eta}|]\!]_L \iff$ either η_1 or η_2 is in $\frac{1}{p^\ell} \cdot \mathbb{N}$,
2. $f^{p^e \left(\langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e \right)} \notin \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$, and
3. $f^{p^e \left(\langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e + \frac{1}{p^e} \right)} \in \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$.

Proof of Theorem 9.14. If $L = \infty$, it follows from Theorem 7.6 that $\mathbf{fpt}_{\mathfrak{m}}(f) = |\boldsymbol{\eta}|$. For the remainder of this proof, we will assume that $L < \infty$ and that both η_1 and η_2 are non-zero (since if one were zero, they would add without carrying). The assertion that $1 \leqslant \ell \leqslant L$ is the content of the first point of Lemma 9.17. It follows from Theorem 7.6 that

$$\mathbf{fpt}_{\mathfrak{m}}(f) \geqslant \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 + \eta_2 \rangle_L \,, \tag{9.18.1}$$

where the last equation in (9.18.1) follows from Lemma 9.17.

We will first assume that neither $\langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0\right)$ nor $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)$ is in Interior $\boldsymbol{P}$. By means of contradiction, suppose that the inequality in (9.18.1) is strict. It then follows from Lemma 4.6 that $\langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_L \geqslant \langle \eta_1 + \eta_2 \rangle_L$, and Lemma 7.7 then shows that

$$f^{p^L \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^L]}.$$

However, this contradicts Lemma 9.17, and thus we have equality in (9.18.1).

It remains to prove the third point. The assertions regarding $\varepsilon > 0$ is the content of the first point of Lemma 9.18, which also shows that

(9.18.2)

$$p^e \left(\langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e \right) = \max \left\{ N : f^N \notin \mathfrak{m}^{[p^e]} \right\} = p^e \langle \mathbf{fpt}_{\mathfrak{m}}(f) \rangle_e \text{ for all } e \geqslant L,$$

where the last equality holds by Lemma 4.6. Finally, dividing (9.18.2) by p^e and taking the limit as $e \to \infty$ shows that $\mathbf{fpt}_{\mathfrak{m}}(f) = \langle \eta_1 + \eta_2 \rangle_L + \varepsilon$. □

4 Proof of the main technical lemmas

In this section we prove the two lemmas used in the proof of Theorem 9.14.

Lemma 9.17. Let $\ell \leqslant L < \infty$ be as in Theorem 9.14. Then

1. $1 \leqslant \ell \leqslant L$ and
2. $\langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 + \eta_2 \rangle_L$.
3. Furthermore,

$$\begin{aligned} f^{p^L \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^L]} &\iff f^{p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1} \notin \mathfrak{m}^{[p^\ell]} \\ &\iff \left\{ \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0 \right), \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell} \right) \right\} \cap \text{Interior } \boldsymbol{P} \neq \varnothing. \end{aligned}$$

Proof. We will first show that $L \geqslant 1$. If $L = 0$, then by definition we have that $c_1(\eta_1) + c_1(\eta_2) \geqslant p$. This implies that $\eta_1 + \eta_2 > \langle \eta_1 \rangle_1 + \langle \eta_2 \rangle_1 = \frac{c_1(\eta_1)+c_1(\eta_2)}{p} \geqslant 1$, contradicting the assumption $\eta_1 + \eta_2 \leqslant 1$. We have that $\ell \leqslant L$ by definition, so it remains to show that that $\ell \neq 0$. If $\ell = 0$, then by definition we have that $c_e(\eta_1) + c_e(\eta_2) = p - 1$ for $1 \leqslant e \leqslant L$, from which it follows that $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L = \sum_{e=1}^{L} \frac{p-1}{p} = \frac{p^L - 1}{p^L}$. By definition of L, we have that $c_{L+1}(\eta_1) + c_{L+1}(\eta_2) \geqslant p$, and so

$$\eta_1 + \eta_2 > \langle \eta_1 \rangle_{L+1} + \langle \eta_2 \rangle_{L+1} = \langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{c_{L+1}(\eta_1) + c_{L+1}(\eta_2)}{p^{L+1}} \geqslant \frac{p^L - 1}{p^L} + \frac{1}{p^L} = 1,$$

which again contradicts the assumption that $\eta_1 + \eta_2 \leqslant 1$.

