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A RATIONAL STRIP THEORY OF SHIP MOTIONS: PART I

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ABSTRACT

The exact ideal-fluid boundary-value problem is formulated for a ship forced to heave and pitch sinusoidally in otherwise calm water. This problem is then simplified by applying three restrictions: 1) the body must be slender; 2) the motions must be small in amplitude compared with ship beam or draft; 3) the frequency of oscillation, ω , must be high, viz., $\omega = O(\varepsilon^{-1/2})$, where ε is the slenderness parameter. The hydrodynamic problem is then recast as a singular perturbation problem which is solved to order $\delta\varepsilon^2$ by the method of matched asymptotic expansions. (δ is a motion-amplitude parameter.) Formulas are derived for the hydrodynamic heave force and pitch moment, from which added-mass and damping coefficients can be easily obtained. The latter are similar but not identical to those used in several other versions of "strip theory;" in particular, the forward-speed effects have the symmetry required by the theorem of Timman and Newman, a result which has not been realized in previous versions of strip theory. In order to calculate the coefficients by the formulas derived, it is necessary to solve numerically a set of boundary-value problems in two dimensions, namely, the problem of a cylinder oscillating vertically in the free surface. At least two practical procedures are available to this purpose.

CONTENTS

PREFACE

This is a report of a project which began early in 1966 when both authors were on the staff of the David Taylor Model Basin (now a part of the Naval Ship Research and Development Center). The initial work was done primarily by Tuck. His contribution included the remarkable theorem in Appendix A, without which the simple final formulas could not have been obtained. He established much of the general approach to the problem, including the demonstration that the restriction to high frequencies was the key to deriving a strip theory from slender-body theory.

The project has suffered from rather erratic attention since October, 1966, when Tuck left DTMB. It has been carried along since then mostly by Ogilvie. He worked out the analysis of the near- and far-field problems, including the equivalent applied-pressure problem described in Appendix D, and showed how the pieces fit together in the matching of the expansions. The section on calculation of force and moment was the result of a joint effort while Tuck was at the California Institute of Technology. Ogilvie completed the project to the stage reported here while on the staff of the University of Michigan.

Because of the distance separating the authors, it has been impossible to collaborate on the writing of the report. Therefore the text is the work of the first-named author; he is responsible for any errors which remain.

Besides the acknowledgment which appears on the title page, mention should be made of the institutional support received from the David Taylor Model Basin while the authors were employed there, and from the University of Michigan and the California Institute of Technology.

Personally, it is a pleasure to thank Professor J. N. Newman (Massachusetts Institute of Technology, formerly of DTMB) for his helpful comments in the early stages of the project. Also, we acknowledge our appreciation to Mr. Young T. Shen (University of Michigan) for pointing out an important error and for his critical reading of the final manuscript.

Finally, it is appropriate to express our debt to Professor B. V. Korvin-Kroukovsky of Stevens Institute of Technology. We started this project with the avowed intent of finding a mathematical justification for the facts which he knew to be true about ship motions in waves. The fact that we hope actually to have improved on his results does not detract from his achievement many years ago. Mathematically inclined ship hydrodynamicists have been slow to recognize Professor Korvin-Kroukovsky's keen insight into the ship-motions problem. We acknowledge here that this project would not even have been started were it not for the remarkable results of his insight.

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PRINCIPAL NOTATION

a_{ij}	added-mass coefficient (See Equation (17).)
$a_{ij}^{(0)}$	value of a_{ij} for $U = 0$
b_{ij}	damping coefficient (See Equation (17).)
$b_{ij}^{(0)}$	value of b_{ij} for $U = 0$
C_0	line of intersection of S_0 and undisturbed free surface
$F_i(t)$	$i = 3$: vertical component of hydrodynamic force on ship; $i = 5$: hydrodynamic pitch moment on ship (positive bow-up)
g	gravitation acceleration
$h(x,y,t)$	function defining body in unsteady-motion problem
$h_0(x,y)$	function defining body in steady-motion problem
$H_0^{(k)}(z)$	Hankel functions (Bessel functions of the third kind)
$\underline{i}$, $\underline{j}$, $\underline{k}$	unit vectors in directions of x- , y- , z- axes, respectively
$J_0(z)$	Bessel function of the first kind
L	length of ship
$\underline{m}$	the vector: $-(\underline{\nu}\cdot\nabla)\underline{v}$
m_i	$i = 1,2,3$: the x- , y- , z- component of $\underline{m}$; $i = 4,5,6$: the x- , y- , z- component of $-(\underline{\nu}\cdot\nabla)(\underline{r}\times\underline{v})$
n_i	$i = 1,2,3$: the x- , y- , z- component of $\underline{\nu}$; $i = 4,5,6$: the x- , y- , z- component of $\underline{r}\times\underline{\nu}$
n	n_3
O , o	See below under "Miscellaneous Conventions."
$\underline{r}$	position vector, (x,y,z)
S	wetted surface of hull

S_0	wetted surface of hull at rest in still water
t	time variable
T_{ij}	transfer-function matrix between $\xi_j(t)$ and $F_i(t)$
$T_{ij}^{(k)}$	components in a decomposition of T_{ij} ; See Equation (49).
U	forward speed of ship
$\underline{V}$	$(1/U) \times$ fluid velocity in steady-motion problem, equal to $\nabla[x + \chi(x,y,z)]$
x , y , z	Cartesian coordinates
$y_0(x)$	waterplane half-beam at x
δ	motion-amplitude parameter
$\delta(x)$	Dirac delta function
ε	slenderness parameter
$\zeta(x,y,t)$	free-surface displacement in general problem
$\eta(x,y)$	free-surface displacement in steady-motion problem
$\theta(x,y,t)$	$\zeta(x,y,t) - \eta(x,y)$
κ	g/U^2
μ	fictitious (Rayleigh) viscosity
ν	ω^2/g
$\underline{\nu}$	unit normal vector to S or S_0
$\xi_3(t)$	heave variable (in units of length, positive upwards)
$\xi_5(t)$	pitch variable (in radians, positive bow up)
ρ	density of water (mass per unit volume)
$\sigma(x)$, $\sigma(x,t)$	source density per unit length in line distribution of sources

τ	$\omega U/g$
$\phi(x,y,z,t)$	velocity potential in general problem and in zero-speed problem
$\Phi_i(x,y,z)$	a normalized potential function (See (35) and (42a).)
$\Phi(x,y,z)$	$\Phi_3(x,y,z)$ (See (13) and (35).)
$\chi(x,y,z)$	$(1/U) \times$ perturbation-velocity potential in steady-motion problem
$\psi(x,y,z,t)$	time-dependent part of velocity potential, equal to $\phi(x,y,z,t) - Ux - U\chi(x,y,)$
$\Psi_i(x,y,z)$	a normalized potential function (See (36) and (42b).)
ω	radian frequency of heave, pitch oscillations
$\Omega_i(x,y,z)$	a normalized potential function (See (37) and (42c).)

MISCELLANEOUS CONVENTIONS

1) Coordinates and orientation: The ship moves in the direction of the negative x -axis. (Free stream moves toward positive x .) z is measured upwards, y to starboard.

2) Indicial notation: In a six-degree-of-freedom system, denote the ship displacements by $\xi_j(t)$, $j = 1,\ldots,6$. ξ_1 , ξ_2 , ξ_3 denote translational displacements in x- , y- , z- directions, respectively. ξ_4 , ξ_5 , ξ_6 denote rotations about these axes, respectively, in a right-handed sense. All added-mass and damping coefficients are denoted by a_{ij} and b_{ij} . See Equation (17) for interpretation.

3) Unit normal vector: Always directed out of fluid

4) Time dependence: Always taken in the form: $e^{i\omega t}$

5) Velocity Potential: Velocity equals positive gradient of potential.

6) Fourier transforms: Denoted by an asterisk. For example,

$$\sigma^*(k) = \int_{-\infty}^{\infty} dx\; e^{-ikx}\sigma(x) \;;$$

$$\phi^{**}(k,l;z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx\, dy\; e^{-i(kx + \ell y)}\phi(x,y,z) \;.$$

7) Order notation:

a) " $y = O(x)$ " means: $|y/x| < M$ as $x \to 0$, where M is a constant not depending on x .

b) " $y = O(xz)$ " means: $|y/xz| < M$ as $x \to 0$ and/or $z \to 0$, where M is a constant not depending on x and/or z , respectively.

c) " $y = o(x)$ " means: $|y/x| \to 0$ as $x \to 0$.

See Erdelyi (1956) for further definitions and properties.

INTRODUCTION

For several years, both authors have been involved with the problem of predicting ship motions in waves. Some of this effort was reported in two papers presented at the Fifth Symposium on Naval Hydrodynamics, sponsored by the Office of Naval Research in 1964*. In one of these papers, an attitude was adopted and expressed quite specifically that "rational" methods were being sought rather than empirical formulas. The word "rational" was used to imply that one should start with an appropriate boundary-value problem, simplify it with reasonable initial assumptions, and solve to obtain formulas for predicting ship motions. The other paper was clearly based on a similar, if tacit, attitude.

It was somewhat apparent then and it has become steadily more apparent since that more success was being realized by the less "rational" methods. These more empirical methods may all be loosely categorized as "strip theory," in that boundary-value problems are initially formulated which are meaningful in a two-dimensional sense only, that is, in planes perpendicular to the mean direction of travel, and the two-dimensional solutions are then adjusted to include certain three-dimensional and forward-speed effects. A considerable amount of physical insight goes into making these corrections, and the work of Korvin-Kroukovsky (1955) is really a *tour de force* in engineering analysis of an incredibly difficult problem.

Recognizing that strip theory has been really quite successful in predicting ship motions, we set ourselves the task of formulating a "rational" basis for strip theory. We started with a complete and exact boundary-value problem (assuming, of course, an ideal fluid) and then sought a set of simplifying

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*Ogilvie (1964), Newman and Tuck (1964)

assumptions which would reduce the results more or less to strip theory. The goal was not just to provide a pedantic justification for strip theory nor to satisfy a mathematician's sense of aesthetics. While both of these might be worthwhile to some extent, it was hoped that the resulting theory would not be identical to existing strip theory, for the latter does have some failings. In particular, every ship-motions theory based on the strip-theory analysis violated a proven symmetry theorem concerning the coupling between pitch and heave. Furthermore, the numerical results from strip theory were not perfect, even in an engineering sense. It was hoped that a more fundamental approach to strip theory would help to correct these deficiencies, while retaining those properties of strip theory which have been proven experimentally to be valid.

The present paper reports the first phase of this effort. Herein is formulated the problem of a ship which is moving with finite forward speed, forced (by some external means) to undergo heave and pitch motions. We start with the full, non-linear, free-surface problem, which is of course completely intractable, and we seek a systematic procedure for simplifying the problem to the point where it can be solved numerically.

There are three restrictions which we have had to apply in order to achieve the simplification:

1) The geometry of the ship must be restricted in a slender-body sense, i.e., the beam and draft must be small compared with length, and the shape and dimensions of cross-sections must vary slowly along the length.

2) The motions must have amplitudes small compared with the cross-section dimensions of the ship. This requirement permits the development of a theory which is linear in the amplitudes of motion.

3) The frequency of oscillation in heave and pitch must be large. Waves of this order of magnitude of frequency (which could exist on an otherwise calm, infinite ocean) would have a wavelength comparable with the beam of the ship.

It is the third restriction which causes the lowest-order approximation to degenerate exactly into the strip-theory results. It does not imply that the complete theory (including incident waves) will be valid only for very short waves. One must note that the frequency under consideration is the actual frequency of oscillation; it corresponds to a frequency of encounter, not to the frequency of any wave in an earth-fixed reference frame.

There is another assumption which is no more than implied in the actual analysis, but it sets very real limitations on the applicability of the results. We assume throughout that the Froude number is a quantity which is of order unity with respect to the slenderness parameter. It can be shown that this, in turn, implies that the ship has a high forward speed, not because a Froude number near unity is large in the naval architect's estimation, but because it is equivalent to requiring that the ship move faster than the waves generated by its oscillations. It is well-known, for example, that any linear analysis becomes invalid near the speed at which the ship-generated waves have a group velocity equal to ship speed. We are considering cases in which ship speed is always considerably higher than the group velocity of the generated waves.

The solution is based on a perturbation analysis, valid as $\varepsilon \longrightarrow 0$, where ε is the slenderness parameter. In the case of no oscillations at all, that is, for steady forward speed, the velocity potential for the perturbation of the uniform flow is found to be of order ε^2 . When the ship oscillates, the velocity potential is of order $\varepsilon^{3/2}$, and so this part of the potential dominates the steady-motion potential. Moreover, this part corresponds precisely to strip theory, in the most restricted sense: it satisfies the Laplace equation in two dimensions; there are no three-dimensional effects and no forward-speed effects. The body boundary condition and the free-surface boundary condition are exactly what we would obtain in formulating a free-surface problem for an infinitely long cylinder (of cross-section identical to a cross-section of the ship) oscillating in the free surface. Thus the largest part of the potential arises because of the oscillation, and this part does not depend on forward speed in any way.

At first sight, this appears to be quite remarkable. It has already been commented that our theory is really a theory for high speeds, and yet the leading-order potential does not depend on speed at all. If we formulate a zero-speed problem *ab initio* with the same assumption regarding frequency, we again obtain the strict strip theory as the first approximation. (In this case, we would really be dealing with waves of length comparable to ship beam.) Perhaps one is inclined to speculate that, if frequency of oscillation is high enough, strip theory might give the correct first approximation for all forward speeds, for it gives reasonable results at zero speed and it is constructed to give valid predictions at high speeds.

This cannot be, of course, for the solution will be singular at some speed (for a given frequency of oscillation). The conditions for the singular solution are often specified by saying that $\tau = 1/4$, where $\tau = \omega U/g$. Physically, one must expect that two quite different patterns of solution are possible: (a) At zero speed, there will be radiated waves going out in all directions, including the forward direction. (b) At high speed, there will be essentially no radiation ahead of the ship. The condition $\tau = 1/4$ really marks the boundary between these two regimes, and clearly, for very large ω , the speed at which the singular behavior occurs approaches zero. But it cannot equal zero.

We must conclude from these considerations that the analysis breaks down at some low speed. Since our lowest-order solution does not exhibit a singular behavior at any speed, we must expect that the higher-order solutions will be singular, and the lowest-order solution may be just as invalid as if the singularity had appeared straightaway at the first step.

It has been observed several times in recent years that practically any analysis gives reasonable predictions of ship motions at zero speed, provided only that the analysis treats buoyancy effects properly. Strip theory does treat buoyancy effects correctly and completely, and so strip theory can be expected to give reasonable predictions at zero speed. Therefore, for purposes of convenience, we shall frequently refer to the strip theory results as "zero-speed results." This practice will not be incorrect, but one should not assume that there is a smooth variation in all quantities when one gradually changes speed from zero up to high values.

The strip-theory results alone are not very interesting. Not only are they rather trivial in meaning, but they also give generally fairly poor predictions. Therefore we carry the solution of the oscillation problem through another order of magnitude, and it turns out that this part of the velocity potential is of order ε^2 . Some interesting results are found:

(1) We obtain effects of interactions between the oscillatory motion of the ship and the incident uniform stream. Part of this interaction can be identified with certain of Korvin-Kroukovsky's forward-speed effects, but we find other terms. In particular, we find that there are just the right interactions so that the Timman-Newman (1962) symmetry theorem is satisfied. This is one of the most important results of the paper. Until now, the users of strip theory have had only intuitive, physical arguments on which to base their calculations of forward-speed effects, and it was known that at least one term in their equations of motion had to be wrong. Now we can show that we have results consistent with the symmetry theorem*, and it is evident that the intuitive approach missed some aspects of the problem.

(2) We find also a new interaction effect, unlike any reported before. This is an interaction between the steady flow around the ship and the waves caused by the oscillations.

*Actually, our results show a stronger symmetry than proven by Timman and Newman. See the discussion in the next section.

To some extent, it is analogous to the interaction discussed in the previous paragraph, which is discovered only when we consider the effect of the oscillating ship on the steady flow. Now we find that the time-dependent displacement of the free surface also interacts with the steady flow around the ship. The force and moment on the ship caused by this interaction are of the same order of magnitude in our scheme as the ship forward-speed interaction. We have not yet obtained any numerical results for this effect, and so we can not assess its practical importance. A large part of this paper is devoted to the analysis leading up to this interaction. As will be seen, there is considerable difficulty encountered in simply formulating the appropriate boundary-value problems.

What we really want from all of this analysis are some formulas by which we can compute added mass and damping coefficients. We have a sequence of boundary-value problems, and the solutions of these problems must be combined appropriately according to the perturbation scheme used. Then the pressure must be computed and integrated over the hull surface in such a way that the required coefficients can be identified.

It appears at first that the force and moment cannot be calculated until after we have solved numerically a host of boundary-value problems. Fortunately, as it turns out, the calculation of force and moment is not really so complicated. By a considerable amount of mathematical manipulation, we shall be able to eliminate from the final formulas practically all of the velocity potentials which will have arisen in the boundary-value problem. In order to use the final formulas, one must solve just one kind of boundary-value problem:

> For any cross-section of the ship, consider an infinitely long cylinder of the same shape and dimensions which is oscillating vertically in the free surface. Find the velocity potential which satisfies the kinematic boundary condition on the body and the usual condition in two dimensions for outgoing waves on the free surface.

This problem must be solved for enough cross-sections that the solutions can be computed in a fairly smooth manner along the entire hull. Such problems have been solved numerically in at least two quite different ways, and computer programs exist for carrying out computations of the potential and the pressure over the surface of the body. (See Appendix B.)

Considering the complexity of the analysis, we find it quite startling to see how the final results are really both simple and elegant. To make the results of this analysis accessible to those who do not care to labor through the lengthy details, the next section contains a precise statement of the problem, a brief summary of the major steps, and a listing of the final formulas.

FORMULATION OF THE PROBLEM; MAJOR RESULTS

We assume that the ship is moving with constant speed U in the direction of the negative x -axis. The z -axis is upwards, and the y -axis extends to starboard. The origin of coordinates is located in the undisturbed free surface at midship, so that the undisturbed incident flow appears to be a streaming flow in the positive- x direction.

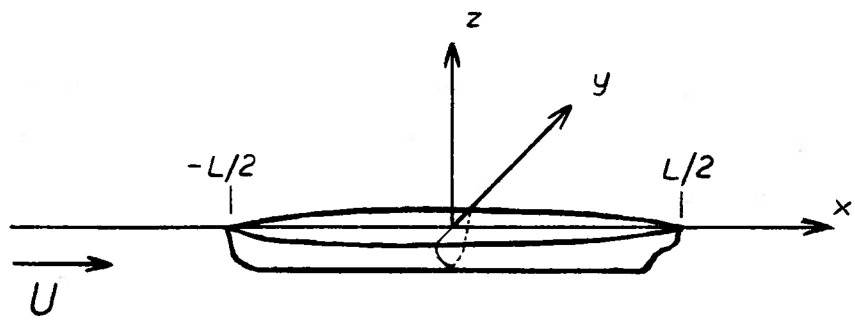

FIGURE 1.

Let the surface of the ship be specified mathematically by the equation:

$$z - h(x,y,t) = 0 , \qquad (1)$$

and let the free surface be given by:

$$z - \zeta(x,y,t) = 0 . \qquad (2)$$

It is assumed that the fluid velocity can be represented as the positive gradient of a scalar potential function, $\phi(x,y,z,t)$, which satisfies*:

(L), the Laplace equation, in the fluid domain:

$$\phi_{xx} + \phi_{yy} + \phi_{zz} = 0 ; \qquad (3)$$

*See, for example, Wehausen and Laitone (1960).

(A), the dynamic free-surface condition:

$$g\zeta + \phi_t + \frac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] = \frac{1}{2}U^2 , \quad \text{on} \quad z = \zeta(x,y,t) ; \qquad (4)$$

(B), the kinematic free-surface condition:

$$\phi_x\zeta_x + \phi_y\zeta_y - \phi_z + \zeta_t = 0 , \qquad \text{on} \quad z = \zeta(x,y,t) ; \qquad (5)$$

(H), the kinematic body condition:

$$\phi_x h_x + \phi_y h_y - \phi_z + h_t = 0 , \qquad \text{on} \quad z = h(x,y,t) ; \qquad (6)$$

(R), a radiation condition, which will be discussed in detail presently.

The hull condition, (H), can be stated in another way which is particularly convenient in the ensuing analysis. Let $\partial/\partial n$ denote the operation of taking a directional derivative normal to and into a cylinder with the same cross-section as the ship at a given section. Then it is easily shown that Equation (6) is equivalent to:

$$\text{(H)} \quad \frac{\partial\phi}{\partial n} = \frac{\phi_x h_x + h_t}{\sqrt{1+h_y^2}} , \qquad \text{on} \quad z = h(x,y,t) . \qquad (6')$$

We note that $\partial\phi/\partial n$ is a component of velocity in the cross plane.

