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CATENARY MOORING LINES WITH NONLINEAR DRAG AND TOUCHDOWN

by

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ABSTRACT

Analytical expressions modeling the tension distribution and geometric deformation along catenary mooring line, including touchdown effect and nonlinear drag, are developed . The motions of a catenary mooring line in the horizontal plane are studied thoroughly. Due to the complexity of the modeling equations, solution can be achieved only by iterative methods. Approximate values for recursion in the solution of the horizontal tension in the catenary are derived based on the first order approximation of the geometric properties of the line. Expressions for the nonlinear drag in the horizontal plane of the catenary are derived based on energy dissipation principles. The resulting analytical expressions serve to calculate the forces in the catenary due to drag in the horizontal plane. The nonlinear drag forces are then recast into a form suitable for mooring applications.

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NOMENCLATURE

A	reference point denoting the upper point of the catenary
B	reference point denoting the mooring point of the catenary at the sea floor
C_D	drag coefficient for the catenary
CG	center of gravity of the vessel
DICAS	Differentiated Compliance Anchoring System(s)
d	horizontal length of the undeformed catenary
D_{eff}	effective diameter of the catenary
F_a, F_A	horizontal drag force in the direction parallel to the motion of the catenary
F_l, F_L	horizontal drag force in the direction perpendicular to the motion of the catenary
F_{XD}	damping force in the direction of motion of the vessel
F_{YD}	damping force perpendicular to the direction of motion of the vessel
h	water depth (or vertical projection of the catenary)
ℓ	horizontally projected length of the suspended portion of the catenary
ℓ'	horizontal distance between mooring and attachment points
ℓ_{eff}	length of the suspended catenary
ℓ_T	total length of the catenary
M_{ZD}	damping moment about the Z-axis
P	vertical force per unit catenary length
P_{RT}	horizontal pretension in the catenary line
r	yaw angular velocity of the vessel
R_H	horizontal rigidity (stiffness) of the catenary
s	arclength coordinate of the suspended portion of the catenary
SMS	Spread Mooring System(s)
$T(s)$	tension distribution of the catenary for $0 \le s \le \ell_{eff}$
TMS	Turret Mooring System(s)
T_o	horizontal tension of the catenary
T_v	vertical tension of the catenary
u	forward velocity of vessel with respect to water
v	lateral velocity of vessel with respect to water
$(x(s), y(s))$	horizontal geometric distribution of catenary line for $0 \le s \le \ell_{eff}$
(x, y, z)	reference frame of catenary line at the point of contact with sea floor

(X,Y,Z)	reference coordinate frame of the vessel
(x_c,y_c)	horizontal geometric distribution of the catenary from (x_m,y_m,z_m)
(x_o,y_o,z_o)	coordinates of zero slope of the catenary
(x_m,y_m,z_m)	coordinates of the mooring point at the sea floor
(x_p,y_p)	body fixed coordinates of the ith fairlead
(x_T,y_T,z_T)	coordinates of the upper attachment point of the catenary
$z(s)$	vertical geometric distribution of catenary line for $0 \le s \le \ell_{eff}$

Greek Symbols

β	angle between the X-axis of the vessel and the catenary, measured counterclockwise
γ	angle between the ξ-axis and the catenary, measured counterclockwise
$\gamma_{(x,y)}$	angle between catenary endpoints measured with respect to the (x,y) plane
γ_p	energy dissipation function of the mooring line
$\theta(s)$	angle of the catenary with respect to the horizontal plane at point s
θ_ℓ	angle between the upper endpoint of the mooring line and the horizontal plane
(ξ,η)	earth-fixed reference frame
ρ	water density
ψ	yaw (or drift) angle

Special Symbols

$\overline{(\bullet)}$	value of $(\bullet)$ before displacement of the catenary
$\delta(\bullet)$	first order variation of $(\bullet)$
$\Delta(\bullet)$	displacement of $(\bullet)$ with respect to its initial position

I. INTRODUCTION

In deep water operations, several types of lines can be used for the purpose of towing, mooring and anchoring. These include nonlinear elastic strings (nylon, polyester), catenary chains, and steel cables. A number of mooring and anchoring systems, such as Turret Mooring Systems (TMS), Differentiated Compliance Anchoring Systems (DICAS), and general Spread Mooring Systems (SMS), use a hybrid combination of mooring lines during operations, especially in deep waters. A combination of different types of mooring lines is sometimes required to decrease the overall weight of the lines, thus reducing the vertical force on the moored vessel..

Different types of quasistatic and dynamic models of towing and mooring lines have been extensively studied in recent years [1, 3, 5-8, 10, 11], with various degrees of variations in the complexity of the models. Quasistatic models can be adapted to study the slow motion dynamics of towing, mooring, and anchoring systems [10]. In this work, an analytical model for the quasistatic analysis of deep-water mooring/anchoring catenary chains is developed. This model includes touchdown effects and nonlinear drag. The model developed is two-dimensional due to the nature of the catenary deformation. Thus, bottom friction due to off-plane motion of the catenary is not taken into account.

Catenary chains are heavy compared to other types of mooring lines, have in-plane deformation, are nearly fully submerged and thus have high hydrodynamic resistance. They are commonly used in most mooring and anchoring situations.

The equations for the catenary are developed in Chapter II with a special emphasis on deriving analytical expressions for the horizontal tension at the top of the catenary. These expressions require an iterative solution. Two methods for solution are developed in this work. Expressions for the tension distribution along the catenary as well as the geometry of the catenary are also derived. In Chapter III, analytical expressions for the drag force components on the catenary are developed using energy dissipation functions. The horizontal plane nonlinear drag forces due to the catenary exerted on the vessel are obtained based on the slow motion dynamics of the mooring/anchoring vessel. Finally, such expressions are recast in a form suitable for implementation in a dynamical mooring/anchoring model.

II. CATENARY MOORING LINES

The typical equations for inextensible catenaries used in towing applications cannot be applied in deep water mooring operations because they do not take into account touchdown effects and nonlinear drag. The main focus of this chapter is to develop analytical expressions for the horizontal tension at the top of the catenary using the catenary relations for mooring applications. Additional expressions for the tension distribution along the catenary, and the catenary geometry, are also derived.

2.1. Catenary Equations

Figure 1 shows the geometry of a deep water catenary line. The origin of its reference frame (x, y, z) is located at the point of contact of the catenary with the ground (x_o, y_o, z_o), which corresponds to the point of zero slope. In this figure, $\vec{x} = (x, y)$ represents the horizontal plane of the catenary; (x_m, y_m, z_m) is the mooring point on the sea floor; (x_T, y_T, z_T) is the attachment point of the catenary on the vessel; ℓ_{eff} is the length of the suspended catenary; ℓ' is the horizontal distance between the mooring point and the attachment point; ℓ is the horizontally projected length of the suspended catenary; d is the horizontally projected length of the undeformed catenary; and h is water depth. In addition, s is the arclength coordinate of the suspended catenary, with its origin at the point of zero slope (x_o, y_o, z_o)=(0,0,0); and $\theta(s)$ is the angle of the catenary with respect to the (x, y) plane at point s. The geometric configuration of the catenary is given by $x(s)$, $y(s)$, $z(s)$ and $\theta(s)$, for $0 \le s \le \ell_{eff}$.

The total length of the catenary is given by the geometric relation:

$$\ell_T = \ell_{eff} + d \ , \tag{1}$$

and the total horizontally projected length of the catenary is given by

$$\ell' = \ell + d \ . \tag{2}$$

By introducing the following nondimensional quantities:

$$X_1 = \frac{P\ell}{T_o}\ , \qquad L = \frac{P\ell_{eff}}{T_o}\ , \qquad H = \frac{Ph}{T_o}\ , \tag{3}$$

where P is the vertical force per unit catenary length and T_o is the horizontal tension in the (x, y) plane, the dimensionless deep sea mooring catenary equations are:

$$X_1 = \sinh^{-1}(L)\ , \tag{4}$$
$$H_1 = \cosh(X_1) - 1\ . \tag{5}$$

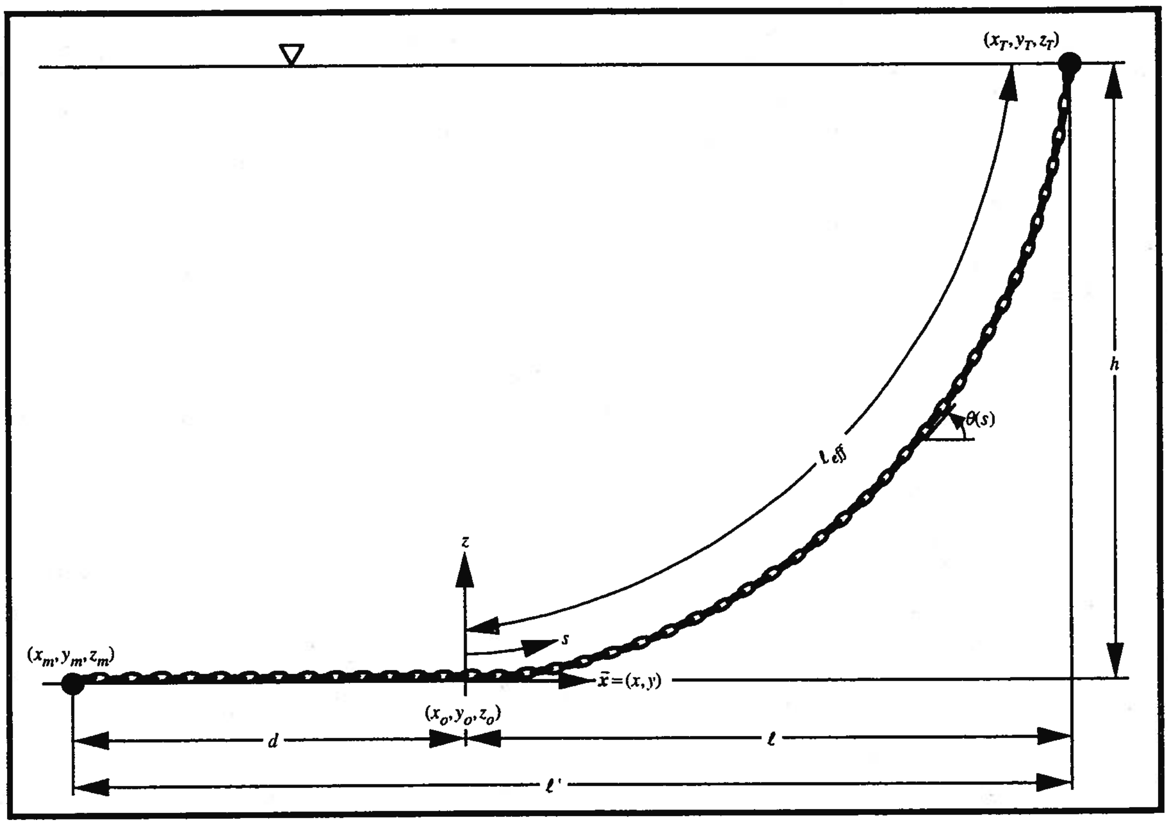

Figure 1: Geometry of deep water catenary

Expressions (3), (4) and (5), along with the geometric constraints (1) and (2), are the governing expressions for deep water catenaries. The total horizontally projected length of the catenary ℓ' is known, provided that the position of the upper endpoint of the catenary in the horizontal plane (x_T, y_T) is known with respect to the mooring point (x_m, y_m) by the relation:

$$\ell' = \sqrt{(x_m - x_T)^2 + (y_m - y_T)^2}\ . \tag{6}$$

In addition, the water depth h, and the total length of the mooring line ℓ_T are constant and known, whereas the horizontal tension T_o, tension distribution $T(s)$, and configuration of the catenary ($x(s)$, $y(s)$, $z(s)$, and $\theta(s)$) are unknown.

