Quantum aspects of massive gravity

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Abstract

We consider the effect of quantum interactions on Pauli–Fierz massive
gravity. With generic graviton cubic interactions, we observe that the 1-
loop counterterms do not conform to the tree level structure of Pauli–Fierz
action, resulting in the reappearance of the sixth mode ghost. Then to explore
the quantum effects to the full extent, we calculate the resummed graviton
propagator with an arbitrary interaction and analyze its complete structure,
from which a minimal condition for the absence of the ghost is obtained.

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1. Introduction

As a first step to reconcile gravity and quantum physics, or to understand the quantum nature
of gravity, quantum corrections to general relativity (GR) have been extensively studied. The
works on induced gravity [1] can be considered as an approach to matter loop corrections
to Newton's constant, whereas 1-loop corrections to the massless graviton propagator were
obtained for various types of fields: scalars [2], gauge bosons [3], fermions [4], and even
gravitons themselves [5].

On a different front of the gravitational research, the theory of massive gravity has been
an interesting topic for both theoretical and phenomenological reasons. On the theory side,
it has been studied how to deal with the classical pathologies of massive gravity, the most
famous one being the van Dam–Veltman–Zakharov (vDVZ) discontinuity [6]: adding a mass
term to the linearized Einstein–Hilbert action seems to be a natural way of giving mass to a
graviton. But doing so breaks the general coordinate invariance (GCI) of GR, so that a massive
theory ends up with more degrees of freedom (DOFs) than the massless one. Requiring that
none of these extra DOFs have any pathology, one is forced to choose Pauli–Fierz (PF) theory
[7]. Then the coupling of the extra scalar DOF to the sources remains finite even in the limit
of vanishing graviton mass. That is, no matter how small the graviton mass is, the massive
theory is finitely different from the massless one.

To elucidate, let us look at the linearized massive gravity action with a generic mass term in a flat 4D background:

$$S_{m_g,a} = \int d^4x \left\{ \partial_\alpha h^{\alpha\mu} \partial_\beta h^\beta_\mu - \frac{1}{2} \partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \frac{1}{2} \partial_\alpha h \partial^\alpha h - \partial_\alpha h^{\mu\alpha} \partial_\mu h - \frac{m_g^2}{2} (h^{\mu\nu} h_{\mu\nu} - ah^2) \right\},$$
(1)

with $h = \eta^{\mu\nu} h_{\mu\nu}$ and $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. The corresponding tree level propagator is

$$\mathbf{P}^{(0)}_{m_g,a} = \underbrace{\frac{i}{k^2 + m_g^2} \left(-\frac{\mathbf{I}_1}{2} + \frac{\mathbf{I}_2}{2} \right)}_{H2} + \underbrace{\frac{i}{k^2 + m_g^2} \frac{\mathbf{I}_1}{6}}_{H0_1} + \underbrace{\frac{-i}{k^2 + \frac{4a-1}{2(1-a)} m_g^2} \frac{\mathbf{I}_1}{6}}_{H0_2} + \cdots,$$
(2)

where the $\mathbf{I}_i$'s are a complete set of tensor bases with four indicies, whose definitions will be given in section 2. The propagator has a GR-like helicity-2 pole($H2$) and two helicity-0 ones($H0_1$ and $H0_2$), while $\cdots$ is terms that vanish upon contraction with conserved sources. In the $m_g \to 0$ limit, (2) becomes $\frac{i}{k^2} \left(-\frac{\mathbf{I}_1}{2} + \frac{\mathbf{I}_2}{2} \right) + \cdots$, which is the same as the massless graviton propagator of GR, and hence we do not have any discontinuity problem. But unfortunately (1) has a ghost DOF; (2) shows that $H0_2$ has a negative coupling. In fact, it is this ghost that cancels the other scalar in the massless limit, allowing a smooth transition to the massless theory.

By choosing $a = 1$ the ghost mode decouples because its mass diverges, and we obtain PF theory:

$$\mathbf{P}^{(0)}_{m_g} = \frac{i}{k^2 + m_g^2} \left(-\frac{\mathbf{I}_1}{2} + \frac{\mathbf{I}_2}{2} \right) + \frac{i}{k^2 + m_g^2} \frac{\mathbf{I}_1}{6} + \cdots.$$
(3)

But then $H0_1$ survives the $m_g \to 0$ limit, creating an untraversable gap between the massive and the massless theories. How or if we can remove this discontinuity has been an active subject of research [8, 9].

The phenomenological reason to study massive gravity is a possibility of solving (a part of) the cosmological constant(Λ) problem by modifying gravity at large distances. Among many proposed solutions to the Λ problem, the infrared(IR) modification of gravity is the idea that gravity behaves differently at large distance scales compared to short distances. Let us assume that by some mechanism, e.g., [10], we succeed in achieving vanishing Λ. The next step is to reconcile our zero Λ with the small but non-zero Λ calculated from the observational data. The 'observed' Λ [11] is obtained with the assumption that GR holds at all scales. Then we can imagine that if the characteristic of the 'real' gravity is different from that of GR, interpreting the data with the right gravity may give an explanation to the accelerated expansion of the Universe without Λ. But this idea gets severely constrained by experiments and observations which confirm the validity of GR from mm to the solar system scale. Therefore, the desired modification of gravity should be consistent with GR at short distances, while getting weaker than GR at large distances in order to mimic Λ. PF theory meets our demands in the IR, but fails to satisfy the short distance criterion because of the vDVZ discontinuity.

While interesting enough already at the classical level, not much attention has been paid to the quantum aspects of massive gravity theories. The simplest case where we might expect to see some quantum effects is a real massive scalar field, ϕ, minimally coupled to gravity:

$$S_\phi = \frac{1}{2} \int d^4x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2).$$
(4)

By expanding it in M_P^{-1}, using $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}$ with M_P the Planck mass, we can get the terms describing the quantum interactions between ϕ and the graviton $h_{\mu\nu}$. But as is well known

[1, 12], the effective action obtained by integrating out ϕ (or any matter field in general) takes the form of

$$\sim \int d^4x \sqrt{-g}(\Lambda + aR + b_1 R^2 + b_2 R_{\mu\nu}R^{\mu\nu} + \cdots). \tag{5}$$

Since this result is independent of the specifics of the gravity sector, it holds for massive gravity as well as for GR. That is, PF theory gets no more unusual or unexpected contributions from matter loops than GR does, which leads us to considering graviton loops in order to see something interesting. Therefore, in this paper, we will explore what the combination of 'quantum effects' and 'massive gravity' can offer, by investigating loop corrections to PF theory from the graviton with a generic cubic interaction (section 2) and the graviton with an arbitrary interaction (section 3).

2. Graviton loop corrections to PF

The tree level quadratic action of PF massive gravity in a flat background is

$$S_{\text{PF}} = \int d^4x \left\{ \partial_\alpha h^{\alpha\mu} \partial_\beta h_\mu^\beta - \frac{1}{2}\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + \frac{1}{2}\partial_\alpha h \partial^\alpha h - \partial_\alpha h^{\mu\alpha}\partial_\mu h - \frac{m_g^2}{2}(h^{\mu\nu}h_{\mu\nu} - h^2) \right\}$$

$$= \frac{1}{2} \int \frac{d^4k}{(2\pi)^4} \hat{h}^{\mu\nu} \left\{ (k^2 + m_g^2)\left(\mathbf{I}_{1,\mu\nu;\lambda\rho} - \frac{\mathbf{I}_{2,\mu\nu;\lambda\rho}}{2}\right) + \frac{\mathbf{I}_{3,\mu\nu;\lambda\rho}}{2} - \mathbf{I}_{4,\mu\nu;\lambda\rho} \right\} \hat{h}^{\lambda\rho}, \tag{6}$$

with $\hat{h}^{\mu\nu}$ the Fourier transform of $h^{\mu\nu}$, from which we obtain the tree level massive graviton propagator,

$$\mathbf{P}_{m_g,\mu\nu;\lambda\rho}^{(0)} = \frac{i}{k^2 + m_g^2}\left(-\frac{\mathbf{I}_{1,\mu\nu;\lambda\rho}}{3} + \frac{\mathbf{I}_{2,\mu\nu;\lambda\rho}}{2} + \frac{\mathbf{I}_{3,\mu\nu;\lambda\rho}}{2m_g^2} - \frac{\mathbf{I}_{4,\mu\nu;\lambda\rho}}{3m_g^2} + \frac{2\mathbf{I}_{5,\mu\nu;\lambda\rho}}{3m_g^4}\right). \tag{7}$$