Now we introduce two small parameters, both in a somewhat vague manner:

a) ε , the slenderness parameter, may be considered the ratio of maximum beam (or draft) to length, or it may be interpreted in other precise ways. Its smallness expresses the notion that the ship varies gradually in shape and size along its length.

b) δ , the motion-amplitude parameter, is any convenient measure of the smallness of the ship oscillations. The only requirement which it absolutely must satisfy is that, as it approaches zero, the oscillatory motion of all points of the ship uniformly approaches zero.

We use the slenderness parameter, ε , to formalize mathematically our concept of slenderness. Far away from the ship, at distances comparable with ship length, the details of the ship shape cannot be detected, and one can be aware only that there is a disturbance concentrated near a line, that part of the x -axis between $-L/2$ and $+L/2$. Analytically, we must treat the nature of the disturbance as unknown. Therefore, initially, we should find a farfield solution which would be appropriate for <u>any</u> line-concentrated disturbance. It turns out to be sufficient to assume that the disturbance is caused by a line of pulsating sources plus a line of steady sources. (The latter vanish in the zero-speed problem.) As $\varepsilon \longrightarrow 0$, the ship shrinks down to a line, for both beam and draft approach zero, and the disturbance vanishes altogether. The problem, of course, is to determine how the disturbance behaves for very small values of ε .

Near the ship, the disturbance depends critically on the details of ship shape. Consider the blown-up view of a small part of the ship shown in Figure 2. Since the cross-section varies slowly along the length of the ship, the flow is predominantly the longitudinal component, which is nearly equal to the free-stream velocity, on top of which is superposed a small transverse component depending primarily on the rate of

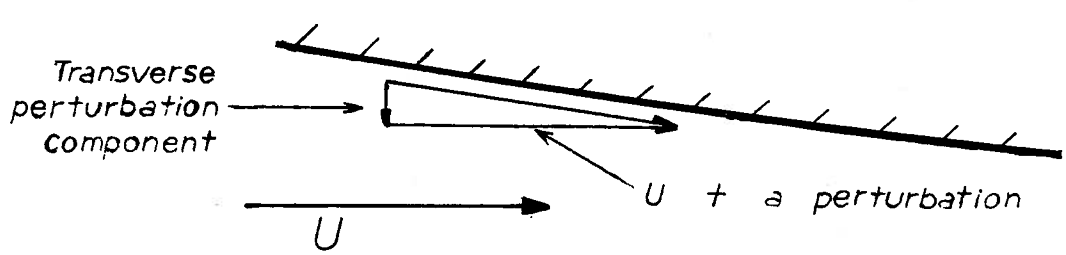

FIGURE 2.

change of cross-section and on the transverse motion of the section of the ship. It is plausible to assume that these transverse components of fluid flow will be much larger than the longitudinal perturbations of the incident flow.

This physical picture does not differ significantly from the description devised in the 1920's and applied to airship problems. Modern slender-body theory formalizes the description however, and this is done by means of one major assumption:

> Derivatives of flow variables in the transverse direction are larger than longitudinal derivatives by an order of magnitude with respect to the slenderness parameter.

For example, the velocity components are assumed to be related in an order-of-magnitude sense as follows:

$$\frac{\partial\phi}{\partial x} = O(\phi) \;;\quad \frac{\partial\phi}{\partial y} = O(\phi/\varepsilon) \;;\quad \frac{\partial\phi}{\partial z} = O(\phi/\varepsilon) \;,$$

where ϕ is the potential of the perturbation velocity component. Symbolically, we may write:

$$\partial/\partial x = O(1) \;;\quad \partial/\partial y = O(\varepsilon^{-1}) \;;\quad \partial/\partial z = O(\varepsilon^{-1}) \text{ as } \varepsilon \longrightarrow 0 \;. \tag{7}$$

We should note that, in (6'), the operator $\partial/\partial n$ is a directional derivative operator in the transverse plane, and so it also has the property:

$$\partial/\partial n = O(\varepsilon^{-1}) \;. \tag{7'}$$

These properties are valid only in the near field, that is, in the region in which $r = (y^2 + z^2)^{1/2} = O(\varepsilon)$. This assumption allows us to order various quantities according to their order of magnitude with respect to ε . Then, since we are developing an asymptotic theory, valid as $\varepsilon \longrightarrow 0$, we neglect all except the lowest order terms.

We make one more assumption about orders of magnitude, namely, that all oscillations are sinusoidal at high frequency. We shall use the exponential form of the sine function, $e^{i\omega t}$, and, since the operator $\partial/\partial t$ is then equivalent to multiplication by $i\omega$, time differentiation also changes the order of magnitude of the quantity operated upon. To be specific, we assume that:

$$\omega = O(\varepsilon^{-1/2}) , \tag{8}$$

which then implies that

$$\partial/\partial t = O(\varepsilon^{-1/2}) . \tag{8'}$$

Finally, we consider that Froude number, $U/\sqrt{gL}$, is $O(1)$ in terms of ε. When convenient, we shall treat U, g, and L separately as quantities which are $O(1)$.

We are now ready to reconsider the body boundary definition, as given in (1). If there is no heave or pitch motion, we replace $h(x,y,t)$ by $h_0(x,y)$, i.e.,

$$z - h_0(x,y) = 0 . \tag{9a}$$

We denote the heave and pitch variables by $\xi_3(t)$ and $\xi_5(t)$*. Positive ξ_3 represents an upwards heave displacement, and positive ξ_5 represents a bow-up pitch rotation. For small pitch angles, the body surface can then be defined:

$$z - h_0(x,y) - \xi_3(t) + x\xi_5(t) = 0 . \tag{9b}$$

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*We use subscripts 3 and 5 because these are natural when we come to study six degrees of freedom. Thus ξ_1, ξ_2, and ξ_3 denote translations of the ship along the x, y, and z axes, and ξ_4, ξ_5, and ξ_6 denote rotations about these axes, respectively.

As stated in the Introduction, we require that the amplitudes be small compared with ship beam and draft. The "smallness" of the motion amplitudes is symbolized by the parameter δ . In order that the latter may not depend on the slenderness parameter, ε , we assume that

$$\xi_j(t) = O(\delta\varepsilon) . \tag{10}$$

This assumption guarantees that the motion will vanish as $\delta \longrightarrow 0$, even though ε remains finite. It also guarantees that the motion amplitudes will be small compared with beam, even as $\varepsilon \longrightarrow 0$.

All that remains in formulating the problem is to assume that all dependent variables can be expressed by asymptotic expansions in terms of ε and δ , substitute these expansions into all conditions, and reorder the terms with respect to the small parameters. The details will be found in the appropriate sections.

The Zero-Speed Problem. As a prelude to the major problem, we consider first the case of heave and pitch motions at zero forward speed. No new results are obtained for this case, but certain difficulties with the radiation condition are already evident here, and the treatment of this case adds much perspicuity to the analysis of the general problem.

In the far field, the solution can be represented as the flow caused by a line distribution of pulsating sources. This disturbance causes waves which radiate outwards in all directions. The velocity potential for an arbitrary line distribution of pulsating sources is known, and we use that solution to find the flow behavior near the singular line. It turns out to be of the form:

$$\phi(x,y,z,t) \sim 4\pi i\sigma(x)\; e^{i\omega t + \nu z - i\nu|y|}\,{}^{*} .$$

*Only the real part of this expression is implied. We shall generally suppress the symbol "Re".

Here, $\sigma(x)e^{i\omega t}$ is the density of sources at x on the x -axis, and $\nu = \omega^2/g$. This result clearly represents an outgoing wave in two dimensions, but the amplitude of the wave is not known unless $\sigma(x)$ is known.

In the near field, the velocity potential is found to satisfy the following conditions:

$$\text{(L)} \qquad \phi_{yy} + \phi_{zz} = 0 \qquad \text{in} \quad z < 0 \; ; \tag{12a}$$

$$\text{(F)} \qquad g\phi_z + \phi_{tt} = 0 \qquad \text{on} \quad z = 0 \; ; \tag{12b}$$

$$\text{(H)} \qquad \phi_n = \frac{\dot{\xi}_3(t) - x\dot{\xi}_5(t)}{\sqrt{1 + h_{0_y}^2}} \qquad \text{on} \quad z - h_0(x,y) = 0 \; ; \tag{12c}$$

Condition (L) is just the Laplace equation in two dimensions; slender-body theory converts the 3-D problem into a 2-D problem. (F) is the ordinary linearized free-surface boundary condition which leads to the prediction of gravity waves. (H) is the usual kinematic body condition which applies if a two-dimensional cylinder is oscillating vertically with speed $\dot{\xi}_3(t) - x\dot{\xi}_5(t)$. It should be emphasized that these three conditions are not exact; they apply to the first term in an asymptotic expansion of the velocity potential.

The near-field problem expressed by the above conditions is incomplete in one important respect: Nothing is said about the behavior at infinity. As is well-known, this means that the solution is not unique. For example, there might be incoming ambient waves. Of course, it is much more reasonable to assume that there are only outgoing waves at large distance from the body. Previous workers with strip theory have assumed this, and it is correct. But it is a violation of the principles of the method of matched asymptotic expansions to make such an assumption. The solution of the above problem need not be applicable at large distance from the body, and in fact it is <u>not</u> valid far away, for we know that far away there are waves going out in all directions, not just laterally. The only justification for this radiation condition is that it agrees with the near-field limit of the far-field solution. The potential for a line of pulsating sources satisfies a radiation condition that is physically meaningful, and near the singular line it represents (approximately) just two-dimensional outgoing waves. It is the latter fact which tells us that the near-field solution should represent outgoing waves in two dimensions at infinity.

The above analysis would be rather pedantic except for one fact: In the analysis of the forward-speed problem, there arises a near-field boundary-value problem for which the condition at infinity is far from obvious. It will be clear in a later section that one must solve the far-field problem before the near-field problem can be satisfactorily formulated. The process will not be substantially different from that used in the zero-speed case, just more difficult and tedious.

Once we have obtained a near-field radiation condition from the far-field problem, we have no further use for the latter. We want to find the force on the oscillating ship, and this requires using the near-field solution to predict pressure on the hull surface. The details may be found in the appropriate section of this report; here we present only the results.

For convenience, we introduce a normalized potential function. Let $\Phi(x,y,z)$ satisfy the conditions:

$$\text{(L)} \qquad \Phi_{yy} + \Phi_{zz} = 0 , \quad \text{in} \quad z < 0 ; \tag{13a}$$

$$\text{(F)} \qquad \Phi_z - \nu\Phi = 0 , \quad \text{on} \quad z = 0 , \text{ where } \quad \nu = \omega^2/g ; \tag{13b}$$

$$\text{(H)} \qquad \Phi_n = 1/\sqrt{1 + h_{0_y}^2} \quad \text{on} \quad z - h_0(x,y) = 0 . \tag{13c}$$

In the zero speed problem, the potential function for the near field can now be represented by:

$$\phi(x,y,z,t) = + i\omega\Phi(x,y,z)\, [\xi_3(t) - x\xi_5(t)] . \tag{14}$$

The convenience of introducing the new potential function should now be clear: it is independent of the heave and pitch motions. We must find Φ for a number of cross-sections of the ship; having done that, we can immediately write down the velocity potential for any given heave and pitch motions or, for that matter, for unknown motions, using the notation in (14).

We express the force as follows:

$$F_i(t) = \sum_j T_{ij}\xi_j(t) \ . \tag{15}$$

We may look on the matrix T_{ij} as a transfer function which transforms motion variables into force components. The indices i and j have the values 3 and 5 in the present analysis. We shall find that, for zero forward speed, the transfer function is given by:

$$T_{33}^{(0)} = \rho\omega^2 \int_{S_0} dS \ n \ \Phi \ ; \tag{16a}$$

$$T_{55}^{(0)} = \rho\omega^2 \int_{S_0} dS \ x^2 \ n \ \Phi \ ; \tag{16b}$$

$$T_{35}^{(0)} = T_{53}^{(0)} = -\rho\omega^2 \int_{S_0} dS \ x \ n \ \Phi \ . \tag{16c}$$

(The upper index, $^{(0)}$, has been introduced to point out that these results are valid at zero speed only.) Here, n is an abbreviation for $1/\sqrt{1 + h_{0y}^2}$. The integrations are to be carried out over the surface of the hull at its undisturbed position.

Combining the three formulas above with the previous formula for force, we see that knowledge of the velocity potential, $\Phi(x,y,z)$, provides knowledge of the hydrodynamic*

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*Hydrostatic force and moment have not been included in the above formulas. These must be computed (by elementary methods) and added to the above.

force and moment as well, if only we know the heave and pitch variables. The latter are, of course, the major unknowns in a ship-motion problem. In well-known fashion, one formulates a pair of coupled differential equations for $\xi_3(t)$ and $\xi_5(t)$ to be solved for these two unknowns. The hydrodynamic force and moment, as computed above, are equivalent to the terms which are usually written as added-mass- and damping-coefficient terms. In a conventional form, we can rewrite the above results:

$$\begin{aligned} F_i(t) &= -\sum_j [a_{ij}\ddot{\xi}_j(t) + b_{ij}\dot{\xi}_j(t)] \\ &= \sum_j [\omega^2 a_{ij} - i\omega b_{ij}]\xi_j(t) \ . \end{aligned} \qquad (17)$$

The quantity a_{ij} is the added mass coefficient in the equation for the i-th mode of motion, giving the force (moment) due to the j-th mode of motion. A similar interpretation applies to the damping coefficients, b_{ij} . Comparison with the previous results shows that:

$$a_{33}^{(0)} + (1/i\omega)b_{33}^{(0)} = \rho\int_{S_0} dS\ n\ \Phi \ ; \qquad (18a)$$

$$a_{55}^{(0)} + (1/i\omega)b_{55}^{(0)} = \rho\int_{S_0} dS\ x^2 n\ \Phi \ ; \qquad (18b)$$

$$\begin{aligned} a_{35}^{(0)} + (1/i\omega)b_{35}^{(0)} &= a_{53}^{(0)} + (1/i\omega)b_{53}^{(0)} \\ &= -\rho\int_{S_0} dS\ x\ n\ \Phi \ . \end{aligned} \qquad (18c)$$

Thus we have obtained explicit formulas for added-mass and damping coefficients. Furthermore, numerous workers have developed computer programs for finding just this function Φ and for computing the integrals in these formulas. (See Appendix B.)

<u>The Forward-Speed Problem</u>. The results collected above for the zero-speed problem are rather trivial in that they have been obtained many times before by simpler methods.

Furthermore, they represent the most primitive kind of a strip theory. Now we turn to the forward-speed case, in which some new results are obtained. For the purposes of the present section, we mention only the general approach to the problem, and then we proceed immediately to the final formulas.

We first set up the steady-motion problem. That is, we allow the motion-amplitude parameter, δ , to be zero, and we formulate the hydrodynamic problem for steady forward motion under the usual assumptions of slender-body theory. Its solution is to be considered valid in the asymptotic sense as $\varepsilon \longrightarrow 0$. In fact, we can obtain a sequence of problems, leading to successively better approximations to the exact solution. The asymptotic series thus obtained has been considered in some detail by Tuck (1965), and nothing new is added to it here.

Then, in effect, we subtract the steady-motion solution from the exact solution of the complete problem and investigate how the remainder behaves for infinitesimal values of δ . It appears readily that all conditions on this remainder are linear in δ , and so the solution is itself linear in terms of the motion amplitudes. This problem, linear in δ , is still rather complicated in terms of the slenderness parameter, ε , and so its solution is again reduced to the finding of an asymptotic solution in terms of ε .

In other words, the complete solution is expressed first as a series in δ and then each term is expressed as a series in ε . Only the first two terms with respect to δ will be considered, that is, the terms which are independent of and linear in δ . These are carried consistently to the same order of magnitude in terms of ε . Just as in the zero-speed problem, we use the method of matched asymptotic expansions to determine the ε -dependence.

Although the analysis takes many pages, the ultimate formulas for added-mass and damping coefficients are rather simple. We give them here:

$$\left.\begin{aligned}
a_{33} &= a_{33}^{(0)} \; ; \qquad\qquad b_{33} = b_{33}^{(0)} \; ; \\
a_{55} &= a_{55}^{(0)} \; ; \qquad\qquad b_{55} = b_{55}^{(0)} \; ; \\
a_{35} &= a_{35}^{(0)} + (U/\omega^2)\, b_{33}^{(0)} - \underline{\mathrm{Im}} \left[(2\rho\omega U/g) ⨍_F dS\, \Phi^2 \right] ; \\
b_{35} &= b_{35}^{(0)} - U\, a_{33}^{(0)} \qquad - \underline{\mathrm{Re}} \left[(2\rho\omega^2 U/g) ⨍_F dS\, \Phi^2 \right] ; \\
a_{53} &= a_{53}^{(0)} - (U/\omega^2)\, b_{33}^{(0)} + \underline{\mathrm{Im}} \left[(2\rho\omega U/g) ⨍_F dS\, \Phi^2 \right] ; \\
b_{53} &= b_{53}^{(0)} + U\, a_{33}^{(0)} \qquad + \underline{\mathrm{Re}} \left[(2\rho\omega^2 U/g) ⨍_F dS\, \Phi^2 \right] .
\end{aligned}\right\} \tag{19}$$

A few things have yet to be explained in these formulas, but we notice immediately that all terms except those involving the integrals depend only on the zero-speed added-mass and damping coefficients. In other words, having calculated the zero-speed coefficients, it is a trivial matter to obtain most of the terms needed above.

The integral terms involve just the same potential function that we have already discussed. Presumably the latter has been determined in the zero-speed problem. But here the integration is to be carried out over the undisturbed free surface.

A bar has been drawn through the integral sign to call attention to the fact that the integral does not really exist as written. A special interpretation is required. If we investigate the function Φ , we find that it has an oscillatory behavior as $|y| \longrightarrow \infty$, and so it cannot be integrated to infinity in a straightforward way. But let us assume that the oscillatory behavior of Φ can be expressed:

$$\Phi(x,y,0) \sim f(x)\, e^{-i\nu|y|} \; . \tag{20}$$

If we square this expression and subtract it from Φ^2 , the difference has a well-defined integral, and it is essentially that integral that we imply in the formulas above. To be precise, we define the integral as follows:

$$\int_F dS\, \Phi^2 = \int_{-L/2}^{L/2} dx \int_{y_0(x)}^{\infty} dy\, [\Phi^2 - f^2 e^{-2i\nu y}]$$
$$- \frac{i}{2\nu} \int_{-L/2}^{L/2} dx\, f^2 e^{-2i\nu y_0(x)} \quad , \qquad (21)$$

where $y_0(x)$ is the half-beam at x . Thus, we subtract from the integrand just enough to remove its bad behavior at infinity; this yields an unwanted contribution at the lower limit, $y = y_0(x)$, which is removed by the single-integral term.

In order to facilitate comparison of these formulas with those obtained by others, we exhibit below the corresponding formulas as expressed by Gerritsma (1966). The latter are representative of the results obtained by all who have followed the approach of Korvin-Kroukovsky (1955). At the left, in parenthesis, we indicate our notation for the coefficient given in the same line.

(a_{33}) $\quad a = \int m'(x)\, dx$;

(b_{33}) $\quad b = \int N'(x)\, dx$;

(a_{55}) $\quad A = \int m'(x)\, x^2\, dx + \frac{U}{\omega^2} \int N'(x)\, x\, dx + \frac{U^2}{\omega^2}\, a$;

(b_{55}) $\quad B = \int N'(x)\, x^2\, dx$;

(a_{35}) $\quad d = \int m'(x)\, x\, dx + \frac{U}{\omega^2}\, b$;

(a_{53}) $\quad D = \int m'(x)\, x\, dx$;

$$(b_{35}) \quad e = \int N'(x)\ x\ dx - U\ a\ ;$$

$$(b_{53}) \quad E = \int N'(x)\ x\ dx + U\ a\ .$$

The integrals appearing here can be compared directly to those we used above in defining the zero-speed coefficients. In fact, Gerritsma's $(a + b/i\omega)$ is exactly the same as our $[a_{33}^{(0)} + b_{33}^{(0)}/i\omega]$.

Three of the eight coefficients are given by the same formulas in both analyses:

$$a_{33} = a\ ; \quad b_{33} = b\ ; \quad b_{55} = B\ .$$

The forward-speed corrections in A (corresponding to our a_{55}) do not appear in our analysis. If there exists a rational basis for these corrections, one must assume that they are higher-order quantities in our perturbation scheme and are thus negligible.