To find the horizontal tension T_o, catenary equations (4) and (5) are squared as follows:

$$L^2 = \sinh^2(X_1), \qquad H^2 = \cosh^2(X_1) - 2\cosh(X_1) + 1, \tag{7}$$

and then substracted from one another to find an expression relating L and H:

$$L^2 - H^2 = \sinh^2(X_1) - \cosh^2(X_1) + 2\cosh(X_1) - 1 = 2[\cosh(X_1) - 1] = 2H. \tag{8}$$

Thus,

$$L = \sqrt{H^2 + 2H}\ . \tag{9}$$

In dimensional form, equation (9) provides a relation between the suspended length of the catenary ℓ_{eff} and the horizontal tension T_o :

$$\ell_{eff} = \sqrt{h\left(h + 2\frac{T_o}{P}\right)}\ . \tag{10}$$

A second dimensional expression that relates ℓ_{eff} and T_o is found from equation (4):

$$\ell_{eff} = \frac{T_o}{P}\sinh\left(\frac{P\ell}{T_o}\right)\ . \tag{11}$$

By combining equations (10) and (11), an expression relating T_o to the horizontally projected length of the suspended catenary ℓ is obtained:

$$\boxed{\frac{T_o}{P}\sinh\left(\frac{P\ell}{T_o}\right) = \sqrt{h\left(h + 2\frac{T_o}{P}\right)}\ .} \tag{12}$$

2.2. Solution for the Horizontal Tension in the (x, y) Plane

An exact analytical solution for the horizontal tension T_o can be directly obtained provided both ℓ' and ℓ are known by combining the geometric constraints (1) and (2) of the system such that:

$$\ell_T - \ell_{eff} - \ell' + \ell = 0. \tag{13}$$

Then, relation (13) can be combined with (10) to find an expression in terms of known quantities ℓ_T, ℓ', ℓ and h which can be readily solved for T_o:

$$\ell_T - \ell' + \ell = \sqrt{h\left(h + 2\frac{T_o}{P}\right)} \ . \tag{14}$$

In the equation above, the only unknown is T_o, and thus the analytical expression for the horizontal mooring line tension is given by:

$$T_o = \frac{P\left[\left(\ell_T - \ell' + \ell\right)^2 - h^2\right]}{2h} \ . \tag{15}$$

In general, however, the distance ℓ' is known (depends solely on the position of the endpoints of the catenary as shown in (6)), but ℓ is unknown. In such a case, T_o must be obtained iteratively. Expression (12) alone does not suffice for calculating T_o because it involves two unknowns in this case (T_o and ℓ).

Iterative expressions for the solution to the horizontal tension can be obtained as a function of a single unknown (i.e. T_o) by combining expressions (1) and (11) as follows:

$$\ell_T - \frac{T_o}{P}\sinh\left(\frac{P\ell}{T_o}\right) - \ell' + \ell = 0 \ . \tag{16}$$

The expression above, along with an additional expression that relates T_o and ℓ are used to obtain a single equation in terms of T_o. There are two methods to calculate T_o if ℓ is unknown:

Method 1

An expression relating T_o and ℓ is derived from equation (12)

$$\ell = \frac{T_o}{P}\sinh^{-1}\left(\frac{P}{T_o}\sqrt{h\left(h+2\frac{T_o}{P}\right)}\right), \tag{17}$$

which, combined with the constraint relation (16), yields a single equation that can be solved for T_o:

$$\sinh\left(\frac{P}{T_o}\left(\sqrt{h\left(h+2\frac{T_o}{P}\right)}+\ell'-\ell_T\right)\right)-\frac{P}{T_o}\sqrt{h\left(h+2\frac{T_o}{P}\right)}=0\ . \tag{18}$$

Letting $\alpha = \dfrac{P}{T_o}$, expression (18) can be recast as follows:

$$\boxed{\sinh\left(\alpha\left(\sqrt{h\left(h+\frac{2}{\alpha}\right)}+\ell'-\ell_T\right)\right)-\alpha\sqrt{h\left(h+\frac{2}{\alpha}\right)}=0\ .} \tag{19}$$

A non-trivial solution to equation (19) is obtained by iterating for α, and can be computed numerically using a Newton-Rapson algorithm [2] of the form:

$$\alpha_{j+1} = \alpha_j - \frac{f(\alpha_j)}{f'(\alpha_j)}\ , \tag{20}$$

where

$$f(\alpha_j) = \sinh\left(\alpha_j\left(\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}+\ell'-\ell_T\right)\right)-\alpha_j\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}\ , \tag{21}$$

and

$$f'(\alpha_j)=\frac{\left[h(\alpha_j+1)+\alpha_j(\ell'-\ell_T)\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}\right]\cosh\left(\alpha_j\left(\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}+\ell'-\ell_T\right)\right)}{\alpha_j\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}}$$

$$-\frac{h(\alpha_j h+1)}{\alpha_j\sqrt{h\left(h+\frac{2}{\alpha_j}\right)}} . \qquad (22)$$

Method 2

In this method, equation (5) is dimensionalized

$$\frac{Ph}{T_o}=\cosh\left(\frac{P\ell}{T_o}\right)-1 \ , \qquad (23)$$

to obtain the following expression of ℓ in terms of the unknown T_o:

$$\ell=\frac{T_o}{P}\cosh^{-1}\left(\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)\right) . \qquad (24)$$

Then, equation (24) is combined with the constraint (16) to arrive to an equation with a single unknown, T_o, of the form:

$$\ell_T-\frac{T_o}{P}\sinh\left(\cosh^{-1}\left(\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)\right)\right)-\ell'+\frac{T_o}{P}\cosh^{-1}\left(\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)\right)=0 \ . \qquad (25)$$

A direct iterative solution to the equation above is mathematically tedious, but it can be simplified accordingly to obtain a simple hyperbolic function in terms of the unknown T_o. Letting $x=\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)$, and using the hyperbolic relation $\cosh^{-1}(x)=\sinh^{-1}\sqrt{x^2+1}$ [2], the expression above can be expanded to yield:

$$\ell_T - \frac{T_o}{P}\sinh\left(\sinh^{-1}\sqrt{\left(\frac{P}{T_o}\right)^2\left(h+\frac{T_o}{P}\right)^2 - 1}\right) - \ell' + \frac{T_o}{P}\cosh^{-1}\left(\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)\right) = 0 \ , \tag{26}$$

and then simplified in the form,

$$\cosh^{-1}\left(\frac{P}{T_o}\left(h+\frac{T_o}{P}\right)\right) + \frac{P}{T_o}(\ell_T - \ell') - \sqrt{\left(\frac{Ph}{T_o}\right)^2 + 2\frac{Ph}{T_o}} = 0 \ . \tag{27}$$

Equation (27) can be further recast as

$$\cosh\left(\sqrt{\left(\frac{Ph}{T_o}\right)^2 + 2\frac{Ph}{T_o}} - \frac{P}{T_o}(\ell_T - \ell')\right) - \frac{Ph}{T_o} - 1 = 0. \tag{28}$$

Again, letting $\alpha = \dfrac{P}{T_o}$, the expression above becomes:

$$\cosh\left(\sqrt{(\alpha h)^2 + 2\alpha h} - \alpha(\ell_T - \ell')\right) - \alpha h - 1 = 0 \ . \tag{29}$$

Equation (29) can be solved for α using a Newton-Rapson algorithm of the form (20) with

$$f(\alpha_j) = \cosh\left(\sqrt{\left(\alpha_j h\right)^2 + 2\alpha_j h} - \alpha_j(\ell_T - \ell')\right) - \alpha_j h - 1 \ , \tag{30}$$

and

$$f'(\alpha_j) = \left[h\left(\frac{\alpha_j h + 1}{\sqrt{\left(\alpha_j h\right)^2 + 2\alpha_j h}}\right) - (\ell_T - \ell')\right]\sinh\left(\sqrt{\left(\alpha_j h\right)^2 + 2\alpha_j h} - \alpha_j(\ell_T - \ell')\right) - h \ . \tag{31}$$

Expressions (19) and (29), which have been obtained with different methods, are a function of a single unknown (α), and can be solved to find the tension in the (x,y) plane T_o. Once the value for α has been obtained, T_o can be calculated from the relation:

$$T_o = \frac{P}{\alpha} \ . \tag{32}$$

2.3. Approximation of the Iteration Values for the Horizontal Tension

Equations (19) or (29) can be solved readily by iteration if an appropriate initial value for α in sequence (20) is provided. In this section, an approximation for α based on the first order expansions of the mooring line geometric properties is derived.

Consider an initial known horizontal pretension P_{RT} of the catenary line in the (x, y) plane. The values of the geometric properties of the catenary can be found readily as follows:

$$\bar{\ell}_{eff} = \sqrt{h\left(h + 2\frac{P_{RT}}{P}\right)} \ , \tag{33}$$

$$\bar{\ell} = \frac{P_{RT}}{P}\sinh^{-1}\left(\frac{P\bar{\ell}_{eff}}{P_{RT}}\right) \ , \tag{34}$$

$$\bar{d} = \ell_T - \bar{\ell}_{eff} \ , \tag{35}$$

$$\bar{\ell}' = \bar{\ell} + \bar{d} \ . \tag{36}$$

The overbar in expressions (33)-(36) indicates the value of the geometric properties when $T_o = P_{RT}$.

As the catenary line is displaced a small amount in the (x, y) plane, the geometry of the catenary changes, thus producing a change in T_o. To find the horizontal tension after the displacement of the catenary takes place, an approximate initial value for α must be given for either equation (19) or (29) to converge in the iteration process. Such value can be found from the equations of either of the two methods developed in Section 2.2. Letting

$$\alpha' = \frac{T_o}{P} = \frac{1}{\alpha} \ , \tag{37}$$

and using the equations from Method 2 of Section 2.2, relations (16) and (23) are recast in terms of α' as follows:

$$\ell_T - \alpha' \sinh\left(\frac{\ell}{\alpha'}\right) - \ell' + \ell = 0 \ , \tag{38}$$

$$\cosh\left(\frac{\ell}{\alpha'}\right) = \frac{h}{\alpha'} + 1 \ . \tag{39}$$

By squaring the two expressions above and substracting (38) from (39), we obtain an expression of the form:

$$\alpha'^2 \cosh^2\left(\frac{\ell}{\alpha'}\right) - \alpha'^2 \sinh^2\left(\frac{\ell}{\alpha'}\right) = h^2 + 2\alpha' h + \alpha'^2 - (\ell_T - \ell' + \ell)^2 \ , \tag{40}$$

which can be simplified to yield

$$\alpha' = \frac{(\ell_T - \ell' + \ell)^2}{2h} - \frac{h}{2} \ . \tag{41}$$

The value of α' can be solved for readily if both ℓ' and ℓ are known (i.e. $\ell' = \overline{\ell}'$ and $\ell = \overline{\ell}$). When the upper end of the mooring line is offset by a small horizontal amount, however, the values for ℓ' and ℓ change nonproportionately, depending on the amount of the displacement. Consequently, equation (41) can be solved directly only if ℓ is known.

