Borel singularities at small x

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Abstract.
   D.I.S. at small Bjorken x is considered within the dipole cascade formalism. The running coupling in impact parameter space is introduced in order to parametrize effects that arise from emission of large size dipoles. This results in a new evolution equation for the dipole cascade. Strong coupling effects are analyzed after transforming the evolution equation in Borel (b) space. The Borel singularities of the solution are discussed first for the universal part of the dipole cascade and then for the specific process of D.I.S. at small x. In the latter case the leading infrared renormalon is at $b = 1/\beta_0$ indicating the presence of $1/Q^2$ power corrections for the small-x structure functions.

INTRODUCTION

   Small-x D.I.S. is a typical example of the class of processes known as semi-hard processes. They are characterized by the presence of two large scales ordered as $\sqrt{s} \gg Q \gg \Lambda_{\mathrm{QCD}}$. Since the momentum transfer Q involved is large, these processes are amenable to perturbative QCD treatment. The resummation of leading logarithmic corrections of the energy, or $\ln(1/x)$ for D.I.S., is performed by the well known BFKL equation [1]. It is also known that for small enough x the LLA(x) result obtained à la BFKL, although infrared finite, receives contributions from low transverse momentum regions, where observables become sensitive to non-perturbative corrections. It is this region of low transverse momenta that will be considered here.

   QCD factorization in the small-x regime can be formulated either in transverse momentum space or in impact parameter space through the introduction of the dipole cascade in the s-channel [2], [3]. It is the latter formulation that will be used here. Its merit, apart from the simplicity of the final results, is that it organizes the perturbative expansion in terms of sequential soft gluon emissions in the s-channel, which in the Coulomb gauge have a clear probabilistic interpretation (to be contrasted with gluon ladders exchanged in the

t-channel used in the regge description). For fixed α_s this formulation has been shown to be equivalent to BFKL. The dipole approach is more suitable for introducing the running coupling because of its radiative nature. Just like in timelike cascades, relevant for jet evolution, the scale in the coupling will be assumed to be the virtuality of the emitted gluon. Once the dipole evolution equation with running coupling is constructed, its solutions are studied in Borel space and their singularities are identified. This analysis concerns the universal part of the dipole cascade which is independent of the external particles involed in the process (i.e. independent of impact factors in the BFKL formalism). Finally, for the specific process of D.I.S. at small x it is shown that the leading infrared renormalon (Borel singularity on the positive semi-axis) occurs at $b\beta_0 = 1$, which indicates power corrections for the structure functions of $\mathcal{O}(\Lambda^2_{\mathrm{QCD}}/Q^2)$.

DIPOLE EVOLUTION EQUATION WITH RUNNING COUPLING

Consider small-x D.I.S. in the rest frame of the nucleon target. The QCD factorization theorem for the structure functions takes the form [2]

$$F_{T,L}(x, Q^2) = \frac{Q^2}{4\pi\alpha_{em}} \int_0^1 dz \int d^2\mathbf{r}\, \Phi^{(0)}_{T,L}(z, r)\, \sigma_{d,N}(Y = \ln(z/x), \mathbf{r}). \quad (1)$$

Here, $\Phi^{(0)}_{T,L}(z, r)$ is the $\mathcal{O}(\alpha_{em})$ transition probability for $\gamma^\star \to q\bar{q}$ and $\sigma_{dN}(Y, \mathbf{r})$ is the q-$\bar{q}$ dipole-nucleon total cross section. The LLA(x) corrections in this formalism are generated by the sequential emission of soft gluons with strictly ordered rapidities. In the large N_c limit the emission of a soft gluon can be thought of as the production of a pair of dipoles each one of which can become the parent for further emissions. The whole cascade can be described in terms of the dipole density $n(Y, r, \rho)$ [4] which satisfies the evolution equation

$$\frac{\partial}{\partial Y} n(Y, r, \rho) = \int_0^\infty dr' \mathcal{K}(r, r')\, n(Y, r', \rho), \quad n(Y = 0, r, \rho) = r\,\delta(r - \rho). \quad (2)$$

The kernel $\mathcal{K}$ is calculated in perturbation theory to $\mathcal{O}(\alpha_s)$ from the $q\bar{q} \to q\bar{q}g$ process plus the corresponding virtual non-radiative process. In terms of the dipole density n, the cross section σ_{dN} becomes

$$\sigma_{dN}(Y, \mathbf{r}) = \int \frac{d^2\rho}{2\pi\rho^2}\, n(Y, r, \rho)\, \sigma_0(\rho, m_N). \quad (3)$$

All the large $\ln(1/x)$ corrections are contained in n, which is projectile and target independent, whereas σ_0 is the cross section for absorption of dipole of transverse size ρ by the nucleon target. σ_0 contains information about the

nucleon size and it is typically beyond the reach of perturbation theory, unless the nucleon is simulated by a small size onium state.