If we ignore for the moment the integral terms in our coupling coefficients, we find that our new formulas agree with Gerritsma's except in one case: His D lacks the forward-speed correction found in our a_{53} . In this case, we can state with considerable confidence that the usual strip theory must be wrong; this correction has simply been overlooked. There are two reasons for our confidence in such an assertion:

1) Timman and Newman (1962) have proven that a_{35} and a_{53} (or d and D) must have the symmetry that our results exhibit*.

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*As mentioned earlier, our results go somewhat beyond the Timman-Newman symmetry theorem. Those authors showed that the coupling coefficients can be expressed:

$$a_{35} = a_{35}^{I} + a_{35}^{II}$$

where a_{35}^{I} is the pitch-to-heave added-mass coefficient at zero speed (called $a_{35}^{(0)}$ in this report), and a_{35}^{II} gives the additional coupling due to forward speed. They prove that $a_{35}^{I} = a_{53}^{I}$ and that $a_{35}^{II} = -\ a_{53}^{II}$, the latter being true only for a ship which is symmetrical fore-and-aft. In our formulas, the latter result is true regardless of whether the ship has such symmetry. Such an outcome undoubtedly results from the

2) Experiments by W. E. Smith (1966) at Delft, presented in Figure 3 below, show that the D -coefficient has a fairly strong speed dependence. The points in the figure represent his experimental results, and the two curves show calculated values of:

a) $D = a_{53}^{(0)}$,

b) $a_{53}^{(0)} - Ub_{33}^{(0)}/\omega^2$.

Curve a), the broken line, is clearly in poor agreement with his experiments, whereas the second curve shows rather good agreement. It must be noted that our calculations do not include the integral term in a_{53} , and so the comparison is not conclusive. Nevertheless, inclusion of one forward-speed effect brings about such a dramatic improvement in prediction that one has some basis for hoping that the other term (the integral) has little effect in computation of the a_{53} coefficient.

Some authors (e.g., Gerritsma (1967)) have camouflaged their lack of symmetry in the d - D coefficients by combining the unsymmetrical parts with the buoyancy-force coefficients. However, one must compute the total force and subtract the buoyancy terms; the Timman-Newman symmetry theorem applies to the remainder.

Unfortunately, we have not yet computed any numerical values for the integrals in the formulas for the coupling coefficients. We can only note that at least these terms satisfy the Timman-Newman symmetry theorem.

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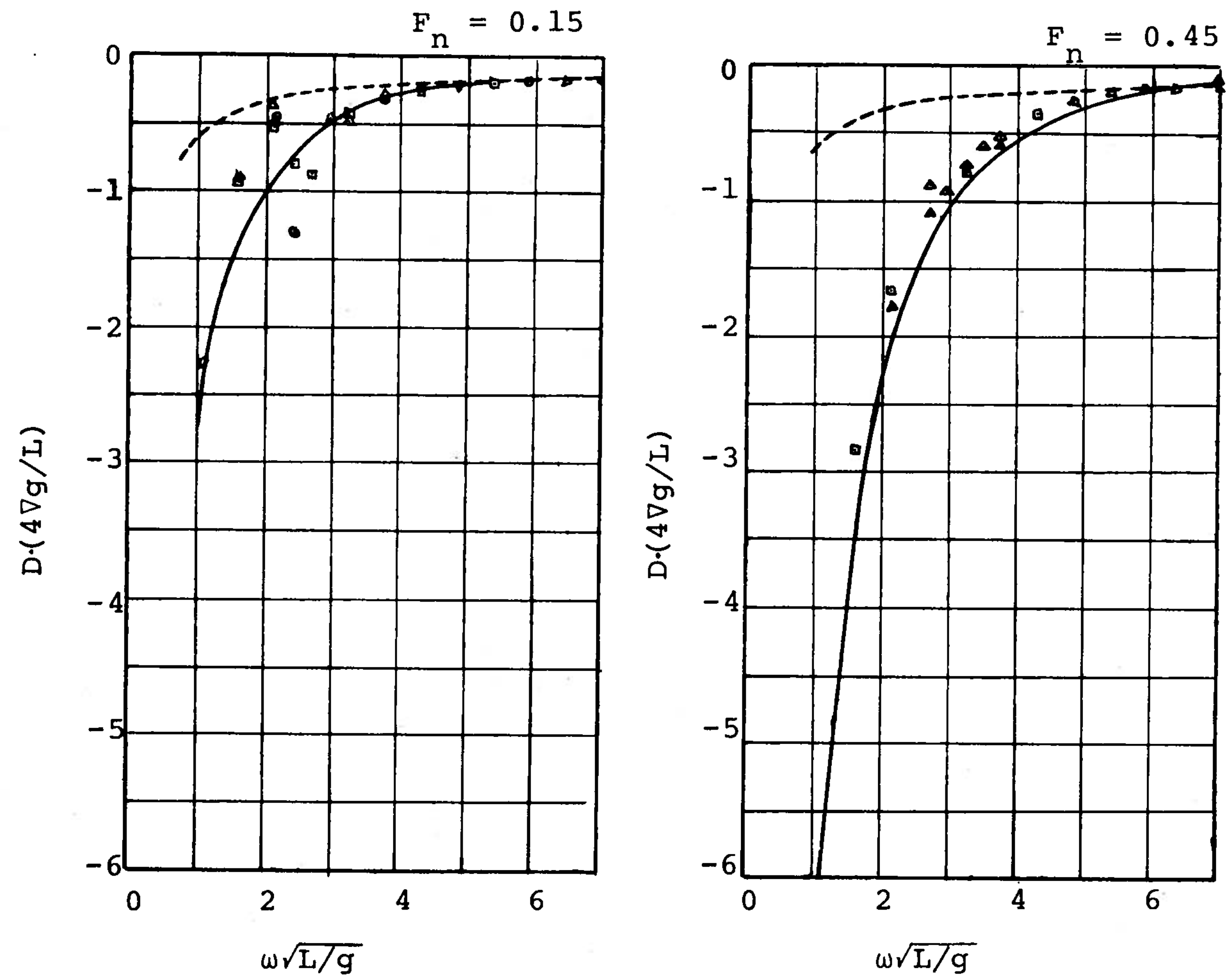
approximations which we make. However one is inclined to speculate that it may be true in any linear theory. Compare, for example, the well-known result that linearized wave-resistance theory gives the same result for a ship going backwards or forwards, regardless of hull symmetry (or lack of it).



---- Curve a): $D = a_{53}^{(0)}$

—— Curve b): $a_{53}^{(0)} - Ub_{33}^{(0)}/\omega^2$

FIGURE 3.

THE ZERO-SPEED PROBLEM

The purpose of this section is just to show how the far-field solution provides a radiation condition for the near-field problem. The fact that the final formulas give acceptable accuracy for the added-mass and damping coefficients must be considered as good luck. Our analysis is based on the assumption that $\omega = O(\varepsilon^{-1/2})$, and waves with this frequency will have length comparable with ship beam. It has already been commented that, at zero speed, a correct treatment of buoyancy effects is all that is needed to yield good results, and so our answers happen to be valid for wavelengths comparable to ship length, even though they are based on a short-wave hypothesis.

The far-field problem. At distances which are $O(1)$ from the ship, the detail of the ship is lost; we see what amounts to a singular line generating a pulsating disturbance. In general, we might assume that there are pulsating sources, dipoles, quadripoles, etc., along the singular line. In practice, we need to include singularities with just enough generality so that we can match the resulting potential function to the near-field potential function; a line of sources suffices. Accordingly, we assume that there is a line distribution of sources of density $\sigma(x)e^{i\omega t}$ spread along the line $y = z = 0$, $-L/2 < x < L/2$. In the absence of the free surface, the source distribution would have a velocity potential:

$$-\int_{-L/2}^{L/2} \frac{d\xi\ \sigma(\xi)e^{i\omega t}}{[(x-\xi)^2 + y^2 + z^2]^{1/2}} \quad .$$

It is necessary to modify this potential function to account for the presence of the free surface. It is easily shown that the free-surface conditions can be linearized, for, as $\varepsilon \longrightarrow 0$, the disturbance vanishes altogether. It is not at all clear which of the linear terms should be retained. The reason for this uncertainty is that the differential operators have various order-of-magnitude effects in various physical regions, and one must make extremely perceptive assumptions in order to obtain the proper free-surface conditions. We avoid this difficulty by including (inconsistently) all of the linear free-surface terms. One could afterwards make consistent asymptotic estimates of the solution, if that appeared desirable.

The solution of this problem has been given by Ursell (1962):

$$\phi(x,y,z,t) = \underline{\mathrm{Re}}\left[\phi(x,y,z)\ e^{i\omega t}\right] , \tag{22}$$

where

$$\phi(x,y,z) = -\ 2\int_{-L/2}^{L/2} d\xi\sigma(\xi) \int_{0}^{\infty} \frac{k dk}{k-\nu}\ e^{kz}\ J_0(k\sqrt{(x-\xi)^2+y^2}) \ . \tag{22'}$$

The inner integral is a contour integral, indented at the pole as indicated. The function in (22) is the solution of Equations (3), (4), and (5) after they are linearized. It does not, of course, satisfy the body boundary condition, (6), nor can it generally be made to do so. It does satisfy a proper radiation condition, viz., it represents outgoing waves at infinity. The above solution can also be constructed by a slight modification of Equation (13.17") in Wehausen and Laitone (1960).

Equation (22') contains too much information to be useful to us; it is valid all the way from the line of sources out to infinity. We want the inner expansion of this solution, that is, its asymptotic form in the case of $r = \sqrt{y^2+z^2} = O(\varepsilon)$. Before finding this, we note that we can first assume that $y = O(1)$, and re-order quantities according to their orders of magnitude. This still leaves us with an outer-region solution, but it is just as good as the original solution for our purposes, for we do not need the information about what is happening at infinity. Since $\nu = O(\varepsilon^{-1})$, and thus $\nu|y| = O(\varepsilon^{-1})$, we are considering a region which is many wavelengths removed from the line of singularities, even though it is at a distance which is $O(1)$.

Let $R = [(x-\xi)^2 + y^2]^{1/2}$. We note that $R = O(1)$ even if $x = \xi$. Also let

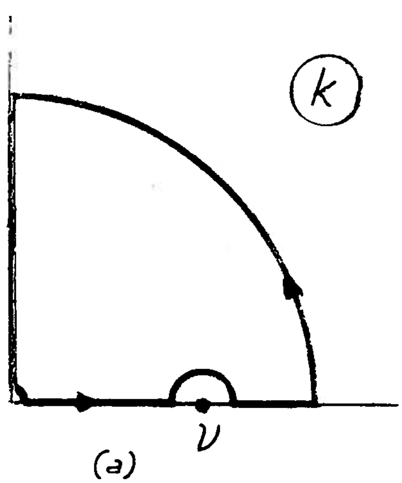
$$I = \int_0^\infty \frac{kdk}{k-\nu} e^{kz} J_0(kR) .$$

We substitute for $J_0(kR)$:

$$J_0(kR) = \tfrac{1}{2}[H_0^{(1)}(kR) + H_0^{(2)}(kR)] ,$$

where $H_0^{(j)}(kR)$ is a Hankel function (Bessel function of the third kind). If kR is considered to be a complex variable, $H_0^{(1)}(kR)$ becomes exponentially small as $\underline{\mathrm{Im}}\{kR\} \longrightarrow +\infty$, and $H_0^{(2)}(kR)$ becomes exponentially small as $\underline{\mathrm{Im}}\{kR\} \longrightarrow -\infty$. Therefore we write I as two separate integrals, each containing one of the Hankel functions. For the first, we close the contour as shown in Figure 4a; since the pole is outside the enclosed region, and since the integral along the quarter-circle contributes vanishingly little (as its radius goes to infinity), the first integral is:

$$I^{(1)} = \frac{1}{2}\int_0^{i\infty} \frac{kdk}{k-\nu} e^{kz} H_0^{(1)}(kR) .$$



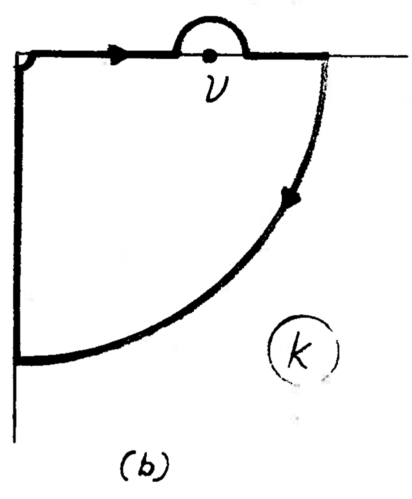


FIGURE 4.

The second integral is closed below, thus encircling the pole. Therefore,

$$I^{(2)} = \frac{1}{2}\int_0^{-i\infty} \frac{kdk}{k-\nu}\, e^{kz}\, H_0^{(2)}(kR) - \pi i\nu e^{\nu z}\, H_0^{(2)}(\nu R) \; .$$

We combine the two integrals and change the variable of integration, obtaining:

$$I = -\pi i\nu e^{\nu z}\, H_0^{(2)}(\nu R) + \frac{1}{\pi}\int_0^{\infty} dk\; k\; K_0(kR)\left[\frac{e^{ikz}}{k+i\nu} + \frac{e^{-ikz}}{k-i\nu}\right] \; .$$

Because of the change of variable, we have been able to rewrite the Hankel functions in terms of the K -function, a modified Bessel function of the second kind.

Now $K_0(kR)$ is a positive function, and so we have the estimate:

$$\left|\int_0^{\infty} \frac{dk\; k\; K_0(kR)\; e^{\pm ikz}}{k \pm i\nu}\right| \le \frac{1}{\nu}\int_0^{\infty} dk\; k\; K_0(kR)$$

$$= \frac{1}{\nu R^2} = O(\varepsilon) \; .$$

Thus,

$$I = -\pi i\nu e^{\nu z}\, H_0^{(2)}(\nu R) + O(\varepsilon) \; .$$

Furthermore, we can use the asymptotic expression for the remaining Hankel function, which yields:

$$I = -i\sqrt{\frac{2\pi\nu}{R}}\; e^{-i(\nu R-\frac{\pi}{4})}\; e^{\nu z}\left[1+O(\varepsilon)\right] = O(\varepsilon^{-1/2}) \; .$$

Returning to the double integral, we now have the following:

$$\phi(x,y,z) = -2\sqrt{2\pi\nu}\; e^{\nu z-i\pi/4}\int_{-L/2}^{L/2} d\xi\; \frac{\sigma(\xi)e^{-i\nu R}}{R^{1/2}}\left[1+O(\varepsilon)\right] \; .$$

The integrand is in just the proper form for applying the method of stationary phase, since ν is a large parameter. For $-L/2 < x < L/2$, the contribution to the integral at the point of stationary phase will dominate the contributions from the ends of the interval (See Erdelyi (1956)), and so, applying this method, we obtain:

$$\phi(x,y,z) \sim 4\pi i\sigma(x)\, e^{\nu(z-i|y|)} \quad . \tag{23}$$

For the real potential, we have:

$$\phi(x,y,z,t) \sim \mathrm{Re}\{4\pi i\sigma(x)\, e^{\nu z}\, e^{i(\omega t-\nu|y|)}\} \quad . \tag{23'}$$

Equation (23') clearly predicts outgoing waves in two dimensions. It is valid at distances from the source line where $y = O(1)$, but no more simplification is possible if we now let $y = O(\varepsilon)$, and so Equation (23') also expresses the inner expansion of the outer expansion.

A simple physical explanation can be given for the above result. If a wave generator has dimensions which are very large compared with wavelength, the waves can be sharply focused. This is true whether one is studying acoustic waves, electromagnetic waves, or water waves. Our wave generator (the ship) has length L, and the wavelength is $O(\varepsilon)$ by comparison, and so our system falls into this general category. At quite considerable distances, one may expect the waves to be still propagating uni-directionally, and this is just what happens. Figure 5 is a photograph showing this phenomenon in a wave tank. The waves are being generated by a small wavemaker in the short side of the tank. The length of the wavemaker is about one-eighth of the tank width, but it is much longer than the length of the generated waves.

In the above solution, as given in (22), the potential was represented simply as a superposition of the potentials of the sources on the line. In the forward-speed problem, an entirely different representation will be used, and it is helpful to

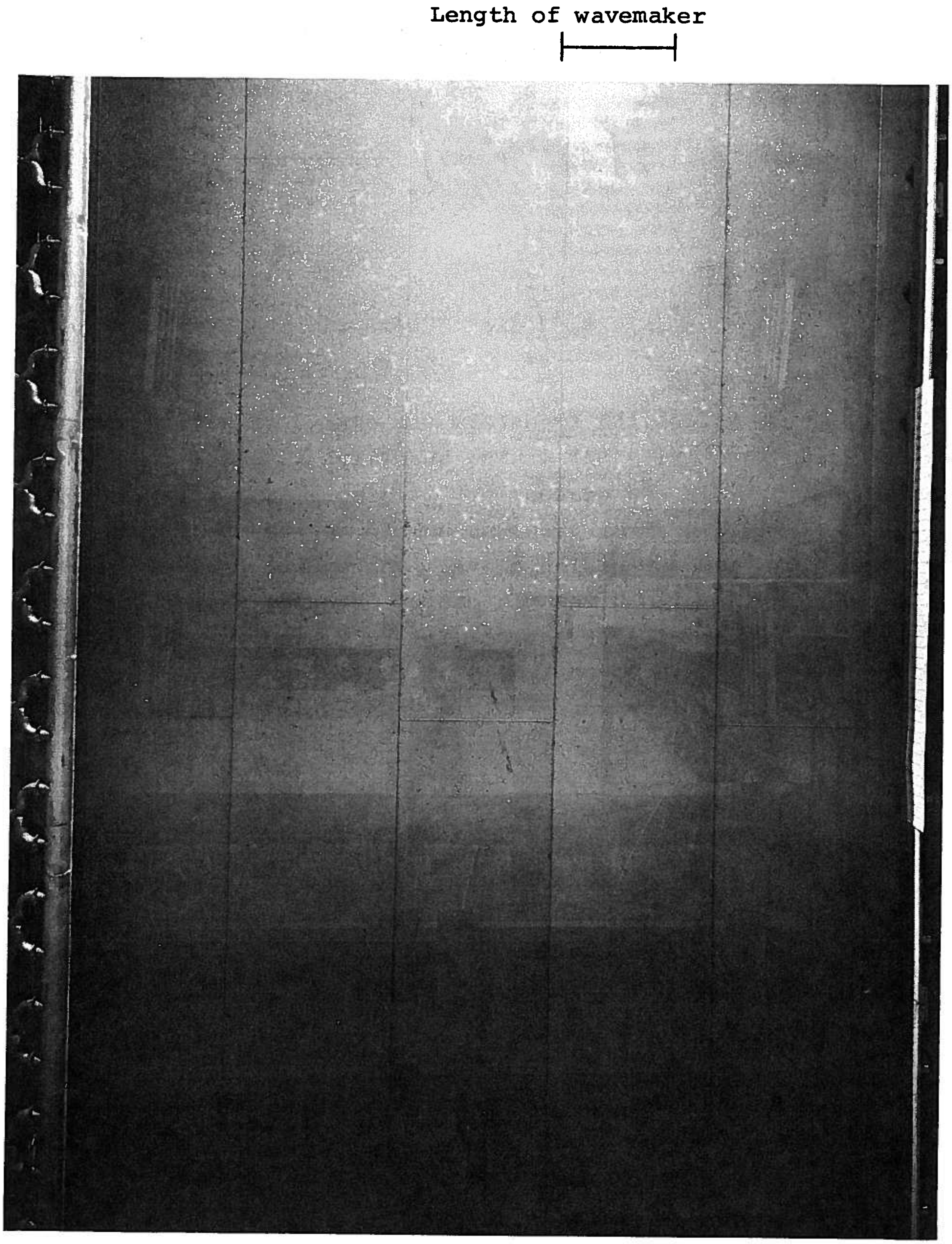


Figure 5. Note relative lengths of wavemaker and of waves.

point out now how that analysis will proceed, by considering here the case of zero forward speed.

The solution will be given as an inverse double Fourier transform:

$$\phi(x,y,z) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dk e^{ikx}\sigma^*(k)\int_{-\infty}^{\infty}\frac{d\ell e^{i\ell y+z\sqrt{k^2+\ell^2}}}{\sqrt{k^2+\ell^2}-\frac{1}{g}(\omega-i\mu)^2}, \qquad (24)$$

where

$$\sigma^*(k) = \int_{-\infty}^{\infty} dx e^{-ikx}\sigma(x) .$$

In order to define the transform of $\sigma(x)$, we simply set $\sigma(x) = 0$ for $|x| > L/2$. The quantity μ is discussed in Appendix C; for now we consider it simply as a parameter which, as it approaches zero, shows us how to define the improper integral in the expression above. The solution given in (24) will not be derived here. It can be obtained from the forward-speed result by simply setting $U = 0$. (See Equation (C3).)

