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OPTIMUM DESIGN OF STATICALLY INDETERMINATE FRAMES BY MEANS OF NON–LINEAR PROGRAMMING

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SUMMARY

This report presents a computer program and numerical examples comcerning automated optimum design of statically indeterminate structures with specified comfigurations. The analysis of the structure is formulated according to the displacement method. Cross-sectional shapes and sizes are selected as free variables. Weight or cost may be selected as object function.

Computer times required to solve moderately complex problems may easily become excessive; hence, it is mandatory to seek methods to

a) improve the strategy of the search for the optimum
b) reduce the numerical computations required in each step during the search.

Several avenues for further research on these questions are outlined.

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CONTENTS

1. INTRODUCTION

During the first fifteen years of the computer-age great efforts have been made to computerize the analyses of engineering structures. It is, however, only recently that comparable efforts have been made in the more important and difficult field of computer aided design. Recent developments of time sharing systems combined with remote terminals using teletypewriters as well as graphical devices have stimulated tremendously the interest in man-machine modes of operation, by which the user may interact with the computer and thus more successfully use it as a design tool. Already computer capabilities exist which allow the designer to solve structural problems by communication with the computer through graphical displays only |1|. This type of computer usage will become increasingly important in the years ahead. Indeed, it is believed that in the forseeable future most of the complex design problems of practical life can be solved efficiently only through some sort of active interaction between computer and designer.

In spite of this, we are presently also experiencing a rapidly growing interest in fully automated design capabilities that only require the designer to state his problem whereupon the computer does the complete design. No conflict arises in these apparently opposing trends. The future designer will, while interacting with the computer, want to administer a library of programs that automatically perform various parts of the design job. The task of the designer himself should primarily consist of supervision, establishment of priorities between conflicting requirements and other types of decision making.

In this paper some aspects concerning the development of fully automated design capabilities are dealt with. It is understood that such automated design capabilities should be able to derive the best structure, judged by certain prescribed criterion. Hence any feasible solution is not accepted as the final design. A number of algorithms and programs have recently been developed to solve different types of design problems of this kind. The search for the optimum is usually performed by means of some methematical programming technique. References |2-4| present examples of the application of linear, nonlinear and dynamic programming, respectively.

2. PRESENTATION OF THE DESIGN PROBLEM

The following discussion is limitted to designing plane frames and beams. Extensions to three-dimensional frames and grillages pose no extra theoretical difficulties. Cases in which the geometry of the structure in terms of span lengths and member incidences is fixed are studied. Again these limitations can be removed at the expense of complicating the problem slightly. The structure will be analyzed by means of the theory of elasticity.

The aim is to develop automated procedures by which to select the cross-sectional properties of the members of frames such as those shown in Fig. 1. While in Fig. 1a the members are prismatic, Fig. 1b shows an example of non-prismatic members. Here the members may be described by means of the section properties at two or more sections and a prescribed variation between these sections.

Some typical cross-sectional forms are shown in Fig. 2. For the T-sections (Fig. 2b,d) it may be assumed that the area of the plate flange is fixed since it is usually determined from considerations other than the frame action. Hence the sets of free variables for each of these sections may in most cases be selected as indicated in the respective figures. The number of variables for eash member type vary between two (for the glued laminated beam) and four (for the reinforced concrete beam with rectangular cross-section). In many instances practical or economical considerations may require that several members be identical. Hence the number of member types may be considerably less than the total number of members.

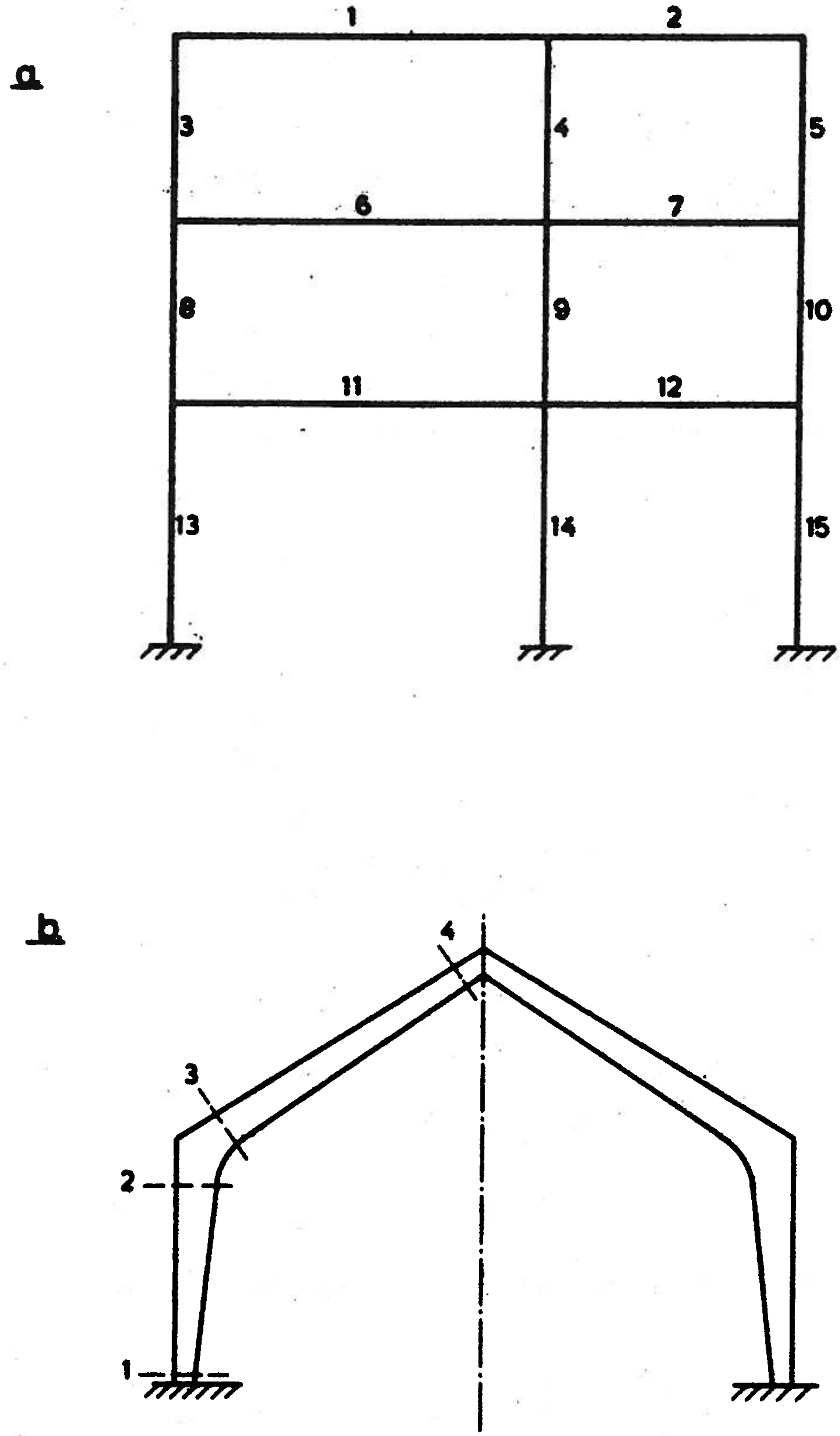
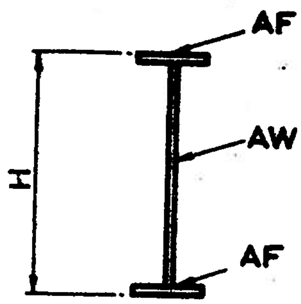
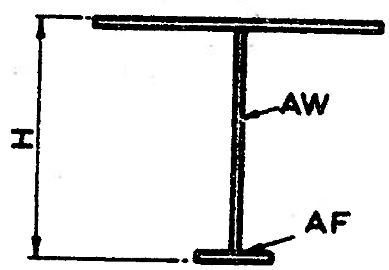


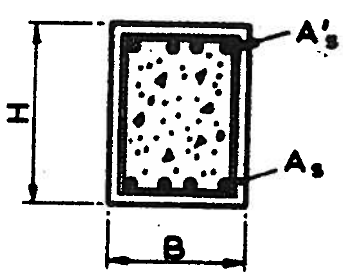
Fig. 1 Typical Frames



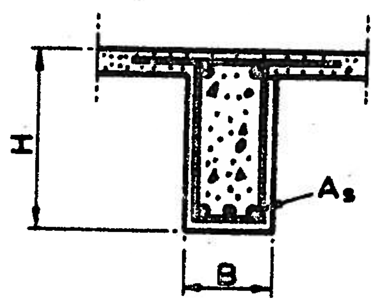
a. Symmetrical Steel Section



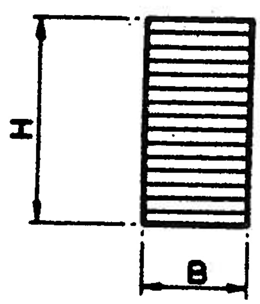
b. Unsymmetrical Steel Section



c. Reinforced Concrete Beam with Rectangular Cross Section.



d. Reinforced Concrete Beam with T-Section.



e. Laminated Timber Beam with Rectangular Cross Section.

Fig. 2 Typical Member Cross Sections

A member type array such as

JK = 3, 4, 1, 2, 1, 3, 4, 1, 2, 1, 3, 4, 1, 2, 1

would express the following requirements (see Fig. 1a):

Members 1, 6 and 11 are of type 3
" 2, 7 and 12 " " " 4
" 4, 9 and 14 " " " 2
" 3, 5, 8, 10, 13 and 15 are of type 1.

The structure under consideration should be designed for a multiple of loading conditions. Stresses and deflections are required to stay within certain specified limits. Buckling of columns should be considered. Secondary requirements such as upper and lower limits on dimensions should also be provided for.

The goal is to derive the optimum design according to some predetermined criterion. In aircraft and space vehicles minimum weight structures are often sought. In civil engineering designs minimum cost of construction is usually the goal. Aesthetical considerations also should receive proper attention, but this is probably easier to achieve by the introduction of proper restrictions rather than by trying to incorporate such items into the object function. In shipbuilding, both weight and cost must be considered simultaneously since excess of weight means reduced efficiency of the ship in terms of load carrying capacity.

Nonlinear programming methods have been used with considerable success to solve various problems of structural optimization |3,5|. A discussion of some methods of searching for the optimum is presented in reference |6|. It should be noticed that the search generally requires a large number of slightly different structures to be studied. The number of different structures that must be investigated in order to find the optimum generally will increase with the square of the number of free variables. For each step in the search a fairly complete analysis of the structure has to be performed to determine (at least approximately) maximum stresses and deflections. As the structure itself increases in complexity the number of variables - and hence the number of steps in the search, as well as the amount of computations required to determine stresses and deflections in each step - increase rapidly, to the point that the available methods easily become impracticable. Therefore, algorithms which reduce as far as possible the number of steps required as well as the amount of computations involved in each step must be sought. These questions are discussed further in Section 3 and 4. The attention is now turned to the possibility of decreasing the number of free variables. It is suggested that this can be rather efficiently accomplished by means of the following two-stage strategy:

Stage 1. After the set of initial design variables has been selected the structure is analyzed for all loading conditions. Now each member type is treated separately and the optimum cross-sectional shape is determined for the force distribution initially determined. Since this optimization only involves two to four variables and no new analysis of the force distribution, it requires only moderate computer time. Restrictions on overall displacements of the frame are disregarded at this stage.