To prove the second point, we may assume that $\ell < L$. By definition of ℓ and L, $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \sum_{e=\ell+1}^{L} \frac{p-1}{p^e}$. From this, we may conclude that

$$\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L + \frac{1}{p^L} = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \sum_{e=\ell+1}^{L} \frac{p-1}{p^e} + \frac{1}{p^L} = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}.$$

The assertion that $\langle \eta_1 \rangle_L + \langle \eta_2 \rangle_L = \langle \eta_1 + \eta_2 \rangle_L$ follows from Lemma 4.17.

We now prove the last point. As $\langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} = \langle \eta_1 + \eta_2 \rangle_L$, we have that

$$f^{p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1} = f^{p^\ell \left(\langle \eta \rangle_\ell + \langle \eta \rangle_\ell + \frac{1}{p^\ell} \right)} = f^{p^\ell \langle \eta_1 + \eta_2 \rangle_L}.$$

From this, we see that

$$f^{p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1} = f^{p^\ell \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^\ell]} \iff \left(f^{p^\ell \langle \eta_1 + \eta_2 \rangle_L} \right)^{p^{L-\ell}} = f^{p^L \langle \eta_1 + \eta_2 \rangle_L} \notin \mathfrak{m}^{[p^L]}.$$

We now prove the remaining equivalence. We will begin by supposing that

$$\left\{ \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0 \right), \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell} \right) \right\} \cap \operatorname{Interior} \boldsymbol{P} \neq \varnothing.$$

Without loss of generality, we assume that $\boldsymbol{\sigma} := \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0 \right) = \left(\langle \eta_1 \rangle_\ell + \frac{1}{p^\ell}, \langle \eta_2 \rangle_\ell \right)$ is in the interior of $\boldsymbol{P}$. We gather some important properties of $\boldsymbol{\sigma}$ below:

1. We have that $\boldsymbol{\sigma} \in \frac{1}{p^\ell} \cdot \mathbb{N}^2$, and that $|\boldsymbol{\sigma}| = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$.

2. By definition of $\ell \leqslant L$, we have that $c_e(\eta_1) + c_e(\eta_2) \leqslant p - 1$ for $e \leqslant \ell$, with this inequality strict for $e = \ell$. This shows that the coordinates of $p^\ell \boldsymbol{\sigma} \in \mathbb{N}^2$ add without carrying, and by Lemma 4.14, we have that

$$\binom{p^\ell |\boldsymbol{\sigma}|}{p^\ell \boldsymbol{\sigma}} \neq 0 \bmod p.$$

3. Finally, as $\boldsymbol{\sigma} \in \operatorname{Interior} \boldsymbol{P}$, it follows that $p^\ell \sigma_1 \nu_i + p^\ell \sigma_2 \omega_i = p^\ell (\nu_i, \omega_i) * \boldsymbol{\sigma} < p^\ell$ for all $1 \leqslant i \leqslant m$, so that $\boldsymbol{x}^{p^\ell \sigma_1 \boldsymbol{\nu} + p^\ell \sigma_2 \boldsymbol{\omega}} \notin \mathfrak{m}^{[p^e]}$.

The first two points above, combined with Lemma 9.9, show that $\boldsymbol{x}^{p^\ell \sigma_1 \boldsymbol{\nu} + p^\ell \sigma_2 \boldsymbol{\omega}}$ is contained in Support $\left(f^{p^\ell \langle \nu_1 \rangle_\ell + p^\ell \langle \nu_2 \rangle_\ell + 1}\right)$, while the last point shows that $\boldsymbol{x}^{p^\ell \sigma_1 \boldsymbol{\nu} + p^\ell \sigma_2 \boldsymbol{\omega}} \notin \mathfrak{m}^{[p^\ell]}$. From this we may conclude that $f^{p^\ell \langle \nu_1 \rangle_\ell + p^\ell \langle \nu_2 \rangle_\ell + 1} \notin \mathfrak{m}^{[p^\ell]}$.

