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PROBABILISTIC TORSIONAL VIBRATION ANALYSIS OF A MARINE DIESEL ENGINE SHAFTING SYSTEM: AN ALTERNATIVE APPROACH TO THE LEVEL CROSSING PROBLEM

by

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ABSTRACT

The level crossing problem is treated and the first excursion probability is presented for the case of a cyclostationary random process. A new method is then introduced, where the cyclostationary process is replaced with an equivalent stationary random process. The latter is defined to either (1) render *exactly* the same upcrossing rate at a given threshold as the time averaged upcrossing rate of the cyclostationary process, or (2) have an *envelope process* with the same upcrossing rate as the envelope of the cyclostationary process. Both these processes are assumed to have the same probability of upcrossing a specific threshold. An analytical expression is derived for this upcrossing probability for a certain form of a cyclostationary process and compared with existing results. The agreement between this approach and the "exact" but time-consuming "Markov approach" presented in earlier work of the authors is found to be excellent.

1. INTRODUCTION AND BACKGROUND

In previous work [Nikolaidis, Perakis and Parsons,1985,1987], a probabilistic torsional vibration analysis of a marine diesel shafting system has been introduced. An important feature of that problem is that the excitation statistics are varying with time. For example, when viewed on a time scale shorter than the combustion cycle time, the mean and standard deviation of the gas torque are higher at the time of combustion of the air-fuel mixture than at any other time within the combustion cycle. Therefore, the excitation is a cyclostationary ([Franks, 1969], [Ogura, 1971], [Gardner and Franks, 1975], [Papoulis, 1983]) random process. A statistical model for the cylinder excitation that takes into account the cylinder-to-cylinder and cycle-to-cycle variability has been developed [Nikolaidis, Perakis, and Parsons, 1985]. The cylinder gas torque is modeled as an amplitude modulated process with amplitudes which are jointly Gaussian stationary processes [Nikolaidis, Perakis, and Parsons, 1989].

The vibratory shear stress at a resonance, R(t), has been modeled as a Gaussian random process of the form [Nikolaidis, Perakis and Parsons, 1985, 1987]:

$$R(t)=X(t)\cos\omega t+Y(t)\sin\omega t, \qquad (1)$$

where X(t) and Y(t) are Gaussian stationary processes with known statistics and $\omega=\rho\omega_0$ is the resonant natural frequency of the shafting system, where ρ is the resonant order of the excitation Fourier representation and ω_0 is the singular frequency of rotation of the shaft. Finally, the frequency content of X(t) and Y(t) is low compared to ω. Process R(t) is stationary [Nikolaidis, Perakis and Parsons, 1989]. We will also assume here that the means of processes X(t) and Y(t) are zero. For nonzero mean processes the methods presented here can easily be adjusted. We can, in addition, assume that *random variables* X(t) and Y(t) are independent at time t. If they are not, we can easily transform them to the principal axii, where they will be independent. If, in addition to the above assumptions, it is also true that:

$$R_{XX}(t)=R_{YY}(t) \text{ and } R_{XY}(-t)=-R_{XY}(t),$$

then it can be shown that R(t) is also a stationary random process [Nikolaidis, Perakis and Parsons, 1989].

The level crossing problem for a general random process remains unsolved. Results for certain special cases of stationary processes have been found ([Blake and Lindsey, 1973], [Nikolaidis, Perakis and Parsons, 1989]). In some cases, independence of the local maxima of the random process is assumed ([Rice, 1944, 1945], [Cramer and Leadbetter, 1967], [Ochi, 1973]). In one other case, the mean frequency of upcrossings is calculated and, a theorem by Cramer [Cramer and Leadbetter, 1967] which states that the number of upcrossings of a random process is a random variable that is asymptotically Poisson distributed as the threshold increases is used [Naess, 1982]. In some other cases, the local maxima of the process are assumed to form a Markov Chain ([Epstein, 1949], [Robert, 1968], [Ochi, 1979], [Naess, 1983]). The nonstationary case has also been examined ([Lin, 1970], [Vanmarche, 1969], [Yang, 1972], [Yang, 1973], [Krenk, 1979]).

The two approaches to the level crossing problem for a cyclostationary process of the form of equation (1) have been developed and compared in [Nikolaidis, Perakis and Parsons, 1989]. The first approach approximates the maxima of the process with the values of its envelope process at the time these maxima occur and assumes that these constitute a discrete-time Markov process. The second approach, on the other hand, is based on the assumption that the maxima of the process are approximately equal to the corresponding maxima of the envelope process and that the envelope crossings form a

Poisson process. By the latter, it is implied that the maxima of the envelope process are statistically independent.

In this report the same level crossing problem, for a cyclostationary random process of the form of (1), has been considered, but it is treated in an alternative way. This is motivated by the computationally intensive nature of the earlier cyclostationary process results. Since it is relatively easy to find the probability that a stationary Gaussian random process crosses a specific level within a prespecified time interval, we will define a stationary process as "equivalent" to the original cyclostationary one using two different approaches.

In the first, hereafter called "direct approach", the expected upcrossing rate (for a specific threshold) of the original cyclostationary process is calculated and then averaged over the specified time interval. The "equivalent" stationary process is then defined as having an expected upcrossing rate equal to this value. We finally assume that both the original and the "equivalent" processes have the same upcrossing probability. In the second, hereafter called the "envelope approach", the "equivalent" stationary process is defined as having an upcrossing rate (for a specific threshold) equal to the expected rate (for the same threshold) of the envelope of the cyclostationary process. The upcrossing probabilities of these two processes are, again, considered equal. In both cases, this probability is estimated for a process of the form (1), that satisfies all our assumptions (and is therefore stationary) using the Markov approach, as developed in [Nikolaidis, Perakis and Parsons, 1989].

2. DIRECT APPROACH

2.1. Expected upcrossing rate for a given threshold

The expected upcrossing rate for a specified threshold ρ_o is [Rice, 1944, 1945]:

$$v_{\rho_o}^{+} = \int_0^{\infty} x f_{R\dot{R}}(\rho_o, x) dx, \tag{2}$$

where $f_{R\dot{R}}(r,\dot{r})$ is the joint pdf of the process and its derivative.

To evaluate $v_{\rho_o}^{+}$, we first derive an expression for $f_{R\dot{R}}(r,\dot{r})$. $\{R(t),t\geq 0\}$ and $\{\dot{R}(t),t\geq 0\}$ are zero-mean, jointly Gaussian random processes, since:

$$\dot{E}[R(t)] = E[X(t)]\cos\omega t + E[Y(t)]\sin\omega t,$$

$$E[\dot{R}(t)] = E[\dot{X}(t)]\cos\omega t + E[\dot{Y}(t)]\sin\omega t - \omega E[X(t)]\sin\omega t + \omega E[Y(t)]\cos\omega t,$$

and both $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ have been assumed to be zero-mean, stationary, Gaussian random processes.

The autocorrelation function of R(t) is:

$$R_{RR}(t_1,t_2) = E[R(t_1)R(t_2)]$$

$$= R_{XX}(t_1 - t_2)\cos\omega t_1\cos\omega t_2 + R_{YX}(t_1 - t_2)\sin\omega t_1\cos\omega t_2 + \qquad (3)$$

$$+ R_{XY}(t_1 - t_2)\cos\omega t_1\sin\omega t_2 + R_{YY}(t_1 - t_2)\sin\omega t_1\sin\omega t_2$$

By differentiating (3) with respect to t_2, we get:

$$\frac{\partial R_{RR}(t_1,t_2)}{\partial t_2} = R_{R\dot{R}}(t_1,t_2) = R_{X\dot{X}}(t_1 - t_2)\cos\omega t_1\cos\omega t_2 -$$

$$- \omega R_{XX}(t_1 - t_2)\cos\omega t_1\sin\omega t_2 +$$

$$+ R_{Y\dot{X}}(t_1 - t_2)\sin\omega t_1\cos\omega t_2 - \omega R_{YX}(t_1 - t_2)\sin\omega t_1\sin\omega t_2 + \qquad (4)$$

$$+ R_{X\dot{Y}}(t_1 - t_2)\cos\omega t_1\sin\omega t_2 + \omega R_{XY}(t_1 - t_2)\cos\omega t_1\cos\omega t_2 +$$

$$+ R_{Y\dot{Y}}(t_1 - t_2)\sin\omega t_1\sin\omega t_2 + \omega R_{YY}(t_1 - t_2)\sin\omega t_1\cos\omega t_2$$

By differentiating (4) with respect to t_1, we get:

$$\frac{\partial^2 R_{RR}(t_1,t_2)}{\partial t_1 \partial t_2} = R_{\dot{R}\dot{R}}(t_1,t_2) =$$

$$= R_{\dot{X}\dot{X}}(t_1 - t_2)\cos\omega t_1 \cos\omega t_2 - \omega R_{X\dot{X}}(t_1 - t_2)\sin\omega t_1 \cos\omega t_2 -$$

$$- \omega R_{\dot{X}X}(t_1 - t_2)\cos\omega t_1 \cos\omega t_2 + \omega^2 R_{XX}(t_1 - t_2)\sin\omega t_1 \sin\omega t_2 +$$

$$+ R_{\dot{Y}\dot{X}}(t_1 - t_2)\sin\omega t_1 \cos\omega t_2 + \omega R_{Y\dot{X}}(t_1 - t_2)\sin\omega t_1 \sin\omega t_2 -$$

$$- \omega R_{\dot{Y}X}(t_1 - t_2)\sin\omega t_1 \sin\omega t_2 - \omega^2 R_{YX}(t_1 - t_2)\cos\omega t_1 \sin\omega t_2 +$$

$$+ R_{\dot{X}\dot{Y}}(t_1 - t_2)\cos\omega t_1 \sin\omega t_2 - \omega R_{X\dot{Y}}(t_1 - t_2)\sin\omega t_1 \sin\omega t_2 + \qquad (5)$$

$$+ \omega R_{\dot{X}\dot{Y}}(t_1 - t_2)\cos\omega t_1 \cos\omega t_2 - \omega^2 R_{\dot{X}\dot{Y}}(t_1 - t_2)\sin\omega t_1 \cos\omega t_2 +$$

$$+ R_{\dot{Y}Y}(t_1 - t_2)\sin\omega t_1 \sin\omega t_2 + \omega R_{Y\dot{Y}}(t_1 - t_2)\cos\omega t_1 \sin\omega t_2 +$$

$$+ \omega R_{\dot{Y}Y}(t_1 - t_2)\sin\omega t_1 \cos\omega t_2 + \omega^2 R_{YY}(t_1 - t_2)\cos\omega t_1 \cos\omega t_2$$