First we examine the poles of the integrand. Since we shall presently allow $\mu \longrightarrow 0$, we find easily that the approximate positions of the poles are:

$$\ell = \pm\ell_0 \simeq \pm\sqrt{(\nu^2-k^2) - 4i\omega^3\mu/g^2} .$$

For $|k| < \nu$, we can let $\mu \longrightarrow 0$ and the ℓ-integral will be taken along a contour indented as shown in Figure 6a. For $|k| > \nu$, the poles are on the imaginary axis, even for $\mu = 0$, and so the contour need not be indented at all, as shown in Figure 6b.

In what follows, we shall assume that $y > 0$. It is readily checked that the final results hold for $y < 0$ if we replace y by $|y|$.

First we consider the case $|k| < \nu$. Define the integral I:

$$I = \int_{-\infty}^{\infty}\frac{d\ell e^{i\ell y+z\sqrt{k^2+\ell^2}}}{\sqrt{k^2+\ell^2} - \nu} .$$

We form a closed contour as shown in Figure 7. The integrand is analytic inside and on the contour except at the pole, $\ell = -\ell_0 = -\sqrt{\nu^2-k^2}$. The integrals along the circular arcs vanish as the radii of the arcs go to infinity. Thus, the integral I can be expressed:

$$I = -\frac{2\pi i\nu}{\sqrt{\nu^2-k^2}} e^{-i\sqrt{\nu^2-k^2}y + \nu z}$$

$$+ \int_{|k|}^{\infty} d\ell e^{-\ell y} \left[\frac{e^{iz\sqrt{\ell^2-k^2}}}{\sqrt{\ell^2-k^2}+i\nu} + \frac{e^{-iz\sqrt{\ell^2-k^2}}}{\sqrt{\ell^2-k^2}-i\nu} \right],$$

where the first term is the residue contribution, and the second term comes from the integrations along the imaginary axis.

An upper bound on the integrals is easily obtained:

$$\left| \int_{|k|}^{\infty} \frac{d\ell e^{-\ell y \pm iz\sqrt{\ell^2-k^2}}}{\sqrt{\ell^2-k^2} \pm i\nu} \right| \leq \frac{1}{\nu} \int_{|k|}^{\infty} d\ell e^{-\ell y} = \frac{e^{-|k|y}}{\nu y} = O(\varepsilon) ,$$

since $y = O(1)$. If $|k| = o(\varepsilon^{-1})$, we have:

$$I = -2\pi i e^{\nu(z-iy)} \left[1 + o(1)\right] .$$

If $|k| = O(\varepsilon^{-1})$, a much stronger statement is possible:

$$\left| \int_{|k|}^{\infty} \frac{d\ell e^{-\ell y \pm iz\sqrt{\ell^2-k^2}}}{\sqrt{\ell^2-k^2} \pm i\nu} \right| = O(e^{-1/\varepsilon}) ,$$

and so:

$$I = -\frac{2\pi i\nu}{\sqrt{\nu^2-k^2}} e^{\nu z-i\sqrt{\nu^2-k^2}y} \left[1 + O(e^{-1/\varepsilon})\right] .$$

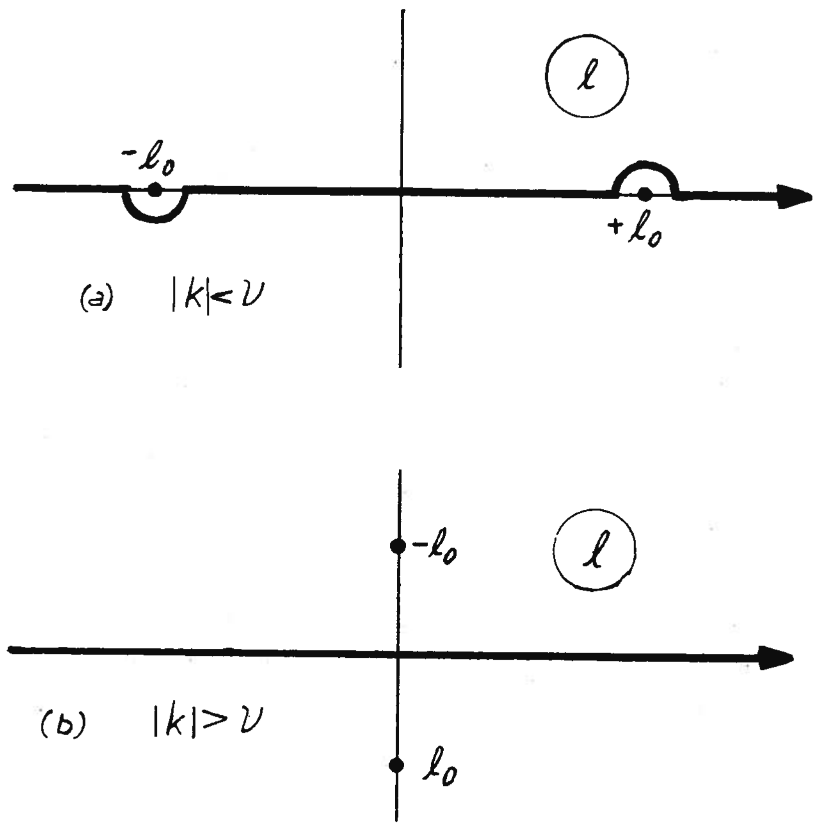


FIGURE 6.

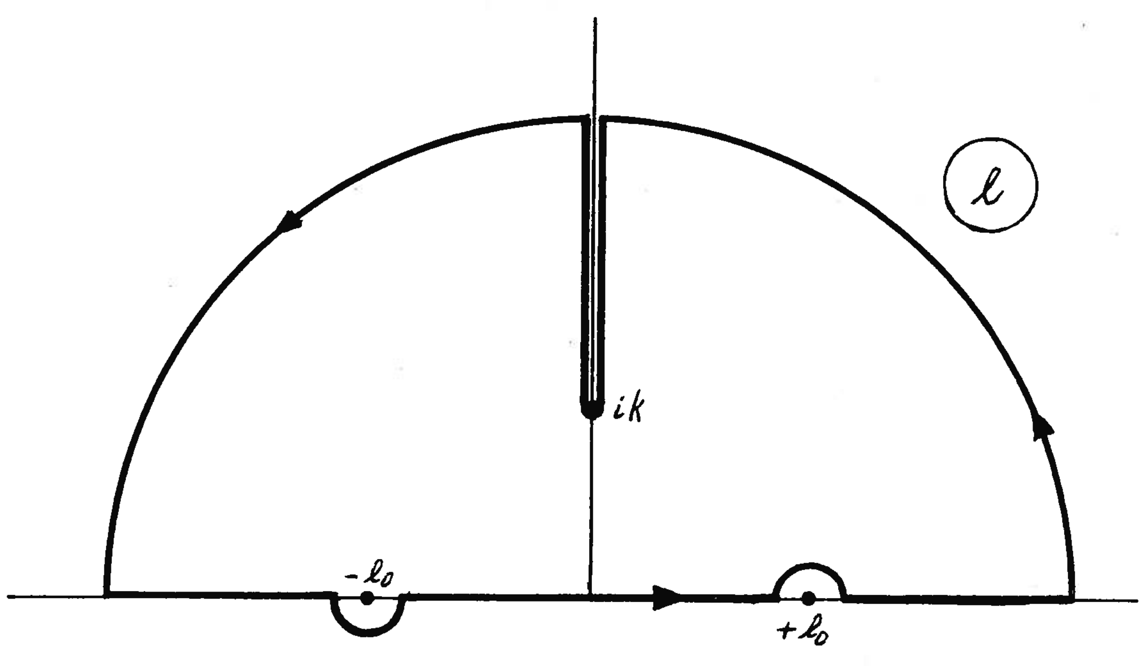


FIGURE 7.

In the case of $|k|>\nu$, the stronger estimate is again found to be valid. Thus, we can write for all cases,

$$I \sim - \frac{2\pi i\nu}{\sqrt{\nu^2-k^2}} e^{\nu z-i\sqrt{\nu^2-k^2}y} ,$$

where we take the positive square root if $\nu > |k|$, and we take $\sqrt{\nu^2-k^2} = -i\sqrt{k^2-\nu^2}$ if $\nu < |k|$.

The potential function is now given by:

$$\phi(x,y,z) \sim 2i\nu e^{\nu z}\int_{-\infty}^{\infty} dk e^{ikx}\sigma^*(k) \frac{e^{-i\sqrt{\nu^2-k^2}y}}{\sqrt{\nu^2-k^2}} . \tag{25}$$

For small $|k|$, we can approximate

$$\frac{e^{-i\sqrt{\nu^2-k^2}y}}{\sqrt{\nu^2-k^2}} \quad \text{by} \quad \frac{e^{-i\nu y}}{\nu} .$$

If it were legitimate to use this approximation for all k , the integral would reduce simply to the inverse transform of $\sigma^*(k)$, that is, to $\sigma(x)$. In fact, it probably is legitimate to do this. If $\sigma(x)$ were an analytic function of x , its transform would act like $e^{-|k|}$ as $k \longrightarrow \infty$, and so any contribution to the integral from the neighborhood of $|k| = \nu$ would contain a factor $e^{-|k|} = O(e^{-\nu}) = O(e^{-1/\varepsilon})$. Thus the singularity in the integrand would have negligible effect on the value of the integral. Now $\sigma(x)$ is certainly not analytic, since it is identically zero outside of the interval $-L/2 < x < L/2$. So let us consider how $\sigma^*(k)$ is likely to behave near $|k| = \nu$. The source density is probably proportional to cross-section area, for any given frequency and amplitude of oscillation. If the ship is pointed (neither bluff nor cusped), this means that, near $x = L/2$,

$$\sigma(x) \propto \begin{cases} (x-L/2)^2 , & x < L/2 , \\ 0 , & x > L/2 . \end{cases}$$

The transform of a function with such a singularity behaves like

$$\sigma^*(k) \quad \propto \quad e^{-ikL/2}/k^3$$

for large k . (See Lighthill (1958).) The same argument applies at $x = -L/2$. Thus, one may guess that $\sigma^*(k) = O(\varepsilon^{-3})$ near $|k| = \nu$. The contribution to the integral from the neighborhood of the singularity will again be negligibly small, although it will not be $O(e^{-1/\varepsilon})$. Even for a blunt body, the same conclusion will hold, although the basic assumptions of slender-body theory cast some doubt on any results obtained in such a case.

Since we apparently need to consider only the contribution to the integral from moderate values of $|k|$, we use the simple approximation of the integrand given previously, and we find that:

$$\phi(x,y,z) \sim 2ie^{\nu(z-iy)} \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k)$$

$$= 4\pi i \; \sigma(x) \; e^{\nu(z-iy)} \; .$$

This is identical to the result in (23), which gives additional credence to the plausible assumptions made above about $\sigma^*(k)$.

<u>The near-field problem.</u> We use the ordinary methods of slender-body theory to simplify the conditions appropriate to the near-field problem. The details will be omitted, for one may refer to the later sections and simply set $U = 0$ therein. If $\delta = 0$, there is no fluid motion at all, and so one may expect that the first problem will be linear in δ . The first term in the asymptotic series for the velocity potential will here be denoted by:

$$\underline{\text{Re}}\{\phi(x,y,z)e^{i\omega t}\} \ .$$

The function $\phi(x,y,z)$ satisfies the following conditions in the near field:

$$\text{(L)} \quad \phi_{yy} + \phi_{zz} = 0 \ , \qquad \text{in} \quad z < 0 \ ;$$

$$\text{(F)} \quad \phi_z - \nu\phi = 0 \ , \qquad \text{on} \quad z = 0 \ ;$$

$$\text{(H)} \quad \phi_n e^{i\omega t} = \frac{\dot{\xi}_3(t) - x\dot{\xi}_5(t)}{\sqrt{1 + h_{0_y}^2}} \ , \qquad \text{on} \quad z = h_0(x,y) \ .$$

These should be compared with Equations (12). The discussion following the latter is again appropriate. Here we only remind the reader that the problem set by the above conditions is not complete; we must add a radiation condition, given by (23). Then the problem is properly posed, and one may proceed to solving it—most likely by numerical methods. (See Appendix B.)

STEADY FORWARD MOTION

The use of ordinary slender-body theory for solving the problem of steady forward motion of a ship is in rather ill repute. Neither author has been completely able to accept this judgment, but aside from matters of faith there is good reason to expect that the much maligned slender-body theory may serve as a satisfactory basis for treating the ship-motion problem, namely, that it has proved rather accurate in other problems involving primarily the prediction of forces normal to the mean direction of travel. Thus, in the prediction of sinkage and trim in shallow water*, it has given good results except near the critical speed for shallow-water problems. More relevant, the present theory is based on slender-body theory, and the formulas for added-mass and damping coefficients appear to be at least as good as those obtained by strip theory, which are known to be fairly accurate.

In this section, we do no more than collect a few results which will be needed. Details may be found in Tuck (1965).

Let the velocity potential be expressed:

$$\phi(x,y,z) = U[x + \chi(x,y,z)] \quad . \tag{26}$$

The first term in an asymptotic expansion for $\chi(x,y,z)$ in the near field satisfies the conditions:

(L) $\chi_{yy} + \chi_{zz} = 0 ;$

(A) $\frac{g}{U^2}\zeta = -\chi_x - \frac{1}{2}\chi_y^2 ,$ on $z = 0 ;$

(B) $\chi_z = 0$ on $z = 0 ;$

(H) $\frac{\partial\chi}{\partial n} = \frac{h_{0_x}}{\sqrt{1 + h_{0_y}^2}}$ on $z = h_0(x,y)$.

*See, for example, Tuck (1966) and Tuck (1967).

At large distance from the body, the potential behaves like $a(x)\log r$ plus a function of x only, where $r = \sqrt{y^2+z^2}$, and $a(x)$ is proportional to the longitudinal rate of change of cross-section area. This "radiation" condition must be derived by matching the near-field solution to a far-field solution. Otherwise, the boundary-value problem is set by conditions (L), (B), and (H). Condition (A) serves to predict the free-surface shape once the boundary-value problem has been solved.

The following facts will all be needed later:

(1) $O(\chi) = O(\chi_x) = O(\zeta) = O(\varepsilon^2)$.

(2) $O(\chi_y) = O(\chi_z) = O(\varepsilon)$.

(3) $O(\chi_{yy}) = O(\chi_{zz}) = O(1)$.

(4) $\chi_y(x,y_0(x),0) = y_0'(x)$, where $y_0(x)$ is the half-beam on the plane of the undisturbed free surface.

We also call attention to the properties of $\chi(x,y,z)$ which are demonstrated in Appendix A.

THE GENERAL PROBLEM: THE NEAR FIELD

The general formulation has been given in Equations (1) to (6'). Now we introduce the following notation:

$$\phi(x,y,z,t) = Ux + U\chi(x,y,z) + \psi(x,y,z,t) , \tag{27}$$

in which $\chi(x,y,z)$ has the same meaning as in the preceding section. Thus, $\psi(x,y,z,t)$ includes everything that must be added to the steady-motion potential function. The hull surface is again defined mathematically:

$$z - h_0(x,y) - \xi_3(t) + x\xi_5(t) = 0 \tag{9b}$$

It proves convenient to define also:

$$\zeta(x,y,t) = \eta(x,y) + \theta(x,y,t) , \tag{28}$$

where $\eta(x,y)$ is the free-surface disturbance in the steady-motion problem (the ζ of the previous section), and $\theta(x,y,t)$ includes all further disturbances found in the present section. The problem now is to find the conditions to be satisfied by $\psi(x,y,z,t)$ and $\theta(x,y,t)$.

We make the assumptions about orders of magnitude stated in (7), (7'), (8), (8'), and (10). It is fairly trivial to prove that

$$\text{(L)} \qquad \psi_{yy} + \psi_{zz} = 0 . \tag{29}$$

Thus, we again have a potential problem in two dimensions. Fortunately, (29) is valid for the first two terms in an asymptotic expansion of $\psi(x,y,z,t)$.

We use the body boundary condition in the form (6):

$$\text{(H)} \qquad 0 = [U(1+\chi_x) + \psi_x][h_{0_x} - \xi_5] + [U\chi_y + \psi_y]\, h_{0_y}$$
$$- [U\chi_z + \psi_z] + (\dot{\xi}_3 - x\dot{\xi}_5) \tag{30}$$
$$\text{on} \quad z = h_0(x,y) + \xi_3 - x\xi_5 \;.$$

It is inconvenient to have to apply this condition on the instantaneous position of the hull, and so we assume that the expression in (30) can be expanded in a Taylor series about the mean hull position. To carry this out, we apply the operator

$$\{1 + (\xi_3 - x\xi_5)\; \partial/\partial z + \ldots\}$$

to (30) and evaluate everything on $z = h_0(x,y)$. The result is:

$$0 = \underset{[\varepsilon]}{Uh_{0_x}} - \underset{[\varepsilon]}{U\chi_z} + \underset{[\varepsilon]}{U\chi_y h_{0_y}} + \underset{[\varepsilon^{1/2}\delta]}{\psi_y h_{0_y}} - \underset{[\varepsilon^{1/2}\delta]}{\psi_z} + \underset{[\varepsilon^{1/2}\delta]}{(\dot{\xi}_3 - x\dot{\xi}_5)}$$

$$- \underset{[\varepsilon\delta]}{U\xi_5} + \underset{[\varepsilon\delta]}{U(\xi_3 - x\xi_5)}(h_{0_y}\chi_{yz} - \chi_{zz}) + \ldots ,$$
$$\text{on} \quad z = h_0(x,y) .$$

Under each term we have indicated its order of magnitude. This can only be done with information yet to be obtained, but it gives some extra clarity here to see the relative orders of magnitude, and so we take this liberty. Some terms have been dropped already because they subsequently appear to be of higher order of magnitude.

In the steady-motion problem, the body boundary condition can be written:

$$0 = Uh_{0_x} - U\chi_z + U\chi_y h_{0_y} \,, \quad \text{on} \quad z = h_0(x,y) \,,$$

and so we subtract these three terms from the previous result, leaving:

$$0 = \psi_y h_{0_y} - \psi_z + (\dot{\xi}_3 - x\dot{\xi}_5)$$

$$- U\xi_5 + U(\xi_3 - x\xi_5)(h_{0_y}\chi_{yz} - \chi_{zz})$$

$$\text{on} \quad z = h_0(x,y) \,.$$

We can now write this condition in our canonical form (Cf. (6')):

$$\frac{\partial\psi}{\partial n} = \frac{\psi_z - h_{0_y}\psi_y}{\sqrt{1+h_{0_y}^2}}$$

$$= \underset{[\varepsilon^{1/2}\delta]}{\frac{\dot{\xi}_3 - x\dot{\xi}_5}{\sqrt{1+h_{0_y}^2}}} + \underset{[\varepsilon\delta]}{\frac{U(\xi_3 - x\xi_5)(h_{0_y}\chi_{yz} - \chi_{zz}) - U\xi_5}{\sqrt{1+h_{0_y}^2}}} \quad \text{on} \quad z = h_0(x,y) \,. \qquad (31)$$

It is evident that we are carrying along two different orders of magnitude, the two differing by $O(\varepsilon^{1/2})$; we shall continue to do so. The lower-order term is, by itself, identical to the normal velocity component in the zero-speed problem, as given in (12c). The higher-order term leads to interactions between the oscillations and the forward motion. Equation (31) can also be obtained by utilizing the body boundary condition formulated by Timman and Newman (1962).

We perform the same analysis on the two free-surface conditions, (3) and (4). For these, we assume that the potential function can be expanded in a Taylor series about $z = 0$. The results are:

(A) $\psi_t + g\theta = - U\psi_x - U(\chi_y \psi_y)$; (32)

$[\varepsilon\delta]$ $[\varepsilon^{3/2}\delta]$

(B) $\psi_z - \theta_t = U\theta_x + U\chi_y \theta_y + U\chi_{yy}\theta$. (33)

$[\varepsilon^{1/2}\delta]$ $[\varepsilon\delta]$

Both are to be applied on $z = 0$. We can eliminate $\theta(x,y,t)$ from the two conditions, the result consistent to order $O(\varepsilon\delta)$ being:

(F) $\psi_{tt} + g\psi_z = - 2U\psi_{tx} - 2U\chi_y \psi_{ty} - U\chi_{yy}\psi_t$ (34)

$[\varepsilon^{1/2}\delta]$ $[\varepsilon\delta]$

Equation (31) suggests that $\psi(x,y,z,t)$ will have an asymptotic expansion starting with a term which is $O(\varepsilon^{3/2}\delta)$, followed by a term which is $O(\varepsilon^2\delta)$. There is nothing in (L) or (F) to contradict this suggestion, and, as we shall find, it is completely compatible with the far-field solution. It can be shown that the unsteady free-surface disturbance, $\theta(x,y,t)$, is $O(\varepsilon\delta)$.

As in the zero-speed problem (See Equations (13) and (14) and the accompanying discussion), it is convenient to define normalized potential functions which do not depend on the motion variables. In the forward-speed problem, we find it desirable to define several sets of such potential functions. First, we note the following definitions from Appendix A:

$$n_3 \cong (1+h_{0_y}^2)^{-1/2} ;$$

$$n_5 \cong - x(1+h_{0_y}^2)^{-1/2} ;$$

$$m_3 \cong (\chi_{yz} h_{0_y} - \chi_{zz})(1+h_{0_y}^2)^{-1/2} ;$$

$$m_5 \cong [-1 - x\ (\chi_{yz} h_{0_y} - \chi_{zz})](1+h_{0_y}^2)^{-1/2} .$$

The quantities n_i are the **six** components of a generalized unit normal vector, and **the** m_i **are related** to the rate of change on the hull of the **vector** $\underline{v}$, the fluid velocity in the steady-motion problem. With this notation, we define the potential functions, $\Phi_i(x,y,z)(i = 3,5)$:

$$\left.\begin{aligned} \Phi_{i_n} &= n_i && \text{on} \quad z = h_0(x,y) \; ; \\ \Phi_{i_z} - \nu\Phi_i &= 0 && \text{on} \quad z = 0 \; . \end{aligned}\right\} \qquad (35)$$

Since $\dot{\xi}_j = i\omega\xi_j$, we can show that the potential function $\psi - \sum_j (i\omega\Phi_j\xi_j)$ has a normal derivative on the body given by the second term on the right-hand side of (31), and on the free surface (that is, on $z = 0$) it satisfies the same condition, (34), as ψ alone. Thus, we still have two nonhomogeneous boundary conditions to satisfy. We divide the job between two sets of potentials. Define Ψ_i and Ω_i by these problems:

$$\left.\begin{aligned} \Psi_{i_n} &= m_i && \text{on} \quad z = h_0(x,y) \; ; \\ \Psi_{i_z} - \nu\Psi_i &= 0 && \text{on} \quad z = 0 \; ; \end{aligned}\right\} \qquad (36)$$

$$\left.\begin{aligned} \Omega_{i_n} &= 0 && \text{on} \quad z = h_0(x,y) \; ; \\ \Omega_{i_z} - \nu\Omega_i &= -(1/g)[2\Phi_{i_x} + 2\chi_y\Phi_{i_y} + \chi_{yy}\Phi_i] && \text{on} \quad z = 0 \; . \end{aligned}\right\} \qquad (37)$$

If we define:

$$\psi(x,y,z,t) = \sum_j \; [i\omega\Phi_j + U\Psi_j - \omega^2 U\Omega_j]\xi_j(t) \; , \qquad (38)$$

we find that it satisfies all boundary condition previously imposed on ψ .

The first set of new functions, Φ_i , is obviously very closely related to the Φ introduced in (13). In fact, the present Φ_3 is identical to the previous Φ , and our Φ_5 can be shown to be just $-x\Phi$. Thus, the Φ_i -problem is really just the zero-speed problem again.

The second set of new functions, Ψ_i , satisfies the ordinary homogeneous linearized free-surface boundary condition, and on the body it supplies the effects of interactions between oscillations and the incident stream.

The third set is unlike either of the other two. Ω_i has zero normal derivative on the hull, but on the free surface it is nonhomogeneous in a way not encountered before in our analysis. In fact, if we were to apply a pressure distribution,

$$p_i(x,y,t) = i\rho\omega U\xi_i[2\Phi_{i_x} + 2\chi_y\Phi_{i_y} + \chi_{yy}\Phi_i] \; , \tag{39}$$

on the free surface, we would obtain exactly the same boundary condition for a velocity potential as the condition on $-\omega^2 U\Omega_i\xi_i$.

One can make an analogy between the problems for Ψ_i and Ω_i . The former arises because we must satisfy the body boundary condition on the instantaneous position of the body surface. In the steady-motion problem, χ satisfied a condition on the undisturbed position of the hull, and so χ must now be corrected. This correction can be obtained approximately by making χ satisfy (31), in which the second term on the right-hand side leads to the effect being considered. Thus, Ψ_i is an oscillatory correction to the steady-motion potential which comes about because the latter satisfies a condition on the wrong boundary.

The situation with Ω_i is similar but slightly more complicated. There is, first of all, a kinematic effect at the free surface entirely comparable to that on the body. If we were to correct for just this effect, the right-hand side of (34) would contain the terms

$$- U\psi_{tx} - U\chi_y\psi_{ty} - U\chi_{yy}\psi_t = g[U\theta_x + U\chi_y\theta_y - U\chi_{zz}\theta] .$$

These terms have enough direct similarity to the pertinent terms of (31) to show the analogy. (The basic difference in nature between the surface of the body and the free surface accounts for the superficial difference between the results.) In addition, however, there is a dynamic condition on the free surface, and this leads to a further effect of the same kind. Because the steady-motion potential satisfies the dynamic condition on the wrong surface, one must add an oscillatory pressure distribution to offset the error. This pressure has the effect, as it turns out, of doubling two of the three terms written above. We conclude that the potential functions Ω_i represent an interaction effect which is just as legitimate as that represented by Ψ_i .

None of the above boundary-value problems is complete yet, for we have not provided radiation conditions. We might proceed intuitively and suppose that the potentials should represent outgoing waves at infinity. We would, in fact, be correct--with respect to Φ_i and Ψ_i only. This will be discussed further when we come to match the near-field solution to the far-field solution.

Even without any considerations of matching asymptotic expansions, we can see immediately that there is trouble with the Ω_i -problem. Let us suppose that we have solved the χ -problem and the Φ_i -problem, the latter indeed representing outgoing waves. Then the problem for Ω_i involves a fixed impermeable body ($\Omega_{i_n} = 0$ on body) and an imposed pressure distribution on the free surface, specified in Equation (39). One might consider that this is a well-known solved problem. (See Equation (21.17), Wehausen and Laitone (1960).) But the formula for the answer requires, first of all, that the pressure distribution be absolutely integrable, and ours is not, for it oscillates sinusoidally all the way to infinity. There is still more serious trouble. If the pressure were set equal to zero, the boundary condition would allow for the existence of "free waves" of wave number ν . The equivalent pressure as given above includes one part which has exactly the same time and space periodicity as the free waves. The situation is quite analogous to driving an undamped linear oscillator at its natural frequency; no steady-state solution is possible and, starting from rest, the response grows in time without limit. In the present problem, one may consider that the free surface provides the mechanism of an oscillator, undamped except for the possibility of radiation damping. Since the pressure excitation extends to infinity without abatement, even radiation damping cannot stabilize the resulting motion. It appears that no solution is possible for the Ω_i -problem.

However, in Appendix D, we formulate a problem which can be solved and which, after certain limiting operations, can be considered to be equivalent to the present problem. Neglecting for the moment the presence of the body, we assume that there is an applied pressure distribution on the free surface given by:

$$p(x,y,t) = \underline{Re}\left[p(x,y)\ e^{i\omega t}\right] \quad , \tag{40}$$

where

$$p(x,y) = p_0(x)\ e^{-i\nu|y|} \quad . \tag{40'}$$

This artificial problem includes the essence of our difficulty*. It is shown in the Appendix that a possible solution can be obtained which has an outer expansion given by:

$$\phi(x,y,t) \sim \frac{\omega}{\rho g} e^{\nu z} \underline{\mathrm{Re}} \left[i p_0(x) (-z + i|y|) e^{i(\omega t - \nu|y|)} \right] . \quad (41)$$

If we interpret $p_0(x) e^{-i\nu|y|}$ as the oscillatory part of (39), this asymptotic behavior should apply also to the terms in ψ containing Ω_i. Physically, this outer expansion appears to represent outgoing waves which have an amplitude increasing linearly with distance from the origin. (Compare the analogy with the undamped spring-mass system.) The fact that this outer expansion is ill-behaved at infinity causes no concern, because it is the outer expansion of an inner expansion, and the latter is not expected to be valid at infinity. All that we must insist upon is that it should match the inner expansion of an outer expansion, and this it does.

<u>Outer expansion of the inner expansion</u>. Since we have defined:

$$\psi(x,y,z,t) = \sum_j [i\omega\Phi_j + U\Psi_j - \omega^2 U\Omega_j]\xi_j(t) , \quad (38)$$

the outer expansion of the time-dependent near-field solution can be expressed in terms of the outer expansions of Φ_j, Ψ_j, and Ω_j. We now assume that the first two have outer expansions given by:

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*After solving this problem, we could calculate the resulting normal derivative of the potential function on the body surface, then formulate a second problem involving a nonhomogeneous boundary condition on the body and a homogeneous free-surface condition. The second problem would be identical in form to the problems for Φ_i and Ψ_i.

$$i\omega\Phi_j(x,y,z)\xi_j(t) \sim a_j(x)\; e^{\nu z}e^{i(\omega t - \nu|y|)} = O(\varepsilon^{3/2}\delta) \;, \qquad (42a)$$

$$U\Psi_j(x,y,z)\xi_j(t) \sim b_j(x)\; e^{\nu z}e^{i(\omega t - \nu|y|)} = O(\varepsilon^{2}\delta) \;, \qquad (42b)$$

where $a_j(x)$ and $b_j(x)$ can be determined from the (numerical) solution of the relevant near-field problems. The justification for this assumption comes in the matching to be performed presently. The orders of magnitude shown are based on all of the available near-field information.

The functions Ω_j will have one component in their outer expansions which are of the same form as those in (42a) and (42b). The only difficulty arises because of the term Φ_{j_x} in (37). In view of the discussion leading up to (41), we see that we can set:

$$p_0(x) = 2\rho U\; a_j'(x) \;. \qquad (43)$$

The velocity potential for such an applied-pressure problem would satisfy:

$$\phi_z - \nu\phi = -2i\tau a_j'(x)\; e^{i(\omega t - \nu|y|)} \qquad \text{on} \quad z = 0 \;.$$

From (37) and (42a) we see that:

$$-\omega^2 U\xi_j(t)[\Omega_{jz} - \nu\Omega_j] = -i\tau[2\Phi_{jx} + 2\chi_y\Phi_{j_y} + \chi_{yy}\Phi_j]i\omega\xi_j(t)$$

$$\sim\; -2i\tau a_j'(x)\; e^{i(\omega t - \nu|y|)} \;.$$

Thus we can identify the asymptotic behavior of $-\omega^2 U\xi_j\Omega_j$ with the asymptotic behavior of the potential in the artificial applied-pressure problem. In addition to the growing-wave behavior, there will be a simple outgoing wave of constant amplitude, and so we take as the asymptotic form of Ω_j :

$$-\omega^2 U\Omega_j(x,y,z)\xi_j(t) \sim \underset{[\varepsilon^2\delta]}{c_j(x)}\ e^{\nu z}e^{i(\omega t - \nu|y|)}$$
$$\qquad -2i\tau \underset{[\varepsilon\delta]}{a_j'(x)}\ e^{\nu z}(z - i|y|)\ e^{i(\omega t - \nu|y|)} \ . \tag{42c}$$

For $\psi(x,y,z,t)$ we have:

$$\psi(x,y,z,t) \sim \sum_j \left[\underset{[\varepsilon^{3/2}\delta]}{a_j(x)} - 2i\tau \underset{[\varepsilon\delta]}{a_j'}(z - i|y|)\right] e^{\nu z}\ e^{i(\omega t - \nu|y|)} \ . \tag{44}$$

This is a two-term outer expansion of a two-term inner expansion. (Note that we consider $y = O(1)$ in deciding where to cut off this expansion. Thus the growing-wave term is the leading term, and the terms containing $b_j(x)$ and $c_j(x)$ are both $O(\varepsilon^2\delta)$, that is, negligible.)

MATCHING THE EXPANSIONS

In Appendix C, we present the details of the far-field problem. It is necessary to obtain a far-field solution with only sufficient generality that it can be matched to the near-field solution. The solution in the appendix pertains to the flow around a line of pulsating sources. It is not necessary to include any higher-order singularities. In fact, there is reason to suppose that singularities of any kind could be used (provided they have the proper lateral symmetry), for the inner expansion of the far-field solution is shown to represent outgoing waves and it is not at all evident that the nature of those waves depends on the precise kind of singularities assumed.

In any case, having assumed a line of sources of density $\sigma(x)\, e^{i\omega t}$, we show in Appendix C that the far-field solution is:

$$\phi(x,y,z)\, e^{i\omega t} = -\frac{1}{\pi} e^{i\omega t} \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k) \int_{-\infty\, C(k)}^{\infty} \frac{d\ell \exp i\ell y + z\sqrt{k^2 + \ell^2}}{\sqrt{k^2 + \ell^2} - \frac{1}{g}(\omega + Uk)^2},$$

which has a two-term inner expansion:

$$4\pi i\, e^{\nu z} \left[\sigma(x) - 2i\tau(z - i|y|)\sigma'(x)\right] e^{i(\omega t - \nu|y|)} . \tag{45}$$

This inner expansion must match the outer expansion of the near-field solution, given in (44). We see that one need only set:

$$4\pi i \sigma(x) = \sum_j a_j(x) \tag{46}$$

and both terms match. Thus, we can select the source strength on the basis of the solution of the Φ_j -problems, and the strange growing-wave terms automatically match. It is this result that justifies much of what went before, viz., the interpretation adopted for the Ω_j -problems and the form assumed for the asymptotic behavior of Φ_j , Ψ_j , and Ω_j .

It may now be noted that $\sigma(x) = O(a_j) = O(\varepsilon^{3/2}\delta)$. In (45), the two terms are $O(\varepsilon^{3/2}\delta)$ and $O(\varepsilon^2\delta)$, respectively. The second term is higher order than the first, which contrasts with the result found in (44). This occurs because $y = O(\varepsilon)$ in (45), which is an inner expansion of an outer expansion, and $y = O(1)$ in (44), which is an outer expansion of an inner expansion.

Having found the outer solution and matched it to the inner solution, we can henceforth ignore it, for it has served its only purpose, namely, to provide a radiation condition on the inner solution. It may well be appreciated now that the boundary-value problem set by (37) could not be solved without the insight provided by the outer solution. The whole interpretation of the applied-pressure problem and thus of the Ω_j -problem could not be justified (or perhaps even suspected) without the knowledge that the 2-D flow appears to be a superposition of ordinary outgoing waves plus growing outgoing waves at large distance from the ship.

FORCE AND MOMENT ON THE SHIP

In principle, the computation of this section is very simple. We use the near-field expression for the velocity potential, substitute it into the Bernoulli equation, multiply the pressure by appropriate direction cosines and lever arms, and integrate over the hull surface. In effect, this is all that we do.

But we want the results to be in a usable form, and this requires that, at the least, (1) we should keep to a minimum the number of boundary-value problems which must be solved numerically and (2) we should avoid having to perform longitudinal differentiations on these velocity potentials. (The reason for the second demand is that the numerical solutions all pertain to boundary-value problems in two dimensions; derivatives with respect to x would have to be found by processes of numerical differentiation.) It takes a considerable amount of manipulation to satisfy these demands, and, as one might suspect, the Ω_j -functions cause most of the difficulty.

Before starting on this program, let us gather together a number of formulas which will be needed. The potential function is given by:

$$\phi(x,y,z,t) = Ux + U\chi(x,y,z) + \psi(x,y,z,t) \ . \qquad (27)$$

The first two terms satisfy the conditions of the steady-motion problem. For large $|y|$,

$$\chi(x,y,z) \propto \log(y^2 + z^2)^{1/2} + \text{a function of } x \ . \qquad (47)$$

If we set $\underline{v} = \nabla[x + \chi(x,y,z)]$, then $\underline{v}$ has the property:

$$\int_{S_0} n_i(\underline{v}\cdot\nabla\phi)dS = -\int_{S_0} m_i\phi dS \ , \qquad (48)$$

as proven in Appendix A. In Equation (48), ϕ may be any differentiable scalar function of (x,y,z), but, in our application of the theorem, ϕ will represent Φ_3 or Φ_5. (See (35), which is repeated below.)

The time-dependent part of the potential function has the following form in the near field:

$$\psi(x,y,z,t) = \sum_j \left[i\omega\Phi_j + U\Psi_j - \omega^2 U\Omega_j\right]\xi_j(t) , \qquad (38)$$

where the sum is taken over $j = 3,5$, $\xi_3(t)$ and $\xi_5(t)$ are the heave and pitch variables, respectively, and the 2-D potential functions Φ_j, Ψ_j, and Ω_j are defined by the conditions:

$$\Phi_{j_n} = n_j \text{ , on } z = h_0(x,y) \text{ ; } \Phi_{j_z} - \nu\Phi_j = 0 \text{ , on } z = 0 \text{ ; } \qquad (35)$$

$$i\omega\Phi_j(x,y,z)\xi_j(t) \sim a_j(x)\, e^{\nu z} e^{i(\omega t - \nu|y|)} \text{ , as } |y| \to \infty \text{ ; } \qquad (42a)$$

$$\Psi_{j_n} = m_j \text{ , on } z = h_0(x,y) \text{ ; } \Psi_{j_z} - \nu\Psi_j = 0 \text{ , on } z = 0 \text{ ; } \qquad (36)$$

$$U\Psi_j(x,y,z)\xi_j(t) \sim b_j(x)\, e^{\nu z} e^{i(\omega t - \nu|y|)} \text{ , as } |y| \to \infty \text{ ; } \qquad (42b)$$

$$\Omega_{j_n} = 0 \text{ on } z = h_0(x,y) \text{ ; }$$
$$\Omega_{j_z} - \nu\Omega_j = -(1/g)\left[2\Phi_{j_x} + 2\chi_y\Phi_{j_y} + \chi_{yy}\Phi_j\right] \text{ on } z = 0 \text{ ; } \qquad (37)$$

$$-\omega^2 U\Omega_j(x,y,z)\xi_j(t) \sim c_j(x)\, e^{\nu z} e^{i(\omega t - \nu|y|)}$$
$$-2i\tau a_j'(x)\, e^{\nu z}(z - i|y|)\, e^{i(\omega t - \nu|y|)} ,$$
$$\text{as } |y| \to \infty . \qquad (42c)$$

The orders of magnitude of the various quantities should be recalled too:

$$\Phi_j = O(\varepsilon) \; ; \qquad \Psi_j = O(\varepsilon) \; ;$$

$$\Omega_j = O(\varepsilon^2) \; ; \qquad \chi = O(\varepsilon^2).$$

The properties of n_j and m_j are listed in Appendix A.

The first step in calculating the force and moment is to express the pressure in the fluid. From the Bernoulli equation, we obtain the estimate:

$$-\frac{p}{\rho} = gz + \psi_t + \frac{1}{2}\left[(U+U\chi_x+\psi_x)^2 + (U\chi_y+\psi_y)^2 + (U\chi_z+\psi_z)^2 - U^2\right]$$

$$\sim \underset{[\varepsilon]}{gz} + \underset{[\varepsilon\delta]}{\psi_t} + \underset{[\varepsilon^{3/2}\delta]}{U\psi_x} + \underset{[\varepsilon^{3/2}\delta]}{U\chi_y\psi_y} + \underset{[\varepsilon^{3/2}\delta]}{U\chi_z\psi_z} \,,$$

the second result being valid after we drop all quantities which are $o(\varepsilon^{3/2}\delta)$. Note that we shall be evaluating the pressure on the hull, and so the hydrostatic pressure term is $O(\varepsilon)$. We could substitute the coordinates of a point on the hull, $z = h_0(x,y) + \xi_3(t) - x\xi_5(t)$, into the above expression and we could then integrate directly to find the force and moment. However, it is much more convenient to be able to evaluate the pressure on the undisturbed or mean position of the hull. To do this, we assume that the pressure expression can be expressed as a Taylor series about the undisturbed position of the hull. To a consistent approximation, we find that:

$$-\frac{p}{\rho} - gz = +g(\xi_3 - x\xi_5) + \psi_t + U\psi_x + U\chi_y\psi_y + U\chi_z\psi_z \,.$$

The first term gives just the ordinary buoyancy restoring force and moment, and so we shall ignore it hereafter.

