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THE LOGIC AND MATHEMATICS OF NON-ANNUAL COMPOUNDING APPLIED TO INVESTMENT DECISIONS

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INTRODUCTION

Many of us have grown so used to living with compound interest that we have forgotten its underlying principles. In particular, those of us who deal in long-term investment decisions often grow too accustomed to the simplifying assumption of annual compounding. When confronted with situations in which non-annual compounding must be understood, we often find ourselves confused. All of us tend to be too unthinking about the mathematics of compound interest. The purpose of this paper is to stimulate clear thinking about non-annual compounding and to suggest when and where its application may be worth the trouble.

SYMBOLS AND ABBREVIATIONS

CR : capital recovery factor
e : base of natural logarithms $\cong 2.718$
i : see r_1
k : an artificial constant
LL : log log
ln : natural log
M : compounding periods per annum
N : number of years
P : investment, principal, present worth
P_X : residual debt X years before end of loan period
A_M : annual returns in M uniform, equally spaced amounts
A_∞ : annual returns with continuous cash flow
r_1 : effective interest rate per annum
r_M : nominal interest rate per annum associated with M compounding periods per annum
r_∞ : nominal interest rate per annum associated with continuous compounding
F : future amount
(CA-i%-N) : single compound amount factor for an interest rate of i% per compounding period and N compounding periods
X : years before end of loan period
∞ : infinity

I. SINGLE PAYMENT RELATIONSHIPS

GENERAL REMARKS

Suppose a usurous banker offers to lend you $10,000 for 20 years at 12-percent annual interest, but forgets to stipulate in writing that the interest is to be compounded annually. That is, you find yourself

with an agreement to repay the loan with simple, rather than compound, interest. This is a most unusual arrangement because simple interest is essentially illogical in that it gives insufficient emphasis to the time-value of money. That is, we must concern ourselves not only with how much rent is paid, but when. Simple interest ignores the <u>when</u>.

At the end of the 20-year loan period, you would owe him an amount equal to the rent per year (12 percent of the principal) multiplied by the number of years--plus, of course, the principal:

$$F = N\ i\ P + P$$
$$F = P(1 + N\ i)$$
$$F = \$10{,}000\ (1 + 20 \times 0.12) = \underline{\$34{,}000}$$

where

F = future sum
N = number of years
i = interest rate year
P = principal

Now, if our friendly banker had been more perspicacious, he would have specified annual compounding and you would have owed him

$$F = (CA - i\% - N)P$$
$$F = (CA - 12\% - 20)\ 9.646 \times \$10{,}000 = \underline{\$96{,}460}$$

where

(CA - i% - N) = single compound amount factor for an interest rate i per compounding period and N compounding periods

$$(CA - i\% - N) = (1 + i)^N$$

Note that we have defined CA in such a way that we can use our standard interest formulas and tables for periods other than yearly. Suppose our banker had asked for 12 percent interest per year but compounded semi-annually. Now our interest rate per compounding period is cut in half but the number of periods is doubled. Let us see how much you would owe at the end of 20 years:

$$F = (CA - 6\% - 40)\ 10.286 \times \$10{,}000 = \underline{\$102{,}860}$$

Or, for compounding quarterly:

$$F = (CA - 3\% - 80)\ 10.641 \times \$10,000 = \underline{\$106,410}$$

and for compounding monthly:

$$F = (CA - 1\% - 240)\ 10.87 \times \$10,000 = \underline{\$108,700}$$

We can see that shortening the compounding period progressively increases the future amount, F. The rate of increase, however, is declining. When carried to the ultimate frequency, we find we are compounding continuously. Using methods we will develop later, we would find the limiting future amount to be about $110,500.

Table 1 summarizes the future amounts obtained in each case above.

TABLE 1

INFLUENCE OF FREQUENCY OF COMPOUNDING ON AMOUNT OWED AFTER 20 YEARS FOR A $10,000 LOAN AT 12 PERCENT ANNUAL INTEREST

Compounding Period	Future Amount
(without compounding)	$ 34,000
annual	96,460
semi-annual	102,860
quarterly	106,410
monthly	108,700
(continuous)	110,500

DISCRETE NON-ANNUAL COMPOUNDING

By now you will understand that, although we have used the same annual interest rate in each case, we have not given equal weight to the time-value of money. Annual interest rate (or nominal rate as it is often called) is clearly a worthless measuring device when non-annual compounding is involved. There is no other finite unit of time that would be any better. Because the year is an arbitrary unit, some purists prefer continuous compounding--which also yields neater mathematical manipulations in certain applications. We shall stick to the year as our standard, however. The year is as good a time period as any, and all of us are in the habit of calibrating our judgment of fair rates on this arbitrary unit of time. What we need then is an artificial annual interest rate that can be applied to non-annual compounding after borrower and lender have agreed upon some time value of money

calibrated on the standard annual basis. Conversely, this artificial annual rate can be used to compare two or more loan plans involving differing payment periods. We call this artificial annual rate effective interest and abbreviate it here as r_1. We define effective interest as the interest rate that, if compounded annually, would produce the same future amount as that produced by the same principal compounded non-annually at a given nominal rate. This definition is admittedly a tough one to follow, so let us illustrate what we mean.

In the earlier example, we showed that a $10,000 principal would grow to $108,700 if compounded monthly at 12 percent nominal rate for 20 years. What interest rate compounded annually would give us the same amount? If we find this, we will have the effective rate, r_1. This will give us an exact idea of the time-value of money implied in the terms of the loan. We know the principal (P) and the amount owed (F) after 20 years at an unknown annual interest rate compounded annually (r_1). In equation form we have

$$F = (CA - r_1\% - 20)P$$

$$\$108,700 = (CA - r_1\% - 20)\ \$10,000$$

$$(CA - r_1\% - 20) = \frac{\$108,700}{\$\ 10,000} = 10.870$$

also,

$$(1 + r_1)^{20} = 10.870$$

A little jiggery pokery with the LL scale of your slide rule will show that r_1 equals 12.65 percent. If you do not communicate with LL scales, you can reach the same conclusion by plotting values of CA versus interest rate from your interest tables, or you can use logarithms if so inclined. In any event, our conclusion is that a nominal 12 percent rate compounded monthly is equivalent in value to a 12.65 percent rate compounded annually.

Figure 1 illustrates the general case. It plots the growth of future amounts (F) under three different loan plans, all based on the same time-value of money. Since all plans have the same time-value of money, the future amounts will all be equal at the end of each year. (They will differ slightly at other times but that detail ordinarily need not concern us.)