Hence:

$$\sigma_R^2 = E[R(t)R(t)] = R_{RR}(t,t) = \sigma_X^2 \cos^2\omega t + \sigma_Y^2 \sin^2\omega t,$$

$$\sigma_{\dot{R}}^2 = E[\dot{R}(t)\dot{R}(t)] = R_{\dot{R}\dot{R}}(t,t) =$$

$$= \sigma_{\dot{X}}^2 \cos^2\omega t + \sigma_{\dot{Y}}^2 \sin^2\omega t + \omega^2 \sigma_X^2 \sin^2\omega t + \omega^2 \sigma_Y^2 \cos^2\omega t$$

and:

$$\sigma_{R\dot{R}} = E[R(t)\dot{R}(t)] = -\omega\sigma_X^2 \sin\omega t \cos\omega t + \omega\sigma_Y^2 \sin\omega t \cos\omega t =$$

$$= \left[\left(\sigma_Y^2 - \sigma_X^2\right)/2\right]\omega \sin 2\omega t,$$

since $\sigma_{X\dot{X}} = \sigma_{Y\dot{Y}} = 0$ and we assume that, at time t, *random variables* X(t) and Y(t) are independent and, therefore,

$$\sigma_{XY} = \sigma_{\dot{X}Y} = \sigma_{X\dot{Y}} = \sigma_{\dot{X}\dot{Y}} = 0.$$

We, now, define the correlation coefficient as:

$$\rho \equiv \rho_{R\dot{R}} = \sigma_{R\dot{R}} / \sigma_R \sigma_{\dot{R}} \quad , \quad -1 \le \rho \le 1 \tag{6}$$

Finally:

$$f_{R\dot{R}}(r,\dot{r}) = \frac{1}{2\pi\sigma_R\sigma_{\dot{R}}\sqrt{1-\rho^2}} \exp\left\{ -\frac{1}{1-\rho^2}\left(\frac{r^2}{2\sigma_R^2} - \frac{\rho r\dot{r}}{\sigma_R\sigma_{\dot{R}}} + \frac{\dot{r}^2}{2\sigma_{\dot{R}}^2} \right) \right\}$$

We can now evaluate:

$$\nu^+_{\rho_o} = \int_0^\infty x f_{R\dot{R}}(\rho_o, x)\, dx$$

If we define:

$$a_1 = \frac{1}{1-\rho^2}\,\frac{\rho_o^2}{2\sigma_R^2} \quad , \quad a_2 = \frac{1}{1-\rho^2}\,\frac{\rho\rho_o}{\sigma_R\sigma_{\dot{R}}} \quad , \tag{7}$$

$$a_3 = \frac{1}{1-\rho^2}\,\frac{1}{2\sigma_{\dot{R}}^2} \quad , \quad c = \frac{1}{2\pi\sigma_R\sigma_{\dot{R}}\sqrt{1-\rho^2}}$$

Then:

$$f_{R\dot{R}}(\rho_o, \dot{r}) = c \exp\left[-\left(a_3\dot{r}^2 - a_2\dot{r} + a_1 \right) \right]$$

The exponent becomes:

$$-a_3\left(\dot{r}^2 - \frac{a_2}{a_3}\dot{r}\right) - a_1 = -a_3\left[\dot{r}^2 - \frac{a_2}{a_3}\dot{r} + \left(\frac{a_2}{a_3}\right)^2\right] + a_3\left(\frac{a_2}{2a_3}\right)^2 - a_1 =$$

$$= -\frac{\left(\dot{r} - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)} + a_3\left({a_2}/{2a_3}\right)^2 - a_1$$

Therefore:

$$v^+_{\rho_o} = \int_o^\infty x\, f_{R\dot{R}}(\rho_o, x)\,dx = c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\int_o^\infty x\exp\left[\frac{\left(x - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)}\right]dx =$$

$$= c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\int_o^\infty \exp\left[-\frac{\left(x - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)}\right]\left(x - {a_2}/{2a_3}\right)dx +$$

$$+ c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\frac{a_2}{2a_3}\int_o^\infty \exp\left[-\frac{\left(x - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)}\right]dx =$$

$$= -c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\frac{1}{2a_3}\exp\left[-\frac{\left(x - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)}\right]\Bigg|_o^\infty +$$

(8)

$$+ c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\frac{a_2}{(2a_3)^{3/2}}\sqrt{2\pi}\,\frac{1}{\sqrt{2\pi}\,\frac{1}{\sqrt{2a_3}}}\int_o^\infty \exp\left[-\frac{\left(x - {a_2}/{2a_3}\right)^2}{2\left({1}/{2a_3}\right)}\right]dx =$$

$$= c\exp\left[a_3\left({a_2}/{2a_3}\right)^2 - a_1\right]\frac{1}{2a_3}\exp\left[-a_3\left({a_2}/{2a_3}\right)^2\right] +$$

$$+ c\exp\left[a_3\left(\frac{a_2}{2a_3}\right)^2 - a_1\right]\sqrt{2\pi}\,\frac{a_2}{(2a_3)^{3/2}}\,\frac{1}{2} =$$

$$= \frac{c}{2a_3}\exp(-a_1) + \frac{\sqrt{2\pi}\,a_2 c}{2(2a_3)^{3/2}}\exp\left[a_3\left(\frac{a_2}{2a_3}\right)^2 - a_1\right]$$

where a_1, a_2, a_3, and c are given by (7).

2.2. Expected upcrossing rate for the "equivalent" stationary process

In order for our random process $\{R(t), t \geq 0\}$, as defined by (1), to be stationary, the assumptions of section 1 must be satisfied. In this limiting case:

$$\mu_R = 0$$

$$R_{RR}(0) = \sigma_R^2 = \sigma_X^2\cos^2\omega t + \sigma_Y^2\sin^2\omega t = \sigma_X^2$$

Moreover, [Crandall and Mark, 1973]:

$$\nu_{\rho_o}^{+} = \frac{1}{2\pi}\sqrt{-\frac{R''_{RR}(0)}{R_{RR}}}\;\exp\left(-\frac{\rho_o^2}{2R_{RR}(0)}\right) \qquad (9)$$

where $R_{RR}(\tau)$ is the autocorrelation function of (stationary) process $\{R(t), t \geq 0\}$ given by:

$$R_{RR}(\tau) = E[R(t+\tau)R(t)] =$$

$$= E[[X(t+\tau)\cos\omega(t+\tau) + Y(t+\tau)\sin\omega(t+\tau)][X(t)\cos\omega t + Y(t)\sin\omega t]] =$$

$$= R_{XX}(\tau)\cos\omega(t+\tau)\cos\omega t + R_{YX}(\tau)\sin\omega(t+\tau)\cos\omega t + \qquad (10)$$

$$+ R_{XY}(\tau)\cos\omega(t+\tau)\sin\omega t + R_{YY}\sin\omega(t+\tau)\sin\omega t$$

Since X(t) and Y(t) have been assumed independent at $\tau=0$, hence $R_{XX}(0)=R_{YY}(0)$ and $R_{XY}(0)=R_{YX}(0)=0$, (10) then yields:

$$R_{RR}(0) = R_{XX}(0)\cos^2\omega t + R_{YY}(0)\sin^2\omega t = R_{XX}(0)$$

By differentiating (10) twice, we get:

$$\begin{aligned} R''_{RR}(\tau) = & R''_{XX}(\tau)\cos\omega(t+\tau)\cos\omega t - 2\omega R'_{XX}(\tau)\sin\omega(t+\tau)\cos\omega t - \\ & - \omega^2 R_{XX}(\tau)\cos\omega(t+\tau)\cos\omega t + \\ & + R''_{YX}(\tau)\sin\omega(t+\tau)\cos\omega t + 2\omega R'_{YX}(\tau)\cos\omega(t+\tau)\cos\omega t - \\ & - \omega^2 R_{YX}(\tau)\sin\omega(t+\tau)\cos\omega t + \\ & + R''_{XY}(\tau)\cos\omega(t+\tau)\sin\omega t - 2\omega R'_{XY}(\tau)\sin\omega(t+\tau)\sin\omega t - \\ & - \omega^2 R_{XY}(\tau)\cos\omega(t+\tau)\sin\omega t + \\ & + R''_{YY}(\tau)\sin\omega(t+\tau)\sin\omega t + 2\omega R'_{YY}(\tau)\cos\omega(t+\tau)\sin\omega t - \\ & - \omega^2 R_{YY}(\tau)\sin\omega(t+\tau)\sin\omega t \end{aligned} \tag{11}$$

When $\tau=0$,

$$R_{XY}(0) = R_{YX}(0),$$

$$R_{X\dot{X}}(0) = -R'_{XX}(0) = R_{Y\dot{Y}}(0) = -R'_{YY}(0) = 0,$$

$$R'_{XY}(0) = - R'_{XY}(0) = - E[X(t)\,\dot{Y}(t)] = - E[X(t)]\,E[\dot{Y}(t)] = 0 = R'_{YX}(0)$$

and:

$$R_{YX}(\tau) = R_{XY}(-\tau), \text{ hence } R''_{YX}(\tau) = R''_{XY}(-\tau) \text{ and } R''_{YX}(0) = R''_{XY}(0),$$

But, for R(t) to be stationary, conditions of section 1, also, imply:

$$R_{XY}(\tau) = - R_{XY}(-\tau), \text{ or } R''_{XY}(\tau) = - R''_{XY}(-\tau) \text{ or}$$

$$R''_{XY}(0) = - R''_{XY}(0), \text{ and}$$

$$R''_{XY}(0) = R''_{YX}(0) = 0.$$

Hence, at τ=0, (11) yields:

$$R''_{RR}(0) = R''_{XX}(0) - \omega^2 R_{XX}(0)$$

Then by substituting this into (9), we obtain:

$$\nu_{\rho_{os}}^{+} = \frac{1}{2\pi}\sqrt{\omega^2 - \frac{R''_{XX}(0)}{\sigma_X^2}}\ \exp\left[-\rho_o^2 \big/ 2\sigma_x^2\right] \tag{12}$$

2.3. Time averaged, expected upcrossing rate for a cyclostationary process

We now integrate (8) over the given time interval T, to get the average (over time) expected upcrossing rate for the prespecified threshold ρ_o:

$$\bar{\nu}_{\rho_{os}}^{+} = \frac{1}{T}\int_0^T \nu_{\rho_{oc}}^{+}(t)dt \tag{13}$$

This integration has been performed numerically.