To conclude the proof, suppose that $f^{p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1} \notin \mathfrak{m}^{[p^\ell]}$. Our aim now is to show that either $\langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, 0\right)$ or $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)$ is in the interior of $\boldsymbol{P}$. By assumption, there exists an element $\boldsymbol{k} \in \mathbb{N}^2$ with $|\boldsymbol{k}| = p^\ell \langle \eta_1 \rangle_\ell + p^\ell \langle \eta_2 \rangle_\ell + 1$ such that $\boldsymbol{x}^{k_1 \boldsymbol{\nu} + k_2 \boldsymbol{\omega}}$ is contained in Support(f) but *not* in $\mathfrak{m}^{[p^\ell]}$. Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2) := \frac{1}{p^\ell} \cdot \boldsymbol{k}$. We summarize some important properties of $\boldsymbol{\alpha}$:

4. $\boldsymbol{\alpha} \in \frac{1}{p^\ell} \cdot \mathbb{N}$ and $|\boldsymbol{\alpha}| = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$.

5. As $\boldsymbol{x}^{k_1 \boldsymbol{\nu} + k_2 \boldsymbol{\omega}} \notin \mathfrak{m}^{[p^\ell]}$, every entry of $k_1 \boldsymbol{\nu} + k_2 \boldsymbol{\omega}$ is strictly less than p^ℓ, which shows that $\boldsymbol{\alpha} \in \operatorname{Interior} \boldsymbol{P}$.

By (4) above, it is not possible that both $\alpha_1 \leqslant \langle \eta_1 \rangle_\ell$ and $\alpha_2 \leqslant \langle \eta_2 \rangle_\ell$. Without loss of generality, we will assume that $\alpha_2 \geqslant \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$, and substituting this into (4) forces the inequality $\alpha_1 \leqslant \langle \eta_1 \rangle_\ell$. By (4) again, we see there exists $N \in \mathbb{N}$ such that

6. $\boldsymbol{\alpha} + \left(\frac{N}{p^\ell}, -\frac{N}{p^\ell}\right) = \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)$.

We now show that $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) \in \text{Interior}\, P$. By Lemma 4.6, $\langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} \geqslant \eta_2$, so that

$$\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) \in \boldsymbol{P} \iff \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) \in {}^\star\boldsymbol{P}. \tag{9.18.3}$$

By (9.18.3), to show that $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) \in \text{Interior}\, \boldsymbol{P}$, it suffices to show that

(9.18.4)
$$\begin{aligned}\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) &\in \text{Interior}\, \boldsymbol{P} \cap {}^\star\boldsymbol{P} \\ &= \left\{\, \boldsymbol{s} \in \mathbb{R}^2_{\geqslant 0} : s_2 \geqslant \boldsymbol{\eta}_2 \text{ and } (\nu_i, \omega_i) * \boldsymbol{s} < 1 \;\forall\; (\nu_i, \omega_i) \text{ with } \nu_i < \omega_i \,\right\}.\end{aligned}$$

In (9.18.4), we have used the assumption that Support(f) minimally defines $\boldsymbol{P}$, along with the description of ${}^\star\boldsymbol{P}$ given in Lemma 9.8. By (6) above,

$$\begin{aligned}(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)\right) &= (\nu_i, \omega_i) * \left(\boldsymbol{\alpha} + \left(\frac{N}{p^\ell}, -\frac{N}{p^\ell}\right)\right) \\ &= (\nu_i, \omega_i) * \boldsymbol{\alpha} + (\nu_i, \omega_i) * \left(\frac{N}{p^\ell}, -\frac{N}{p^\ell}\right) \\ &= (\nu_i, \omega_i) * \boldsymbol{\alpha} + \frac{N}{p^\ell}\,(\nu_i - \omega_i)\,.\end{aligned} \tag{9.18.5}$$

Finally, as $\boldsymbol{\alpha} \in \text{Interior}\, \boldsymbol{P}$, it follows from (9.18.5) that

$$(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right)\right) < (\nu_i, \omega_i) * \boldsymbol{\alpha} \; < 1 \text{ for all } \nu_i < \omega_i,$$

which shows that the desired containment from (9.18.4) holds. $\square$

Lemma 9.18. If $\varepsilon := \sup\left\{ r : \left\{ \langle \boldsymbol{\eta} \rangle_\ell + \left(r, \frac{1}{p^\ell}\right), \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, r\right) \right\} \cap \boldsymbol{P} \neq \varnothing \right\} \; > \; 0$, then