At this point we must define a few abbreviations. To help purge our minds of old confusions, we shall eschew the abbreviation i because its meaning tends to be ambiguous in this context. We have

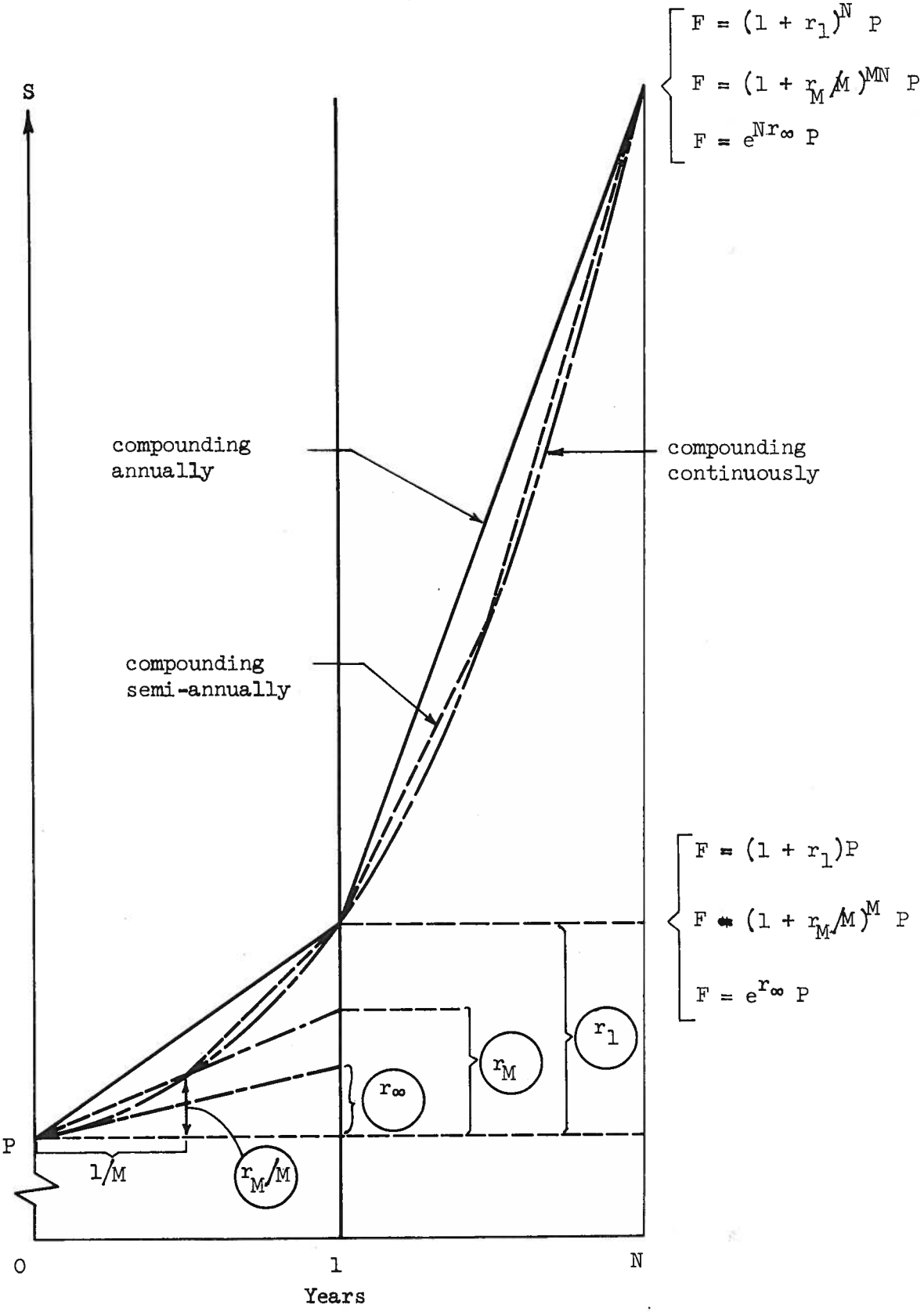


Figure 1. Growth Pattern for Future Amounts with Different Compounding Periods but Same Time-Value of Money (Note: letters in circles are proportional values).

already defined r_1 as the effective interest rate. Remember that this is our basic measure of the time-value of money. It is the rent money per year as a percent of the amount owed, compounded annually. If non-annual compounding is used, the nominal interest rate (r_M) is given on an annual basis but is always divided by the compounding periods per year (M) before using. As can be seen in Figure 1, the nominal rate describes the initial slope of the line to which it applies. As mentioned earlier, our standard interest formulas and tables can be extended to cover non-annual compounding if we substitute r_M/M for i and MN (the number of compounding periods) for N (the number of years). Thus, the standard equation for CA

$$CA = (1 + i)^N$$

becomes

$$CA = (1 + \frac{r_M}{M})^{MN}$$

You may recall that we applied this equation, without having spelled it out, when we calculated the future amount owed on a $10,000 loan for 20 years at 12 percent nominal interest ($r_M = 0.12$) compounded semi-annually (M = 2):

$$F = (CA - \frac{r_M}{M} - MN)P$$

$$\frac{r_M}{M} = \frac{0.12}{2} = 0.06$$

$$MN = 2 \times 20 = 40$$

$$(CA - 6\% - 40) = (1 + \frac{r_M}{M})^{MN} = 10.286$$

$$F = (CA - 6\% - 40)\ 10.286 \times \$10,000 = \$102,860 \text{ (as before)}$$

(CA - 6% - 40) can also be found from interest tables. Look up the value for 6 percent annual interest and 40 years. It is, fundamentally, the factor for 6 percent interest per compounding period and 40 compounding periods.

Now let us look again at Figure 1. We can generalize that the future amount, F, at the end of the n^{th} year is:

$$F = (1 + \frac{r_M}{M})^{MN} P$$

For annual compounding, $M = 1$, so we have

$$F = (1 + r_1)^N P$$

But, since by definition F is the same in each case:

$$(1 + \frac{r_M}{M})^{MN} P = (1 + r_1)^N P$$

$$(1 + \frac{r_M}{M})^{M} = (1 + r_1)$$

This leads to the general equation for effective interest rate:

$$r_1 = (1 + \frac{r_M}{M})^M - 1 \qquad (1)$$

If the effective rate is specified and we want to find the nominal rate, we can rearrange equation (1) as follows:

$$\frac{r_M}{M} = (1 + r_1)^{\frac{1}{M}} - 1 \qquad (1a)$$

Let us try equation (1) on the earlier problem of finding the effective rate for a 12 percent nominal rate compounded monthly:

$$r_M = 0.12$$

$$M = 12$$

$$\frac{r_M}{M} = \frac{0.12}{12} = 0.01$$

We now find the effective rate from equation (1):

$$r_1 = (1 + \frac{r_M}{M})^M - 1$$

$$r_1 = (1 + 0.01)^{12} - 1 = 1.1265 - 1 = \underline{12.65 \text{ percent}}$$

which agrees with the previous value and furnishes pragmatic proof of the validity of equation (1).