3. "ENVELOPE" APPROACH

3.1. Expected upcrossing rate for the envelope process

As mentioned in [Nikolaidis, Perakis and Parsons, 1989], the expected upcrossing rate of a threshold is given by:

$$\nu_{\rho_o}^{+} = \int_0^\infty x f_{\rho\dot{\rho}}(\rho_o, x)dx \tag{15}$$

where $f_{\rho\dot{\rho}}(\cdot\ ,\cdot)$ is the joint pdf of the envelope process and its derivative.

In [Nikolaidis, Perakis and Parsons, 1989], an expression is derived for $\nu^{+}_{\rho_o}$, in the general case, where $\rho(t)$ is the envelope of a nonstationary process, even though R(t) is not. In this case $\nu^{+}_{\rho_o}$ is given by [Nikolaidis, Perakis and Parsons, 1989]:

$$\nu^{+}_{\rho_o} = \int_0^{2\pi}\Bigg\{\frac{\rho_o^2}{(2\pi)^{3/2}\,|\Lambda'|^{1/2}B'^{1/2}L'}\exp\left[-\frac{1}{2}F' + \frac{E'^2}{8B'}\right] - \tag{16}$$

$$- \frac{\rho_o^2 M'}{4\pi\,|\Lambda'|^{1/2}B'^{1/2}L'^{3/2}}\exp\left[\frac{1}{2}F' + \frac{E'^2}{8B'} + \frac{M'^2}{8L'}\right]\cdot\left[1 - \Phi\left(\frac{M'}{2L'^{1/2}}\right)\right]\Bigg\}\,d\vartheta,$$

where Λ' is the following covariance matrix of the *random vector process* $\left\{\left\{X(t), Y(t), \dot{X}(t), \dot{Y}(t)\right\}^T, t \geq 0\right\}$:

$$\Lambda' = \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & 0 & \sigma_{X\dot{Y}} \\ \sigma_{XY} & \sigma_Y^2 & \sigma_{\dot{X}Y} & 0 \\ 0 & \sigma_{\dot{X}Y} & \sigma_{\dot{X}}^2 & \sigma_{\dot{X}\dot{Y}} \\ \sigma_{Y\dot{X}} & 0 & \sigma_{\dot{X}\dot{Y}} & \sigma_{\dot{Y}}^2 \end{bmatrix}, \tag{17}$$

with:

$$B' = \rho_o^2(\lambda'_{33}\sin^2\vartheta + \lambda'_{44}\cos^2\vartheta - \lambda'_{34}\sin 2\vartheta),$$

$$D' = -\lambda'_{33}\rho_o\sin 2\vartheta + \lambda'_{44}\rho_o\sin 2\vartheta + 2\lambda'_{34}\rho_o\cos 2\vartheta,$$

$$E' = -\lambda'_{13}\rho_o^2\sin 2\vartheta + 2\lambda'_{13}\mu_X\rho_o\sin\vartheta + 2\lambda'_{14}\rho_o^2\cos^2\vartheta - 2\lambda'_{14}\mu_X\rho_o\cos\vartheta -$$

$$- 2\lambda'_{23}\rho_o^2\sin^2\vartheta + 2\lambda'_{23}\rho_o\mu_Y\sin\vartheta + \lambda'_{24}\rho_o^2\sin 2\vartheta - 2\lambda'_{24}\mu_Y\rho_o\cos\vartheta,$$

$$F' = \lambda'_{11}(\rho_o\cos\vartheta - \mu_X)^2 + \lambda'_{22}(\rho_o\sin\vartheta - \mu_Y)^2 + 2\lambda'_{12}(\rho_o\cos\vartheta - \mu_X)(\rho_o\sin\vartheta - \mu_Y),$$

$$G' = 2\lambda'_{13}\rho_o\cos^2\vartheta - 2\lambda'_{13}\mu_X\cos\vartheta + \lambda'_{14}\rho_o\sin2\vartheta - 2\lambda'_{14}\mu_X\sin\vartheta + \lambda'_{23}\rho_o\sin2\vartheta -$$

$$- 2\lambda'_{23}\mu_Y\cos\vartheta + 2\lambda'_{24}\rho_o\sin^2\vartheta - 2\lambda'_{24}\mu_Y\sin\vartheta, \quad (18)$$

$$H' = \lambda'_{33}\cos^2\vartheta + \lambda'_{44}\sin^2\vartheta + \lambda'_{34}\sin2\vartheta,$$

$$L' = H' - D'^2/4B' \quad ,$$

$$M' = G' - D'E'/2B' \quad ,$$

λ_{ij}' being the elements of the inverse of the covariance matrix Λ' and Φ the probability distribution function of the "Unit Normal" random variable.

3.2. Expected upcrossing rate for the envelope of a stationary random process

In this limiting case, conditions for stationarity hold, in addition to the assumptions made in section 1.

Since we assume that X(t) and Y(t) are independent at time t, then:

$$\sigma_{X\dot{Y}} = R_{X\dot{Y}}(0) = E[X(t)\dot{Y}(t)] = E[X(t)]\cdot E[\dot{Y}(t)] = 0$$

Similarly:

$$\sigma_{XY} = \sigma_{\dot{X}Y} = \sigma_{\dot{X}\dot{Y}} = 0$$

And:

$$\sigma_{\dot{X}}^2 = R_{\dot{X}\dot{X}}(0) = -R''_{XX}(0) = -R''_{YY}(0) = R_{\dot{Y}\dot{Y}} = \sigma_{\dot{Y}}^2$$

Consequently, the covariance matrix becomes:

$$\Lambda' = \begin{bmatrix} \sigma_X^2 & 0 & 0 & 0 \\ 0 & \sigma_X^2 & 0 & 0 \\ 0 & 0 & \sigma_{\dot{X}}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\dot{X}}^2 \end{bmatrix}$$

The determinant of the above is:

$$|\Lambda'| = (\sigma_X \sigma_{\dot{X}})^4$$

And the inverse of the covariance matrix:

$$[\Lambda']^{-1} = \begin{bmatrix} 1/\sigma_X^2 & 0 & 0 & 0 \\ 0 & 1/\sigma_X^2 & 0 & 0 \\ 0 & 0 & 1/\sigma_{\dot{X}}^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_{\dot{X}}^2 \end{bmatrix} \qquad (19)$$

Therefore, by substituting the elements of (19) into the (18), cofficients, B' through M', become:

$$B' = \rho_o^2 / \sigma_{\dot{X}}^2$$

$$D' = E' = 0$$

$$F' = \rho_o^2 / \sigma_X^2$$

$$H' = 1 / \sigma_{\dot{X}}^2$$

$$L' = H' = {}^{1}\!/_{\sigma_{\dot X}^2}$$

$$M' = 0$$

Subsequently, (16) becomes:

$$\nu_{\rho_{os}}^{+} = \int_0^{2\pi} \frac{\rho_o^2}{(2\pi)^{3/2}\sigma_X^2\sigma_{\dot X}^2\left(\rho_o/_{\sigma_{\dot X}}\right)\left(1/_{\sigma_{\dot X}^2}\right)} \exp\left[-\frac{1}{2}\frac{\rho_o^2}{\sigma_X^2}\right] d\vartheta =$$

$$= \frac{\rho_o \sigma_{\dot X}}{\sqrt{2\pi}\,\sigma_X^2} \exp\left[-\frac{\rho_o^2}{2\sigma_X^2}\right] = \qquad (20)$$

$$= \frac{\rho_o\sqrt{-R''_{XX}(0)}}{\sqrt{2\pi}\,R_{XX}(0)} \exp\left[-\frac{1}{2}\frac{\rho_o^2}{R_{XX}(0)}\right]$$

3.3. <u>Expected upcrossing rate for the envelope of a cyclostationary random process</u>

Since we assumed that, at time t, X(t) and Y(t) are independent, normal random processes, the following relations hold:

$$\sigma_{XY} = \sigma_{YX} = 0,$$

$$\sigma_{X\dot Y} = E[X(t)\,\dot Y(t)] - E[X(t)]\,E[\dot Y(t)] = E[X(t)]\,E[\dot Y(t)] - E[X(t)]\,E[\dot Y(t)] = 0 \qquad (21)$$

$$\sigma_{\dot X Y} = \sigma_{\dot X \dot Y} = 0$$

Therefore, the covariance matrix (17) becomes:

$$\Lambda' = \begin{bmatrix} \sigma_X^2 & 0 & 0 & 0 \\ 0 & \sigma_Y^2 & 0 & 0 \\ 0 & 0 & \sigma_{\dot X}^2 & 0 \\ 0 & 0 & 0 & \sigma_{\dot Y} \end{bmatrix} \qquad (22)$$

The determinant of this diagonal matrix is,

$$|\Lambda'| = \sigma_X^2 \sigma_{\dot{X}}^2 \sigma_Y^2 \sigma_{\dot{Y}}^2 \tag{23}$$

while the inverse matrix is:

$$[\Lambda]^{-1} = \begin{bmatrix} 1/\sigma_X^2 & 0 & 0 & 0 \\ 0 & 1/\sigma_Y^2 & 0 & 0 \\ 0 & 0 & 1/\sigma_{\dot{X}}^2 & 0 \\ 0 & 0 & 0 & 1/\sigma_{\dot{Y}}^2 \end{bmatrix} \tag{24}$$

