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Generalizations of the vector coherent state method

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ABSTRACT The introduction of a set of intrinsic coordinates to give an explicit construction of the intrinsic states of vector coherent state theory has greatly simplified earlier attempts to generalize this theory to include the construction of vector coherent state realizations of operators other than the group generators. The group $U(3) \supset U(2) \times U(1)$ is used as a prototype. The construction of irreducible tensor operators with specific shift properties is illustrated with a number of examples. These show how the Wigner calculus for a higher symmetry group can be expressed solely in terms of the recoupling coefficients of the core subgroup and the simple K-matrix elements of vector coherent state theory.

2.1 Introduction

In the past few years vector coherent state (VCS) theory [1]–[6] and its associated K-matrix technique [1,7,8] have been used to great advantage to evaluate explicit expressions for the matrix representations of higher rank Lie algebras. VCS theory can be applied whenever the generators of an algebra can be divided into *(1)* a "core" subalgebra, a subalgebra containing the Cartan subalgebra of the full algebra, *(2)* a set of m raising operators, and *(3)* a set of m adjoint lowering operators. VCS theory then gives a very explicit method of construction of the irreducible representations of the full algebra from the irreducible representations of the subalgebra by an inductive process, or in the language of quantum theory by a vector-coupling process which couples the "collective" or "orbital" degrees of freedom with the "intrinsic" (or "spin", or "internal") degrees of freedom.

In VCS theory the "collective" excitations are realized in terms of polynomials in a set of m complex Bargmann space variables, z_i, $i = 1, ..., m$. The "intrinsic" states form a d-dimensional vector, where d is the dimension of the irreducible representation of the subalgebra which is used to induce the irreducible representation of the full algebra. Matrix elements of the generators of the algebra are then extremely simple and follow from a knowledge of the recoupling (Racah) coefficients of the subalgebra and the matrix elements of the intrinsic components of the generators. These follow from a knowledge of the generator matrix elements of the subalgebra. Like the matrix elements of the "spin" operators they do not require knowledge of the "intrinsic" or "spin" degrees of freedom. Generator matrix elements can thus be evaluated without the introduction of intrinsic or internal coordinates. For the detailed evaluation of the full Wigner-Racah calculus of higher rank algebras it becomes necessary to find the matrix elements of simple operators lying outside the group algebra. In a recent attempt to generalize VCS theory [9]–[11] coherent state realizations of such operators have also been given

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in terms of vector-coupled intrinsic operators and collective $\mathbf{z}$-space operators.

Unlike the intrinsic components of the group generators, however, the intrinsic components of operators lying outside the group algebra can now change the irreducible representation of the core subgroup, and their matrix elements are no longer known *a priori*. In references [9]–[11] an attempt was made to define such intrinsic operators through their actions on the intrinsic state vectors without the explicit introduction of a set of intrinsic or internal coordinates. The process is complicated by the fact that such an intrinsic operator when acting on a component of an intrinsic state will make connections not only to pure intrinsic states in a different irreducible representation but as well to some simple collective or $\mathbf{z}$-space excitations in this new irreducible representation.

Despite those difficulties, a method was devised in references [9]–[11] whereby the intrinsic components of many simple operators were defined through their nonzero reduced matrix elements. Such operators are thus defined through a table of reduced matrix elements with purely intrinsic states on the left (bra) side and both purely intrinsic and mixed intrinsic – collective states on the right (ket) side. Although the whole procedure is cumbersome, this generalized VCS theory was used to calculate many of the simple Wigner coefficients of the neutron-proton quasispin group,[9] SO(5)⊃U(2), of the Sp(6)⊃U(3) fermion dynamical symmetry group [10], and of the canonical branch of the unitary group [11], U(3)⊃U(2)×U(1). This method led to expressions for the Wigner coefficients which involve only the K-normalization factors of VCS theory and recoupling coefficients for the core subalgebra.

In principle therefore, the Wigner-Racah calculus of a higher rank symmetry algebra follows from the known Wigner-Racah calculus of a simpler subalgebra. In practice, however, this version of the generalized VCS method is somewhat difficult to apply because of the lack of an explicit construction of the intrinsic states and the intrinsic components of the operators. In a very recent development LeBlanc [12], using the work of Bouwknegt, McCarthy, and Pilch [13] on two-dimensional conformal field theories, has shown how the concepts of vertex and screening operators can be used to solve these difficulties in a very elegant way.

In this work a set of intrinsic coordinates q_i, $i = 1, \ldots, \ell$ is introduced (where ℓ is the rank of the full group), together with their canonically conjugate momenta, p_i. The q_i are used to construct the highest weight components of the intrinsic states. The intrinsic components of general tensor operators are constructed through the q_i's and p_i's and the Bargmann variables of the core subalgebra. These constructions then lead directly to expressions for Wigner coefficients in terms of the K-normalization factors of VCS theory and recoupling coefficients of the core subalgebra.

It is the purpose of this presentation to give a brief review of LeBlanc's method [12], in particular to show how it ties in with the earlier attempts to generalize VCS theory and how it eliminates all the difficulties of the earlier attempts. Although the new method is very general, it will be applied to the canonical group chain $\mathrm{U}(n)\supset\mathrm{U}(n-1)\times\mathrm{U}(1)$, particularly to the simplest nontrivial case with $n = 3$. Marcos Moshinsky's contributions to this basic problem are well known, particularly through his seminal work in references [14] and [15]. It is indeed a pleasure to dedicate this review to Marcos Moshinsky.

2.2 Intrinsic and collective variables for U(3).

The unitary group, $U(3) \supset U(2) \times U(1)$, furnishes one of the simplest examples for the application of VCS theory [6,16,17]. Generators are denoted by E_{ij}, with $i,j = 1,2,3$. The U(3) state vectors are specified by the Gel'fand basis

$$\left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle .$$

The $(m_{13} - m_{23} + 1)$ intrinsic states are to be abbreviated by $|\Lambda >$ with

$$|\Lambda\rangle = \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{13} & & m_{23} & \\ & & m_{11} & & \end{array} \right\rangle , \qquad m_{13} \geq m_{11} \geq m_{23}.$$

The VCS z-space functional realizations of state vectors $|\Psi\rangle$ are given in terms of the z-space variables z_{13}, z_{23} [for general, $U(n)$, in terms of z_{in}, $i = 1, \ldots n-1$] by

$$\langle\Lambda|e^{\mathbf{z}\cdot\mathbf{E}}|\Psi\rangle, \qquad \mathbf{z}\cdot\mathbf{E} = z_{13}E_{13} + z_{23}E_{23}. \tag{1}$$

Operators X are given by their VCS realizations $\Gamma(X)$, through

$$\begin{aligned} \Gamma(X)\langle\Lambda|e^{\mathbf{z}\cdot\mathbf{E}}|\Psi\rangle &= \langle\Lambda|e^{\mathbf{z}\cdot\mathbf{E}}X|\Psi\rangle = \langle\Lambda|(e^{\mathbf{z}\cdot\mathbf{E}}Xe^{-\mathbf{z}\cdot\mathbf{E}})e^{\mathbf{z}\cdot\mathbf{E}}|\Psi\rangle \\ &= \langle\Lambda|\{X + [(\mathbf{z}\cdot\mathbf{E}),X] + \tfrac{1}{2}[(\mathbf{z}\cdot\mathbf{E}),[(\mathbf{z}\cdot\mathbf{E}),X]] + \cdots\}e^{\mathbf{z}\cdot\mathbf{E}}|\Psi\rangle. \end{aligned} \tag{2}$$

It is now extremely useful to introduce a set of intrinsic variables, to be distinguished from the collective variables z_{13}, z_{23}. For this purpose it is convenient to first replace $|\Lambda\rangle$ by the single highest weight state $|hw\rangle$ (with $m_{11} = m_{13}$), and $\mathbf{z}\cdot\mathbf{E}$ by the raising-generator function

$$\mathbf{z}'\cdot\mathbf{E} = z'_{12}E_{12} + z'_{13}E_{13} + z'_{23}E_{23}. \tag{3}$$

This is tantamount to a replacement of the core subalgebra U(2) + U(1) by the simple Cartan subalgebra itself. This seeming retrogression makes possible the introduction of a set of intrinsic variables. These are to include a set of hermitian q_i (with conjugate hermitian p_i), such that

$$[p_i, q_i] = -i\,\delta_{ij}, \qquad i,j = 1,2,3, \tag{4}$$

where the highest weight state is to be specified by

$$|hw\rangle \;=\; \exp[i(m_{13}q_1 + m_{23}q_2 + m_{33}q_3)]|0\rangle. \tag{5}$$

