# ANALYZING THE DIAGRAMMATIC REGISTER IN GEOMETRY TEXTBOOKS: TOWARD A SEMIOTIC ARCHITECTURE

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*Diagrams are key resources for students when reasoning in geometry. Over the course of the 20th century, diagrams in geometry textbooks have evolved from austere collections of strokes and letters to become diverse arrays of symbols, labels, and differently styled visual parts. Diagrams are thus multisemiotic texts that present meanings to students across a range of communication systems. We propose a scheme for analyzing how geometric diagrams function as resources for mathematical communication in terms of four semiotic systems: type, position, prominence, and attributes. The semiotic architecture we propose draws on research in systemic functional linguistics (Halliday, 2004; O'Halloran, 2005) and suggests a framework for analyzing how geometry diagrams function as mathematical texts.*

Keywords: Geometry, Classroom Discourse, Curriculum Analysis

## Introduction

Diagrams are key resources for students when reasoning in geometry. Research shows that different properties of a diagram facilitate different types of student interaction (Herbst, 2004; Laborde, 2004). Laborde (2004) draws a distinction along the lines of spatiographical versus theoretical properties of a diagram. A geometric figure can be defined as a set of points that satisfy some given properties (e.g., an isosceles triangle ABC with sides AB and BC congruent). A diagram is a visual representation of a figure. As representation, the diagram also represents the properties that define the figure (e.g., the strokes for AB and BC are the same length)—these are the theoretical properties of the diagram. But the diagram also has other properties that are not necessitated by the definition of the figure but contingent to its diagrammatic representation (e.g., the stroke for side AC is either longer, shorter, or just as long as the other two). The spatiographical properties are these visual properties of the objects in a diagram (Laborde, 2004 p. 160). Diagrams support the reading of spatiographical and theoretical properties, and these readings may even contradict each other. A diagram is thus an ambiguous representation of a set of geometric objects and their relations.

Laborde's distinction between spatiographical and theoretical properties of a diagram is related to Herbst's distinction between the empirical and the representational modes of interacting with diagrams (Herbst, 2004). A mode of interaction concerns an actor (e.g., a student), a diagram (or other representation), and a geometric object (e.g., what the diagram is a representation of). In the empirical mode of interaction, a student acts on a diagram by looking at it, measuring its different parts, or drawing in it, only constrained "…by the actual features of the physical drawing and the operational constraints of the physical instruments of interaction" (Herbst, 2004, p. 130). In the representational mode of interaction, the known theoretical properties of the geometric objects the diagram represents constrain the actions on the diagram.

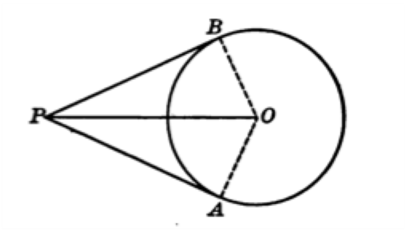
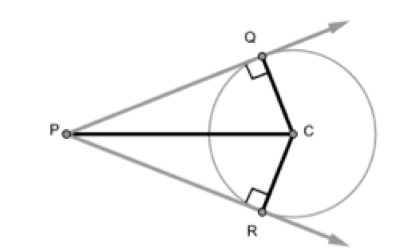
The spatiographical-theoretical difference in diagrammatic properties combined with the empirical-representational difference in modes of diagrammatic interaction points to an underlying duality between a diagram (as an actual physical thing) and the geometric objects it represents. Mathematicians in the 19th century purged this duality from formal mathematics by rejecting arguments that depended on properties of diagrams in favor of arguments that were completely contained in propositions that could be stated solely with linguistic and logical symbols (Barwise & Ecthemendy, 1996; Greaves, 2002; Miller, 2007). But teachers of high school geometry are not only obligated to their discipline, they are also obligated to their students (Herbst & Chazan, 2011). And on that count, diagrams are not only instrumental at representing the relationships that hold between abstract objects like points, lines, and planes, but also are powerful tools for people to solve problems (Larkin & Simon, 1987; Duval, 1995). So even when the diagrams of Euclidean geometry were under siege by mathematicians, teachers of school geometry—rather than follow suit and banish diagrams from their classrooms—had to devise strategies for facilitating student interactions with diagrams. The way that diagrams are presented in 20th century textbooks provides one record of how these facilitation strategies evolved. The goal of this paper is to present a semiotic architecture or scheme for analyzing geometry diagrams that builds on a systemic functional perspective.