By substituting (24) into (18), we get:

$$B' = \rho_o^2\left(\frac{\sin^2\vartheta}{\sigma_{\dot{X}}^2} + \frac{\cos^2\vartheta}{\sigma_{\dot{Y}}^2}\right),$$

$$D' = -\rho_o\frac{\sin 2\vartheta}{\sigma_{\dot{X}}^2} + \rho_o\frac{\sin 2\vartheta}{\sigma_{\dot{Y}}^2}, \tag{25}$$

$$E' = 0 \ , \ F' = \frac{\rho_o^2\cos^2\vartheta}{\sigma_X^2} + \frac{\rho_o^2\sin^2\vartheta}{\sigma_Y^2} \ ,$$

$$G' = 0 \ , \ H' = \frac{\cos^2\vartheta}{\sigma_{\dot{X}}^2} + \frac{\sin^2\vartheta}{\sigma_{\dot{Y}}^2} \ ,$$

$$L' = H' - D'^2/4B' \ , \ M' = 0$$

Hence, (16) is simplified as follows:

$$\nu_{\rho_{oc}}^{+} = \int_0^{2\pi} \frac{\rho_o^2}{(2\pi)^{3/2}\sigma_X\sigma_Y\sigma_{\dot{X}}\sigma_{\dot{Y}}\sqrt{B'L'}} \exp\left[-F'/2\right] d\vartheta \tag{26}$$

where B′, L′, F′, are functions of ϑ, given in [Nikolaidis, Perakis and Parsons, 1989].

4. CALCULATION OF THE UPCROSSING PROBABILITY

Our problem is to derive an expression for the probability that random process $\{R(t), t \geq 0\}$ exceeds a specified threshold, ρ_o, at least once during a time interval [0,T]. The assumptions set forth in section 1 still hold for R(t). Furthermore, we assume that R(t) is twice continuously differentiable.

By setting the upcrossing rates, in either (12) and (13), or (20) and (26) (pending on the approach we follow), equal to one another, we get:

$$\nu_{\rho_{os}}^{+} = \overline{\nu}_{\rho_{oc}}^{+} \text{ or } \nu_{\rho_{oc}}^{+} \tag{27}$$

(Relation (27) contains on the left side the autocorrelation of the "equivalent" stationary process and on the right side the autocorrelation functions of $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$ that form the original process, which are known.)

We will try to solve (27) for $R_{XX_s}(0)$ and $R''_{XX_s}(0)$, the subscript "s" denoting that $\{X_s(t), t \geq 0\}$ is such that the random process $\{R_s(t), t \geq 0\}$ is stationary.

Hence, we will estimate the upcrossing probability of a stationary random process $\{R_s(t), t \geq 0\}$. This is distributed as $N(0, \sigma_X)$, at time t, since it is a combination of two zero-mean normal random variables.

The autocorrelation function $R_{XX_s}(t)$ can be evaluated using (27). (We may observe that (27) has two unknowns, $R_{XX_s}(0)$ and $R''_{XX_s}(0)$, and we only have one equation. However, we may assume a certain form for the autocorrelation function, an approach that is developed later in this report in section 5. This will provide the additional required equation.

The envelope $\rho(t)$ of a stationary random process [Nikolaidis, Perakis and Parsons, 1989], is defined as the square root of the sum of the squares of the process and its Hilbert transform:

$$\hat{R}_s(t) = X_s(t)\sin\omega t - Y_s(t)\cos\omega t,$$

$R_s(t)$ is $\pi/2$ out of phase with $R_s(t)$. Then:

$$\rho(t) = \sqrt{X_s^2(t) + Y_s^2(t)} \qquad (28)$$

The local maxima of $R_s(t)$ can be approximated by the values of the envelope process $\rho(t)$ at the time they occur [Nikolaidis, Perakis and Parsons, 1989]. The first and second order statistical properties of the envelope random process have already been derived ([Epstein, 1951] [Nikolaidis and Perakis, 1985]).

The Markov approach we will use here to estimate the upcrossing probability is based on this envelope approximation. The procedure is the same as described in detail in [Nikolaidis, Perakis and Parsons, 1989]. It results in:

$$P[\text{Upcrossing in } [\,0,T\,]\,] = 1 - c\, b^{\nu-1}\ , \qquad (29)$$

where:

$$c = \int_0^{\rho_o} f_{\rho(t)}(\rho)d\rho \qquad (30)$$

$$b = \frac{1}{c}\int_0^{\rho_o}\int_0^{\rho_o} f_{\rho(t+\tau)\rho(t)}(\rho_1, \rho_2)d\rho_1 d\rho_2, \qquad (31)$$

$$\nu = T/\tau + 1, \qquad (32)$$

$$\tau = 2\pi/\omega \qquad (33)$$

The above is derived after considering a discrete-time, two state Markov process taking values at time steps 0, τ, 2τ,... A step is defined to occur when the envelope process does not exceed the threshold ρ_o, while the other when it does. The Markov property is then assumed. (Since R(t) is narrowband, the time interval between two successive local maxima is approximately constant and denoted by τ.) The rest of the symbols were defined previously.

(In the same paper it is also argued that, if the local maxima of R(t) are assumed independent, then (29) becomes:

$$P[\text{Upcrossing in } [\,0,T\,]\,] = 1 - c^{v} \,, \tag{34}$$

which leads to an overestimation of the upcrossing probability.)

5. EXAMPLE

As an example, we will assume that normal processes {X(t), $t \geq 0$} and {Y(t), $t \geq 0$} have, in addition to zero means, covariance (and , consequently, autocorrelation) and cross-covariance (cross-correlation) functions given by [Nikolaidis, Perakis and Parsons, 1989]:

$$\begin{aligned} R_{XX}(\tau) &= R_{XX}(0)\exp\left[-\kappa\tau^{2}\right] \\ R_{YY}(\tau) &= R_{YY}(0)\exp\left[-\kappa\tau^{2}\right] \\ C_{XY}(\tau) &= C_{XY}(0)\exp\left[-\kappa\tau^{2}\right] \end{aligned} \tag{35}$$

A reasonable value for κ is $\kappa = 0.1$ [Nikolaidis, Perakis and Parsons, 1989]. Since at $\tau = 0$, X(t) and Y(t) have been assumed independent, we conclude from (35), that:

$$R_{XY}(\tau) = 0, \text{ for all } \tau \tag{36}$$

and, therefore, X(t) and Y(t+τ) are orthogonal and, as zero-mean Gaussian random processes, independent for every $0 \leq t < \infty$.

In addition, we assume that the autocorrelation function of $X_s(t)$ is also given by a similar expression:

$$R_{XX_s}(\tau) = R_{XX_s}(0)\exp\left[-\kappa\tau^{2}\right] \tag{37}$$

By differentiating (37) twice, we get:

$$R'_{XX_s}(\tau) = -2\kappa\tau\, R_{XX_s}(0)\, \exp\left[-\kappa\tau^2\right] \tag{38}$$

and,

$$R''_{XX_s}(\tau) = -2\kappa\, R_{XX_s}(0)\, \exp\left[-\kappa\tau^2\right] + (2\kappa\tau)^2 R_{XX_s}(0)\, \exp\left[-\kappa\tau^2\right] = \tag{39}$$

$$= \left(2\kappa\tau^2 - 1\right) 2\kappa\, R_{XX_s}(0)\, \exp\left[-\kappa\tau^2\right]$$

Therefore:

$$R''_{XX_s}(0) = -2\kappa\, R_{XX_s}(0) \tag{40}$$

The left side of (27) then becomes:

$$\rho_o \sqrt{\frac{\kappa}{\pi R_{XX_s}(0)}}\, \exp\left[-\rho_o^2 \Big/ \left[2R_{XX_s}(0)\right]\right] = \nu_{\rho_{oc}}^{+}\ , \tag{41}$$

where $\nu_{\rho_{oc}}^{+}$ is known and must be calculated using the autocorrelations of the original process $\{X(t), t \geq 0\}$ and $\{Y(t), t \geq 0\}$.

In fact:

$$\sigma_{\dot{X}}^2 = R_{\dot{X}\dot{X}}(0) = -R''_{XX}(0) = 2\kappa\, R_{XX}(0)\ , \tag{42}$$

$$\sigma_{\dot{Y}}^2 = R_{\dot{Y}\dot{Y}}(0) = -R''_{YY}(0) = 2\kappa\, R_{YY}(0)\ , \tag{43}$$

both (42) and (43) having been derived similarly to (40).

Then, using (25), we obtain:

$$B' = \frac{\rho_o^2}{2\kappa}\left[\frac{\sin^2\vartheta}{R_{XX}(0)} + \frac{\cos^2\vartheta}{R_{YY}(0)}\right]\ ,$$

$$D' = -\frac{\rho_o}{2\kappa}\left[\frac{\sin 2\vartheta}{R_{XX}(0)} - \frac{\sin 2\vartheta}{R_{YY}(0)}\right] ,$$

$$F' = \rho_o^2\left[\frac{\cos^2\vartheta}{R_{XX}(0)} + \frac{\sin^2\vartheta}{R_{YY}(0)}\right] , \qquad (44)$$

$$H' = \frac{1}{2\kappa}\left[\frac{\cos^2\vartheta}{R_{XX}(0)} + \frac{\sin^2\vartheta}{R_{YY}(0)}\right] = \frac{F'}{2\kappa\rho_o^2} ,$$

$$L' = H' - D'^2/_{4B'} = F'/\left(2\kappa\rho_o^2\right) - D'^2/_{4B'}$$

or the equally useful (and less complicated):

$$B' = \rho_o^2/_{2\kappa}\left[\sin^2\vartheta/_{R_{XX}(0)} + \cos^2\vartheta/_{R_{YY}(0)}\right] ,$$

$$D' = -\rho_o/_{2\kappa}\left[\sin 2\vartheta/_{R_{XX}(0)} - \sin 2\vartheta/_{R_{YY}(0)}\right] , \qquad (45)$$

$$F' = \rho_o^2\left[\cos^2\vartheta/_{R_{XX}(0)} + \sin^2\vartheta/_{R_{YY}(0)}\right] ,$$

$$L' = F'/\left(2\kappa\rho_o^2\right) - D'^2/(4B') .$$

We now solve (41) for $R_{XX_s}(0)$.