The hydrodynamic force (moment) can now be written:

$$F_i(t) = \int_{S_0} n_i \; p \; dS \; ,$$

where $i = 3$ gives the vertical force and $i = 5$ gives the pitch moment. We substitute the pressure expression into the integral (omitting the buoyancy term), and we use the theorem of Appendix A to rewrite the result:

$$F_i(t) = -\rho \int_{S_0} dS \; \Big(\sum_j \left[n_i (i\omega)^2 - Um_i (i\omega) \right] \Phi_j \xi_j$$

$$+ Un_i \sum_j \left[(i\omega) \Psi_j + (i\omega)^3 \Omega_j \right] \xi_j \Big) \; .$$

It is convenient to break this into several parts, as follows:

$$F_i(t) = \sum_j T_{ij} \xi_j(t) \; , \qquad (49)$$

where

$$T_{ij} = T_{ij}^{(0)} + T_{ij}^{(1)} + T_{ij}^{(2)} \; , \qquad (49')$$

and

$$T_{ij}^{(0)} = -\rho (i\omega)^2 \int_{S_0} dS \; n_i \Phi_j \; ; \qquad (49a)$$

$$T_{ij}^{(1)} = -\rho (i\omega) \; U \int_{S_0} dS [n_i \Psi_j - m_i \Phi_j] \; ; \qquad (49b)$$

$$T_{ij}^{(2)} = -\rho (i\omega)^3 U \int_{S_0} dS \; n_i \Omega_j \; . \qquad (49c)$$

We now treat each of these components in turn.

$\underline{T_{ij}^{(0)}}$. It is easily seen from (49) that the first, $T_{ij}^{(0)}$, is the only one that does not depend on U , and so it yields the zero-speed formulas. No simplification is really possible. Comparison of (35) and the result in Appendix A: $n_5 = -xn_3[1 + O(\varepsilon^2)]$, shows that

$$\Phi_5(x,y,z) \cong -x\Phi_3(x,y,z) \quad . \tag{50}$$

Using these facts, we write out $T_{ij}^{(0)}$ for the four cases at hand:

$$T_{33}^{(0)} = -\rho(i\omega)^2 \int_{S_0} dS \; n_3 \; \Phi_3 \; ; \tag{51a}$$

$$T_{55}^{(0)} = -\rho(i\omega)^2 \int_{S_0} dS \; n_3 \; x^2 \; \Phi_3 \; ; \tag{51b}$$

$$T_{35}^{(0)} = T_{53}^{(0)} = \rho(i\omega)^2 \int_{S_0} dS \; n_3 \; x\Phi_3 \; . \tag{51c}$$

$\underline{T_{ij}^{(1)}}$. In order not to carry along extra constants, let us define:

$$T_{ij}^{(1)} = -\rho(i\omega) \; U \; I_{ij}^{(1)} \quad , \tag{52}$$

where

$$I_{ij}^{(1)} = \int_{S_0} dS \; (n_i\Psi_j - m_i\Phi_j) \quad .$$

Using (35) and (36), we replace n_i and m_i in this integral by Φ_{i_n} and Ψ_{i_n} , respectively:

$$I_{ij}^{(1)} = \int_{S_0} dS \, (\Phi_{i_n} \Psi_j - \Psi_{i_n} \Phi_j) \ .$$

<u>Case 1</u>. Let $i = j$. The integral is in just the form for applying Green's theorem, except that the surface of integration must be closed. We add to the surface S_0 three more surfaces: the undisturbed free surface, F_0 , a vertical control surface far away, S_∞ , and a horizontal control surface at great depth. If the above integral is taken over all of these surfaces, the result must be zero. Obviously, the surface at great depth contributes nothing, and so we have:

$$I_{ii}^{(1)} = -\int_{F_0} dS \, (\Phi_{i_n} \Psi_i - \Psi_{i_n} \Phi_i) - \int_{S_\infty} dS \, (\Phi_{i_n} \Psi_i - \Psi_{i_n} \Phi_i) \ .$$

(Note that we are applying Green's theorem to potential functions defined in two dimensions only. Thus, we need no control surfaces parallel to the y-z plane.) The integral over F_0 contributes nothing, for $\Phi_{i_n} = \Phi_{i_z}$ on $z = 0$ (and similarly for Ψ_i), and so the integrand is zero if we substitute from (35) and (36). Similarly, we can use the asymptotic estimates of Φ_i and Ψ_i for large $|y|$ to show that the integrals over the surface S_∞ vanish. Thus,

$$T_{33}^{(1)} = T_{55}^{(1)} = 0 \ . \qquad (53a)$$

<u>Case 2</u>. For $i \neq j$, the result is different. We add and subtract a quantity, as follows:

$$I_{35}^{(1)} = \int_{S_0} dS\ (\Phi_{3_n}\Psi_5 - \Psi_{3_n}\Phi_5)$$

$$= \int_{S_0} dS\ (\Phi_{3_n}\Psi_5 - \Phi_3\Psi_{5_n}) + \int_{S_0} dS\ (\Phi_3\Psi_{5_n} - \Psi_{3_n}\Phi_5) \ .$$

The first integral contributes nothing, as can be shown by an argument like that used in Case 1. In the second integral, we substitute from (36):

$$I_{35}^{(1)} = \int_{S_0} dS\ (m_5\Phi_3 - m_3\Phi_5) \ .$$

From Appendix A, we draw the fact that $m_5 = -(n_3 + xm_3)[1 + O(\varepsilon^2)]$, which we substitute into the last equation, obtaining:

$$I_{35}^{(1)} = -\int_{S_0} n_3\Phi_3\ dS \ .$$

A similar analysis can be performed for $I_{53}^{(1)}$, and we find that, ultimately:

$$T_{35}^{(1)} = T_{53}^{(1)} = \rho(i\omega)\ U \int_{S_0} n_3\Phi_3\ dS = -\frac{U}{i\omega}\ T_{33}^{(0)} \ . \qquad (53b)$$

$\underline{T_{ij}^{(2)}}$. The problem here is quite different from those we have just studied, for the functions Ω_j satisfy a homogenous condition on the hull and a nonhomogeneous condition on the free surface. Moreover, Ω_j has a more complicated behavior at infinity.

Again, we define an integral $I_{ij}^{(2)}$:

$$T_{ij}^{(2)} = -\rho(i\omega)^3 U \, I_{ij}^{(2)} ,$$

where

$$I_{ij}^{(2)} = \int_{S_0} dS \; n_i \Omega_j = \int_{S_0} dS \; (\Phi_{i_n} \Omega_j - \Phi_i \Omega_{j_n}) .$$

We have used (35) to introduce Φ_i , and the extra term which we have added is equal to zero, since $\Omega_{j_n} = 0$ on S_0 , by (37). As before, we use Green's theorem to rewrite this integral in terms of integrals over F_0 and S_∞ :

$$I_{ij}^{(2)} = -\int_{F_0} dS \; (\Phi_{i_n} \Omega_j - \Phi_i \Omega_{j_n}) - \int_{S_\infty} dS \; (\Phi_{i_n} \Omega_j - \Phi_i \Omega_{j_n}) . \quad (54)$$

The surface S_∞ cannot really be at infinity, of course, but we assume that it is far enough away (at $|y| = R$) that we may use the estimate in (42c).

First we consider the integral over F_0 . We use the boundary conditions on Φ_i and Ω_i ((35) and (37)):

$$- \int_{F_0} dS \; (\Phi_{i_n} \Omega_j - \Phi_i \Omega_{j_n}) = - \int_{F_0} dS \Big[\nu \Phi_i \Omega_j - \nu \Phi_i \Omega_j + \Phi_i \Big(\frac{2}{g} \Phi_{j_x} + \frac{2}{g} \chi_y \Phi_{j_y} + \frac{1}{g} \chi_{yy} \Phi_j \Big) \Big]$$

$$= -\frac{2}{g} \int_{-L/2}^{L/2} dx \int_{y_0(x)}^{R} dy \; \Phi_i (2\Phi_{j_x} + 2\chi_y \Phi_{j_y} + \chi_{yy} \Phi_j) .$$

Since the problem is symmetric about $y = 0$, we have reduced the domain of integration to one side of the ship, and we have introduced the function $y_0(x)$ to denote the half-beam at any x. The term containing χ_{yy} can be integrated once by parts:

$$\int_{y_o(x)}^{R} dy\ \chi_{yy}\Phi_i\Phi_j = \chi_y\Phi_i\Phi_j \Big|_{y_o(x)}^{R} - \int_{y_o(x)}^{R} dy\ \chi_y \left[\Phi_{i_y}\Phi_j + \Phi_i\Phi_{j_y}\right] .$$

When we take note of (47) and also use the fact that $\Phi_5 = -x\Phi_3$, we find that in all cases the terms containing χ cancel each other except for one contribution from the lower limit in the integration by parts, and we have left:

$$- \int_{F_0} dS\ (\Phi_{i_n}\Omega_j - \Phi_i\Omega_{j_n}) = \frac{2}{g} \int_{-L/2}^{L/2} dx\,[\chi_y\Phi_i\Phi_j]_{y=y_o(x)} - \frac{4}{g} \int_{-L/2}^{L/2} dx \int_{y_o(x)}^{R} dy\ \Phi_i\Phi_{j_x} .$$

In the integral over S_∞, we must use the asymptotic estimates for Φ_i and Ω_i, as given in (42a) and (42c). We assume that we can differentiate these estimates, and, noting that $\partial/\partial n = \partial/\partial y$, we obtain:

$$- \int_{S} dS\ (\Phi_{i_n}\Omega_j - \Phi_i\Omega_{j_n}) = \frac{2i}{g\nu} e^{-2i\nu R} \int_{-L/2}^{L/2} dx\ a_i(x)\ a_j'(x) .$$

We combine this with the previous result:

$$I_{ij}^{(2)} = \frac{2}{g} \int_{-L/2}^{L/2} dx\,[\chi_y\Phi_i\Phi_j]_{y=y_o(x)} - \frac{4}{g} \int_{-L/2}^{L/2} dx \int_{y_o(x)}^{R} dy\ \Phi_i\Phi_{j_x} + \frac{2i}{g\nu} e^{-2i\nu R} \int_{-L/2}^{L/2} dx\ a_i(x) a_j'(x) . \qquad (55)$$

Clearly, we have trouble with the last term if we try to let R go to infinity. However, it may be observed that there is a similar trouble with the second term, for the integrand oscillates with undiminishing amplitude as y goes to infinity. Perhaps it is not surprising that these two difficulties cancel each other, and we now show this. For large y, we have:

$$\phi_i \phi_{j_x} \sim a_i(x) a_j'(x) \; e^{2\nu(z - iy)} \; .$$

We subtract and add just this quantity in the integrand of the second term of (55):

$$I_{ij}^{(2)} = \frac{2}{g} \int_{-L/2}^{L/2} dx \, [\chi_y \Phi_i \Phi_j]_{y=y_0(x)} - \frac{4}{g} \int_{-L/2}^{L/2} dx \int_{y_0(x)}^{R} dy \, [\Phi_i \Phi_{j_x} - a_i a_j' e^{-2i\nu y}]$$
$$+ \frac{2i}{g\nu} \int_{-L/2}^{L/2} dx \; a_i(x) a_j'(x) \; e^{-2i\nu y_0(x)} \; .$$

In the last equation, we can now set $R = \infty$ without further complications.

There is one final simplification which is possible because χ_y has the value $y_0'(x)$ when it is evaluated on S_0 at $z = 0$. (This follows from the wall-sidedness of the ship.) First we consider the case $i = j$. We note that:

$$\frac{d}{dx} \int_{y_0(x)}^{\infty} dy \left[\Phi_i^2 - a_i^2 e^{-2i\nu y}\right] = -y_0'(x) \left[\Phi_i^2 - a_i^2 e^{-2i\nu y}\right]_{y = y_0(x)}$$
$$+ 2 \int_{y_0(x)}^{\infty} dy \left[\Phi_i \Phi_{i_x} - a_i a_i' e^{-2i\nu y}\right] \; .$$

Then we have for $I_{ii}^{(2)}$:

$$I_{ii}^{(2)} = \frac{2}{g} \int_{-L/2}^{L/2} dx \left(y_0'(x) \left[\Phi_i^2 \right]_{y = y_0(x)} + \frac{i}{\nu} e^{-2i\nu y_0(x)} a_i(x) a_i'(x) \right.$$

$$- y_0'(x) \left[\Phi_i^2 - a_i^2(x) e^{-2i\nu y} \right]_{y = y_0(x)}$$

$$\left. - \frac{d}{dx} \int_{y_0(x)}^{\infty} dy \left[\Phi_i^2 - a_i^2(x) e^{-2i\nu y} \right] \right)$$

$$= \frac{2}{g} \int_{-L/2}^{L/2} dx \left[\frac{i}{\nu} e^{-2i\nu y_0(x)} a_i(x) a_i'(x) + y_0'(x) a_i^2(x) e^{-2i\nu y_0(x)} \right] .$$

We see that two terms have canceled each other, and the last term became the longitudinal integral of a perfect differential. If the ship ends are such that the slender-body assumptions are not grossly violated there, that last term must vanish. Finally, we perform an integration by parts on the first term, and under the same assumptions as before we find that:

$$I_{ii}^{(2)} = 0 ,$$

and so:

$$T_{ii}^{(2)} = 0 . \qquad (56a)$$

For the case that $i \neq j$, we do not get zero. Again, we use the fact that $\Phi_5 \cong -x\Phi_3$, which also implies that $a_5 \cong -xa_3$ (See (42a).) The steps are similar to those above, the only variation coming in the results of the partial integrations with respect to z. We note first that, for $i = 3$ and $j = 5$,

$$\frac{d}{dx}\int_{y_0(x)}^{\infty} dy\left[\Phi_3\Phi_5 - a_3(x)a_5(x)e^{-2i\nu y}\right] = -\,y_0'(x)\Big[\Phi_3\Phi_5 - a_3(x)a_5(x)e^{-2i\nu y}\Big]_{y=y_0(x)}$$

$$+\,2\int_{y_0(x)}^{\infty} dy\left[\Phi_3\Phi_{5_x} - a_3(x)a_5'(x)e^{-2i\nu y}\right]$$

$$+\int_{y_0(x)}^{\infty} dy\left[\Phi_3^2 - a_3^2(x)e^{-2i\nu y}\right] \quad .$$

(The last term on the right-hand side has an opposite sign if we take $i = 5$ and $j = 3$. It is this term which leads finally to a non-zero result.) The rest of the steps will be omitted, since they are nearly identical to those in the analysis just preceding. We obtain finally:

$$I_{35}^{(2)} = \frac{2}{g}\int_{-L/2}^{L/2} dx \left(\int_{y_0(x)}^{\infty} dy\left[\Phi_3^2 - a_3^2(x)e^{-2i\nu y}\right] - \frac{i}{2\nu}a_3^2(x)e^{-2i\nu y_0(x)}\right) ,$$

and the same expression with opposite sign for $I_{53}^{(2)}$.
For $T_{ij}^{(2)}$ we have:

$$T_{35}^{(2)} = -T_{53}^{(2)} = -\frac{2\rho(i\omega)^3 U}{g}\int_{-L/2}^{L/2} dx \left(\int_{y_0(x)}^{\infty} dy\left[\Phi_3^2 - a_3^2(x)e^{-2i\nu y}\right] - \frac{i}{2\nu}a_3^2(x)e^{-2i\nu y_0(x)}\right) .$$

The integral expression is exactly what we denoted by

$$\int_F dS\ \Phi_3^2$$

in (19) and (21). Thus we have come to our final analytical results.

All that remains is to identify these results with the ordinary added-mass and damping coefficients. Recall that we have omitted the buoyancy restoring forces from our results. Therefore, the quantity which we labeled $F_i(t)$ in (49) includes just the force components which are associated with ship velocity and acceleration, viz.,

$$F_i(t) = \sum_j T_{ij}\xi_j(t) = -\sum_j [a_{ij}\ddot{\xi}_j(t) + b_{ij}\dot{\xi}_j(t)]$$

$$= \sum_j [\omega^2 a_{ij} - (i\omega)b_{ij}]\xi_j(t) .$$

Thus we have:

$$T_{ij} = \omega^2[a_{ij} + b_{ij}/i\omega] ;$$

$$a_{ij} = (1/\omega^2)\ \underline{\mathrm{Re}}\left[T_{ij}\right] ;$$

$$b_{ij} = -\ (1/\omega)\ \underline{\mathrm{Im}}\left[T_{ij}\right] .$$

The complete expression of these results has been written out in (18) and (19).

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APPENDIX A. VECTOR DEFINITIONS AND RELATIONS

In formulating the near-field problem and in deriving force formulas, we have used the components of two generalized vectors, n_i and m_i . (See, for example, Equations (35) and (36).) In this appendix, we provide a complete definition of these vectors, show some of their properties, and then prove a theorem relating n_i and m_i . The theorem is essential in deriving the final, simple formulas for force on the ship.

We make a number of definitions:

1) S_0 is the hull surface in its undisturbed position, bounded above by the plane $z = 0$. It is assumed to be wall-sided at the free surface.

2) C_0 is the line of intersection of S_0 and the plane $z = 0$.

3) $\underline{\nu}$ is a unit vector normal to S_0 , directed into the hull.

4) $\underline{v}$ is a vector function with the properties:

a) $\nabla \cdot \underline{v} = 0$ in the fluid domain,

b) $\nabla \times \underline{v} = 0$ in the fluid domain, and

c) $\underline{\nu} \cdot \underline{v} = 0$ on S_0 .

In the applications of our theorem, $\underline{v}$ is the (normalized) fluid velocity in the steady-forward-motion problem, that is, $\underline{v} = \nabla(x + \chi(x,y,z))$.

5) n_i , $i = 1,\ldots,6$, is given by:

$$n_1\underline{i} + n_2\underline{j} + n_3\underline{k} = \underline{\nu} \; ;$$

$$n_4\underline{i} + n_5\underline{j} + n_6\underline{k} = \underline{r}\times\underline{\nu} \; , \text{ where } \; \underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \; .$$

6) m_i , $i = 1,\ldots,6$, is given by:

$$m_1\underline{i} + m_2\underline{j} + m_3\underline{k} = -(\underline{\nu}\cdot\nabla)\underline{v} \equiv \underline{m} \; ;$$

$$m_4\underline{i} + m_5\underline{j} + m_6\underline{k} = -(\underline{\nu}\cdot\nabla)(\underline{r}\times\underline{v}) = \underline{r}\times\underline{m} + \underline{v}\times\underline{\nu} \; .$$

For convenience in the main problem (and for the future), we list here some expressions for n_i and m_i , including slender-body approximations:

$$n_1 = \frac{-h_{0_x}}{\sqrt{1 + h_{0_x}^2 + h_{0_y}^2}} = \frac{-h_{0_x}}{\sqrt{1 + h_{0_y}^2}}\,[1 + O(\varepsilon^2)] = O(\varepsilon) \;;$$

$$n_2 = \frac{-h_{0_y}}{\sqrt{1 + h_{0_x}^2 + h_{0_y}^2}} = \frac{-h_{0_y}}{\sqrt{1 + h_{0_y}^2}}\,[1 + O(\varepsilon^2)] = O(1) \;;$$

$$n_3 = \frac{1}{\sqrt{1 + h_{0_x}^2 + h_{0_y}^2}} = \frac{1}{\sqrt{1 + h_{0_y}^2}}\,[1 + O(\varepsilon^2)] = O(1) \;;$$

$$n_4 = yn_3 - zn_2 = \frac{y + zh_{0_y}}{\sqrt{1 + h_{0_y}^2}}\,[1 + O(\varepsilon^2)] = O(\varepsilon) \;;$$

$$n_5 = zn_1 - xn_3 = -xn_3[1 + O(\varepsilon^2)] = O(1) \;;$$

$$n_6 = xn_2 - yn_1 = xn_2[1 + O(\varepsilon^2)] = O(1) \;;$$

$$m_1 = -\chi_{xx}n_1 - \chi_{xy}n_2 - \chi_{xz}n_3 = O(\varepsilon) \;;$$

$$m_2 = -(\chi_{yy}n_2 + \chi_{yz}n_3)[1 + O(\varepsilon^2)] = O(1) \;;$$

$$m_3 = -(\chi_{zy}n_2 + \chi_{zz}n_3)[1 + O(\varepsilon^2)] = O(1) \;;$$

$$m_4 = \chi_y n_3 - \chi_z n_2 + ym_3 - zm_2 = O(\varepsilon) \;;$$

$$m_5 = -(n_3 + xm_3)[1 + O(\varepsilon^2)] = O(1) \;;$$

$$m_6 = (n_2 + xm_2)[1 + O(\varepsilon^2)] = O(1) \;.$$

We now prove:

Theorem. Let $\phi(x,y,z)$ be a differentiable scalar function. Then, for $i = 1,\ldots,6$, the following is true:

$$\int_{S_0} [m_i\phi + n_i(\underline{v}\cdot\nabla\phi)]dS = -\int_{C_0} d\ell n_i \phi(\underline{k}\cdot\underline{v}),$$

where C_0 is followed in a counterclockwise direction, looking down on the ship.