The intrinsic space inner product is defined such that $\langle hw|hw\rangle = 1$. Note also that $p_i|hw\rangle = m_{i3}|hw\rangle$. The $\mathbf{z}'$-space realizations of the generators follow from the $\mathbf{z}'$-analog of Eq. (2). Final expressions are now somewhat more complicated due to the noncommutability of E_{12} and E_{23}. Using the simple form of the Baker-Campbell-Hausdorff relation valid for this case

$$e^{\mathbf{z}'\cdot\mathbf{E}} \;=\; e^{z'_{12}E_{12}}e^{z'_{23}E_{23}}e^{(z'_{13}-\frac{1}{2}z'_{12}z'_{23})E_{13}} \;=\; e^{z'_{23}E_{23}}e^{z'_{12}E_{12}}e^{(z'_{13}+\frac{1}{2}z'_{12}z'_{23})E_{13}}, \tag{6}$$

and the abbreviation $\partial/\partial z'_{ab} \equiv \partial'_{ab}$, we are led to

$$E_{13}e^{\mathbf{z}'\cdot\mathbf{E}} = e^{\mathbf{z}'\cdot\mathbf{E}}E_{13} = \partial'_{13}e^{\mathbf{z}'\cdot\mathbf{E}}, \tag{7}$$

and

$$\begin{aligned} E_{12}e^{\mathbf{z}'\cdot\mathbf{E}} &= (\partial'_{12} + \tfrac{1}{2}z'_{23}\partial'_{13})e^{\mathbf{z}'\cdot\mathbf{E}}, \\ e^{\mathbf{z}'\cdot\mathbf{E}'}E_{12} &= (\partial'_{12} - \tfrac{1}{2}z'_{23}\partial'_{13})e^{\mathbf{z}'\cdot\mathbf{E}}, \end{aligned} \tag{8}$$

where the right form of Eq. (8) leads to

$$\Gamma(E_{12}) = (\partial'_{12} - \tfrac{1}{2}z'_{23}\partial'_{13}). \tag{9}$$

The left form is needed in the commutator expansion of the lowering operators [see Eq. (2)] *e.g.*:

$$\Gamma(E_{21}) = (m_{13} - m_{23})z'_{12} - z'_{13}\partial'_{23} + z'_{12}(\tfrac{1}{2}z'_{23}\partial'_{23} - z'_{12}\partial'_{12} - \tfrac{1}{2}z'_{13}\partial'_{13} - \tfrac{1}{4}z'_{12}z'_{23}\partial'_{13}) \tag{10}$$

[see Eq. (4.2) of Ref. [12] for the full set of $\Gamma(E_{ij})$]. The intrinsic and collective variables can now be untangled by a transformation to new variables z_{ab},

$$z_{12} = z'_{12}, \qquad z_{23} = z'_{23}, \qquad z_{13} = z'_{13} - \tfrac{1}{2}z'_{12}z'_{23}, \tag{11}$$

leading to the standard VCS realization [6,16] of U(3)

$$\begin{aligned} \Gamma(E_{13}) &= \partial_{13}, & \Gamma(E_{23}) &= \partial_{23}, \\ \Gamma(E_{12}) &= \mathbb{E}_{12} - z_{23}\partial_{13}, & \Gamma(E_{21}) &= \mathbb{E}_{21} - z_{13}\partial_{23}, \\ \Gamma(E_{11}) &= \mathbb{E}_{11} - z_{13}\partial_{13}, & \Gamma(E_{22}) &= \mathbb{E}_{22} - z_{23}\partial_{23}, \\ \Gamma(E_{33}) &= \mathbb{E}_{33} + z_{\alpha 3}\partial_{\alpha 3}, \\ \Gamma(E_{31}) &= z_{\alpha 3}\mathbb{E}_{\alpha 1} - z_{13}\mathbb{E}_{33} - z_{13}z_{\alpha 3}\partial_{\alpha 3}, \\ \Gamma(E_{32}) &= z_{\alpha 3}\mathbb{E}_{\alpha 2} - z_{23}\mathbb{E}_{33} - z_{23}z_{\alpha 3}\partial_{\alpha 3}, \end{aligned}$$

(12*a*), (12*b*), (12*c*)

where repeated Greek indices are summed from 1 to 2.

The intrinsic operators $\mathbb{E}_{ij}$ which generate the intrinsic U(2) + U(1) subalgebra can now be given explicitly in terms of intrinsic variables p_i and z_{12} by

$$\begin{aligned} \mathbb{E}_{12} &= \partial_{12}, \\ \mathbb{E}_{11} &= p_1 - z_{12}\partial_{12}, \qquad \mathbb{E}_{22} = p_2 + z_{12}\partial_{12}, \qquad \mathbb{E}_{33} = p_3, \\ \mathbb{E}_{21} &= z_{12}(p_1 - p_2 - z_{12}\partial_{12}). \end{aligned} \tag{13}$$

The $(m_{13} - m_{23} + 1)$ components of the intrinsic state are given in the intrinsic variables q_i and z_{12} by

$$\left| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{13} & & m_{23} & \\ & & m_{11} & & \end{matrix} \right\rangle = \frac{z_{12}^{(m_{13}-m_{11})}}{\sqrt{(m_{13} - m_{11})!}} e^{i(m_{13}q_1 + m_{23}q_2 + m_{33}q_3)}|0\rangle. \tag{14}$$

The full VCS state vector is given in terms of a polynomial in the collective variables z_{13}, z_{23},

$$Z^{[0,-w]}_{M_{11}}(\mathbf{z}) \equiv Z^{\frac{1}{2}w}_{M=M_{11}+\frac{w}{2}}(\mathbf{z}) = \frac{(-z_{23})^{\frac{w}{2}+M}(z_{13})^{\frac{w}{2}-M}}{\sqrt{(\frac{w}{2}+M)!(\frac{w}{2}-M)!}} \tag{15}$$

by

$$\left|\begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{matrix}\right\rangle = (-1)^{\Phi}\left[Z^{[0,-w]}(\mathbf{z}) \times |[m_{13}m_{23}]\rangle\right]^{[m_{12}m_{22}]}_{m_{11}}, \tag{16}$$

where

$$w = m_{13} + m_{23} - m_{12} - m_{22}, \tag{17}$$

and where the square bracket denotes SU(2) or ordinary angular momentum vector coupling, with a right to left coupling order. The U(2) notation, $[0,-w]$, may be preferred to the simple angular momentum notation, $\frac{1}{2}w$, to highlight the antispinor character of z_{13}, z_{23}. Note that $(-z_{23})$ and (z_{13}) are the $M = +\frac{1}{2}$ and $-\frac{1}{2}$ components of the spin $\frac{1}{2}$ or $[0,-1]$ tensor. On the other hand $(\partial_{13}, \partial_{23})$ transform as the $(+\frac{1}{2}, -\frac{1}{2})$ components of a [1,0] tensor. (The phase Φ, with $\Phi = \frac{1}{2}(m_{13}-m_{23}) - \frac{1}{2}w - \frac{1}{2}(m_{12}-m_{22})$ is based on the Biedenharn-Louck [18] phase convention; see the summary to Ref. [11].)

The operators $\Gamma(\mathbf{0})$ of Eq. (12) are non-unitary but can be transformed to unitary form $\gamma(\mathbf{0})$ by the standard VCS transformation $\gamma(\mathbf{0}) = K^{-1}\Gamma(\mathbf{0})K$. For the U($n$) groups the K operators are represented by simple 1×1 matrices and thus serve as simple normalization factors [6,16]. For U(3), with $[m_{13},\ m_{23},\ m_{33}] \equiv [\mathbf{m}_3]$ and $[m_{12}\, m_{22}] \equiv [\mathbf{m}_2]$,

$$K([\mathbf{m}_3],[\mathbf{m}_2]) = \sqrt{\frac{(m_{13}-m_{33}+1)!\,(m_{23}-m_{33})!}{(m_{12}-m_{33}+1)!\,(m_{22}-m_{33})!}}. \tag{18}$$

Final matrix elements in a U(3)$\supset$U(2)$\supset$U(1) basis thus follow from the $\mathbf{z}$-space – intrinsic space matrix elements in the basis (16), multiplied by the K-factor for the initial state ket and the inverse K^{-1}-factor for the final state bra. (Alternatively, as in Ref. [12], the unitarization can be effected by including the K-factor in the state vector definition.) Similarly, pure intrinsic operator matrix elements in the intrinsic basis (14) require the U(2) K-factors for ket and K^{-1} factors for bra where the U(2) K-factor is given by

$$K\begin{pmatrix} [m_{13}\, m_{23}] \\ m_{11} \end{pmatrix} = \sqrt{\frac{(m_{13}-m_{23})!}{(m_{11}-m_{23})!}}. \tag{19}$$

If operators other than the group generators can be expressed in terms of the *collective* operators z_{13}, z_{23}, ∂_{13}, ∂_{23} and the *intrinsic* operators q_i, p_i, z_{12}, and ∂_{12} in a vector-coupled form, their matrix elements then follow directly through simple vector-coupling formulae.