Stage 2. Next all the member cross-sectional shapes are fixed with shapes as determined in stage 1, and there remains only one variable for each member type. This stage then involves a search for the optimum combination of member sizes. As the ratios between the different member sizes change, the internal force distribution in the frame also changes. Thus, this stage involves numerous re-analyses of the structure.

Admittedly the two-stage strategy does not necessarily yield the true optimum for the problems initially started. However, if the initial design is not too remote from the final result, the solution obtained should be close enough to the optimum for practical purposes. An even better result could be obtained if the end result were then used as the initial design in a new cycle involving stages 1 and 2. This approach would probably still be considerably more efficient than a straightforward, simultaneous treatment of all the variables.

3. SEARCH FOR THE OPTIMUM

Mathematically the optimization problem may be formulated as follows |6|:

$$\text{optimize } m = f(y_1, \ldots, y_n) \qquad (1)$$

subject to the conditions:

$$h_j(y_1 \ldots y_n, \sigma_1 \ldots \sigma_t) = 0 \qquad j = 1 \ldots t \qquad (2)$$

$$g_i(y_1 \ldots y_n, \sigma_1 \ldots \sigma_t) \geq 0 \qquad i = 1 \ldots n_c \qquad (3)$$

where

$y_1 \ldots y_n$ = the design variables

$\sigma_1 \ldots \sigma_t$ = the behaviour variables

Eq. (1) presents the criterion, also called the object function. Eqs. (2) and (3) correspondingly represent the analysis equations (equilibrium and compatibility conditions) and the restrictions (stress and deflection limitations etc.), also called the constraints. The inequalities (3) divide the design space into a feasible region, where Eqs. (3) are satisfied, and an unfeasible region.

In practical structural design problems the optimum solution will always be located on the border of the feasible region, i.e. one or several of the stress and deflection limitations etc. will govern the design. This characteristic about the optimum design has been utilized by many investigators who have developed search procedures by which to travel as closely as possible along the boundary between the feasible and unfeasible regions. While this approach is ideally suited for linear programming problems, considerable difficulties arise when the g_i-functions are highly nonlinear, as in most cases of structural design. Rather than trying to solve the above described constrained minimization problem, it has been found advantageous to transform the original problem into that of minimizing the following function:

$$P(y_1 \ldots y_n, \sigma_1 \ldots \sigma_t, r_k) = f(y_1 \ldots y_n) + r_k \sum_{i=1}^{n_c} \frac{1}{g_i(y_1 \ldots y_n, \sigma_1 \ldots \sigma_t)} \qquad (4)$$

for a sequense of decreasing values of the parameter r_k. The second term on the right-hand side of Eq. (4) may be interpreted as a penalty term which tends toward infinity as soon as one or several of the g_i-functions approach zero, i.e., as one approaches the border of the feasible region. Starting the search inside the feasible region, the penalty terms provide the means to stay inside, if a suitable search technique is used.

Until now the equality conditions expressed by Eqs. (2) have been disregarded. Theoretically they could be treated in the same manner as the inequalities by adding the following new penalty term to the right-hand side of Eq. (4)

$$\sum_{j=1}^{t} r_k^{-\frac{1}{2}} h_j^2(y_1 \ldots y_n, \sigma_1 \ldots \sigma_t)$$

By a suitable selection of r_k-values it might be possible to force the h_j-values close enough to zero at the minimum value of the P-function to regard Eqs. (2) as satisfied. In this approach both design variables $(y_1 \ldots y_n)$ and behaviour variables $(\sigma_1 \ldots \sigma_t)$ must be considered as free variables. Eqs. (2) usually will not be satisfied for the trial designs prior to reaching the optimum. The great advantage in this method is that it does not really matter much whether Eqs. (2) are linear or nonlinear. Therefore, this method should be equally well suited to handle cases with nonlinear as well as linear structural behaviour. One of the major disadvantages lies in the great number (n+t) of free variables.

Furthermore, practical experiments seem to indicate that the available algorithms are inefficient when equality constraints are present. If Eqs. (2) are linear it seems to be most efficient to solve these equations directly to obtain

$$\sigma_k = c_k(y_1 \ldots y_n) \qquad k = 1 \ldots t \tag{5}$$

thus eliminating t free variables. This problem is discussed in the next section of the paper.

Once the problem has been transformed into a sequence of unconstrained minimizations, numerous search techniques exist to chose between. Some of these utilize gradient directions, while others do not require the evaluation of gradients. In either case the search is successively performed along a number of different directions S^i, such that

$$y^{i+1} = y^i + \lambda_i S^i \tag{6}$$

where

y = the n-dimensional vector of design variables

S^i = the i-th direction

λ^i = the i-th step length

y^i = the i-th starting value

y^{i+1}=the design vector corresponding to the minimum of the object function along the current direction S^i.

The search along any particular direction S^i is one-dimensional and to locate the minimum point along this direction does not pose any theoretical problem. It is, however, important that the step length λ_i be found with as few trial points as possible. The Golden Section method of search described in |6| recognizes this fact. A combination of the Golden Section search and quadratic interpolation has been applied. Quadratic interpolation is used whenever the search is performed at some distance from the constraints (see Fig. 3a). If a constraint is encountered (see Fig. 3b), the value of the function at this point (P_3) becomes very large. A Golden Section method of search is then more suitable than the quadratic interpolation.

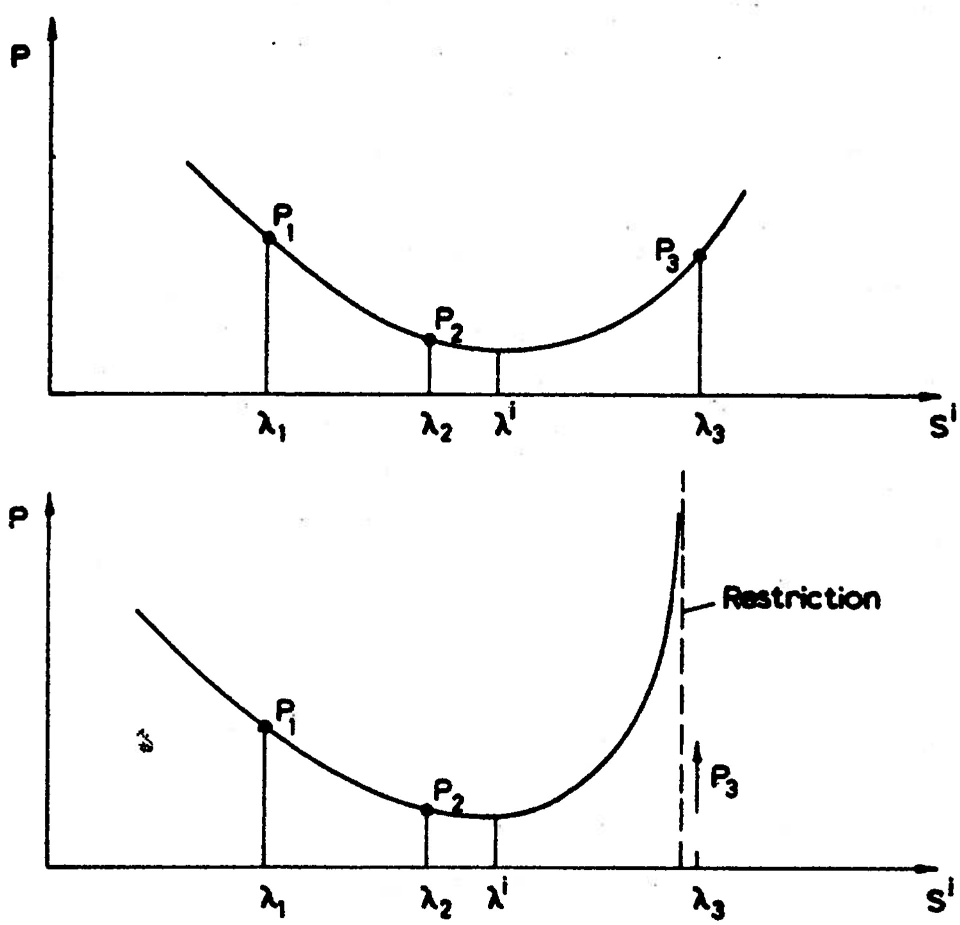


FIG. 3. Search for minimum along a line.

The basic differences between various methods of search lies in the manner by which the directions S^i are created. Kowalik |6| describes in detail two different methods, both of which have been applied with success to problems of structural optimization, i.e.,

a) A method employing conjugate directions which does require the evaluation of gradients (the Variable Metric method).
b) Powells Direct Search method, which does not require the evaluation of gradients.

One of the principal problems common to all of the available methods lies in the great number of evaluations of the P-function (4) which are required. Powell's method in combination with standard methods for finding the minimum along a line may be expected to call for about $10n^2$ function evaluations for each response surface (r_k-value). Here, n is the number of free variables. The number of response surfaces may typically be four to six.

A comparable estimate would suggest that the number of function evaluations using the variable metric method might be approximately 10n to 15n for each response surface. In addition, in this method it is necessary to find the vector of gradients to the response surface at approximately 1.5n points. If the gradients are determined by means of the forward difference method,

$$g_i = \left.\frac{\partial P}{\partial y_i}\right|_{y=\mathbf{y}} \cong \frac{P(\mathbf{y}+\Delta y_i)-P(\mathbf{y})}{\Delta y_i} \qquad i = 1 \ldots n \qquad (7)$$

This requires additionally $1.5n^2$ function evaluations since, to find the gradient vector

$$\mathbf{g} = \{g_1 \ldots g_i \ldots g_n\}\Big|_{y=\mathbf{y}}$$

n function evaluations for points around $\mathbf{y}$ will have to be evaluated.

Each function evaluation requires, strictly speaking, that the analysis equations (4) are established, and then solved again for a new set of design variables. For highly redundant structures this part of the algorithm easily becomes very time consuming and every effort must be made to minimize the time required. Physically the procedure outlined above corresponds to a large number of analyses of slightly modified structures. Quite often the modification involves change of only one of the member types of which the frame consists. This is true when gradients are evaluated, and also for a considerable proportion of the steps involved in Powell's method (directions S^1 and S^2 in Fig. 4). But a number of steps are also taken along other directions, which usually involve changes in several or all of the variables simultaneously. In the next section different approaches to the modification problem are discussed.

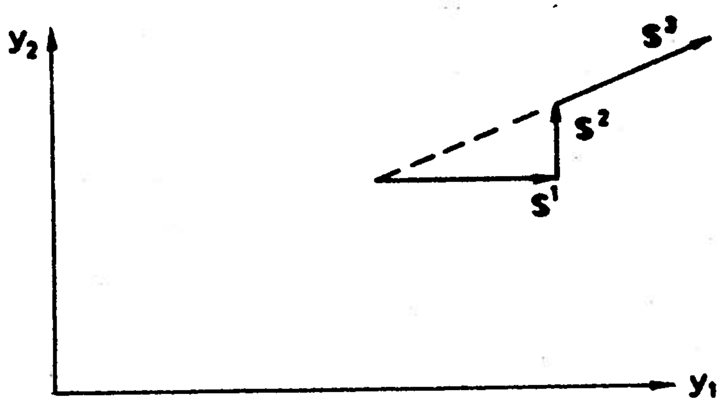


FIG. 4. Search directions.