The introduction of the running coupling occurs at the level of the basic branching process $q\bar{q} \to q\bar{q}g$ and with scale $\alpha_s(\mathbf{k}^2)$, with $\mathbf{k}$ the transverse momentum of the emitted soft gluon. The derivation and the form of the kernel with running coupling are given in ref. [5]. Here it is more fruitful to consider the dipole evolution equation in Borel space. The Borel image $\tilde{n}$ of n with respect to $\alpha_s(Q^2)$ is defined as

$$n(Y, r, \rho; \alpha_s(Q^2)) = \int_0^\infty db\, \tilde{n}(Y, r, \rho; b)\, e^{-b/\alpha_s(Q^2)}\,. \tag{4}$$

This leads to the following evolution equation in Borel space.

$$\frac{\partial}{\partial Y}\tilde{n}(Y, r, \rho; b) = \int_0^\infty dr' \int_0^b db' \tilde{\mathcal{K}}(r, r'; b')\, \tilde{n}(Y, r', \rho; b - b')\,, \tag{5}$$

with boundary condition

$$\tilde{n}(Y = 0, r, \rho; b) = r\, \delta(r - \rho)\, \delta(b)\,. \tag{6}$$

The evolution kernel in Borel space is

$$\begin{aligned}
\tilde{\mathcal{K}}(r, r'; b) &= \frac{N_c}{\pi}\left(\frac{Q^2 r^2}{4}\right)^{b\beta_0} \frac{\Gamma(-b\beta_0)}{\Gamma(1 + b\beta_0)}\delta(r - r') \\
&\quad + 2\frac{N_c}{\pi^2}\frac{1}{r'}\left(\frac{Q^2 r'^2}{4}\right)^{b\beta_0} \int_0^1 \frac{d\omega}{\omega^{1/2}(1 - \omega)^{1/2}} \frac{\Gamma(1 - \omega b\beta_0)\,\Gamma(1 - (1 - \omega)b\beta_0)}{\Gamma(1 + \omega b\beta_0)\,\Gamma(1 + (1 - \omega)b\beta_0)} \\
&\quad \times \left\{\left(\frac{r_>^2}{r'^2}\right)^{b\beta_0 - 1}{}_2F_1\left(1 - b\beta_0, 1 - b\beta_0; 1; \frac{r_<^2}{r_>^2}\right)\right. \\
&\quad + \left(\frac{r^2 - r'^2}{r_>^2}\right)\left(\frac{r_>^2}{r'^2}\right)^{\omega b\beta_0}{}_2F_1\left(1 - \omega b\beta_0, 1 - \omega b\beta_0; 1; \frac{r_<^2}{r_>^2}\right) \\
&\quad \left. - \left(\frac{r_>^2}{r'^2}\right)^{\omega b\beta_0}{}_2F_1\left(-\omega b\beta_0, -\omega b\beta_0; 1; \frac{r_<^2}{r_>^2}\right)\right\}\,,
\end{aligned} \tag{7}$$

where $r_< = \min(r, r')$ and $r_> = \max(r, r')$ and $\beta_0 = (1/4\pi)[(11/3)N_c + (2/3)N_f]$. The first term in (7) comes from virtual corrections where no new dipole is emitted. The rest comes from real emission. Note that in the infrared limit, where there is emission of dipoles of large size, $r' = r_> \gg r = r_<$, the hypergeometric functions are analytic. This makes $\tilde{K}$ suitable for studying the infrared limit analytically and numerically.

BOREL SINGULARITIES AND POWER CORRECTIONS

Inspecting the kernel (7) it is seen that for fixed parent dipole size r there is a potential singularity at $b = 0$. However, using the properties of the hypergeometric functions it can be shown that the $b = 0$ singularity is of ultraviolet origin and cancels between the real and the virtual part. Near $b = 0$ the kernel is analytic. To exhibit the singularity structure, the kernel is convoluted with test functions $(r^2)^\gamma$. These are known to be eigenfunctions of the kernel in the fixed coupling case. After defining the function $\chi(\gamma, b)$ as

$$\int_0^\infty dr' \, \tilde{\mathcal{K}}(r, r'; b)(r'^2)^\gamma = \frac{N_c}{\pi}\chi(\gamma, b) \left(\frac{Q^2 r^2}{4}\right)^{b\beta_0} (r^2)^\gamma, \tag{8}$$