Note that the effective rate, r_1, is not affected by the number of years involved. It is also worth noting that equation (1) is applicable to those rare cases where compounding periods are greater than one year. If, for example, compounding occurs on alternate years and the nominal rate is 12 percent:

$$M = \frac{1}{2}$$

$$r_M = 0.12$$

$$\frac{r_M}{M} = \frac{0.12}{\frac{1}{2}} = 0.24$$

$$r_1 = (1 + \frac{r_M}{M})^M - 1 = (1 + 0.24)^{1/2} - 1 = 1.114 - 1 = \underline{11.4 \text{ percent}}$$

In a case such as this, the future amount, F, when compounded biannually at 12 percent would be the same as with 11.4 percent compounded annually. This would be true only at the end of every other year, however.

Here is another example. Suppose you are offered a loan at 1 percent a week, compounded weekly. What is the effective rate of interest?

$$\frac{r_M}{M} = 0.01$$

$$M = 52$$

$$r_1 = (1 + \frac{r_M}{M})^M - 1 = (1 + 0.01)^{52} - 1 = (1.01)^{52} - 1 = 1.68 - 1$$

$$= \underline{68 \text{ percent}}$$

Run, don't walk, to your nearest Better Business Bureau.

CONTINUOUS COMPOUNDING

Next let us take the extreme case and derive the effective interest rate for continuous compounding. Equation (1) bears a resemblance to the expression for e, the base of natural logarithms:

$$e = (1 + \frac{1}{k})^k \text{ as } k \to \infty \qquad (2)$$

We next arbitrarily generate this relationship:

$$k = \frac{M}{r_M}$$

then

$$M = r_M k$$

Substituting into equation (1):

$$r_1 = (1 + \frac{r_M}{M})^{r_M k} - 1$$

$$r_1 = (1 + \frac{1}{k})^{r_M k} - 1$$

With continuous compounding, M approaches infinity and r_M approaches r_∞. We can then substitute e from equation (2) into the above:

$$r_1 = e^{r_\infty} - 1 \tag{3}$$

$$e^{r_\infty} = r_1 + 1 \tag{3A}$$

$$r_\infty = \ln(1 + r_1) \tag{3B}$$

If we substitute the value of r_1 from equation (3) into the standard expression relating future and present amounts, we have:

$$F = (1 + r_1)^N P = (1 + e^{r_\infty} - 1)^N P = e^{Nr_\infty} P \tag{3C}$$

which is shown in Figure 1.

Let us now return to our initial problem and solve for the future amount when a loan of $10,000 is compounded continuously for 20 years. Equation (3C) gives us

$$F = e^{Nr_\infty} P = 2.718^{20 \times 0.12} \$10,000 = 2.718^{2.4} \$10,000 =$$

$$11.05 \times \$10,000 = \$110,500$$

which is the amount shown in Table 1.

The term r_∞ describes the slope of the continuous compounding line (see Figure 1) at the origin. This is the nominal annual rate for continuous compounding. It is sometimes called the "force of interest." Equations (3), (3A), and (3B) relate this nominal rate to the effective rate. Let us illustrate the use of these equations next by finding the effective rate (r_1) for a force of interest of 12 percent:

$$r_\infty = 0.12$$

$$r_1 = e^{r_\infty} - 1 = (2.718)^{0.12} - 1 = 1.127 - 1 = \underline{12.7 \text{ percent}}$$

This means that 12 percent compounded continuously is equivalent in value to 12.7 percent compounded annually.

Next we can illustrate these principles by solving for the force of interest (r_∞) corresponding to an effective rate of 12 percent.

$$r_1 = 0.12$$

$$e^{r_\infty} = r_1 + 1 = 0.12 + 1 = 1.12$$

$$(2.718)^{r_\infty} = 1.12$$

$$r_\infty = \underline{11.33 \text{ percent}}$$

Figure 2 shows how nominal rates and effective rates vary for different compounding periods.

Admittedly, there is little practical use for continuous compounding applied to single-payment loans. The ideas, however, are necessary to the analysis of the realistic situation of virtually continuous cash flow from a business venture. That topic forms part of the next chapter.

II. MULTIPLE PAYMENT RELATIONSHIPS

GENERAL REMARKS

In many business investments, cash returns occur almost continuously. Obviously, $1,000 per year flowing back in a continuous stream is more desirable than $1,000 collected at each year's end. Most investment decision-makers ignore this difference and simplify their work with the standard assumption that all cash flows are concentrated at the end of the year during which they occur. This simplification is usually safe since it applies to cash flowing out as well as in. Moreover, any resulting error applies in equal measure to all alternatives. Nevertheless, there are occasional situations where continuous compounding deserves careful analysis. For example, one investment opportunity might involve daily cash returns while an alternative opportunity would produce periodic returns at annual intervals. In deciding between these two, you would want to know how to convert them to the common basis of effective interest. The purpose of this chapter is to lend such understanding.

When we pay or collect periodic returns that include interest payments, we are using compound interest. That is, we recognize not only how much rent money is due, but when.

