THE WAVE RESISTANCE OF A SURFACE PRESSURE DISTRIBUTION IN UNSTEADY MOTION

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TRANSLATORS' NOTE:-

In this paper the symbols used for the hyperbolic functions have the following meanings:

\[
\begin{align*}
\text{sh} & \equiv \sinh \\
\text{ch} & \equiv \cosh \\
\text{th} & \equiv \tanh
\end{align*}
\]
We consider the problem of non-steady motion of a surface pressure distribution of arbitrary form \( p(x, y) \) over an area \( \Omega \). A coordinate system \((0, x, y, z)\) is fixed on the undisturbed surface, where \( x \) is in the direction of motion and \( z \) is vertically upwards.

The disturbance potential which is caused by this pressure distribution is defined by the solution of Laplace's Equation, \( \Delta \phi = 0 \) with the following boundary conditions:

On the free surface:

\[
\frac{\partial^2 \phi}{\partial t^2} + \frac{\partial \phi}{\partial x} + \gamma^2 \frac{\partial^2 \phi}{\partial x^2} - 2\gamma \frac{\partial \psi}{\partial x} \frac{\partial \phi}{\partial t} \frac{\partial \phi}{\partial x} = -x \frac{\partial \mu}{\partial t} + \gamma \frac{\partial \mu}{\partial x}
\]

at \( z = 0 \). (1)

On the bottom:

\[
\frac{\partial \phi}{\partial x} \bigg|_{z = -H} = 0
\]

at \( z = -H \). (2)

In expressions (1) and (2) all the variables are non-dimensionalized with respect to \( l \), the half-length of the pressure distribution (see Fig. 1), \( p_0 \), the characteristic pressure, and \( g \), the acceleration of gravity. The disturbance potential \( \phi \) can be represented by the following integral form:

\[
\phi(x, y, z, t) = \frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ A(t) e^{-\sqrt{\alpha^2 + \beta^2} z} + B(t) e^{\sqrt{\alpha^2 + \beta^2} z} \right] e^{i(\alpha x + \beta y)} d\alpha d\beta
\]
We consider the case when \( \frac{\partial p}{\partial t} = 0 \). Let us assume

\[
\mathbf{X} \cdot \frac{\partial \mathbf{u}}{\partial x} = \frac{1}{4\pi^2} \int \int d\alpha d\beta \int \int \mathbf{X} \cdot \frac{\partial p}{\partial x_0} (x_0, y_0) e^{-i\left[k(x-x_0)^2 + \beta(y-y_0)^2\right]} dx_0 dy_0.
\]

Making the substitutions

\[\alpha = \kappa \cos \theta, \quad \beta = \kappa \sin \theta,\]

we get

\[
y(x, y, z) = \frac{1}{4\pi^2} \int \int \kappa dk \int d\theta \left[A(t) e^{i \kappa x} + B(t) e^{-i \kappa x}\right] e^{i \kappa \left[(x-x_0) \cos \theta + (y-y_0) \sin \theta\right]} \kappa
\]  

and

\[
x \cdot \frac{\partial \mathbf{u}}{\partial x} = \int \int \kappa dk \int d\theta \int \int \frac{\partial p}{\partial x_0} e^{i \kappa \left[(x-x_0) \cos \theta + (y-y_0) \sin \theta\right]} dx_0 dy_0.
\]

Fig. 1

Substituting (3) and (4) in (1) and (2), and equating the integrands (because the equality must hold for all \( k \) and \( \theta \)), then expressing \( B \) in terms of \( A \) from equation (2) and substituting in (1), we obtain
\[ \frac{d^2A}{dt^2} + \kappa \Lambda \Lambda H + \Lambda \Lambda (i \kappa \cos \theta)^2 - 2 \Lambda \frac{dA}{dt} (i \kappa \cos \theta) - A \frac{d \nu}{dt} (i \kappa \cos \theta) = \]
\[ = (1 + e^{2 \kappa t})^{-1} \int_\mathbb{R} \int_\mathbb{R} \frac{\partial u}{\partial x_0} e^{-i \kappa (x_0 \cos \theta + y_0 \sin \theta)} dx_0 dy_0. \quad (5) \]

