Convection, Stability, and Low Dimensional Dynamics

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Abstract. Recent developments concerning the connection between notions of hydrodynamic stability—usually associated with stationary laminar flows—and dynamics, most notably turbulent fluid flows, are reviewed. Based on a technical device originally introduced by Hopf in 1941, a rigorous mathematical relationship between criteria for nonlinear energy stability and bounds on global transport by steady, unsteady, or even turbulent flows, has been established. The optimal "marginal stability" criteria for the best bound leads to a novel variational problem, and the differential operator associated with the stability condition generates an adapted basis in which turbulent flow fields may naturally be decomposed. The application and implications of Galerkin truncations in these bases to produce low dimensional dynamical systems models is discussed in the context of thermal convection in a saturated porous layer.

INTRODUCTION

Hydrodynamic stability is usually considered a characteristic of steady flow whereas turbulence is the antithesis of steady flow. The goals of this presentation are, first, to describe a physical and mathematical relationship between stability and turbulence and, second, to propose that this connection may lead to some new insights regarding low dimensional models of turbulent dynamics. Such stability considerations in the context of turbulence have been applied to many fundamental problems in fluid dynamics including convective heat flux, momentum flux across a shear layer, and mass flux in pipe and channel flows. In each case the basic observation is that global transport is controlled by the thickness of a laminar boundary layer near a surface driving the flow by heating or shearing, and in each case the basic physical assumption is that the layer thickness adjusts itself so that, considered as a system unto itself,

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the layer is marginally stable. This idea has been around for a long time. The issue of stability for convective thermal boundary layers was raised by Malkus [1] in the 1950's. In the early 1960's, Howard [2] used a marginal stability argument to derive a quantitative scaling relationship, including prefactor estimate, for global heat transport in a fluid confined between parallel plates. There is now a mathematically rigorous way to implement these ideas starting from a mathematical idea introduced by Hopf in the 1940's [3].

Hopf's motivation was to derive *a priori* bounds on the time averaged rate of energy dissipation for solutions of the Navier-Stokes equations in the presence of rigid boundaries, a necessary technical step in the construction of weak solutions to the nonlinear partial differential equations. Hopf derived finite estimates for the energy dissipation rate in some some general cases which were, however, so unrealistically large as to be physically irrelevant. But as established in the 1990's [4–6], Hopf's method can be applied to moderately simple geometries where sharper calculations lead to a rigorous derivation of Kolmogorov-type scaling estimates for turbulent transport. A careful formulation of Hopf's techniques leads naturally to novel variational problems for the optimal estimates [7,8], and these recent technical developments have been the focus of a number papers: the method has been generalized for use with time-dependent boundary conditions [9] and more complex geometries [10,11], and there have been improvements in the anaysis and formulation of the optimization problem [12,13]. For the problem of turbulent transport in relatively symmetric geometries [14,15], the variational problem makes contact with the large body of work, initiated by Howard [16,17] and Busse [18,19] in the 1960's and 1970's, on bounds for statistically stationary turbulence. Much recent experimental work has focused on convection in a fluid layer [20–22], but in this talk I will outline these ideas in the context of the problem of thermal convection of an incompressible Newtonian fluid in a saturated porous layer [23–26]. The next section describes the set-up and presents the model, with a brief review of the system's stability and dynamics given in the following section. The subsequent section contains a brief discussion of heuristic and rigorous connections between stability and turbulence, and in the final section implications for low order dynamical systems models are discussed.

CONVECTION IN A POROUS LAYER

Consider a fluid-saturated layer of porous material occupying a region with horizontal surfaces parallel to the $x-y$ plane and thickness h in the vertical (z) direction. The layer is heated from below so that the bottom of the slab is held at tempertature $T_0 + \delta T$ while the top is cooled uniformly to tempertature T_0. The horizontal planes at $z = 0$ and $z = h$ are impenetrable to the fluid and, for the purposes of this presentation, we consider periodic boundary conditions in the (finite) horizontal directions. We use Darcy's law to model the relationship between the incompressible velocity vector field $\mathbf{u}(\mathbf{x},t) = (u,v,w)$ and the local temperature in the slab, $T(\mathbf{x},t)$, presumed to evolve according to the convection-diffusion equation. Then the dynamical equations are

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \kappa\Delta T,\ \frac{\nu}{K}\mathbf{u} = -\nabla p + g\alpha\mathbf{k}T,\ \nabla\cdot\mathbf{u} = 0, \tag{1}$$

where κ is the thermal diffusivity of the fluid-material mixture, ν is the fluid viscosity, K is the Darcy permeability coefficient, $\mathbf{k}$ is the vertical unit vector, g is the acceleration of gravity, α is the thermal expansion coefficient, and $p(\mathbf{x},t)$ is the pressure as determined by incompressibility condition. Neglect of inertial terms in the velocity equation corresponds to the infinite Darcy-Prandtl number limit of the model [23,24].