We observe that (41) can be written alternatively, by denoting $\sigma^2 \equiv R_{XX_s}(0)$, as :

$$\frac{\rho_o}{\sqrt{2\pi}} \cdot \frac{\sqrt{2\kappa}}{\sigma} \exp\left[-\frac{\rho_o^2}{2\sigma^2}\right] = \nu_{\rho_{oc}}^{+} , \text{or}$$

$$\frac{\rho_o}{\sqrt{2}\,\sigma} \exp\left[-\left(\frac{\rho_o}{\sqrt{2}\,\sigma}\right)^2\right] = \frac{\nu_{\rho_{oc}}^{+}}{\sqrt{2\kappa}}\sqrt{\pi} , \text{or} \qquad (41')$$

$$x \exp\left[-x^2\right] = \nu ,$$

if we set:

$$x = {}^{\rho_o}\!/\!_{(\sqrt{2}\,\sigma)}$$

and:

$$\nu = \nu_{\rho_{oc}} \stackrel{+}{\cdot} \sqrt{\pi/2\kappa} \quad .$$

Let us examine for a minute (41′). We, eventually, want to solve it for x. If we plot:

$$h(x) = x \exp\left[-x^2\right]$$

(see figure 5.1) we see that it has a maximum. This can easily be found by differentiating h(x). The first derivative is:

$$h(x) = \exp\left[-x^2\right] + x(-2x)\exp\left[-x^2\right] = \exp\left[-x^2\right]\left(1 - 2x^2\right)$$

This becomes zero only at $x = 1/\sqrt{2}$.

The second derivative is, then:

$$h''(x) = -2x \exp\left[-x^2\right] - 2x^2 \exp\left[-x^2\right](-2x) - \exp\left[-x^2\right] 4x =$$

$$= -6x \exp\left[-x^2\right] - 4x^3 \exp\left[-x^2\right] =$$

$$= -2x \exp\left[-x^2\right]\left(3 + 2x^2\right)$$

This is always negative for any x>0 (including $x = 1/\sqrt{2}$). Therefore, at $x = 1/\sqrt{2}$ we indeed have a maximum.

There, the value of h(x) is:

$$h\left(1/\sqrt{2}\right)=\frac{1}{\sqrt{2}}\exp\left[-\frac{1}{2}\right]=\frac{1}{\sqrt{2e}} .$$

We can conclude from the above and by looking again at fig. 5.1, that $h(x)=\nu$ was two solutions (that must be calculated numerically), when:

$$\nu<(2e)^{-1} ,$$

one double solution, when:

$$\nu=(2e)^{-1} ,$$

(this is the point x that maximizes h(x), i.e., $x=1/\sqrt{2}$) and no solutions, when:

$$\nu>(2e)^{-1}$$

These values of ν correspond to:

$$\nu_{\rho_{oc}^{+}}=\nu\sqrt{\frac{2\kappa}{\pi}}<\sqrt{\kappa/en} ,$$

$$\nu_{\rho_{oc}^{+}}=\sqrt{\kappa/en}$$

and,

$$\nu_{\rho_{oc}^{+}}>\sqrt{\kappa/en} .$$

To solve (41′) numerically (when such a solution actually exists) we search in both intervals $(0, 1/\sqrt{2}]$ and $(1/\sqrt{2}, \infty)$. Therefore, we generally find two solutions for x. The corresponding values for σ are:

$$\sigma_{1,2}=\frac{\rho_o}{\sqrt{2}\,x_{1,2}}$$

($\sigma=\rho_o$, in the case, where the $x_1=x_2=1/\sqrt{2}$)

We define, then, as σ of the "equivalent" process, the:

$$\sigma \equiv \sigma_2 = \frac{\rho_o}{\sqrt{2}\, x_2} \;, \tag{46}$$

since x_1, which belongs to the interval $(0, 1/\sqrt{2})$, is not suitable for two reasons. First, it takes values in a very small interval so the (numerical, at least) results acquired may lack accuracy, and second, its value is so small, at times, that it returns unreasonably high values for the variance of the "equivalent" random process. (Note that when there is no solution, we cannot apply this approach.)

Assume, now, that this solution is:

$$\sigma^2 \equiv R_{XX_s}(0) = g\left\{ \kappa, \rho_o, \nu_{\rho_{oc}}^{+} \left[R_{XX}(0), R_{YY}(0); \rho_o\right] \right\} \tag{47}$$

Then, (37) yields:

$$R_{XX_s}(\tau) = \sigma^2 \exp\left[-\kappa\tau^2\right] \tag{48}$$

Therefore, we now have to find the upcrossing probability for a threshold ρ_o of a stationary process $\{R_s(t), t \geq 0\}$. At any time t, $R_s(t)$ is a random variable, distributed as $N(0,\sigma)$ and given by:

$$R_s(t) = X_s(t)\cos\omega t + Y_s(t)\sin\omega t \;, \tag{49}$$

where $\{X_s(t), t \geq 0\}$ and $\{Y_s(s), s \geq 0\}$ are zero-mean, stationary, Gaussian random processes independent for any t, $s \geq 0$, with identical autocorrelation functions given by (47). This probability, as previously discussed, is given by (29).

From [Mikhalevsky, 1982], we get the following, for our example (setting $\rho_{XY}=0$, $\sigma_X^2=\sigma_Y^2=\sigma^2$ and $\mu_X=\mu_Y=0$):

$$f_{\rho(t)}(\rho) = \frac{\rho}{2\pi\sigma^2} \exp\left[-\frac{1}{2}\left(\frac{\rho^2}{2\sigma^2} + \frac{\rho^2}{2\sigma^2}\right)\right] \int_o^{2\pi} d\varphi = \tag{50}$$

$$= \rho/\sigma^2 \ \exp\left[-\rho^2/2\sigma^2\right] \quad , \rho \geq o$$

This, as expected, is a Rayleigh distribution.

From [Nikolaidis and Perakis, 1985], we get the second order pdf for the envelope process:

$$f_{\rho(t)\rho(t+\tau)}(\rho_1,\rho_2) = \frac{\rho_1\rho_2}{4\pi^2 \, |\Lambda_X|^{1/2}} \int_o^{2\pi}\int_o^{2\pi} \exp\,[-\frac{1}{2}(A + B\cos\vartheta_1 + C\sin\vartheta_1 +$$

$$+ D\cos2\vartheta_1 + E\sin2\vartheta_1 + F\cos\vartheta_2 + G\sin\vartheta_2 + H\cos2\vartheta_2 + I\sin2\vartheta_2 + \tag{51}$$

$$+ J\cos\vartheta_1\cos\vartheta_2 + K\sin\vartheta_1\cos\vartheta_2 + L\sin\vartheta_1\sin\vartheta_2 + M\cos\vartheta_1\sin\vartheta_2)\,]\,d\vartheta_1 d\vartheta_2,$$

$$\rho_1,\rho_2 \geq 0,$$

where:

$$A = \lambda'_{11}\,\rho_1^2/2 + \mu_X^2\,\lambda'_{11} + 2\mu_X\mu_Y\,\lambda'_{12} + 2\lambda'_{13}\,\mu_X^2 + 2\lambda'_{14}\,\mu_X\mu_Y +$$

$$+ \lambda'_{22}\,\rho_1^{\,2}/2 + \mu_Y^2\,\lambda'_{22} + 2\lambda'_{23}\,\mu_X\mu_Y + 2\lambda'_{24}\,\mu_Y^2 + \lambda'_{33}\,\rho_2^{\,2}/2 +$$

$$+ \mu_X^2\,\lambda'_{33} + 2\lambda'_{34}\,\mu_X\mu_Y + \rho_2^2\lambda_{44}/2 + \mu_Y^2\lambda_{44},$$

$$B = -2\lambda'_{11}\,\mu_X\rho_1 - 2\lambda'_{12}\,\mu_Y\rho_1 - 2\lambda'_{13}\,\mu_X\rho_1 - 2\lambda'_{14}\,\mu_Y\rho_1,$$

$$C = -2\lambda'_{12}\,\mu_X\rho_1 - 2\lambda'_{22}\,\mu_Y\rho_1 - 2\lambda'_{23}\,\mu_X\,\rho_1 - 2\lambda'_{24}\,\mu_Y\rho_1,$$

$$D = \lambda'_{11} \rho_1^2/2 - \lambda'_{22} \rho_1^2/2 \;, \quad E = \lambda'_{11} \rho_1^2 \tag{52}$$

$$F = -2\lambda'_{3} \mu_X \rho_2 - 2\lambda'_{23} \mu_Y \rho_2 - 2\lambda'_{33} \mu_X \rho_2 - 2\lambda'_{34} \mu_Y \rho_2 ,$$

$$G = -2\lambda'_{14} \mu_X \rho_2 - 2\lambda'_{24} \mu_Y \rho_2 - 2\lambda'_{34} \mu_X \rho_2 - 2\lambda'_{44} \mu_Y \rho_2 ,$$

$$H = \lambda'_{33} \rho_2^2/2 - \lambda'_{44} \rho_2^2/2 ,$$

$$I = \lambda'_{34} \rho_2^2 \;, J = 2\lambda'_{13} \rho_1 \rho_2 ,\; K = 2\lambda'_{23} \rho_1 \rho_2 \;,\; L = 2\lambda'_{24} \rho_1 \rho_2 \;,$$

$$M = 2\lambda'_{14} \rho_1 \rho_2$$

and λ'_{ij} are the elements of the inverse $[\Lambda_X]$, $[\Lambda_X]$ itself being defined as:

$$[\Lambda_X] = \begin{bmatrix} C_X(0) & C_{XY}(0) & C_X(\tau) & C_{YX}(\tau) \\ C_{YX}(0) & C_Y(0) & C_{XY}(\tau) & C_Y(\tau) \\ C_X(\tau) & C_{XY}(\tau) & C_X(0) & C_{XY}(0) \\ C_{YX}(\tau) & C_Y(\tau) & C_{XY}(0) & C_Y(0) \end{bmatrix} \tag{53}$$

with:

$$C_X(\tau) \equiv E\left[\left(X(t+\tau) - \mu_X\right)\left(X(t) - \mu_X\right)\right] ,$$

$$C_Y(\tau) \equiv E\left[\left(Y(t+\tau) - \mu_Y\right)\left(Y(t) - \mu_Y\right)\right] ,$$

$$C_{XY}(\tau) \equiv E\left[\left(X(t+\tau) - \mu_X\right)\left(Y(t) - \mu_Y\right)\right] ,$$

$$C_{YX}(\tau) \equiv E\left[\left(Y(t+\tau) - \mu_Y\right)\left(X(t) - \mu_X\right)\right] ,$$