## Theoretical Framework

The systemic functional perspective originated with M.A.K. Halliday's analysis of grammar, whereby he posited that grammar consists of a set of systems (e.g., transitivity, mood, conjunction) that offer choices for writers and speakers to realize meanings (Halliday, 2004). This perspective has been extended to levels above grammar (e.g., discourse) and below grammar (e.g., phonology) as well as to nonlinguistic semiotic systems (e.g., images; Kress & van Leuven, 1996; O'Toole, 1994).

Mathematical communication employs various semiotic systems to make meanings, in particular language, symbols, and visuals (Lemke, 2003; O'Halloran, 2005 ). Duval (2006) uses *register* to refer to these semiotic systems. Building on that use of register, in their study of the types of translation tasks that students might be assigned in the geometry class, Weiss and Herbst (2008) argue that the diagrams of high school geometry comprise a distinct mathematical register—the *diagrammatic register*. The symbols of this register are "…pictures of (idealized) 'real' things…"—e.g., circles, points, parallel lines—together with the system of "markup signs"—e.g., congruence, perpendicularity, and parallelism markings, labels—that encode the properties of those objects or permit references to those objects (Weiss & Herbst, 2008, p. 19).

The system of markup signs for geometric properties in diagrams and the norms that govern which geometric properties the diagram represents are products of the 20th century. The diagrams of Euclid and Descartes, as well as those in early 20th century plane geometry textbooks—particularly those from the Era of the Text and the Era of the Originals (Herbst, 2002)—were collections of strokes (for lines, line segments, and circles) and letters (for points). The diagrammatic register in these early textbooks of plane geometry lacked many of the features that one would expect from the diagrams in mainstream textbooks from the later 20th and 21st centuries. Figure 1 illustrates these differences.

|   |   |
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| Diagram from Wentworth's *Plane Geometry* (1913, p. 107) | Replication of diagram from Merrill *Geometry* (1990, p. 366) |

**Figure 1: Figure 1: Comparison of two geometry diagrams**

Figure 1 shows two different diagrams from two different textbooks (Wentworth, 1913 and Foster et al., 1990, respectively) that accompany the statement of the same theorem: that the external tangents drawn from a point to a circle form segments (PB, PA and PQ, PR respectively) that are congruent. The diagrams in Figure 1 have commonalities. For example, each figure shows the segments one might use to prove the tangent segments theorem, and each figure labels the points one would expect to use in the proof (P, B, O, A and P, Q, C, R). Yet the diagrams in Figure 1 are also clearly different: the later diagram shows marked right angles at PQC and PRC and uses different colors (the lighter lines are blue, the darker lines are red) for different strokes, while the earlier diagram uses different styles of lines (BO, BA are dashed) and uniform choices for the thickness of strokes. The changes observed between these diagrams suggest that readers of diagrams need to be able to interpret and integrate different semiotic systems as they interact with diagrams. The semiotic architecture presented below aims to characterize these and other systems used in the visual display of geometric diagrams.

**Mode of Inquiry**

The architecture proposed here builds on O'Halloran's survey of the literal, symbolic, and visual semiotic systems of mathematics (O'Halloran, 2005). O'Halloran defines diagrams as a distinct genre of mathematical visual image, using "diagram" in the "broadest sense" to include "…Venn diagrams, geometrical figures, and other figures such as those found in graph theory and topology" (O'Halloran, 2005 p. 133). We draw on and extend O'Halloran's framework to the specific genre of geometric diagrams, which we operationally define as the 2-dimensional, visual representations that accompany proofs (or proof problems) in plane geometry textbooks.

The proposed architecture synthesizes theoretical and empirical inquiries. Our proposed scheme for analyzing diagrams draws on the literature that conceptualizes visual images as coherent semiotic systems. As such, we take as a first principle that geometric diagrams obey a visual grammar and that differences in how parts of diagrams are presented are purposeful and exist to communicate specific information to a modal viewer of a diagram. From the systemic functional perspective, there are choices available when creating any communicative text, and these choices can be described in terms of the different systems available with which the text (the diagram, in our case) can fulfill communicative functions. In this paper our main contribution is to identify those systems—we leave our considerations of function for a later time.

