Laminar and Turbulent Dissipation in Shear Flow with Suction

Charles R. Doering*, Edward A. Spiegel† and Rodney A. Worthing*

*Department of Mathematics, University of Michigan[1]
Ann Arbor, Michigan 48109-1109
†Department of Astronomy, Columbia University
New York, New York 10027

Abstract. The rate of viscous energy dissipation in a shear layer of incompressible Newtonian fluid with injection and suction is studied by means of exact solutions, nonlinear and linearized stability theory, and rigorous upper bounds. For large enough values of the injection angle a steady laminar flow is nonlinearly stable for all Reynolds numbers, while for small but nonzero angles the laminar flow is linearly unstable at high Reynolds numbers. The upper bound on the energy dissipation rate—valid even for turbulent solutions of the Navier-Stokes equations—scales precisely the same as that in the steady laminar solution with regard to the viscosity in the vanishing viscosity limit. Both the laminar dissipation and the upper bound on turbulent dissipation display scaling in which the energy dissipation rate becomes independent of the viscosity for high Reynolds numbers. Hence the laminar energy dissipation rate and the largest possible turbulent energy dissipation rate for flows in this geometry differ by just a prefactor that depends only on injection angle. This result establishes the sharpness of the upper bound's scaling in the vanishing viscosity limit for these boundary conditions, and this system provides an analytic illustration of the delicacy of corrections to scaling (e.g., logarithmic terms as appearing in the "law of the wall") to perturbations in the boundary conditions.

INTRODUCTION

Turbulent fluid dynamics presents some of the most challenging unsolved problems in theoretical physics and applied mathematics. Even so, there are quantities we may hope to be able to bound when we cannot compute them in detail. Here we describe some of the ways of doing this within the context of a flow in a shear layer with fluid injection and suction at the boundaries. Via rigorous mathematical analysis we obtain bounds on the rate of viscous energy dissipation, valid for turbulent flows as well as for any laminar (steady or unsteady) flows, that may be

[1]) This research was supported in part by National Science Foundation awards to the University of Michigan.

CP502, *Stochastic and Chaotic Dynamics in the Lakes: STOCHAOS*,
edited by D. S. Broomhead, E. A. Luchinskaya, P.V. E. McClintock, and T. Mullin

directly compared to that in an exact solution for a flow with the same boundary conditions.

The quantitative study of upper bounds for turbulence was largely initiated by Howard in the early 1960s with his formulation of a variational principle for bounds on the bulk averaged rate of energy dissipation ε for statistically stationary flows [1]. This theory was developed by Busse and collaborators [2] and applied to a variety of shear flow and convection problems. In this paper we utilize the "background field" method that has its foundation in a mathematical device introduced in 1941 by Hopf [3]. Hopf's approach produces estimates on long time averages of bulk dissipation without any statistical hypotheses. It has been applied to incompressible turbulence in the Navier-Stokes equations [4–11], convection in a fluid layer [12–14] including infinite Prandtl number convection with [15] and without [16] rotation, and convection in a fluid saturated porous layer [17]. When applied to problems with sufficient geometrical symmetry so that Howard's statistical stationarity hypotheses may be invoked, these two approaches share a related mathematical structure [18] and have in many cases produced similar estimates [19].

For general geometries Wang [20] used Hopf's approach to show that ε is typically bounded independent of the viscosity (ν) when $\nu \to 0$ as long as there is no flux at the boundaries, i.e., if the flow is driven by shearing alone at the rigid boundaries. If there is flux (a normal component) at the boundary, then Hopf's original estimates, where the *a priori* bound varies exponentially with the Reynolds number (Re), are generally the only known limits. We find that the upper bound for the specific problem under consideration *with* mass flux at the rigid boundaries does not exhibit such an unphysical exponential dependence on the viscosity or the Reynolds number.