For our example, where $\{X(t), t\geq 0\}$ and $\{Y(t), t\geq 0\}$ are zero-mean, independent (for any time t), stationary, Gaussian random processes with identical autocorrelation functions given by (47) (and zero cross-correlation function), we get:

$$C_X(\tau) = R_{XX_s}(\tau) = \sigma^2 \exp\left[-\kappa\tau^2\right], \ \mu_X = 0$$

$$C_Y(\tau) = R_{YY_s}(\tau) = \sigma^2 \exp\left[-\kappa\tau^2\right], \ \mu_Y = 0 \qquad (55)$$

$$C_{XY}(\tau) = R_{XY_s}(\tau) = C_{YX}(\tau) = R_{YX_s}(\tau) = 0 ,$$

Therefore, by defining $r_t = R_{XX_s}(\tau)/\sigma^2$, we obtain:

$$\Lambda_X = \begin{bmatrix} \sigma^2 & 0 & r_\tau\sigma^2 & 0 \\ 0 & \sigma^2 & 0 & r_\tau\sigma^2 \\ r_\tau\sigma^2 & 0 & \sigma^2 & 0 \\ 0 & r_\tau\sigma^2 & 0 & \sigma^2 \end{bmatrix} \qquad (56)$$

The determinant of this matrix is:

$$\left|\Lambda_X\right| = \sigma^8\left(1-r_\tau\right)^2 \qquad (57)$$

After some algebra, we find that the inverse of the covariance matrix is:

$$\left[\Lambda_X\right]^{-1} = \begin{bmatrix} \frac{1}{\sigma^2\left(1 - r_\tau^2\right)} & 0 & -\frac{r_\tau}{\sigma^2\left(1 - r_\tau^2\right)} & 0 \\ 0 & \frac{1}{\sigma^2\left(1 - r_\tau^2\right)} & 0 & -\frac{r_\tau}{\sigma^2\left(1 - r_\tau^2\right)} \\ -\frac{r_\tau}{\sigma^2\left(1 - r_\tau^2\right)} & 0 & \frac{1}{\sigma^2\left(1 - r_\tau^2\right)} & 0 \\ 0 & -\frac{r_\tau}{\sigma^2\left(1 - r_\tau^2\right)} & 0 & \frac{1}{\sigma^2\left(1 - r_\tau^2\right)} \end{bmatrix} \qquad (58)$$

With $\lambda'_{ij} = \lambda'_{ji}$, for every ij = 1,2,3,4,

$$\lambda'_{11} = \lambda'_{22} = \lambda'_{33} = \lambda'_{44} = \left[\sigma^2\left(1 - r_\tau^2\right)\right]^{-1} \qquad (59)$$

$$\lambda'_{12} = \lambda'_{14} = \lambda'_{23} = \lambda'_{34} = 0\ ,$$

$$\lambda'_{13} = \lambda'_{24} = -r_\tau\left[\sigma^2\left(1 - r_\tau^2\right)\right]^{-1}$$

After the above observations, (52) becomes:

$$A = \frac{\rho_1^2}{\sigma^2\left(1 - r_\tau\right)^2} + \frac{\rho_2^2}{\sigma^2\left(1 - r_\tau\right)^2}\ ,$$

$$B = C = D = E = F = G = H = I = 0\ , \qquad (60)$$

$$J = L = -\frac{2 r_\tau \rho_1 \rho_2}{\sigma^2\left(1 - r_\tau^2\right)}\ ,\ K = M = 0\ .$$

Finally, (51) becomes:

$$A = p^2 - q^2 + a^2 - b^2,$$

$$B = 2(pq + ab), \tag{69}$$

$$C = p^2 + q^2 - a^2 - b^2,$$

$$D = -2\,(ap + bq)$$

and $I_m(z)$ is the m^{th} order Bessel function of an imaginary argument.

In our special case, we get:

$$m = a = b = 0,$$

$$A = p^2 - q^2,$$

$$B = 2pq,$$

$$C = p^2 + q^2 = \alpha^2,$$

$$D = 0.$$

Hence:

$$\int_0^{2\pi} \exp\left[p\cos\vartheta_2 + q\sin\vartheta_2\right] d\vartheta_2 = 2\pi I_0(\alpha) \tag{70}$$

But, again from [Gradshtein and Ryzhik, 1965], we find:

$$I_0(\alpha) = J_0\left(\exp\left[\frac{\pi}{2} i\right]\alpha\right), \quad -\pi < \arg\alpha \leq \pi/2$$

(here $\arg\alpha = 0$), where J_0 is the 0^{th} order Bessel function. It can also be found [Gradshtein and Ryzhik, 1965]:

$$J_0\left[\exp\left[\left(\pi/2\right) i\right]\alpha\right] = \sum_{k=0}^{\infty} (-1)^k \frac{\left[\exp\left[\left(\pi/2\right) i\right]\alpha\right]^{2k}}{2^{2k}\,(k!)^2}, \tag{71}$$

Since:

$$\exp\left[\left(\pi/2\right) i\right] = \cos\left(\pi/2\right) + i \sin\left(\pi/2\right) = i,$$

we get:

$$I_o(\alpha) = J_o\left[\exp\left[\left(\pi/2\right) i\right] \alpha\right] = \sum_{k=0}^{\infty} (-1)^k \frac{\left(i^2\right)^k \alpha^{2k}}{2^{2k} (k!)^2} = \sum_{k=0}^{\infty} (-1)^{2k} \left(\frac{\alpha}{2}\right)^{2k} \frac{1}{k!} =$$

$$= \sum_{k=0}^{\infty} \frac{\left(\alpha/2\right)^{2k}}{(k!)^2} \qquad (72)$$

Therefore, (67) yields:

$$I = \int_o^{2\pi} 2\pi \sum_{k=0}^{\infty} \frac{\left(\alpha/2\right)^{2k}}{(k!)^2} d\vartheta_1 = 2\pi \sum_{k=0}^{\infty} \frac{\left(\alpha/2\right)^{2k}}{(k!)^2} \int_o^{2\pi} d\vartheta_1 =$$

$$= (2\pi)^2 \sum_{k=0}^{\infty} \frac{\left(\alpha/2\right)^{2k}}{(k!)^2} \quad .$$

Hence, (64) finally becomes:

$$f_{\rho(t+\tau)\rho(t)} d\rho_1, d\rho_2 = \frac{\rho_1 \rho_2}{\sigma^4\left(1 - r_\tau^2\right)} \exp\left[-\frac{\rho_1^2 + \rho_2^2}{2\sigma^2\left(1 - r_\tau^2\right)}\right] \sum_{k=0}^{\infty} \frac{\left[\frac{r_\tau \rho_1 \rho_2}{2\sigma^2\left(1 - r_\tau^2\right)}\right]^{2k}}{(k!)^2} \; ,$$

$$\rho_1, \rho_2 \geq 0 \, . \qquad (73)$$

Now, we can integrate (73) over ρ_1 and ρ_2 from 0 to ρ_o, to get b. But, first, we will rewrite (73) in a more convenient form:

$$f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2) = \sum_{k=0}^{\infty}\left[\frac{r_\tau}{2\sigma^2(1-r_\tau^2)}\right]^{2k}\frac{(\rho_1\rho_2)^{2k+1}}{\sigma^4(1-r_\tau^2)(k!)^2}\exp\left[-\frac{\rho_1^2+\rho_2^2}{2\sigma^2(1-r_\tau^2)}\right] =$$

$$= \sum_{k=0}^{\infty} C_k \rho_1^{2k+1}\rho_2^{2k+1}\exp[-\gamma\rho_1^2]\exp[-\gamma\rho_2^2]\ ,\ \rho_1,\rho_2 \ge 0\ , \qquad (74)$$

where:

$$C_k = \frac{r_\tau^{2k}}{\left[2\sigma^2(1-r_\tau^2)\right]^{2k}}\cdot\frac{1}{(k!)^2}\cdot\frac{1}{\sigma^4(1-r_\tau^2)} = \frac{\beta}{(k!)^2}(\gamma r_\tau)^{2k},$$

$$\gamma = \left[2\sigma^2(1-r_\tau^2)\right]^{-1}\ ,\quad \beta = \left[\sigma^4(1-r_\tau^2)\right]^{-1} \qquad (75)$$

And:

$$\int_0^{\rho_0}\int_0^{\rho_0} f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2)d\rho_1 d\rho_2 = \sum_{k=0}^{\infty} C_k\left(\int_0^{\rho_0}\exp[-\gamma\rho_1^2]\rho_1^{2k+1}d\rho_1\right)\left(\int_0^{\rho_0}\exp[-\gamma\rho_2^2]\rho_2^{2k+1}d\rho_2\right) =$$

$$= \sum_{k=0}^{\infty} C_k I_k^2\ , \qquad (76)$$

Then,

$$I_k \equiv \frac{1}{2}\int_0^{\rho_0}\rho^{2k}\exp[-\gamma\rho^2]d(\rho^2) = \frac{1}{2}\int_0^{\rho_0}(\rho^2)^k\exp[-\gamma\rho^2]d(\rho^2) = \quad ,(t=\rho^2\ge 0)$$

$$= \frac{1}{2}\int_0^{\rho_0^2} t^k\exp[-\gamma t]dt, \qquad k=0,1,2,\ldots \qquad (77)$$

This can be found to be [Gradshtein and Ryzhik, 1965]:

where:

$$l_2 \equiv \frac{b - x + 0.5}{\sqrt{x}} \ ,$$

$$l_1 \equiv \frac{a - x - 0.5}{\sqrt{x}} \ .$$

The term $\gamma\rho_o^2$ in (80) is usually large enough to allow us to write:

$$\int_0^{\rho_o}\int_0^{\rho_o} f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2)\, d\rho_1 d\rho_2 = \frac{\beta}{(2\gamma)^2}\sum_{k=0}^{\infty}\left\{r_\tau^k\left[1 - \Phi(l_2) + \Phi(l_1)\right]\right\}^2 ,$$

where:

$$l_2 \equiv \frac{k - \gamma\rho_o^2 + 0.5}{\sqrt{\gamma\rho_o^2}} \ ,$$

$$l_1 \equiv \frac{-\gamma\rho_o^2 - 0.5}{\sqrt{\gamma\rho_o^2}}$$

and $\Phi(\cdot)$ is the Unit Normal pdf N(0,1).

