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Fractional boundary for the Gott-Hiscock string

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A fractional boundary condition is used to join the Gott-Hiscock string to a Levi-Civita vacuum. The use of a fractional derivative generates Israel boundary layers whose density depends on the order of the fractional derivative. Variable boundary layers for the same two bounding space–times can be studied. The string angular deficit depends on the order of the fractional deficit. © *2005 American Institute of Physics.* [DOI: 10.1063/1.1863692]

I. INTRODUCTION

The Gott-Hiscock[1,2] solution describes a constant density string matched to a vacuum Levi-Civita space–time with angular deficit. Two of the most common boundary matching conditions for non-null boundaries are due to Lichnerowicz and Darmois. These conditions have been discussed by Bonnor and Vickers.[3] Both conditions involve matching a metric and some metric derivatives across the boundary. The Darmois condition matches the metric and its second fundamental forms on the boundary while the Lichnerowicz condition matches the metric and its first derivatives on the boundary. The resulting space–time exterior to the string has an angular deficit related to the mass per unit length of the string interior. For general space—times, if a derivative, or extrinsic curvature, match is not possible, the most common procedure is to use the Israel[4] formalism, defining a surface boundary layer stress energy through the jump in the extrinsic curvature between space–times on either side of the boundary. Other space–times, whose extrinsic curvature was not continiuous across the boundary, could have been used for the exterior match, creating an Israel layer on the string boundary.

The Israel technique provides information about the stress energy content of the bounding surface layer. While the Israel surface layer can be used to describe the jump in the extrinsic curvature across a space–time boundary, it will not distinguish surface layers of different densities or structures that might bound the same two spacetimes. Within the Israel formalism, the only way of varying the boundary layer properties is to vary the bounding space–times.

One way of modifying the boundary conditions to include different kinds of boundary surfaces for the same bounding space–times is to generalize the Lichnerowicz boundary conditions to a fractional derivative matching and then to use the Israel formalism to describe the resulting surface layer. This creates boundary layers whose stress energy content depends on the order of the fractional derivative and allows the study of variable boundary structures between the same two space–times. In the next section, we apply this fractional matching method to the Gott-Hiscock string. The hallmark of string behavior, the angular deficit, also depends on the order of the fractional derivative. The change in the angular deficit introduced by the fractional match is discussed.

II. FRACTIONAL BOUNDARY CONDITION

A. The metrics

Consider a string oriented along the z axis with constant stress energy content

$$T_t^t = T_z^z = -\varepsilon. \tag{1}$$

From the field equations, the interior metric[2] is

$$ds^2 = -dt^2 + d\rho^2 + dz^2 + \rho^{*2}\sin^2\left(\frac{\rho}{\rho^*}\right)d\varphi^2, \tag{2}$$

where $\rho^* = (8\pi\epsilon)^{-1/2}$ and the string is assumed to be flat at $\rho = 0$.

The exterior vacuum space–time is

$$ds^2 = -dT^2 + dr^2 + dZ^2 + a^2r^2d\varphi^2. \tag{3}$$

B. The nonfractional boundary match

The metric match at $\rho = \rho_o$, $r = r_o$ provides the condition

$$ar_o = \rho^* \sin\left(\frac{\rho_o}{\rho^*}\right).$$

The extrinsic curvature is defined as[5]

$$K_{ij} = n_{(\alpha;\beta)}e_i^\alpha e_j^\beta,$$

where the e_i^α are the tangents to the boundary hypersurface with normal vector n_α. Calculating the extrinsic curvatures one finds in the interior

$$K_{\varphi\varphi} = \rho^* \sin\left(\frac{\rho_o}{\rho^*}\right)\cos\left(\frac{\rho_o}{\rho^*}\right). \tag{4}$$