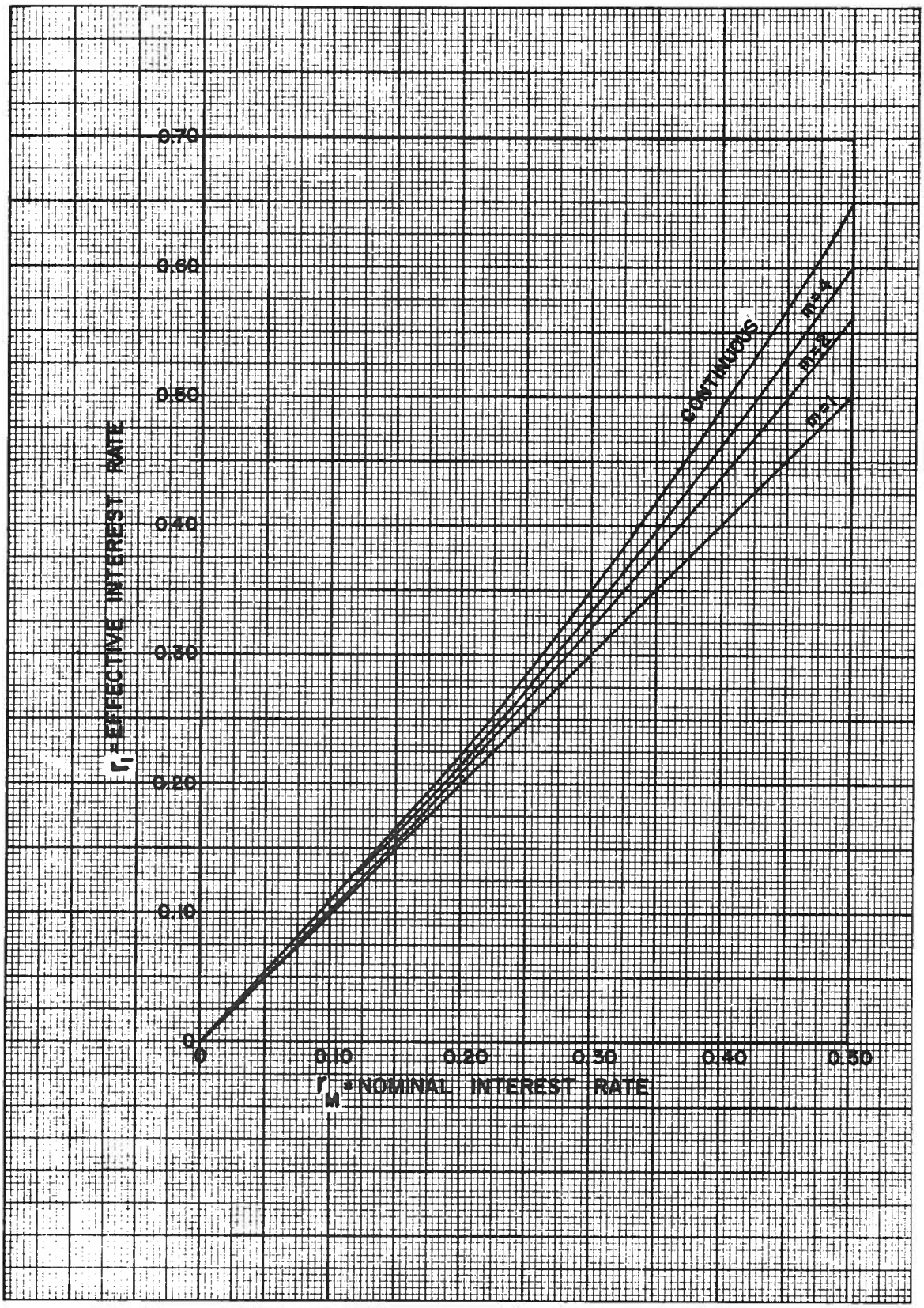


Figure 2. Nominal vs Effective Interest Rates for Various Compounding Periods.

DISCRETE RETURNS

We shall start with uniform annual returns, then look at increasingly frequent returns, and finally tackle continuous returns. Figure 3 shows how uniform annual returns reduce a debt to zero over a given period of time. Two cases are shown: one with interest charges, and one without. The latter corresponds to simple straight-line depreciation; for our purposes we can look upon it as the limiting case of capital recovery at zero interest. The other pattern is more general and is the one we want to study. Notice that the debts remaining at any time in the two plans coincide nowhere except at the extreme terminal points, at beginning and end of the loan period.

Figure 4 shows two payment plans, both based on the same time-value of money, (which was not the case in Figure 3). One plan has annual payments, the other semi-annual. Remember that we have specified that the time-value of money be the same in each case. This requires that the amount still owed at the end of each year (after periodic payment) be always the same. The semi-annual payment plan can achieve this requirement at a lower nominal rate simply because the lender gets some of his money back sooner. In Figure 4, the term A_M is a uniform annual amount returned in M equal payments. The amount of each payment is A_M/M. As was the case with single payments, our interest equations can be generalized by mentally converting i to an interest rate per compounding period and converting N to the number of compounding periods. That is, substitute r_M/M for i and MN for N in our standard equations and tables.

Now suppose we have an annual payment plan involving an interest rate r_1 and we want to find what annual interest rate (r_M) would be appropriate for a second payment plan involving more frequent returns (M per year). We want to keep the time-value of money the same in each case. Figure 5 shows the first year's debt pattern for the two plans (assuming for the moment that $M = 2$). At the end of the first year, just before making the year-end payment, the debt for the annual payment plan will be

$$F = (1 + r_1)P$$

Now if we pretend for a moment that the borrower in the second plan had been unable to make his payment at the end of the first six months, his debt would have started to increase at that moment by the extent of the interest owed. In short, his debt would have compounded. And, remembering that the time-value of money is held the same for both plans, his debt at the end of the first year would by definition be the same as for the first plan. The compounded amount would be

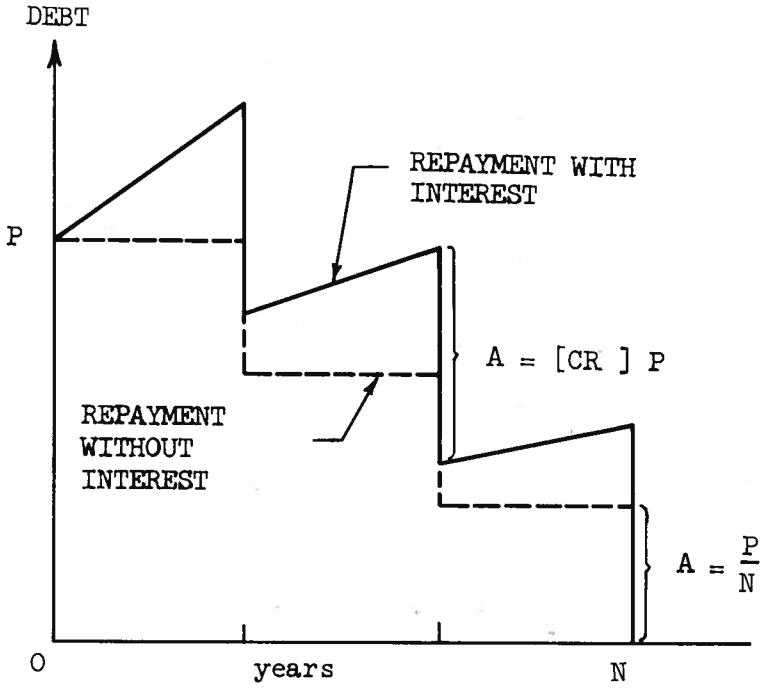


Figure 3. Debt Reduction With and Without Interest.