If we take the limit of (80) as $\rho_o \to \infty$, we find that since $\rho_o^\lambda \exp[-\gamma\rho_o^2] \to 0$, for any λ, the term inside the brackets tends to (1/2) $(k!/\gamma^{k+1})$ and the infinite sum becomes:

$$\sum_{k=1}^{\infty}(\gamma r_\tau)^{2k}\frac{\beta}{(k!)^2}\frac{1}{4}\frac{(k!)^2}{\gamma^{2k+2}} = \sum_{k=1}^{\infty}\frac{\beta}{4\gamma^2}r_\tau^{2k} = \tag{81}$$

$$= \frac{\beta}{4\gamma^2}\left[\sum_{k=0}^{\infty}\left(r_\tau^2\right)^k - 1\right] = \frac{\beta}{4\gamma^2}\left(\frac{1}{1 - r_\tau^2} - 1\right) = \frac{\beta r_\tau^2}{4\gamma^2\left(1 - r_\tau^2\right)}$$

While the first term of (80) yields as $\rho_o \to \infty$, $\beta/(2\gamma)^2$. Combining the above, we get:

$$\int_0^{\rho_o}\int_0^{\rho_o} f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2)\,d\rho_1 d\rho_2 = \lim_{\rho_o\to\infty} \int_0^{\rho_o}\int_0^{\rho_o} f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2)\,d\rho_1 d\rho_2 =$$

$$= \frac{\beta}{4\gamma^2} + \frac{\beta r_\tau^2}{4\gamma^2\left(1 - r_\tau^2\right)} = \frac{\beta}{4\gamma^2\left(1 - r_\tau^2\right)} \qquad (82)$$

Substituting in the above β and γ from (75), we conclude that:

$$\frac{\beta}{4\gamma^2\left(1 - r_\tau^2\right)} = \frac{4\sigma^4\left(1 - r_\tau^2\right)^2}{4\sigma^4\left(1 - r_\tau^2\right)\left(1 - r_\tau^2\right)} = 1\,,$$

which verifies that, indeed, $f_{\rho(t+\tau)\rho(t)}(\rho_1,\rho_2)$ is a valid pdf.

Finally, the probability of at least one upcrossing in the interval [0,T] is given by (29).

Using the above results, we developed a computer code and ran several different cases. Specifically, we varied the variances of $\{X(t), t\geq 0\}$ and $\{Y(t), t\geq 0\}$, the threshold ρ_o and the frequency ω.

We used the same data as in [Nikolaidis, Perakis, and Parsons, 1989] in order to compare our results. Hence, the variances were selected in such a way that the RMS of the process (defined as: $[(\sigma_X^2+\sigma_Y^2)/2]^{1/2}$) remains constant, equal to 5 ksi. Therefore, the variances ranged in the set:

{(25,25), (20,30), (15,35), (10,40), (5,45)}, (in (ksi)^2),

the threshold, in the set:

{20, 22.5, 25, 27.5, 30, 32.5}, (in ksi)

and the frequency, in the set:

"exact" Markovian approach, for which results were available, so that we could compare the two approaches. An almost entirely analytical solution was derived, involving an infinite summation. We ran a numerical example and compared the results with the ones of [Nikolaidis, Perakis, and Parsons, 1989]. The agreement was excellent.

The main problem of this method and generally of all reliability approaches) is to determine the number of terms in the infinite summation that are required for an accurate result. We can overcome this by keeping a "sufficiently long" number of terms before truncation. In the above example, for instance, we kept 30,000 terms. The increase in computational effort is negligible.

What is very important is that while the exact "Markov" approach [Nikolaidis, Perakis, and Parsons, 1989] is computationally very time consuming, ours is much more efficient and equally accurate. Therefore, at least for the range of frequencies tested it is a preferable approach.

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9. APPENDIX

Probability of upcrossing a specific threshold, for different variances and frequencies.

PROBABILITY OF UPCROSSING A SPECIFIC THRESHOLD

Frequency: 15.00000

Threshold: 20.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.239E-03	.250E+02	.3786E+00
20.00000	30.00000	0.339E-03	.262E+02	.4904E+00
15.00000	35.00000	0.632E-03	.287E+02	.7165E+00
10.00000	40.00000	0.111E-02	.315E+02	.8918E+00
5.00000	45.00000	0.177E-02	.342E+02	.9714E+00

Threshold: 22.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.322E-04	.250E+02	.6160E-01
20.00000	30.00000	0.567E-04	.266E+02	.1061E+00
15.00000	35.00000	0.138E-03	.295E+02	.2394E+00
10.00000	40.00000	0.295E-03	.325E+02	.4433E+00
5.00000	45.00000	0.545E-03	.356E+02	.6623E+00

Threshold: 25.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.332E-05	.250E+02	.6512E-02
20.00000	30.00000	0.772E-05	.269E+02	.1509E-01
15.00000	35.00000	0.252E-04	.301E+02	.4859E-01
10.00000	40.00000	0.667E-04	.334E+02	.1237E+00
5.00000	45.00000	0.146E-03	.366E+02	.2510E+00

Threshold: 27.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.265E-06	.250E+02	.5172E-03
20.00000	30.00000	0.857E-06	.272E+02	.1678E-02
15.00000	35.00000	0.386E-05	.306E+02	.7559E-02
10.00000	40.00000	0.129E-04	.341E+02	.2516E-01
5.00000	45.00000	0.339E-04	.375E+02	.6476E-01

Threshold: 30.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.163E-07	.250E+02	.3162E-04
20.00000	30.00000	0.775E-07	.274E+02	.1509E-03
15.00000	35.00000	0.494E-06	.311E+02	.9664E-03
10.00000	40.00000	0.214E-05	.347E+02	.4196E-02
5.00000	45.00000	0.686E-05	.383E+02	.1341E-01

Threshold: 32.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.776E-09	.250E+02	.1494E-05
20.00000	30.00000	0.571E-08	.277E+02	.1104E-04
15.00000	35.00000	0.530E-07	.315E+02	.1031E-03
10.00000	40.00000	0.304E-06	.352E+02	.5932E-03
5.00000	45.00000	0.121E-05	.390E+02	.2365E-02

Frequency: 25.00000

Threshold: 20.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.239E-03	.250E+02	.3796E+00
20.00000	30.00000	0.339E-03	.262E+02	.4914E+00
15.00000	35.00000	0.632E-03	.287E+02	.7172E+00

10.00000	40.00000	0.111E-02	.315E+02	.8920E+00
5.00000	45.00000	0.177E-02	.342E+02	.9715E+00

Threshold: 22.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.322E-04	.250E+02	.6199E-01
20.00000	30.00000	0.567E-04	.266E+02	.1067E+00
15.00000	35.00000	0.138E-03	.295E+02	.2404E+00
10.00000	40.00000	0.295E-03	.325E+02	.4443E+00
5.00000	45.00000	0.545E-03	.356E+02	.6632E+00

Threshold: 25.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.332E-05	.250E+02	.6576E-02
20.00000	30.00000	0.772E-05	.269E+02	.1522E-01
15.00000	35.00000	0.252E-04	.301E+02	.4892E-01
10.00000	40.00000	0.667E-04	.334E+02	.1244E+00
5.00000	45.00000	0.146E-03	.366E+02	.2519E+00

Threshold: 27.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.265E-06	.250E+02	.5243E-03
20.00000	30.00000	0.857E-06	.272E+02	.1698E-02
15.00000	35.00000	0.386E-05	.306E+02	.7632E-02
10.00000	40.00000	0.129E-04	.341E+02	.2536E-01
5.00000	45.00000	0.339E-04	.375E+02	.6517E-01

Threshold: 30.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab

25.00000	25.00000	0.163E-07	.250E+02	.3218E-04
20.00000	30.00000	0.775E-07	.274E+02	.1533E-03
15.00000	35.00000	0.494E-06	.311E+02	.9788E-03
10.00000	40.00000	0.214E-05	.347E+02	.4241E-02
5.00000	45.00000	0.686E-05	.383E+02	.1353E-01

Threshold: 32.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.776E-09	.250E+02	.1527E-05
20.00000	30.00000	0.571E-08	.277E+02	.1125E-04
15.00000	35.00000	0.530E-07	.315E+02	.1047E-03
10.00000	40.00000	0.304E-06	.352E+02	.6012E-03
5.00000	45.00000	0.121E-05	.390E+02	.2393E-02

Frequency: 50.00000

Threshold: 20.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.239E-03	.250E+02	.3804E+00
20.00000	30.00000	0.339E-03	.262E+02	.4922E+00
15.00000	35.00000	0.632E-03	.287E+02	.7180E+00
10.00000	40.00000	0.111E-02	.315E+02	.8924E+00
5.00000	45.00000	0.177E-02	.342E+02	.9716E+00

Threshold: 22.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.322E-04	.250E+02	.6224E-01
20.00000	30.00000	0.567E-04	.266E+02	.1070E+00
15.00000	35.00000	0.138E-03	.295E+02	.2410E+00
10.00000	40.00000	0.295E-03	.325E+02	.4451E+00
5.00000	45.00000	0.545E-03	.356E+02	.6639E+00