In the exterior the extrinsic curvatures are

$$K_{tt} = 0,$$

$$K_{\varphi\varphi} = a^2r_o, \tag{5}$$

$$K_{zz} = 0.$$

The difference in the extrinsic curvatures across the boundary is

$$\langle K_{\phi\phi}\rangle = K_{\phi\phi}(\text{vacuum}) - K_{\phi\phi}(\text{interior}) = \left(a^2r_o - \rho^* \sin\left(\frac{\rho_o}{\rho^*}\right)\cos\left(\frac{\rho_o}{\rho^*}\right)\right),$$

$$\langle K\rangle = K_\varphi^\varphi(\text{vacuum}) - K_\varphi^\varphi(\text{interior}) = \frac{1}{r_o} - \frac{1}{\rho^*}\frac{\cos(\rho_o/\rho^*)}{\sin(\rho_o/\rho^*)} = \frac{1}{r_o} - \frac{1}{a}\frac{\cos(\rho_o/\rho^*)}{r_o}.$$

In the Israel formalism,[4] the stress energy of the boundary layer is

$$8\pi S_{ab} = -\{\langle K_{ab}\rangle - h_{ab}\langle K_c^c\rangle\},$$

$$8\pi S_{tt} = \langle K\rangle h_{oo} = \frac{\cos(\rho_o/\rho^*) - a}{ar_o},$$

$$8\pi S_{zz} = \frac{a - \cos(\rho_o/\rho^*)}{ar_o},$$

$$8\pi S_{\varphi\varphi} = 0. \tag{6}$$

If the extrinsic curvatures match, there is no stress energy in the boundary. The match of the extrinsic curvature provides the same condition as a match of the $g_{\phi\phi}$ derivatives across the string boundary,

$$a = \cos\left(\frac{\rho_o}{\rho^*}\right).$$

C. The fractional match

In the fractional boundary match, the metric matching conditions are the same as in the smooth boundary match:

$$ar_o = \rho^* \sin\left(\frac{\rho_o}{\rho^*}\right). \tag{7}$$

Instead of a match of the extrinsic curvatures, a fractional derivative match is used with the regular partial derivative, ∂_i replaced by an (α) order Caputo fractional derivative as described in the Appendix.

1. Fractional derivatives for the interior metric

In the interior, since the Caputo derivative of a constant is zero, the only function to consider is $f=\rho^{*2}\sin^2(\rho/\rho^*)$. Using α as the fractional index we have that the αth fractional derivative is

$$D^{(\alpha)}_{\rho,L}\left(\rho^{*2}\sin^2\left(\frac{\rho}{\rho^*}\right)\right) = \rho^* S_{\rho_o}(\alpha,\rho^*,\rho_o).$$

The form of the function $S_{\rho_o}(\alpha,\rho^*,\rho_o)$ depends on the range of the fractional derivative order. For $\alpha<1$ we have

$$S_{\rho_o}(\alpha,\rho^*,\rho_o) = \frac{(\rho^*/2)^{1-\alpha}}{\Gamma(1-\alpha)}\left[\sin\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*} z^{-\alpha}\cos(z)dz - \cos\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*} z^{-\alpha}\sin(z)dz\right]. \tag{8}$$

For $\alpha>1$ the function is

$$S_{\rho_o}(\alpha,\rho^*,\rho_o) = \frac{(\rho^*/2)^{1-\alpha}}{\Gamma(2-\alpha)}\left[\sin\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*} z^{1-\alpha}\sin(z)dz + \cos\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*} z^{1-\alpha}\cos(z)dz\right]. \tag{9}$$

The two can be related by an integration by parts and the use of a gamma function recursion relation. For $\alpha=1$, the usual first derivative is recovered.

2. Fractional derivatives for the exterior metric

In the exterior it is the right-handed derivative which should be applied. The exterior fractional derivative of the metric function is

$$D^{(\alpha)}_{r,R}(a^2r^2) = \frac{\Gamma(3)}{\Gamma(3-\alpha)}a^2r^{2-\alpha}. \tag{10}$$

TABLE I. Values of the function $h(\alpha,\rho_o/\rho^*)=\cos(\rho_o/\rho^*)-a$ as α and ρ_o/ρ^* vary. ρ_o is the radius of the string in the internal metric. $\rho^*=(8\pi\epsilon)^{(-1/2)}$ where ϵ is the constant internal energy density of the string. Complex values are indicated by C. When $\alpha=1$, $h(\alpha,\rho_o/\rho^*)=0$, providing the usual derivative matching condition.