2.3 The U(3) fundamental tensors.

With the introduction of the intrinsic operators of Section 2.2 it is now possible to give very explicitly constructions for the U(3) irreducible tensor operators with very specific shift properties. For U(3) these can be specified by the usual double Gel'fand pattern, where the $[\mathbf{M}_3] \equiv [M_{13}M_{23}M_{33}]$-tensor

$$\left\langle\begin{matrix} & & \Gamma_{11} & & \\ & \Gamma_{12} & & \Gamma_{22} & \\ M_{13} & & M_{23} & & M_{13} \\ & M_{12} & & M_{22} & \\ & & M_{11} & & \end{matrix}\right\rangle \equiv T^{\Gamma}_{M}([\mathbf{M}_3]) \tag{20}$$

induces shifts in the irreducible representations $[\mathbf{m}_3] \equiv [m_{13}m_{23}m_{33}]$ given by the upper pattern

$$\begin{aligned} m'_{13} - m_{13} &\equiv \Delta_1 = \Gamma_{11}, \\ m'_{23} - m_{23} &\equiv \Delta_2 = \Gamma_{12} + \Gamma_{22} - \Gamma_{11}, \\ m'_{33} - m_{33} &\equiv \Delta_3 = M_{13} + M_{23} + M_{33} - \Gamma_{12} - \Gamma_{22}. \end{aligned} \tag{21}$$

The lower pattern labels $M_{12}M_{22}$, M_{11} on the other hand give the shifts in the U(2) $\supset$ U(1) subgroup labels. Such a tensor will be named a *unit*-tensor if its SU(3)-reduced matrix element has the value +1. (In the language of Refs. [12] and [13] these are the so-called vertex operators for U(3).) A U(3) irreducible tensor operator of maximal shift ($\Gamma_{ij} = M_{i3}$) and maximal weight ($M_{ij} = M_{i3}$) is given by the simple exponential operator $\exp[i(\mathbf{M}_3 \cdot \mathbf{q})]$. To construct the remaining shift operators it is useful to introduce the so-called "screening charges" of Refs. [12] and [13]. These are operators s_{ij} ($i < j$) which shift the U(n) labels by one unit out of row i into row j but are $U(n-1)$-scalars, the shift in the intrinsic labels being compensated by a corresponding shift in the collective variables in the VCS realizations. To achieve this end it is simplest to consider the left action of such operators and consider the left action VCS-realization $\rho(X)$ of the raising generators $X = E_{ij}$ ($i < j$), via

$$\rho(X)\langle hw|e^{\mathbf{z}'\cdot\mathbf{E}}|\Psi\rangle = \langle hw|(-X)e^{\mathbf{z}'\cdot\mathbf{E}}|\Psi\rangle. \tag{22}$$

After the untangling transformation (11) this leads to

$$\rho(E_{12}) = -\partial_{12}, \qquad \rho(E_{13}) = -\partial_{13}, \qquad \rho(E_{23}) = -(\partial_{23} - z_{12}\partial_{13}). \tag{23}$$

Note that the minus sign in Eq. (22) is required to preserve the commutator algebra of the E_{ij}. The *simple* screening charges (corresponding to the two *simple* roots) are then

$$\begin{aligned} s_{12} &= e^{-i(q_1-q_2)}\rho(E_{12}) = e^{-i(q_1-q_2)}(-\partial_{12}), \\ s_{23} &= e^{-i(q_2-q_3)}\rho(E_{23}) = e^{-i(q_2-q_3)}(-\partial_{23} + z_{12}\partial_{13}). \end{aligned} \tag{24}$$

Note the shift in the intrinsic variables and the compensating shift in the collective variables, and note that the s_{ij} commute with the U(2) subalgebra, $\Gamma(E_{ij})$ with $i,j = 1,2$. Note also that the derived s_{13} is given by $[s_{12},\ s_{23}]$.

Shift tensors of the type (20) with shifts of type (21) can then be derived from the shift tensors with maximal weight and maximal shift via the "equivariance condition"

$$(s_{ij})^{m'_{i3}-m'_{j3}+1}\, T^{\Gamma}_{hw}([\mathbf{M}_3]) = T^{\Gamma'}_{hw}([\mathbf{M}_3])\,(s_{ij})^{m_{i3}-m_{j3}+1}, \tag{25}$$

where the shift Γ' follows from the shift

$$\Gamma \;:\; [m_{13}\; m_{23}\; m_{33}] \;\rightarrow\; [m'_{13}\; m'_{23}\; m'_{33}]$$

via the Weyl reflection r_{ij} corresponding to the simple s_{ij}

$$\Gamma' \;:\; r_{ij} * [m_{13}\; m_{23}\; m_{33}] \;\rightarrow\; r_{ij} * [m'_{13}\; m'_{23}\; m'_{33}],$$

e.g. if $T([\mathbf{M}_3])$ is a U(3) fundamental tensor with $[\mathbf{M}_3] = [100]$ and if the shift Γ is given by $(\Delta_1\Delta_2\Delta_3) = (010)$ so that $[\mathbf{m}'_3] = [m_{13},\ m_{23}+1,\ m_{33}]$; then the shift

$$\begin{aligned} r_{12} * [m_{13}m_{23}m_{33}] &= [m_{23}-1,\ m_{13}+1,\ m_{33}] \rightarrow r_{12} * [m_{13},\ m_{23}+1,\ m_{33}] \\ &= [m_{23},\ m_{13}+1,\ m_{33}] \end{aligned}$$

corresponds to a Γ' given by $(\Delta'_1\Delta'_2\Delta'_3) = (100)$; whereas the shift

$$\begin{aligned} r_{23} * [m_{13}\ m_{23}\ m_{33}] &= [m_{13}\ m_{33}-1\ m_{23}+1] \rightarrow r_{23} * [m_{13}\ m_{23}+1\ m_{33}] \\ &= [m_{13}\ m_{33}-1\ m_{23}+2] \end{aligned}$$

corresponds to a Γ' given by $(\Delta'_1\Delta'_2\Delta'_3) = (001)$. This leads to the two relations

$$\begin{aligned} (s_{12})^{m_{13}-m_{23}} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle &= \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle (s_{12})^{m_{13}-m_{23}+1}, \\ (s_{23})^{m_{23}-m_{33}+2} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle &= \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle (s_{23})^{m_{23}-m_{33}+1}. \end{aligned} \tag{26}$$

Since the maximal-shift maximal-weight operator with $\Gamma_{11} = \Gamma_{12} = 1,\ \Gamma_{22} = 0$, has the simple value e^{iq_1} the remaining two maximal-weight operators follow from these equations

$$\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle = e^{iq_1}s_{12} = e^{iq_2}(-\partial_{12}), \tag{27}$$

$$\begin{aligned} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array} \right\rangle &= e^{iq_1}(s_{12}s_{23} + s_{13}(p_2 - p_3)) \\ &= e^{iq_3}\{\partial_{12}\partial_{23} - \partial_{13}(p_2 - p_3 + 1 - z_{12}\partial_{12})\}. \end{aligned} \tag{28}$$

Lower weight components of these shift operators follow from commutator relations, such as

$$\left[\Gamma(E_{31}),\ T^{\Gamma}_{hw}([100])\right] = \langle \ell w|E_{31}|hw\rangle\ T^{\Gamma}_{\ell w}([100]), \quad \text{with } |\ell w\rangle = \left| \begin{array}{ccc} 0 & & 0 \\ & 0 & \end{array} \right\rangle, \tag{29}$$

through the well-known simple generator matrix elements [6,16]. This leads to the $(\Delta_1\Delta_2\Delta_3) = (100)$ fundamental shift tensors

$$\begin{aligned} &\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle = e^{iq_1}, \qquad \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right\rangle = z_{12}e^{iq_1}, \\ &\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle = e^{iq_1}(z_{13} + z_{12}z_{23}). \end{aligned} \tag{30a}$$

Note that the first two components are built from purely intrinsic operators. These intrinsic $(\Delta_1\Delta_2\Delta_3) = (100)$-operators will be given the intrinsic shift label $\$_1$. Their SU(2) angular momentum tensor character is $(\$_1)^{\frac{1}{2}}_{+\frac{1}{2}} = e^{iq_1}$, $(\$_1)^{\frac{1}{2}}_{-\frac{1}{2}} = z_{12}e^{iq_1}$, so that

$$
\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle = (\$_1)^{\frac{1}{2}}_{+\frac{1}{2}}, \qquad
\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right\rangle = (\$_1)^{\frac{1}{2}}_{-\frac{1}{2}},
$$
$$
\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle = \sqrt{2}[Z^{\frac{1}{2}}(\mathbf{z}) \times (\$_1)^{\frac{1}{2}}]^0_0, \tag{30b}
$$

where the square bracket vector-coupling of the intrinsic and collective spin $-\frac{1}{2}$ tensors of the last component are given in right to left coupling order. The $(\Delta_1\Delta_2\Delta_3) = (010)$ shift tensors can be expressed in the same way in terms of intrinsic shift operators $\$_2$,

$$
\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle = (\$_2)^{\frac{1}{2}}_{+\frac{1}{2}} \quad , \quad
\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right\rangle = (\$_2)^{\frac{1}{2}}_{-\frac{1}{2}},
$$
$$
\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle = \sqrt{2}[Z^{\frac{1}{2}}(\mathbf{z}) \times (\$_2)^{\frac{1}{2}}]^0_0, \tag{31a}
$$

where

$$
(\$_2)^{\frac{1}{2}}_{+\frac{1}{2}} \equiv -e^{iq_2}\partial_{12}, \qquad (\$_2)^{\frac{1}{2}}_{-\frac{1}{2}} = e^{iq_2}(p_1 - p_2 - z_{12}\partial_{12}). \tag{31b}
$$

In general such operators are not unit tensors since their SU(3) reduced matrix elements are not unity. The explicit form of the operators makes it easy to calculate the SU(3)-reduced matrix elements; *e.g.*,

$$
\begin{aligned}
&\left\langle \begin{array}{ccccc} m_{13} & & m_{23}+1 & & m_{33} \\ & m_{13} & & m_{23}+1 & \\ & & m_{13} & & \end{array} \middle| \, e^{iq_2}(p_1 - p_2 - z_{12}\partial_{12}) \, \middle| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{13} & & m_{23} & \\ & & m_{13} & & \end{array} \right\rangle \\
&= (m_{13} - m_{23}) \\
&= \langle \tfrac{1}{2}(m_{13} - m_{23})\tfrac{1}{2}(m_{13} - m_{23}), \tfrac{1}{2} - \tfrac{1}{2} \,\big|\, \tfrac{1}{2}(m_{13} - m_{23} - 1)\tfrac{1}{2}(m_{13} - m_{23} - 1)\rangle(+1) \\
&\quad \times \langle\langle [m_{13}, m_{23}+1, m_{33}] \| \left[\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \end{array} \right] \| [m_{13}m_{23}m_{33}] \rangle\rangle,
\end{aligned} \tag{31c}
$$

where the $(+1)$ is the trivial value of the $SU(3) \supset SU(2)$ reduced Wigner coefficient connecting highest weight states. With the value of the SU(2) Wigner coefficient, the SU(3)-reduced matrix element is seen to be $[(m_{13} - m_{23})(m_{13} - m_{23} + 1)]^{1/2}$. Note that the SU(3)-reduced matrix element is given a double-bar, double caret notation to distinguish it from an SU(2)-reduced matrix element which is to be given the ordinary double-bar symbol. The $(\Delta_1\Delta_2\Delta_3) = (001)$ shift tensor of Eq. (28) appears to be more complicated; but it can be given in simple SU(2)-tensor form through the commutator relation [12]

$$\{\partial_{12}\partial_{23} - \partial_{13}(p_2 - p_3 + 1 - z_{12}\partial_{12})\} \;=\; [\partial_{13},\ \Omega] \;=\; [(\partial)^{\frac{1}{2}}_{+\frac{1}{2}},\ \Omega], \tag{32}$$

where Ω is the U(2) scalar operator

$$\begin{aligned}\Omega \;&=\; z_{\alpha 3}\partial_{\beta 3}\not{E}_{\alpha\beta} - (\not{E}_{11} + \not{E}_{22} - \not{E}_{33} + 1)z_{\alpha 3}\partial_{\alpha 3} \\ &=\; -2(\mathbf{I}^{\mathrm{intr}}\cdot\mathbf{I}^{\mathrm{coll.}}) - \left[\tfrac{1}{2}(\not{E}_{11} + \not{E}_{22} - 2\not{E}_{33}) + 1\right] z_{\alpha 3}\partial_{\alpha 3},\end{aligned} \tag{33}$$

where repeated Greek indices are summed from 1 to 2, and where $\mathbf{I}^{\mathrm{intr}}$, *e.g.*, is given by $(I^{\mathrm{intr}}_{+},\ I^{\mathrm{intr}}_{0},\ I^{\mathrm{intr}}_{-}) = \left(\not{E}_{12},\ \tfrac{1}{2}(\not{E}_{11} - \not{E}_{22}),\ \not{E}_{21}\right)$. This leads to the (unnormalized) shift tensors

$$\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle = e^{iq_3}[(\partial)^{\frac{1}{2}}_{+\frac{1}{2}},\ \Omega], \qquad \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \right\rangle = e^{iq_3}[(\partial)^{\frac{1}{2}}_{-\frac{1}{2}},\ \Omega],$$

$$\begin{aligned}\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle &= e^{iq_3}\{(\not{E}_{22} - \not{E}_{33} + 1 - z_{23}\partial_{23})(\not{E}_{11} - \not{E}_{33} - z_{13}\partial_{13}) \\ &\qquad -(\not{E}_{12} - z_{23}\partial_{13})(\not{E}_{21} - z_{13}\partial_{23})\} \\ &= e^{iq_3}\{[\tfrac{1}{2}(\not{E}_{11} + \not{E}_{22}) - \not{E}_{33} - \tfrac{1}{2}z_{\alpha 3}\partial_{\alpha 3} + 1][\tfrac{1}{2}(\not{E}_{11} + \not{E}_{22}) - \not{E}_{33} - \tfrac{1}{2}z_{\beta 3}\partial_{\beta 3}] - \mathbf{I}^{\mathrm{tot}}\cdot\mathbf{I}^{\mathrm{tot}}\}\end{aligned} \tag{34}$$

Equations (30), (31) and (34) give the components of all U(3) fundamental tensors in very explicit form in terms of intrinsic and collective variables. It is now of interest to compare these with earlier attempts to build a generalized VCS theory for operators lying outside the Lie algebra. In the earlier work[11] the U(3) fundamental tensors were given by the oscillator creation operators, $\alpha^{\dagger}_{ip}$, with $i = 1, 2, 3$, p = particle index (where $E_{ij} = \Sigma_p \alpha^{\dagger}_{ip}\alpha_{jp}$). Eq. (2), with $X \;=\; \alpha^{\dagger}_{ip}$, led to

$$\Gamma\left((\alpha^{\dagger}_{p})^{\frac{1}{2}}_{m}\right) \;=\; (\not{\phi}^{\dagger}_{p})^{\frac{1}{2}}_{m}, \tag{35a}$$

$$\Gamma\left((\alpha^{\dagger}_{p})^{0}_{0}\right) \;=\; (\not{\phi}^{\dagger}_{p})^{0}_{0} + \sqrt{2}[Z^{1/2}(\mathbf{z}) \times (\not{\phi}^{\dagger}_{p})^{\frac{1}{2}}]^{0}_{0}, \tag{35b}$$

with $\{\alpha^{\dagger}_{1p},\ \alpha^{\dagger}_{2p},\ \alpha^{\dagger}_{3p}\} \;\equiv\; \{(\alpha^{\dagger}_{p})^{\frac{1}{2}}_{+\frac{1}{2}},\ (\alpha^{\dagger}_{p})^{\frac{1}{2}}_{-\frac{1}{2}},\ (\alpha^{\dagger}_{p})^{0}_{0}\}$, and where the slashed operators $\not{\phi}^{\dagger}$ were intrinsic operators. Like the intrinsic generators these were defined solely through their SU(3) reduced matrix elements, the intrinsic nature of the operator dictating that these have only purely intrinsic states in the left (bra) side of the matrix elements. (Matrix elements of such intrinsic operators are given in Ref. [11] for all Moshinsky polynomials of $\alpha^{\dagger}$.) It is now very interesting to note that these intrinsic operator matrix elements depended on the nature of the shift, *e.g.*, if the operator adds zero squares to the third row of the U(3) tableau, *i.e.* for shifts with $\Delta_3 \;=\; 0$, the intrinsic operator $(\not{\phi}^{\dagger}_{p})^{0}_{0}$ has zero reduced matrix elements. In this case, therefore, Eqs. (35*a*) and (34*b*), with $(\not{\phi}^{\dagger}_{p})^{0}_{0}$ effectively missing, have exactly the same structure as Eqs. (30*b*) and (31). However, the very explicit construction of the $(\not{S}_1)$ and $(\not{S}_2)$ operators in terms of the intrinsic p_i, q_i, z_{12}, and ∂_{12} is a tremendous advantage. With shift $\Delta_3 \neq 0$, both the intrinsic operators $(\phi_p^\dagger)_0^0$ and $(\phi_p^\dagger)_m^{\frac{1}{2}}$ have non-zero matrix elements. Moreover, they connect purely intrinsic states on the left to mixed intrinsic – collective states on the right, destroying some of the simplicity of the intrinsic – collective vector-coupling calculus. We note that Eq. (34) has some of the same characteristics since the Ω-operator is a function of both intrinsic and collective variables. The very explicit form of the operators, however, now makes it possible to express their matrix elements in the standard vector coupling form. With the eigenvalue difference $\Omega([\mathbf{m}_3], [\mathbf{m}_2]) - \Omega([\mathbf{m}_3], [\mathbf{m}_2']) \equiv (\Omega - \Omega')$,