Now let us introduce the function
\[ A_1 = \Lambda e^{-i \kappa \cos \theta \cdot \int \nu dt}. \quad (6) \]
To determine \( A_1 \) we note that it satisfies the differential equation
\[ \frac{d^2A_1}{dt^2} + \Lambda_1 \kappa \Lambda \Lambda H = \hat{f}(k, \theta, t), \quad (7) \]
where
\[ \hat{f}(k, \theta, t) = e^{-i \kappa \cos \theta \cdot \int (1 + e^{2 \kappa t})^{-1} \int_\mathbb{R} \int_\mathbb{R} \frac{\partial u}{\partial x_0} e^{-i \kappa (x_0 \cos \theta + y_0 \sin \theta)} dx_0 dy_0}. \quad (8) \]
and
\[ s = \int \nu dt. \quad (8a) \]

Solving (7) with zero initial conditions,
\[ t = 0 \quad A_1 = 0; \quad \frac{dA_1}{dt} = 0 \]
(equivalent to \( y = 0 \) and \( \frac{\partial y}{\partial t} = 0 \)), by the method of variation of the integration constant, we get
\[ A_1 = \left( \frac{1}{\delta} \int_0^t \cos \delta t dt \right) \sin \delta t - \left( \frac{1}{\delta} \int_0^t \sin \delta t dt \right) \cos \delta t + \]
\[ + \left( - \frac{1}{\delta} \int_0^t \cos \delta t dt \right) \sin \delta t + \left( \frac{1}{\delta} \int_0^t \sin \delta t dt \right) \cos \delta t, \quad (9) \]
where \( \delta = \kappa \Lambda \Lambda H \).
Therefore the disturbance potential is

\[ y (x, y, z, t) = \frac{1}{4\pi} \int_0^\infty \int_{-\infty}^\infty d\theta \cdot A_i \cdot e^{ik \cos \theta \cdot s \cdot t} \times \]

\[ e^{-ik(2z-x)} \left[ e^{i \alpha} + e^{i \beta} \right] e^{ik(x \cos \theta + y \sin \theta)} \]

\[ x \cos \theta \]

(10)

The wave resistance is obtained from

\[ R = -\iint_{\Omega} \rho u(x, y) \frac{\partial \zeta}{\partial x} \, dx \, dy, \]

(11)

assuming that \( R \) is equal to the horizontal component of the pressure acting on the area \( \Omega \). In formula (11) all variables are dimensional. In non-dimensional form it would be

\[
\frac{R}{\rho_0 l^2} = -\iint_{\Omega} \rho u(x, y) \frac{\partial \zeta}{\partial x} \, dx \, dy; \\
\frac{\partial \zeta}{\partial x} = -x \frac{\partial \rho u}{\partial x} + \nu \frac{\partial^2 y}{\partial x^2} - \frac{\partial^2 y}{\partial x \partial t} \quad \text{at} \quad x = 0; \\
\frac{R}{\rho_0 l^2} = -\iint_{\Omega} x \rho u \frac{\partial \rho u}{\partial x} \, dx \, dy + \iint_{\Omega} \rho u \nu \frac{\partial^2 y}{\partial x^2} \, dx \, dy - \iint_{\Omega} \rho u \frac{\partial^2 y}{\partial x \partial t} \, dx \, dy = -J_1 + J_2 - J_3.
\]

(12)

In the last equation the first term is the horizontal component of the pressure, the second is equal to the resistance during steady motion, and the third term occurs only because of the unsteadiness. We can obtain

\[ J_2 \iint_{\Omega} \int_0^\infty \int_{-\infty}^\infty d\theta \cdot A_i \cdot (1 + e^{ik \cos \theta}) \cdot \Theta (k, \theta) \cdot e^{ik \cos \theta \cdot s \cdot t}, \]

\[ \Theta (k, \theta) = \iint_{\Omega} \rho u \cdot e^{i \alpha (x \cos \theta + y \sin \theta)} \, dx \, dy; \]

(13)

(14)
and

$$J_3 = \frac{1}{4\pi^2} \int_{-\pi}^{\pi} k dk \int_{-\pi}^{\pi} d\theta \frac{\partial A_1}{\partial t} (1 + e^{2i\kappa}) e^{i\kappa \cos \theta \cdot S} e^{i\kappa \cos \theta} \Theta(k, \theta) + J_2.$$

(15)