1. $0 < \varepsilon \leqslant [\![|\boldsymbol{\eta}|]\!]_L$, with $\varepsilon = [\![|\boldsymbol{\eta}|]\!]_L \iff$ either η_1 or η_2 is in $\frac{1}{p^\ell} \cdot \mathbb{N}$,
2. $f^{p^e\left(\langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e\right)} \notin \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$, and
3. $f^{p^e\left(\langle \eta_1 + \eta_2 \rangle_L + \langle \varepsilon \rangle_e + \frac{1}{p^e}\right)} \in \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$.

Proof. We prove the first assertion. Suppose that $\langle \boldsymbol{\eta} \rangle_\ell + \left(r, \frac{1}{p^\ell}\right)$ is in $\boldsymbol{P}$. It follows from the fact that $\boldsymbol{\eta}$ is a maximal point of $\boldsymbol{P}$ and Lemma 9.17 that

$$\langle |\boldsymbol{\eta}| \rangle_L + r = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} + r = \left| \langle \boldsymbol{\eta} \rangle_\ell + \left(r, \frac{1}{p^\ell}\right) \right| \leqslant |\boldsymbol{\eta}| = \langle |\boldsymbol{\eta}| \rangle_L + [\![|\boldsymbol{\eta}|]\!]_L ,$$

which shows that r, and consequentially ε, is less than or equal to $[\![|\boldsymbol{\eta}|]\!]_L$.

Suppose that $\varepsilon = [\![|\boldsymbol{\eta}|]\!]_L$ and that $\langle \boldsymbol{\eta} \rangle_\ell + \left([\![|\boldsymbol{\eta}|]\!]_L , \frac{1}{p^\ell}\right) \in \boldsymbol{P}$. Note that

$$\left| \langle \boldsymbol{\eta} \rangle_\ell + \left([\![|\boldsymbol{\eta}|]\!]_L , \frac{1}{p^\ell}\right) \right| = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell} + [\![\eta_1 + \eta_2]\!]_L = \langle \eta_1 + \eta_2 \rangle_L + [\![\eta_1 + \eta_2]\!]_L ,$$

which shows that $\langle \boldsymbol{\eta} \rangle_\ell + \left([\![|\boldsymbol{\eta}|]\!]_L , \frac{1}{p^\ell}\right)$ is a maximal point of $\boldsymbol{P}$. As $\boldsymbol{\eta}$ is the unique such point of $\boldsymbol{P}$, we have that $\boldsymbol{\eta} = \langle \boldsymbol{\eta} \rangle_\ell + \left([\![|\boldsymbol{\eta}|]\!]_L , \frac{1}{p^\ell}\right)$. This shows that $\eta_2 = \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$, which by Lemma 4.6 shows that $\eta_2 \in \frac{1}{p^\ell} \cdot \mathbb{N}$.

Conversely, without loss of generality, suppose that $\eta_1 \in \frac{1}{p^\ell} \cdot \mathbb{N}$, so that $[\![\eta_1]\!]_\ell = \frac{1}{p^\ell}$ by Lemma 4.6. We have already seen that $\varepsilon \leqslant [\![\eta_1 + \eta_2]\!]_L$, and to prove equality it suffices to show that $\langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, [\![\eta_1 + \eta_2]\!]_L\right) \in \boldsymbol{P}$. Observe that

$$\begin{aligned} \langle \eta_1 + \eta_2 \rangle_L + [\![\eta_1 + \eta_2]\!]_L &= \eta_1 + \eta_2 \\ &= \langle \eta_1 \rangle_\ell + [\![\eta_1]\!]_\ell + \langle \eta_2 \rangle_\ell + [\![\eta_2]\!]_\ell \\ &= \langle \eta_1 \rangle_\ell + \frac{1}{p^\ell} + \langle \eta_2 \rangle_\ell + [\![\eta_2]\!]_\ell \\ &= \langle \eta_1 + \eta_2 \rangle_L + [\![\eta_2]\!]_\ell . \end{aligned}$$