$\mathbf{I}_i$ is a complete set with which a symmetric second rank tensor can be expanded:

$$\mathbf{I}_{1,\mu\nu;\lambda\rho} = \eta_{\mu\nu}\eta_{\lambda\rho}, \quad \mathbf{I}_{2,\mu\nu;\lambda\rho} = \eta_{\mu\lambda}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\lambda},$$

$$\mathbf{I}_{3,\mu\nu;\lambda\rho} = \eta_{\mu\lambda}k_\nu k_\rho + \eta_{\mu\rho}k_\nu k_\lambda + (\mu \leftrightarrow \nu), \tag{8}$$

$$\mathbf{I}_{4,\mu\nu;\lambda\rho} = \mathbf{I}_{41,\mu\nu;\lambda\rho} + \mathbf{I}_{42,\mu\nu;\lambda\rho}, \quad \mathbf{I}_{41,\mu\nu;\lambda\rho} = \eta_{\mu\nu}k_\lambda k_\rho, \quad \mathbf{I}_{42,\mu\nu;\lambda\rho} = k_\mu k_\nu \eta_{\lambda\rho},$$

$$\mathbf{I}_{5,\mu\nu;\lambda\rho} = k_\mu k_\nu k_\lambda k_\rho.$$

Hereafter, we omit spacetime indices on $\mathbf{I}_i$ and $\mathbf{P}_{m_g}$ for the sake of simplicity. Note that we do not have to worry about fixing a gauge nor introducing Faddeev–Popov ghost, because the general covariance is explicitly broken by the graviton mass terms. To (6), we add a generic cubic interaction:

$$S_{\text{int}} = \int d^4x \frac{\lambda}{(2!)^3 3!} \frac{m_g^2}{2M_P}\left(\alpha h_{\nu_1}^{\mu_1} h_{\nu_2}^{\nu_1} h_{\mu_1}^{\nu_2} + \beta h^{\mu\nu}h_{\mu\nu}h + \gamma h^3\right), \tag{9}$$

which gives a 3-graviton vertex:

$$V_g^{(3)} = -\frac{i\lambda}{(2!)^3 3!} \frac{m_g^2}{2M_P}\left(\alpha\eta_{\mu_1\nu_3}\eta_{\nu_1\mu_2}\eta_{\nu_2\nu_3} + \beta\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3} + \gamma\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3}\right.$$

$$\left. + \text{symmetrization in } \mu\nu + \text{permutation in } 123\right). \tag{10}$$

With the necessary building blocks ready, let us calculate loops. The details are given in appendix A. The first is the tadpole:

$$\left.\begin{array}{c}\text{(tadpole diagram)}\end{array}\right.^{\mu\nu} = -\frac{1}{\epsilon}\frac{5i\lambda m_g^4}{192\pi^2 M_P}(3\alpha + 4\beta)\eta_{\mu\nu} + \text{finite}, \tag{11}$$

which is non-zero for a generic choice of the interaction, and therefore would imply that we have chosen the wrong background. But we can use the freedom of choosing α, β and γ, and

$$\beta = -\tfrac{3}{4}\alpha \tag{12}$$

renders (11) to vanish. With this choice, vertex (10) becomes

$$V_g^{(3)} = -\mathrm{i}\lambda \frac{m_g^2}{16 M_P}(\alpha \mathbf{J}_1 - \alpha \mathbf{J}_2 + 8\gamma \mathbf{J}_3), \tag{13}$$

where $\mathbf{J}_{1(2,\,3)}$ is the symmetrization and permutation of $\eta_{\mu_1\nu_3}\eta_{\nu_1\mu_2}\eta_{\nu_2\mu_3}(\eta_{\mu_1\mu_2}\eta_{\nu_1\nu_2}\eta_{\mu_3\nu_3},$ $\eta_{\mu_1\nu_1}\eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3})$.

Without the tadpole, the only 1-loop contribution to the 2-point function is

$$\mu\nu \,\, \varhexagon \,\, \lambda\rho \;\; = \frac{1}{\epsilon}\frac{5\mathrm{i}\lambda^2 m_g^2}{13824\pi^2 M_P^2}\alpha^2\big\{\big(7k^2 - m_g^2\big)\mathbf{I}_1 - \big(5k^2 - 2m_g^2\big)\mathbf{I}_2 + 18\mathbf{I}_3 - 18\mathbf{I}_4\big\}$$

$$+\, \mathcal{O}\!\left(\frac{k^4}{M_P^2}\right) + \text{finite.} \tag{14}$$

We can immediately see that it does not conform to (6). Reconstructing the full counterterms out of (14) reveals the trouble it causes:

$$\hat{h}^{\mu\nu}\big\{\big(7k^2 - m_g^2\big)\mathbf{I}_1 - \big(5k^2 - 2m_g^2\big)\mathbf{I}_2 + 18\mathbf{I}_3 - 18\mathbf{I}_4\big\}\hat{h}^{\lambda\rho}$$
$$\Rightarrow 7\partial_\alpha h \partial^\alpha h - 10\partial_\alpha h_{\mu\nu}\partial^\alpha h^{\mu\nu} + 72\partial_\alpha h^{\alpha\mu}\partial_\beta h_\mu^\beta - 36\partial_\alpha h^{\mu\alpha}\partial_\mu h - m_g^2(4h^{\mu\nu}h_{\mu\nu} - h^2)$$
$$=\, -33(\dot{h}_{00})^2 + \cdots. \tag{15}$$

As shown in [8] using the Arnowitt–Deser–Misner-decomposed version of (6),

$$S_{\mathrm{PF}} = \int \mathrm{d}^4x \,\left\{\pi^{ij}\dot{h}_{ij} - \left(\pi_{ij}^2 - \frac{1}{2}\pi_{ii}^2\right) + 2h_{0i}\partial_j\pi^{ij} + \frac{1}{2}h_{00}\big(\partial_i^2 h_{ii} - \partial_i\partial_j h_{ij}\big) + {}^3R|_{h^2}\right.$$
$$\left. -\, \frac{m_g^2}{4}\big(h_{ij}^2 - h_{ii}^2 - 2h_{0i}{}^2 + 2h_{00}h_{ii}\big)\right\}, \tag{16}$$

with π_{ij} being the conjugate momentum to h_{ij} and ${}^3R$ being the curvature scalar constructed with h_{ij}, PF theory comes to have five healthy DOFs because among the 10 DOFs of $h_{\mu\nu}$, h_{00} and h_{0i} are non-dynamical, and furthermore the action is linear in h_{00}. That is, being a Lagrange multiplier, h_{00} provides a constraint to eliminate another DOF, the sixth ghost mode. What we see from (15) is that quantum effects remove such a feature, incurring the reappearance of the ghost DOF.

A similar outcome is obtained with the 3-point function:

$$\overset{\mu_1\nu_1}{\underset{\mu_2\nu_2 \qquad \mu_3\nu_3}{\varhexagon}} \;\; = \frac{1}{\epsilon}\frac{5\mathrm{i}\lambda^3 m_g^4}{442368\pi^2 M_P^3}\alpha^2\left\{13\alpha\mathbf{J}_1 - \frac{1}{3}(35\alpha + 32\gamma)\mathbf{J}_2 + (11\alpha + 16\gamma)\mathbf{J}_3\right\}$$

$$+\, \text{terms with external momenta} + \text{finite.} \tag{17}$$

No nontrivial choice of α and γ can make (17) conform to the tree level vertex (13).