4. MODIFICATION TECHNIQUES

4.1 Introduction

Assume that the frame under consideration has been analyzed for a basic set of dimensions using the displacement method, such that the inverse of the stiffness matrix is known, and

$$r = K^{-1}R \tag{8}$$

where

r = an m×ℓ matrix of nodal point displacements
K = an m×m stiffness matrix
R = an m×ℓ matrix of nodal point loads
m = number of degrees of freedom
ℓ = number of loading cases

The cross-sections of one or more of the members in the frame are next changed by certain amounts such that the new stiffness matrix for the structure is

$$K_m = K + dK \tag{9}$$

Since inversion of matrices is rather time consuming, the goal is to find an expression for K_m^{-1} without inversion. Formally the matrix dK may be written as follows (see Appendix I)

$$dK = \sum_{i=1}^{M} a_i^T k_{in} a_i \tag{10}$$

where

k_{in} = change in the stiffness matrix of member (i)
a_i = a matrix related to the geometry of the structure
M = total number of members in the frame

In the following we shall present four different approaches to the solution of the modification problem.

4.2 Method 1 - Mathematical Approximation

One of the most straightforward methods is to make the following series expansion

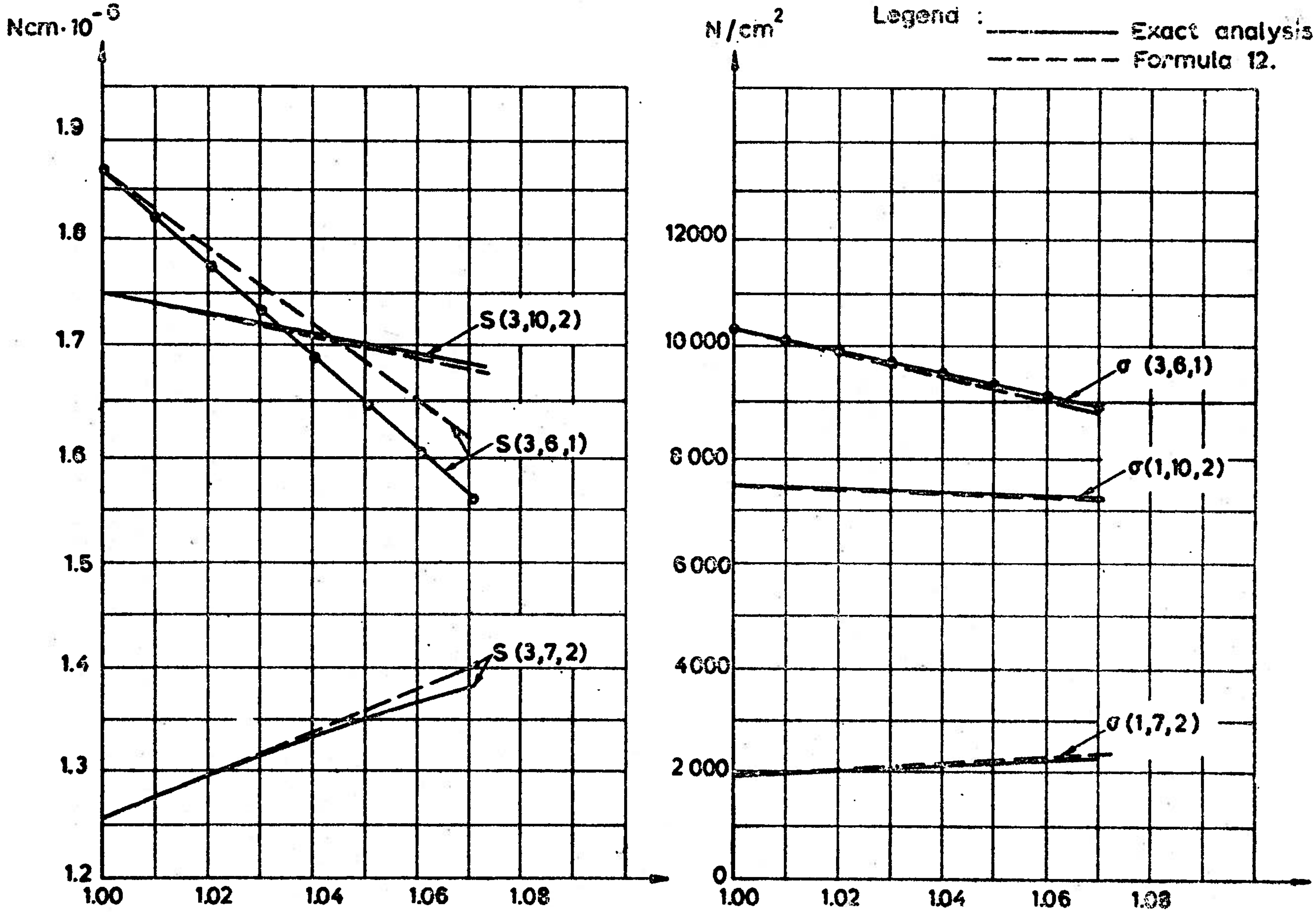
$$(K + dK)^{-1} = K^{-1} + K^{-1} \sum_{j=1}^{\infty} (-dKK^{-1})^j \tag{11}$$

and retain only two terms on the right-hand side of the equation,

$$K_m^{-1} = (K + dK)^{-1} \cong K^{-1} - K^{-1}dKK^{-1} \tag{12}$$

Figure 5 compares the changes in some typical member end forces and stresses of the frame shown in Fig. 1a, as found by means of Eq. (12) and the exact values. In this example the design variables AF, AW and H (see Fig. 2) of members 1, 6 and 11 were increased simultaneously, as indicated along the abscissae in the figure, while the other members remained unchanged.

The frame was loaded by two combinations of evenly distributed vertical loads on the floors and horizontal nodal point loads. For a five percent increase in the member properties mentioned above, corresponding to approximately 16 percent increase in the moments of inertia, the maximum



Increase in AF, AW and H for Members No. 1,6 and 11

Notation : S(m,n,q) = Member End Force No. m for Member No. n in Loading Case q.

FIG. 5 Comparison of results obatined by approximate and exact analyses of modification effects error in stresses calculated on the basis of results obtained using Eq. (12) was less than 2.5 percent. For the points with high stress levels the errors were considerably smaller.

A serious objection toward Eq. (12) is that the evaluation of the right-hand side in its present form is as time consuming as the inversion. However, the computational work can be considerably reduced if only one or a few of the members of the frame have been modified. Let us assume for a moment that only member (i) has been changed, and that this member is connected to nodes S (Start) and E (End). In this case dK may be written as follows:

$$dK_i = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{c|c|c|c} & & & \\ \hline & b_i^{SS} & & b_i^{SE} \\ \hline & (b_i^{SE})^T & & b_i^{EE} \\ & & & \end{array}\right]\begin{array}{l} \\ r_1 \\ r_2 \\ \\ \end{array} \qquad (13)$$

$$\qquad\qquad\qquad\qquad r_1 \qquad\qquad\qquad r_2$$

where the b_i-matrices are of dimension (k×k)

k = number of degrees of freedom for each node

r_1 identifies the first position in the displacement vector (r) of terms associated with the displacement of nodal point S, and

r_2 correspondingly refers to the first position of the terms related to nodal point E.

It may be shown that in this case

$$K^{-1}dK_iK^{-1} = B_1^Tb_i^{SS}B_1 + 2B_1^Tb_i^{SE}B_2 + B_2^Tb_i^{EE}B_2 \qquad (14)$$

where

B_1 is the submatrix of K^{-1} consisting of the same rows that b_i^{SS} covers in dK_i (dimension k×m)

B_2 is the submatrix of K^{-1} consisting of the same rows that b_i^{EE} covers in dK_i.

While the direct evaluation of the left-hand side of Eq.(14) requires m^3 multiplications, use of the expression on the right-hand side reduces this number to 3(k+m)km. Additional reduction is achieved in both cases by taking advantage of the resulting product being a symmetric matrix.

If more than one member is changed, Formula (14) must be used repeatedly, since

$$K^{-1}dKK^{-1} = K^{-1}dK_1K^{-1} + K^{-1}dK_2K^{-1} + \ldots \qquad (15)$$

where $$dK = dK_1 + dK_2 + \ldots. \qquad (16)$$

4.3 Method 2 - The Initial Strain Technique

The initial strain technique has been applied extensively to problems involving modifications of structures |7|. Let us assume that element (i) is modified such that its stiffness matrix is changed from k_i to k_i+k_{in}. In order to study the effect of this change, a fictitious initial strain condition is applied to this element, and the magnitudes of the strains are adjusted such that the effect on the displacements of the structure are the same as those of the structural modification. In Appendix I the resulting inverse of the stiffness matrix for the modified structure is shown to take the following form:

$$K_m^{-1} = K^{-1} - K^{-1}dKK^{-1} \qquad (17)$$

where

$$dK = a_i^T q_{in} a_i \qquad (18)$$

and

$$q_{in} = (I + k_{in}a_iK^{-1}a_i^T)^{-1}k_{in} \qquad (19)$$

I is a 2k×2k unit matrix.

Note that Eqs. (17-18) are exact. Also, it is interesting to note that if the second term in the paranthesis of Eq. (19) is neglected, the same approximation is obtained as that presented in Section 4.2- These results are discussed further at the end of the next section.

4.4 Method 3 - The Parallel Element Technique

If the stiffness of member (i) of a frame is increased by a certain amount ($\mathbf{k}_{in}$), the effect of this change on the rest of the structure is the same as if a new member of stiffness $\mathbf{k}_{in}$ were inserted parallel to the original member (i). Algebraically the sum of the stiffnesses of the original and the new element produces the desired stiffness of the modified element. In Appendix I this approach is shown to yield exactly the same equations (17-19) as the initial strain technique.

Hence, the approximation of Expression (12) is exact as long as the modified stiffness matrix $\mathbf{q}_{in}$ (Eq. (19)) is used instead of $\mathbf{k}_{in}$ in formula (10). To find $\mathbf{q}_{in}$ it is necessary to invert a matrix of dimension (2k×2k) but this is a relatively easy task.

If more than one member has changed the total modification must be treated in several steps, each involving the modification of one element only.

4.5 Method 4 - Gauss-Seidel Iterations

Since the nodal point displacements for the almost similar unmodified structure are known, an iterative solution of the equations of equilibrium may be quite efficient. The stiffness matrix of the modified structure is written in the following manner

$$\mathbf{K}_m = \mathbf{L} + \mathbf{D} + \mathbf{U} \qquad (20)$$

where

> $\mathbf{D}$ is the diagonal matrix $\lceil d_1 \dots d_i \dots d_n \rfloor$ and
> $\mathbf{L}$ and $\mathbf{U}$ are the lower and upper matrices, each with zero terms along the diagonals.

The nodal point deflections (r_q) may then be determined by means of the following scheme of iterations

$$\mathbf{D}r_q^j = \mathbf{R} - \mathbf{U}r_q^j - \mathbf{L}r_q^{j-1} \qquad q = 1,2 \dots m \qquad (21)$$

where j is the current iteration cycle and r_q^o is the vector of nodal point displacements for the unmodified structure. This method has the following advantages when compared with the ones presented earlier.