Letting: $\ell' = \overline{\ell}' + \Delta\ell'$ and $\ell = \overline{\ell} + \Delta\ell$, where $\Delta\ell'$ and $\Delta\ell$ are the changes of the geometric properties after the displacement of the upper end of the line, the relative displacement between $\Delta\ell$ and $\Delta\ell'$ can be calculated to find an equation that can be integrated into (41) to solve for the single unknown α'.

The relation between $\Delta\ell$ and $\Delta\ell'$ is derived in Appendix 2 as follows:

$$\frac{\Delta\ell}{\Delta\ell'} \cong \frac{1}{1 - \left[\left(1 - \frac{\overline{\ell}}{\overline{T_o}} \frac{\overline{dT}}{d\ell}\right) \cosh\left(\frac{P\overline{\ell}}{\overline{T_o}}\right) + \left(\frac{1}{P} \frac{\overline{dT}}{d\ell}\right) \sinh\left(\frac{P\overline{\ell}}{\overline{T_o}}\right)\right]} \ , \tag{42}$$

where $\frac{\overline{dT}}{d\ell}$ is the derivative of the horizontal tension T_o with respect to ℓ evaluated at the initial position of the mooring line $\overline{T_o} = T_P$, and is given by (Appendix 1):

$$\frac{\overline{dT}}{d\ell}=\frac{P\sqrt{h\left(h+{}^{2\overline{T}_o}\!/\!_P\right)}\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)}{\sqrt{h\left(h+{}^{2\overline{T}_o}\!/\!_P\right)}\left[\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)-\sinh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\right]+h}\ . \tag{43}$$

Further, substituting relations $\ell=\overline{\ell}+\frac{\Delta\ell}{\Delta\ell'}\Delta\ell'$ and $\ell'=\overline{\ell}'+\Delta\ell'$ into (41), we find an expression for α' of the form:

$$\alpha'=\frac{\left(\ell_T-\overline{\ell}'-\Delta\ell'+\overline{\ell}+\frac{\Delta\ell}{\Delta\ell'}\Delta\ell'\right)^2-h^2}{2h}\ . \tag{44}$$

Expression (42) is substituted into equation (44) to obtain

$$\alpha'=\frac{\left\{1-\left[\left(1-\frac{\overline{\ell}}{\overline{T}_o}\frac{\overline{dT}}{d\ell}\right)\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)+\left(\frac{1}{P}\frac{\overline{dT}}{d\ell}\right)\sinh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\right]\right\}\left\{\left(\overline{\ell}_T-\overline{\ell}'+\overline{\ell}\right)^2-h^2\right\}}{2h\left\{1-\left[\left(1-\frac{\overline{\ell}}{\overline{T}_o}\frac{\overline{dT}}{d\ell}\right)\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)+\left(\frac{1}{P}\frac{\overline{dT}}{d\ell}\right)\sinh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\right]\right\}}$$
$$+\frac{\left[\left(1-\frac{\overline{\ell}}{\overline{T}_o}\frac{\overline{dT}}{d\ell}\right)\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)+\left(\frac{1}{P}\frac{\overline{dT}}{d\ell}\right)\sinh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\right]\left(\overline{\ell}_T-\overline{\ell}'+\overline{\ell}\right)}{h\left\{1-\left[\left(1-\frac{\overline{\ell}}{\overline{T}_o}\frac{\overline{dT}}{d\ell}\right)\cosh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)+\left(\frac{1}{P}\frac{\overline{dT}}{d\ell}\right)\sinh\left(\frac{P\overline{\ell}}{\overline{T}_o}\right)\right]\right\}}\ . \tag{45}$$

In the expression above, α' is the only unknown and can be calculated readily. An appropriate initial approximation for the value of α in equations (19) or (29) is thus given by $\alpha={}^1\!/\!_{\alpha'}$.

2.4. Solution to the Catenary Model

To complete the study of catenaries, the geometry and tension distributions of the catenary line are derived in this section.

The horizontal geometric distribution of the catenary $\vec{x}(s)=(x(s),y(s))$ - starting at the point of zero slope (x_o,y_o,z_o) - is obtained by dimensionalzing equation (4) as follows:

$$\bar{x}(s) = \frac{T_o}{P}\sinh^{-1}\left(\frac{Ps}{T_o}\right), \tag{46}$$

where $0 \le s \le \ell_{eff}$. Denoting the angle between the endpoints of the catenary measured with respect to the (x, y) plane by $\gamma_{(x,y)}$, the horizontal components of the catenary distribution become:

$$x(s) = \left[\frac{T_o}{P}\sinh^{-1}\left(\frac{Ps}{T_o}\right)\right]\cos\gamma_{(x,y)}, \tag{47}$$

$$y(s) = \left[\frac{T_o}{P}\sinh^{-1}\left(\frac{Ps}{T_o}\right)\right]\sin\gamma_{(x,y)}. \tag{48}$$

The above expressions can be incorporated into expressions for the horizontal distribution of the catenary (x_c, y_c) as measured from the mooring point at the sea floor (i.e. from the point (x_m, y_m, z_m)) as follows:

$$x_c = x(d+s) = \left[\ell_T - \ell_{eff} + \frac{T_o}{P}\sinh^{-1}\left(\frac{Ps}{T_o}\right)\right]\cos\gamma_{(x,y)}, \tag{49}$$

$$y_c = y(d+s) = \left[\ell_T - \ell_{eff} + \frac{T_o}{P}\sinh^{-1}\left(\frac{Ps}{T_o}\right)\right]\sin\gamma_{(x,y)}. \tag{50}$$

The vertical distribution $z(s)$, of the catenary, is obtained from the dimensional form of equation (5):

$$z(s) = \frac{T_o}{P}\left[\cosh\left(\frac{P\sqrt{x(s)^2 + y(s)^2}}{T_o}\right) - 1\right] = \frac{T_o}{P}\left[\sqrt{1 + \left(\frac{Ps}{T_o}\right)^2} - 1\right], \tag{51}$$

and the vertical angle between the catenary at any point s ($s \ge 0$) with respect to the horizontal plane (x, y) is given by:

$$\theta(s) = \tan^{-1}\left(\frac{Ps}{T_o}\right). \tag{52}$$

Finally, the tension distribution $T(s)$ along the catenary ($0 \le s \le \ell_{eff}$) is calculated as a function of the horizontal tension of the line T_o, and the vertical tension T_v, where

$$T_v(s) = Ps, \tag{53}$$

as follows:

$$\boxed{T(s) = \sqrt{T_o^2 + T_v(s)^2}\ .} \tag{54}$$

At the upper end of the catenary, where $s = \ell_{eff}$, the vertical tension in the mooring line is given by

$$T_v(s = \ell_{eff}) = P\sqrt{h\left(h + 2\,{}^{T_o}\!/_{P}\right)}\ , \tag{55}$$

based from equation (10), or by

$$T_v(s = \ell_{eff}) = T_o \sinh\left(\frac{P\ell}{T_o}\right)\ , \tag{56}$$

based on equation (11).

Using the expression for T_v in expression (55), the total tension at the upper point of the catenary becomes:

$$T(s = \ell_{eff}) = \sqrt{T_o^2 + P^2 h\left(h + 2\,{}^{T_o}\!/_{P}\right)}\ , \tag{57}$$

which can be reduced further to the form:

$$\boxed{T(s = \ell_{eff}) = T_o + Ph\ .} \tag{58}$$

Similarly, by combining equations (54) and (56), an alternative form for the tension at the top of the mooring line can be obtained:

$$T(s=\ell_{eff})=\sqrt{T_o^{\,2}+T_o^{\,2}\sinh^2\left(\frac{P\ell}{T_o}\right)}\ , \tag{59}$$

or equivalently:

$$T(s=\ell_{eff})=T_o\cosh\left(\frac{P\ell}{T_o}\right)\ . \tag{60}$$

By substituting $s=\ell_{eff}$ in relations (49)-(52), the geometric distribution at the upper end of the catenary is given by:

$$\theta(\ell_{eff})=\tan^{-1}\left(\frac{P\ell_{eff}}{T_o}\right), \tag{61}$$

$$x(d+\ell_{eff})=\left[\ell_T-\ell_{eff}+\frac{T_o}{P}\sinh^{-1}\left(\tan(\theta(\ell_{eff}))\right)\right]\cos\gamma_{(x,y)}\ , \tag{62}$$

$$y(d+\ell_{eff})=\left[\ell_T-\ell_{eff}+\frac{T_o}{P}\sinh^{-1}\left(\tan(\theta(\ell_{eff}))\right)\right]\sin\gamma_{(x,y)}\ , \tag{63}$$

$$z(\ell_{eff})=\frac{T_o}{P}\left[\sqrt{1+\left(\frac{P\ell_{eff}}{T_o}\right)^2}-1\right]=\frac{T_o}{P}\left[\sqrt{1+\tan^2(\theta(\ell_{eff}))}-1\right]\ . \tag{64}$$

Recognizing that $\sqrt{1+\tan^2(x)}=\sec^2(x)=1/\cos(x)$, the expression above can also be written in the form:

$$z(\ell_{eff})=\frac{T_o}{P}\left[\frac{1-\cos(\theta(\ell_{eff}))}{\cos(\theta(\ell_{eff}))}\right]\ . \tag{65}$$

III. MOORING LINE DAMPING

When a dynamical system moves harmonically with a small period of oscillation (of the order of 150 seconds), the mooring lines act quasistatically. The viscous dissipation of the mooring line can be estimated using energy principles. As discussed in Chapter I, deep water catenary chains exhibit high hydrodynamic drag, which becomes increasingly important with respect to the moored vessel resistance as the water depth increases. In this chapter, an analytical model for the nonlinear catenary drag is derived based on energy principles.

3.1. Derivation of Catenary Damping Coefficients

Figure 2 shows the geometry of a displaced catenary in the two-dimensional plane. In this figure, x_A is the horizontal projected distance of the catenary; $\delta(x_A)$ is the horizontal displacement of the top of the catenary (point A); $\delta(v(s))$ is the displacement normal to the catenary line; and F_x is the horizontal force at the upper endpoint of the catenary ($F_x = T_o$).

The objective in this analysis is to relate the displacement $\delta(v(s))$ to the horizontal displacement $\delta(x_A)$ at the top of the mooring line. The resulting expression is incorporated into the formulation of the mooring line damping forces that act on the vessel.