If we interpret terms with derivatives in (6) as $2M_P^2\sqrt{-g}R\big|_{h^2}$ with $g_{\mu\nu} = \eta_{\mu\nu} + \frac{h_{\mu\nu}}{M_P}$, we might well have to consider derivative interactions such as $2M_P^2\sqrt{-g}R\big|_{h^3}$ as well as (9). The case of a more general interaction including the cubic and quartic expansion of $\sqrt{-g}R$ is analyzed in appendix B with the same conclusion.

3. Resummed graviton propagator

We just saw the breakdown of PF theory at the quantum level. But since we tried only one type of interaction, it is still possible that the situation gets better for different types of well designed interactions. To explore the full capability of quantum effect, let us perform a general analysis independent of the details of the interactions.

No matter what the interaction is, the one particle irreducible diagram (1PI), $\mathbf{\Pi}$, for the 2-point function may be written as

$$\mathbf{\Pi}_{\mu\nu;\lambda\rho} = \sum_{j=1}^{5} b_j(k)\mathbf{I}_j \,, \tag{18}$$

because $\mathbf{I}_j$ is a complete basis. In the previous section, we concentrated only on $\mathcal{O}(k^2)$ part of $\mathbf{\Pi}$, but of course there are pieces of $\mathcal{O}(k^4)$ and higher. Being generically nonrenormalizable, the tree level PF theory cannot handle the divergences of higher powers of k. Here we take the idea of [13] and treat the massive gravity as an effective field theory (EFT). That is, we assume the tree level PF action gets complemented by EFT terms, which can absorb divergences from the loops. But regardless of the EFT treatment, $\mathcal{O}(k^0)$ and $\mathcal{O}(k^2)$ divergences must be absorbed by the bare theory. Then, the very first requirement for our interaction is

> $\mathcal{O}(k^0)$ and $\mathcal{O}(k^2)$ divergences of 1PI for the 2-point function
>
> conform to the tree level PF action. (19)

From here on, it is understood that b_i's are the finite parts of the quantum corrections with the divergences taken care of.

To obtain the resummed graviton propagator

$$\mathbf{P}_{m_g} = \mathbf{P}_{m_g}^{(0)} + \mathbf{P}_{m_g}^{(0)} \cdot \mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)} + \mathbf{P}_{m_g}^{(0)} \cdot \left(\mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)}\right)^2 + \cdots, \tag{20}$$

we need to find $\left(\mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)}\right)^n$ as a function of n. By writing

$$\mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)} = \sum_i r_i \mathbf{I}_i, \tag{21}$$

and then

$$\left(\mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)}\right)^{n+1} = \sum_i a_i^{(n)} \mathbf{I}_i = \sum_i r_i \mathbf{I}_i \cdot \sum_j a_j^{(n-1)} \mathbf{I}_j, \tag{22}$$

we can obtain the recurrence relations between $a_i^{(n)}$ and $a_j^{(n+1)}$. Explicit solutions for $a_i^{(n)}$ are given in appendix C, and the resummed propagator is

$$\mathbf{P}_{m_g} = \mathbf{P}_{m_g}^{(0)} + \sum_{n=0}^{\infty} \mathbf{P}_{m_g}^{(0)} \cdot \left(\mathbf{\Pi} \cdot \mathbf{P}_{m_g}^{(0)}\right)^{n+1}$$

$$= -\frac{i}{k^2 + m_g^2 + 2b_2} \frac{\mathbf{I}_1}{3} + \frac{i}{k^2 + m_g^2 + 2b_2} \frac{\mathbf{I}_2}{2} - i\frac{b_5 k^4 + 2(2b_3 + b_4)k^2 + b_1 + 2b_2}{d_{m_g}(k)} \frac{\mathbf{I}_1}{3}$$

$$+ (\mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5), \tag{23}$$

where

$$d_{m_g}(k) = 2b_5 k^6 - \left(4b_1 + 2b_2 - 3m_g^2\right)\left(2b_2 + m_g^2\right)$$

$$+ \left\{8b_3 + b_4(4 + 3b_4) - b_5(3b_1 + 2b_2) + 2b_5 m_g^2\right\}k^4$$

$$+ \left\{2b_1(1 - 6b_3) + 4b_2(1 - 2b_3 - b_4) + 2(4b_3 - b_4)m_g^2\right\}k^2. \tag{24}$$

Comparing the first two terms of (23) with the tree level propagator, (7), we can immediately see that the tree level theory gets renormalized by b_2, which is a progress over the 1-loop analysis where it was not clear which counterterm renormalized what. Next, by rewriting (23) as

$$\mathbf{P}_{m_g} = \frac{i}{k^2 + m_g^2 + 2b_2}\left(-\frac{\mathbf{I}_1}{2} + \frac{\mathbf{I}_2}{2}\right) + \frac{i}{k^2 + m_g^2 + 2b_2}\frac{\mathbf{I}_1}{6} + \frac{-i}{k^2 + \mathcal{M}^2}\frac{\mathbf{I}_1}{6}$$
$$+ (\mathbf{I}_3 + \mathbf{I}_4 + \mathbf{I}_5), \tag{25}$$

with

$$\mathcal{M}^2 = \frac{1}{2\{b_5 k^4 + 2(2b_3 + b_4)k^2 + b_1 + 2b_2\}}\Big[\{3b_4^2 - b_5(3b_1 + 2b_2 - 2m_g^2)\}k^4$$
$$- 2\{6b_1 b_3 + 2b_2(2b_3 + b_4) - m_g^2(4b_3 - b_4)\}k^2$$
$$- (4b_1 + 2b_2 - 3m_g^2)(2b_2 + m_g^2)\Big], \tag{26}$$

we note that (25) takes the same form as the tree level *non*-PF propagator, (2), implying that a generic interaction revives the ghost sixth mode. But (23) shows how to avoid this pathology; the ghost pole can be removed if

$$b_5 k^4 + 2(2b_3 + b_4)k^2 + b_1 + 2b_2 = 0. \tag{27}$$

Therefore, in order to be quantum-safe, 1PI from a desirable interaction should satisfy at least (19) and (27).

4. Discussion

Through straightforward loop calculations and the analysis on the propagator, we showed that although loop corrections from a quantum interaction may spoil PF massive gravity by reviving the sixth mode ghost, we may still be able to have a healthy theory by requiring the allowed interactions to satisfy appropriate conditions.

Then the next task would be to find the right interactions. For this purpose, understanding why the cubic interaction of section 2 failed would be useful. In fact, the results of section 2 should not be surprising, once we realize that in PF the elimination of the sixth mode is achieved by an onshell symmetry. To identify this symmetry, we start with action, (1), with a generic mass term. The equation of motion for the graviton, h, with a source, T, is

$$T_{\mu\nu} = \partial^2 h_{\mu\nu} - \eta_{\mu\nu}\partial^2 h - \partial_\mu\partial_\alpha h_\nu^\alpha - \partial_\nu\partial_\alpha h_\mu^\alpha + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - m_g^2(h_{\mu\nu} - a\eta_{\mu\nu}h).$$
$$\tag{28}$$

On the RHS, terms with derivatives come from the Einstein tensor, and the Bianchi identity guarantees that their contraction with ∂^μ vanishes. Then for a conserved source, i.e. $\partial^\mu T_{\mu\nu} = 0$, we get the following onshell constraint:

$$0 = \partial^\mu h_{\mu\nu} - a\partial_\nu h. \tag{29}$$

Next we vary (1) under the infinitesimal coordinate transformation, $x \rightarrow x + \xi$ or $h_{\mu\nu} \rightarrow h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. Again, terms with derivatives are $\sqrt{-g}R\big|_{h^2}$ and therefore invariant. A remainder of the variation of the mass term is

$$\delta S_{m_g} = m_g^2 \int d^4 x \xi^\nu(\partial^\mu h_{\mu\nu} - a\partial_\nu h). \tag{30}$$

Then a generic massive gravity seems to have GCI if constraint (29) is imposed. Of course this is not true, because we have yet to take into account that the transformed field should also satisfy (29), i.e.

$$0 = \partial^\mu h'_{\mu\nu} - a\partial_\nu h' = (1 - 2a)\partial_\nu\partial_\mu\xi^\mu + \partial^2\xi_\nu. \tag{31}$$

Decomposing ξ into a transverse vector ξ^T and a longitudinal scalar σ such that $\xi_\mu = \xi_\mu^T + \partial_\mu \sigma$, (31) becomes

$$\partial^2 \xi_\nu^T + 2(1-a)\partial_\nu \partial^2 \sigma = 0. \tag{32}$$

Thus, we end up with $\xi^T = 0$ and for $a \neq 1$ σ should also vanish: Non-PF action has no symmetry. But when $a = 1$, which is the case of PF theory, (32) can be satisfied with a nontrivial σ, and PF theory has a residual symmetry parametrized by $\xi_\mu = \partial_\mu \sigma$. Since this symmetry works under the *onshell* constraint (29), it may not be preserved when we go offshell in the loop calculations.

