

EFFECTIVE CHARGES AND EXPANSION PARAMETERS IN QCD

Eric Braaten

Department of Physics, University of Wisconsin,
Madison, WI 53706Jacques P. Leveille[†]Randall Laboratory of Physics, University of Michigan,
Ann Arbor, MI 48109

ABSTRACT

The momentum subtraction scheme MOM has been empirically successful in producing small QCD corrections to physical quantities at one loop order. By explicit calculations, we show that with a suitable shift in the renormalization scale, the minimal subtraction scheme coupling constant $g_{\overline{MS}}$ coincides with typical momentum scheme coupling constants at both one and two loop order.

I. INTRODUCTION

The large size of the QCD coupling constant and the lack of any natural renormalization scheme have prompted theorists to invent a variety of renormalization schemes in an attempt to improve the convergence of perturbation expansions for physical quantities. In the original minimal subtraction (MS) scheme, only the poles in ϵ arising from dimensional regularization are subtracted from divergent quantities. The modified minimal subtraction (\overline{MS}) scheme, which subtracts the constant ($\ln 4\pi - \gamma$) along with the poles, leads to consistently smaller coefficients in the perturbation expansions. Subtracting different constants might lead to still smaller coefficients.

The momentum subtraction scheme MOM is defined¹ so that radiative corrections to a particular component of the triple-gluon vertex at the symmetric point vanish, along with propagator corrections, at some scale μ . Experience has shown that this scheme usually produces even smaller expansion coefficients than the \overline{MS} scheme². The coupling constant $g_{MOM}(\mu)$ is simply the "effective charge" associated with that component of the triple-gluon vertex. The empirical success of the MOM scheme suggests that studying effective charges can be useful in choosing an expansion parameter for QCD.

Before beginning our study of effective charges, we introduce a convenient parameterization for renormalization schemes in Section II. In Section III, we define effective charges for QCD and examine their one-loop corrections. In Section IV, we introduce an approximation for the effective charges and calculate their corrections to two-loop order. Some conclusions are presented in the final section. Interested readers are referred to Ref. 3 for technical details.

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II. A SIMPLE THEOREM

In one loop order, the schemes which subtract constants along with the poles, such as MS, are equivalent to the MS scheme with a different definition of the analytic continuation for dimensionally regularized integrals:

$$\int \frac{d^4k}{(2\pi)^4} + \frac{\mu^{2\epsilon}}{N(\epsilon)} \int \frac{d^Dk}{(2\pi)^D} \quad (1)$$

where $D=4-2\epsilon$ and μ is an arbitrary mass parameter. $N(\epsilon)$ is analytic at $\epsilon=0$ and is unity for conventional minimal subtraction. If we choose $N(\epsilon) = (4\pi)^\epsilon / \Gamma(1-\epsilon)$ and subtract only the poles in ϵ we reproduce the MS scheme at one loop. This is in fact the best definition of the MS scheme since it automatically removes the constant ($\ln 4\pi - \gamma$) from higher orders as well. At only one loop, as in Eq. (1) above, incorporating a factor $1/N(\epsilon)$ is trivially equivalent to shift $\mu \rightarrow \mu e^{-(\ln N/2\epsilon)}$. This result is true in all orders. For $N=1$ the renormalized charge is $\alpha_{MS}(\mu)$ defined by:

$$\mu^{-2\epsilon} \alpha_0 = Z_{MS}(\alpha_{MS}(\mu), \epsilon) \cdot \alpha_{MS}(\mu) \quad (2)$$

Let $\alpha_N(\mu)$ be the renormalized coupling with $N \neq 1$. Since α_0 occurs with each loop integral Eq. (2) leads to:

$$\mu^{-2\epsilon} \alpha_0 N = Z_N(\alpha_N(\mu), \epsilon) \cdot \alpha_N(\mu) \quad (3)$$

But Z_N and Z_{MS} are both defined by subtracting poles in ϵ , hence they are the same functions of their respective arguments. Substituting $\mu \rightarrow \mu e^{-\ln N/2\epsilon}$ in Eq. (2) we deduce that:

$$\alpha_N(\mu) = \alpha_{MS}(\mu e^{-t}) \quad \text{where } t = \left. \frac{\ln N}{2\epsilon} \right|_{\epsilon=0} \quad (4)$$

A simple corollary is that the MS and \overline{MS} coupling constants are related to all orders by:

$$\alpha_{MS}(\mu) = \alpha_{\overline{MS}}(\mu e^{-(\ln 4\pi - \gamma)/2}) \quad (5)$$

The μ dependences are therefore governed by the same β -function.