Letting:

$\delta(F_x)$= variation of the horizontal force T_o at point A,

$R_H = \dfrac{\delta(F_x)}{\delta(x_A)}$ = horizontal rigidity (stifness) of the catenary ,

$\delta(\ell_{eff})$= variation of the catenary suspended length,

$\delta(z_A)$= variation of the vertical displacement of the catenary at point A,

the relation $\delta(z_A) = 0$ (i.e. no vertical variation of the catenary at the upper point of attachment) yields an expression for $\delta(\ell_{eff})$ in terms of $\delta(F_x)$, which can be used with the expression for the horizontal variation $\delta(x_A)$ to determine the horizontal rigidity of the catenary R_H .

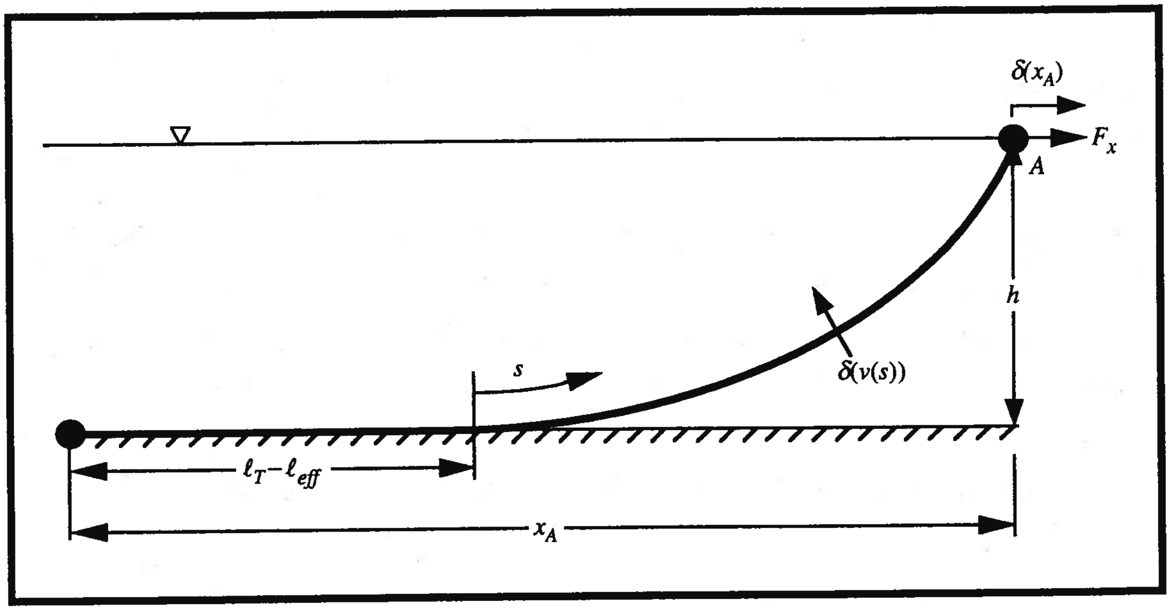

Figure 2: Dispacement of the catenary in the 2-D plane

Letting $F_x\!=\!T_o$, and $z_A\!=\!z(s=\ell_{eff})$ we have, from (51)

$$z_A=\frac{F_x}{P}\left[\sqrt{1+\left(\frac{P\ell_{eff}}{F_x}\right)^2}-1\right]=h\ . \tag{66}$$

Taking the variation of expression (66) above and setting it to zero, we have:

$$\delta(z_A)=\frac{1}{P}\left\{\left[\sqrt{1+\left(\frac{P\ell_{eff}}{F_x}\right)}-1\right]\delta(F_x)+\frac{P^2\ell_{eff}}{F_x}\frac{\left[\delta(\ell_{eff})-(\ell_{eff}/F_x)\delta(F_x)\right]}{\sqrt{1+\left({}^{P\ell_{eff}}\!/_{F_x}\right)^2}}\right\}=0. \tag{67}$$

Further, letting $\theta_\ell=\theta(s=\ell_{eff})$, and observing from (61) that $\dfrac{P\ell_{eff}}{F_x}=\tan(\theta_\ell)$, the expression above can be rewritten as:

$$\delta(z_A)=\frac{1}{P}\left\{\left[\left(\frac{1-\cos(\theta_\ell)}{\cos(\theta_\ell)}\right)-\frac{\sin^2(\theta_\ell)}{\cos(\theta_\ell)}\right]\delta(F_x)+\left[P\sin(\theta_\ell)\right]\delta(\ell_{eff})\right\}=0. \tag{68}$$

Expression (68) can be further simplified to yield an expression for $\delta(\ell_{eff})$ in terms of $\delta(F_x)$:

$$\delta(\ell_{eff}) = \frac{1}{P}\left[\frac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right]\delta(F_x)\ . \tag{69}$$

The expression above can be substituted in an expression for $\delta(x_A)$ (to be derived below) to obtain a relationship that allows us to calculate the horizontal rigidity of the catenary line. Assuming that the mooring line moves in the two-dimensional plane (i.e. setting $\gamma_{(x,y)}$ = 0), we find, from expression (49):

$$x_A = \ell_T - \ell_{eff} + \frac{F_x}{P}\sinh^{-1}\left(\frac{P\ell_{eff}}{F_x}\right)\ . \tag{70}$$

Taking a first order variation of the expression above, the expression for $\delta(x_A)$ in terms of $\delta(\ell_{eff})$ and $\delta(F_x)$ becomes:

$$\delta(x_A) = -\delta(\ell_{eff}) + \frac{1}{P}\left\{\left[\sinh^{-1}\left(\frac{P\ell_{eff}}{F_x}\right)\right]\delta(F_x)\right\} + \frac{\left[\delta(\ell_{eff}) - (\ell_{eff}/F_x)\delta(F_x)\right]}{\sqrt{1+\left({P\ell_{eff}}/{F_x}\right)^2}}\ , \tag{71}$$

which can be simplified further to yield

$$\delta(x_A) = -[1-\cos(\theta_\ell)]\delta(\ell_{eff}) + \frac{1}{P}\left[\sinh^{-1}(\tan(\theta_\ell)) - \frac{P\ell_{eff}}{F_x}\cos(\theta_\ell)\right]\delta(F_x)\ . \tag{72}$$

Relation (69) can be substituted into expression (72) to find the following relation between $\delta(x_A)$ and $\delta(F_x)$:

$$\delta(x_A) = \frac{1}{P}\left[\frac{2(\cos(\theta_\ell)-1)}{\sin(\theta_\ell)} + \sinh^{-1}(\tan(\theta_\ell))\right]\delta(F_x)\ . \tag{73}$$

Expression (73) is a single equation recast in terms of the variations $\delta(x_A)$ and $\delta(F_x)$. The expression for the horizontal rigidity R_H is thus given by:

$$\frac{R_H}{P} = \frac{\delta(F_x)}{P\delta(x_A)} = \frac{1}{\sinh^{-1}(\tan(\theta_\ell)) - 2\left(\frac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right)} \; . \tag{74}$$

For values of $\theta_\ell \geq 45°$, expression (74) above can be asymptotically approximated by:

$$\left.\frac{R_H}{P}\right|_{\theta_\ell \geq 45°} \approx \frac{12}{\theta_\ell^{\,3}} \; . \tag{75}$$

From expressions (69) and (72), an expression that relates $\delta(\ell_{eff})$ in terms of the horizontal displacement $\delta(x_A)$ can also be found:

$$\delta(\ell_{eff}) = \frac{R_H}{P}\left(\frac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right)\delta(x_A) \; . \tag{76}$$

The expression above can be simplified for $\theta_\ell \geq 45°$ as:

$$\left.\delta(\ell_{eff})\right|_{\theta_\ell \geq 45°} \approx \frac{6}{\theta_\ell^{\,2}}\delta(x_A) \; . \tag{77}$$

3.2. Nonlinear Damping Forces on the Catenary

In this section, analytical expressions for the horizontal plane damping forces on the mooring line in the directions parallel and perpendicular to the mooring line motion are derived. This motion is in the catenary deformation plane.

Force in the Direction of the Mooring Line Motion: Let F_a be the drag force that the line adds to the vessel resistance or other oceanic system in its direction of motion. The dissipated power, P_W, due to this phenomenon can be expressed as [9]:

$$P_W = \int_0^{\ell_{eff}} \left\{\frac{1}{2}\rho C_D D\left|\delta(\dot{v}(s))\right|\left[\delta(\dot{v}(s))\right]^2\right\} ds = F_a \delta(\dot{x}_A) \; , \tag{78}$$

where $\delta(\dot{x}_A)$ is a unit velocity at the point of attachment in the direction to the motion of the mooring line, and $\delta(\dot{v}(s))$ is the variation of the speed at each point of the catenary. The drag coefficient C_D is purely a function of the speed $\delta(\dot{v}(s))$. The term in brackets in the equation above represents the energy dissipation at each point of the mooring line.

The relation between the displacements $\delta(v(s))$ and $\delta(x_A)$ (as well as their time derivatives) can be obtained and then substituted into (78) to obtain a relation for the mooring line damping forces in the plane of motion of the catenary.

Observing from Figure 2 that,

$$\delta(v(s)) = -[\sin(\theta(s))]\delta(x(d+s)) + [\cos(\theta(s))]\delta(z(s)), \tag{79}$$

the variation of the displacement $\delta(v(s))$ of the catenary measured from the mooring point at the sea floor can be obtained in terms of θ_ℓ and $\delta(x_A)$. To achieve this, the expressions for $\delta(x(d+s))$ and $\delta(z(s))$ must be expanded in terms of θ_ℓ and $\delta(x_A)$. The differential $\delta(x(d+s))$ is obtained from (62) as follows:

$$\delta(x(d+s)) = -\delta(\ell_{eff}) + \frac{1}{P}\left\{\sinh^{-1}\left(\frac{Ps}{F_x}\right)\delta(F_x) + \frac{1}{\sqrt{1+\left(\frac{Ps}{F_x}\right)^2}}\left[P\delta(s) - \frac{Ps}{F_x}\delta(F_x)\right]\right\} . \tag{80}$$

Substituting expressions (52) and (74) into the equation above, the following relation is obtained:

$$\delta(x(d+s)) = \frac{R_H}{P}\left[\sinh^{-1}(\tan(\theta(s)))\delta(F_x) - \sin(\theta(s))\right]\delta(x_A) + [\cos(\theta(s))]\delta(s) . \tag{81}$$

Letting $\delta(s) = \delta(\ell_{eff})$, given by (76), the expression above can be recast as:

$$\delta(x(d+s)) = \frac{R_H}{P}\left[\sinh^{-1}(\tan(\theta(s))) - (1-\cos(\theta(s)))\frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)} - \sin(\theta(s))\right]\delta(x_A) . \tag{82}$$

Similarly, an expression for $\delta(z(s))$ in terms of $\delta(x_A)$ is obtained by expanding equation (67) in the form:

$$\delta(z(s)) = \frac{1}{P}\left\{\left[\sqrt{1+\left(\frac{Ps}{F_x}\right)^2}-1\right]\delta(F_x) + \frac{Ps}{\sqrt{1+\left(\frac{Ps}{F_x}\right)^2}}\left[\frac{F_x P\delta(s) - Ps\delta(F_x)}{F_x^2}\right]\right\}. \tag{83}$$