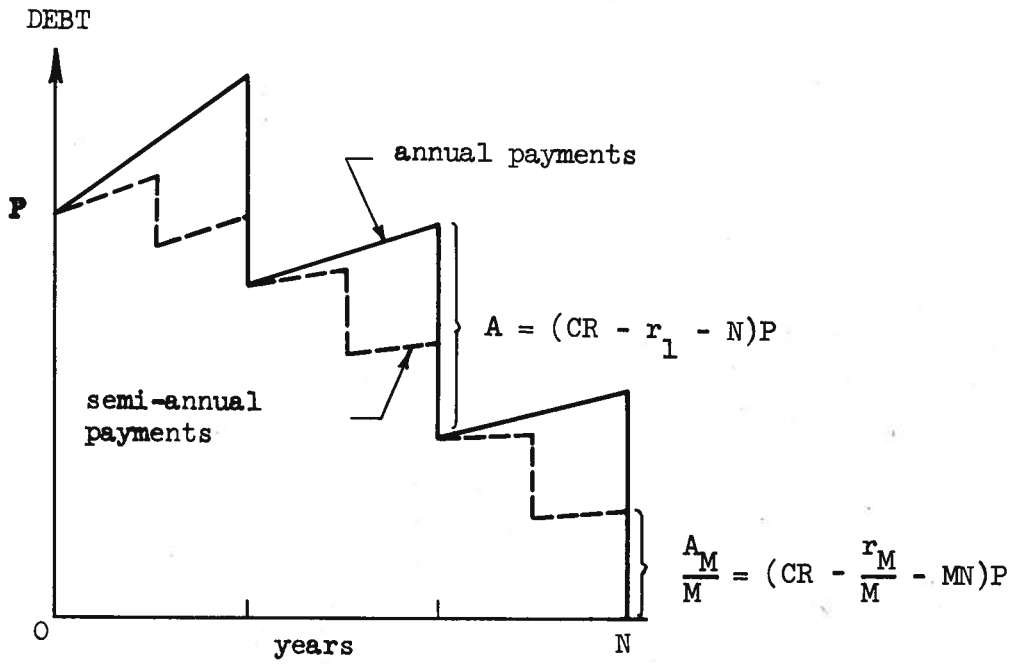


Figure 4. Debt Reduction with Annual and Semi-annual Payments, Time-Value of Money Held Constant.

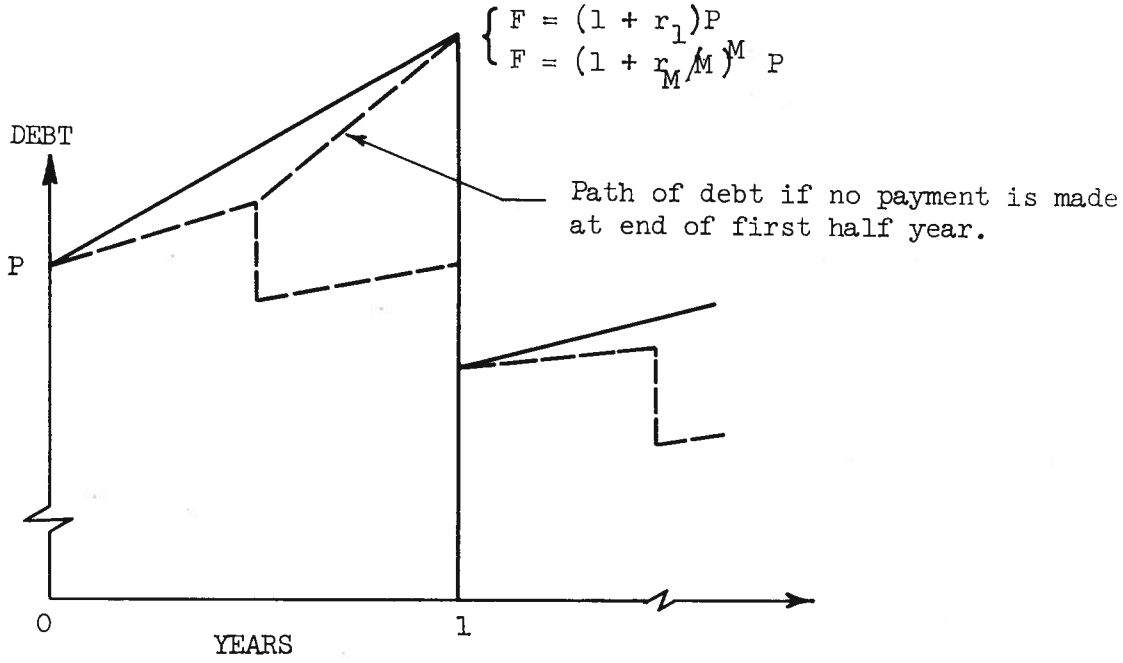


Figure 5. Debt Reduction During First Year.

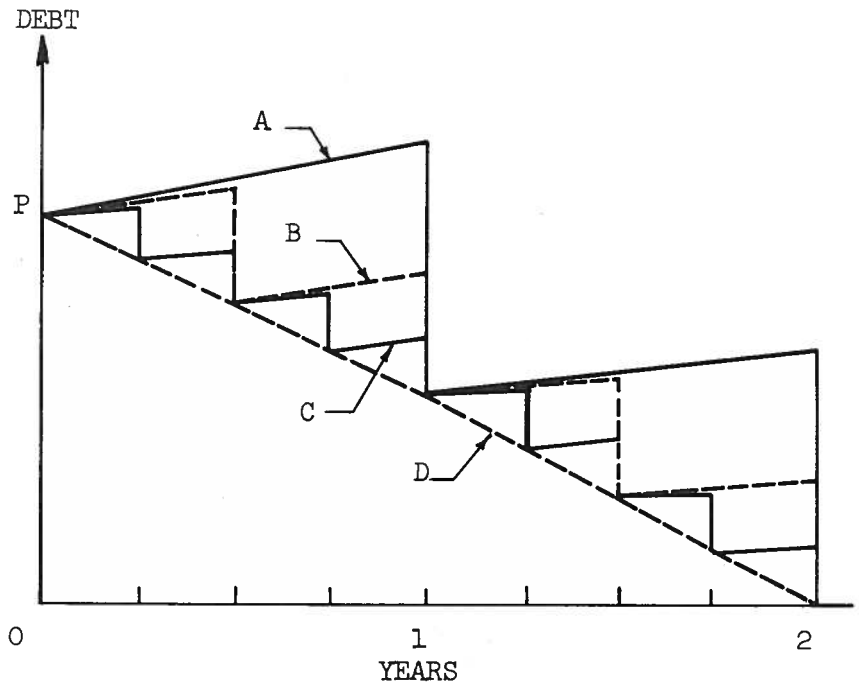


Figure 6. Debt Reduction With Different Frequency of Repayment But All With Same Time-Value of Money.

$$F = (1 + \frac{r_M}{M})^M P$$

and since

$$F = F$$

we have

$$(1 + r_1)P = (1 + \frac{r_M}{M})^M P$$

all of which is exactly the same as with single-payment plans. Therefore our derived relationships are also the same as before:

$$r_1 = (1 + \frac{r_M}{M})^M - 1 \qquad (1)$$

$$\frac{r_M}{M} = (1 + r_1)^{1/M} - 1 \qquad (1A)$$

Let us convince ourselves of the validity of the above by working out three debt patterns and plotting the results. Suppose we start with a $10,000 debt, a 20 percent annual interest rate (r_1), a two-year life, and uniform annual payments.

$$A = (CR - r_1 - N)P = (CR - 20\% - 2)0.6545 \times \$10,000 = \$6545$$

An accountant would do this:

End of Year	Debt at Start of Year	Interest Due	Payment	Reduction In Principal	Remaining Debt
1	$10,000	$2000	$6545	$4545	$5455
2	$ 5,455	$1090	$6545	$5455	0

Line A in Figure 6 shows the debt pattern calculated above.