On the other hand, in many cases of turbulent shear flows along highly symmetric smooth boundaries, the emperical logarithmic friction law [21] (a.k.a. the "law of the wall") suggests that ε scales with corrections proportional to $(\log Re)^{-2}$ as $Re \to \infty$. For the example studied here, we find that the lower estimate of any upper bound on ε provided by the exact solution is itself *above* that predicted by the logarithmic friction law, in accord with the upper bound. Although we have not been able to prove that the steady laminar flow always produces an absolute lower bound on the dissipation, because it is an exact solution it does indeed produce a rigorous lower bound on any mathematical upper bound. Hence this model and the analysis described here is an important example for upper bound theory: the upper bound scaling in the vanishing viscosity limit is sharp; only the prefactor may be improved in the high Reynolds number limit.

In the following we present a complete statement of the problem and a summary of the results of our analysis. The details of the methods of analysis and the calculations, both analytic and computational, can be found in our complete presentation of this project [22].

STATEMENT OF THE PROBLEM AND DEFINITIONS

We consider a layer of an incompressible (unit density) Newtonian fluid with constant kinematic viscosity ν confined between parallel rigid planes separated by distance h. The bottom plate, at $y = 0$, is stationary and the top one at $y = h$ moves with speed U^* in the x-direction. The third direction has coordinate z and the unit vectors are $\mathbf{i}, \mathbf{j}, \mathbf{k}$. The velocity field is designated as $\mathbf{u} = \mathbf{i}u_x + \mathbf{j}u_y + \mathbf{k}u_z$. In addition to the shearing motion imposed by the boundaries, there is a uniform injection of fluid into the layer with speed (flux) V^* on the top plane, and fluid is removed uniformly at the same speed on the bottom plane. The conditions at the (rigid) boundaries are thus

$$\mathbf{u} = -\mathbf{j}V^* \quad \text{at} \quad y = 0, \tag{1}$$

$$\mathbf{u} = \mathbf{i}U^* - \mathbf{j}V^* \quad \text{at} \quad y = h. \tag{2}$$

In the interior, the velocity field and the pressure field $p(\mathbf{x}, t)$ are governed by the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} \tag{3}$$

$$\nabla \cdot \mathbf{u} = 0. \tag{4}$$

We restrict attention to periodic boundary conditions on all dependent variables in the horizontal directions with periods L_x and L_z.

The two dimensionless control parameters of this problem are the Reynolds number

$$Re = \frac{hU^*}{\nu} \tag{5}$$

and the entry angle, θ, given by

$$\tan\theta = \frac{V^*}{U^*} \ . \tag{6}$$

We use the notation

$$\|f\|_2 = \left(\int_0^{L_x} dx \int_0^h dy \int_0^{L_z} dz |f(x,y,z)|^2 \right)^{1/2} , \tag{7}$$

for the L_2 norm, the norm on the Hilbert space of square integrable functions in the layer, and

$$< f > = \limsup_{T \to \infty} \frac{1}{T} \int_0^T f(t) dt. \tag{8}$$

for the largest possible long time average of functions of time.

The rate of viscous energy dissipation is identified from the evolution equations for the kinetic energy in the fluid layer:

$$\frac{d}{dt}\,\frac{1}{2}\|\mathbf{u}(\cdot,t)\|_2^2 = V^*A\,\frac{1}{2}U^{*2} \;+\; V^*\left(\int_{top} p\;dxdz - \int_{bottom} p\;dxdz\right) + U^*\int_{top}\nu\,\frac{\partial u_x}{\partial y}\;dxdz \;-\; \nu\|\nabla\mathbf{u}(\cdot,t)\|_2^2. \tag{9}$$

The terms on the right hand side of this equation are, in order, the net flux of kinetic energy into the layer, the rate of work performed by the injection and suction processes, the power expended shearing the fluid layer, and finally, the power removed by viscous dissipation in the fluid. We thus identify the largest possible long time averaged bulk energy dissipation rate per unit mass in a solution as

$$\varepsilon = \frac{\nu}{hL_xL_z}\left\langle\|\nabla\mathbf{u}\|_2^2\right\rangle. \tag{10}$$