	α					
ρ_o/ρ^*	0.1	0.3	0.7	0.9	1.1	1.2
0.1π	−0.013	−0.011	−0.006	−0.002	0.002	0.005
0.2π	−0.051	−0.043	−0.023	−0.008	0.010	0.021
0.3π	−0.113	−0.097	−0.052	−0.02	0.023	0.05
0.4π	−0.199	−0.172	−0.095	−0.038	0.047	0.107
0.45π	−0.25	−0.218	−0.125	−0.051	0.069	C
0.5π	−0.31	−0.272	−0.163	−0.071	C	C

3. The boundary match

The equation for the boundary match is

$$\frac{\Gamma(3)}{\Gamma(3-\alpha)}a^2 r_o^{2-\alpha}=\rho^* S_{\rho_o}(\alpha,\rho^*,\rho_o).$$

For $\alpha=1$ the usual derivative is obtained, leading to the original derivative matching result

$$a=\cos\left(\frac{\rho_o}{\rho^*}\right). \tag{11}$$

For $\alpha\neq 1$ there is a boundary layer which can be described by the Israel[4] formalism.

III. BOUNDARY STRESS ENERGY

The boundary stress energy, Eq. (6), involves the function

$$h\left(\alpha,\frac{\rho_o}{\rho^*}\right)=\cos\left(\frac{\rho_o}{\rho^*}\right)-a.$$

The two fractional matching equations, Eqs. (7) and (10), can be combined to give an equation for the parameter "*a*,"

$$\alpha<1{:}a^\alpha=\frac{\Psi(\alpha,\rho_o/\rho^*)}{\Gamma(1-\alpha)}\left[\sin\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*}z^{-\alpha}\cos(z)dz-\cos\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*}z^{-\alpha}\sin(z)dz\right], \tag{12}$$

$$\alpha>1{:}a^\alpha=\frac{\Psi(\alpha,\rho_o/\rho^*)}{\Gamma(2-\alpha)}\left[\sin\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*}z^{1-\alpha}\sin(z)dz+\cos\left(\frac{2\rho_o}{\rho^*}\right)\int_0^{2\rho_o/\rho^*}z^{1-\alpha}\cos(z)dz\right], \tag{13}$$

where $\Psi(\alpha,\rho_o/\rho^*)=[\Gamma(3-\alpha)\sin^{\alpha-2}(\rho_o/\rho^*)]/2^{2-\alpha}$. For a smooth match, $h(\alpha,\rho_o/\rho^*)$ should be zero. Using Eq. (12), Eq. (13), and MAPLE, $h(\alpha,\rho_o/\rho^*)$ was evaluated for a range of ρ_o/ρ^* and fractional indices. The results are in Table I.

Examining Table I, one sees in all cases that, as $\alpha=1$ is approached, the stress energy of the layer approaches zero. For $\alpha<1$ the boundary layer has negative density and positive *z*-stress. As ρ_o/ρ^* moves above $\pi/2$, the values become complex. For $\alpha>1$, the boundary layer density is positive and the stress is a *z*-tension. For $\alpha>1$ complex values occur for values of $\rho_o/\rho^*<\pi/2$.

TABLE II. Values of the angular deficit factor "a" as a function of α and ρ_o/ρ^*. ρ_o is the radius of the string in the internal metric. $\rho^*=(8\pi\epsilon)^{(-1/2)}$ where ϵ is the constant internal energy density of the string. Complex values are indicated by C.