Next, let us look at the same debt and the same time-value of money (r_1 = 20 percent) but with payments every six months (M = 2).

$$r_1 = 0.20$$

$$\frac{r_M}{M} = (1 + r_1)^{1/M} - 1$$

$$\frac{r_M}{M} = (1 + 0.20)^{1/2} - 1 = (1.2)^{1/2} - 1 = 1.096 - 1 = 0.096$$

$$\frac{A_2}{M} = (CR - \frac{r_M}{M} - MN)P = (CR - 9.6\% - 4)0.3127 \times \$10,000 = \$3127$$

In tabular form we have:

End of Period	Year	Debt at Start of Period	Interest Due	Payment	Reduction in Principal	Remaining Debt
1	0.5	$10,000	$960	$3127	$2167	$7833
2	1.0	$ 7,833	$751	$3127	$2376	$5457
3	1.5	$ 5,457	$524	$3127	$2603	$2854
4	2.0	$ 2,854	$273	$3127	$2854	0

Line B in Figure 6 shows the debt pattern calculated above.

Finally, let us try it with quarterly payments:

$$r_1 = 0.20$$

$$\frac{r_M}{M} = (1 + r_1)^{1/M} - 1 = (1 + 0.20)^{1/4} - 1 = (1.2)^{1/4} - 1$$

$$= 1.0466 - 1 = 0.0466$$

$$\frac{A_4}{M} = (CR - \frac{r_M}{M} - MN)P$$

$$\frac{A_4}{M} = (CR - 4.66\% - 4 \times 2)0.1526 \times \$10,000 = \$1526$$

In tabular form we have:

End of Period	Year	Debt at Start of Period	Interest Due	Payment	Reduction in Principal	Remaining Debt
1	.25	$10,000	$466	$1526	$1060	$8940
2	.50	8,940	417	1526	1109	7831
3	.75	7,831	365	1526	1161	6670
4	1.0	6,670	311	1526	1215	5455
5	1.25	5,455	254	1526	1272	4183
6	1.50	4,183	194	1526	1332	2851
7	1.75	2,851	133	1526	1393	1458
8	2.0	1.458	68	1526	1458	0

These figures provide Line C in Figure 6. Clearly, as M increases we approach Line D, the locus of the residual debt at start of each payment period. Line D, then, applies to continuous compounding. Note

that it is not a straight line. Before investigating continuous returns, let us illustrate how equations (1) and (1A) can be used when returns are known and we want to derive effective interest rates.

Suppose we have two investment opportunities and must choose between them. Each has a first cost of $10,000, and a five-year life and zero disposal value. Alternative A returns $3000 at the end of each year. Alternative B returns $2900 per year in equal amounts ($1450) every six months. We want to find for each case the effective interest rate (r_1) as our criterion of desirability.

Alternative A

$$(CR - r_1\% - 5) = \$3,000/\$10,000 = 0.30$$

From plots of CR versus interest rate:

$$r_1 = \underline{15.3 \text{ percent}}$$

Alternative B

$$(CR - \frac{r_2}{2} - 2 \times 5) = \frac{A_M}{P} = \frac{\$\ 1,450}{\$10,000} = 0.145$$

as before:

$$\frac{r_2}{2} = 7.4 \text{ percent}$$

Applying equation (1):

$$r_1 = (1 + \frac{r_M}{M})^M - 1 = (1 + 0.074)^2 - 1 = \underline{15.7 \text{ percent}}$$

Conclusion: Alternative B is slightly better. Note in the above that we did not succumb to the temptation of finding the nominal rate (r_M). Remember the dictionary definition of nominal: "being so in name only." Its numerical value (in this case 14.8 percent) is, by itself, a totally useless number.

CONTINUOUS RETURNS

Next we shall look at the extreme case of non-annual uniform returns: the continuous cash flow. From a practical point of view, daily net cash flow from a business venture would approximate this case. We want to learn how to find the effective interest rate (r_1) for a given continuous cash flow produced by a given investment. We can most readily do this by starting with a known interest rate and then solving

for the corresponding annual return. If you will recall, our general equation for periodic returns is:

$$\frac{A_M}{M} = (CR - \frac{r_M}{M} - MN)P$$

where

A_M = returns per year subdivided into M equal amounts
M = payment periods per year
N = years in the loan or investment
P = investment or loan
r_M = nominal annual interest rate

In the above, A_M/M represents the cash return per period and r_M/M represents the interest rate per payment period.

Going back to our fundamental equation for capital recovery factor, we have:

$$CR = \frac{\frac{r_M}{M}(1 + \frac{r_M}{M})^{MN}}{(1 + \frac{r_M}{M})^{MN} - 1}$$

also

$$CR = \frac{\frac{A_M}{M}}{P}$$

therefore

$$\frac{\frac{A_M}{M}}{P} = \frac{\frac{r_M}{M}(1 + \frac{r_M}{M})^{MN}}{(1 + \frac{r_M}{M})^{MN} - 1}$$

cancelling 1/M from each side and multiplying by P:

$$A_M = \frac{r_M(1 + \frac{r_M}{M})^{MN}}{(1 + \frac{r_M}{M})^{MN} - 1} P \qquad (4)$$

With continuous compounding, M approaches infinity and we can re-introduce the concept of:

$$e = (1 + 1/k)^k \text{ as } k \to \infty$$

If we set $k = M/r_\infty$

where

r_∞ = nominal annual interest rate for continuous compounding

then,

$$M = r_\infty k$$

and

$$(1 + r_\infty/M)^{MN} = (1 + r_\infty/r_\infty k)^{r_\infty kN}$$

$$= (1 + 1/k)^{r_\infty kN}$$

$$= e^{r_\infty N}$$

Substituting into equation (4) and converting all M subscripts to ∞ to indicate the extreme case of continuous compounding, we have:

$$A_\infty = \frac{r_\infty e^{r_\infty N}}{e^{r_\infty N} - 1} P \qquad (5)$$

and, as developed in the chapter on single payments:

$$r_1 = e^{r_\infty} - 1 \qquad (3)$$

We now have another of those equations in which knowing X we can find Y, but not vice versa. The practical approach, of course, is to make a parametric analysis, assuming a methodical series of X's and plotting the resulting values of Y. We are then ready to work in either direction. This has been done with equations (3) and (5). Figure 7 is the result. We can illustrate its use with a simple example: Find the effective interest rate (r_1) for an investment of $10,000 that returns $2800 annually in a continuous stream for five years.