The parameter t in Eq. (4) provides a convenient parameterization for renormalization schemes at one-loop order. For example the MOM scheme can be parameterized by the value t_{MOM} for which:

$$\alpha_{MOM}(\mu) = \alpha_{MS}(\mu e^{-t_{MOM}}) + O(\alpha^3) \quad (6)$$

$$t_{MOM} = \overline{t_{MS}} + 1 + \frac{69-32N_f}{33-2N_f} \frac{1}{24}$$

with N_f the number of quark flavors and $I=2.344$. This scale shifted MS coupling constant will produce the same small one loop corrections for physical process as $g_{MOM}(\mu)$. Having fixed this g_{MOM} at one loop, we might wonder if the shifted minimal coupling constant will still (approximately) reproduce g_{MOM} at two loop order. We will show this to be approximately true.

III. EFFECTIVE CHARGES

We have devised a simple way of parameterizing the coupling constants in various schemes using a parameter t . In this section we begin a study of the effective charges of the theory. We might first wonder what effective charges have to do with the expansion parameters. Consider a massless, spinless theory. Assume that a three point vertex occurs inside some Feynman diagram for an infrared finite quantity. The vertex represents a factor g , each leg a propagator $1/p_i^2$. The radiative corrections result in the replacement $g \rightarrow \Gamma(p_1, p_2, p_3)$, the 1PI three point vertex, and $1/p_i^2 \rightarrow 1/p_i^2 (1+\pi(-p_i^2))^{-1}$. Since each propagator joins two vertices, we can associate "half" of the propagator corrections with each vertex. This leads to the definition of an "effective vertex":

$$\Gamma^{eff}(p_1, p_2, p_3) = \frac{\Gamma(p_1, p_2, p_3)}{(1+\pi(-p_1^2))^{1/2} (1+\pi(-p_2^2))^{1/2} (1+\pi(-p_3^2))^{1/2}} \quad (7)$$

Let the characteristic scale of the physical quantity be μ . What region of the integration over the loop momenta p_1, p_2, p_3 is the most important? Because of the assumed infrared finiteness the contributions from the region $p_i \ll \mu^2$ must cancel. The contributions from the ultraviolet region $p_i^2 \gg \mu^2$ is removed by renormalization. Consequently the dominant contribution will come from the region $p_i^2 \sim \mu^2$. Hence a natural definition for an effective charge is:

$$g^{eff}(\mu) = \Gamma^{eff}(p_1, p_2, p_3) \Big|_{\text{symmetric point}} \quad (8)$$

where the symmetric point is defined by $p_1^2 = p_2^2 = p_3^2 = -\mu^2$. Since many of the higher order propagator and vertex corrections are automatically absorbed into this charge, it is a natural candidate for an expansion parameter. It is the analog of the MOM coupling constant in this simple theory.

There are difficulties in extending the definition of effective charges to QCD. First there are three distinct three-point vertices (as well as a 4-point vertex, which we will ignore). Each vertex contains several Lorentz and Dirac indices, whereas an effective charge is a scalar quantity. The Lorentz indices can be removed by contracting the vertex with a "polarization vector" for each gluon leg. A natural basis of polarization vectors for a gluon leg of momentum p in a three-point vertex consists of a longitudinal (L) vector parallel to p , a vector orthogonal to p but in the plane of the vertex (P), and two vectors normal to the vertex plane (N). The quark vertex, when contracted with a polarization vector ϵ_μ , still contains

Dirac indices. These can be eliminated by a projection onto $\not{\epsilon}$ using the Dirac trace. There are 6 effective charges α_i^{eff} for QCD which can be defined in this way. They can be denoted by the polarizations of their gluon legs: $i = \text{PPP}, \text{NNP}, \text{LLP}, \text{qqN}, \text{ggP}, \text{ggL}$, where q and g refer to quark and ghost legs respectively.