1. The method is equally well suited whether one or more members have been modified.
2. Although the method yields approximate results, any desired degree of accuracy can be obtained, and inaccuracies from previous modification analyses are not carried along, as when the method outlined in Section 4.2 is used.

4.6 Choice Between the Methods

Numerical experiments will be necessary in order to judge the relative efficiencies of the methods presented above. However, the following results can reasonably be expected.

If a gradient method os search is used, the evaluation of gradients involves a series of modifications, each representing a change of one member type only. (This may mean that more than one member is changed.) The change Δy_i (see Eq. (7)) in any one member may be arbitrarily small, say one percent of the original value. Method 1 should be ideally suited to evaluate the effect of such changes.

When the gradients have been found, the search will usually follow a direction which involves a simultaneous change in many or all of the members. Under such conditions Method 4 is probably suitable.

Therefore, a combination of Methods 1 and 4 may possibly be efficient if gradient methods are used.

When Powell's method of direct search is used, the same combination of methods will probably also prove satisfactory. It may, however, be necessary to make complete reanalyses at certain intervals to correct for the accummulating inaccuracies of Method 1 for large series of steps along the coordinate axes. Method 2 or 3 may become superior to Method 1 in this case because it would give exact results, and hence, should not require complete reanalyses.

It should be noted, however, that since the search for the optimum solution is constantly performed at some distance from the constraint surfaces, high accuracy of analysis is not a critical requirement when Powell's method is used. The requirements with respect to accuracy are believed to be more severe for gradient methods.

5. PRESENTATION OF COMPUTER PROGRAM AND NUMERICAL EXAMPLE

A computer program has been written that automatically performs the two-stage optimization described in the Section 3. This pilot program applies Powell's Direct Search method. The analysis of the redundant structure is carried out according to the matrix formulation of the displacement method. In the second stage of optimization a complete new analysis of the structure was only made whenever the relative change in the magnitude of the vector of variables,

$$|\mathbf{y}| = \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}} \tag{22}$$

was over 4 percent since the previous complete analysis. Here,

y_i = design variable i

n = number of free variables in stage 2.

If this relative change was less than one percent no new determination of force distribution was undertaken. For intermediate values the effect of member changes was determined by means of the approximate relationship given by Eq. (12).

In its present form the program assumes prismatic members and checks member end stresses at all of the nodes. For each member the stresses are also checked at the point of zero shear, or at midspan if there is no point of zero shear. In order to make the program useful for practical purposes some modifications must be made such that member end stresses are calculated at the critical sections rather than at their theoretical end points at the nodes. For frames such as the one presented in the following numerical example, additional stiffnesses due to brackets and overlapping of adjoining members should also be incorporated.

Fig. 6a shows an outline of a typical transverse frame in a tanker of approximately 150,000 tons displacement. This frame was designed by means of the program with the following assumptions.

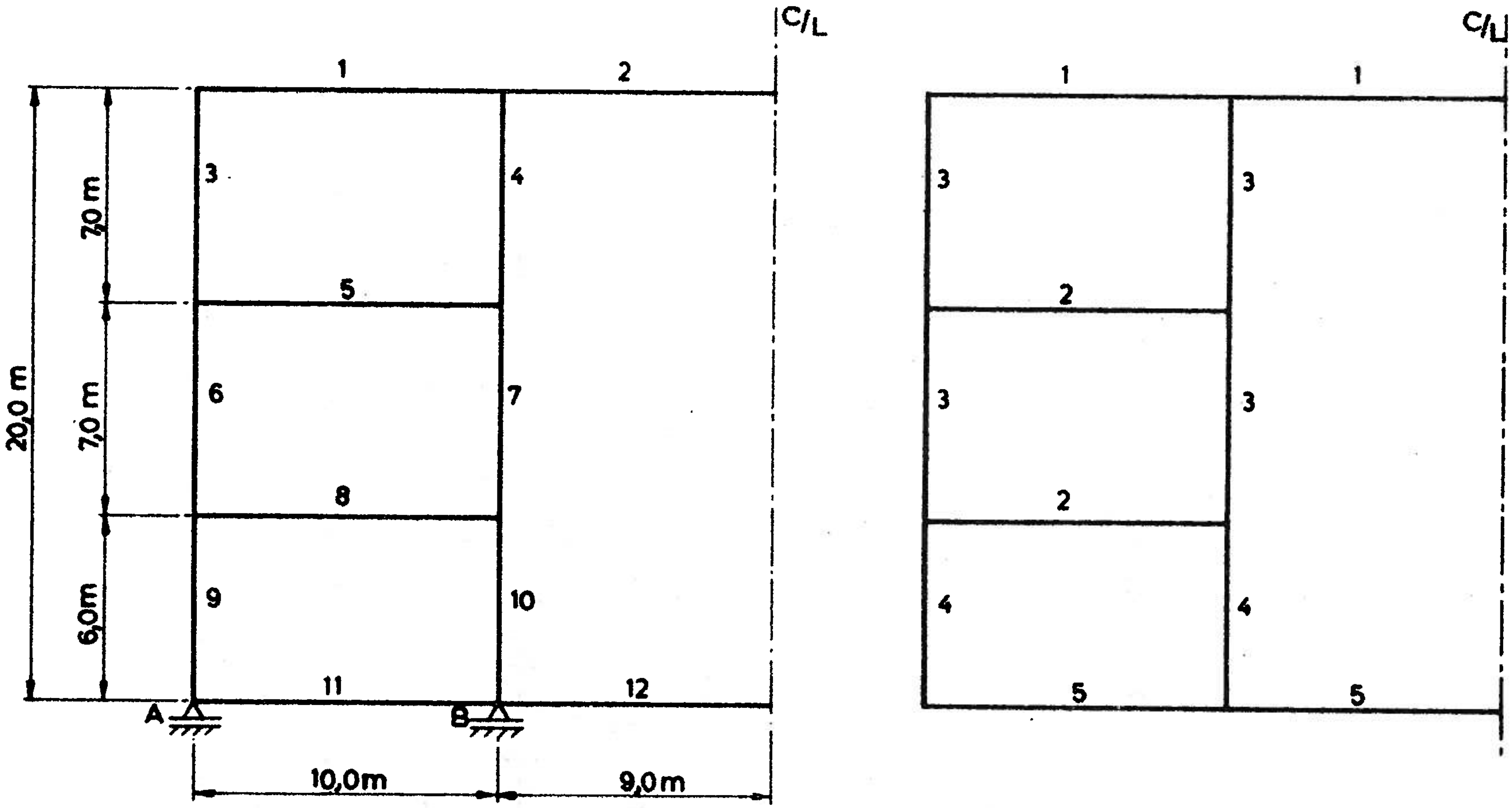
1. Loading conditions. Four separate conditions as shown in Figs. 7a-d.
2. Member types. Five different member types as shown in Fig. 6b. Members of type 2 have symmetrical I-sections of the type shown in Fig. 2a, while all of the other members have cross-sectional shapes such as the one shown in Fig. 2b. Plate flange areas are given as input.
3. Support conditions. The frame as well as the loadings are symmetrical and the frame is assumed to be supported in the vertical direction on the ship sides (A) and the longitudinal bulkheads (B), see Fig. 6a.
4. Allowable stresses. To compensate approximately for the fact that member end stresses were computed at the nodes rather than at some distance away from the nodes, the allowable stresses were selected rather high, viz.:

   1800 kp/cm^2 in tension and compression
   1200 kp/cm^2 in shear
5. Object function. In the present case weight minimization was sought. For all members with plate flanges, the weight of stiffeners on the webs was accounted by the following formula

$$W_w = \gamma A_w \qquad \text{for } H/t_w \leq 50$$

$$W_w = \gamma A_w\{1 + 10^{-4}(H/t_w - 50)^2\} \qquad \text{for } H/t_w > 50 \qquad (23)$$

where



a. Member Numbers and Span Lengths

b. Members Types

FIG. 6 Transverse Frame in Tanker.