A simple exact steady solution of the problem is the laminar flow

$$\begin{aligned} u_x(y) &= U^*\,\frac{1-e^{-V^*y/\nu}}{1-e^{-Re\tan\theta}} \\ u_y &= -V^* \\ u_z &= 0. \end{aligned} \tag{11}$$

For $\theta = 0$ this solution reduces to plane Couette flow $\mathbf{u}_{Couette} = \mathbf{i}U^*y/h$. For $0 < \theta < 90°$ and not-too-small values of Re—in particular, for $Re\tan\theta >> 1$—the flow has a classical boundary layer structure near the suction plane at $y = 0$. That is, the velocity deviates from a nearly constant bulk flow only in a layer of thickness

$$\delta_{laminar} = \frac{\nu}{V^*} = \frac{h}{Re\tan\theta} \tag{12}$$

above the suction boundary.

The energy dissipation rate in the steady solution is

$$\varepsilon_{laminar} = \frac{v}{h}\int_0^h\left(\frac{\partial u_x}{\partial y}\right)^2 dy = \frac{U^{*3}}{h}\frac{\tan\theta}{2\tanh(\frac{1}{2}Re\tan\theta)} \tag{13}$$

Not unexpectedly, this expression reduces to the energy dissipation rate in planar Couette flow as $Re \to 0$ or $\theta \to 0$:

$$\lim_{Re\tan\theta\to 0}\varepsilon_{laminar} = \varepsilon_{Couette} = \nu\frac{U^{*2}}{h^2}. \tag{14}$$

Large values of the Reynolds number may be achieved in various ways for the flow geometry considered here, but in this study we concentrate on two just particular extreme limits. First note that for fixed velocities and geometry, the Reynolds number increases as the viscosity decreases: $Re \to \infty$ as $\nu \to 0$. The laminar flow has two distinct simple limits for vanishing viscosity. For $\theta = 0$, $\mathbf{u}_{Couette}$ does not depend on the viscosity at all and so retains its structure as $\nu \to 0$. But for $\theta \neq 0$, the vanishing viscosity limit is the constant flow field

$$\lim_{\nu \to 0} \mathbf{u}_{laminar} = \mathbf{i}U^* - \mathbf{j}V^* \quad (\theta \neq 0). \tag{15}$$

That is, the limiting velocity vector field is constant parallel flow in the bulk, continuous at the injection boundary, and *discontinuous* at the suction boundary. In other words, $\delta_{laminar} \to 0$ as $Re \to \infty$ at fixed $\theta \neq 0$ and h.

For $\theta = 0$, $\varepsilon_{Couette}$ vanishes as $\nu \to 0$. But for $\theta \neq 0$, the discontinuity in the flow at the suction boundary results in a residual dissipation in the limit of small viscosity:

$$\lim_{\nu \to 0} \varepsilon_{laminar} = \frac{\tan\theta}{2} \frac{U^{*3}}{h} \qquad (\theta \neq 0). \tag{16}$$

This residual dissipation in the fluid layer with suction is a manifestation of what may be referred to as ν^0 scaling in the vanishing viscosity limit, i.e., the energy dissipation rate is nonvanishing and independent of the viscosity as $Re \to \infty$. Dimensional analysis then insists that ε be composed of the cube of the velocity scale divided by a length scale in the system, precisely the content of equation (16) with the prefactor depending on details of the geometry of the flow, i.e., the angle θ. Such scaling is often associated with high Reynolds number energy dissipation in the presence of a turbulent energy cascade, but this flow is an example of a steady laminar flow in which the dissipation takes place on an ever smaller length scale ($\delta_{laminar}$) which disappears in the zero viscosity limit. One of the major points of this work is to compare this particular energy dissipation rate to an upper limit on the energy dissipation rate valid for *any* solutions of the Navier-Stokes equations, even solutions corresponding to turbulent flows.