These effective charges have been calculated to one-loop order in covariant gauges by Celmaster and Gonsalves¹. In the MS scheme, allowing for a shift in the scale of μ , they have the form:

$$\alpha_i^{\text{eff}}(\mu, \xi) = \alpha_{\text{MS}}(\mu e^{-t}) + A_i(\xi, t) \alpha_{\text{MS}}(\mu e^{-t})^2 + \dots \quad (9)$$

The coefficients A_i depend weakly on the gauge parameter ξ for gauges with $|\xi| \lesssim 1$, so we only consider the Landau gauge, $\xi=0$. Note that the MOM coupling constant is by definition the effective charge $\alpha^{\text{eff}}(\mu, \xi=0)$. Fig. 1 shows the variations of the coefficients A_i with NNP

the renormalization scheme parameter t . They are all close to zero for $t=t_{\text{MOM}}$, which is comforting since it indicates that the MOM

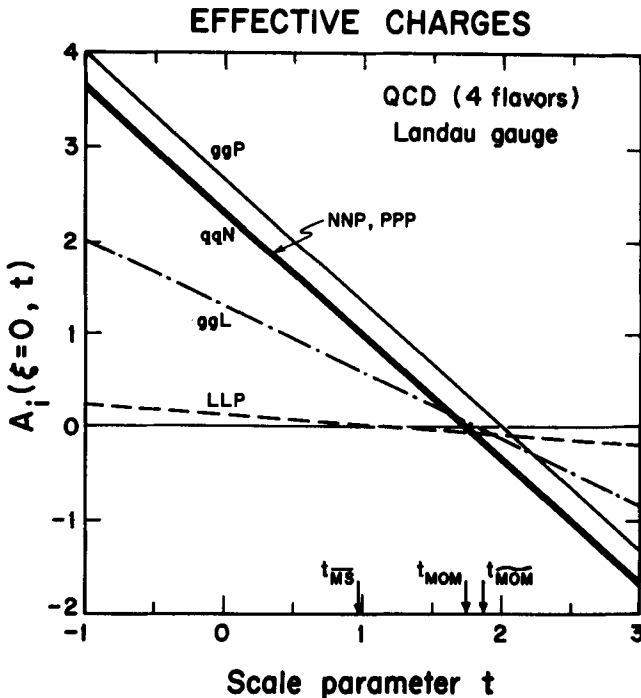


Fig. 1. Order- α^2 coefficients $A_i(0, t)$ in the MS scheme for the effective charges α_i^{eff} as a function of the scale parameter t for QCD with four flavors of massless quarks. The solid, dash-dot, and dashed lines represent charges with zero, one, and two longitudinal gluon legs, respectively.

coupling constant does not depend strongly on the choice of the particular effective charge $g_{\text{eff}}^{\text{NNP}}$.

We have shown that the MOM scheme is characterized at one loop, not only by small corrections to many physical quantities, but also by small corrections to the effective charges. This suggests that, by studying the effective charges in higher order, we can learn how to define a coupling constant which will be a good expansion parameter for QCD.

IV. TWO LOOP CHARGES

To continue the analysis we must compute the effective charges at two loop order. There is a technical problem. The evaluation of the three point vertices to two loop order, at the symmetric point is extremely tedious. A much easier calculation is a three point vertex with one leg at zero momentum, which is of the same degree of difficulty as the calculation of a propagator. We know that a vertex is infrared finite if one external leg is set to zero. Because of this infrared finiteness we might expect that a good approximation to an effective charge is to replace the three point vertex at the symmetric point, by the vertex with one leg at zero momentum; e.g.

$$g_1^{\text{eff}} = \frac{\Gamma_1(p_1, p_2, 0) \Big|_{p_1=p_2=-\mu^2}}{(1 + \pi(-\mu^2))^{3/2}} \quad (10)$$

We must be careful however, because the vertex consists of Lorentz tensors $T_{\mu\dots}$ multiplied by scalar coefficients. While the vertex remains infrared finite as $p_3 \rightarrow 0$, some of the coefficients can diverge like $\log(p_3^2)$ if the corresponding tensors vanish. Therefore, when defining approximate effective charges as in Eq. (10), we must avoid vertex corrections which correspond to tensors that vanish as $p_3 \rightarrow 0$.

We illustrate the quality of our approximation in Fig. (2) for QCD with no quarks using existing one loop calculations⁴. This is done by calculating vertices with $p_3^2 = \eta\mu^2$ and sending $p_3 \rightarrow 0$ with $\eta \rightarrow 0$. This way we interpolate smoothly between the symmetric point $\eta=1$ and the zero momentum point $\eta=0$. The charges are labelled as before with the leg p_3 in parentheses. The solid lines represent the charges which are guaranteed to remain finite as p_3 vanishes, while the dotted lines are coefficients of tensor components which vanish when $\eta=0$. We see that if the charge is infrared finite it is very flat versus η , hence the approximation Eq. (10) should be very reliable.