Proof. We prove the theorem first for $i = 1,2,3$. One of the standard forms (see Milne-Thomson (1968), Section 2.51) of Stokes' theorem is:

$$\int_{S_0} (\underline{\nu}\times\nabla) \times (\phi\underline{v})dS = \int_{C_0} \underline{d\ell}\times\phi\underline{v} .$$

By the use of various vector theorems, the integrand of the surface integral can be rewritten:

$$(\underline{\nu}\times\nabla) \times \phi\underline{v} = \phi\underline{\nu} \times (\nabla\times\underline{v}) + \phi(\underline{\nu}\cdot\nabla)\underline{v} - \underline{\nu}(\underline{v}\cdot\nabla\phi) + (\underline{\nu}\cdot\underline{v})\nabla\phi - \phi\underline{\nu}(\nabla\cdot\underline{v})$$

$$= \phi(\underline{\nu}\cdot\nabla)\underline{v} - \underline{\nu}(\underline{v}\cdot\nabla\phi) ,$$

the last equality following from application of the properties of $\underline{v}$ given previously. Thus, the surface integral is:

$$\int_{S_0} [\phi(\underline{\nu}\cdot\nabla)\underline{v} - \underline{\nu}(\underline{v}\cdot\nabla\phi)]dS = -\int_{S_0} [\phi\underline{m} + \underline{\nu}(\underline{v}\cdot\nabla\phi)]dS .$$

In the line integral, because the ship is wall-sided, we can write:

$$\underline{d\ell} = -d\ell(\underline{k}\times\underline{\nu}) ,$$

and then we again use some standard theorems to show that:

$$(\underline{k}\times\underline{\nu}) \times \phi\underline{v} = -[\underline{\nu}(\underline{k}\cdot\phi\underline{v}) - \underline{k}(\underline{\nu}\cdot\phi\underline{v})]$$

$$= -\underline{\nu}\phi(\underline{k}\cdot\underline{v}) ,$$

the last equality again following from the properties of $\underline{v}$. This gives the theorem for $n = 1,2,3$.

For $i = 4,5,6$, we start with another variation of Stokes' theorem:

$$\int_{S_0} dS[(\underline{\nu}\times\nabla) \times \phi\underline{v}] \times \underline{r} = \int_{C_0} (d\underline{\ell} \times \phi\underline{v}) \times \underline{r} .$$

The integrand of the surface integral can be manipulated into the form

$$[(\underline{\nu}\times\nabla) \times \phi\underline{v}] \times \underline{r} = \phi[\underline{r}\times\underline{m} + \underline{v}\times\underline{\nu}] + (\underline{r}\times\underline{\nu})(\underline{v}\cdot\nabla)\phi ,$$

and the integrand of the line integral into

$$- d\ell \ \phi \ [(\underline{k}\cdot\underline{v})(\underline{\nu}\times\underline{r}) - (\underline{k}\times\underline{r})(\underline{\nu}\cdot\underline{v})] = - d\ell \ \phi \ (\underline{k}\cdot\underline{v})(\underline{\nu}\times\underline{r})$$

which then completes the proof.

For application in this paper, we need only the cases $i = 3,5$. For $i = 3$, the line integral vanishes because the ship is wall-sided and thus $n_3 = 0$ on C_0 . For $i = 5$, the line integral also vanishes, for both $\underline{r}$ and $\underline{\nu}$ lie in the plane $z = 0$ and so $n_5 = \underline{j}\cdot(\underline{r}\times\underline{\nu}) = 0$.

It may be noted that, for other modes of motion $(i = 1,2,4,6)$ the line integral will not necessarily vanish identically, but it will be negligible, for it is higher order than the surface integral. In the slender-body analysis for $\underline{v}$, we obtain $(\underline{k}\cdot\underline{v}) = 0$ for the lowest order approximation of $\underline{v}$. (This is the rigid-wall boundary condition on χ .) Thus, the line integral will start with a term involving the second approximation of $\underline{v}$.

The utility of this theorem lies in the fact that it enables us to convert surface integrals involving <u>derivatives</u> of ϕ into integrals which involve only the <u>values</u> of ϕ . Thus, with ϕ representing Φ_3 or Φ_5 , we avoid the necessity of performing numerical differentiations in the longitudinal direction.

APPENDIX B. NUMERICAL SOLUTIONS OF THE 2-D PROBLEMS

In the formulas presented in (19), it is clear that one must numerically solve the boundary-value problem of a cylinder heaving on the free surface. From this solution, one must obtain, in particular, the value of the potential on the mean position of the body and the value of the potential on the position of the undisturbed free surface. Two different methods have been reported in the literature for solving this problem, and they are described briefly in this appendix.

The first practical solution, valid at arbitrary frequency, for a problem of this type was published by Ursell (1949). He solved the problem for a half-immersed circular cylinder by representing the potential Φ as the sum of an infinite set of multipole potentials, each of which satisfies the free-surface boundary condition and each of which is multiplied by a coefficient such that the complete series satisfies the body boundary condition. Ursell truncated the series after six terms and determined the coefficients by satisfying the boundary condition at nine points on a quarter circle, in a least-squares sense.

Ursell's method has been extended to non-circular cylinders by numerous workers. The method consists of mapping the ship section onto a circle by a conformal mapping of the type:

$$z^* = \zeta + \frac{a_1}{\zeta} + \frac{a_3}{\zeta^3} + \frac{a_5}{\zeta^5} + \ldots ,$$

where $z^* = y + iz$ in our notation; the section is represented in the complex ζ-plane by the circle $|\zeta| = \text{constant}$. The free-surface boundary condition is altered by this operation, so that Ursell's result cannot be used directly. However, a set of multipole potentials can still be written down which satisfy the modified free-surface condition, and so the solution may be completed in a manner similar to that of

Ursell. If the mapping is terminated with the term a_3/ζ^3, the ship sections generated by this two-parameter family are called "Lewis forms," (See Lewis (1929)) and this family has proved remarkably useful since it encompasses forms not unlike many of the commonly-used shapes of ship sections. As an extra advantage, its two parameters a_1 and a_3 can be related explicitly to two more basic parameters of the section, viz., the beam/draft ratio and the section-area coefficient. Hence the possibility exists that if all sections of a ship are adequately mappable by simple Lewis forms, results for all coefficients may be obtained when the only information supplied is beam, draft, and cross-section area curves of the ship. This possibility has been exploited by several workers using Grim's (1960) version of the Ursell method.

The Lewis-form representation may be inadequate for some ship sections which have been proposed or are in use, and so Porter (1960) and Tasai (1959) extended the Ursell method to allow the solution of Φ to be obtained for (in principle) any number of coefficients a_k in the mapping. The major difficulty in this procedure has been the development of a practical technique for finding the a_k's from a given set of offsets. This difficulty has been overcome in various ways by Landweber & Macagno (1965), Bermejo (1965), Smith (1966), and Von Kerczek & Tuck (1966), and the mapping method for solving the boundary-value problems may be considered to be thoroughly tested and proven.

An entirely different method has been developed by Frank (1967). He avoided the mapping problem altogether by using an integral-equation approach. He assumed that the potential could be represented by a distribution of wave sources over the boundary curve $C(x)$ of the section; the varying strength of these sources is an unknown function which must be determined from an integral equation obtained from the boundary condition on the body. This integral equation is solved by standard numerical methods similar to those used by Hess and Smith (1962). It is assumed that there is a constant source strength over each of many discrete small elements of C(x) ; the boundary condition is satisfied at one point of each element, leading to the problem of inverting a matrix of influence coefficients. In principle this method can be applied to sections of quite arbitrary shape, becoming exact as more and more offsets are used. The computer time taken varies as the square of the number of points, and so there is an incentive to use the least possible number of points. In practice, from seven to forty-five offsets have been used, and it has been found that adequate results can often be obtained for conventional sections with just seven offsets.

One defect of Frank's method is that it breaks down at certain discrete frequencies. This is a failing of the integral equation method; it has no apparent physical significance. If a coarse mesh of surface elements is used, the failure spreads into a frequency band around each of the singular frequencies. Fortunately, the frequencies at which this failure occurs can be predicted, so that at least the phenomenon can be anticipated if it occurs in an important frequency region.

One important advantage of Frank's method is that it has been used also for predicting the force due to lateral motions. Thus it is available for the sway-yaw-roll problem. In principle, the mapping technique can probably also be used for this problem.

APPENDIX C. EXPANSIONS OF THE FAR-FIELD VELOCITY POTENTIAL

There are three parts to this appendix: (1) derivation of the velocity potential for a line of pulsating, translating sources; (2) the high-frequency asymptotic evaluation of this potential at distances from the singular line which are $O(1)$; (3) the inner expansion of the expression found in part (2).

Derivation of the velocity potential. Conditions (3), (4), and (5) must all be satisfied asymptotically as $\varepsilon \to 0$, and in addition we must satisfy an appropriate radiation condition. We accomplish the last of these automatically by introducing into (4), the Bernoulli equation, an artificial Rayleigh viscosity. This device is extremely convenient, which is about all that justifies its use.

We assume that μ is a positive constant and introduce into the Euler equation an extra term representing a force opposing the velocity perturbation:

$$\frac{\partial \underline{q}}{\partial t} + (\underline{q}\cdot\nabla)\underline{q} = -\frac{1}{\rho}\nabla p - g\nabla z - \mu(\underline{q} - U\nabla x) ,$$

where $\underline{q}$ is the fluid velocity. It is then easily checked that Equation (4) is modified by the addition of an extra term:

$$g\zeta + \phi_t + \frac{1}{2}[\phi_x^2 + \phi_y^2 + \phi_z^2] + \mu(\phi - Ux) = \frac{1}{2}U^2 .$$

Like (4), this condition is to be satisfied on $z = \zeta(x,y,t)$. At an appropriate point, we allow μ to go to zero. The resulting solution, as is well-known, then has the proper behavior at infinity.

Generally, in the outer solution, one assumes that differentiation with respect to space variables has no effect on the order of magnitude. If we accept such an assumption and also retain our assumption about the effect of time derivatives, we find an unsatisfactory situation: there can be no gravity waves. At great distances, one certainly expects to find waves and nothing else, and so not all of the above assumptions can be retained.

We circumlocute this obstacle in the following way: We include (inconsistently) all terms which could possibly be of importance in the far field and we obtain the solution to this more general problem. The real difficulty is that the far-field includes several regions in which there are different behaviors of the solution. Thus our initial solution covers all of these regions. Then, in the second part of this appendix, we obtain estimates of the solution which are valid for $y = O(1)$ but not at infinity.

In the limit, as $\varepsilon \to 0$, there is no ship at all, and so, for very small values of ε, we may expect that the solution (other than the uniform-stream term) is $o(1)$. Therefore, if we set:

$$\phi(x,y,z,t) = Ux + \underline{\text{Re}}\{\phi(x,y,z)\, e^{i\omega t}\} , \qquad \text{(C1)}$$

we may retain only those terms which are linear in ϕ and ζ. The appropriate boundary condition on ϕ is then found to be:

$$(i\omega + U\frac{\partial}{\partial x} + \mu)^2\phi + g\phi_z = 0 \qquad \text{on } z = 0 ,$$

where we have added an extra term of no consequence involving μ^2.

Let there be a line of sources on $y = 0$, $z = z_0 < 0$, with linear density $\sigma(x,t)$:

$$\sigma(x,t) = \underline{\text{Re}}\{\sigma(x)\, e^{i\omega t}\} . \qquad \text{(C2)}$$

(Starting with sources below the undisturbed free surface simplifies the solution somewhat. We shall presently set $z_0 = 0$.) The method of solution, which will not be presented here in detail, is to replace the Laplace equation by a Poisson equation:

$$\nabla^2 \phi(x,y,z) = 4\pi\sigma(x)\delta(y)\delta(z - z_o) ,$$

form double Fourier transforms* of the differential equation and of the free-surface condition, solve the resulting ordinary differential equation in z , then invert the transforms. The final result, with $z_0 = 0$, is:

$$\phi(x,y,z) = -\frac{1}{\pi} \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k) \int_{-\infty}^{\infty} \frac{d\ell \exp i\ell y + z\sqrt{k^2 + \ell^2}}{\sqrt{k^2 + \ell^2} - \frac{1}{g}(\omega + Uk - i\mu)^2} . \qquad (C3)$$

A slightly different expression of the same solution has been given by Newman and Tuck (1964).

It is at this point that we wish to set $\mu = 0$. However, such a step leaves the inner integral undefined, because there is generally a pole of the integrand which then falls on the real ℓ -axis. We must investigate how such a pole approaches the real axis as $\mu \rightarrow 0$, and then we indent the contour of integration appropriately.

Let us define:

$$I(k) = \int_{-\infty}^{\infty} \frac{d\ell \exp i\ell y + z\sqrt{k^2 + \ell^2}}{\sqrt{k^2 + \ell^2} - \frac{1}{g}(\omega + Uk - i\mu)^2} ; \qquad (C4)$$

---

*Our definition of the transforms is:

$$f^*(k) = \int_{-\infty}^{\infty} dx \, f(x) \, e^{-ikx} ; \quad f(x) = (1/2\pi) \int_{-\infty}^{\infty} dk \, f^*(k) \, e^{ikx} .$$

The other transform is obtained by replacing x and k by y and ℓ , respectively.

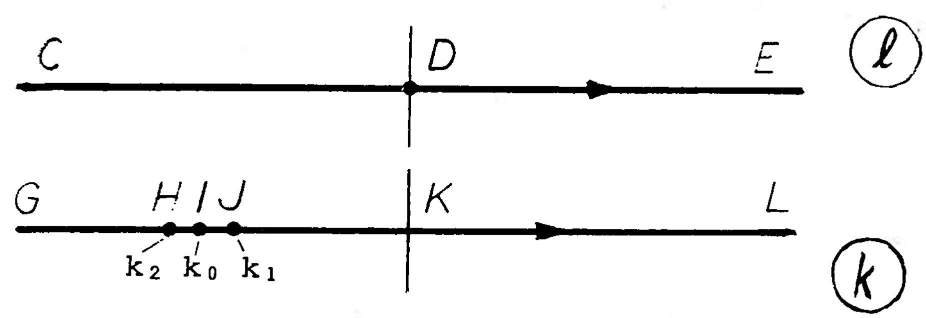
$$k_0 = -\omega/U = -\kappa\tau \; ; \tag{C5a}$$

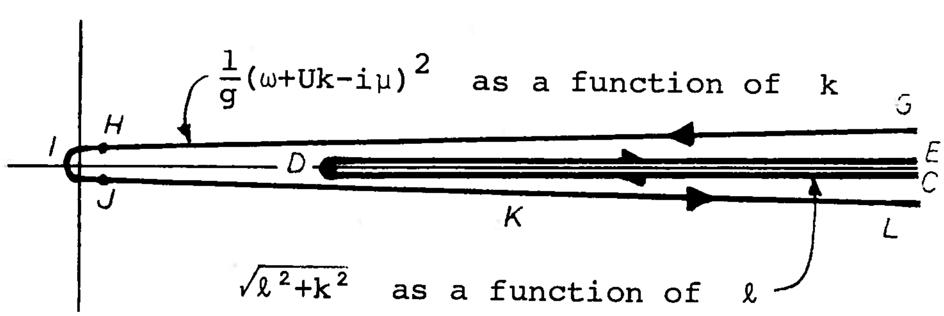
$$k_1 = -\kappa\left[(\tau + \tfrac{1}{2}) - \sqrt{\tau + 1/4}\right] \sim -\kappa(\tau - \sqrt{\tau} + \tfrac{1}{2} - \frac{1}{8\sqrt{\tau}} + \ldots) \; ; \tag{C5b}$$

$$k_2 = -\kappa\left[(\tau + \tfrac{1}{2}) + \sqrt{\tau + 1/4}\right] \sim -\kappa(\tau + \sqrt{\tau} + \tfrac{1}{2} + \frac{1}{8\sqrt{\tau}} + \ldots) \; . \tag{C5c}$$

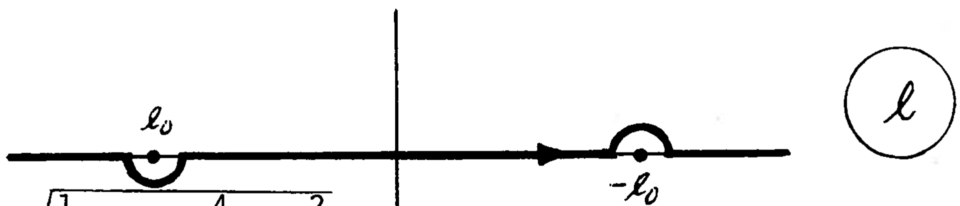
we recall that: $\kappa = g/U^2$; $\tau = \omega U/g$; $\nu = \omega^2/g$.
We may note that, for $k = k_0$, there is no pole at all; this is a special case. There are three cases to be considered: 1) $k_1 < k$; 2) $k_2 < k < k_1$; 3) $k < k_2$. It should be noted that $k_0 = O(\varepsilon^{-1/2})$ and that both k_1 and k_2 differ from k_0 by a quantity which is $O(\varepsilon^{-1/4})$. Furthermore, all three are negative.

Case 1). Since both k and ℓ vary from $-\infty$ to $+\infty$, we show how the two terms in the denominator of the integrand vary:





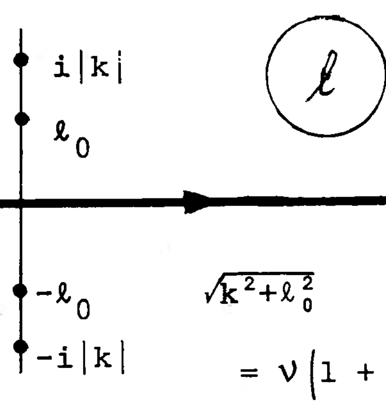
The quantity $\sqrt{\ell^2 + k^2}$ is shown for a particular value of $|k|$. For $k > k_1$ (the present case), the real part of $(1/g)(\omega + Uk - i\mu)^2 > |k|$. Thus, as $\mu \to 0$, there will be two values of ℓ at which the two terms will be equal and therefore there will be two poles on the real ℓ -axis. From the above figures, it is clear that the proper indented contour is as shown here:



$$\ell_0 = -\sqrt{\frac{1}{g^2}(\omega+Uk)^4 - k^2}$$
$$= -\nu\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2}$$

$$\sqrt{k^2+\ell_0^2} = \frac{1}{g}(\omega+Uk)^2$$
$$= \nu(1 + \frac{Uk}{\omega})^2$$

Case 2). For $k_2 < k < k_1$, we find that $\underline{\text{Re}}\{(1/g)(\omega + Uk - i\mu)^2\} < |k|$. Thus, even for $\mu = 0$, there is no pole on the real ℓ -axis. The contour in the ℓ -plane looks like this:

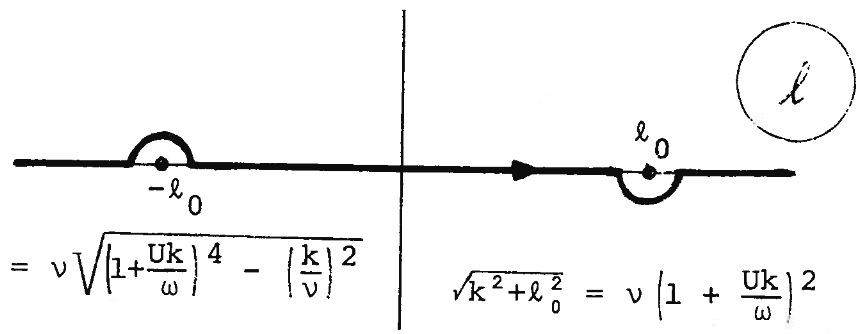


$$\ell_0 = i\nu\sqrt{\left(\frac{k}{\nu}\right)^2 - \left(1+\frac{Uk}{\omega}\right)^4}$$

$$\sqrt{k^2+\ell_0^2}$$
$$= \nu\left(1 + \frac{Uk}{\omega}\right)^2$$

There are still two poles, but they fall on the imaginary axis.

Case 3). For $k < k_2$, the situation is quite comparable to Case 1), except that the poles approach the real ℓ -axis from the opposite directions.