	α					
ρ_o/ρ^*	0.1	0.3	0.7	1	1.1	1.2
0.1π	0.964	0.962	0.957	0.951	0.949	0.946
0.2π	0.86	0.852	0.832	0.809	0.799	0.788
0.3π	0.701	0.684	0.639	0.588	0.565	0.538
0.4π	0.507	0.48	0.404	0.309	0.263	0.202
0.5π	0.308	0.272	0.163	0	C	C

The string angular deficit, the cut in the range of the ϕ coordinate due to the presence of the parameter "a" in the exterior metric, is a hallmark of cosmic string behavior. The angular deficit depends on the energy density of the string.[2] When a fractional boundary condition is used, the internal stress energy of the string does not change but the energy density of the boundary layer affects the angular deficit since it must be included in the total energy density of the string. The angular deficit can be defined by

$$\delta\varphi = 2\pi[1 - a]. \tag{14}$$

The values of "a" for various α and ρ_o/ρ^* were calculated and are found in Table II. In Table II, the $\alpha=1$ column values are calculated from the standard nonfractional match. The linear density of the string is not affected by the boundary layer but the angular deficit is, increasing in value as α increases. For $\alpha<1$, the angular deficit is less than the zero-boundary-layer, $\alpha=1$, value because the negative boundary layer density is decreasing the overall mass density of the string. For $\alpha>1$ the deficit is larger than the zero-boundary-layer value because the positive boundary layer mass increases the string density.

One possible way to interpret the variation of the parameter "a" and the surface energy density is through a packing fraction. The packing fraction is normally defined as the ratio of the area covered by a tiling to the total area being tiled. Adapting this to the surface layer we can consider the ratio of the $\alpha\neq 1$ surface area to the $\alpha=1$ surface area which is just the ratio of the "a" values in each row of Table II to the "a" value for $\alpha=1$:

$$pf(\alpha) = \frac{2\pi a(\alpha\neq 1)Lr_o}{2\pi a(\alpha=1)Lr_o} = \frac{a(\alpha\neq 1)}{a(\alpha=1)}.$$

For $\alpha<1$, the layer with its negative density matter seems overpacked. For $\alpha>1$ this ratio can be used to make some qualitative comments about the energy distribution. From Table II, generally one notes that the packing fraction decreases as α increases for a given ρ_o/ρ^* and that the packing fraction decreases as the ρ_o/ρ^* increases. The latter effect can be simply explained by examining cylinders of constant radius and varying internal energy density. As the internal energy density decreases and the ratio ρ_o/ρ^* goes up, the packing fraction decreases as there is less energy to distribute over the surface. The first effect is more difficult to motivate. One possible explanation comes from assuming a specific tiling mechanism for the boundary layer and then using the packing fraction variation to motivate differences within the tiling. For example, if the surface layer is tiled with an Apollonian packing,[6,7] circles with increasing smaller circles packed in the interstitial regions, the packing fraction is given roughly by

$$pf = 1 - (r(\min)/r(\max))^{2-d}$$

where $r(\max)$ is the radius of the largest circle in the packing, $r(\min)$ is the smallest circle in the tiling, and $d\sim 1.3$ is the fractal dimension[8] of the tiling in flat space. Using this, the variation in

the packing fraction as a function of α can be explained as a variation in the range of granule sizes making up the surface layer.

IV. DISCUSSION

In this note, we have suggested using a Caputo space fractional derivative to generate variable density Israel surface layers on the Gott-Hiscock string bounded by a Levi-Civita vacuum. The method generates, for $\alpha>1$, a family of positive energy, increasingly dense boundary layers between the constant density string interior and the vacuum exterior. For $\alpha<1$, the energy density is negative. The Caputo derivatives can be more restrictive than the Riemann-Liouville forms since the derivatives of the matching metric functions must exist to use the Caputo definition. The Riemann-Liouville form would give an additional nonzero matching condition for the constant metric functions which would identify r_o and ρ_o.

When joining many hypersurfaces, it is common practice to match extrinsic curvatures. Since the extrinsic curvature is a Lie derivative, it can be calculated from simple partial derivatives and can be fractionally generalized. Because the string boundary, as defined in the bounding space–time, had a well-defined unit normal and derivatives, a simple fractional extrinsic curvature matching would have given the same result as the fractional derivative matching. However, null surfaces, for example, do not have a well-defined extrinsic curvature and junction conditions outside of an extrinsic curvature matching have been developed.[9–11] Mars and Senovilla[12] have discussed junction conditions using a rigged metric connection for general hypersurfaces. Hayward[13] has discussed an action for nonsmooth boundaries. Using fractional derivatives as a method for generating variable density boundary layers apart from the second fundamental form might instead be a way of giving physical meaning to possible fractional extrinsic curvature definitions.