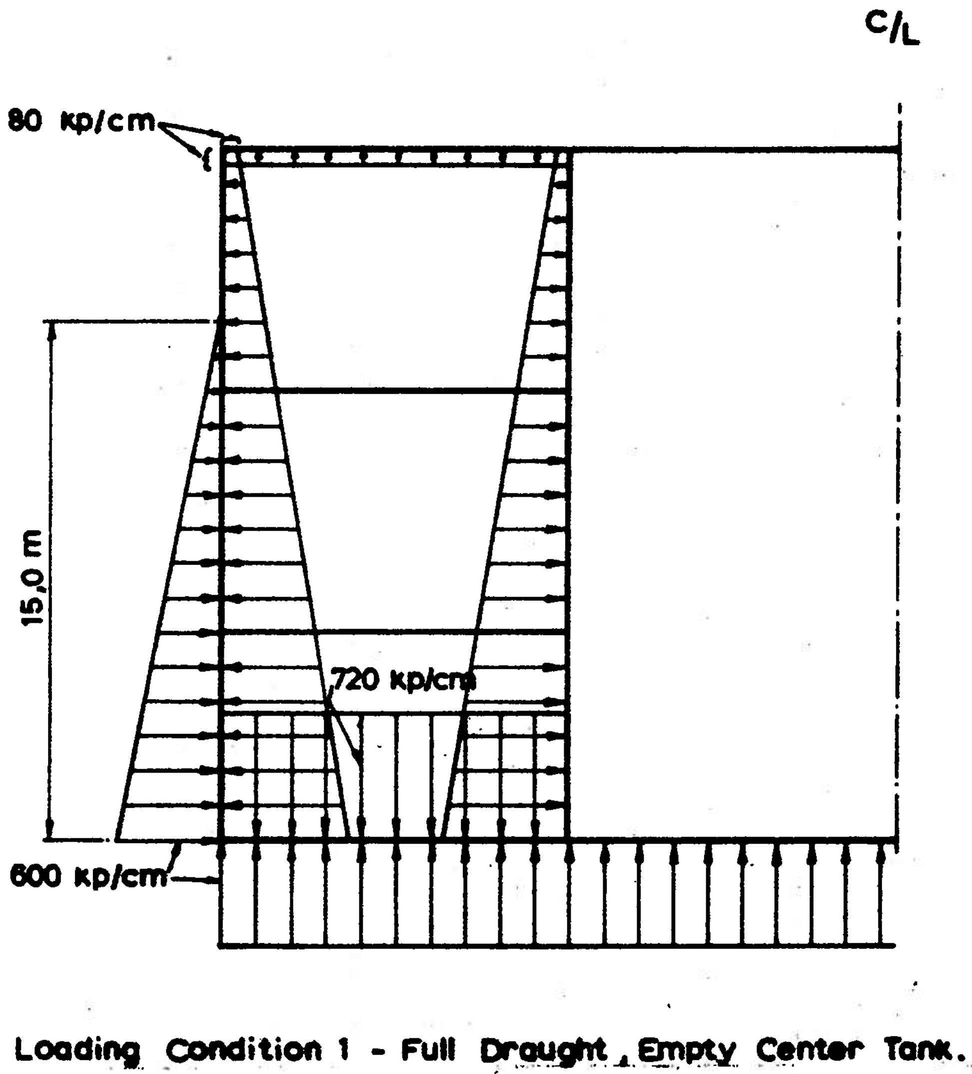
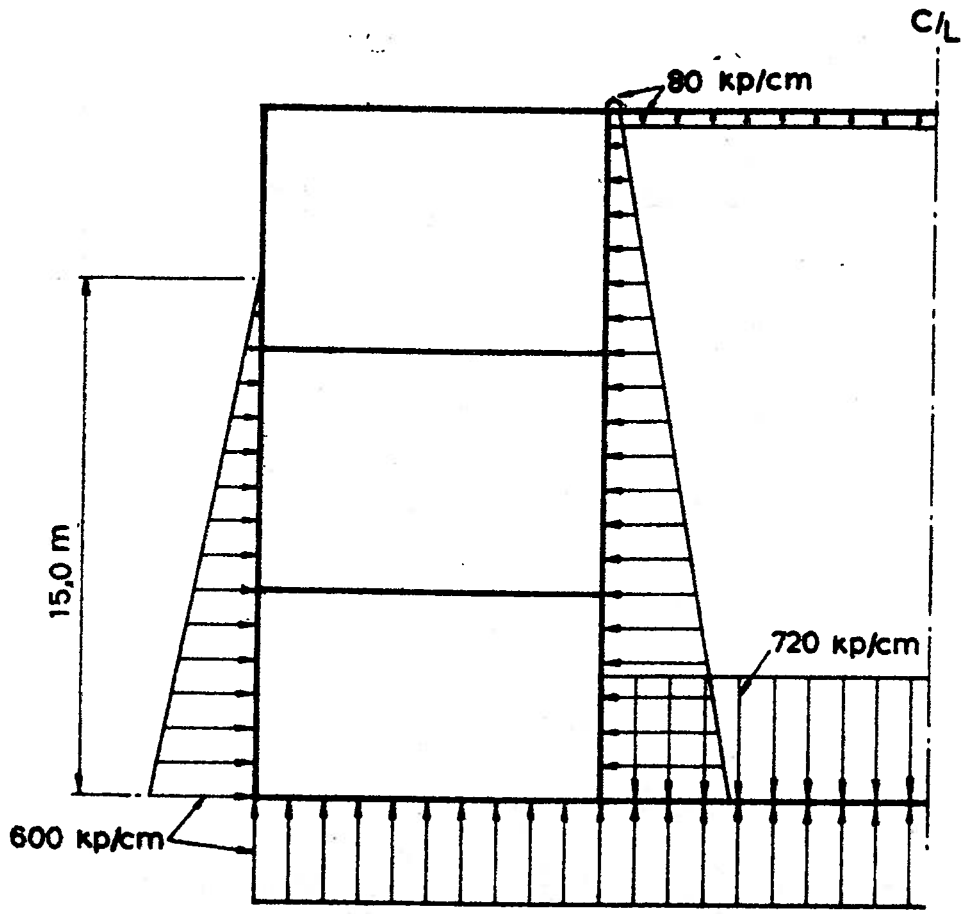
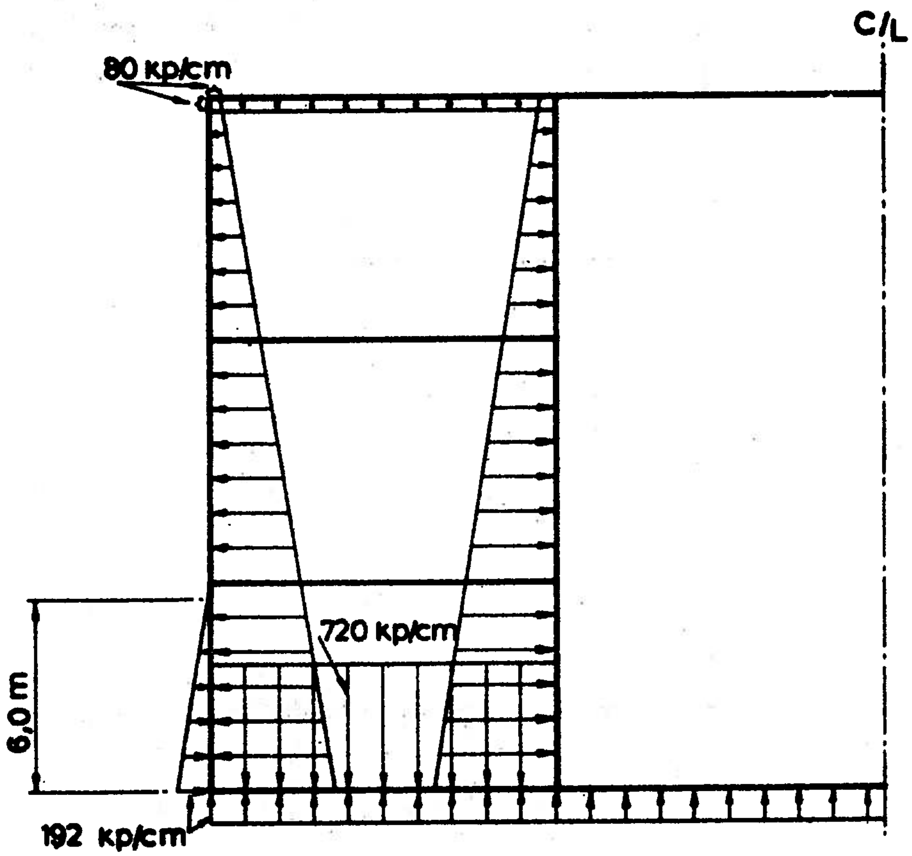


FIG. 7a Typical Frame Loading.



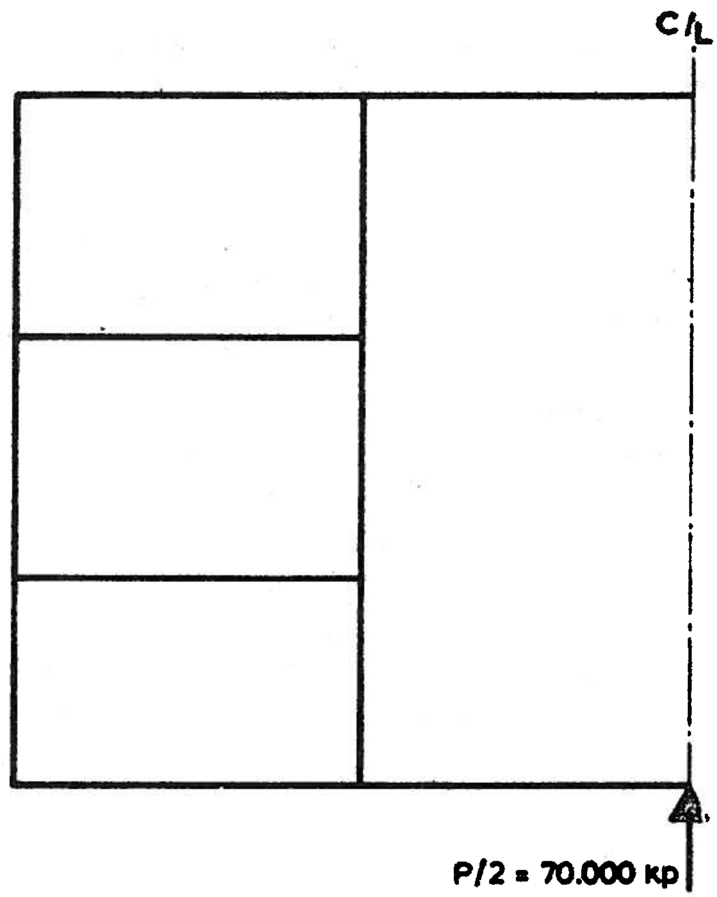
Loading Condition 2 - Full Draught , Empty Side Tanks.

FIG. 7b Typical Frame Loading.



Loading Condition 3 - Light Draught, Full Side Tanks

FIG. 7c Typical Frame Loading.



Loading Condition 4 - Docking with Empty Tanks.

FIG. 7d Typical Frame Loading.

W_w = weight of web per unit length

γ = specific weight of steel

t_w = web thickness

These formulae are only meant to incorporate approximately the trènd of increased volume of stiffeners with increased slenderness of the webs. For beams with a symmetrical cross-section an upper limit on H/t_w was prescribed as part of the input.

A summary of input dimensions as well as the obtained solution is presented in Table 1. Fig. 8 presents an outline of the resulting design and also shows the governing normal stresses. In loading condition 1 the maximum shearing stress in member 10 was 1170 kp/cm². When the search procedure was discontinued after 10 minutes of computing time, the optimum was not completely reached, as can be seen from the fact that the maximum stresses were still lower than the allowable values for several member types. The output from the computer showed that the initial design presented as input did not represent a feasible solution. All of the dimensions of members of type 4 had to be increased by 4.3 percent in order to enter the feasible region of the design space. A corresponding increase of 16.3 percent was required for members of type 5.

6. CRITICAL EVALUATION

The numerical example just presented demonstrates that it is possible to develop computer programs for completely automated design of medium sized frame structures. Multiple loading conditions are treated simultaneously with little extra effort. In the present example deflection and buckling limitations were not incorporated, but this could have been done easily and might have yielded interesting results since a fully stressed design would not necessarily be the optimum in that case.