Therefore, further efforts to find a quantum-safe theory of massive gravity can be directed in two different ways.

(i) We may try to construct a nonlinear completion of PF where the sixth mode is removed by a *full* symmetry.
(ii) We can attempt to directly find a quantum interaction whose 1PI satisfies e.g., (19) and (27).

With various versions [14] of the completion of PF already at hand, it would be straightforward to pursue (i), which in the end might lead us to the right interaction sought after in (ii).

Acknowledgments

Appendix A. Graviton loops

All the 1-loop diagrams constructed with the 3-graviton vertex $V_g^{(3)}$ (10) are drawn in figure A1. Evaluating them is straightforward but laborious and tedious. The tadpole diagram,

$$
\begin{aligned}
\text{(a)} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} V_g^{(3)}(\alpha\beta; \mu_1\nu_1; \mu_2\nu_2) \mathbf{P}^{(0)}_{m_g \mu_1\nu_1;\mu_2\nu_2}(p), \\
&= \frac{\lambda m_g^2}{M_p} \eta_{\mu\nu} \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{1}{p^2 + m_g^2} \left\{ \frac{3\alpha + (d+2)\beta + 3d\gamma}{9 m_g^4 d} p^4 \right. \\
&\quad \left. + \frac{(3d+1)\alpha + (d^2 - d + 4)\beta - 2(d-3)d\gamma}{6} \left(\frac{p^2}{m_g^2 d} + \frac{1}{2} \right) \right\} \\
&= -\frac{1}{\epsilon} \frac{5\mathrm{i}\lambda m_g^4}{192\pi^2 M_P} (3\alpha + 4\beta)\eta_{\mu\nu} + \text{finite}, \tag{A.1}
\end{aligned}
$$

fixes $\beta = -\frac{3}{4}\alpha$. Then

$$
\begin{aligned}
\text{(b)} &= \int \frac{\mathrm{d}^4 p}{(2\pi)^4} V_g^{(3)}(\mu\nu; \mu_1\nu_1; \mu_4\nu_4) \mathbf{P}^{(0)}_{m_g \mu_1\nu_1;\mu_2\nu_2}(p) V_g^{(3)}(\lambda\rho; \mu_2\nu_2; \mu_3\nu_3) \mathbf{P}^{(0)}_{m_g \mu_3\nu_3;\mu_4\nu_4}(p+k) \\
&= -\frac{\lambda^2 m_g^4}{M_P^2} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \int_0^1 \mathrm{d}x \frac{1}{\left\{ l^2 + m_g^2 + x(1-x)k^2 \right\}^2} \\
&\quad \left[\left(\frac{7\alpha^2 + 32\alpha\gamma - 128\gamma^2}{576 m_g^2} (1 - 2x(1-x))k^2 + \frac{(\alpha - 2\gamma)(\alpha + 4\gamma)}{18} \right) \mathbf{I}_1 \right.
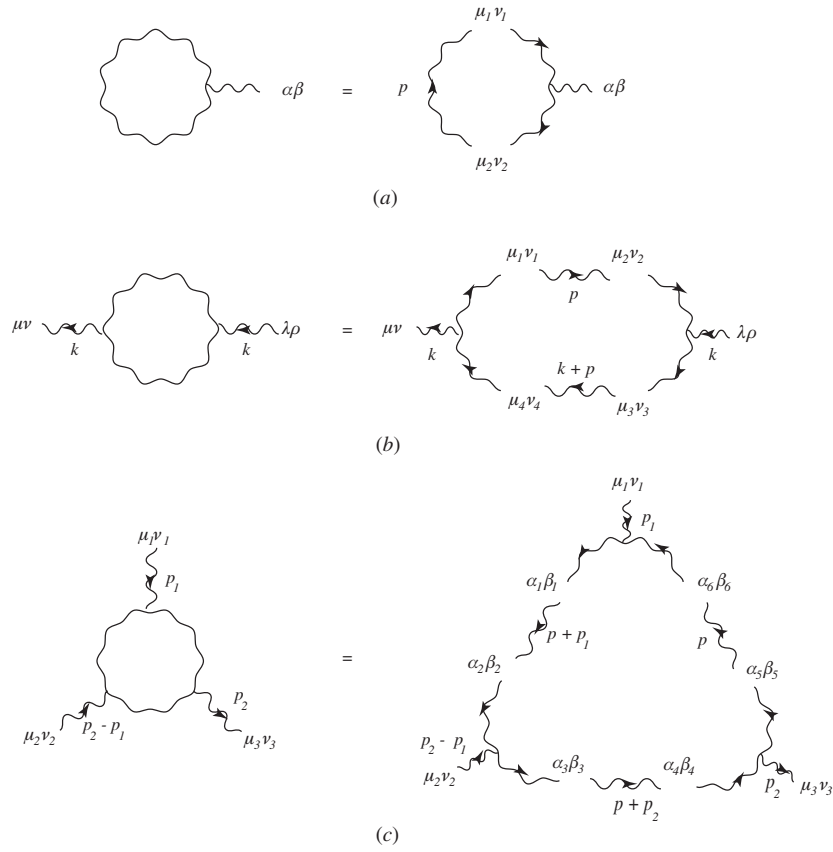\end{aligned}
$$

(a)

(b)

(c)

Figure A1. Graviton 1-loops.

$$
-\frac{\alpha^2}{8}\left(\frac{3(1-2x(1-x))}{8m_g^2}k^2+1\right)\mathbf{I}_2-\frac{\alpha^2}{32m_g^2}(1-2x(1-x))(\mathbf{I}_3-\mathbf{I}_4)
$$

$$
+\left\{\left(\frac{(25-58x(1-x))\alpha^2-32(5-7x(1-x))\alpha\gamma+128(5-9x(1-x))\gamma^3}{1152m_g^2}k^2\right.\right.
$$

$$
\left.+\frac{(\alpha-2\gamma)(\alpha+4\gamma)}{18}\right)\mathbf{I}_1
$$

$$
-\frac{\alpha^2}{8}\left(\frac{5-16x(1-x)}{16m_g^2}k^2+1\right)\mathbf{I}_2-\frac{\alpha^2}{192m_g^2}(5-16x(1-x))\mathbf{I}_3
$$

$$
\left.+\frac{(15-56x(1-x))\alpha^2+64x(1-x)\alpha\gamma}{576m_g^2}\mathbf{I}_4\right)\right\}\frac{l^2}{m_g^2}
$$

$$
+\left\{\left(-\frac{(17-50x(1-x))\alpha^2+64(5-8x(1-x))\alpha\gamma-1536(1-3x(1-x))\gamma^3}{3456m_g^2}k^2\right.\right.
$$

$$
\left.+\frac{19\alpha^2-48\alpha\gamma+192\gamma^2}{576}\right)\mathbf{I}_1
$$