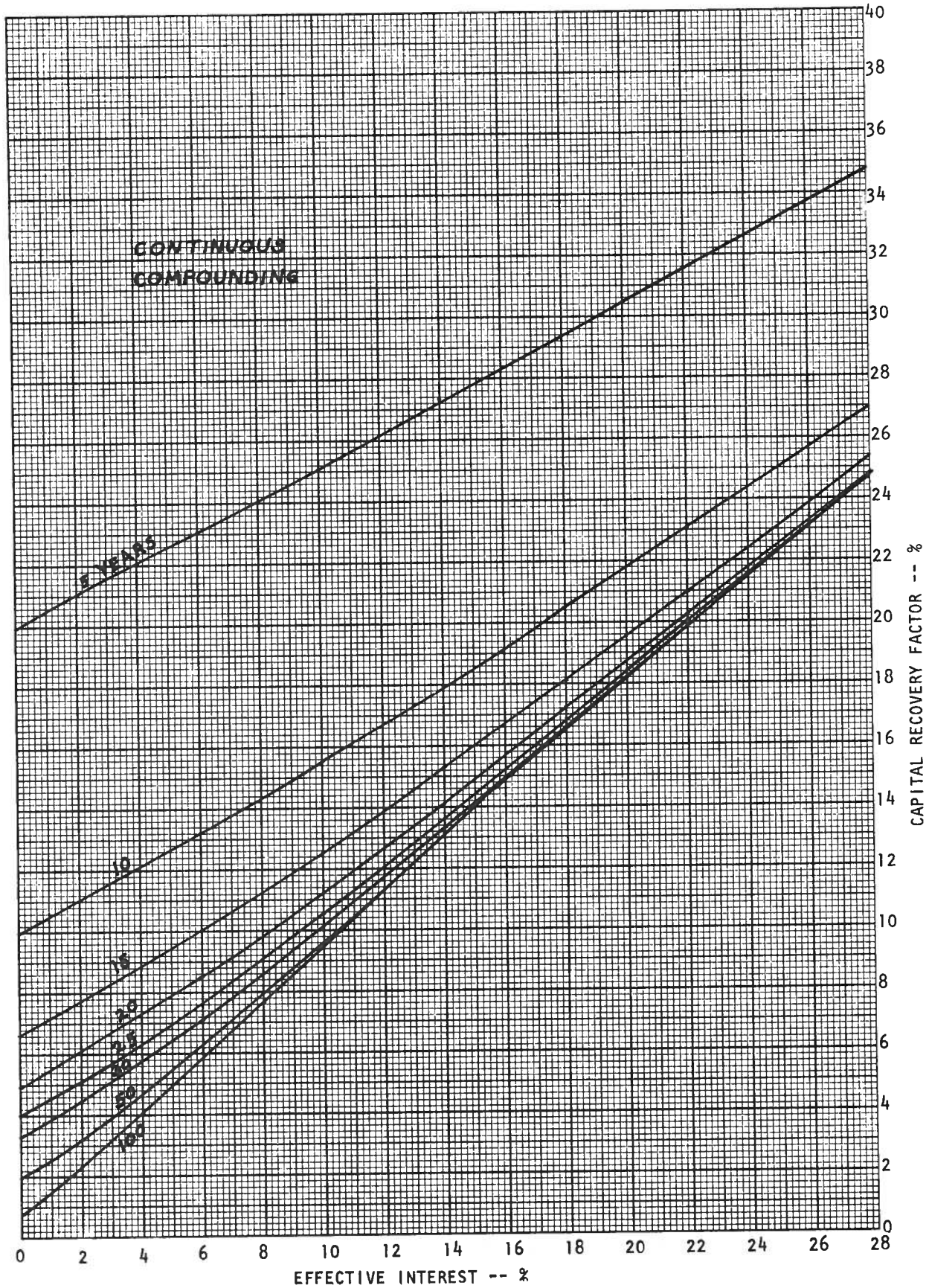
$N = 5$

$P = \$10,000$

$A_\infty = \$\ 2,800$

$$CR = \frac{A_\infty}{P} = \frac{\$\ 2,800}{\$10,000} = 0.28$$

$r_1 =$ 15.3 percent (from Figure 7)


CONTINUOUS
COMPOUNDING
5 YEARS
10
15
20
25
30
50
100
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
EFFECTIVE INTEREST -- %
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28
30
32
34
36
38
40
CAPITAL RECOVERY FACTOR -- %

We can check this procedure by inverting the problem. Let us stipulate an effective interest of 15.3 percent, an investment of $10,000 and a five-year life. If the returns are uniform and continuous, how much must they be per annum?

$$r_1 = 15.3 \text{ percent}$$
$$P = \$10,000$$
$$N = 5$$
$$e^{r_\infty} = r_1 + 1 = 0.153 + 1 = 1.153$$

therefore,

$$r_\infty = 0.1423$$

Applying equation (5):

$$A_\infty = \frac{r_\infty e^{r_\infty N}}{e^{r_\infty N} - 1} P = \frac{0.1423\ (1.153)^5}{(1.153)^5 - 1} \$10,000$$

$$= 0.28 \quad \$10,000 = \underline{\$2800}$$

Since $2800 was the annual return that we used in the previous example, the validity of our equations is confirmed.

GENERAL EQUATION FOR RESIDUAL DEBT

To complete our understanding of uniform, non-annual periodic returns we should develop the general equation for the locus of the residual amounts owed at the beginning of each payment period. As we have shown, this curve is in fact the residual debt line for continuous compounding. We shall refer to the residual debt as P_X, where X is the number of years remaining in the loan period.

Holding the time-value of money constant, the residual amount (P_X) could be paid off in uniform periodic amounts (A_M/M) calculated in the classical manner:

$$\frac{A_M}{M} = (CR - \frac{r_M}{M} - MX)P_X$$

but

$$\frac{A_M}{M} = (CR - \frac{r_M}{M} - MN)P$$

so

$$(CR - \frac{r_M}{M} - MX)P_X = (CR - \frac{r_M}{M} - MN)P$$

therefore,

$$P_X = \frac{(CR - \frac{r_M}{M} - MN)}{(CR - \frac{r_M}{M} - MX)} P \qquad (6)$$

Equation (6) is a simple, convenient tool which hardly needs clarification with a numerical example. Quite simply, the residual debt equals the initial debt multiplied by the capital recovery factor based on the initial life of the debt and divided by the capital recovery factor based on the remaining life. So we can now move on to the case of continuous compounding. First we substitute the basic expressions for both CR's above:

$$P_X = \frac{\frac{r_M/M\ (1 + r_M/M)^{MN}}{(1 + r_M/M)^{MN} - 1}}{\frac{r_M/M\ (1 + r_M/M)^{MX}}{(1 + r_M/M)^{MX} - 1}} P$$