The wave resistance can be evaluated from

$$\frac{R}{\rho_o \Delta} = \frac{1}{\pi} \int_{-\pi}^{\pi} \chi \rho_u \frac{\partial \rho_u}{\partial x} dx dy - \frac{1}{4\pi^2} \int_{-\pi}^{\pi} k dk \int_{-\pi}^{\pi} \frac{\partial A_1}{\partial t} (1 + e^{2i\kappa}) \chi (i\kappa \cos \theta) e^{i\kappa \cos \theta \cdot S} \Theta(k, \theta) d\theta.$$

(16)

On the basis of (9)

$$\frac{\partial A_1}{\partial t} = c_{10} \delta \cos \delta t - c_{20} \delta \sin \delta t + \cos \delta \int f \cos \delta dt + \sin \delta \int f \sin \delta dt.$$

(17)

And from (8)

$$\int f \cos \delta dt = \int f ve^{-i\kappa \cos \theta \cdot S} \cos \delta t (1 + e^{2i\kappa}) \int \frac{\partial \rho_u}{\partial x} e^{-i\kappa (x \cos \theta + y \sin \theta)} dx dy dt =$$

$$= \chi (1 + e^{2i\kappa})^{-1} \Theta^* (k, \theta) \int \frac{\partial \rho_u}{\partial x} e^{-i\kappa \cos \theta \cdot S} \cos \delta dt.$$

(18)

where

$$\Theta^* (k, \theta) = \int \frac{\partial \rho_u}{\partial x} e^{-i\kappa (x \cos \theta + y \sin \theta)} dx dy.$$

(19)

Identically

$$\int f \sin \delta dt = \int f ve^{-i\kappa \cos \theta \cdot S} \sin \delta dt.$$

(20)

If the motion started from rest, i.e. $v \rightarrow 0, t \rightarrow 0$, then, using (8), (9), (18) and (20),

$$\frac{\partial A_1}{\partial t} = \chi (1 + e^{2i\kappa})^{-1} \Theta^* (k, \theta) \Phi^* (k, \theta, t),$$

(21)

where

$$\Phi^* (k, \theta, t) = \left[ \cos \delta t \int ve^{-i\kappa \cos \theta \cdot S} \cos \delta dt + \sin \delta t \int ve^{-i\kappa \cos \theta \cdot S} \sin \delta dt \right].$$

(22)

$$\frac{R}{\rho_o \Delta} = \frac{1}{\pi} \int_{-\pi}^{\pi} \chi \rho_u \frac{\partial \rho_u}{\partial x} dx dy - \frac{1}{4\pi^2} \int_{-\pi}^{\pi} k dk \int_{-\pi}^{\pi} \frac{\partial A_1}{\partial t} (1 + e^{2i\kappa}) \Theta^* (k, \theta) \Theta(k, \theta) \Phi(k, \theta, t) d\theta.$$

(23)
and
\[
\Phi(\kappa, \theta, t) = e^{i \kappa \cos \theta \cdot t} \left[ \cos \delta t \int ve^{-i \kappa \sin \theta} \cos \delta t dt + 
+ \sin \delta t \int ve^{-i \kappa \cos \theta} \sin \delta t dt \right].
\]

(24)

Let us now use this solution to calculate the wave resistance for the case of a constant pressure distribution acting on a rectangular area with \( \mu = b/l \) moving at a constant speed over deep water. Let us assume that the speed changes according to Fig. 2 below.

![Fig. 2](image)

For \( t \to \infty \), the required functions become:

\[
\Phi(\kappa, \theta, t) = \frac{\nu^t (-i \kappa \cos \theta)}{\delta^x - \kappa^2 \cos^2 \theta \cdot \nu^2},
\]

\[
\Theta^*(\kappa, \theta) = \frac{4 i \rho_u \sin(\kappa \cos \theta) \sin(\kappa \mu \sin \theta)}{\kappa \sin \theta} + i \kappa \cos \theta \overline{\Theta}(\kappa, \theta),
\]

\[
\Theta(\kappa, \theta) = 4 \rho_u \frac{\sin(\kappa \mu \sin \theta) \cdot \sin(\kappa \cos \theta)}{\kappa \sin \theta \cdot \kappa \cos \theta},
\]

\[
\Theta^* = i 32 \rho_u^2 \frac{\sin^2(\kappa \mu \sin \theta) \sin^2(\kappa \cos \theta)}{\kappa^2 \sin^4 \theta \cdot \kappa \cos \theta},
\]