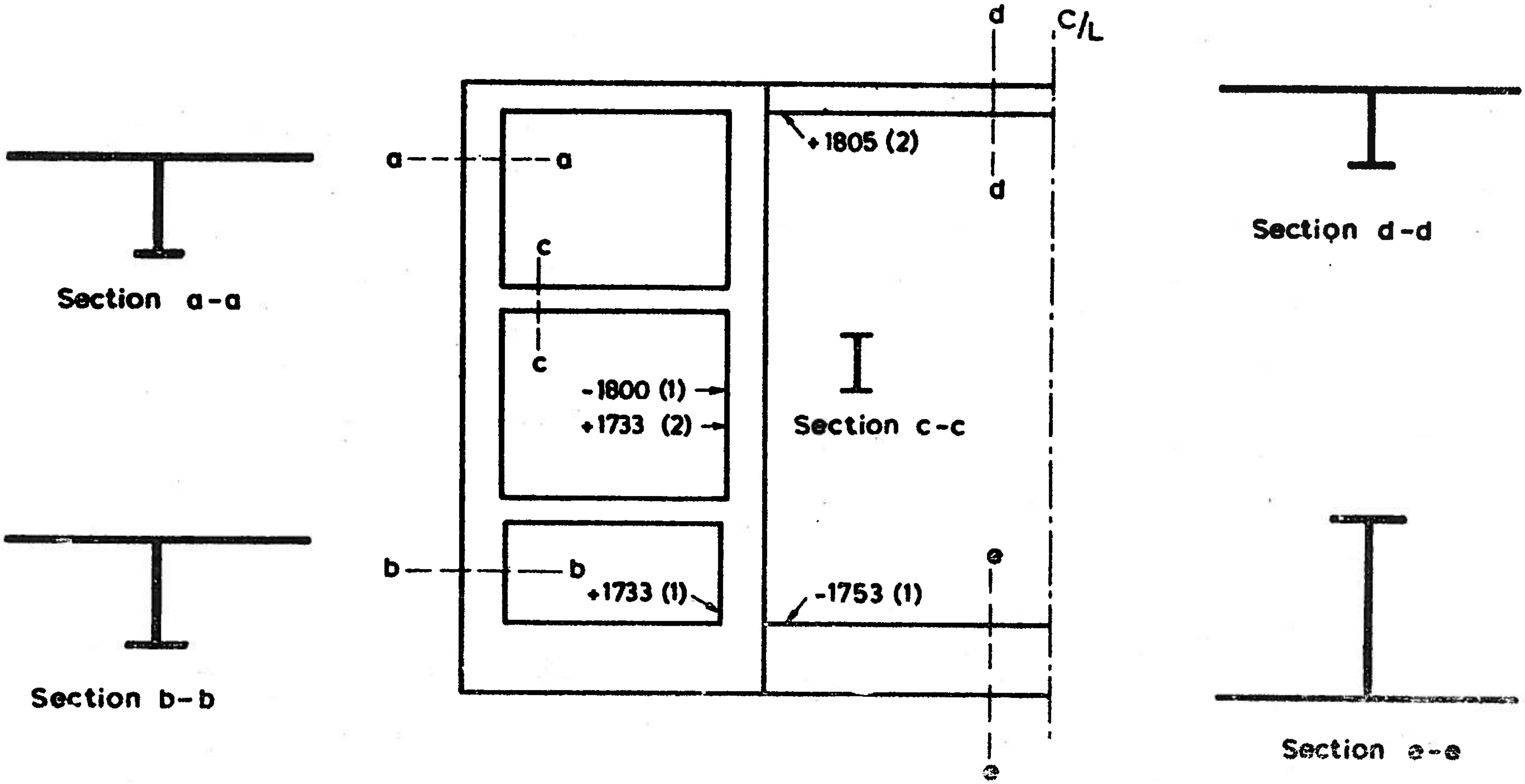


FIG. 8 Dasign Obtained by Computer.

TABLE 1. Summary of Results of Tanker Frame Optimization

Member	Plate Flange Area cm^2	Initial design: Flange Area cm^2	Initial design: Web Area cm^2	Initial design: Web Height cm	Initial design: Total Area cm^2	Optimum design: Flange Area cm^2	Optimum design: Web Area cm^2	Optimum design: Web Height cm	Optimum design: Total Area cm^2	Percentage Change in Total Area
1	700	40	360	180	1100	109	88.7	88.3	998	- 9.2
2	700	40	360	180	1100	109	88.7	88.3	998	- 9.2
3	600	40	360	180	1000	104	181	123	885	-11.5
4	500	40	360	180	900	104	181	123	785	-12.7
5	0	40	180	80	260	85.7	104	70.5	275	+ 5.7
6	600	40	360	180	1000	104	181	123	885	-11.5
7	500	40	360	180	900	104	181	123	785	-12.7
8	0	40	180	80	260	85.7	104	70.5	275	+ 5.7
9 x)	600	70	600	250	1270	230	299	139	1129	- 3.2
10 x)	500	70	600	250	1170	230	299	139	1029	- 3.5
11 xx)	800	70	700	280	1570	189	689	235	1678	+ 6.9
12 xx)	800	70	700	280	1570	189	689	235	1678	+ 6.9

x) Input values had to be increased by 4.7 percent in order to become acceptable.

xx) Input values had to be increased by 16.3 percent in order to become acceptable.

Weight of initial design 0.972

Weight of optimum design 0.928

It may be questioned whether in the example presented the search really proceeded toward the truly optimum solution judging by the established criterion. The consistency of the results could have been studied by means of several parallel runs using different initial designs. This was not done for the present example, but for several smaller design problems of a similar type such reruns showed that although the resulting designs might differ slightly, the final values of the object function varied little, provided that the number of steps in the different loops of the search pattern and the convergency criteria were properly chosen.

It is estimated that full convergency of the example presented with the selected convergency criteria would have required less than 15 minutes of computing time on the IBM 360/67 computer. Although this time would not be too bad, it shows that for problems which are considerably larger the computer costs could easily become excessive. However, there are many promising possibilities of increasing the efficiency of the presently available program. These include

a) Improving the search techniques by means of one or several of the following means:
   1) Use of some type of gradient directions
   2) Selection of optimum combinations of step lengths, maximum number of steps and other convergency criteria
   3) Intelligent usage of extrapolation techniques.

b) Improving the efficiency of the techniques of analysis and reanalysis of the redundant structures. Several alternative modification techniques are available, and their relative advantages should be studied. Optimum use should also be made of available information about sparseness and bandedness of the matrices in order to reduce the number of arithmetic operations.

Through further work along these lines the computing time can be reduced considerably, thus making it feasible to solve design problems which are correspondingly larger than that treated in the numerical example presented here.

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APPENDIX I

Matrix Formulation of Two-Dimensional Frame Analysis

A. The Beam Element

The frame is considered as a system of beam elements which are connected at the ends. In the present derivations straight beam elements with uniform cross-sections are considered. Shear deflections are disregarded, but could easily have been incorporated by a slight expansion of the program.

For each beam element a local cartesian coordinate system is selected, the x-axis coinciding with the neutral axis (or any other desired longitudinal axis) of the beam. The plane of the frame coincides with the x-y plane.

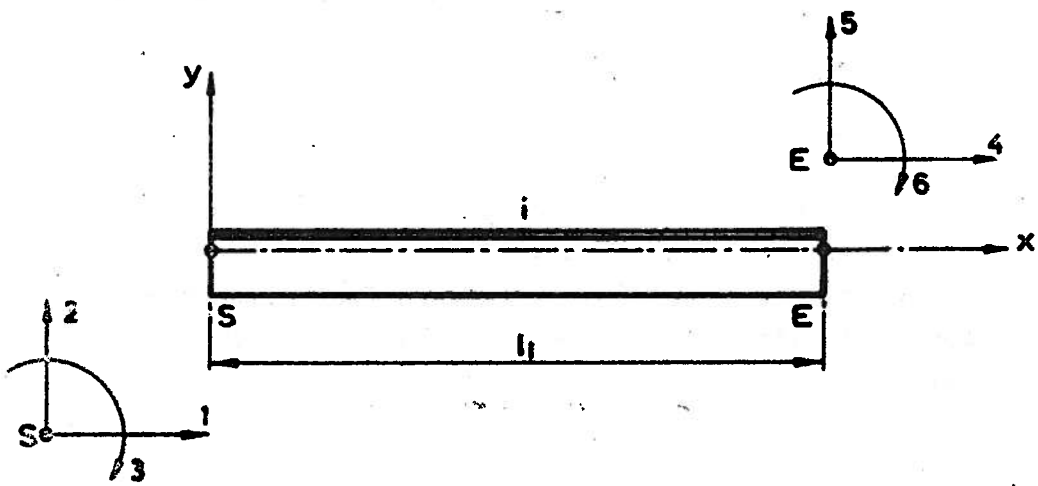


FIG. I.1 Beam Element - Definition of Positive Directions (Three Degrees of Freedom)

Fig. I.1 shows beam element i. This element is connected with other members of the frame at the nodes S (Start) and E (End). The end displacements of the beam are described by means of the vector

$$v_i = \{v_1, v_2, v_3, v_4, v_5, v_6\}_i \qquad (I.1)$$

where $v_1 - v_3$ are related to node S and $v_4 - v_6$ are related to node E as shown in Fig. I.1. The corresponding end force vector is

$$S_i = \{S_1, S_2, S_3, S_4, S_5, S_6\} \qquad (I.2)$$

and the stiffness matrix (k_i) relating end forces to end displacements is

$$k_i = \begin{vmatrix} EA/\ell & 0 & 0 & -EA/\ell & 0 & 0 \\ 0 & 12EI/\ell^3 & -6EI/\ell^2 & 0 & -12EI/\ell^3 & -6EI/\ell^2 \\ 0 & -6EI/\ell^2 & 4EI/\ell & 0 & 6EI/\ell^2 & 2EI/\ell \\ -EA/\ell & 0 & 0 & EA/\ell & 0 & 0 \\ 0 & -12EI/\ell^3 & 6EI/\ell^2 & 0 & 12EI/\ell^3 & 6EI/\ell^2 \\ 0 & -6EI/\ell^2 & 2EI/\ell & 0 & 6EI/\ell & 4EI/\ell \end{vmatrix}_i \qquad (I.3)$$

such that

$$S_i = k_i v_i \qquad (I.4)$$

When a continuous beam on unyielding supports is analysed, only end rotations need to be considered, and the problem reduces to one of a single degree of freedom at each node (as compared with three degrees of freedom in the case just presented).

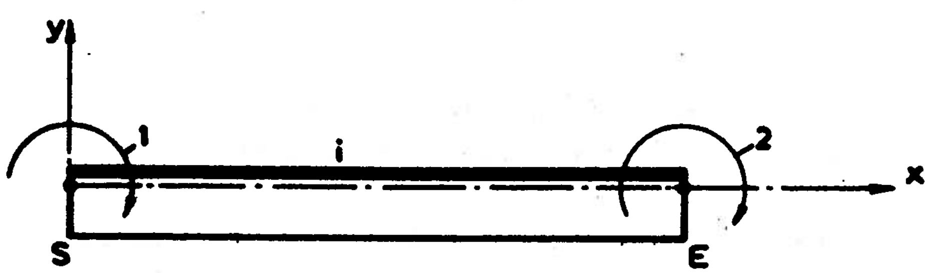


FIG. I.2 Beam Element - Definition of Positive Directions (One Degree of Freedom)

In the case of one degree of freedom the end displacement and the end force vectors are selected, as shown in Fig. I.2, such that

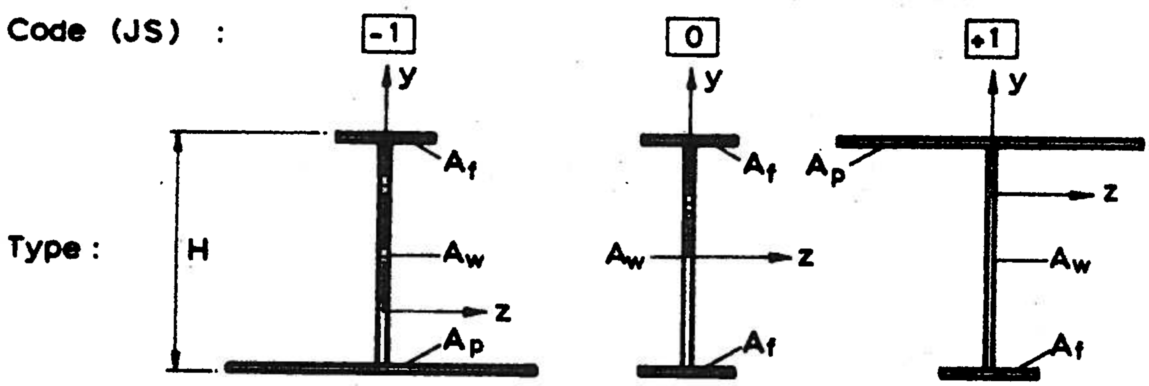
$$\mathbf{v}_i = \{v_1, v_2\}_i \tag{I.1a}$$

$$\mathbf{S}_i = \{S_1, S_2\}_i \tag{I.2a}$$

and the stiffness matrix reduces to

$$\mathbf{k}_i = \begin{vmatrix} 4EI/\ell & 2EI/\ell \\ 2EI/\ell & 4EI/\ell \end{vmatrix} \tag{I.3a}$$

In the present study we have considered three different types of cross-sections, as shown in Fig. I.3.