We developed the architecture by analyzing diagrams in 20th century geometry textbooks published by mainstream publishers. These included textbooks from MacMillan, McGraw-Hill, Merrill, Glencoe, and Ginn and Company publishing houses. In total, 30 textbooks—the earliest published in1899, the latest published in 2004—were examined to develop the structure and categories in the architecture reported below. The work proceeded in iterations, much in line with the tradition of design research (Lesh & Sriraman, 2005). We sketched an initial scheme for analyzing the differences in the diagrams—drawing on O'Halloran's work—then revised and fleshed out this scheme as necessary to account for new types of variation that were encountered by analyzing the diagrams in different textbooks. Through the course of these iterations, we identified four semiotic systems—type, position, prominence, and attributes—that we offer as a first approximation to characterize the diagrammatic register.

**Semiotic Systems in Geometric Diagrams: Type, Position, Prominence, Attributes**

We propose four functional semiotic systems to describe the range of variation in geometry diagrams. These systems are referred to as the *type*, *position*, *prominence*, and *attributes* systems. Our of use "system" concords with its use in functional grammar: systems contain the paradigmatic ordering of a language (a "what-could-go-instead-of-what relation", Halliday, p. 23). The systems we identify inventory the choices that are available when creating a geometry diagram.

For the purpose of this analysis, the *parts* of a geometry diagram are graphical analogues of morphemes in written language (Engelhardt, 2002). In geometric diagrams, there are parts that represent geometric objects (e.g., dots, strokes, regions) and parts that represent geometric (and potentially other mathematical) properties (e.g., congruence, parallelism, perpendicularity, movement). The parts of a diagram can be differentiated analogously to the way that *free* and *bound* morphemes are differentiated in linguistics (Engelhardt, 2002, p. 24). The free parts—like free morphemes—are those that can appear on their own (e.g., dots, strokes), while the bound parts—like bound morphemes—are those that can only appear with others (e.g., congruence hashmarks, parallelism arrows). In general, we have found that free parts represent geometric objects while bound parts represent geometric properties, though this division is not without exception. For example, a "partially limited" region may be considered a part of a diagram that represents an angle, yet this part cannot visually occur independently of strokes (and potentially dots). The free-part/bound-part distinction helps to structure the Type system.

**The Type system.** The Type system categorizes the different parts of a geometric diagram according to their visual identity. The Type system is thus a way of taking stock of the lexical elements of a given diagrammatic text. Figure 2 shows some of the choices in the Type system.

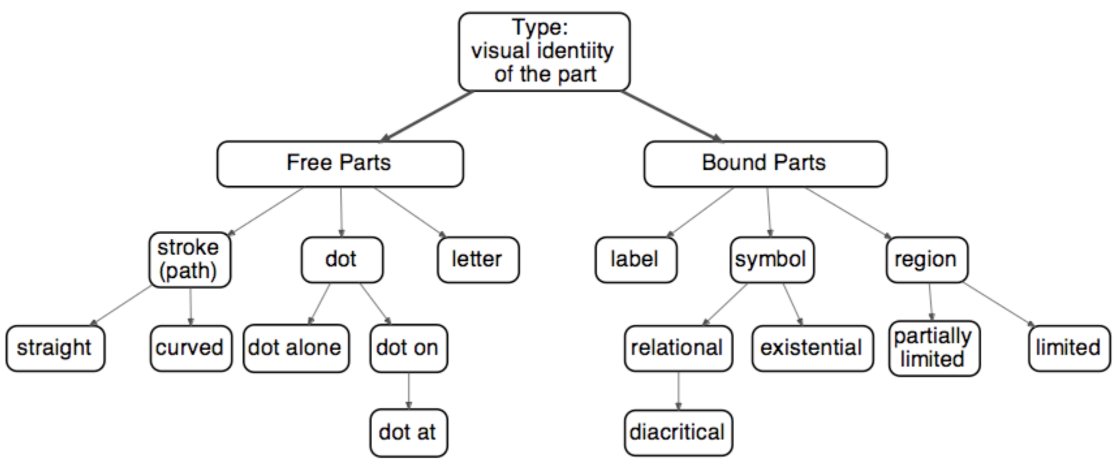



**Figure 2: The Type System**

The divisions in the Type system are visual, not geometric. Keeping visual properties distinct from the geometric properties allows one to study how the different geometric properties are represented as visual parts in diagrams.  Thus, lines, segments, and rays are all examples of {strokes: straight} and are visually of the same kind. The Attribute system (see below) encodes the marked geometric differences in parts that present as the same type of stroke. “Letter” is included as a sublevel of the free parts on account of the fact that, in older textbooks in particular, a letter can appear on its own in a diagram—either on a stroke or in space—and thereby signal the existence of a geometric point where the letter occurs (see Figure 7, below). “Dot” is the visual category that corresponds to geometric points. The “dot alone” – “dot on” – “dot at” distinctions captures the visual differences between a dot that is independent of other parts (dot alone), a dot that is on some other part (dot on), and a dot that is at the intersection of several parts (dot at).