The accuracy of this approximation at one-loop order suggests a momentum subtraction renormalization scheme in which vertex corrections are subtracted at the zero-momentum point. Such a scheme is much easier to extend to higher orders than the MOM scheme, because it avoids the calculation of symmetric point vertex corrections. One such scheme MOM was defined in Ref. 3 and a similar scheme has been advocated by Gupta and Dhar.⁵

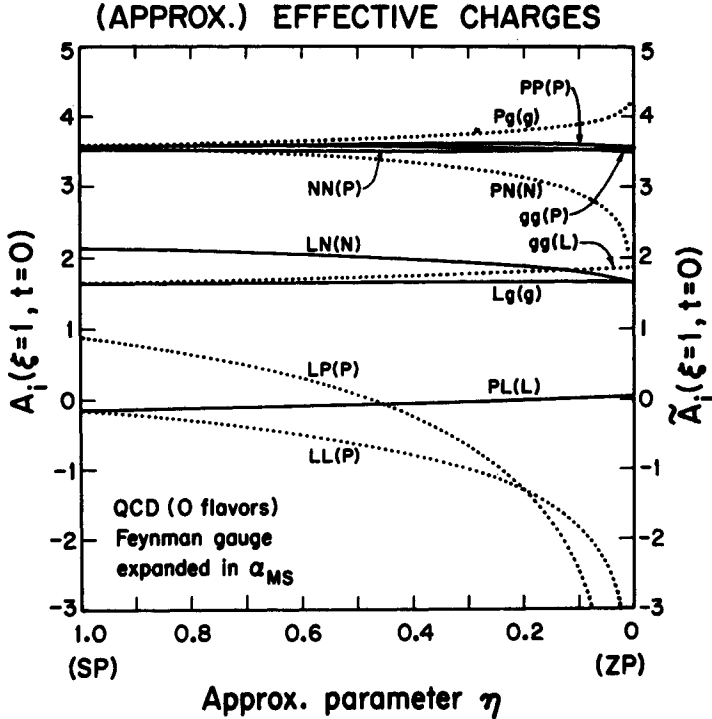


Fig. 2. Order- α^2 coefficients in the MS scheme for the effective charges in the Feynman gauge, with symmetric point vertex corrections approximated by their values at the η -point, as a function of η for QCD with no quarks. Solid (dotted) lines represent charges whose vertex corrections are (are not) guaranteed to be finite at $\eta=0$.

We have calculated the approximate effective charges for QCD to two-loop order in the Feynman gauge ($\xi=1$). They can be expanded as in Eq. (9):

$$\alpha_i^{\text{eff}}(\mu, \xi=1) = \alpha_{\text{MS}}(\mu e^{-t}) + \tilde{A}_i(1, t) \alpha_{\text{MS}}(\mu e^{-t})^2 + \tilde{B}_i(1, t) \alpha_{\text{MS}}(\mu e^{-t})^3 + \dots$$

where t parameterizes the minimal subtraction renormalization schemes. As shown in Fig. 3, the coefficients \tilde{A}_i are linear in t and, like the A_i 's in Fig. 1, they cluster around zero for $t \approx t_{\text{MOM}}$. The coefficients \tilde{B}_i , which are parabolas in t , are shown in Fig. 4. Surprisingly, they all attain their minima, with values close to zero, at $t \approx t_{\text{MOM}}$. We suggest that this will be true of the effective charges as well, and therefore the scale-shifted MS coupling constant $\alpha_{\text{MS}}(\mu e^{-t})$ for $t \approx t_{\text{MOM}}$ should be a good expansion parameter for QCD through two loop order.