Notations: H = Beam height

A_w = Web area

A_f = Area of regular flange

A_p = Area of plate flange

FIG. I.3 Types of Cross-Sections and Notations.

The external load on the beam element may be one of the types shown in Table I.1. The fixed end member forces acting on beam (i) are denoted

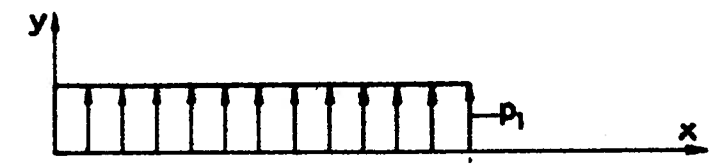
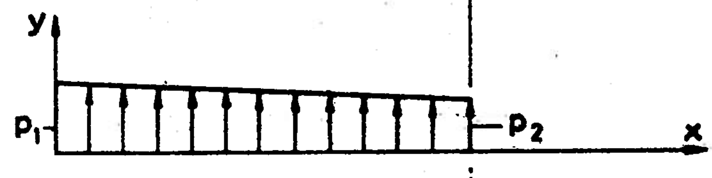
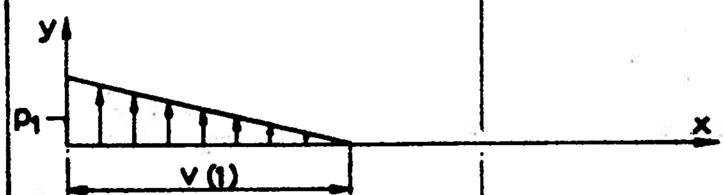
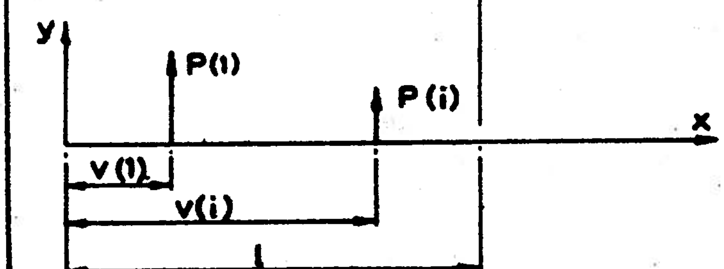
$$\mathbf{S}_i^F = \{S_1^F, S_2^F, S_3^F, S_4^F, S_5^F, S_6^F\} \tag{I.5}$$

in the case with three degrees of freedom, and correspondingly

$$S_i^F = \{S_1^F,\ S_2^F\} \qquad \text{(I.5a)}$$

in the case of one degree of freedom. In equation (I.5a) the vector S_i^F only contains the fixed-end moments.

TABLE I.1 Types of Beam Loadings Considered

Type of load	Code (IND)	NOL[x)]	Input
	-1	1	p_1
	0	2	p_1, p_2
	0	1	p_1, v(1)
	+1	max.4	P(1)..P(i) v(1)..v(i)

x) NOL = No. of load values that must be specified.

B. Frame Topology and Nodal Point Displacements

In the case with three degrees of freedom the displacements of an arbitrary nodal point (j) may be described by the nodal point displacement vector

$$r_j = \{r_1, r_2, r_3\}_j \qquad (I.6)$$

where r_1, r_2 and r_3 are displacements in the global coordinate directions 1, 2 and 3, respectively, see Fig. I.4.

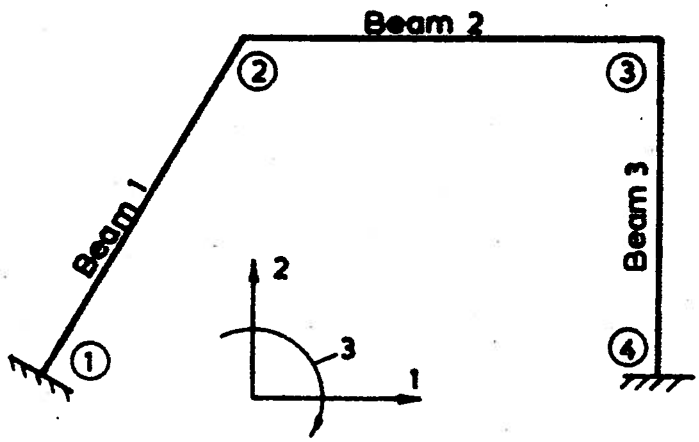


FIG. I.4 Simple Frame

Let

$$r = \{r_1, r_2 \ldots r_j \ldots r_{JO}\} \qquad (I.7)$$

be the vector comprising all nodal point displacements (JO = no. of joints), and

$$R = \{R_1, R_2 \ldots R_j \ldots R_{jo}\} \qquad (I.8)$$

correspondingly denote all nodal point loads. Further

$$v = \{v_1, v_2, v_3 \ldots v_M\} \qquad (I.9)$$

$$S = \{S_1, S_2, S_3 \ldots S_M\} \qquad (I.10)$$

be the collection of all member end displacements and member end forces (M = no. of members).

The topology of the frame is uniquely determined by means of the matrix $\mathbf{a}$, which relates member end displacements to joint displacements

$$\mathbf{v} = \mathbf{a}\mathbf{r} \tag{I.11}$$

This equation may also be written as follows

$$\mathbf{v} = \begin{vmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \cdot \\ \cdot \\ \mathbf{v}_M \end{vmatrix} = \begin{vmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \cdot \\ \cdot \\ \mathbf{a}_M \end{vmatrix} \mathbf{r} \tag{I.11a}$$

C. Derivation of Stiffness Matrix for the Structure

Applying the principle of virtual work the following relationship between nodal point loads and member end forces is found

$$\mathbf{R} = \mathbf{a}^T\mathbf{S} \tag{I.12}$$

From Eqs. (I.4) and (I.9-10)

$$\mathbf{S} = \lceil \mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_M \rfloor \mathbf{v} \tag{I.13}$$

or

$$\mathbf{S} = \mathbf{k}\mathbf{v} \tag{I.13a}$$

where

$$\mathbf{k} = \lceil \mathbf{k}_1, \mathbf{k}_2 \ldots \mathbf{k}_M \rfloor \tag{I.14}$$

is a diagonal matrix.

Eqs. (I.11), (I.12) and (I.13) now yield

$$\mathbf{R} = \mathbf{a}^T\mathbf{k}\mathbf{a}\mathbf{r} = \mathbf{K}\mathbf{r} \tag{I.15}$$

where

$$\mathbf{K} = \mathbf{a}^T\mathbf{k}\mathbf{a} \tag{I.16}$$

is the stiffness matrix for the structure. Eq. (I.16) may also be written as

$$\mathbf{K} = \sum_{i=1}^{M} \mathbf{a}_i^T\mathbf{k}_i\mathbf{a}_i \tag{I.16a}$$

A substantial saving in computations can be achieved by partitioning the matrix a_i as shown below. Consider beam element i in Fig. I.5. The member end displacements may be expressed by the following equation (see Eq.(I.11a))

$$v_i = \begin{vmatrix} a_{i1} & a_{i2} & \cdots & a_{iS} & \cdots & a_{iE} & \cdots \end{vmatrix} \begin{vmatrix} r_1 \\ r_2 \\ \cdot \\ r_S \\ \cdot \\ \cdot \\ r_E \end{vmatrix} \qquad \text{(I.17)}$$

where

S = node at the start of member i
E = node at the end of member i

With k degrees of freedom at each node, the submatrices a_{ij} have the dimension (2k, k), while r_j denoting the displacement of node j is of dimension (k, ℓ), where ℓ is the number of loading cases considered.

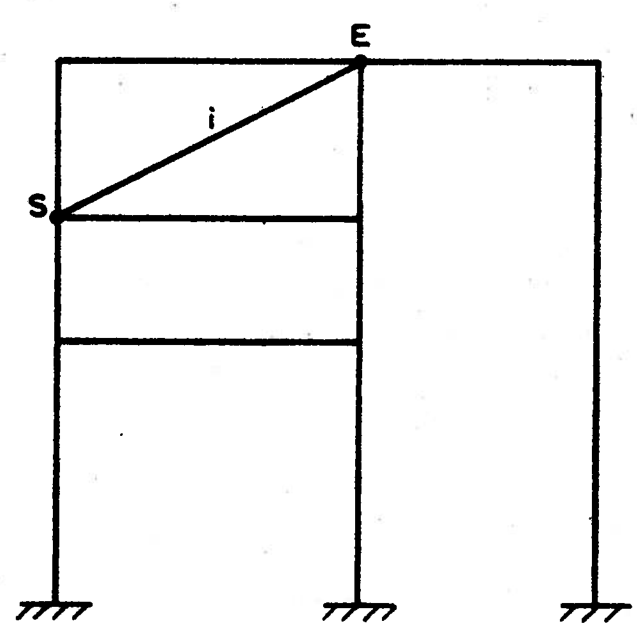


FIG. I.5 Typical Frame.

In Eq. (I.17) all submatrices a_{ij} are nullmatrices except for j = S and j = E. Hence Eq.(I.17) may be written in the following compressed form

$$v_i = a_{iS} r_S + a_{iE} r_E \qquad \text{(I.17a)}$$

Correspondingly each term in the sum on the right-hand side of Eq. (I.16a) may be written as follows

$$a_i^T K_i a_i = \left[\begin{array}{c|c|c} & & \\ \hline & K_i^{SS} & K_i^{SE} \\ & & \\ \hline & K_i^{ES} & K_i^{EE} \\ & & \end{array} \right] \begin{array}{l} \\ \leftarrow \text{Row } k(S-1)+1 \\ \\ \leftarrow \text{Row } k(E-1)+1 \\ \\ \end{array} \qquad \text{(I.18)}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \text{Column} & \text{Column} \\ k(S-1)+1 & k(E-1)+1 \end{array}$$

where

$$K^{SS} = a_{iS}^T k_i a_{iS}$$

$$K_i^{SE} = (K_i^{ES})^T = a_{iS}^T k_i a_{iE} \qquad \text{(I.19)}$$

$$K_i^{EE} = a_{iE}^T k_i a_{iE}$$

In the derivations just given the k first node displacements $r_1 \ldots r_k$ refer to node 1, the next values $r_{k+1} \ldots r_{2k}$ to node 2, and so on. The submatrices K_i^{SS}, K_i^{SE} etc. are of dimension (kxk). They may be added directly into the stiffness matrix K of the structure as shown in Eq. (I.18).

D. Equilibrium Conditions

Using Eqs. (I.5) and (I.12), the following expression may be derived for the unbalanced nodal point forces caused by the loads on the members

$$R^F = a^T S^F \qquad \text{(I.20)}$$

The effective nodal point loads are then

$$R^E = R - R^F \tag{I.21}$$

The corresponding nodal point displacements may be found by the substitution of R^E into the left-hand side of equation (I.15):

$$R^E = Kr \tag{I.15a}$$

E. Supports

The K-matrix of Eq. (I.15a) is singular since not until now have any support conditions been imposed on the structure. A certain number of support constraints are available, such that

$$r_q = 0 \quad \text{for } q = ISUP(I), \quad I = 1, 2, \ldots LLS$$

where

LLS = total number of zero displacement conditions
ISUP(I)= an array of integers corresponding to the numbers of the displacements that are required to be zero.