The Reynolds number defined in (2.7) also becomes large for fixed velocities and viscosity as the layer thickness increases: $Re \to \infty$ as $h \to \infty$. The semi-infinite layer is a trivial limit for $\theta = 0$ at fixed U^* because the rate of strain in Couette flow vanishes as $h \to 0$. But for $\theta \neq 0$, the laminar solution with suction on the boundary has the nontrivial limit

$$\lim_{h \to \infty} \mathbf{u}_{laminar} = \mathbf{i}U^*(1 - e^{-y/\delta_{laminar}}) - \mathbf{j}V^* \quad (\theta \neq 0). \tag{17}$$

The energy dissipation rate per unit horizontal area of the suction boundary is then

$$\varepsilon'_{laminar} = \lim_{h \to \infty} \nu \int_0^h \left(\frac{\partial u_x}{\partial y}\right)^2 dy = \frac{\tan\theta}{2} U^{*3} \tag{18}$$

independent of the viscosity.

SUMMARY OF RESULTS

We explored the stability characteristics of the steady laminar solution in the $Re-\theta$ plane. Energy stability theory [23] was used to search for sufficient conditions for absolute stability. We found that for sufficiently large values of the injection angle ($\theta > 3°$) or sufficiently small values of the Reynolds number ($Re < 82$), the steady laminar flow is indeed absolutely stable. On the other hand we used linear stability theory [24] to establish sufficient conditions for instability. We found that for sufficiently small but nonzero angles ($0 < \theta < .001°$) the laminar flow is linearly unstable at high Reynolds numbers. This is consistent with a previous stability analysis for the semi-infinite layer [25]. Instability of the steady laminar flow does not prove that turbulence necessarily follows, but it is highly suggestive that turbulent flows may appear at high Reynolds numbers with sufficiently small injection angles.

Using the background filed method, we proved [22] that for any flow with Reynolds numbers $Re \geq 2\sqrt{2}$,

$$\varepsilon \leq \varepsilon_B \equiv \frac{1}{2\sqrt{2}} \frac{U^{*3}}{h} + \frac{2\sqrt{2}}{3} \frac{V^{*2}U^*}{h} \left(1 - \frac{3\sqrt{2}}{2} \frac{\nu}{U^*h}\right). \tag{19}$$

This is the explicit upper bound on the energy dissipation rate valid for any solution, steady, unsteady or turbulent, of the Navier-Stokes equations with these boundary conditions. In terms of the velocity scale U^* and the injection angle θ, the upper bound is

$$\varepsilon_B = \frac{1}{2\sqrt{2}} \left(1 + \frac{8}{3}\tan^2\theta \left(1 - \frac{3\sqrt{2}}{2}\frac{1}{Re}\right)\right) \frac{U^{*3}}{h}. \tag{20}$$

This estimate displays the ν^0 scaling at high Reynolds numbers:

$$\varepsilon_B \sim \frac{1}{2\sqrt{2}} \left(1 + \frac{8}{3}\tan^2\theta\right) \frac{U^{*3}}{h} \quad \text{as} \quad Re \to \infty. \tag{21}$$

Note also that the upper bound on the energy dissipation rate per unit horizontal area of the suction boundary is finite in the limit of a semi-infinite fluid layer:

$$\varepsilon' \leq \varepsilon'_B \equiv \lim_{h\to\infty} h\varepsilon_B = \frac{1}{2\sqrt{2}} \left(1 + \frac{8}{3}\tan^2\theta\right) U^{*3}. \tag{22}$$