$$\begin{aligned} (\Omega - \Omega') &= -(m_{22} - m_{33}), \qquad \text{for } [m_{12}' \, m_{22}'] = [m_{12} + 1, \, m_2], \\ (\Omega - \Omega') &= -(m_{12} - m_{33} + 1), \quad \text{for } [m_{12}' \, m_{22}'] = [m_{12}, \, m_{22} + 1], \end{aligned} \tag{36}$$

the SU(2) reduced matrix elements of the $\Delta_3 = 1$ shift tensor now has the standard form, *e.g.*

$$\left\langle \begin{matrix} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12}' & & m_{22}' & \end{matrix} \right\| \left\langle \begin{matrix} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \end{matrix} \right\rangle \left\| \begin{matrix} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{matrix} \right\rangle$$

$$= (-1)^{\frac{1}{2}(m_{12}' - m_{22}' - m_{12} + m_{22} - 1)} \, (\Omega - \Omega') \, \langle (w-1) \, \| \partial^{\frac{1}{2}} \| w \rangle$$

$$\times \begin{bmatrix} [m_{13} m_{23}] & [0, -w] & [m_{12} m_{22}] \\ [00] & [10] & [10] \\ [m_{13} m_{23}] & [0, -(w-1)] & [m_{12}' m_{22}'] \end{bmatrix} \frac{K([m_{13} \, m_{23} \, m_{33}], \, [m_{12} \, m_{22}])}{K([m_{13} \, m_{23} \, m_{33} + 1], \, [m_{12}' \, m_{22}'])}. \tag{37}$$

From Eqs. (30) and (31) it can be seen that all matrix elements have this standard form. They are given in terms of *(1)* a K-factor ratio, *(2)* an SU(2) or ordinary angular momentum recoupling coefficient, and *(3)* SU(2) reduced matrix elements of simple collective and/or intrinsic operators which are now very easy to evaluate because these operators are given in very explicit form. In the special case of Eq. (37), the collective SU(2) reduced matrix element has the value $\langle (w-1) \| \partial^{\frac{1}{2}} \| w \rangle = -\sqrt{w+1}$; the dependence on intrinsic operators sits solely in the Ω eigenvalue difference; and the 9–j unitary recoupling coefficient with one [00] representation collapses to an ordinary unitary Racah coefficient. In standard angular momentum notation it is the U-coefficient $U(\frac{\lambda}{2} \, \frac{w}{2} \, I' \, \frac{1}{2}; \, I \frac{w-1}{2})$, with $\frac{\lambda}{2} = \frac{1}{2}(m_{13} - m_{23})$, $I = \frac{1}{2}(m_{12} - m_{22})$, $I' = \frac{1}{2}(m_{12}' - m_{22}')$.

2.4 More complicated shift tensors.

The techniques used to calculate the U(3) fundamental [100]-tensors can now be used to calculate more challenging tensors. The elementary [110]-tensors follow from the maximal-shift maximal-weight tensor $e^{i(q_1+q_2)}$ and can be expressed in vector-coupled form. The (110)-shift tensors are:

$$\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\rangle = e^{i(q_1+q_2)}, \qquad \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & m & & \end{array} \right\rangle = e^{i(q_1+q_2)} Z^{\frac{1}{2}}_{\pm\frac{1}{2}}(\mathbf{z}). \tag{38}$$

The (101)-shift tensors are:

$$\begin{aligned} \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\rangle &= -\sqrt{2} e^{iq_3} [\partial^{\frac{1}{2}} \times (\mathcal{S}_1)^{\frac{1}{2}}]^0_0, \\ \left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & m & & \end{array} \right\rangle &= e^{iq_3} \{ [(\mathcal{S}_1)^{\frac{1}{2}}_{\pm\frac{1}{2}}, \Omega] - (\mathcal{S}_1)^{\frac{1}{2}}_{\pm\frac{1}{2}} (p_2 - p_3) \}. \end{aligned} \tag{39}$$

The (011)-shift tensors are:

$$\begin{aligned} \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \right\rangle &= -\sqrt{2} e^{iq_3} [\partial^{\frac{1}{2}} \times (\mathcal{S}_2)^{\frac{1}{2}}]^0_0, \\ \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & m & & \end{array} \right\rangle &= e^{iq_3} \{ [(\mathcal{S}_2)^{\frac{1}{2}}_{\pm\frac{1}{2}}, \Omega] - (\mathcal{S}_2)^{\frac{1}{2}}_{\pm\frac{1}{2}} (p_1 - p_3 + 1) \}, \end{aligned} \tag{40}$$

where upper (or lower) subscripts apply for $m = 1$ (or 0). The operators are again in a form from which their matrix elements lead to the basic structure built from (1) a K-factor ratio, (2) an SU(2) recoupling coefficient, and (3) simple collective and/or intrinsic SU(2)-reduced matrix elements (see Ref. [12]).

More complicated tensors can be built from repeated coupling of the basic [100] and [110]-tensors. It is instructive to consider the octet [210]-tensors, where a multiplicity occurs for the first time. The equivariance condition, (25), leads to two independent solutions [12] for [210]-tensors with shifts $(\Delta_1\Delta_2\Delta_3) = (111)$. Alternatively, we could consider the symmetric or antisymmetric combination of the coupled tensors

$$\left(\left[T^{\Gamma}([100]) \times T^{\Gamma'}([110]) \right]^{[210]}_{\mathrm{M}} \pm \left[T^{\Gamma'}([110]) \times T^{\Gamma}([100]) \right]^{[210]}_{\mathrm{M}\,\cdot} \right), \tag{41}$$

where these can be written explicitly via Eqs. (30)–(34) and (38)–(40). The antisymmetric combination leads to the simple commutators, *e.g.*, the commutator

$$\left[\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle, \left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle \right] \begin{aligned} &= -e^{i(q_1+q_2+q_3)} (\partial_{12} - z_{23}\partial_{13}) \\ &= -e^{i(q_1+q_2+q_3)} \Gamma(E_{12}), \end{aligned} \tag{42}$$

while

$$\left[\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\rangle, \left\langle\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array}\right\rangle\right] \begin{aligned} &= -e^{i(q_1+q_2+q_3)}\partial_{13} \\ &= -e^{i(q_1+q_2+q_3)}\Gamma(E_{13}). \end{aligned} \tag{43}$$

Moreover, the same result is obtained if the shifts (100) in Γ and (011) in Γ' are replaced by (010) in Γ and (101) in Γ', or by (001) in Γ and (110) in Γ', thus clearly showing that the antisymmetric combination of coupled tensors lead to the group generators, multiplied by the shift factor $e^{i(q_1+q_2+q_3)}$. The $e^{i(q_1+q_2+q_3)}\Gamma(E_{ij})$ can thus be identified with the upper pattern

$$\left[\begin{array}{ccccc} & & 1 & & \\ & 1 & & 1 & \\ 2 & & 1 & & 0 \end{array}\right].$$

The symmetric combination (or a symmetric coupling of [100] and [200]-tensors to resultant [210]), will lead to the independent

$$\left[\begin{array}{ccccc} & & 1 & & \\ & 2 & & 0 & \\ 2 & & 1 & & 0 \end{array}\right]$$

upper pattern operator. To gain SU(3)-coefficients orthogonal to those associated with the generators, $e^{i(q_1+q_2+q_3)}\Gamma(E_{ij})$, it is necessary to take an $[\mathbf{m}_3]$-dependent linear combination of the symmetric and antisymmetric coupled tensors. This "orthogonalization problem" seems to be endemic to all cases with multiplicity.