The physical heat current vector is proportional to

$$\mathbf{J} = \mathbf{u}T - \kappa\nabla T. \tag{2}$$

A quantity of particular interest is the space-time averaged vertical heat flux,

$$\langle\mathbf{k}\cdot\mathbf{J}\rangle = \langle J_3\rangle = \left\langle \mathbf{k}\cdot\mathbf{u}T - \kappa\frac{\partial T}{\partial z}\right\rangle = \langle wT\rangle + \kappa\frac{\delta T}{h}, \tag{3}$$

as a function of the applied temperature gradient and other system parameters.

These equations are naturally nondimensionalized in terms of the vertical length scale h, the thermal diffusion time scale $\frac{h^2}{\kappa}$, and the overall temperature drop δT to become

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \Delta T,\ \mathbf{u} = -\nabla p + Ra\mathbf{k}T,\ \nabla\cdot\mathbf{u} = 0, \tag{4}$$

with boundary conditions

$$T = 1, w = 0 \text{ when } z = 0, \text{ and } T = 0, w = 0 \text{ when } z = 1. \tag{5}$$

The Rayleigh number $Ra = \frac{g\alpha\delta T K h}{\nu\kappa}$ is the nondimensional measure of the applied temperature drop. The nondimensional heat flux is the Nusselt number Nu, the ratio of the total heat flux to the purely conductive heat flux:

$$Nu \equiv \left\langle J_3^{nondimensional}\right\rangle = \frac{\left\langle J_3^{dimensional}\right\rangle}{\kappa\frac{\delta T}{h}} = 1 + \langle wT\rangle. \tag{6}$$

Note that both the velocity vector field and the pressure field are linear (albeit nonlocal) functionals of the instantaneous temperature distribution. A simple computation based on the equations of motion allows us to express the Nusselt number in terms of the mean squared temperature gradient:

$$Nu = \left\langle|\nabla T|^2\right\rangle. \tag{7}$$

The pure conduction state is the solution with no flow and a linear temperature profile: $\mathbf{u}_{cond} = 0$, $T_{cond} = 1 - z$. For low values of Ra it is both linearly and nonlinearly stable, so the onset of convection in this model is a forward pitchfork bifurcation.

This is seen by writing an arbitrary solution of the initial value problem as $T(\mathbf{x}, t) = T_{cond}(z) + \theta(\mathbf{x}, t)$ where the "fluctuation" temperature field θ satisfies homogeneous boundary conditions on the top and bottom of the slab. The linearized stability analysis of the pure conduction profile $T_{cond}(z)$ consists of looking for solutions to the linearized evolution operator (dropping $\mathbf{u} \cdot \nabla\theta$) with a time dependence $\sim e^{-\lambda t}$:

$$\lambda\theta = -\Delta\theta - w, \ \Delta w = Ra(\Delta - \partial_z^2)\theta. \tag{8}$$

This is a linear constant coefficient system and it is straightforward to see that all the eigenvalues are real with $\lambda > 0$ for $Ra < Ra_c^{lin} = 4\pi^2 \lesssim 40$, in which case there are no unstable modes. Linearized stability gives a sufficient criteria for *instability*, so strictly speaking this shows that the pure conduction solution looses stability at $Ra_c^{lin} = 4\pi^2$.

Nonlinear stability is established via the energy method [27,28]. The fully nonlinear equation of motion for a fluctuation away from the pure conduction solution is

$$\frac{\partial\theta}{\partial t} + \mathbf{u} \cdot \nabla\theta = \Delta\theta + w, \ \Delta w = Ra(\Delta - \partial_z^2)\theta, \tag{9}$$

and multiplying the θ-equation by θ and integrating over the slab making use of the boundary homogeneous conditions leads to

$$\frac{d}{dt}\frac{1}{2}\int \theta^2 d\mathbf{x} = -\int(|\nabla\theta|^2 - w\theta)d\mathbf{x}. \tag{10}$$