**The Position system.** Type system provides a scheme for identifying the participants (taxonomically speaking) that a given diagram provides for forming geometric clauses (i.e., mathematical statements about what is happening in a given diagram). The Position system captures how those different participants relate to each other spatiographically; position accounts for where parts are located relative to one another and how those parts are oriented relative to the frame of reference of the page. Figure 3 shows the relations in the Position system.

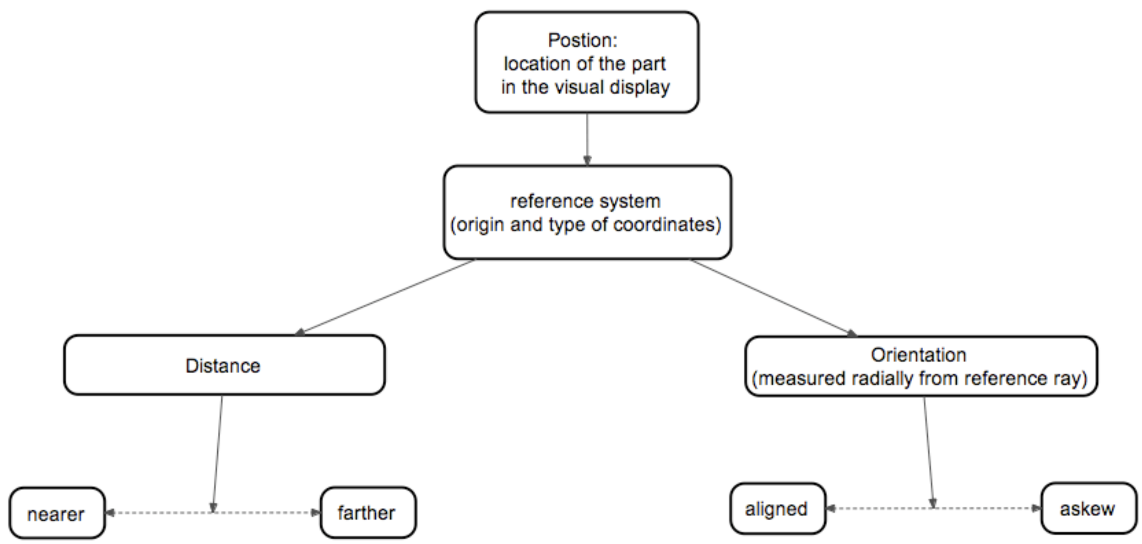



**Figure 3: The Position System**

The first level in the Position system is a frame of reference (e.g., radial, rectangular), followed by sublevels of distance (visual space between parts) and orientation (heading of the part relative to a set of reference axes) that depend on the chosen reference system.

**The Prominence system.** Prominence refers to the visual prominence of a part in the display (O’Halloran, 2005, p. 136). Figure 3 shows the architecture for the Prominence system as it applies to plane geometry diagrams.