2.5 Totally symmetric shift tensors.

Finally, to illustrate the new techniques further, we shall give the details for the calculation of the totally symmetric U(3) $[\Delta 00]$-tensors with arbitrary shifts $(\Delta_1\Delta_2\Delta_3)$, $\Delta = \Delta_1 + \Delta_2 + \Delta_3$. The highest weight component of such a tensor is given by

$$\begin{aligned} &\left\langle\begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ \Delta & & 0 & & 0 \\ & & hw & & \end{array}\right\rangle \\ &= \left(\left\langle\begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array}\right\rangle\right)^{\Delta_1} \left(\left\langle\begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array}\right\rangle\right)^{\Delta_2} \left(\left\langle\begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & & hw & & \end{array}\right\rangle\right)^{\Delta_3}, \end{aligned} \tag{44}$$

where the hw [100]-tensors raised to the Δ_i^{th} power are given by e^{iq_1} and Eqs. (27) and (28), respectively. The only lower weight components needed for the most general $SU(3) \supset SU(2)$ Wigner coefficient can be obtained by taking the commutator of $\Gamma(E_{31})$ with the above tensor k times in succession. This process of taking k commutators in succession, when divided by the generator matrix element product leads to the lower weight tensor component

$$\left\langle \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ \Delta & & 0 & & 0 \\ & \Delta-k & & 0 & \\ & & \Delta-k & & \end{array}\right\rangle = \sum_{\substack{k_i \\ k_1+k_2+k_3=k}} \left[\binom{\Delta}{k}\right]^{-\frac{1}{2}} \binom{\Delta_1}{k_1}\binom{\Delta_2}{k_2}\binom{\Delta_3}{k_3}$$

$$\times \left(\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\rangle\right)^{\Delta_1-k_1} \left(\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}\right\rangle\right)^{k_1} \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\rangle\right)^{\Delta_2-k_2} \tag{45}$$

$$\times \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}\right\rangle\right)^{k_2} \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\rangle\right)^{\Delta_3-k_3} \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}\right\rangle\right)^{k_3},$$

where we have used

$$\prod_{i=0}^{k-1} \left\langle \begin{array}{ccccc} \Delta & & 0 & & 0 \\ & \Delta-i-1 & & 0 & \\ & & \Delta-i-1 & & \end{array} \middle| E_{31} \middle| \begin{array}{ccccc} \Delta & & 0 & & 0 \\ & \Delta-i & & 0 & \\ & & \Delta-i & & \end{array}\right\rangle = \sqrt{\frac{\Delta!\,k!}{(\Delta-k)!}}. \tag{46}$$

It is now useful to recouple the operators

$$\left(\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array}\right\rangle\right)^{\Delta_1-k_1} \left(\left\langle \begin{array}{ccccc} & & 1 & & \\ & 1 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array}\right\rangle\right)^{k_1} \tag{47}$$

$$= e^{iq_1(\Delta_1-k_1)}(\sqrt{2}[Z^{\frac{1}{2}} \times (\$_1)^{\frac{1}{2}}]^0_0)^{k_1} = \sqrt{\frac{(k_1)!\,(\Delta_1+1)}{(\Delta_1-k_1+1)}}\,\left[Z^{\frac{k_1}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}\right]^{\frac{1}{2}(\Delta_1-k_1)}_{\frac{1}{2}(\Delta_1-k_1)}$$

where we have used successive SU(2) recoupling coefficients of 9–j type with $J_{12} = J_{34} = J = 0$, together with $U(\frac{(\Delta_1-k_1)}{2}\frac{k_1}{2}\,\frac{(\Delta_1-k_1)}{2}\frac{k_1}{2};\,\frac{\Delta_1}{2}\,0)$, and the z-space polynomial buildup relation, $[Z^{\frac{a}{2}} \times Z^{\frac{b}{2}}]^{\frac{c}{2}}_m = \delta_{c(a+b)}\,\sqrt{[c!/a!b!]}\,Z^{\frac{c}{2}}_m$. Finally, the SU(2) recoupling transformation

$$\left[\left[Z^{\frac{k_2}{2}} \times (\$_2)^{\frac{\Delta_2}{2}}\right]^{\frac{1}{2}(\Delta_2-k_2)} \times \left[Z^{\frac{k_1}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}\right]^{\frac{1}{2}(\Delta_1-k_1)}\right]^{\frac{1}{2}(\Delta_1+\Delta_2-k_1-k_2)}_{\frac{1}{2}(\Delta_1+\Delta_2-k_1-k_2)}$$

$$= \sqrt{\frac{(\Delta_1-k_1+1)\,(\Delta_2-k_2+1)\,(\Delta_1+\Delta_2+1)}{(\Delta_1+1)\,(\Delta_2+1)\,(\Delta_1+\Delta_2-k_1-k_2+1)}\,\frac{(k_1+k_2)!}{k_1!\,k_2!}} \tag{48}$$

$$\times \left[Z^{\frac{1}{2}(k_1+k_2)} \times \left[(\$_2)^{\frac{\Delta_2}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}\right]^{\frac{1}{2}(\Delta_1+\Delta_2)}\right]^{\frac{1}{2}(\Delta_1+\Delta_2-k_1-k_2)}_{\frac{1}{2}(\Delta_1+\Delta_2-k_1-k_2)},$$

when combined with Eq. (47) and its Δ_2, k_2 analog, leads to a k_1, k_2-independent result for fixed $k_1 + k_2 = k - k_3$. The $k_1(k_2)$-sum over binomial coefficients is therefore trivial and leads to the vector coupled form

$$\left\langle \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ \Delta & & 0 & & 0 \\ & \Delta-k & & 0 & \\ & & \Delta-k & & \end{array}\right\rangle$$

$$
\begin{aligned}
&= \left[\binom{\Delta}{k}\right]^{-\frac{1}{2}} \sum_{k_3=0}^{k} \sqrt{\frac{(\Delta_1+\Delta_2)!\,(\Delta_1+\Delta_2+1)!}{(\Delta_1+\Delta_2-k+k_3)!\,(\Delta_1+\Delta_2-k+k_3+1)!\,(k-k_3)!}} \\
&\times \binom{\Delta_3}{k_3} \left[Z^{\frac{1}{2}(k-k_3)} \times \left[(\not{S}_2)^{\frac{\Delta_2}{2}} \times (\not{S}_1)^{\frac{\Delta_1}{2}}\right]^{\frac{\Delta_1+\Delta_2}{2}} \right]^{\frac{1}{2}(\Delta_1+\Delta_2-k+k_3)}_{\frac{1}{2}(\Delta_1+\Delta_2-k+k_3)} \\
&\times \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle \right)^{\Delta_3-k_3} \left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle \right)^{k_3} .
\end{aligned}
\tag{49}
$$

The operator $\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle$ is a U(2)-scalar. From Eq. (34) and the simple eigenvalues of $(\mathbf{I}^{\text{tot}} \cdot \mathbf{I}^{\text{tot}})$ and $\frac{1}{2}(\not{E}_{11} + \not{E}_{22} - 2\not{E}_{33} - z_{\alpha 3}\partial_{\alpha 3})$ it can be seen that

$$
\begin{aligned}
&\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+1 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right| \left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle \\
&= (m_{12}-m_{33}+1)\,(m_{22}-m_{33}) = \frac{K^2([\mathbf{m}_3'],\,[\mathbf{m}_2])}{K^2([\mathbf{m}_3],\,[\mathbf{m}_2])}\,(m_{13}-m_{33}+1)\,(m_{23}-m_{33}),
\end{aligned}
\tag{50}
$$

with $[\mathbf{m}_3'] = [m_{13}\, m_{23}\, m_{33}+1]$ [see Eq. (18)]. The k_3 successive applications of this operator then yield

$$
\begin{aligned}
&\left(\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{array} \right\rangle \right)^{k_3} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle = \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+k_3 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle \\
&\times \frac{K^2([m_{13}\, m_{23}\, m_{33}+k_3],\,[\mathbf{m}_2])}{K^2([m_{13}\, m_{23}\, m_{33}],\,[\mathbf{m}_2])}\, \frac{(m_{13}-m_{33}+1)!\,(m_{23}-m_{33})!}{(m_{13}-m_{33}+1-k_3)!\,(m_{23}-m_{33}-k_3)!} .
\end{aligned}
\tag{51}
$$

Next, the eigenvalue difference $(\Omega - \Omega')$ of Eq. (36) can be put in the form

$$
\begin{aligned}
&\Omega([\mathbf{m}_3],\,[\mathbf{m}_2]) - \Omega([\mathbf{m}_3],\,[\mathbf{m}_2']) \\
&= -\frac{K^2([m_{13}\, m_{23}\, m_{33}+1],\,[m_{12}'\, m_{22}'])}{K^2([m_{13}\, m_{23}\, m_{33}],\,[m_{12}\, m_{22}])}\,(m_{13}-m_{33}+1)\,(m_{23}-m_{33}).
\end{aligned}
\tag{52}
$$

Through the use of Eqs. (34) and (52), the $(\Delta_3 - k_3)$-fold action of the tensor

$$
\left\langle \begin{array}{ccccc} & & 0 & & \\ & 0 & & 0 & \\ 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{array} \right\rangle \equiv (\Upsilon)
$$

can thus be replaced by

$$
\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+\Delta_3 \\ & \bar{m}_{12} & & \bar{m}_{22} & \\ & & m_{11}+\Delta_3-k_3 & & \end{array} \right| (\Upsilon)^{\Delta_3-k_3} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+k_3 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle
$$

$$
\begin{aligned}
&= (-1)^{\Delta_3-k_3}\,\frac{K^2([m_{13}\;m_{23}\;m_{33}+\Delta_3],\;[\bar{m}_{12}\;\bar{m}_{22}])}{K^2([m_{13}\;m_{23}\;m_{33}+k_3],\;[m_{12}\;m_{22}])} \\
&\times \frac{(m_{13}-m_{33}+1-k_3)!\,(m_{23}-m_{33}-k_3)!}{(m_{13}-m_{33}+1-\Delta_3)!\,(m_{23}-m_{33}-\Delta_3)!} \qquad (53)\\
&\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+\Delta_3 \\ & \bar{m}_{12} & & \bar{m}_{22} & \\ & & m_{11}+\Delta_3-k_3 & & \end{array} \right| (\partial_{13})^{\Delta_3-k_3} \left| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+k_3 \\ & m_{12} & & m_{22} & \\ & & m_{11} & & \end{array} \right\rangle
\end{aligned}
$$

with $\bar{m}_{12}+\bar{m}_{22} \;=\; m_{12}+m_{22}+\Delta_3-k_3$.