The upper bound on the energy dissipation rate—valid even for turbulent solutions of the Navier-Stokes equations—scales precisely the same as that in the steady laminar solution with regard to the viscosity as $\nu \to 0$. The laminar dissipation and the upper bound on turbulent dissipation exhibit a residual dissipation in the vanishing viscosity limit when $\theta \neq 0$; both $\varepsilon_{laminar}$ and ε_B obey

$$\varepsilon \sim \frac{U^{*3}}{h}\mathcal{F}(\theta) \quad \text{as} \quad Re \to \infty. \tag{23}$$

Hence the turbulent bound and high Reynolds number laminar energy dissipation for flows in this geometry differ only by just a prefactor that depends only on the injection angle. This establishes the sharpness of the upper bound's scaling for these boundary conditions. This system provides an mathematically acessible example of the delicacy of corrections to high Reynolds number scaling—such as logarithmic terms as appearing in the law of the wall—to perturbations in the boundary conditions. This observation is consistent with the sensitivity of logarithmic corrections to wall roughness or other disturbances in turbulent shear flows [26,27].

REFERENCES

1. Howard, L. N., *J. Fluid Mech.* **17**, 405 (1963); *Ann. Rev. Fluid Mech.* **4**, 473 (1972).
2. Busse, F. *Adv. Applied Mech.* **18**, 77 (1978).
3. Hopf, E., *Math. Annalen* **117**, 764 (1941).
4. Doering, C. R., and Constantin, P., *Phys. Rev. Lett.* **69**, 1648 (1992).
5. Marchioro, C., *Physica D* **74**, 395 (1994).
6. Doering, C. R., and Constantin, P., *Phys. Rev. E* **49**, 4087 (1994).
7. Constantin, P. and Doering, C. R., *Physica D* **82**, 221 (1995).
8. Gebhardt, T., Grossmann, S., Holthaus, M. and Löhden, M., *Phys. Rev. E* **51**, 360 (1995).
9. Constantin, P. and Doering, C. R., *Phys. Rev. E* **51**, 3192 (1995).
10. Nicodemus, R., Grossmann, S. and Holthaus, M., *Physica D* **101**, 178 (1997).
11. Nicodemus, R., Grossmann, S. and Holthaus, M., *J. Fluid Mech.* **363**, 281 (1998).
12. Doering, C. R., and Constantin, P., *Phys. Rev. E* **53**, 5957 (1996).
13. Constantin, P. and Doering, C. R., *Nonlinearity* **9**, 1049 (1996).
14. Doering, C. R., and Hyman, J. H., *Phys. Rev. E* **55**, 7775 (1997).
15. Constantin, P., Hallstrom, C., and Putkaradze, V., *Physica D* **125** 275 (1999).
16. Constantin, P. and Doering, C. R., *J. Stat. Phys.* **94**, 159 (1999).
17. Doering, C. R., and Constantin, P., *J. Fluid Mech.* **376**, 263 (1998).
18. Kerswell, R., *Physica D* **121**, 175 (1998).
19. Kerswell, R., *Physica D* **100**, 355 (1997).
20. Wang, X., *Physica D* **99**, 555 (1997).
21. Prandtl, L., *Angew. Math. Mech.* **5**, 136 (1925). For a recent experimental realization, see Lathrop, D. P., Feinberg, J. and Swinney, H. L. *Phys. Rev. A* **46** 6390 (1992).
22. Doering, C. R., Spiegel, E. A., and Worthing, R. A., preprint, 1999.
23. Joseph, D. D., *Stability of Fluid Motions*, New York: Springer, 1976.
24. Drazin, P. and Reid, W., *Hydrodynamic Stability*, Cambridge, England: Cambridge University Press, 1981.
25. Hocking, L., *Quart. J. Mech. & Appl. Math.* **28**, 341 (1974).
26. See for example, Vennard, J., and Street, R., *Elementary Fluid Mechanics*, New York: John Wiley & Sons, 1980, ch. 9, pp. 372-380.
27. Cadot, O., Couder, Y., Daerr, A., Douady, S. and Tsinober, A., *Phys. Rev. E.* **56**, 427 (1997).