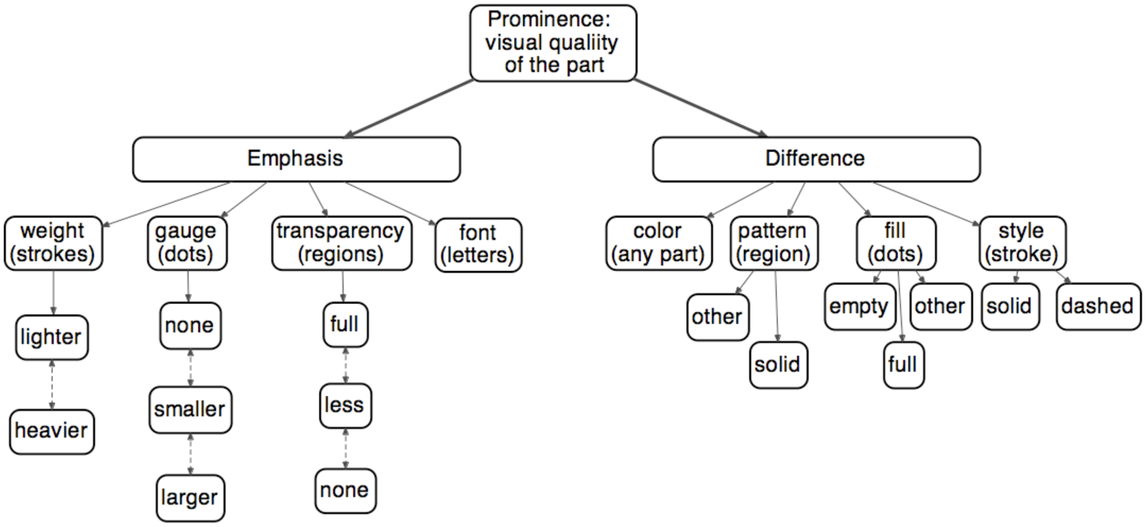



**Figure 4: The Prominence System**

There are emphasis and difference subsystems. Emphasis communicates the visual emphasis of a part, through choices for weight (strokes), gauge (dots), transparency (regions), and style (letters and symbols). Difference communicates the visual difference of a part, through choices for color (all parts), pattern (regions), fill (dots), and style (stroke). The interaction of these different systems is evident in Figure 1 (reproduced below), where circle C (right frame) is given less emphasis relative to strokes PQ, PR and PC, by virtue of its lighter weight, yet linked to PQ and PR—while being set apart from PC—through choices in color.

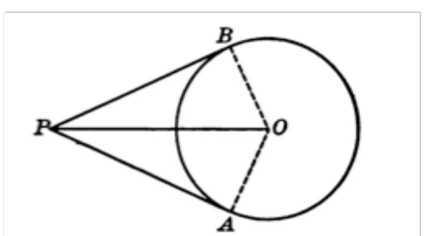
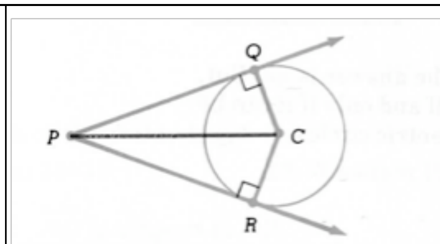
|  |  |
|---|---|
| Diagram from Wentworth's *Plane Geometry* (1913, p. 107) | Diagram from Merrill *Geometry* (Foster et al., 1990, p. 366) |

**Figure 5: Reproduction of Figure 1**

The use of the Prominence system to mark parts of a diagram evolved over the course of the 20th century. For example, in Merrill's *Geometry* (1990), blue is the typical choice for strokes; other colors (orange, red, purple) are used to mark auxiliary or otherwise special lines. This suggests that PC (darker in the right frame of Figure 1) is an auxiliary line. Contrast this to the use of the Prominence system in Wentworth's *Geometry* (1913). In this text, solid strokes were typical and the dashed style was used to mark auxiliary lines. Notice that the radii in the left frame of Figure 1—OB and OA—are marked as different from the other strokes through a choice in style (dashed). This suggests that the radii, rather than PO, are the auxiliary lines in this diagram. These examples show that the Prominence system can signal the properties of the geometric objects that are represented (by parts) in the diagram; how exactly such properties correlate to choices in gauge, fill, weight, style, and color is an area for further study.

**The Attributes system.** The principal system that communicates the geometric properties of a diagram is the Attributes system, shown in Figure 4.