A smooth boundary with well-defined derivatives is not a necessary condition for the use of a fractional derivative matching. The Riemann-Liouville derivatives can be applied to functions which themselves do not have a well defined derivatives. For example, using the Riemann-Liouville form, Rocco and West[14] showed that the continuous but nondifferentiable Wierstrass function had a well-defined fractional derivative. Kolwankar and Gangal[15] also discussed the fractional derivatives of the Weierstrass function. Fractional derivatives have seen increasing use in the development of fractional kinetics,[16] particularly in discussions of anomalous transport processes.[17] While the method presented in this note generates a family of surface layers of varying densities between the same two bounding static space–times, the use of fractional derivatives in boundary matching could reflect an underlying boundary matter structure that was built by an anomalous transport process. A single tiling was used as an example of the relation between possible matter distributions and the order of the fractional derivative. The results from other tilings or sums over tilings is an interesting question for investigation.

APPENDIX: FRACTIONAL OPERATIONS

The idea of a fractional derivative arose in 1695 when L'Hopital[18] asked Leibniz about the meaning of $d^n y/dx^n$ for $n=\frac{1}{2}$. The ideas of fractional calculus have attracted the attention of many of the same people that are associated with the development of physics: Lagrange, Laplace, Fourier, Liouville, Riemann, and Weyl, among others.[19] Since the first conference on fractional calculus in 1974, the applications to physics have grown enormously, describing phenomena such as the modeling of viscoelastic phenomena[20] and fractional matter transport.[17] There have been many definitions of fractional operations because of the diverse array of applications. Some of the fractional differentiation definitions are (a) the left-handed Riemann-Liouville form,[18]

$$D_a^\nu f(x) = \frac{d^m}{dx^m}\left\{ \frac{1}{\Gamma(m-\nu)} \int_a^x f(y)(x-y)^{m-\nu-1} dy \right\},$$

and (b) the right-handed Riemann-Liouville form,[18]

$$ {}_bD^{\nu}f(x) = \frac{d^m}{dx^m}\left\{ \frac{(-1)^m}{\Gamma(m-\nu)} \int_x^b f(y)(y-x)^{m-\nu-1} dy \right\}, $$

where, in both cases, m is the smallest integer bigger than ν. One very interesting thing about this form of the fractional derivative is that the derivative of a constant is not zero. For example, the left-handed derivative of "1" is

$$ D_a^{\nu}[1] = \frac{d^2}{dx^2}\left\{ \frac{1}{\Gamma(2-\nu)} \int_0^x (x-y)^{1-\nu} dy \right\} = \frac{x^{-\nu}\Gamma(1)}{\Gamma(1-\nu)}, $$

where the definition of the beta function,

$$ B(z,w) = \int_0^1 x^{z-1}(1-x)^{w-1} dx = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, $$

and the recursion relation for the gamma function have been used. There are other fractional forms, with modified definitions. The Caputo[19] fractional derivatives are similar to the Riemann-Liouville derivatives except that the derivative appears inside of the integral; $f^{(m)}(y)$ is the mth derivative of $f(x)$. The left-hand Caputo[19] derivative is

$$ {}_CD^{\nu}{}_a[f(x)] = \frac{1}{\Gamma(m-\nu)} \int_a^x (x-y)^{m-\nu-1} f^{(m)}(y) dy. $$

The right-hand derivative is similar. Butzer and Westphal[21] credit Liouville with the definition. In this paper we shall use the Caputo form.

The notation that we will use is

$$ \partial_i -> D_{iL}^{(\nu)}, $$

$$ \partial_i -> D_{iR}^{(\nu)}, $$

where ν the fractional order, has been placed in parentheses to distinguish it from a tensor index, i is the index of the coordinate derivative, and L, R denote the left- or right-hand derivatives.

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