From this, we conclude that

$$[\![\eta_1 + \eta_2]\!]_L = [\![\eta_2]\!]_\ell . \tag{9.18.6}$$

Finally, from (9.18.6), we have that

$$\langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, [\![\eta_1 + \eta_2]\!]_L\right) = \langle \boldsymbol{\eta} \rangle_\ell + \left(\frac{1}{p^\ell}, [\![\eta_2]\!]_\ell\right) = \left(\langle \eta_1 \rangle_\ell + \frac{1}{p^\ell}, \langle \eta_2 \rangle_\ell + [\![\eta_2]\!]_L\right) = \boldsymbol{\eta} \in \boldsymbol{P}.$$

We will now show that $f^{p^{e+L}\left(\langle \eta_1+\eta_2\rangle_L+\langle\varepsilon\rangle_{e+L}\right)} \notin \mathfrak{m}^{[p^e]}$ for all $e \geqslant 1$. We know that either $\langle \boldsymbol{\eta}\rangle_\ell + \left(\frac{1}{p^\ell}, \varepsilon\right)$ or $\langle \boldsymbol{\eta}\rangle_\ell + \left(\varepsilon, \frac{1}{p^\ell}\right)$ is contained in $\boldsymbol{P}$, and without loss of generality we will assume that $\langle \boldsymbol{\eta}\rangle_\ell + \left(\frac{1}{p^\ell}, \varepsilon\right) \in \boldsymbol{P}$. Fix $e \geqslant L$, and consider the element

(9.18.7) $$\boldsymbol{\sigma} := \left(\langle \eta_1\rangle_\ell + \frac{1}{p^\ell}, \langle \eta_2\rangle_\ell + \langle \varepsilon\rangle_e\right) \in \frac{1}{p^e}\cdot \mathbb{N}.$$

We summarize some important properties of $\boldsymbol{\sigma}$ below:

1. $|\boldsymbol{\sigma}| = \langle \eta_1\rangle_\ell + \langle \eta_2\rangle_\ell + \frac{1}{p^\ell} + \langle\varepsilon\rangle_e = \langle \eta_1+\eta_2\rangle_L + \langle \varepsilon\rangle_e$.

2. By definition of $\ell \leqslant L$, the integers $p^\ell\left(\langle\eta_1\rangle_\ell + \frac{1}{p^\ell}\right)$ and $p^\ell\langle\eta_2\rangle_\ell$ add without carrying. As $e \geqslant L \geqslant \ell$, the integers $p^e\left(\langle\eta_1\rangle_\ell + \frac{1}{p^\ell}\right)$ and $p^e\langle\eta_2\rangle_\ell$ also add without carrying. Furthermore, the integers $p^e\left(\langle\eta_1\rangle_\ell + \frac{1}{p^\ell}\right)$ and $p^e\langle\eta_2\rangle_\ell$ are both greater than or equal to $p^{e-\ell}$, and hence also greater than or equal to p^{e-L}. By Lemma 9.17 we know that $\langle\varepsilon\rangle_e < \varepsilon \leqslant [\![\eta_1+\eta_2]\!]_L \leqslant \frac{1}{p^L}$, so that $p^e\langle\varepsilon\rangle_e < p^{e-L}$. It follows that the entries of $p^e\boldsymbol{\sigma}$ add without carrying, so that $\binom{p^e|\boldsymbol{\sigma}|}{p^e\boldsymbol{\sigma}} \neq 0 \bmod p$ by Lemma 4.14.

3. Finally, as $\langle\varepsilon\rangle_e < \varepsilon$, it follows that $\boldsymbol{\sigma} \in \operatorname{Interior} \boldsymbol{P}$.

By Lemma 9.9, the first two points above show that the monomial $\boldsymbol{x}^{p^e\sigma_1\cdot\boldsymbol{\nu}_1 + p^e\sigma_2\cdot\boldsymbol{\omega}}$ is contained in the support of $f^{p^e\left(\langle\eta_1+\eta_2\rangle_L+\langle\varepsilon\rangle_e\right)}$, while the third point shows that this same monomial is not contained in $\mathfrak{m}^{[p^e]}$. From this, we may conclude that $f^{p^e\left(\langle\eta_1+\eta_2\rangle_L+\langle\varepsilon\rangle_e\right)} \notin \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$.