the expression of $\chi(\gamma, b)$ turns out to be

$$\chi(\gamma, b) = \frac{\Gamma(-b\beta_0)}{\Gamma(1 + b\beta_0)}$$
$$+ \frac{\Gamma(-\gamma - b\beta_0)}{\Gamma(1 + \gamma + b\beta_0)} \frac{1}{\pi} \int_0^1 \frac{d\omega}{\omega^{1/2}(1-\omega)^{1/2}} \frac{\Gamma(1 - \omega b\beta_0)}{\Gamma(1 + \omega b\beta_0)} \frac{\Gamma(1 - (1-\omega)b\beta_0)}{\Gamma(1 + (1-\omega)b\beta_0)}$$
$$\times \left[\frac{\Gamma(1+\gamma)}{\Gamma(-\gamma)} \frac{\Gamma(b\beta_0)}{\Gamma(1 - b\beta_0)} - 2\frac{\Gamma(1 + \gamma + (1-\omega)b\beta_0)}{\Gamma(1 - \gamma - (1-\omega)b\beta_0)} \frac{\Gamma(1 + \omega b\beta_0)}{\Gamma(1 - \omega b\beta_0)}\right], \tag{9}$$

where the first term comes from virtual correction and the second from real emission. The virtual contribution to $\chi(\gamma, b)$ contains a series of poles at $b\beta_0 = 1, 2, 3 \ldots$ which are identified with the IR renormalons and correspond to power corrections of $\mathcal{O}((m_N^2/Q^2)^n)$, $n = 1, 2, \ldots$ Note that these poles are independent of the specific form of the test function. This set of (familiar) poles result from the exponentiation of soft radiation. In addition, there is a series of poles at $b\beta_0 = n - \gamma$, $n = 0, 1, 2 \ldots$ generated by the $\Gamma(-\gamma - b\beta_0)$ dependence of the real contribution to $\chi(\gamma, b)$. For $\mathrm{Re}(\gamma) \geq m$ these poles correspond to IR renormalons for $n > m$. Their IR origin is established by observing that these singularities arise from the $r' > r$ integration region of eq. (5), where the offspring dipole is emitted with size larger than the parent dipole. Taken at face value the γ-dependent poles indicate the presence of $\mathcal{O}((m_N^2/Q^2)^{n-\gamma})$ power corrections.

Scale invariance is manifestly broken by the introduction of the running coupling and $(r^2)^\gamma$ are not eigenfuctions of the kernel. It is worth noting though that $\chi(\gamma, b)$ admits power series expansion around the conformal point $b = 0$ of the form

$$\chi(\gamma, b) = \chi(\gamma) + b\beta_0\chi^{(1)}(\gamma) + \mathcal{O}(b^2\beta_0^2), \tag{10}$$

$$\chi^{(1)}(\gamma) = -\frac{1}{\gamma}\chi(\gamma) - 2\Psi(1)\chi(\gamma) + \frac{1}{2}\chi(\gamma)^2 + \frac{1}{2}\chi'(\gamma). \tag{11}$$

The $\mathcal{O}(1/b)$ singular terms have cancelled as anticipated and the $\mathcal{O}(b^0)$ term is the BFKL spectral function $\chi(\gamma) = 2\Psi(1) - \Psi(\gamma) - \Psi(1-\gamma)$.

Eq. (5) is an integral equation in b of the Volterra type, hence for bounded kernel there are no eigenfunctions. However it can be solved formally by iteration. It can then be shown [6] that for fixed γ and (say) $0 < \gamma < 1$ subsequent iterations of the kernel do not change the position of the leading IR renormalon, which is a branch cut at $b = \gamma/\beta_0$. This would signal the presence of γ-dependent power corrections of $\mathcal{O}(m_N^{2\gamma}/Q^{2\gamma})$, and for $\gamma = 1/2$, the saddle point in the conformal (BFKL) limit, this results in $1/Q$ corrections [7].

However, as eqs. (1) and (3) indicate, the small-x structure functions are determined by the convolution of the dipole density n with $\Phi^{(0)}$ and σ_0. Test functions of the form $(r^2)^\gamma$, $\gamma > 0$, do not have an IR cutoff for $r \to \infty$, whereas both $\Phi^{(0)}$ and σ_0 do. Specifically, convolution with σ_0 will constrain the emission of arbitrarily large dipoles down to scale $R_N \sim m_N^{-1}$, where R_N is the characteristic length scale or size of the nucleon. Even though σ_0 cannot be calculated in perturbation theory, to study its effect on the Borel singularities of σ_{dN} it is enough to model $\sigma_0(\rho, m_N)$ by a function that regularizes it in the infrared through the scale R_N. A simple choice would be to approximate the nucleon by an onium state of size R_N. Then, via numerical integration of eq. (5), it can be seen that the leading IR renormalon is a branch cut at $b\beta_0 = 1$, signaling the presence of $\mathcal{O}(m_N^2/Q^2)$ power corrections for the structure functions at small x [6]. This is consistent with the expectation from Wilson OPE. We expect this formalism to yield information about the effect of the IR region on the perturbative pomeron intercept.

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