As long the right hand side is negative, arbitrarily large deviations from the pure conduction state will vanish as $t \to \infty$. This will be the case if

$$0 < \lambda \equiv min \int(|\nabla\theta|^2 - w\theta)d\mathbf{x}, \tag{11}$$

where the minimization is over all test functions $\theta(\mathbf{x})$ satisfying homogeneous boundary conditions, normalization $\int \theta^2 d\mathbf{x} = 1$, and where $w(\mathbf{x})$ is constrained to be the linear functional of θ defined by the solution of the Poisson. After Fourier transforming in the horizontal plane, the Euler-Lagrange equations for this variational problem of nonlinear energy stability are found to be

$$\lambda\hat{\theta}(z) = (-\partial_z^2 + k^2)\hat{\theta} - \frac{1}{2}\hat{w} - \frac{1}{2}k^2\hat{v}, \ (-\partial_z^2 + k^2)\hat{v} = Ra\ \hat{\theta}, \tag{12}$$

where k is the magnitude of the horizontal wavenumber, the "eigenvalue" λ is the Lagrange multiplier enforcing the normalization, and $\hat{v}(z)$, the Lagrange multiplier enforcing the constraint $(-\partial_z^2 + k^2)\hat{w} = Ra\ k^2\hat{\theta}$, satisfies homogeneous Dirichlet conditions at $z = 0$ and $z = 1$. Solutions of these differential equation corresponds precisely to solutions of the linearized stability equations, so the critical Rayleigh number Ra_c^{nonlin} of nonlinear stability—ensuring absolute stability for $Ra < Ra_c^{nonlin}$—corresponds exactly to the critical Rayleigh number Ra_c^{lin}. In terms of the heat transport, these stability analyses imply

$$Nu = 1 \text{ for } Ra \leq Ra_c^{nonlin} = Ra_c^{lin} = 4\pi^2 \equiv Ra_c. \tag{13}$$

Stationary roles appear immediately beyond the onset of convective motion, and they may be studied by means of amplitude equations for $Ra \gtrsim Ra_c$ not too far above transition. Subsequent bifurcations, patterns, and dynamics depend on details of the container shape and sidewall conditions but eventually, for sufficiently high Ra, one expects to enter what may be referred to as a turbulent regime. Convective turbulence may be identified by the loss of coherent flow structures correlated over long ranges (i.e., the disappearance of rolls). This is accompanied by the formation of thermal boundary layers across which most of the (horizontally averaged) temperature drop occurs, and an isothermal (on average) core where the heat is transported by rising or falling "plumes" or "blobs" of hot or cold fluid that have broken away from the hot or cold boundary layers. Scaling with is observed in convective turbulence, and the boundary layer thicknesses δ varies inversely as a power of Ra.

A chief aim of theories of convection is to predict the relationship between the heat current and the applied temperature drop, i.e., the function $Nu(Ra)$. The $Nu - Ra$ relationship in the turbulent regime is determined by the $\delta - Ra$ scaling because the heat transport through each horizontal layer must be the same in a statistically stationary situation. Near the boundaries where $w \to 0$, the vertical heat current $J_3 \approx -\frac{\partial T}{\partial z} \approx \frac{1}{2\delta}$ because of the temperature drop of $\frac{1}{2}$ across the layer of thickness δ. Hence $Nu = \frac{1}{2\delta}$.

The observed [25] high Rayleigh number scaling is $\delta \sim Ra^{-1}$, i.e.,

$$Nu \sim Ra \text{ for } Ra >> Ra_c \tag{14}$$

We refer to this particular scaling, $Nu \sim Ra$, as *Kolmogorov* scaling for this problem because in this situation the physical heat flux is independent of the microscopic heat conduction coefficient κ. Indeed, $Nu \sim Ra$ implies

$$\left\langle J_3^{dimensional} \right\rangle \sim \kappa \frac{\delta T}{h} Ra = \frac{g\alpha(\delta T)^2 K}{\nu}. \tag{15}$$

This is in direct analogy to the fundamental Kolmogorov scaling hypothesis for incompressible Navier-Stokes turbulence that the rate of energy dissipation is independent of the molecular viscosity at high Reynolds numbers.

STABILITY, TURBULENCE AND BOUNDS

Howard's marginal stability argument [2] applied to convection in a porous layer starts with the observation that most of the horizontally averaged temperature drop occurs across thermal boundary layers in which the fluid is effectively at rest due to the impenetrable boundaries. How thick are the boundary layers? If they are so thin as to be stable when considered as convection layers, then they may grow. If they are so thick that they are ustable as convection layers, then they will break up and shed a plume. Thus one expects the average thickness to self-adjust so that the boundary layer is just marginally stable. This means that the Rayleigh number based on half the total temperature drop across a layer of (dimensional) thickness δ is just about equal to the critical Rayleigh number:

$$O(40) = \frac{g\alpha(\frac{1}{2}\delta T)K\delta}{\nu\kappa} = \frac{1}{2}Ra\frac{\delta}{h} \tag{16}$$

Hence the nondimensionalized boundary layer thickness is expected to scale like $\delta \sim Ra^{-1}$ so that

$$Nu \approx c\ Ra \tag{17}$$

with a coefficient $c = O(.01)$.