The solution can now be written:

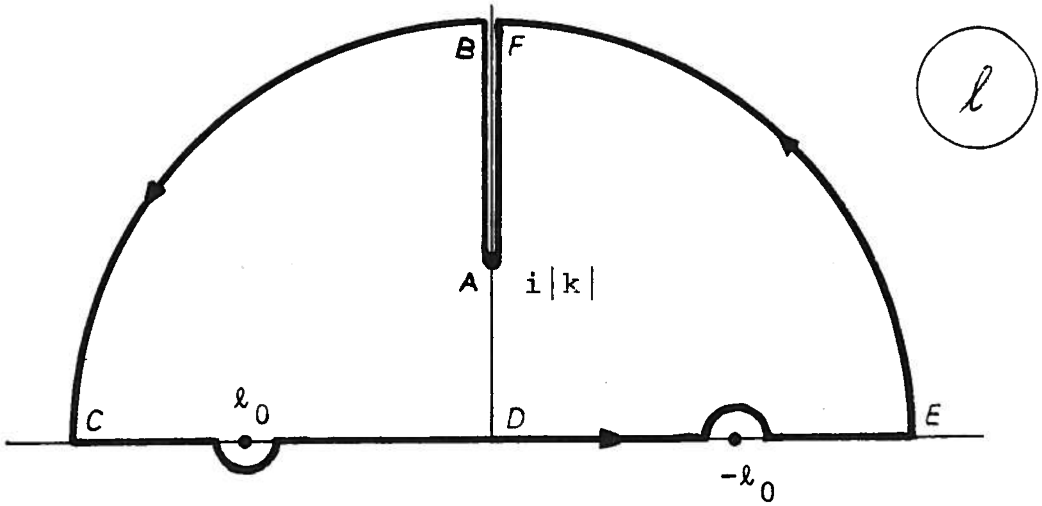
$$\phi(x,y,z) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dke^{ikx}\sigma^*(k)\int_{C(k)\,-\infty}^{\infty} \frac{d\ell \exp\left[i\ell y + z\sqrt{k^2 + \ell^2}\right]}{\sqrt{k^2 + \ell^2} - \frac{1}{g}(\omega + Uk)^2} \quad , \qquad (C6)$$

where it is understood that the contour of the ℓ -integration is taken appropriately according to the value of k . There are no special difficulties with the k -integration which need be discussed.

The potential for $y = O(1)$. For convenience, we shall assume that $y > 0$. A simple modification of the following procedure shows that we can set $y = |y|$ in the final results and they will then be valid for all $|y| = O(1)$.

We consider again the three cases defined above and obtain for each an estimate of the integral $I(k)$.

Case 1). $(k_1 < k)$ The integrand of I is analytic in the upper half-space except on the imaginary axis above $\ell = i|k|$, and the factor $e^{i\ell y}$ is well-behaved for $y > 0$. Therefore we complete the contour as shown:



We now have:

$$I = 2\pi i \ \mathrm{Res}(\ell_0) - \int_{FAB} \frac{d\ell \ \exp\left[i\ell y + z\sqrt{k^2 + \ell^2}\right]}{\sqrt{k^2 + \ell^2} - \frac{1}{g}(\omega + Uk)^2} \ .$$

The residue can be computed easily by l'Hospital's rule:

$$\mathrm{Res}(\ell_0) = - \frac{\left(1 + \frac{Uk}{\omega}\right)^2}{\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2}} \ \exp\left[-i\nu\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2} + \nu z\left(1 + \frac{Uk}{\omega}\right)^2\right] .$$

The integral down and back the imaginary axis can be rewritten:

$$-\int_{FAB} = i\int_{|k|}^{\infty} \frac{d\ell \ \exp\left[-\ell y + iz\sqrt{\ell^2 - k^2}\right]}{i\sqrt{\ell^2 - k^2} - \frac{1}{g}(\omega + Uk)^2} - i\int_{|k|}^{\infty} \frac{d\ell \ \exp\left[-\ell y - iz\sqrt{\ell^2 - k^2}\right]}{-i\sqrt{\ell^2 - k^2} - \frac{1}{g}(\omega + Uk)^2} \ .$$

For $k = o(\varepsilon^{-1/2})$, the following estimate of the integrals is valid:

$$\left|\int_{|k|}^{\infty} \frac{d\ell \, \exp\left[-\ell y \pm iz\sqrt{\ell^2 - k^2}\right]}{\pm i\sqrt{\ell^2 - k^2} - \frac{1}{g}(\omega + Uk)^2}\right| \leq \int_{|k|}^{\infty} \frac{d\ell \, e^{-\ell y}}{\sqrt{(\ell^2 - k^2 + \frac{1}{g^2}(\omega + Uk)^4}}$$

$$= \frac{1}{\nu}\int_{|k|}^{\infty} d\ell \, e^{-\ell y}[1 + o(1)] = O(\varepsilon) \; .$$

For $\varepsilon^{-1/2} = O(k)$, a stronger estimate is possible:

$$\left|\int_{|k|}^{\infty} \frac{d\ell \, \exp\left[-\ell y \pm iz\sqrt{\ell^2 - k^2}\right]}{\pm i\sqrt{\ell^2 - k^2} - \frac{1}{g}(\omega + Uk)^2}\right| \leq e^{-|k|y}\int_{|k|}^{\infty} \frac{d\ell \, \exp[-(\ell - |k|)y]}{\sqrt{\ell^2 - k^2}}$$

$$= O(\varepsilon^{-1/\sqrt{\varepsilon}}) \; .$$

In either case, then, the integrals are $O(\varepsilon)$, and we find that:

$$I = -\frac{2\pi i\left(1 + \frac{Uk}{\omega}\right)^2}{\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2}} \exp \nu\left[-iy\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2} + z\left(1 + \frac{Uk}{\omega}\right)^2\right] + O(\varepsilon) \; .$$

<u>Case 2)</u>. We complete the contour as before, but now the pole is inside the contour, on the imaginary axis. The stronger estimate above for the integrals along the imaginary axis can again be obtained, leaving only the residue term:

$$I = \frac{2\pi\left(1 + \frac{Uk}{\omega}\right)^2}{\sqrt{\left(\frac{k}{\nu}\right)^2 - \left(1 + \frac{Uk}{\omega}\right)^4}} \exp \nu\left[-y\sqrt{\left(\frac{k}{\nu}\right)^2 - \left(1 + \frac{Uk}{\omega}\right)^4} + z\left(1 + \frac{Uk}{\omega}\right)^2\right] + O(\varepsilon) \; .$$

Again, we may note that, although this case includes the special case that $\ell_0 = i|k|$, there is no difficulty, for in fact the pole vanishes as $k \to k_0$ and $\ell_0 \to i|k_0|$.

Case 3). The procedure is the same, and we find that:

$$I = \frac{2\pi i\left(1 + \frac{Uk}{\omega}\right)^2}{\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2}} \exp \nu\left[iy\sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2} + z\left(1 + \frac{Uk}{\omega}\right)^2\right] + O(\varepsilon) \ .$$

We can collect all three cases together, as follows. Define:

$$f(k) = \sqrt{\left(1 + \frac{Uk}{\omega}\right)^4 - \left(\frac{k}{\nu}\right)^2} \ ,$$

where we take a branch cut in the k -plane between k_2 and k_1 , and we use the value of the root on the lower side of the cut. Then:

$$I = -2\pi i \frac{\left(1 + \frac{Uk}{\omega}\right)^2}{f(k)} \exp \nu\left[-iyf(k) + z\left(1 + \frac{Uk}{\omega}\right)^2\right] + O(\varepsilon) \ ,$$

and:

$$\phi(x,y,z) = 2i \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k) \cdot \frac{\left(1 + \frac{Uk}{\omega}\right)^2}{f(k)} \exp\left[i\nu y f(k) + \nu z\left(1 + \frac{Uk}{\omega}\right)^2\right] + O(\varepsilon) \qquad \text{(C7)}$$

It may be noted that, if we define:

$$k_3 = -\kappa\left[\left(\tau - \frac{1}{2}\right) - \sqrt{\frac{1}{4} - \tau}\right] ,$$

$$k_4 = -\kappa\left[\left(\tau - \frac{1}{2}\right) + \sqrt{\frac{1}{4} - \tau}\right] ,$$

then:

$$f(k) = (1/\kappa\tau)^2 \sqrt{(k - k_1)(k - k_2)(k - k_3)(k - k_4)} \; .$$

Thus we see that there are four square-root singularities in $f(k)$, but two of them are complex.

Formula (C7) should be compared with (25), the latter pertaining to the zero-speed problem. We now make the same assumption that we did there, viz., that $\sigma^*(k)$ drops off rapidly enough with large $|k|$ that we can use a small- $|k|$ approximation for the integral I . For $k = o(\varepsilon^{-1/2})$, we have:

$$\left(1 + \frac{Uk}{\omega}\right)^4 = [1 + o(1)]^4 = O(1) \; ,$$

$$\left(\frac{k}{\nu}\right)^2 = o(\varepsilon) \; ,$$

and so:

$$f(k) = \left(1 + \frac{Uk}{\omega}\right)^2 [1 + O(\varepsilon)] \; .$$

We can now simplify (C7) considerably:

$$\phi(x,y,z) \sim 2i \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k) e^{\nu(z - iy)\left(1 + \frac{Uk}{\omega}\right)^2} \; . \qquad \text{(C8)}$$

This is as far as we can go with the estimate for $y = O(1)$.

<u>The inner expansion of the far-field expansion</u>. We now assume that $y = O(\varepsilon)$. Then the exponential can be rewritten:

$$e^{\nu(z - iy)\left(1 + \frac{Uk}{\omega}\right)^2} = e^{(z - iy)(\nu + 2\tau k + k^2/\kappa)}$$

$$= e^{\nu(z - iy)}\left[1 + (z - iy)\left(2\tau k + \frac{k^2}{\kappa}\right) + \ldots\right] \; .$$

Noting the following Fourier-transform properties,

$$\sigma(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx}\sigma^*(k) \quad ,$$

$$\sigma'(x) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} k\sigma^*(k) \quad ,$$

$$\sigma''(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} k^2\sigma^*(k) \quad ,$$

we substitute the approximation for the exponential into the integral and obtain:

$$\phi(x,y,z) \sim 2i\ e^{\nu(z - iy)}\left[\int_{-\infty}^{\infty} dk e^{ikx}\sigma^*(k) + 2\tau(z - iy)\int_{-\infty}^{\infty} dk e^{ikx} k\sigma^*(k) + \frac{1}{\kappa}(z - iy)\int_{-\infty}^{\infty} dk e^{ikx} k^2\sigma^*(k) + \ldots\right]$$

$$= 4\pi i\ e^{\nu(z - iy)}\left[\sigma(x) - 2i\tau(z - iy)\sigma'(x) - \frac{1}{\kappa}(z - iy)\sigma''(x) + \ldots\right] .$$

Each successive term in the braces is $O(\varepsilon^{1/2})$ times the preceding, and so it is not proper for us to keep all three terms. (We have previously dropped terms which were $O(\varepsilon)$ times the leading term in various expressions.) But it is consistent to keep two terms, and they are just the terms we need:

$$\phi(x,y,z) \sim 4\pi i\ e^{\nu(z - i|y|)}\left[\sigma(x) - 2i\tau(z - i|y|)\sigma'(x)\right] . \quad \text{(C9)}$$

We have replaced y by $|y|$, and this result can be used as the inner expansion of the outer expansion on either side. When combined with the time factor, this solution represents a sum of ordinary outgoing waves plus outgoing waves increasing linearly in amplitude. This is the result desired.

APPENDIX D. THE APPLIED-PRESSURE PROBLEM

In order to make sense out of one part of the near-field problem, it is necessary to be able to interpret the following problem:

Given a two-dimensional pressure field applied on the free surface:

$$p(x,t) = \underline{\text{Re}}\left[p(x)\ e^{j\omega t}\right] ,$$

where

$$p(x) = p_0 e^{-j\nu|x|} ,$$

$$\nu = \omega^2/g = O(\varepsilon^{-1}) ,$$

find the velocity potential. The undisturbed free surface is here taken as the x -axis, with positive y measured upwards. Since this problem is related to the original near-field problem, the result must be valid in particular for $|z| = |x + iy| = O(\varepsilon)$, as $\varepsilon \to 0$, which means that $\nu|z| = O(1)$. We shall then need to find the outer expansion of this inner solution, at which point we shall consider that $|z| = O(1)$ and $\nu|z| = (\varepsilon^{-1})$.

This problem is not well-posed as stated above. If we look up the general solution of such problems in, say, Wehausen and Laitone (1960), we find that the expressions there are not even defined for the given pressure field. Therefore we define a preliminary problem which <u>can</u> be solved. Let:

$$p(x) = p_0 e^{-\mu|x| - jk|x|} ,$$

where μ is a real positive constant, and we require that $k \neq \nu$. After finding the velocity potential for this case, we shall let $\mu \to 0$ and, after that, $k \to \nu$. In this way, we obtain a well-defined solution, but it should be emphasized that it is only a solution. We could obtain other solutions by a different choice of limiting operations. The special virtue of our result is that it provides an inner solution which can be matched to the outer solution.

It may be noted that we have used " j " as the imaginary unit, contrary to our usual practice here. The reason is that we shall follow the convention used in this problem by Wehausen and Laitone(1960) and others, namely, using two imaginary quantities, i and j. Each is equal to $(-1)^{1/2}$, but they cannot be combined. That is, we cannot write: $ij = -1$. The point of this device is to allow the use of the complex notation, $e^{j\omega t}$, for the time dependence while we also use functions of a complex variable, $z = x + iy$. When indicating " Re " and " Im " for the real and imaginary parts of a function, we must always suffix an i or a j to the symbol to show whether we mean the real or imaginary part with respect to i or j.

For the benefit of readers who may not be familiar with this technique of using two imaginary units, we list here a few simple properties which are easily proven:

1) $(1 - ij)(1 + ij) = 0$.
2) $(1 - ij)(a + jb) = (1 - ij)(a + ib)$.
3) $(1 + ij)(a + jb) = (1 + ij)(a - ib)$.
4) $(1 + ij)^2 = 2(1 + ij)$.
5) $(1 - ij)^2 = 2(1 - ij)$.
6) $e^{\pm i\alpha} = (1/2)[(1 \mp ij)e^{j\alpha} + (1 \pm ij)e^{-j\alpha}]$.
7) $e^{\pm j\beta} = (1/2)[(1 \mp ij)e^{i\beta} + (1 \pm ij)e^{-i\beta}]$.

It should also be noted that contour integrals in the z-plane should involve just functions including the unit i. For example,

$$\int_z^\infty \frac{ds\ e^{-jks}}{s} = \frac{1}{2}\int_z^\infty \frac{ds}{s}\left[(1 + ij)e^{iks} + (1 - ij)e^{-iks}\right] .$$

We now proceed to the solution of the problem. Let the real velocity potential be expressed:

$$\phi(x,y,t) = \underline{Re}_i\left[f(z,t)\right] ,$$

where

$$f(z,t) = \underline{Re}_j\left[f(z)e^{j\omega t}\right] .$$

The velocity potential can then be written in terms of $f(z)$, and we find that the appropriate expression is:

$$f(z) = e^{-i\nu z}\left[-\frac{j}{\pi\rho g}\int_{-\infty}^{+\infty} ds\ p(s)\int_\infty^z \frac{d\zeta e^{i\nu\zeta}}{s - \zeta} - \frac{(1 - ij)\omega}{g}\int_{-\infty}^{\infty} ds\ p(s)e^{i\nu s}\right] . \qquad \text{(D1)}$$

Using the Plemelj formula (Muskhelishvili (1953)), we can readily check that this result satisfies all conditions in the linearized problem. Our formulation of the problem differs from that in Wehausen and Laitone (1960) in that we use $e^{j\omega t}$ to express the time dependence, rather than $e^{-j\omega t}$. We should be able to obtain the above result from that in Wehausen and Laitone (1960) by changing their j to $-j$, but their result is apparently in error throughout by a factor of (-1) .

For $x \to \pm\infty$, the above result reduces to

$$f(z) \to -e^{\nu y}e^{\mp j\nu x}\left[\frac{\omega(1 \mp ij)}{\rho g}\right]\int_{-\infty}^{\infty} ds\ p(s)\ e^{\pm j\nu s} .$$

The integral here can be evaluated simply for the assumed pressure distribution. Its value is

$$\frac{2p_0(\mu + jk)}{\nu^2 + (\mu + jk)^2} ,$$

which obviously is infinite if both $\mu = 0$ and $k = \nu$. This demonstrates clearly that we must proceed carefully.

First we let $\mu \rightarrow 0$ in the expression for $f(z)$. There is no difficulty with the single-integral term in Equation (D1), but the double-integral term requires some manipulation. We divide the infinite interval of the s -integration into two semi-infinite intervals, $-\infty$ to 0 , and 0 to $+\infty$, and we perform one integration by parts with respect to s . We can then carry out all of the required operations and let $\mu \rightarrow 0$ with well-defined results. We obtain for $f(z)$:

$$f(z) = -\frac{2\omega p_0 k}{\pi\rho g(k^2 - \nu^2)} e^{-i\nu z} \left[\int_z^\infty \frac{ds\ e^{i\nu s}}{s} - \pi j(1 - ij) \right]$$

$$+ \frac{j\omega p_0}{\pi\rho g(k^2 - \nu^2)} \left[(i\nu - jk)\ e^{jkz} \int_z^\infty \frac{ds\ e^{-jks}}{s} \right.$$

$$\left. - (i\nu + jk)\ e^{-jkz} \int_{-z}^\infty \frac{ds\ e^{-jks}}{s} \right] .$$

Each contour integral is carried out either above or below the x -axis; the contour may not cross the axis.

In order to perform the second limit operation, namely, letting $k \rightarrow \nu$, we let

$$k = \nu - \delta , \quad |\delta/\nu| << 1 .$$

We also assume that $|\delta z| << 1$, which implies that $|z|$

is bounded, and so our results will not be valid as $|z| \to \infty$. There follows a tremendous amount of rather tedious algebra, all for the purpose of identifying how the various quantities depend on δ, as $\delta \to 0$. The integrals can be expressed in terms of standard exponential integrals:

$$E_1(z) = \int_z^\infty \frac{dt\ e^{-t}}{t} ,$$

in which we assume that there is a branch cut from $-\infty$ to 0 along the negative real axis. (See Abramowitz and Stegun (1964)) We note the following intermediate results:

$$\int_z^\infty \frac{ds\ e^{i\nu s}}{s} = \int_{-i\nu z}^{-i\infty} \frac{ds\ e^{-s}}{s} = E_1(-i\nu z) , \quad \text{for} \quad x > 0 ,$$

$$= E_1(-i\nu z) + 2\pi i , \quad \text{for} \quad x < 0 ;$$

$$\int_z^\infty \frac{ds\ e^{-i\nu s}}{s} = E_1(i\nu z) ;$$

$$\int_{-z}^\infty \frac{ds\ e^{i\nu s}}{s} = E_1(i\nu z) ;$$

$$\int_{-z}^\infty \frac{ds\ e^{-i\nu s}}{s} = E_1(-i\nu z) - 2\pi i , \quad \text{for} \quad x > 0 ,$$

$$= E_1(-i\nu z) , \quad \text{for} \quad x < 0 .$$

Also,

$$E_1(\pm ikz) = E_1(\pm i\nu z) + \frac{\delta}{\nu} e^{-i\nu z} + O(\delta^2) ;$$

$$e^{\pm ikz} = e^{\pm i\nu z}(1 \mp i\delta z + O(\delta^2)) .$$

Using all of these facts, we obtain finally:

$$f(z) = \frac{\omega p_0}{2\pi\rho g\nu}\left[e^{-i\nu z}E_1(-i\nu z)(-1 - 2i\nu z) + e^{i\nu z}E_1(i\nu z) - 2 + e^{-i\nu z}(j + i \text{ sgn } x)(1 + 2i\nu z) + O(\delta)\right] .$$

This is the inner solution--valid for $|z| = O(\varepsilon)$. Since $\nu = O(\varepsilon^{-1})$, we note that all terms included here are of the same order of magnitude. We also note that the solution breaks down if we formally let $|z| \rightarrow \infty$. This should not be a matter for concern, for it should be recalled that the near-field solution is not expected to be valid at infinity.

We now want to find the outer expansion of this near-field solution. To do so, we consider $|z| = O(1)$, which allows us to use the asymptotic formula for the exponential integral for large magnitude of the argument:

$$E_1(z) \sim \frac{e^{-z}}{z}\left[1 - \frac{1!}{z} + \frac{2!}{z^2} + \ldots\right] .$$

Keeping terms consistently of just the lowest order of ε , we obtain:

$$f(z) \sim \frac{\omega p_0}{\rho g} iz(j + i \text{ sgn } x)\, e^{-i\nu z}$$

$$= \frac{\omega p_0 j}{\rho g}(1 \mp ij)(-y \pm jx)\, e^{\nu y \pm j\nu x} , \quad \text{for} \quad x \gtrless 0 .$$

We may note that, when differentiated, this expression gives two kinds of terms, (a) a wave-like term proportional to $e^{-i\nu z}$, and (b) a term which is a product of z times $e^{-i\nu z}$.

Finally, we recombine this result with the time dependence and obtain the real velocity potential:

$$\phi(x,y,t) \sim \frac{\omega}{\rho g} e^{\nu y} \underline{\mathrm{Re}}_j [j p_0(-y \pm jx)\, e^{j(\omega t \pm \nu x)}]$$

This is the outer expansion of the inner expansion. It represents a sum of (a) ordinary outgoing waves and (b) outgoing waves with amplitude increasing linearly in x.

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13. ABSTRACT

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