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$$+ \frac{\alpha^2}{18} \left(\frac{1 + 22x(1-x)}{64 m_g^2} k^2 - 1 \right) \mathbf{I}_2 - \frac{\alpha^2}{576 m_g^2} (21 - 50x(1-x)) \mathbf{I}_3$$

$$+ \frac{(71 - 230x(1-x))\alpha^2 - 64(1 - 10x(1-x))\alpha\gamma}{1728 m_g^2} \mathbf{I}_4 \Biggr) \frac{l^4}{m_g^4}$$

$$+ \Biggl\{ \left(-\frac{65(1-2x)^2 \alpha^2 - 64(11 - 45x(1-x))\alpha\gamma + 768(3 - 10x(1-x))\gamma^3}{6912 m_g^2} k^2 \right.$$

$$+ \frac{5\alpha^2 - 32\alpha\gamma + 128\gamma^2}{576} \Biggr) \mathbf{I}_1$$

$$+ \frac{\alpha^2}{96} \left(\frac{7 + 20x(1-x)}{72 m_g^2} k^2 - 1 \right) \mathbf{I}_2 - \frac{\alpha^2}{3456 m_g^2} (19 - 60x(1-x)) \mathbf{I}_3$$

$$+ \frac{5(7 - 36x(1-x))\alpha^2 - 64(2 - 15x(1-x))\alpha\gamma}{3456 m_g^2} \mathbf{I}_4 \Biggr\} \frac{l^6}{m_g^6}$$

$$- \frac{(\alpha^2 - 24\alpha\gamma + 96\gamma^2)\mathbf{I}_1 + \alpha^2 \mathbf{I}_2}{864} \frac{l^8}{m_g^8} + \mathcal{O}(k^4) \Biggr]$$

$$= \frac{\alpha^2}{\epsilon} \frac{5\mathrm{i}\lambda^2 m_g^2}{13824\pi^2 M_P^2} \Bigl\{ (7k^2 - m_g^2)\mathbf{I}_1 - (5k^2 - 2m_g^2)\mathbf{I}_2 + 18\mathbf{I}_3 - 18\mathbf{I}_4 \Bigr\} + \mathcal{O}\left(\frac{k^4}{M_P^2} \right) + \text{finite},$$

$$(A.2)$$

$$(c) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} V_g^{(3)}(\mu_1\nu_1; \alpha_1\beta_1; \alpha_6\beta_6) \mathbf{P}_{m_g \alpha_1\beta_1; \alpha_2\beta_2}^{(0)}(p + p_1)$$

$$V_g^{(3)}(\mu_2\nu_2; \alpha_2\beta_2; \alpha_3\beta_3) \mathbf{P}_{m_g \alpha_3\beta_3; \alpha_4\beta_4}^{(0)}(p + p_2) V_g^{(3)}(\mu_3\nu_3; \alpha_4\beta_4; \alpha_5\beta_5) \mathbf{P}_{m_g \alpha_5\beta_5; \alpha_6\beta_6}^{(0)}(p)$$

$$= \frac{\lambda^3 m_g^6}{256 M_P^3} \int \frac{\mathrm{d}^d l}{(2\pi)^d} \frac{1}{(l^2 + m_g^2)^3} \Biggl\{ \alpha^3 \mathbf{J}_1 - \alpha^3 \mathbf{J}_2 + \frac{31\alpha^3 - 96\alpha^2\gamma + 768\alpha\gamma^2 - 2048\gamma^3}{27} \mathbf{J}_3$$

$$+ \left(\frac{3\alpha^3}{2m_g^2} \mathbf{J}_1 - \frac{3\alpha^3}{2m_g^2} \mathbf{J}_2 + \frac{31\alpha^3 - 96\alpha^2\gamma + 768\alpha\gamma^2 - 2048\gamma^3}{18 m_g^2} \mathbf{J}_3 \right) l^2$$

$$+ \left(\frac{7\alpha^3}{6 m_g^4} \mathbf{J}_1 - \frac{67\alpha^3 - 32\alpha^2\gamma}{54 m_g^4} \mathbf{J}_2 + \frac{21\alpha^3 + 32\alpha^2\gamma - 384\alpha\gamma^2 + 1024\gamma^3}{18 m_g^4} \mathbf{J}_3 \right) l^4$$

$$+ \left(\frac{5\alpha^3}{6 m_g^6} \mathbf{J}_1 - \frac{28\alpha^3 - 44\alpha^2\gamma}{27 m_g^6} \mathbf{J}_2 + \frac{101\alpha^3 + 264\alpha^2\gamma - 4224\alpha\gamma^2 + 11264\gamma^3}{108 m_g^6} \mathbf{J}_3 \right) l^6$$

$$+ \left(\frac{13\alpha^3}{36 m_g^8} \mathbf{J}_1 - \frac{181\alpha^3 - 512\alpha^2\gamma}{324 m_g^8} \mathbf{J}_2 + \frac{77\alpha^3 - 400\alpha^2\gamma + 1152\alpha\gamma^2 - 3072\gamma^3}{108 m_g^8} \mathbf{J}_3 \right) l^8$$

$$+ \left(\frac{5\alpha^3}{72 m_g^{10}} \mathbf{J}_1 - \frac{97\alpha^3 - 416\alpha^2\gamma}{648 m_g^{10}} \mathbf{J}_2 + \frac{53\alpha^3 - 496\alpha^2\gamma + 2304\alpha\gamma^2 - 6144\gamma^3}{216 m_g^{10}} \mathbf{J}_3 \right) l^{10}$$

$$+ \left(\frac{\alpha^3}{162 m_g^{12}} \mathbf{J}_1 - \frac{3\alpha^3 - 16\alpha^2\gamma}{162 m_g^{12}} \mathbf{J}_2 + \frac{\alpha^3 + 48\alpha^2\gamma - 576\alpha\gamma^2 + 1536\gamma^3}{162 m_g^{12}} \mathbf{J}_3 \right) l^{12} \Biggr\}$$

$$+ \text{terms with } p_1 \text{ and } p_2$$

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$$= \frac{\alpha^2}{\epsilon} \frac{5i\lambda^3 m_g^4}{442368\pi^2 M_P^3} \left\{ 13\alpha \mathbf{J}_1 - \frac{1}{3}(35\alpha + 32\gamma)\mathbf{J}_2 + (11\alpha + 16\gamma)\mathbf{J}_3 \right\}$$
$$+ \text{ terms with } p_1 \text{ and } p_2 + \text{finite}. \tag{A.3}$$

Note that in the second equality of (A.3), instead of $\left(l^2 + m_g^2\right)^{-3}$ there should have been $\left\{l^2 + m_g^2 + x(1-x)p_1^2 + y(1-y)p_2^2 - 2xyp_1 \cdot p_2\right\}^{-3}$ with the appropriate integrations over the Feynman parameters x and y. But here we are only interested in the contribution in the form of the tree level 3-point vertex, and take the shortcut of ignoring any dependence on the external momenta p_1 and p_2.