The expression above can further be written as

$$\delta(z(s)) = \frac{1}{P}\left\{\frac{(1-\cos(\theta(s)))}{\cos(\theta(s))}R_H\delta(x_A) + \cos(\theta(s))\left[\frac{P^2 s\delta(s)}{F_x} - \left(\frac{Ps}{F_x}\right)^2 R_H\delta(x_A)\right]\right\}, \tag{84}$$

which can be simplified further to yield

$$\delta(z(s)) = \frac{R_H}{P}\left[\frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)}\sin(\theta(s)) - 1 + \cos(\theta(s))\right]\delta(x_A). \tag{85}$$

Notice that $\delta(x(\ell_{eff})) = \delta(x_A)$; $\delta(z(\ell_{eff})) = \delta(z_A) = 0$; $\delta(x(0)) = \delta(z(0)) = 0$, since the variation at the point of touchdown is of second order in $\delta(x_A)$. After substitution of expressions (82) and (85) into (79), the first order variation $\delta(v(s))$ can be written as:

$$\begin{aligned}\delta(v(s)) = \frac{R_H}{P}\Bigg\{&\left[\frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)}(1-\cos(\theta(s))) - \sinh^{-1}(\tan(\theta(s))) + \sin(\theta(s))\right]\sin(\theta(s)) \\ &+\left[\frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)}\sin(\theta(s)) - 1 + \cos(\theta(s))\right]\cos(\theta(s))\Bigg\}\delta(x_A),\end{aligned} \tag{86}$$

or

$$\delta(v(s)) = \frac{R_H}{P}\left[\frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)}\sin(\theta(s)) + 1 - \cos(\theta(s)) - \sinh^{-1}(\tan(\theta(s)))\right]. \tag{87}$$

The final form of $\delta(v(s))$ can be recast as follows:

$$\delta(v(s)) = \frac{R_H}{P}[f(\theta(s))]\delta(x_A) \ , \tag{88}$$

where

$$f(\theta(s)) = \frac{(1-\cos(\theta_\ell))}{\sin(\theta_\ell)}\sin(\theta(s)) + 1 - \cos(\theta(s)) - \sinh^{-1}(\tan(\theta(s))) \ . \tag{89}$$

Time derivatives of expression (88) can be derived in order to obtain an expression that directly relates $\delta(\dot{v}(s))$ to $\delta(\dot{x}_A)$. Since only the slow motions of the catenary are considered, the horizontal stiffness of the catenary R_H as well as the function in expression (89) above do not change in time. Thus,

$$\delta(\dot{v}(s)) = \frac{R_H}{P}[f(\theta(s))]\delta(\dot{x}_A) \ . \tag{90}$$

Expression (90) above can be substituted into (78) to obtain an expression for the damping force in the direction of the mooring line motion as follows:

$$F_a = \frac{1}{2}\rho C_D(\dot{x}_A) D_{eff} h \gamma_p |\delta(\dot{x}_A)| \delta(\dot{x}_A), \tag{91}$$

In expression (91), ρ is the water density, D_{eff} is the effective diameter of the catenary, h is the depth of immersion of the catenary, and γ_p is a function of the energy dissipation:

$$\gamma_p = \left(\frac{R_H}{P}\right)^3 \frac{1}{h}\int_0^{\ell_{eff}} \left\{ |f(\theta(s))| f^2(\theta(s)) \right\} ds \ . \tag{92}$$

By observing that $d\theta = \frac{P}{F_x}\cos^2(\theta)$, expression (92) above can be written in terms of θ_ℓ as follows:

$$\gamma_p = \left(\frac{R_H}{P}\right)^3 \frac{\cos(\theta_\ell)}{(1-\cos(\theta_\ell))}\int_0^{\theta_\ell} \left\{ \frac{|f(\theta)| f^2(\theta)}{\cos^2(\theta)} \right\} d\theta \ . \tag{93}$$

For values of $\theta_\ell \le 65^\circ$, expression (93) can be approximated by

$$\gamma_P\big|_{\theta_\ell \le 65^\circ} \approx \frac{\theta_\ell^{\,5}}{560}\left(\frac{R_H}{P}\right)^3 . \tag{94}$$

The first order expansion of the energy dissipation function (93) with respect to θ_ℓ is shown in Appendix 3.

Force Perpendicular to the Horizontal Mooring Line Motion: Let $\delta(\dot{y}_A)$ be the variation of the velocity in the mooring line in the direction perpendicular to the motion of the catenary line. The damping force on the catenary in this direction is given by [9]:

$$F_l = \frac{1}{2}\rho C_D(\dot{y}_A) D_{eff}\hbar\left|\delta(\dot{y}_A)\right|\delta(\dot{y}_A) , \tag{95}$$

where $\hbar$ is a function of the displacement of the catenary, defined as follows:

If we consider that the complete suspended catenary acquires a velocity $\dot{y}_A$ perpendicular to the direction of motion, by ignoring the bottom friction, then

$$\hbar = \ell_{eff} . \tag{96}$$

Equation (96) can be given in terms of h and θ_ℓ (Appendix 4) as follows:

$$\ell_{eff} = h\left[\frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)}\right], \tag{97}$$

and, thus,

$$\hbar = h\frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)} . \tag{98}$$

Substituting (98) into (95) we find an expression for the damping force in the normal direction of motion as follows:

$$F_l = \frac{1}{2}\rho C_D(\dot{y}_A) D_{eff} h \frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)} \left|\delta(\dot{y}_A)\right| \delta(\dot{y}_A) \; . \tag{99}$$

If we consider, however, that the catenary has no displacement at its bottom, then the expression for $\hbar$ is given by:

$$\hbar = \int_0^h \frac{h'}{h} \frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)} dh' = \frac{h}{2}\left[\frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)}\right] , \tag{100}$$

and the expression for the damping force becomes:

$$F_l = \frac{1}{4}\rho C_D(\dot{y}_A) D_{eff} h \frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)} \left|\delta(\dot{y}_A)\right| \delta(\dot{y}_A) \; . \tag{101}$$

3.3. Nonlinear Damping Forces on the Vessel

Figure 3 shows the horizontal plane geometry of a moored vessel with a deep water catenary line. In this figure, (ξ, η) is the earth-fixed reference frame; (x, y) is the catenary reference frame as measured from the point of contact of the chain to the ground (x_o, y_o); (X, Y, Z) is the body reference frame imbedded at the center of gravity of the vessel (*CG*); and (x', y') is the coordinate system of the catenary with its origin at the attachment point on the vessel, x' measured parallel to the horizontal plane of the line and y' normal to x'. In addition, (x_p, y_p) are the body-fixed coordinates of the attachment point on the vessel; ψ is the drift angle; γ is the angle between the ξ-axis and the catenary, measured counterclockwise; ℓ' is the distance between the mooring point *B* and the attachment point *A* on the vessel, also shown in Figure 1; F_A and F_L are the drag forces in the directions parallel and perpendicular to the mooring line motion respectively.

The drag force on the mooring line in the catenary coordinate system $(\vec{i}', \vec{j}')$ can be expressed as:

$$\vec{F}_m = F_A \vec{i}' + F_L \vec{j}' \; , \tag{102}$$

where, from expressions (91) and (95):

$$F_A = \frac{1}{2}\rho C_{DX} D_{eff} h \gamma_p |\dot{x}'_A| \dot{x}'_A \ , \tag{103}$$

$$F_L = \frac{1}{4}\rho C_{DY} D_{eff} h \frac{\sin\theta_\ell}{1-\cos\theta_\ell} |\dot{y}'_A| \dot{y}'_A \ . \tag{104}$$

In the expressions above, C_{DX} and C_{DY} are the drag coefficients in the x' and y' directions of motion measured with respect to the velocities at the top of the catenary line.

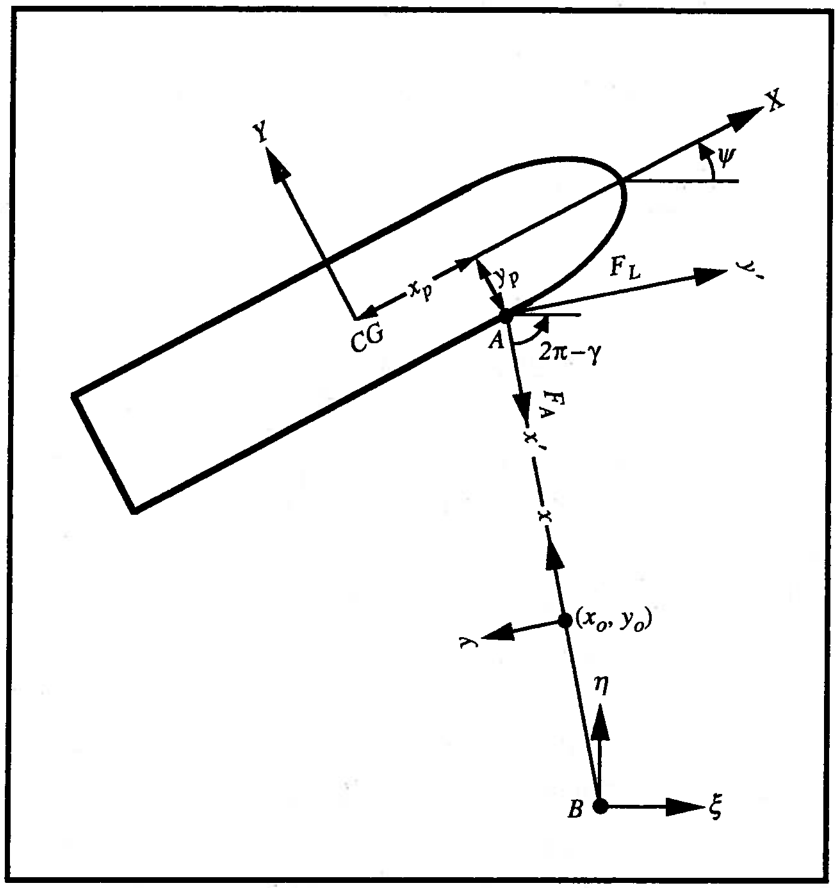

Figure 3. Geometry of the mooring system

The coordinate transformation between the vessel coordinate system $(\vec{I}, \vec{J})$ and the mooring line coordinate system $(\vec{i}', \vec{j}')$ in the horizontal plane is given by:

$$\vec{i}' = \cos\beta\vec{I} + \sin\beta\vec{J}, \qquad \vec{j}' = -\sin\beta\vec{I} + \cos\beta\vec{J} \ ,$$

where β is the angle from $\vec{i}'$ to $\vec{I}$ measured counterclockwise ($\beta = \gamma - \psi$).

Therefore, the drag forces in the (X,Y) coordinate system are given by

$$F_{XD} = F_A \cos\beta - F_L \sin\beta \ , \tag{105}$$
$$F_{YD} = F_A \sin\beta + F_L \cos\beta \ . \tag{106}$$

The expressions for the velocity components in the mooring line also can be recast in the $(\vec{I},\vec{J})$ frame with respect to the center of gravity of the vessel (*CG*). Notice from Figure 3 that the upper endpoint of the mooring line is located at the point (x_p, y_p) as measured with respect to the (X,Y) coordinate frame.