(25)
\[ \frac{R}{p_0} = -\frac{\lambda}{4\pi^2} \int_0^\pi d\theta \int_0^{\infty} dx \frac{x}{\sin^2(\kappa \cos \theta) \sin^2(\kappa \cos \theta)} \frac{32 \rho u^2 \sin^2(\kappa u \sin \theta) \sin^2(\kappa \cos \theta)}{\kappa^2 \sin \theta \kappa \cos \theta} \times \]

\[ \times \frac{\psi^2(i \kappa \cos \theta)}{\delta^2 \kappa \cos^2 \theta \cdot \psi^2} d\theta \]

Substituting

\[ \lambda = \kappa; \quad \alpha = \kappa \cos \theta, \]

we obtain

\[ \frac{R}{p_0} = -\frac{8 \rho u^2}{\pi^2} \int_0^\infty d\lambda \int_0^\infty \frac{i \sin^2(\mu \lambda^2 - \alpha^2) \sin^2 \alpha \cdot \lambda}{(\lambda^2 - \alpha^2)} \frac{\lambda \alpha}{\sqrt{\lambda^2 - \alpha^2}} \cdot \]

\[ \cdot \]

(26)

The integrand has four poles on the real axis, at \( \alpha = \pm \lambda \)
and \( \alpha = \pm \sqrt{\lambda}/\nu \) (Fig. 3, below). To calculate the inner integral we use the theorem of residues. For the contour of integration we take the semicircle \( C_\rho \) of infinitely large radius in the upper half of the complex plane \( \alpha \), going underneath the poles as shown. Substituting \( \lambda = t^2/\nu^2 \) in (26), we obtain finally, as a particular case of (23), the established formula of B.P. Bolshakov.

Fig. 3
\[ R = \frac{16 \rho^2 \nu^2}{\pi \delta y_0} \int_1^\infty \sin^2 \frac{x}{2} \sin^2 \frac{\mu \kappa \sqrt{t-1}}{2} \frac{dt}{(t-1)^{3/2}}, \]  

(27)

where \[ \kappa = \frac{2 \rho \delta}{y_0}. \]

**TWO-DIMENSIONAL PROBLEM:**

In this case the velocity potential can be expressed as

\[ \psi(x, x, t) = \int_0^\infty \left[ (A \cos \kappa x + C \sin \kappa x) \sin \kappa x + 
(B \cos \kappa x + D \sin \kappa x) \sinh \kappa x \right] \, d\kappa, \]  

(28)

where \( A, B, C, D \) are unknown functions of \( t \) and \( k \). For the solution we employ (1) and (2). Assuming \( \partial p/\partial t = 0 \), the right-hand side of (1) can be written as a Fourier integral:

\[ \chi \frac{\partial u}{\partial x} = \chi \int_0^\infty \frac{\partial p}{\partial u} \cos \kappa (u - x) \, du = \]

\[ = \chi \int_0^\infty f_1(k, t) \cos \kappa x \, d\kappa \int_0^\infty f_2(k, t) \sin \kappa x \, d\kappa, \]

where

\[ f_1(k, t) = \chi \int_0^1 p_0 \kappa \sin \kappa u \, du; \quad f_2(k, t) = -\chi \int_0^1 p_0 \kappa \cos \kappa u \, du. \]

After substituting (28) into (2) we obtain the following system of equations:
\[
\begin{align*}
A \operatorname{Sh} \kappa H &= B \operatorname{ch} \kappa H \\
C \operatorname{Sh} \kappa H &= D \operatorname{ch} \kappa H
\end{align*}
\]
(29)

\[
\begin{align*}
B &= A \operatorname{th} \kappa H \\
D &= C \operatorname{th} \kappa H
\end{align*}
\]
(30)

Substituting (28) in (1) and equating the coefficients of \( \cos \kappa x \) and \( \sin \kappa x \) we obtain the following system:

\[
\begin{align*}
\frac{\partial^2 A}{\partial t^2} - \kappa B - \kappa^2 v^2 A - 2 \nu \kappa \frac{\partial C}{\partial t} - \kappa \frac{du}{dt} c &= f_1(\kappa, t) \\
\frac{\partial^2 C}{\partial t^2} + \kappa D - \kappa^2 v^2 C + 2 \nu \kappa \frac{\partial A}{\partial t} + \kappa \frac{du}{dt} a &= f_2(\kappa, t)
\end{align*}
\]
(31)