Appendix B. Graviton loops with a more general interaction

An interaction more general than (9) may contain derivatives, and a reasonable way to introduce such interactions is to expand $\sqrt{-g}R$ to higher orders in h. That is, we now investigate

$$S_{m_g} = S_{\text{PF}} + 2M_P^2 \int \mathrm{d}^4x \left\{ [\sqrt{-g}R]|_{h^3} + [\sqrt{-g}R]|_{h^4} \right.$$
$$- \frac{m_g^2}{4(2!)^3 3! M_P^3} \left(x_1 h_\nu^\mu h_\sigma^\nu h_\mu^\sigma + x_2 h_{\mu\nu} h^{\mu\nu} h + x_3 h^3\right)$$
$$- \frac{m_g^2}{4(2!)^4 4! M_P^4} \left(y_1 h_\nu^\mu h_\sigma^\nu h_\lambda^\sigma h_\mu^\lambda + y_2 (h_{\mu\nu} h^{\mu\nu})^2 + y_3 h_\nu^\mu h_\sigma^\nu h_\mu^\sigma h + y_4 h_{\mu\nu} h^{\mu\nu} h^2 + y_5 h^4\right) \right\}$$
$$= S_{\text{PF}} + \int \mathrm{d}^4x \left(\frac{h^{\mu_1\nu_1} h^{\mu_2\nu_2} h^{\mu_3\nu_3}}{M_P} V_g^{(3)}{}_{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3} \right.$$
$$+ \left. \frac{h^{\mu_1\nu_1} h^{\mu_2\nu_2} h^{\mu_3\nu_3} h^{\mu_4\nu_4}}{M_P^2} V_g^{(4)}{}_{\mu_1\nu_1;\mu_2\nu_2;\mu_3\nu_3;\mu_4\nu_4} \right). \tag{B.1}$$

Now that M_P^{-1} plays the role of the coupling, we need the cubic and quartic vertices in order to get loops of $\mathcal{O}\left(M_P^{-2}\right)$. $\sqrt{-g}R\big|_{h^3}$ and $\sqrt{-g}R\big|_{h^4}$ are $\mathcal{O}(h^3)$ and $\mathcal{O}(h^4)$ parts of $\sqrt{-g}R$, whose explicit form can be found at, e.g., [15], and then we obtain $V_g^{(3)}$ and $V_g^{(4)}$ straightforwardly.

To find loop corrections to S_{PF}, we path-integrate over h, while $\mathcal{O}(h^3)$ and $\mathcal{O}(h^4)$ terms provide quantum interactions:

$$Z[J] = \int \mathcal{D}h \exp\left[i\left(S_{m_g} + \int \mathrm{d}^4x J^{\mu\nu} h_{\mu\nu}\right) \right]$$
$$= \mathcal{N} \exp\left[i \int \mathrm{d}^4z \left(\frac{iV_g^{(3)}}{M_P} \frac{\delta^3}{\delta J(z)^3} + \frac{V_g^{(4)}}{M_P^2} \frac{\delta^4}{\delta J(z)^4} \right) \right]$$
$$\times \exp\left[\frac{i}{2} \int \mathrm{d}^4x \mathrm{d}^4y J(x)\left(-i\tilde{\mathbf{P}}_{m_g}^{(0)}(x-y)\right) J(y) \right], \tag{B.2}$$

where $\mathcal{N}$ is a normalization constant and $\tilde{\mathbf{P}}_{m_g}^{(0)}$ is the inverse Fourier transform of the tree level PF graviton propagator, (7). With

$$\tilde{\mathbf{P}}_{m_g}^{(0)} = \text{\~{}\~{}\~{}}, \quad i \int \mathrm{d}^4x J(x)\left(-i\tilde{\mathbf{P}}_{m_g}^{(0)}(x-y)\right) = \times\text{\~{}\~{}}, \tag{B.3}$$

we can expand (B.2) diagrammatically. Up to $\mathcal{O}\!\left(M_P^{-3}\right)$,

$$3i \underbrace{}_{(a)} + 6 \underbrace{}_{(b)} - \frac{1}{2}\Big(18 \underbrace{}_{(c)} + 18 \underbrace{}_{(d)}\Big), \tag{B.4}$$

where $(a) \sim (d)$ are the loop parts of corresponding diagrams. The linear tadpole, (a), is

$$(a) = \frac{5im_g^4}{768\pi^2 M_P\epsilon}(-2 + 3x_1 + 4x_2)\eta_{\alpha\beta} + \text{finite}. \tag{B.5}$$

In order not to have a tadpole, we have to fix $x_2 = \frac{2-3x_1}{4}$, which in turn makes (c) vanish. The rest of the loops are

$$(b) = \frac{5im_g^2}{13824\pi^2 M_P^2\epsilon}\big\{(18k^2 + (4y_1 - 4y_2 + 27y_3 + 36y_4)m_g^2)\mathbf{I}_1$$
$$- \big(9k^2 - (9 + 19y_1 + 44y_2)m_g^2\big)\mathbf{I}_2 + 9\mathbf{I}_3 - 18\mathbf{I}_4\big\} + \text{finite}, \tag{B.6}$$

$$(d) = \frac{5im_g^2}{1990656\pi^2 M_P^2\epsilon}\big\{\big((600 + 312x_1 + 63x_1^2)k^2 - (376 - 348x_1 + 9x_1^2)m_g^2\big)\mathbf{I}_1$$
$$- \big((272 + 456x_1 + 45x_1^2)k^2 - 2(376 - 348x_1 + 9x_1^2)m_g^2\big)\mathbf{I}_2$$
$$+ 18(28 + 4x_1 + 9x_1^2)\mathbf{I}_3 - 2(428 + 102x_1 + 81x_1^2)\mathbf{I}_4\big\} + \mathcal{O}\!\left(\frac{k^4}{M_P^2}\right) + \text{finite}. \tag{B.7}$$

Applying $\delta^2/(i\delta J^{\mu\nu})(i\delta J^{\lambda\rho})$ to (B.4) gives the 1-loop correction to the 2-point 1PI:

$$12(b) - 18(d) = \frac{5im_g^2}{110592\pi^2 M_P^2\epsilon}\Big[\big\{3(376 - 104x_1 - 21x_1^2)k^2$$
$$- \big(348x_1 - 9x_1^2 - 8(47 + 48y_1 - 48y_2 + 324y_3 + 432y_4)\big)m_g^2\big\}\mathbf{I}_1$$
$$- \big\{(592 - 456x_1 - 45x_1^2)k^2 - 2(56 + 348x_1 - 9x_1^2 + 912y_1 + 2112y_2)m_g^2\big\}\mathbf{I}_2$$
$$+ 18(20 - 4x_1 - 9x_1^2)\mathbf{I}_3 - 2(436 - 102x_1 - 81x_1^2)\mathbf{I}_4\Big] + \mathcal{O}\!\left(\frac{k^4}{M_P^2}\right) + \text{finite}. \tag{B.8}$$

Then,
$$\hat{h}^{\mu\nu}\big\{12(b) - 18(d)\big\}\hat{h}^{\lambda\rho}$$
$$\propto \big\{9(k^0)^2(40 - 81x_1 + 33x_1^2) - (k^i)^2(56 - 600x_1 - 27x_1^2) + \cdots\big\}h_{00}{}^2 + \cdots, \tag{B.9}$$
which does not vanish for any x_1.

Appendix C. Summing up 1PI's

First of all, we need to know various contractions between $\mathbf{I}_i$'s, which are worked out in table C1. Here tensors on the leftmost column with $(\alpha\beta; \mu\nu)$ index structure are multiplied into those on the top row with $(\mu\nu; \lambda\rho)$, and $\mathbf{I}_{41} = \eta_{\alpha\beta}k_\mu k_\nu$ or $\eta_{\mu\nu}k_\lambda k_\rho$, $\mathbf{I}_{42} = k_\alpha k_\beta \eta_{\mu\nu}$ or $k_\mu k_\nu \eta_{\lambda\rho}$. A graviton self energy, $\mathbf{\Pi}_{\alpha\beta;\lambda\rho}$, has the general form of

$$\mathbf{\Pi} = \sum_{j=1}^{5} b_j(k)\mathbf{I}_j, \tag{C.1}$$

because $\mathbf{I}$'s are a complete basis.

Table C1. Tensor multiplication table.

	$\mathbf{I}_1$	$\mathbf{I}_2$	$\mathbf{I}_3$	$\mathbf{I}_{41}$	$\mathbf{I}_{42}$	$\mathbf{I}_5$
$\mathbf{I}_1$	$4\mathbf{I}_1$	$2\mathbf{I}_1$	$4\mathbf{I}_{41}$	$4\mathbf{I}_{41}$	$k^2\mathbf{I}_1$	$k^2\mathbf{I}_{41}$
$\mathbf{I}_2$	$2\mathbf{I}_1$	$2\mathbf{I}_2$	$2\mathbf{I}_3$	$2\mathbf{I}_{41}$	$2\mathbf{I}_{42}$	$2\mathbf{I}_5$
$\mathbf{I}_3$	$4\mathbf{I}_{42}$	$2\mathbf{I}_3$	$2k^2\mathbf{I}_3 + 8\mathbf{I}_5$	$4\mathbf{I}_5$	$4k^2\mathbf{I}_{42}$	$4k^2\mathbf{I}_5$
$\mathbf{I}_{41}$	$k^2\mathbf{I}_1$	$2\mathbf{I}_{41}$	$4k^2\mathbf{I}_{41}$	$k^2\mathbf{I}_{41}$	$k^4\mathbf{I}_1$	$k^4\mathbf{I}_{41}$
$\mathbf{I}_{42}$	$4\mathbf{I}_{42}$	$2\mathbf{I}_{42}$	$4\mathbf{I}_5$	$4\mathbf{I}_5$	$k^2\mathbf{I}_{42}$	$k^2\mathbf{I}_5$
$\mathbf{I}_5$	$k^2\mathbf{I}_{42}$	$2\mathbf{I}_5$	$4k^2\mathbf{I}_5$	$k^2\mathbf{I}_5$	$k^4\mathbf{I}_{42}$	$k^4\mathbf{I}_5$

C.1. GR

Let us look at the case of GR first. To do the summation

$$\mathbf{P} = \mathbf{P}^{(0)} + \mathbf{P}^{(0)} \cdot \mathbf{\Pi} \cdot \mathbf{P}^{(0)} + \mathbf{P}^{(0)} \cdot (\mathbf{\Pi} \cdot \mathbf{P}^{(0)})^2 + \cdots, \tag{C.2}$$

with the graviton propagator in the de Donger gauge,

$$\mathbf{P}^{(0)} = \frac{i}{k^2}\left(-\frac{\mathbf{I}_1}{2} + \frac{\mathbf{I}_2}{2}\right), \tag{C.3}$$

we should find the n-dependence of $(\mathbf{\Pi} \cdot \mathbf{P}^{(0)})^n$. Using table C1, we get

$$\mathbf{\Pi} \cdot \mathbf{P}^{(0)} = \sum_i r_i \mathbf{I}_i, \tag{C.4}$$

with

$$r_1 = \frac{b_1 + b_2}{k^2} + \frac{b_4}{2}, \quad r_2 = -\frac{b_2}{k^2}, \quad r_3 = -\frac{b_3}{k^2},$$
$$r_{41} = -\frac{b_4}{k^2}, \quad r_{42} = \frac{2b_3 + b_4}{k^2} + \frac{b_5}{2}, \quad r_5 = -\frac{b_5}{k^2}. \tag{C.5}$$

Then, by writing

$$(\mathbf{\Pi} \cdot \mathbf{P}^{(0)})^{n+1} = \sum_i a_i^{(n)} \mathbf{I}_i = \sum_i r_i \mathbf{I}_i \cdot \sum_j a_j^{(n-1)} \mathbf{I}_j, \tag{C.6}$$

we obtain the recurrence relations between $a_i^{(n)}$ and $a_j^{(n+1)}$:

$$a_1^{(n+1)} = (4r_1 + 2r_2 + r_{41}k^2)a_1^{(n)} + 2r_1 a_2^{(n)} + (r_1 + r_{41}k^2)k^2 a_{42}^{(n)}, \tag{C.7}$$

$$a_2^{(n+1)} = 2r_2 a_2^{(n)}, \tag{C.8}$$

$$a_3^{(n+1)} = 2r_3 a_2^{(n)} + 2(r_2 + r_3 k^2)a_3^{(n)}, \tag{C.9}$$

$$a_{41}^{(n+1)} = 2r_{41}a_2^{(n)} + 4(r_1 + r_{41}k^2)a_3^{(n)} + (4r_1 + 2r_2 + r_{41}k^2)a_{41}^{(n)} + (r_1 + r_{41}k^2)k^2 a_5^{(n)}, \tag{C.10}$$

$$a_{42}^{(n+1)} = (4r_3 + 4r_{42} + r_5 k^2)a_1^{(n)} + 2r_{42}a_2^{(n)} + (2r_2 + (4r_3 + r_{42} + r_5 k^2)k^2)a_{42}^{(n)}, \tag{C.11}$$

$$a_5^{(n+1)} = 2r_5 a_2^{(n)} + 4(2r_3 + r_{42} + r_5 k^2)a_3^{(n)}$$
$$+ (4r_3 + 4r_{42} + r_5 k^2)a_{41}^{(n)} + (2r_2 + (4r_3 + r_{42} + r_5 k^2)k^2)a_5^{(n)}. \tag{C.12}$$

(C.8) and (C.9) are trivial to solve. With $a_2^{(n)}$ and $a_3^{(n)}$ determined, the remaining equations are grouped into two sets of coupled equations: (C.7) and (C.11) and (C.10) and (C.12). Each

of these sets can be solved in the same way as solving the Fibonacci sequence. The complete answer is rather lengthy:

$$a_1^{(n)} = \frac{\gamma_+ Y_-^{(n)} - \gamma_- Y_+^{(n)}}{\gamma_+ - \gamma_-}, \quad a_2^{(n)} = \frac{1}{2}(2r_2)^{n+1}, \quad a_3^{(n)} = \frac{(2r_2 + 2r_3 k^2)^{n+1} - (2r_2)^{n+1}}{2k^2},$$

$$a_{41}^{(n)} = \frac{\gamma_+ X_-^{(n)} - \gamma_- X_+^{(n)}}{\gamma_+ - \gamma_-}, \quad a_{42}^{(n)} = \frac{Y_+^{(n)} - Y_-^{(n)}}{\gamma_+ - \gamma_-}, \quad a_5^{(n)} = \frac{X_+^{(n)} - X_-^{(n)}}{\gamma_+ - \gamma_-}, \tag{C.13}$$

where

$$\gamma_\pm = \frac{-\alpha_1 + \alpha_2' \pm \sqrt{(\alpha_1 - \alpha_2')^2 + 4\alpha_1'\alpha_2}}{2\alpha_1'}, \tag{C.14}$$

$$X_\pm^{(n)} = (r_{41} + \gamma_\pm r_5)\sigma_\pm^n + \rho_{1\pm}\frac{\sigma_\pm^n - (2r_2)^n}{\sigma_\pm - 2r_2} + \rho_{2\pm}\frac{\sigma_\pm^n - (2r_2 + 2r_3 k^2)^n}{\sigma_\pm - (2r_2 + 2r_3 k^2)}, \tag{C.15}$$

$$Y_\pm^{(n)} = (r_1 + \gamma_\pm r_{42})\sigma_\pm^n + \rho_\pm\frac{\sigma_\pm^n - (2r_2)^n}{\sigma_\pm - 2r_2}, \tag{C.16}$$

with

$$\alpha_1 = 4r_1 + 2r_2 + r_{41}k^2, \quad \alpha_2 = (r_1 + r_{41}k^2)k^2,$$
$$\alpha_1' = 4r_3 + 4r_{42} + r_5 k^2, \quad \alpha_2' = 2r_2 + (4r_3 + r_{42} + r_5 k^2)k^2,$$
$$\sigma_\pm = \frac{\alpha_1 + \alpha_2' \pm \sqrt{(\alpha_1 - \alpha_2')^2 + 4\alpha_1'\alpha_2}}{2}, \quad \rho_\pm = 2r_2(r_1 + \gamma_\pm r_{42}), \tag{C.17}$$
$$\rho_{1\pm} = -2r_2\left\{r_{41} + \gamma_\pm r_5 + \frac{2}{k^2}(r_1 + \gamma_\pm(2r_3 + r_{42}))\right\},$$
$$\rho_{2\pm} = 4\left(\frac{r_2}{k^2} + r_3\right)\{r_1 + r_{41}k^2 + \gamma_\pm(2r_3 + r_{42} + r_5 k^2)\}.$$

Then,

$$\mathbf{P}^{(0)} \cdot (\mathbf{\Pi} \cdot \mathbf{P}^{(0)})^{n+1}$$
$$= \frac{i}{2(\gamma_+ - \gamma_-)k^2}\left[\frac{\rho_+(2^n r_2^n - \sigma_+^n)(2\gamma_- - k^2)}{2r_2 - \sigma_+} - \frac{\rho_-(2^n r_2^n - \sigma_-^n)(2\gamma_+ - k^2)}{2r_2 - \sigma_-}\right.$$
$$\left. - (2r_2)^{n+1}(\gamma_+ - \gamma_-) + \gamma_+^n(r_1 + r_{42}\gamma_+)(2\gamma_- - k^2) - \gamma_-^n(r_1 + r_{42}\gamma_-)(2\gamma_+ - k^2)\right]\mathbf{I}_1$$
$$+ \frac{1}{2k^2}(2r_2)^{n+1}\mathbf{I}_2 + (\mathbf{I}_3, \ \mathbf{I}_4, \ \mathbf{I}_5). \tag{C.18}$$

As this is merely a geometric series, summing them up is straightforward. Plugging (C.14) and (C.17), we finally get

$$\mathbf{P} = \mathbf{P}^{(0)} + \sum_{n=0}^{\infty} \mathbf{P}^{(0)} \cdot (\mathbf{\Pi} \cdot \mathbf{P}^{(0)})^{n+1}$$
$$= -\frac{i}{k^2 + 2b_2}\frac{\mathbf{I}_1}{3} + \frac{i}{k^2 + 2b_2}\frac{\mathbf{I}_2}{2} - i\frac{2b_5 k^4 + (1 + 8b_3 + 4b_4)k^2 + 2(b_1 + 2b_2)}{d(k)}\frac{\mathbf{I}_1}{3} + (\mathbf{I}_3, \ \mathbf{I}_4, \ \mathbf{I}_5), \tag{C.19}$$

with

$$d(k) = b_5 k^6 + \left\{2 + 4b_3 - 2b_4(2 - 3b_4) - 2b_5(3b_1 + 2b_2)\right\}k^4$$
$$- 8\left\{b_1(1 + 3b_3) + b_2(2b_3 + b_4)\right\}k^2 - 8b_2(2b_1 + b_2). \tag{C.20}$$

Since GR has a full GCI, $\mathbf{\Pi}$ should be invariant under $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, i.e.

$$0 = \int \mathrm{d}^4 x \left\{ \mathbf{\Pi}(h'_{\mu\nu}) - \mathbf{\Pi}(h_{\mu\nu}) \right\} = -2 \int \mathrm{d}^4 x \, \xi_\mu \partial_\nu \frac{\delta}{\delta h_{\mu\nu}} \mathbf{\Pi}$$

$$\Rightarrow \quad 0 = (b_1 + k^2 b_4) k_\nu \hat{h} + 2(b_2 + k^2 b_3) k_\rho \hat{h}^\rho_\nu + (2b_3 + b_4 + k^2 b_5) k_\nu k_\lambda h_\rho \hat{h}^{\lambda\rho}, \tag{C.21}$$

which gives

$$b_1 + k^2 b_4 = 0, \quad b_2 + k^2 b_3 = 0, \quad 2b_3 + b_4 + k^2 b_5 = 0, \tag{C.22}$$

and (C.19) turns into

$$\mathbf{P} = \frac{\mathrm{i}}{k^2 + 2b_2} \left\{ \frac{-k^2 + b_1}{2k^2 - 3b_1 - 2b_2} \mathbf{I}_1 + \frac{\mathbf{I}_2}{2} + \frac{b_2}{k^4} \mathbf{I}_3 - \frac{b_1 + 2b_2}{k^2(2k^2 - 3b_1 - 2b_2)} \left(\mathbf{I}_4 + \frac{2}{k^2} \mathbf{I}_5 \right) \right\}. \tag{C.23}$$

In GR, the forms of the quadratic action and the corresponding propagator can vary as we change the gauge choice, so that a difference between the resummed propagator and the tree level one may not be a problem as long as there is a general coordinate transformation that connects them.

C.2. PF

For PF theory, we can follow the same steps as those of the GR case. The differences are that there is no Ward identity here and that we use $\mathbf{P}^{(0)}_{m_g}$ instead of $\mathbf{P}^{(0)}$, which changes (C.5) into

$$r_1 = \frac{b_1}{3m_g^2} + \frac{2b_2}{3(k^2 + m_g^2)} + \frac{b_4 k^2}{3m_g^2}, \quad r_2 = -\frac{b_2}{k^2 + m_g^2}, \quad r_3 = -\frac{b_2}{m_g^2(k^2 + m_g^2)} - \frac{b_3}{m_g^2},$$

$$r_{41} = -\frac{2b_1}{3m_g^4} + \frac{2b_2}{3m_g^2(k^2 + m_g^2)} - \frac{b_4(2k^2 + 3m_g^2)}{3m_g^4},$$

$$r_{42} = \frac{2b_2}{3m_g^2(k^2 + m_g^2)} + \frac{4b_3}{3m_g^2} + \frac{b_4}{3m_g^2} + \frac{b_5 k^2}{3m_g^2}, \tag{C.24}$$

$$r_5 = -\frac{4b_2}{3m_g^4(k^2 + m_g^2)} - \frac{8b_3}{3m_g^4} - \frac{2b_4}{3m_g^4} - \frac{b_5(2k^2 + 3m_g^2)}{3m_g^4}.$$

Then, the resummed PF graviton propagator is

$$\mathbf{P}_{m_g} = \mathbf{P}^{(0)}_{m_g} + \sum_{n=0}^{\infty} \mathbf{P}^{(0)}_{m_g} \cdot \left(\mathbf{\Pi} \cdot \mathbf{P}^{(0)}_{m_g} \right)^{n+1}$$

$$= -\mathrm{i} \left\{ \frac{1}{k^2 + m_g^2 + 2b_2} + \frac{b_5 k^4 + 2(2b_3 + b_4) k^2 + b_1 + 2b_2}{d_{m_g}(k)} \right\} \frac{\mathbf{I}_1}{3} + \frac{\mathrm{i}}{k^2 + m_g^2 + 2b_2} \frac{\mathbf{I}_2}{2}$$

$$- \frac{\mathrm{i}}{2k^2} \left(\frac{1}{k^2 + m_g^2 + 2b_2} - \frac{1}{2b_3 k^2 + m_g^2 + 2b_2} \right) \mathbf{I}_3 + \frac{\mathrm{i} n_{m_g}(k)}{(k^2 + m_g^2 + 2b_2) d_{m_g}(k)} \mathbf{I}_4$$

$$+ \frac{\mathrm{i}}{3k^4} \left\{ \frac{2}{k^2 + m_g^2 + 2b_2} - \frac{6}{2b_3 k^2 + m_g^2 + 2b_2} \right.$$

$$\left. - \frac{b_5 k^4 - (6 - 4b_3 - 8b_4) k^2 + 16b_1 + 8b_2 - 12m_g^2}{d_{m_g}(k)} \right\} \mathbf{I}_5, \tag{C.25}$$

where

$$d_{m_g}(k) = 2b_5 k^6 + \big(8b_3 + b_4(4 + 3b_4) - b_5(3b_1 + 2b_2) + 2b_5 m_g^2\big)k^4$$
$$+ \big(2b_1(1 - 6b_3) + 4b_2(1 - 2b_3 - b_4) + 2(4b_3 - b_4)m_g^2\big)k^2$$
$$- (4b_1 + 2b_2 - 3m_g^2)(2b_2 + m_g^2), \tag{C.26}$$

$$n_{m_g}(k) = b_5 k^4 + \big(4b_3 + 3b_4 + b_4^2 - b_1 b_5 + b_5 m_g^2\big)k^2 + b_1(2 - 4b_3)$$
$$+ 2b_2(1 + b_4) - (1 - 4b_3 - b_4)m_g^2. \tag{C.27}$$

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