The SU(2)-reduced matrix element of the totally symmetric shift tensor, including the initial/final-state K-ratio factor, can thus be replaced through ordinary angular momentum recoupling techniques by

$$
\begin{aligned}
&\left\langle \begin{array}{ccccc} m_{13}+\Delta_1 & & m_{23}+\Delta_2 & & m_{33}+\Delta_3 \\ & m'_{12} & & m'_{22} & \end{array} \right\| \\
&\qquad \left\langle \Delta \begin{array}{ccccc} & & \Delta_1 & & \\ & \Delta_1+\Delta_2 & & 0 & \\ & & 0 & & 0 \\ & \Delta-k & & 0 & \\ & & \_\_ & & \end{array} \right\rangle \left\| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33} \\ & m_{12} & & m_{22} & \end{array} \right\rangle \\
&= (-1)^{\Phi}\,\left[\binom{\Delta}{k}\right]^{-\frac{1}{2}} \sum_{k_3=0}^{k} \sqrt{\frac{(\Delta_1+\Delta_2)!\,(\Delta_1+\Delta_2+1)!}{(\Delta_1+\Delta_2-k+k_3)!\,(\Delta_1+\Delta_2-k+k_3+1)!\,(k-k_3)!}} \\
&\times \binom{\Delta_3}{k_3}(-1)^{\Delta_3-k_3}\sqrt{(\Delta_3-k_3)!}\sum_{[\bar{m}_{12}\;\bar{m}_{22}]} \frac{(m_{13}-m_{33}+1)!\,(m_{23}-m_{33})!}{(m_{13}-m_{33}-\Delta_3+1)!\,(m_{23}-m_{33}-\Delta_3)!} \\
&\times \frac{K^2([m_{13}\;m_{23}\;m_{33}+\Delta_3],\;[\bar{m}_{12}\;\bar{m}_{22}])}{K([m_{13}\;m_{23}\;m_{33}],\;[m_{12}\;m_{22}])\;K([m'_{13}\;m'_{23}\;m'_{33}],\;[m'_{12}\;m'_{22}])} \qquad (54)\\
&\times\; U([m_{12}m_{22}]\,[\Delta_3-k_3,0]\,[m'_{12}m'_{22}]\,[\Delta_1+\Delta_2-k+k_3,0];\,[\bar{m}_{12}\bar{m}_{22}]\;[\Delta-k,0]) \\
&\times \left\langle \begin{array}{ccccc} m_{13} & m_{23} & & m_{33}+\Delta_3 & \\ & \bar{m}_{12} & & \bar{m}_{22} & \end{array} \right\| Z^{\frac{1}{2}(\Delta_3-k_3)}\,(\partial) \left\| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+k_3 \\ & m_{12} & & m_{22} & \end{array} \right\rangle \\
&\times \left\langle \begin{array}{ccccc} m_{13}+\Delta_1 & & m_{23}+\Delta_2 & & m_{33}+\Delta_3 \\ & m'_{12} & & m'_{22} & \end{array} \right\| \\
&\quad \left[Z^{\frac{1}{2}(k-k_3)}(\mathbf{z})\left[(\mathcal{S}_2)^{\frac{\Delta_2}{2}}\times(\mathcal{S}_1)^{\frac{\Delta_1}{2}}\right]^{\frac{1}{2}(\Delta_1+\Delta_2)}\right]^{\frac{1}{2}(\Delta_1+\Delta_2-k+k_3)} \\
&\quad \left\| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+\Delta_3 \\ & \bar{m}_{12} & & \bar{m}_{22} & \end{array} \right\rangle
\end{aligned}
$$

with phase factor $\Phi \;=\; m_{22}+\Delta_2-m'_{22}$. Since $\bar{m}_{12}+\bar{m}_{22}$ is fixed by k_3, the additional sum is effectively a single sum over the intermediate angular momentum quantum number $\frac{1}{2}(\bar{m}_{12}-\bar{m}_{22})$. The U-coefficient, written here in U(2) form, is a standard angular momentum unitary-form Racah coefficient. In Eq. (54) we have also expressed the collective operator $(\partial_{13})^{\Delta_3-k_3}$ in terms of the standard collective polynomial

$$
(\partial_{13})^{\Delta_3-k_3} \;=\; \sqrt{(\Delta_3-k_3)!}\;Z^{[\Delta_3-k_3,\,0]}_{M_{11}=\Delta_3-k_3}(\partial) \;\equiv\; \sqrt{(\Delta_3-k_3)!}\;Z^{\frac{1}{2}(\Delta_3-k_3)}_{\frac{1}{2}(\Delta_3-k_3)}(\partial). \qquad (55)
$$

The SU(2) reduced matrix elements in the angular momentum coupled basis (16) are given by

$$
\left\langle \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+\Delta_3 \\ & \bar{m}_{12} & & \bar{m}_{22} & \end{array} \right\| Z^{\frac{1}{2}(\Delta_3-k_3)}(\partial) \left\| \begin{array}{ccccc} m_{13} & & m_{23} & & m_{33}+k_3 \\ & m_{12} & & m_{22} & \end{array} \right\rangle
$$

$$= U([m_{13}\, m_{23}]\, [0,\ -w]\ [\bar{m}_{12}\, \bar{m}_{22}]\ [\Delta_3 - k_3,\ 0];\ [m_{12}\, m_{22}]\ [0,\ -(w - \Delta_3 + k_3)])$$
$$\times\ \langle \tfrac{1}{2}(w - \Delta_3 + k_3) \| Z^{\frac{1}{2}(\Delta_3 - k_3)}(\partial)\ \| \tfrac{1}{2} w \rangle \tag{56a}$$

with the pure z-space SU(2) reduced matrix element

$$\langle \tfrac{1}{2}(w - \Delta_3 + k_3) \| Z^{\frac{1}{2}(\Delta_3 - k_3)}(\partial) \| \tfrac{1}{2} w \rangle \ =\ (-1)^{\Delta_3 - k_3} \sqrt{\frac{(w+1)!}{(w + 1 - \Delta_3 + k_3)!\ (\Delta_3 - k_3)!}}. \tag{56b}$$

Similarly

$$\left\langle \begin{matrix} m_{13} + \Delta_1 & & m_{23} + \Delta_2 & & m_{33} + \Delta_3 \\ & m'_{12} & & m'_{22} & \end{matrix} \right\|$$
$$\times \left[Z^{\frac{1}{2}(k - k_3)}(\mathbf{z}) [(\$_2)^{\frac{\Delta_2}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}]^{\frac{1}{2}(\Delta_1 + \Delta_2)} \right]^{\frac{1}{2}(\Delta_1 + \Delta_2 - k + k_3)}$$
$$\left\| \begin{matrix} m_{13} & & m_{23} & & m_{33} + \Delta_3 \\ & \bar{m}_{12} & & \bar{m}_{22} & \end{matrix} \right\rangle$$
$$= \begin{bmatrix} [m_{13}\, m_{23}] & [0,\ -(w - \Delta_3 + k_3)] & [\bar{m}_{12}\, \bar{m}_{22}] \\ [\Delta_1 + \Delta_2,\ 0] & [0,\ -(k - k_3)] & [\Delta_1 + \Delta_2 - k - k_3,\ 0] \\ [m'_{13}\, m'_{23}] & [0,\ -(w - \Delta_3 + k)] & [m'_{12}\, m'_{22}] \end{bmatrix} \tag{57a}$$
$$\times\ \langle \tfrac{1}{2}(w - \Delta_3 + k) \| Z^{\frac{1}{2}(k - k_3)}(\mathbf{z}) \| \tfrac{1}{2}(w - \Delta_3 + k_3) \rangle$$
$$\times\ \langle \tfrac{1}{2}(m_{13} + \Delta_1 - m_{23} - \Delta_2) \| [(\$_2)^{\frac{\Delta_2}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}]^{\frac{1}{2}(\Delta_1 + \Delta_2)} \| \tfrac{1}{2}(m_{13} - m_{23}) \rangle,$$

with the pure z-space and pure intrinsic space SU(2)-reduced matrix elements

$$\langle \tfrac{1}{2}(w - \Delta_3 + k) \| Z^{\frac{1}{2}(k - k_3)}(\mathbf{z}) \| \tfrac{1}{2}(w - \Delta_3 + k_3) \rangle$$
$$= \sqrt{\frac{(w - \Delta_3 + k)!}{(w - \Delta_3 + k_3)!\ (k - k_3)!}} \tag{57b}$$

and

$$\langle \tfrac{1}{2}(m_{13} + \Delta_1 - m_{23} - \Delta_2) \| [(\$_2)^{\frac{\Delta_2}{2}} \times (\$_1)^{\frac{\Delta_1}{2}}]^{\frac{1}{2}(\Delta_1 + \Delta_2)} \| \tfrac{1}{2}(m_{13} - m_{23}) \rangle$$
$$= \sqrt{\frac{\Delta_1!\ \Delta_2!\ (m_{13} - m_{23})!\ (m_{13} - m_{23} + \Delta_1 + 1)!}{(\Delta_1 + \Delta_2)!\ (m_{13} - m_{23} - \Delta_2)!\ (m_{13} - m_{23} + \Delta_1 - \Delta_2 + 1)!}}. \tag{57c}$$

Since the vector-coupled operator now involves both an intrinsic and a collective component, the unitary 9-j coefficient in Eq. (57a), written again in U(2) form, now will not collapse to a Racah coefficient as in Eq. (56a).

The totally symmetric shift tensor of Eqs. (46) and (54) is not a unit tensor. To convert the SU(2) matrix element of Eq. (54) into an SU(3) $\supset$ SU(2) reduced Wigner coefficient, this matrix element must be divided by the U(3)-reduced (double-caret, double-bar) matrix element of this operator

$$\langle\langle [m_{13} + \Delta_1,\ m_{23} + \Delta_2,\ m_{33} + \Delta_3] \| \begin{bmatrix} & & \Delta_1 & \\ & \Delta_1 + \Delta_2 & & 0 \\ \Delta & & 0 & & 0 \end{bmatrix} \| [m_{13}\, m_{23}\, m_{33}] \rangle\rangle$$
$$= \sqrt{\frac{\Delta_1!\ \Delta_2!\ \Delta_3!}{(\Delta_1 + \Delta_2 + \Delta_3)!}} \prod_{i<j}^{3} \sqrt{\frac{(m_{i3} - m_{j3} + j - i - 1)!}{(m_{i3} - m_{j3} - \Delta_j + j - i - 1)!}} \tag{58}$$

$$\times \prod_{i<j}^{3} \sqrt{\frac{(m_{i3} - m_{j3} + \Delta_i + j - i)!}{(m_{i3} - m_{j3} + \Delta_i - \Delta_j + j - i)!}},$$

where this formula can be derived by a successive application of the buildup relation, starting with $\Delta = 1$,

$$\begin{aligned} &\langle\langle [\mathbf{m}_3 + \boldsymbol{\Delta}] \| [T^{[\Delta-1,\ 00]} \times T^{[100]}]^{[\Delta 00]} \| [\mathbf{m}_3] \rangle\rangle \\ &= U([\mathbf{m}_3][100][\mathbf{m}_3 + \boldsymbol{\Delta}][\Delta - 1,\ 00];\ [\mathbf{m}_3 + \mathbf{1}]\ [\Delta 00]) \\ &\quad \times \langle\langle [\mathbf{m}_3 + \mathbf{1}] \| T^{[100]} \| [\mathbf{m}_3] \rangle\rangle\ \langle\langle [\mathbf{m}_3 + \Delta] \| T^{[\Delta-1,\ 00]} \| [\mathbf{m}_3 + \mathbf{1}] \rangle\rangle. \end{aligned} \tag{59}$$

The U-coefficient is now a U(3) Racah coefficient whose value is known from permutation group techniques (see *e.g.* Eq. (A.9) of Ref. [17]), and the fundamental tensor U(3)-reduced matrix elements are easy to calculate [see, *e.g.*, Eq. (31*c*)]. We note that the result is independent of coupling order and independent of the choice of shift in the representation $[\mathbf{m}_3 + \mathbf{1}]$ for these *totally symmetric* tensors.

Eqs. (54), with (56), (57), and (58) lead to an expression for the SU(3) $\supset$ SU(2) Wigner coefficient for the general coupling of $[\mathbf{m}_3] \times [\Delta 000]$ with $\Delta = \Delta_1 + \Delta_2 + \Delta_3$:

$$\begin{aligned} &\langle [\mathbf{m}_3]\ [\mathbf{m}_2];\ [\Delta 00]\ [\Delta - k,\ 0]\ \|\ [\mathbf{m}_3']\ [\mathbf{m}_2'] \rangle \\ &= (-1)^{m_{22}+\Delta_2-m_{22}'} \sqrt{\frac{(m_{13} - m_{33} + 1)!\ (m_{23} - m_{33})!}{(m_{13} - m_{33} - \Delta_3 + 1)!\ (m_{23} - m_{33} - \Delta_3)!}} \\ &\times \sqrt{\frac{(m_{13} + \Delta_1 - m_{33} - \Delta_3 + 2)!\ (m_{23} + \Delta_2 - m_{33} - \Delta_3 + 1)!}{(m_{13} + \Delta_1 - m_{33} + 2)!\ (m_{23} + \Delta_2 - m_{33} + 1)!}} \\ &\times \sum_{k_3=0}^{k} \sum_{\frac{1}{2}(\bar{m}_{12}-\bar{m}_{22})} \sqrt{\frac{k!\ (\Delta - k)!\ (\Delta_1 + \Delta_2 + 1)!\ \Delta_3!\ (w+1)!\ (w - \Delta_3 + k)!}{(\Delta_1 + \Delta_2 - k + k_3 + 1)\ (w - \Delta_3 + k_3 + 1)}} \\ &\times \frac{1}{k_3!\ (\Delta_3 - k_3)!\ (k - k_3)!\ (\Delta_1 + \Delta_2 - k + k_3)!\ (w - \Delta_3 + k_3)!} \\ &\times \frac{K^2([m_{13}\ m_{23}\ m_{33} + \Delta_3],\ [\bar{m}_{12}\ \bar{m}_{22}])}{K([m_{13}\ m_{23}\ m_{33}],\ [m_{12}\ m_{22}])K([m_{13}'\ m_{23}'\ m_{33}']),\ [m_{12}'\ m_{22}']} \\ &\times \begin{bmatrix} \frac{1}{2}(m_{13} - m_{23}) & \frac{1}{2}(w - \Delta_3 + k_3) & \frac{1}{2}(\bar{m}_{12} - \bar{m}_{22}) \\ \frac{1}{2}(\Delta_1 + \Delta_2) & \frac{1}{2}(k - k_3) & \frac{1}{2}(\Delta_1 + \Delta_2 - k + k_3) \\ \frac{1}{2}(m_{13}' - m_{23}') & \frac{1}{2}(w - \Delta_3 + k) & \frac{1}{2}(m_{12}' - m_{22}') \end{bmatrix} \\ &\times U\Big(\frac{(m_{13} - m_{23})}{2} \frac{w}{2} \frac{(\bar{m}_{12} - \bar{m}_{22})}{2} \frac{(\Delta_3 - k_3)}{2};\ \frac{(m_{12} - m_{22})}{2} \frac{(w - \Delta_3 + k_3)}{2}\Big) \\ &\times U\Big(\frac{(m_{12} - m_{22})}{2} \frac{(\Delta_3 - k_3)}{2} \frac{(m_{12}' - m_{22}')}{2} \frac{(\Delta_1 + \Delta_2 - k + k_3)}{2};\ \frac{(\bar{m}_{12} - \bar{m}_{22})}{2} \frac{(\Delta - k)}{2}\Big) \end{aligned} \tag{60}$$

where the recoupling coefficients are now written in terms of standard angular momentum notation. This result is in agreement with the expression derived by the earlier version of the generalized VCS method (see Eq. (84) of Ref. [11]). Since this relation includes two summations, the new expression is in terms of its complexity comparable to previously known results. Now, however, all SU(3) results are expressed in terms of only two basic ingredients, SU(2)-recoupling coefficients and the simple K-factors of VCS theory.

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