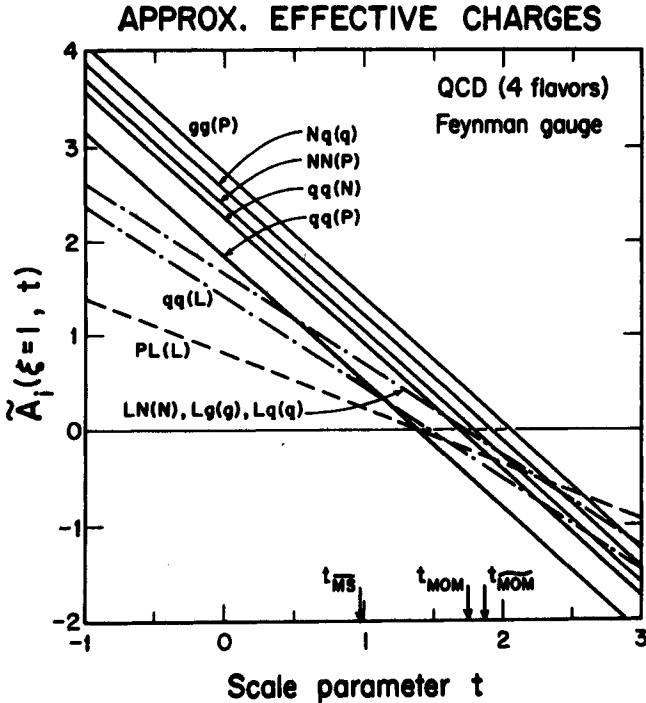


Fig. 3. Order- α^2 coefficients $\tilde{A}_1(\xi=1, t)$ for the approximate effective charges $\tilde{\alpha}_i^{\text{eff}}$ in the Feynman gauge as a function of the scale shift parameter t for QCD with four flavors of massless quarks.

IV. CONCLUSIONS

Intuitively we expect that a renormalization scheme which yields small radiative corrections to the effective charges of QCD will also produce small corrections to physical quantities. This expectation is borne out empirically at one-loop order by the success of α_{MOM} in yielding consistently smaller corrections than the $\overline{\text{MS}}$ or MS schemes. Studying the effective charges in higher orders should therefore help us define a coupling constant which will be a useful expansion parameter for QCD.

The calculation of effective charges can be greatly simplified by approximating symmetric point vertex corrections by their values with one leg at zero-momentum. This approximation is motivated by the infrared-finiteness of the vertex in this limit, and explicit calculations were used to demonstrate its validity at one-loop order. We calculated all these approximate effective charges for QCD to two-loop order. They were all found to have small one and two-loop corrections in the $\overline{\text{MS}}$ scheme with a scale-shifted coupling constant $\alpha_{\overline{\text{MS}}}(\mu e^{-t})$. The required scale-shift is such that it agrees with

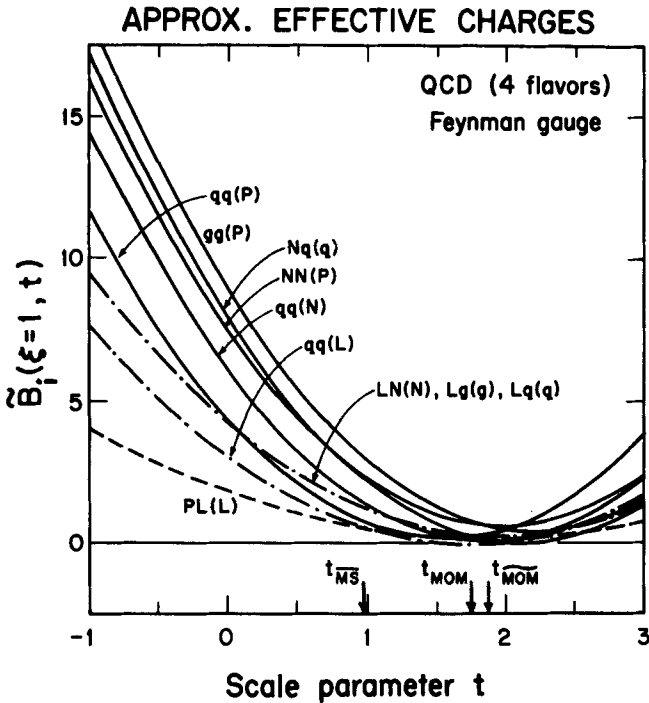


Fig. 4. Order- α^3 coefficients \tilde{B}_i ($\xi=1, t$) for the approximate effective charges \tilde{q}_i^{eff} in the Feynman gauge as a function of the scale-shift parameter t for QCD with four flavors of massless quarks.

$g_{\text{MOM}}(\mu)$ at one-loop order. We therefore expect that this scale-shifted MS coupling constant will also yield small corrections to many physical quantities through two-loop order.

Note that our conclusion is based on calculational evidence at one-loop order, with only plausibility arguments to extend it to higher orders. We also emphasize that this approach to the choice of an expansion parameter says nothing about the choice of scale μ for a given physical quantity. Its choice is left to the art and intuition of the phenomenologist.

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