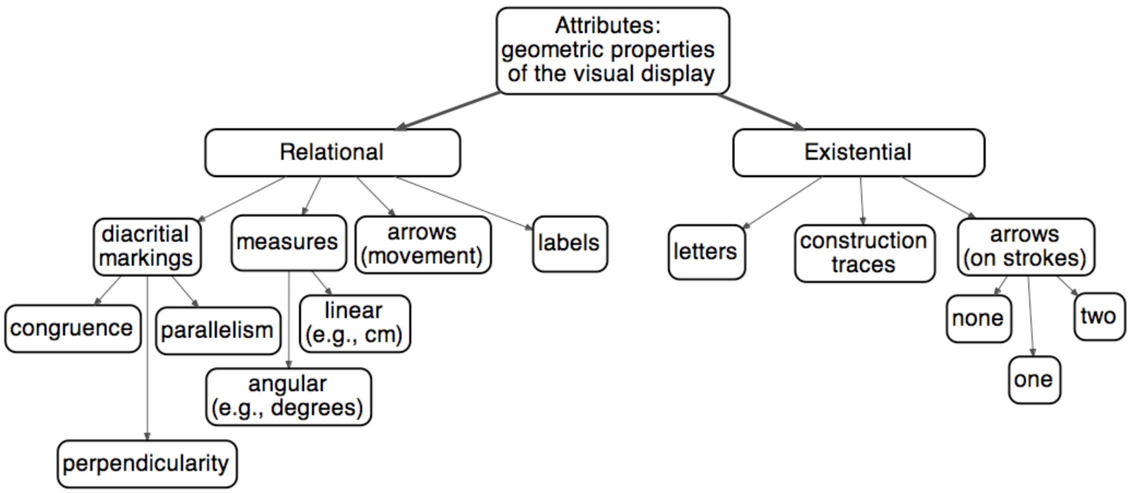



**Figure 6: The Attributes System**

Attributes are *relational* or *existential*; like the word “attribute”, “relational” and “existential” are chosen to draw an analogy to linguistics. In this case, it is the distinction between relational and existential processes (Halliday, 2004). Relational processes serve to “characterize and identify” (Halliday, 2004 p. 210), while existential processes are those “…by which phenomena of all kinds are simply recognized ‘to be’” (Halliday, 2004 p. 171). Similarly, the relational attributes of parts are those diacritical markings, measures, and labels that serve to identify and classify relations that hold among specific parts. These markings are resources in the diagram that help manage the spatiographical-theoretical duality; thus, for the modal viewer, a marked right angle is *right*, regardless of what it might actually look like (and conversely: an unmarked angle that looks right might not be).

Complementing the relational attributes are the existential attributes. Like their linguistic cousins, existential attributes are so named because they actually stipulate the existence of a part in a diagram. Consider, for example, points D, E, C, B, and A in Figure 5, a diagram in Wells and Hart’s *Plane Geometry* (1915). In this diagram, the presence of the letters ‘D’, ‘E’, ‘C’, ‘B’, and ‘A’ positioned at the ends of the straight strokes mark the existence of points on their ends.

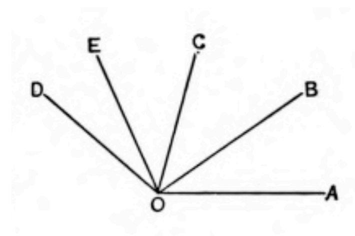



**Figure 7: Diagram from Wells and Hart's *Plane Geometry* (1915, p. 19)**

Arrows serve as existential attributes when they are applied to the ends of straight strokes, as the means of stipulating that a given straight stroke is a line (two arrows) or a ray (one arrow). The right frame of Figure 1 (see above) has examples of these attributes as they are applied to PQ and PR, thereby bringing into existence ray PQ and ray PR (as opposed to bringing into existence a segment or a line). Apart from the relational and existential attributes that apply to single parts, there are also attributes such as captions or arrows (transformational) that apply to the entire diagram or to several parts.

There are further choices within the attributes system that are not depicted in the diagram. Consider the case of labels. An angle may be labeled through its vertex point and points on its component rays; thus, in Figure 5, we could use the labels of points to refer to angle DOA. But angles—like segments—can also be assigned their own labels, as is the case in Figure 6.

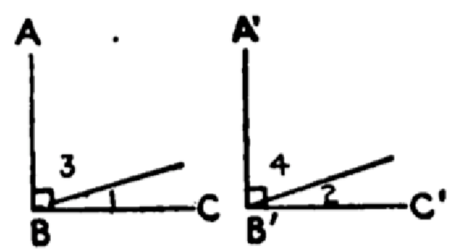



**Figure 8: Diagram from Schultze's *Plane Geometry* (1913, p. 18)**

In Figure 6, we see that angles 1, 2, 3, 4 have their own specific labels. This is just one example of the possible variation. As was the case with the Prominence system, the use of the Attributes system has evolved during the 20$^{th}$ century, and tracing the emergence of this system is an area for further study.

**Summary**

Geometry teachers have been concerned with how to teach students to communicate with geometric diagrams for more than 100 years (Baker, 1902). The evolution of the diagrammatic register in 20$^{th}$ century geometry textbooks speaks to this concern, and the architecture we have proposed in this report is one means through which this evolution can be analyzed. Studying the development of the diagrammatic register in 20$^{th}$ century textbooks will shine a light on how the multiple, ambiguous, and sometimes conflicting roles that diagrams play in student mathematical reasoning are semiotically managed. The work reported here is a step in this direction.

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