We will now show that $f^{p^e\left(\langle\eta_1+\eta_2\rangle_L+\langle\varepsilon\rangle_e+\frac{1}{p^e}\right)} \in \mathfrak{m}^{[p^e]}$ for all $e \geqslant L$. As

$$f^{p^e\left(\langle\eta_1+\eta_2\rangle_L+\langle\varepsilon\rangle_e+\frac{1}{p^e}\right)} = \left(f^{p^L\langle\eta_1+\eta_2\rangle_L}\right)^{p^{e-L}} \cdot f^{p^e\langle\varepsilon\rangle_e+1},$$

it suffices to show that

(9.18.8)
$$\mu_1^{p^{e-L}}\cdot\mu_2 \in \mathfrak{m}^{[p^e]} \text{ for all } \mu_1 \in \operatorname{Support}\left(f^{p^L\langle\eta_1+\eta_2\rangle_L}\right) \text{ and } \mu_2 \in \operatorname{Support}\left(f^{p^e\langle\varepsilon\rangle_e+1}\right).$$

We may assume that $\mu_1 \notin \mathfrak{m}^{[p^L]}$ and $\mu_2 \notin \mathfrak{m}^{[p^e]}$, for otherwise the containment in (9.18.8) holds trivially. Thus, we may write $\mu_1 = \boldsymbol{x}^{a_1\boldsymbol{\nu}+a_2\boldsymbol{\omega}}$ and $\mu_2 = \boldsymbol{x}^{b_1\boldsymbol{\nu}+b_2\boldsymbol{\omega}}$ for some $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_1)$ in $\mathbb{N}^2$ with $|\boldsymbol{a}| = p^L \langle \eta_1 + \eta_2 \rangle_L$ and $|\boldsymbol{b}| = p^e \langle \varepsilon \rangle_e + 1$ such that every entry of $a_1\boldsymbol{\nu} + a_2\boldsymbol{\omega}$ is less than p^L and every entry of $b_1\boldsymbol{\nu} + b_2\boldsymbol{\omega}$ is less than p^e. We set $\boldsymbol{\alpha} := \frac{1}{p^L} \cdot \boldsymbol{a}$ and $\boldsymbol{\beta} := \frac{1}{p^e} \cdot \boldsymbol{b}$, and summarize some important properties of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ below:

4. $\boldsymbol{\alpha} \in \frac{1}{p^L} \cdot \mathbb{N}^2 \cap \operatorname{Interior} \boldsymbol{P}$, and $|\boldsymbol{\alpha}| = \langle \eta_1 + \eta_2 \rangle_L = \langle \eta_1 \rangle_\ell + \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$.
5. $\boldsymbol{\beta} \in \frac{1}{p^e} \cdot \mathbb{N}^2 \cap \operatorname{Interior} \boldsymbol{P}$, and $|\boldsymbol{\beta}| = \langle \varepsilon \rangle_e + \frac{1}{p^e}$.

Under this new notation, the condition appearing in (9.18.8) may be restated as

$$\boldsymbol{x}^{\left(p^{e-L}a_1+b_1\right)\boldsymbol{\nu}+\left(p^{e-L}a_2+b_2\right)\boldsymbol{\omega}} = \boldsymbol{x}^{p^e(\boldsymbol{\alpha}+\boldsymbol{\beta})} \in \mathfrak{m}^{[p^e]}.$$

From this, we see that to conclude the proof, it suffices to show that

$$\boldsymbol{\alpha} + \boldsymbol{\beta} \notin \operatorname{Interior} \boldsymbol{P}. \tag{9.18.9}$$