For a frame with k degrees of freedom at each node, JO nodes and LLS zero displacement conditions, the total number of unknown nodal displacements is

$$KJLS = k \cdot JO - LLS \tag{I.22}$$

The equations of equilibrium expressed by (I.15) are now rearranged such that those corresponding to zero displacement conditions are grouped together below the others. This is achieved by means of a series of interchanges of rows and column.[x)]

If the array ISUP(I) is arranged such that ISUP(I)>ISUP(I+1), I = 1 ...(LLS-1), this interchange can be performed in the following way.

Interchange row ISUP(I) with row (k·JO+1-I), I = 1,...LLS
Interchange column ISUP(I) with column (k·JO+1-I), I = 1,...LLS

[x)] Other methods are also variable, by which this interchange is omitted.

After completing this procedure Eq. (I.15a) may be written as

$$\begin{vmatrix} R_R \\ R_S \end{vmatrix} = \begin{vmatrix} K_R & L \\ M & N \end{vmatrix} \begin{vmatrix} r_R \\ 0 \end{vmatrix} \qquad (I.23)$$

where

0 = Null matrix

R_R, K_R and r_R are the reduced load, stiffness and nodal point displacement matrices, respectively.

R_S = Support reaction matrix

From Eq. (I.23)

$$R_R = K_R r_R \qquad (I.24)$$

from which

$$r_R = K_R^{-1} R_R \qquad (I.25)$$

and

$$R_S = M r_R = M K_R^{-1} R_R \qquad (I.26)$$

F. Determination of Member End Forces

The vector $\{r_R, 0\}$ must be arranged into the original sequence by a number of row interchanges before member end forces the displacement are determined.

Having obtained the nodal displacement vector (r), member end displacements and member end forces are finally obtained by means of the following equations

$$v_i = a_i r = a_{iS} r_S + a_{iE} r_E \qquad (I.27)$$

$$S_i = k_i v_i + S_i^F \qquad (I.28)$$

G. Effect of a New Member on Displacements and Forces

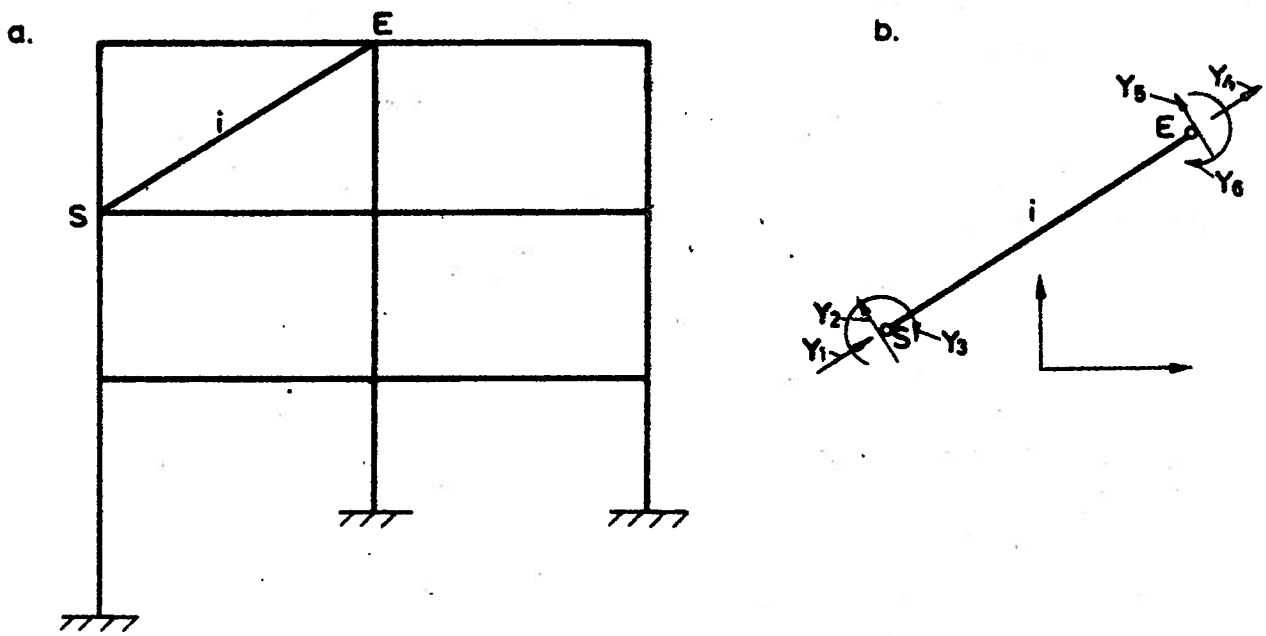


FIG. I.6 Typical Frame with New Member i

In the following an expression is derived for the change in force distribution in the frame shown in Fig. I.6a caused by the addition of an extra member (i), such as shown in the figure. Assume that this new member has a known stiffness expressed by the matrix $\mathbf{k}_{in}$. Furthermore the analysis of the original structure is assumed to have been performed, and the flexibility matrix

$$\mathbf{F} = \mathbf{K}^{-1} \tag{I.29}$$

for this structure is known.

Let

$$\mathbf{Y} = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\} \tag{I.30}$$

be the vector of statically indeterminate forces acting upon the newly introduced member (i). The end displacements of member (i) may be expressed according to Eq. (I.27) in terms of the nodal point displacements of nodes S and E only. The effect of the new member on the original structure may easily be computed since the forces Y may be regarded as external loads. According to Eq. (I.20) these forces are transformed to generalized nodal point loads by means of the relationship

$$R_o = -a_i^T Y \tag{I.31}$$

The generalized forces introduced into the rest of the structure are then given by the expression

$$S_o = kv = kar = kaK^{-1}R_o = -kaK^{-1}a_i^T Y$$

or

$$S_o = -Za_i^T Y \tag{I.32}$$

where

$$Z = kaK^{-1}$$

The resulting member end forces caused by the external loads R as well as the modification are then found from the following expression

$$S = kaK^{-1}R + S_o = Z(R - a_i^T Y) \tag{I.33}$$

and correspondingly the member end displacements

$$v = k^{-1}S = aK^{-1}(R - a_i^T Y) \tag{I.34}$$

The end displacements of the new member are

$$v_{in} = k_{in}^{-1}Y = f_{in}Y \tag{I.35}$$

where

f_{in} = the flexibility matrix of the new member.

The conditions of compatibility between the new member and the adjoining original structure requires that

$$\tilde{Y}^T v_{in} + \tilde{S}_o^T v = 0 \tag{I.36}$$

where

$\tilde{Y}$ is a set of virtual end forces on member (i) and
$\tilde{S}$ is a corresponding set of end forces in the original structure.

Introduction from Eqs. (I.33-35) into (I.36) yields, after some rearrangement

$$Y = (f_{in} + a_i K^{-1} a_i^T)^{-1} a_i K^{-1} R \qquad (I.37)$$

or

$$Y = q_n v_{io} \qquad (I.38)$$

where

$$q_n = (f_{in} + a_i K^{-1} a_i^T)^{-1} \qquad (I.39)$$

is a reduced "stiffness" matrix for the new member, and

$$v_{io} = a_i r = a_i K^{-1} R \qquad (I.40)$$

is the matrix of end displacements computed for the new member (i) when disregarding its own contribution to the stiffness matrix of the structure.

The resulting member end displacements may now be found by back-substitution

$$v = aK^{-1}(R - a_i^T q_n a_i K^{-1} R)$$

and since

$$v = ar$$

$$r = (K^{-1} - K^{-1} dK K^{-1})R$$

or

$$r = K_m^{-1} R \qquad (I.41)$$

where

$$K_m^{-1} = K^{-1} - K^{-1} dK K^{-1} \qquad (I.42)$$

$$dK = a_i^T q_n a_i \qquad (I.43)$$

In summary the analysis of the modified structure then involves the evaluation of q_n and v_{io} according to Eqs. (I.39-40) and dK and K_m^{-1} according to Eqs. (I.43-42). By means of Eq. (I.42) the flexibility matrix of the entire structure has been updated in such a manner that we are immediately ready to perform other modifications if required.

By means of an adressing scheme (similar to that outlined under Section C) the computational effort involved in establishing the matrices

$$a_i K^{-1} a_i^T \quad \text{and} \quad a_i^T q_n a_i$$

may be reduced substantially. Note that the analysis of the modified structure only requires the inversion of a matrix of dimension equal to that of the flexibility matrix of the new member (Eq. (I.39)). Yet, the procedure just outlined yields the exact solution for the modified structure.

H. Initial Strain Approach for Modification Analysis

Consider a statically indeterminate structure for which an analysis has been completed by means of the displacement method, such that the relationship

$$r = K^{-1} R \qquad \text{(I.44)}$$

is known. The cross-section of member (i) is now changed such that the member stiffness is increased by k_{in}. The inverse of the modified stiffness matrix is sought such that

$$r' = K_m^{-1} R \qquad \text{(I.45)}$$

yields the displacement matrix (r') for the modified structure. To this end fictitious initial strains are applied corresponding to member end displacements H_i on element i. The magnitude of H_i is selected such that the resulting effect on the total displacements is the same as that of the structural modification.

The initial strains in member (i) produce the following fixed end member forces

$$S_{iH}^F = -k_i H_i \qquad \text{(I.46)}$$

which according to Eq. (I.20) correspond to the following unbalanced nodal point forces

$$R_H = -a_i^T k_i H_i \qquad \text{(I.47)}$$

The nodal point displacements under external loads (R) combined with initial strains are then according to Eqs. (I.21) and (I.15a)

$$r' = K^{-1}(R - R_H)$$

or

$$r' = K^{-1}(R + a_i^T k_i H_i) \tag{I.48}$$

The end displacements of member (i) are given by the formula

$$v_i = a_i r' = a_i K^{-1}(R + a_i^T k_i H_i) \tag{I.49}$$

and the corresponding member end forces

$$S_i = k_i(v_i - H_i) \tag{I.50}$$

In the modified structure the following relationship must hold

$$S_i = (k_i + k_{in})v_i \tag{I.51}$$

Hence H_i must be selected such that

$$(k_i + k_{in})v_i = k_i(v_i - H_i)$$

or

$$k_{in}v_i = -k_i H_i$$

By introduction from Eq. (I.49),

$$H_i = -k_i^{-1}(I + k_{in}a_i K^{-1}a_i^T)^{-1}k_{in}a_i K^{-1}R$$

which with Eq. (I.48) yields

$$r' = K^{-1}\{I - a_i^T(I + k_{in}a_i K^{-1}a_i^T)^{-1}k_{in}a_i K^{-1}\}R \tag{I.52}$$

Introducing in Eq. (I.52) the following notations

$$q = (I + k_{in}a_i K^{-1}a_i^T)^{-1}k_{in} \tag{I.53}$$

and

$$dK = a_i^T q a_i$$

Eq. (I.52) may also be written in the same form as Eq. (I.41), where again

$$K_m^{-1} = K^{-1} - K^{-1}dKK^{-1} \qquad (I.54)$$

Expression (I.53) is identical to (I.39) since

$$k_{in} = f_{in}^{-1}$$

and

$$A^{-1}B^{-1} = (AB)^{-1}$$

where **A** and **B** are two arbitrary matrices.