Letting u and v be the horizontal velocity components of the vessel at its center of gravity, and r be its rotational velocity, we can calculate the velocity components at the attachment point of the vessel $\vec{u}_A = (u_A,\ v_A,\ r_A)$ with respect to the (X,Y) frame as follows:

$$\vec{u}_A = \vec{u}_G + \vec{\omega} \times \vec{r}_{AG} \ , \tag{107}$$

where $\vec{u}_G$ is the velocity vector of the vessel's center of gravity, $\vec{r}_{AG} = (x_p, y_p, 0)$ is the distance vector between the vessel *CG* and point *A*, and $\vec{\omega} = r$ is the rotational velocity of the ship. Equation (107) can be recast as:

$$\begin{Bmatrix} u_A \\ v_A \\ r_A \end{Bmatrix} = \begin{Bmatrix} u \\ v \\ r \end{Bmatrix} + \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & r \\ x_p & y_p & 0 \end{bmatrix} = \begin{Bmatrix} u - y_p r \\ v + x_p r \\ r \end{Bmatrix} \ . \tag{108}$$

The horizontal velocity of the upper endpoint of the mooring line $\vec{u}_m$ also can be recast in terms of the $(\vec{I},\vec{J})$ frame as follows:

$$\vec{u}_m = \dot{x}'_A \vec{i}\,' + \dot{y}'_A \vec{j}\,' = \dot{x}'_A (\cos\beta \vec{I} + \sin\beta \vec{J}) + \dot{y}'_A (-\sin\beta \vec{I} + \cos\beta \vec{J}) \ , \tag{109}$$

to yield

$$\vec{u}_m = (\dot{x}'_A \cos\beta - \dot{y}'_A \sin\beta)\vec{I} + (\dot{x}'_A \sin\beta + \dot{y}'_A \cos\beta)\vec{J} = u'\vec{I} + v'\vec{J} \ . \tag{110}$$

Equating expressions (108) and (110) we find:

$$\dot{x}'_A \cos\beta - \dot{y}'_A \sin\beta = u - y_p r \ , \tag{111}$$
$$\dot{x}'_A \sin\beta + \dot{y}'_A \cos\beta = v + x_p r \ . \tag{112}$$

Thus, the horizontal velocity components of the mooring line at point A are given in terms of vessel velocities and coordinates as:

$$\boxed{\dot{x}'_A = (u - y_p r)\cos\beta + (v + x_p r)\sin\beta \ ,} \tag{113}$$

$$\boxed{\dot{y}'_A = (v + x_p r)\cos\beta - (u - y_p r)\sin\beta \ .} \tag{114}$$

The damping forces can be expressed in terms of vessel velocities as follows:

$$F_A = \frac{1}{2}\rho C_{DX} D_{eff} h \gamma_p \left|(u - y_p r)\cos\beta + (v + x_p r)\sin\beta\right| \left[(u - y_p r)\cos\beta + (v + x_p r)\sin\beta\right] \ , \tag{115}$$
$$F_L = \frac{1}{4}\rho C_{DY} D_{eff} h \frac{\sin\theta_\ell}{1 - \cos\theta_\ell} \Big\{ \left|(v + x_p r)\cos\beta - (u - y_p r)\sin\beta\right| \left[(v + x_p r)\cos\beta - (u - y_p r)\sin\beta\right] \Big\} \ . \tag{116}$$

The expressions above can be inserted into equations (105) and (106) to obtain the damping forces F_{XD} and F_{YD} with respect to the vessel coordinate system. The moment M_{ZD} on the vessel about the Z-axis due to the damping force components is given by:

$$M_{ZD} = x_p F_{YD} - y_p F_{XD} \ , \tag{117}$$

which can be expanded further to yield:

$$M_{ZD} = F_A\left(x_p \sin\beta - y_p \cos\beta\right) + F_L\left(x_p \cos\beta + y_p \sin\beta\right) \ . \tag{118}$$

Expressions (105), (106) and (118) denote the damping forces and moment on each of the mooring lines. These must be summed up according to the number of mooring lines present in the system. Notice that they act in the same direction as the hydrodynamic forces and moment exerted on the vessel (i.e. resisting its motion). The expressions for the mooring line damping forces and moment therefore must be implemented properly into any mathematical model.

APPENDICES

APPENDIX 1: FIRST ORDER VARIATION OF THE HORIZONTAL TENSION IN TERMS OF THE HORIZONTAL PROJECTED LENGTH OF THE SUSPENDED CATENARY

The first order variation of the horizontal mooring line tension ($T_o = T$), is obtained from equation (12) in terms of the first order variation of the horizontal projected length of the suspended catenary (ℓ). This expression, in turn, can be subsequently expressed as a function of other geometric properties of the catenary. This is achieved by taking the first order variation of ℓ in terms of the variable of interest, such as the horizontal length between the endpoints of the catenary (ℓ') (Appendix 2),.and applying the chain rule of differentiation.

Consider equation (12), which is of the form

$$\ell_{eff} = \frac{T}{P}\sinh\left(\frac{P\ell}{T}\right) = \sqrt{h\left(h + 2\frac{T}{P}\right)} \quad . \tag{A1.1}$$

This expression relates the horizontal tension in the mooring line (T), and the horizontal suspended length of the catenary (ℓ), where P and h are known constants. The derivative of T with respect to ℓ can be derived from equation (A1.1) by taking the derivative of all involved terms in the equation with respect to ℓ as follows:

$$\frac{1}{P}\frac{dT}{d\ell}\sinh\left(\frac{P\ell}{T}\right) + \left(1 - \frac{\ell}{T}\frac{dT}{d\ell}\right)\cosh\left(\frac{P\ell}{T}\right) = \frac{h\dfrac{dT}{d\ell}}{P\sqrt{h\left(h + 2\dfrac{T}{P}\right)}} \quad . \tag{A1.2}$$

In this equation, $\frac{dT}{d\ell}$ is the derivative of the horizontal tension T with respect to the horizontal projected length of the suspended catenary ℓ. The equation above can be further recast as follows:

$$\frac{dT}{d\ell} = \frac{P\cosh\left(\frac{P\ell}{T}\right)}{\left(\frac{P\ell}{T}\right)\cosh\left(\frac{P\ell}{T}\right) - \sinh\left(\frac{P\ell}{T}\right) + \frac{h}{\sqrt{h\left(h + {}^{2T}\!/_{P}\right)}}} \quad . \tag{A1.3}$$

Rearranging further relation (A1.3), the final form for $\frac{dT}{d\ell}$ becomes:

$$\frac{dT}{d\ell} = \frac{P\sqrt{h\left(h + {}^{2T}\!/_{P}\right)}\cosh\left(\frac{P\ell}{T}\right)}{\sqrt{h\left(h + {}^{2T}\!/_{P}\right)}\left[\left(\frac{P\ell}{T}\right)\cosh\left(\frac{P\ell}{T}\right) - \sinh\left(\frac{P\ell}{T}\right)\right] + h} \quad . \tag{A1.4}$$

APPENDIX 2: FIRST ORDER VARIATION OF THE HORIZONTAL PROJECTED LENGTH OF THE SUSPENDED CATENARY IN TERMS OF ITS TOTAL HORIZONTAL LENGTH

The geometry of a deep water catenary mooring line does not change proportionally to a small horizontal displacement of its upper endpoint (i.e. a change in the total horizontal length in the catenary ℓ' does not equal to the change in the horizontal projected length of the suspended part of the catenary ℓ). The relative change between these geometric properties as the mooring line is displaced horizontally by a small amount is derived below.

Figure A2.1 shows the geometry of the catenary before (1) and after (2) the uppermost point of the chain has been displaced by an amount Δx horizontally in the plane of the catenary. In Figure A2.1, $|\Delta x| = \Delta\ell_1$.

From the geometric relations in Figure A2.1, the horizontal components of the catenary have the following relations:

$$\ell' = \bar{\ell}' + \Delta\ell' = \bar{\ell}' + \Delta\ell_1 , \qquad \text{(A2.1)}$$

$$d = \bar{d} + \Delta d = \bar{d} - \Delta\ell_2 , \qquad \text{(A2.2)}$$

$$\ell = \bar{\ell} + \Delta\ell = \bar{\ell} + \Delta\ell_1 + \Delta\ell_2 . \qquad \text{(A2.3)}$$

The overbar in the expressions above denotes the value of the geometric property before the horizontal displacement. In addition, the following relations hold:

$$\ell_T = \bar{\ell}_T + \Delta\ell_T = \text{constant} , \qquad \text{(A2.4)}$$

where

$$\bar{\ell}_T = \bar{\ell}_{eff} + \bar{d} , \qquad \text{(A2.5)}$$

and

$$\Delta\ell_T = \Delta\ell_{eff} + \Delta d = 0 \; . \tag{A2.6}$$

By combining expressions (A2.2) and (A2.6), the following geometric relation is obtained:

$$\Delta\ell_{eff} \approx \Delta\ell_2 \; . \tag{A2.7}$$

The equation above shows that the change in the length of the suspended catenary $\Delta\ell_{eff}$ is equal to the negative of the change in length of the horizontal length of the undeformed catenary $\Delta\ell_2 = -\Delta d$.