By (4) above, it is not possible that both $\alpha_1 \leqslant \langle \eta_1 \rangle_\ell$ and $\alpha_2 \leqslant \langle \eta_2 \rangle_\ell$. Without loss of generality, we will assume that $\alpha_2 \geqslant \langle \eta_2 \rangle_\ell + \frac{1}{p^\ell}$, and (4) above again shows that $\alpha_1 \leqslant \langle \eta_1 \rangle_\ell$. Thus, there exists $N \in \mathbb{N}$ such that

$$\boldsymbol{\alpha} + \left(\frac{N}{p^\ell}, -\frac{N}{p^\ell}\right) = \langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right).$$

For indices i with $\nu_i < \omega_i$, we have that

$$\begin{aligned}(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) + \boldsymbol{\beta}\right) &= (\nu_i, \omega_i) * \left(\boldsymbol{\alpha} + \left(\frac{N}{p^\ell}, -\frac{N}{p^\ell}\right) + \boldsymbol{\beta}\right) \\ &= (\nu_i, \omega_i) * (\boldsymbol{\alpha} + \boldsymbol{\beta}) + \frac{N}{p^\ell}(\nu_i - \omega_i) \\ &< (\nu_i, \omega_i) * (\boldsymbol{\alpha} + \boldsymbol{\beta}).\end{aligned} \tag{9.18.10}$$

By (9.18.10), to demonstrate (9.18.9), it suffices to show that

$$(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) + \boldsymbol{\beta} \right) \geqslant 1 \text{ for some } i \text{ with } \nu_i < \omega_i. \tag{9.18.11}$$

There exists an i with $\nu_i < \omega_i$ by our assumption that η_1 and η_2 are non-zero.

We will now conclude this proof by showing that the desired statement in (9.18.11) holds. As $\boldsymbol{\beta} \in \frac{1}{p^e} \cdot \mathbb{N}^2$, it follows that $\boldsymbol{\beta}$ must be contained the line segment determined by $(0, |\boldsymbol{\beta}|)$ and $(|\boldsymbol{\beta}|, 0)$, and thus must be of the form

$$\boldsymbol{\beta} = (|\boldsymbol{\beta}|, 0) + t\,(-|\boldsymbol{\beta}|, |\boldsymbol{\beta}|) \text{ for some } t \in [0,1]. \tag{9.18.12}$$

If $\nu_i < \omega_i$, then (9.18.12) shows that

(9.18.13)

$$\begin{aligned}
(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) + \boldsymbol{\beta} \right) &= (\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(|\boldsymbol{\beta}|, \frac{1}{p^\ell}\right) + t\,(-|\boldsymbol{\beta}|, |\boldsymbol{\beta}|) \right) \\
&= (\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(|\boldsymbol{\beta}|, \frac{1}{p^\ell}\right) \right) + t|\boldsymbol{\beta}|(\omega_i - \nu_i) \\
&> (\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(|\boldsymbol{\beta}|, \frac{1}{p^\ell}\right) \right)
\end{aligned}$$

Comparing (9.18.11) with (9.18.13), we see that it suffices to show that

$$(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(|\boldsymbol{\beta}|, \frac{1}{p^\ell}\right) \right) \geqslant 1 \text{ for some } i \text{ with } \nu_i < \omega_i.$$

However, as $|\boldsymbol{\beta}| = \langle \varepsilon \rangle_e + \frac{1}{p^\ell} \geqslant \varepsilon$, it suffices to show that

$$(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(\varepsilon, \frac{1}{p^\ell}\right) \right) \geqslant 1 \text{ for some } i \text{ with } \nu_i < \omega_i.$$

By definition, we have that $\langle \boldsymbol{\eta} \rangle_\ell + \left(0, \frac{1}{p^\ell}\right) \in {}^\star\boldsymbol{P}$, and it follows from the definition of ε that $\langle \boldsymbol{\eta} \rangle_\ell + \left(\varepsilon, \frac{1}{p^\ell}\right)$ is not contained in Interior ${}^\star\boldsymbol{P}$. By Lemma 9.8, this shows that

$$(\nu_i, \omega_i) * \left(\langle \boldsymbol{\eta} \rangle_\ell + \left(\varepsilon, \frac{1}{p^\ell}\right) \right) \geqslant 1 \text{ for some } i \text{ with } \nu_i < \omega_i,$$

which allows us to conclude the proof of the Lemma. $\square$

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