Eliminating \( B \) and \( D \) from (31) and (30), we have

\[
\begin{align*}
\frac{\partial^2 A}{\partial t^2} + \left[ \kappa \operatorname{th} \kappa H - \kappa^2 v^2 \right] A - 2 \nu \kappa \frac{\partial C}{\partial t} - \kappa \frac{du}{dt} c &= f_1(\kappa, t) \\
\frac{\partial^2 C}{\partial t^2} - \left[ \kappa \operatorname{th} \kappa H - \kappa^2 v^2 \right] C + 2 \nu \kappa \frac{\partial A}{\partial t} - \kappa \frac{du}{dt} a &= f_2(\kappa, t)
\end{align*}
\]
(32)

Multiplying the second equation by \( i \) and adding to the first, and introducing

\[
A_1 = Ae^{ixs}; \quad C_1 = Ce^{ixs}; \quad s = \int \nu dt
\]

we obtain the following differential equation for \( W_1 \) which is represented as

\[
W_1 = \frac{d^2 W_1}{dt^2} + \kappa \operatorname{th} \kappa W_1 = f(\kappa, t),
\]

where

\[
f(\kappa, t) = e^{ixs} \left[ f_1(\kappa, t) + if_2(\kappa, t) \right].
\]
The solution of this equation with zero initial conditions is

\[ W_t = \frac{\sin \delta t}{\delta} \int f \cos \delta t \, dt - \frac{\cos \delta t}{\delta} \int f \sin \delta t \, dt + \]

\[ + \left( \frac{1}{\delta} \int f \sin \delta t \, dt \right) t=0 \cos \delta t + \frac{1}{\delta} \int f \cos \delta t \, dt \sin \delta t. \]  

(33)

If the motion starts from rest, the two last terms become zero. Then

\[ W_t = \frac{\sin \delta t}{\delta} \int f \cos \delta t \, dt - \frac{\cos \delta t}{\delta} \int f \sin \delta t \, dt. \]  

(34)

The functions \( A, C \) are

\[ A = Re(W_1 e^{-i \kappa s}); \]  

(35)

and

\[ C = Im(W_1 e^{-i \kappa s}). \]  

(36)

The velocity potential is then

\[ \phi(x, z, t) = \int_0^\infty \left\{ Re \left( W_1 e^{-i \kappa s} \right) \left[ ch \kappa z + \frac{\alpha}{\delta} \kappa H_0 \cdot sh \kappa z \right] \cos \kappa x + \right. \]

\[ + \left. Im \left( W_1 e^{-i \kappa s} \right) \left[ ch \kappa z + \frac{\alpha}{\delta} \kappa H_0 \cdot sh \kappa z \right] \sin \kappa x \right\} \, dx, \]  

(37)

where \( W_1 \) is given by (34) and \( s = \int_0^t \, dt \).

The wave resistance is

\[ \frac{R}{\rho u} = \int_{-1}^1 p_u \frac{\partial c}{\partial x} \, dx = \int_{-1}^1 p_u \kappa \frac{\partial p_u}{\partial x} \, dx + \nu \int_{-1}^1 p_u \frac{\partial^2 y}{\partial x^2} \, dx - \]

\[ - \int_{-1}^1 p_u \frac{\partial^2 y}{\partial x \partial t} \, dx \quad \text{at} \quad z = 0. \]  

(38)
Substituting (37) in (38) and reversing the order of integration, we obtain
\[
\frac{R}{\rho_0 l} = -x \int_{-1}^{1} \rho_u \frac{\partial \rho_u}{\partial x} \, dx - v \int_{0}^{\infty} \kappa^2 \text{Re} (\Phi \cdot G) \, d\kappa - \int_{0}^{\infty} \kappa \frac{d}{d\kappa} \left[ \text{Im} (\Phi \cdot G) \right] \, d\kappa,
\]
(39)
where
\[
\Phi (\kappa, t) = W_t e^{-i\kappa s},
\]
\[
G (\kappa) = \int_{-1}^{1} \rho_u e^{-i\kappa x} \, dx.
\]
(40)

Using these solutions for the case of constant speed, we get Lamb's formula for the resistance:
\[
R = \frac{4 \rho_u^2}{\gamma} \sin^2 \frac{\alpha \ell}{\nu^2}.
\]
(41)

The results of the investigation of the wave resistance for the case of a constant acceleration, using (39) and the approximate method of Stationary Phase, are presented in Fig. 4 below. The hump speed for this motion is considerably less than for the case of steady motion.
Formulas (23) and (39) are recommended for the determination of wave resistance for non-steady motion.

REFERENCES:
