# SU(3) TECHNIQUES FOR ANGULAR MOMENTUM PROJECTED MATRIX ELEMENTS IN MULTI-CLUSTER PROBLEMS+

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The theory of integral transforms furnishes a powerful tool for the evaluation of the resonating group kernels needed for cluster model calculations; but the evaluation of matrix elements in an angular momentum coupled basis has proved to be difficult for cluster problems involving more than two fragments. For multi-cluster wave functions SU(3) coupling and recoupling techniques can furnish a tool for the practical evaluation of matrix elements in an angular momentum coupled basis if the several relative motion harmonic oscillator functions are first expressed in SU(3)-coupled form. Of the integral transforms employed in cluster model calculations the Bargmann-Segal (B-S) transform[1] is ideally suited to this technique since oscillator functions in Bargmann space have simple SU(3) coupling properties. The method is illustrated by a 3-cluster problem, such as $^{12}C=\alpha+\alpha+\alpha$, involving three $^1S$ clusters. In this case the kernels, $\mathcal{K}$, are functions of two relative motion Jacobi vectors, $\vec{R}_1$, $\vec{R}_2$; and the integrals in an angular momentum basis are best expressed in terms of

$$\int d\vec{\vec{R}}\, d\vec{R}\, \left[\psi^*(\vec{\vec{R}}_1)^{(n_10)} \times \psi^*(\vec{\vec{R}}_2)^{(n_20)}\right]^{(\lambda\mu)}_{\kappa LM} \mathcal{K} \left[\psi(\vec{R}_1)^{(m_10)} \times \psi(\vec{R}_2)^{(m_20)}\right]^{(\lambda'\mu')}_{\kappa' LM}$$

$$= \sum_{\substack{(\lambda_0\mu_0)\\ \rho_0}} \langle n_1n_2(\lambda\mu)\|\mathcal{K}^{(\lambda_0\mu_0)}\|m_1m_2(\lambda'\mu')\rangle_{\rho_0} \frac{\langle(\lambda\mu)\kappa L;(\mu'\lambda')\kappa' L\|(\lambda_0\mu_0)0\rangle_{\rho_0}}{\sqrt{2L+1}} \tag{1}$$

with $\vec{R}\equiv\vec{R}_1,\vec{R}_2$; $\vec{\vec{R}}\equiv\vec{\vec{R}}_1,\vec{\vec{R}}_2$. Superscripts ( ) indicate SU(3) quantum numbers, and the square brackets denote SU(3) coupling of the relative motion functions. The kernel is imagined to be expanded in terms of SU(3) irreducible tensor components $(\lambda_0\mu_0)$ whose reduced matrix elements appear in (1) in combination with readily available[2] SU(3)$\supset$R(3) Wigner coefficients which carry the angular momentum (L) dependence. The reduced matrix elements are evaluated through an expansion of the B-S transform of the kernel, $\mathcal{K}$,

$$\int d\vec{\vec{R}}\, d\vec{R}\, A(\vec{\vec{K}},\vec{\vec{R}})\, \mathcal{K}\, A^*(\vec{K},\vec{R}) = \sum_\beta c_\beta e^{\rho(\beta)} e^{\sigma(\beta)} e^{\tau(\beta)}$$

$$= \sum_{\substack{n_1n_2m_1m_2\\ (\lambda\mu)(\lambda'\mu')(\lambda_0\mu_0)\rho_0}} \langle n_1n_2(\lambda\mu)\|\mathcal{K}^{(\lambda_0\mu_0)}\|m_1m_2(\lambda'\mu')\rangle_{\rho_0} \tag{2}$$

$$\times \left[\left[P(\vec{\vec{K}}_1)^{(n_10)} \times P(\vec{\vec{K}}_2)^{(n_20)}\right]^{(\lambda\mu)} \times \left[P(\vec{K}_1^*)^{(0m_1)} \times P(\vec{K}_2^*)^{(0m_2)}\right]^{(\mu'\lambda')}\right]^{(\lambda_0\mu_0)}_{\rho_0\, L_0=0}$$

On the one hand the expression is in terms of SU(3)-coupled polynomials in the Bargmann space variables $\vec{K}_1,\vec{K}_2$ which in transformed space correspond to $\vec{R}_1,\vec{R}_2$. On the other hand the value of this B-S transform is known[1] in terms of simple Gaussian functions where antisymmetrization is handled by a sum over double cosets (the $\beta$-sum in (2)). The $\sigma$-factor (expressed in short-hand form in (2)) is an SU(3)-scalar so that its expansion in terms of SU(3)-coupled K-space polynomials is particularly simple

$$e^{\sigma} \equiv \exp\Big(\sum_{i,j=1}^{2} \sigma_{ij}\, \vec{K}_i \cdot \vec{K}_j^{*}\Big) = \sum_{\ell m n j (\lambda_\sigma \mu_\sigma)} \sigma_{11}^{\ell}\, \sigma_{12}^{m}\, \sigma_{21}^{n}\, \sigma_{22}^{j}$$

$$\times\sqrt{\dim(\lambda_\sigma\mu_\sigma)\binom{\ell+m}{\ell}\binom{n+j}{j}\binom{\ell+n}{\ell}\binom{m+j}{j}}\begin{bmatrix}(\ell 0) & (m0) & (\ell+m,0)\\ (n0) & (j0) & (n+j,0)\\ (\ell+n,0) & (m+j,0) & (\lambda_\sigma\mu_\sigma)\end{bmatrix} \qquad (3)$$

$$\times\Big[\big[P(\vec{K}_1)^{(\ell+m,0)} \times P(\vec{K}_2)^{(n+j,0)}\big]^{(\lambda_\sigma\mu_\sigma)} \times \big[P(\vec{K}_1^{*})^{(0,\ell+n)} \times P(\vec{K}_2^{*})^{(0,m+j)}\big]^{(\mu_\sigma\lambda_\sigma)}\Big]^{(00)}_{0}$$

where the 9-($\lambda\mu$) coefficient is equivalent to a simple SU(2) 9-j coefficient. The $\rho$-factor $\Big(\exp\big(\sum_{i,j=1}^{2} \rho_{ij}\, \vec{K}_i \cdot \vec{K}_j\big)\Big)$ is expanded in terms of SU(3)-tensors $\big[P(\vec{K}_1)\times P(\vec{K}_2)\big]^{(\lambda\mu)}$ by similar but somewhat more complicated expansions, (similarly for the $\tau$-factor in terms of $P(\vec{K}^{*})$). The product of the three is then reorganized by SU(3) recoupling transformations into an expansion of final SU(3)-coupled tensors of the form appearing in (2) with coefficients which give the reduced matrix elements needed for (1). The only ingredients needed for the practical exploitation of this technique are a) the $\rho,\sigma,\tau$ matrix elements for the two-body and norm kernels and b) readily available[2] SU(3) Racah coefficients, SU(2) 9-j coefficients with at least two stretched couplings, and a few SU(3)$\supset$R(3) Wigner coefficients.

1. T. H. Seligman and W. Zahn, J. Phys. G2, 79 (1976).
2. J. P. Draayer and Y. Akiyama, J. Math. Phys. 14, 1904 (1973), and Comp. Phys. Commun. 5, 405 (1973).

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