This idea is takes on a rigorous realization in the following theorem [14,26]: Let $\tau(z)$ be a "background" temperature profile that satisfies the physical boundary conditions, $\tau(0) = 1$ and $\tau(1) = 0$. Then the bulk heat transfer is bounded from above by the heat transfer in $\tau(z)$, i.e., $Nu < \int_0^1 \tau'(z)^2 dz$, when the profile is marginally energy stable at parameter value $2Ra$.

In the theorem, energy stability is defined by the natural generalization of the criteria we used for the nonlinear stability analysis of the pure conduction solution. For an arbitrary profile $\tau(z)$ at parameter value $2Ra$, energy stability means

$$0 < \lambda \equiv min \int (|\nabla\theta|^2 + 2\tau'(z)w\theta)d\mathbf{x}, \tag{18}$$

where the minimization is performed with the same constraints as before. For the proof of this theorem see the references [14,26].

Stable profiles may be constructed by considering function $\tau(z)$ which is constant, so $\tau' = 0$, throughout most of the slab. This is incompatible with the boundary conditions, but allowing τ' to vary strongly near the boundaries where the "test" θ's and w's vanish can be made to work. A piecewise linear stable temperature profile produces the rigorous upper bound [26]

$$Nu < \frac{1}{24}Ra \approx .04\ Ra, \tag{19}$$

just an $O(4)$ factor above the observed [24,25] behavior. Optimization, i.e., solving the variational problem for the optimal profile $\tau_{opt}(z)$, improves the upper bound considerably [24,29,15,26].

LOW DIMENSIONAL MODELS

The self-adjoint linear operator associated with the nonlinear stability condition for $\tau_{opt}(z)$ produces a natural basis for the function space of temperature fluctuations $\theta(\mathbf{x},t) = T(\mathbf{x},t) - \tau_{opt}(z)$. Indeed, we may write

$$\theta(\mathbf{x},t) = \sum_{k_x,k_y} e^{ik_x x + ik_y y} \sum_j \Theta_j^{(k_x,k_y)}(z) a_j^{(k_x,k_y)}(t), \tag{20}$$

or more compactly,

$$\theta(\mathbf{x},t) = \sum_n \Theta_n(x,y,z) a_n(t), \tag{21}$$

where the functions $\Theta_n(x,y,z)$ are the eigenfunctions of the linear (albeit nonlocal) operator in the quadratic form in Eq.(18), ordered in ascending order of the magnitude of the eigenvalues. The spectrum of the linear operator of nonlinear stability for the optimal "marginally stable" profile is nonnegative definite, i.e., each $\lambda_n \geq 0$. These eigenvalues contribute to the linear term in the evolution equations for the modal amplitudes, the $a_n(t)$'s, either neutrally (for the marginal modes with $\lambda_n = 0$) or dissipatively (for the modes with $\lambda_n > 0$).

Once the optimal "marginally stable" background, $\tau_{opt}(z)$, profile has been computed, it is natural to look for corrections to the upper bound it produces. In terms of the modal amplitudes of the optimal eigenfunctions, the exact heat transfer is

$$Nu = \int_0^1 \tau'(z)^2 - \sum_{n=0}^{\infty} \lambda_n \left\langle \left| a_n^2 \right| \right\rangle, \tag{22}$$

so knowledge of the exact modal dynamics leads to the exact heat flux. Of course the full set of coupled ode's for the modal amplitudes are of the same complexity as the original nonlinear pde's, but the ordering of the eigenvalues suggests that low dimensional truncations, a.k.a. Galerkin truncations, in this basis may yield useful approximations. Sensible truncations will include the weakly damped modes, i.e., those corresponding to small values of λ_n, with the expectation that the strongly damped modes with large λ_n will be small in magnitude and slaved to the active modes. Certainly the marginally stable ($\lambda_n = 0$) modes should be included in any such dynamical systems model.

Now we can use a result from the variational problem for $\tau_{opt}(z)$ to get some insight into meaningful low dimensional dynamical systems models. It has been observed both theoretically [18,24,14] and numerically [29] that for increasing Ra, the marginal subspace (the span of the eigenvectors with $\lambda_n = 0$) becomes increasingly high dimensional. That is, the eigenvalue 0 becomes increasingly degenerate and the number of independent modes with $\lambda_n = 0$ increases without bound as $Ra \to \infty$. This means that there can be "no free lunch" for low order truncations: if we seek quantitative corrections to the upper bound as in Eq.(22), then such models must necessarily be of increasing complexity, i.e., of higher and higer dimension, as the turbulence intensifies.

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