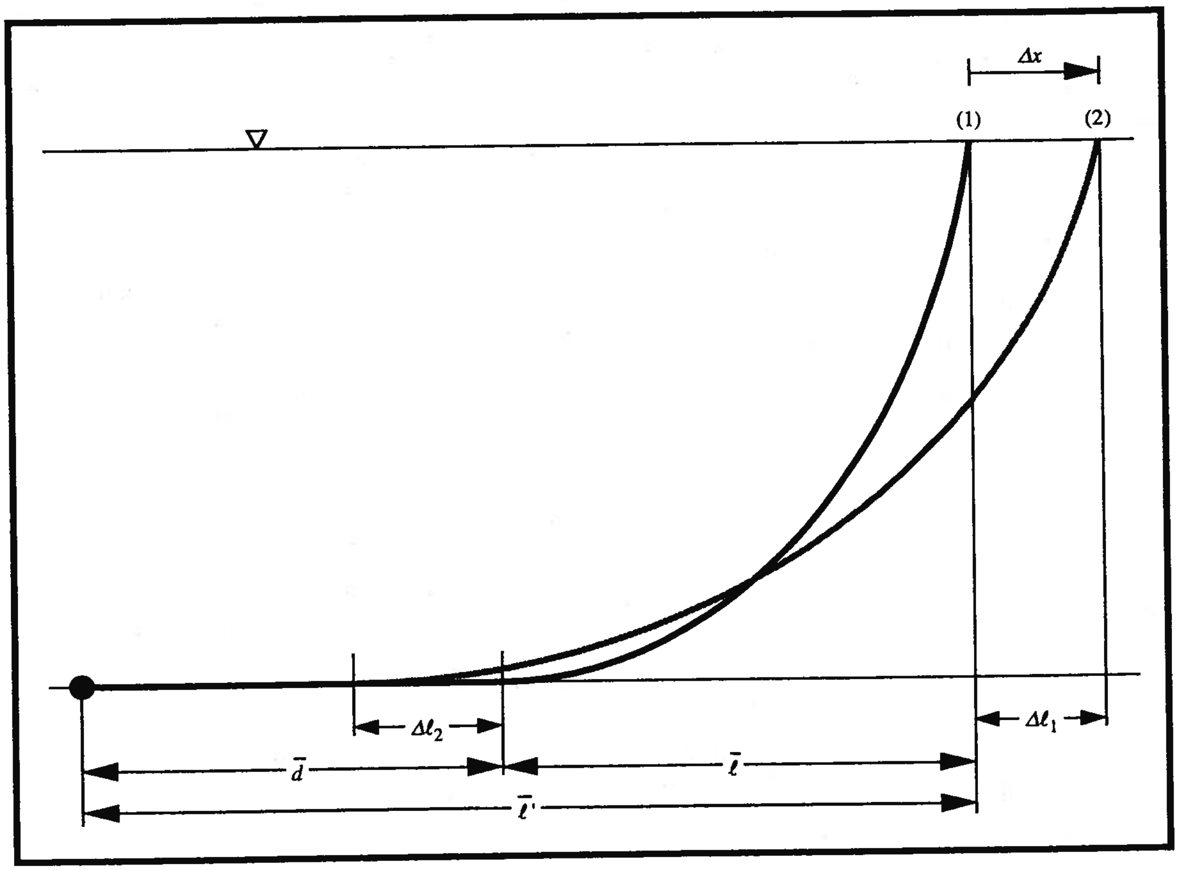

Figure A2.1: Geometry of the displaced catenary

In order to obtain an expression relating the changes between the geometric properties ℓ and ℓ', first consider equation (11)

$$\ell_{eff} = \frac{T}{P}\sinh\left(\frac{P\ell}{T}\right) . \tag{A2.8}$$

Equation (A2.8) can be expanded reeadily in terms of ℓ_{eff}, ℓ, and the horizontal tension (T) as follows:

$$\bar{\ell}_{eff} + \Delta\ell_{eff} = \frac{\bar{T}+\Delta T}{P}\sinh\left(\frac{P(\bar{\ell}+\Delta\ell)}{\bar{T}+\Delta T}\right) . \tag{A2.9}$$

Letting $x = \frac{\Delta T}{\bar{T}}$ (small), and using the binomial expansion relation

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots ,$$

the rightmost term inside the hyperbolic function of equation (A2.9) can be expanded as follows:

$$\frac{P(\bar{\ell}+\Delta\ell)}{\bar{T}+\Delta T} = \frac{P(\bar{\ell}+\Delta\ell)}{\bar{T}\left(1+\Delta T/\bar{T}\right)} = \frac{P}{\bar{T}}(\bar{\ell}+\Delta\ell)\left[1-\frac{\Delta T}{\bar{T}}+\left(\frac{\Delta T}{\bar{T}}\right)^2-\left(\frac{\Delta T}{\bar{T}}\right)^3+\dots\right] . \tag{A2.10}$$

The expression for the derivative of the horizontal tension with respect to the horizontal suspended length of the catenary (i.e. $\frac{dT}{d\ell}=\frac{\Delta T}{\Delta\ell}$) (see Appendix 1), can be incorporated into (A2.10) by letting:

$$\Delta T = \frac{\Delta T}{\Delta\ell}\Delta\ell = \frac{dT}{d\ell}\Delta\ell = T_L\Delta\ell , \tag{A2.11}$$

and the right hand side of expression (A2.10) can be recast in terms of $\Delta\ell$ such that:

$$\frac{P(\bar{\ell}+\Delta\ell)}{\bar{T}+\Delta T} = \frac{P\bar{\ell}}{\bar{T}} + \frac{P}{\bar{T}}\left(1-\frac{T_L}{\bar{T}}\bar{\ell}\right)\Delta\ell + \dots . \tag{A2.12}$$

Substitution of (A2.12) into (A2.9) yields:

$$\bar{\ell}_{eff} + \Delta\ell_{eff} = \frac{\left(\bar{T}+T_L\Delta\ell\right)}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}} + \frac{P}{\bar{T}}\left(1-\frac{T_L}{\bar{T}}\bar{\ell}\right)\Delta\ell\right) + \dots . \tag{A2.13}$$

Using the hyperbolic relation $\sinh(u+v) = \sinh(u)\cosh(v) + \cosh(u)\sinh(v)$, equation (A2.13) above can be written in the form

$$\bar{\ell}_{eff} + \Delta\ell_{eff} = \frac{(\bar{T} + T_L\Delta\ell)}{P}\left\{\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\cosh\left(\frac{P}{\bar{T}}\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)\Delta\ell\right)\right.$$
$$\left. + \cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\sinh\left(\frac{P}{\bar{T}}\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)\Delta\ell\right)\right\} + \ldots . \qquad \text{(A2.14)}$$

The terms on the right hand side of equation (A2.14) can be expanded further in Maclauren Series [2]:

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots + \frac{x^{2n+1}}{(2n+1)!} \qquad |x| < \infty ,$$
$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots + \frac{x^{2n}}{(2n)!} \qquad |x| < \infty ,$$

to yield

$$\bar{\ell}_{eff} + \Delta\ell_{eff} = \frac{(\bar{T} + T_L\Delta\ell)}{P}\left\{\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\left[1 + \frac{1}{2}\left(\frac{P}{\bar{T}}\right)^2\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)^2\Delta\ell^2 + \ldots\right]\right.$$
$$\left. + \cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\left[\left(\frac{P}{\bar{T}}\right)\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)\Delta\ell + \frac{1}{6}\left(\frac{P}{\bar{T}}\right)^3\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)^3\Delta\ell^3 + \ldots\right]\right\} + \ldots . \qquad \text{(A2.15)}$$

Then, the right hand side of equation (A2.15) can be linearized:

$$\bar{\ell}_{eff} + \Delta\ell_{eff} \cong \frac{\bar{T}}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right) + \left[\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)\cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right) + \frac{T_L}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\right]\Delta\ell . \qquad \text{(A2.16)}$$

Recognizing that $\bar{\ell}_{eff} = \frac{\bar{T}}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)$, an expression for $\Delta\ell_{eff}$ can be found from (A2.16) as follows:

$$\Delta\ell_{eff} \cong \left[\left(1 - \frac{T_L}{\bar{T}}\bar{\ell}\right)\cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right) + \frac{T_L}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\right]\Delta\ell . \qquad \text{(A2.17)}$$

In equation (2.17), $\Delta\ell = \Delta\ell_1 + \Delta\ell_2$ and $\Delta\ell_{eff} \cong \Delta\ell_2$. The equation above can be recast in terms of these displacements as follows:

$$\Delta\ell_2 \cong \frac{\left[\left(1-\frac{T_L}{\bar{T}}\bar{\ell}\right)\cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right)+\frac{T_L}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\right]}{1-\left[\left(1-\frac{T_L}{\bar{T}}\bar{\ell}\right)\cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right)+\frac{T_L}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\right]}\Delta\ell_1 \ . \tag{A2.18}$$

To find $\Delta\ell = \Delta\ell_1 + \Delta\ell_2$ in terms of $\Delta\ell' = \Delta\ell_1$ the expression above can be further recast to obtain:

$$\boxed{\frac{\Delta\ell}{\Delta\ell'} \cong \frac{1}{1-\left[\left(1-\frac{T_L}{\bar{T}}\bar{\ell}\right)\cosh\left(\frac{P\bar{\ell}}{\bar{T}}\right)+\frac{T_L}{P}\sinh\left(\frac{P\bar{\ell}}{\bar{T}}\right)\right]} \ . \qquad \text{(A2.19)}}$$

Once the relative change between ℓ and ℓ' is calculated from the equation above, the change in other geometric properties (such as the relative change in position of the endpoints of the catenary, for example) can be obtained using the chain rule of differentiation following the expressions derived in [4].

APPENDIX 3: FIRST ORDER EXPANSION OF THE DRAG TERMS WITH RESPECT TO THE HORIZONTAL PROJECTED LENGTH OF THE SUSPENDED CATENARY

In order to perform stability analysis of the horizontal plane equations of motion in mooring and anchoring, the derivatives of the various terms involved in the drag equations with respect to the velocities (u, v, r) and the position vector (x, y, ψ) are needed. Taking derivatives with respect to the velocity terms is straight forward. Derivatives with respect to the position vector, are, however more difficult to derive.

The derivatives of the various terms involved in the drag equations with respect to the position vector can be obtained from relations derived based on the horizontal distance between the mooring and the attachment points on the catenary ℓ' as shown in [4]. The derivative with respect to the variable ℓ' can be obtained from ℓ as shown in Appendix 2, and the derivative with respect to ℓ can be obtained from θ_ℓ as is shown below.

In this Appendix, the first order expansion of the nonlinear terms in the drag component equations (91) and (95), specifically the horizontal rigidity R_H and the energy dissipation function γ_p, are derived with respect to θ_ℓ.

The expression for θ_ℓ is given from (61) as follows:

$$\theta(\ell_{eff}) = \theta_\ell = \tan^{-1}\left(\frac{P\ell_{eff}}{T}\right), \tag{A3.1}$$

where ℓ_{eff} is given by either equation (10) or (11) as:

$$\ell_{eff} = \sqrt{h\left(h + 2\frac{T}{P}\right)}, \tag{A3.2}$$

or

$$\ell_{eff} = \frac{T}{P}\sinh\left(\frac{P\ell}{T}\right). \qquad \text{(A3.3)}$$

Expression (A3.1) above can be recast in either of the following two forms:

$$\theta_\ell = \tan^{-1}\left(\sinh\left(\frac{P\ell}{T}\right)\right), \qquad \text{(A3.4)}$$

or

$$\theta_\ell = \tan^{-1}\left(\frac{P}{T}\sqrt{h\left(h+2\frac{T}{P}\right)}\right). \qquad \text{(A3.5)}$$

The derivative of θ_ℓ with respect to ℓ can be obtained by taking the appropriate partial derivatives of either expression (A3.4) or (A3.5). From (A3.4) this yields:

$$\frac{d\theta_\ell}{d\ell} = \frac{(P/T)}{\cosh(P\ell/T)}\left[1-\frac{\ell}{T}\frac{dT}{d\ell}\right], \qquad \text{(A3.6)}$$

and from (A3.5):

$$\frac{d\theta_\ell}{d\ell} = -\frac{(Ph/T)}{1+(Ph/T)}\left[\frac{1}{P\sqrt{h(h+2T/P)}}\right]\frac{dT}{d\ell}. \qquad \text{(A3.7)}$$

In expressions (A3.6) and (A3.7), $\frac{dT}{d\ell}$ is the derivative of the horizontal tension T with respect to the horizontal projected length of the suspended catenary ℓ, given in Appendix1 (equation (A1.4)). Both expressions (A3.6) and (A3.7) are equilvalent.