Cancelling r_M/M's and transposing:

$$P_X = \frac{(1 + r_M/M)^{MN}\ [(1 + r_M/M)^{MX} - 1]}{[(1 + r_M/M)^{MN} - 1]\ (1 + r_M/M)^{MX}} P$$

$$P_X = \frac{(1 + r_M/M)^{MX} - 1}{(1 + r_M/M)^{MN} - 1} P(1 + r_M/M)^{M(N-X)} \qquad (7)$$

but since

$$(1 + r_M/M)^M = 1 + r_1$$

we have

$$P_X = \frac{(1 + r_1)^X - 1}{(1 + r_1)^N - 1} P(1 + r_1)^{N-X} \qquad (8)$$

Equation (7) may be used where nominal interest rates are given, equation (8) where effective rates are given. Equation (7) may be more convenient than equation (6) when nominal interest rate per period (r_M/M) or number of periods (MN or MX) are not conveniently found in interest tables. With continuous compounding

$$(1 + r_\infty/M)^M = e^{r_\infty} \tag{9}$$

Substituting into equation (7):

$$P_X = \frac{e^{r_\infty X} - 1}{e^{r_\infty N} - 1} P\, e^{r_\infty (N-X)} \tag{10}$$

which is the general equation for the locus of residual debt.

Here is a sample problem. A $1000 debt is to be paid off in uniform amounts every six months over a three-year period. The effective interest rate is 12 percent. What is the residual debt after the payment at the end of the first year?

$M = 2$
$N = 3$
$P = \$1000$
$r_1 = 12$ percent
$X = 2$

Applying equation (8):

$$P_2 = \frac{(1 + 0.12)^2 - 1}{(1 + 0.12)^3 - 1} \$1000\ (1 + 0.12)^{3-2}$$

$$P_2 = \frac{1.254 - 1}{1.404 - 1} \$1000\ (1.12) = \underline{\$704}$$

Next, suppose the payments are quarterly. If our relationships have been correctly developed, we should find the same residual debt. This is implied by equation (8) which does not contain the variable M. But, just to prove our point, let us convert r_1 to r_M and try our luck with equation (7).

$$r_M/M = (1 + r_1)^{1/M} - 1 \tag{1A}$$

$$r_M/M = (1.12)^{0.25} - 1 = 0.0286$$

then

$$P_2 = \frac{(1.0286)^8 - 1}{(1.0286)^{12} - 1} \$1000\ (1.0286)^4$$

$$P_2 = \frac{1.254 - 1}{1.404 - 1} \$1000\ (1.12) = \underline{\$704}$$
(check)

Finally, we can assume continuous compounding and try equation (10). First we need to find the value of $e^{r\infty}$:

$$e^{r\infty} = r_1 + 1$$

$$e^{r\infty} = 1.12$$

$$P_X = \frac{1.12^2 - 1}{1.12^3 - 1} \quad \$1000\ (1.12)$$

$$P_X = \frac{1.254 - 1}{1.405 - 1} \quad \$1000\ (1.12) = \underset{\text{(check)}}{\$704}$$

Admitting that the three precisely identical answers are perhaps more a matter of lenient middle-aged eyes than good slide rule work, nevertheless, the three identical outcomes prove that you can use any one of a number of different equations when faced with this sort of problem.

III. ANNUAL AND CONTINUOUS RETURNS IN COMBINATION

In making investment decisions, we may often find combinations of continuous and annual cash flows. We can illustrate how to handle these by working a simple problem.

Suppose you are offered the sale of a ferry boat that has an estimated remaining life of ten years. The city government will give you a contract of $200,000 per year to carry all the municipal vehicles that come along. The $200,000 will be paid in a lump sum at the end of each year. In addition, you will have a daily income from general users amounting to $500,000 per year. There will be daily operating costs amounting to $150,000 per year. The annual bill for repairs, and insurance (both paid in lump sums) will be $100,000. The net disposal value at the end of ten years will be $750,000. You want to earn an interest rate of 20 percent before taxes. How much should you pay for the ferry?

When an interest rate, such as the 20 percent above, is stipulated without further qualification, we always infer that an <u>effective annual rate</u> is implied.

$$r_1 = 20 \text{ percent}$$

$$e^{r\infty} = r_1 + 1 = 1.2$$

therefore,

$$r_\infty = 0.1823$$

Our net periodic annual amounts are:

City contract	$200,000
Repairs and insurance	($100,000)
Net	$100,000

Our net continuous amounts per annum are:

Income	$500,000
Operating costs	($150,000)
Net	$350,000

If we discount all future amounts to the present, using appropriate interest rates, the net total figure will tell us how much we can afford to invest.

Item			Present Worth
Periodic returns:	$100,000(SPW-20%-10)4.192	=	$ 419,200
Continuous returns:	$350,000(SPW-18.23%-10)4.459	=	$1,561,000
Disposal:	$750,000(PW-20%-10)0.1615	=	$ 121,100
Net present worth		=	$2,101,300
		rounded to	2,100,000

In the above calculation, the present worth of the continuous returns is found using the nominal interest rate (18.23 percent) and treating the returns as annual. The SPW is found by taking the reciprocal of the CR for 10 years and 18.23 percent interest. Another way of doing it would be to use Figure 7, taking the reciprocal of the CR for the 20 percent effective interest rate.

IV. CONCLUSIONS

At this point, the most appropriate thing to do is to repeat the assertion that the assumption of annual compounding is usually a valid simplification in long-term investment decisions. Nevertheless, there may be times when decisions will be between alternatives, one of which has continuous cash flow characteristics, while the other has widely spaced periodic returns. In these instances, an understanding

of the mathematics of non-annual compounding becomes important. This knowledge is also useful in everyday affairs such as in quarterly or monthly installment plans.

There are several inherently confusing aspects of non-annual compounding. For this reason, we must be more than ever careful in our semantics. Any meaningful discussion in this area must be prefaced with a review of the thoughts behind the key words.

An important concept is the realization that our standard interest formulas are not confined to annual compounding. Our standard interpretation has perhaps given us the mistaken notion that interest rates are always per annum and that compounding periods are always years. We have learned here that the standard interpretation represents only a special case of the general relationships.

The single-payment concepts clearly show that frequent compounding requires lower nominal annual interest rates than would be needed to maintain the same time-value of money with infrequent compounding. In short, interest rate by itself is no measure of the time-value of money; it must be modified with a specific compounding period.

The ultimate frequency of compounding is reached when we compound continuously. This is not merely a mathematical concept, because daily returns from a business enterprise approach this condition. The logic applied to non-annual uniform returns is derived from the simple case of single payments. We have stressed the capital recovery aspects of uniform returns but the ideas could be easily turned in the other direction to analyze sinking funds, etc.

An economic analyst should be able to deal in both nominal and effective interest rates. He should understand how to handle both discrete and continuous returns, and how to find their equivalent interest rates based on annual compounding. These are the topics discussed in this paper. Understanding them is seldom easy, but careful thought and repeated practice can produce facility in the useful application of these ideas.

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