Once the relation between ℓ and θ_ℓ is calculated from either (A3.6) or (A3.7), the first order expansions of the rest of the terms involved in the nonlinear damping equations can be taken directly with respect to θ_ℓ. Those expansions, in turn, can be recast as functions of ℓ via the chain rule of differentiation. Expansions of such terms with respect to ℓ' or any other position vector can be obtained following [4].

As mentioned previously, the first order expansions of the horizontal rigidity (stiffness) R_H and the energy dissipation function γ_p are derived with respect to θ_ℓ.

To find the first order expansion for the horizontal rigidity R_H, expression (74) is expanded with respect to θ_ℓ. Recall from (74):

$$\frac{R_H}{P} = \frac{1}{\sinh^{-1}(\tan(\theta_\ell)) - 2\left(\dfrac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right)} \ . \tag{A3.8}$$

The derivative of R_H/P with respect to θ_ℓ is given by:

$$\frac{d(R_H/P)}{d\theta_\ell} = \frac{2\left[\dfrac{\sin^2(\theta_\ell) - \cos(\theta_\ell) + \cos^2(\theta_\ell)}{\sin^2(\theta_\ell)}\right] - \dfrac{\sec^2(\theta_\ell)}{\sqrt{1+\tan^2(\theta_\ell)}}}{\left[\sinh^{-1}(\tan(\theta_\ell)) - 2\left(\dfrac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right)\right]^2} \ , \tag{A3.9}$$

which can be arranged to obtain:

$$\frac{d(R_H/P)}{d\theta_\ell} = \frac{2\cos(\theta_\ell) - 1 - \cos^2(\theta_\ell)}{\cos(\theta_\ell)\left[\vartheta^2 \sin^2(\theta_\ell) - 4\vartheta \sin(\theta_\ell)(1-\cos(\theta_\ell)) + 4(1-\cos(\theta_\ell))^2\right]} \ . \tag{A3.10}$$

In the equation above,

$$\vartheta = \sinh^{-1}(\tan(\theta_\ell)) \ . \tag{A3.11}$$

To obtain the derivative of the energy dissipation function γ_p with respect to θ_ℓ, consider equation (93), which is repeated below:

$$\gamma_p = \left(\frac{R_H}{P}\right)^3 \frac{\cos(\theta_\ell)}{(1-\cos(\theta_\ell))} \int_0^{\theta_\ell} \left\{\frac{|f(\theta)| f^2(\theta)}{\cos^2(\theta)}\right\} d\theta \ . \tag{A3.12}$$

In the equation above, $f(\theta)$ is given, from (89) as

$$f(\theta)=\left(\frac{1-\cos(\theta_\ell)}{\sin(\theta_\ell)}\right)\sin(\theta)-\sin(\theta)\left[\sinh^{-1}(\tan(\theta))\right]+1-\cos(\theta)\ . \qquad \text{(A3.13)}$$

Equation (A3.12) can be expanded by taking derivatives with respect to θ_ℓ.

The derivative of the integral in equation (A3.12) can be derived using the generalized version of Leibnitz theorem [12], which states that:

$$\frac{dR(\alpha)}{d\alpha}=r(\alpha,b(\alpha))\frac{db(\alpha)}{d\alpha}-r(\alpha,a(\alpha))\frac{da(\alpha)}{d\alpha}+\int_{a(\alpha)}^{b(\alpha)}\frac{\partial r(\alpha,x)}{\partial\alpha}dx\ . \qquad \text{(A3.14)}$$

Letting

$$R(\alpha)=\int_{a(\alpha)}^{b(\alpha)}r(\alpha,x)dx\ , \qquad \text{(A3.15)}$$

where

$$a(\alpha)=0,\ b(\alpha)=\theta_\ell,\ r(\alpha,x)=\frac{|f(\alpha,x)|f^2(\alpha,x)}{\cos^2(x)},\ \text{and}$$

$$f(\alpha,x)=\left(\frac{1-\cos(\alpha)}{\sin(\alpha)}\right)\sin(x)-\sin(x)\left[\sinh^{-1}(\tan(x))\right]+1-\cos(x)\ ,$$

and applying (A3.15), the derivative of (A3.14) with respect to θ_ℓ becomes:

$$\frac{dR(\theta_\ell)}{d\theta_\ell}=\frac{|f(\theta_\ell)|f^2(\theta_\ell)}{\cos^2(\theta_\ell)}+\int_0^{\theta_\ell}\frac{\partial}{\partial\theta_\ell}\left(\frac{|f(\theta)|f^2(\theta)}{\cos^2(\theta)}\right)d\theta\ , \qquad \text{(A3.16)}$$

where

$$\frac{\partial}{\partial\theta_\ell}\left(\frac{|f(\theta)|f^2(\theta)}{\cos^2(\theta)}\right)=\frac{(1-\cos(\theta_\ell)\sin(\theta))}{\cos(\theta)}\left\{\frac{f^2(\theta)sgn(f(\theta))+2|f(\theta)|f(\theta)}{\sin^2(\theta_\ell)}\right\}\ , \qquad \text{(A3.17)}$$

and $sgn(f(\theta))=$ $+1$ if $f(\theta)>0$

$=$ -1 if $f(\theta)<0$

$=$ 0 if $f(\theta)=0$.

By incorporating expressions (A3.9), (A3.16) and (A3.17) into the expansion for equation (A3.12), the derivative of γ_p with respect to θ_ℓ is obtained and given by:

$$\frac{d\gamma_p}{d\theta_\ell} = \left(\frac{R_H}{P}\right)^2 \left\{ \frac{3\cos(\theta_\ell)}{1-\cos(\theta_\ell)} \frac{d(R_H/P)}{d\theta_\ell} - \left(\frac{R_H}{P}\right) \frac{\sin(\theta_\ell)}{(1-\cos(\theta_\ell))^2} \right\} \int_0^{\theta_\ell} \frac{|f(\theta)| f^2(\theta)}{\cos^2(\theta)} d\theta$$
$$+ \left(\frac{R_H}{P}\right)^3 \left(\frac{\cos(\theta_\ell)}{\sin^2(\theta_\ell)}\right) \left\{ \int_0^{\theta_\ell} \frac{\left[f^2(\theta) sgn(f(\theta)) + 2|f(\theta)| f(\theta)\right]}{\cos^2(\theta)} \sin(\theta) d\theta \right\}$$
$$+ \left(\frac{R_H}{P}\right)^3 \left(\frac{1}{1-\cos(\theta_\ell)}\right) \left\{ \frac{|f(\theta_\ell)| f^2(\theta_\ell)}{\cos(\theta_\ell)} \right\} . \qquad \text{(A3.18)}$$

APPENDIX 4: RELATION BETWEEN THE EFFECTIVE LENGTH OF THE CATENARY AND WATER DEPTH

The geometric relationship between the effective length of the catenary ℓ_{eff} and the water depth h (or vertical distance between the catenary endpoints) can be obtained by substituting expression (61) into (66) as follows:

$$h = \frac{F_x}{P}\left[\sqrt{1+\left(\frac{P\ell_{eff}}{F_x}\right)^2} - 1\right] = \frac{F_x}{P}\left[\frac{1-\cos(\theta_\ell)}{\cos(\theta_\ell)}\right]. \tag{A4.1}$$

The expression above can be recast as

$$\frac{1}{h} = \frac{P}{F_x}\left[\frac{\cos(\theta_\ell)}{1-\cos(\theta_\ell)}\right]. \tag{A4.2}$$

By multiplying both sides by ℓ_{eff} into (A4.2) and performing the appropriate trigonometric relations we find:

$$\frac{\ell_{eff}}{h} = \frac{P\ell_{eff}}{F_x}\left[\frac{\cos(\theta_\ell)}{1-\cos(\theta_\ell)}\right] = \tan(\theta_\ell)\left[\frac{\cos(\theta_\ell)}{1-\cos(\theta_\ell)}\right] = \frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)}. \tag{A4.3}$$

The relation between the suspended length of the catenary (ℓ_{eff}) and the water depth (h) can then be simplified in the form:

$$\boxed{\ell_{eff} = h\left[\frac{\sin(\theta_\ell)}{1-\cos(\theta_\ell)}\right].} \tag{A4.4}$$

BIBLIOGRAPHY

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[1] Ansari, K. A., and Khan, N. U., "The Effect of Cable Dynamics on the Station Keeping Response of a Moored Offshore Vessel," Proceedings of the ASME 5th International conference on Offshore Mechanics and Arctic Engineering (OMAE), Vol. III, Tokyo, April 1986, pp. 514-521.

[2] Beyer, W. H., CRC Standard Mathematical Tables, 28th Edition, CRC Press, Ann Arbor, 1990.

[3] Cox, J. V., "Statmoor - A Single Point Mooring Static Analysis Program," Naval Civil Engineering Laboratory, Report No. AD-A119 979, June 1982.

[4] Garza-Rios, L. O., and Bernitsas, M. M., "Stability Criteria for the Slow Motion Nonlinear Dynamics of Towing and Mooring Systems," Report to the University of Michigan/Sea Grant/Industry Consortium in Offshore Engineering, and Department of Naval Architecture and Marine Engineering, University of Michigan, Ann Arbor, Publication No. 332, November 1996.

[5] Huang, S., "Dynamic Analysis of Three-Dimensional Marine Cables," *Ocean Engineering*, Vol. 21, No. 6, Elsevier Science LTD, 1994, pp. 587-605.

[6] Kwan, C. T., and Bruen, F. J., "Mooring Line Dynamics: Comparison of Time Domain, Frequency Domain, and Quasi-Static Analyses," Proceedings of the 23rd Offshore Technology Conference (OTC), Paper No. OTC-6657, Houston, May 1991, pp. 95-108.

[7] Mc Kenna, H. A., and Wong, R. K., "Synthetic Fiber Rope, Properties and Calculations Relating to Mooring Systems," *Deepwater Mooring and Drilling*, ASME Transactions, Ocean Engineering Division, Vol. 7, Dec. 1979, pp. 189-203.

[8] Nakajima, T., Motora, S., and Fujino, M., "On the Dynamic Analysis of Multi-Component Mooring Lines," Proceedings of the 14th Offshore Technology Conference (OTC), Paper No. OTC-4309, Houston, May 1982, pp. 105-120.

[9] Nishimoto, K., Kaster, F., and Aranha, J. A. P., "Full Scale Decay Test of a Moored Tanker, ALAGOAS-DICAS System," Report to Petrobrás, Brazil, February 1996 (in Portuguese).

[10] Papoulias, F. A., and Bernitsas, M. M., "MOORLINE: A Program for Static Analysis of MOORing LINEs," Report to the University of Michigan/Sea Grant/Industry Consortium in Offshore Engineering, and Department of Naval Architecture and Marine Engineering, The University of Michigan, Ann Arbor, Publication No. 309, May 1988.

[11] van den Boom, H. J. J., "Dynamic Behaviour of Mooring Lines," Proceedings of the 4th International Conference on the Behaviour of Offshore Structures (BOSS), Amsterdam, 1985, pp. 359-368.

[12] Wylie, C. R. Jr., Advanced Engineering Mathematics, Third Edition, Mc Graw-Hill, 1961.

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