Threshold: 25.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.332E-05	.250E+02	.6615E-02
20.00000	30.00000	0.772E-05	.269E+02	.1530E-01
15.00000	35.00000	0.252E-04	.301E+02	.4913E-01
10.00000	40.00000	0.667E-04	.334E+02	.1248E+00
5.00000	45.00000	0.146E-03	.366E+02	.2526E+00

Threshold: 27.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.265E-06	.250E+02	.5284E-03
20.00000	30.00000	0.857E-06	.272E+02	.1709E-02
15.00000	35.00000	0.386E-05	.306E+02	.7676E-02
10.00000	40.00000	0.129E-04	.341E+02	.2548E-01
5.00000	45.00000	0.339E-04	.375E+02	.6544E-01

Threshold: 30.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.163E-07	.250E+02	.3250E-04
20.00000	30.00000	0.775E-07	.274E+02	.1546E-03
15.00000	35.00000	0.494E-06	.311E+02	.9860E-03
10.00000	40.00000	0.214E-05	.347E+02	.4267E-02
5.00000	45.00000	0.686E-05	.383E+02	.1360E-01

Threshold: 32.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.776E-09	.250E+02	.1546E-05
20.00000	30.00000	0.571E-08	.277E+02	.1137E-04

15.00000	35.00000	0.530E-07	.315E+02	.1057E-03
10.00000	40.00000	0.304E-06	.352E+02	.6058E-03
5.00000	45.00000	0.121E-05	.390E+02	.2409E-02

Frequency: 75.00000

Threshold: 20.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.239E-03	.250E+02	.3806E+00
20.00000	30.00000	0.339E-03	.262E+02	.4924E+00
15.00000	35.00000	0.632E-03	.287E+02	.7181E+00
10.00000	40.00000	0.111E-02	.315E+02	.8925E+00
5.00000	45.00000	0.177E-02	.342E+02	.9716E+00

Threshold: 22.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.322E-04	.250E+02	.6230E-01
20.00000	30.00000	0.567E-04	.266E+02	.1071E+00
15.00000	35.00000	0.138E-03	.295E+02	.2412E+00
10.00000	40.00000	0.295E-03	.325E+02	.4453E+00
5.00000	45.00000	0.545E-03	.356E+02	.6641E+00

Threshold: 25.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.332E-05	.250E+02	.6623E-02
20.00000	30.00000	0.772E-05	.269E+02	.1531E-01
15.00000	35.00000	0.252E-04	.301E+02	.4918E-01
10.00000	40.00000	0.667E-04	.334E+02	.1249E+00
5.00000	45.00000	0.146E-03	.366E+02	.2527E+00

Threshold: 27.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.265E-06	.250E+02	.5292E-03
20.00000	30.00000	0.857E-06	.272E+02	.1712E-02
15.00000	35.00000	0.386E-05	.306E+02	.7686E-02
10.00000	40.00000	0.129E-04	.341E+02	.2551E-01
5.00000	45.00000	0.339E-04	.375E+02	.6549E-01

Threshold: 30.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.163E-07	.250E+02	.3257E-04
20.00000	30.00000	0.775E-07	.274E+02	.1549E-03
15.00000	35.00000	0.494E-06	.311E+02	.9875E-03
10.00000	40.00000	0.214E-05	.347E+02	.4273E-02
5.00000	45.00000	0.686E-05	.383E+02	.1362E-01

Threshold: 32.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.776E-09	.250E+02	.1550E-05
20.00000	30.00000	0.571E-08	.277E+02	.1140E-04
15.00000	35.00000	0.530E-07	.315E+02	.1059E-03
10.00000	40.00000	0.304E-06	.352E+02	.6068E-03
5.00000	45.00000	0.121E-05	.390E+02	.2412E-02

Frequency: 100.00000

Threshold: 20.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.239E-03	.250E+02	.3807E+00
20.00000	30.00000	0.339E-03	.262E+02	.4925E+00

15.00000	35.00000	0.632E-03	.287E+02	.7182E+00
10.00000	40.00000	0.111E-02	.315E+02	.8925E+00
5.00000	45.00000	0.177E-02	.342E+02	.9716E+00

Threshold: 22.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.322E-04	.250E+02	.6232E-01
20.00000	30.00000	0.567E-04	.266E+02	.1071E+00
15.00000	35.00000	0.138E-03	.295E+02	.2412E+00
10.00000	40.00000	0.295E-03	.325E+02	.4454E+00
5.00000	45.00000	0.545E-03	.356E+02	.6642E+00

Threshold: 25.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.332E-05	.250E+02	.6626E-02
20.00000	30.00000	0.772E-05	.269E+02	.1532E-01
15.00000	35.00000	0.252E-04	.301E+02	.4919E-01
10.00000	40.00000	0.667E-04	.334E+02	.1249E+00
5.00000	45.00000	0.146E-03	.366E+02	.2528E+00

Threshold: 27.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.265E-06	.250E+02	.5295E-03
20.00000	30.00000	0.857E-06	.272E+02	.1713E-02
15.00000	35.00000	0.386E-05	.306E+02	.7689E-02
10.00000	40.00000	0.129E-04	.341E+02	.2552E-01
5.00000	45.00000	0.339E-04	.375E+02	.6551E-01

Threshold: 30.00000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.163E-07	.250E+02	.3259E-04
20.00000	30.00000	0.775E-07	.274E+02	.1550E-03
15.00000	35.00000	0.494E-06	.311E+02	.9880E-03
10.00000	40.00000	0.214E-05	.347E+02	.4275E-02
5.00000	45.00000	0.686E-05	.383E+02	.1362E-01

Threshold: 32.50000

Var(x)	Var(y)	UpcrRate	EquivVar	Up Probab
25.00000	25.00000	0.776E-09	.250E+02	.1551E-05
20.00000	30.00000	0.571E-08	.277E+02	.1140E-04
15.00000	35.00000	0.530E-07	.315E+02	.1059E-03
10.00000	40.00000	0.304E-06	.352E+02	.6071E-03
5.00000	45.00000	0.121E-05	.390E+02	.2413E-02

10. FIGURES

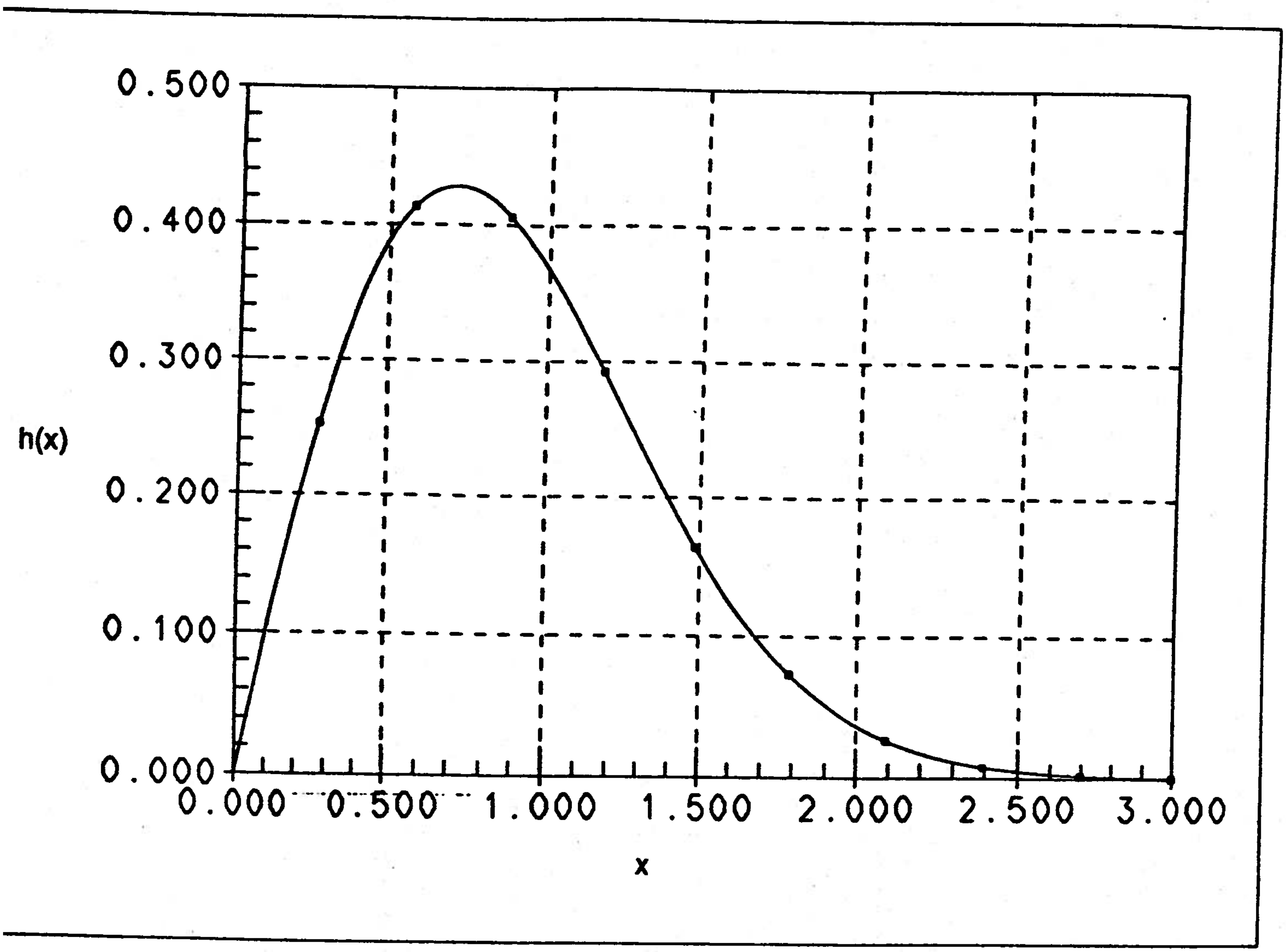
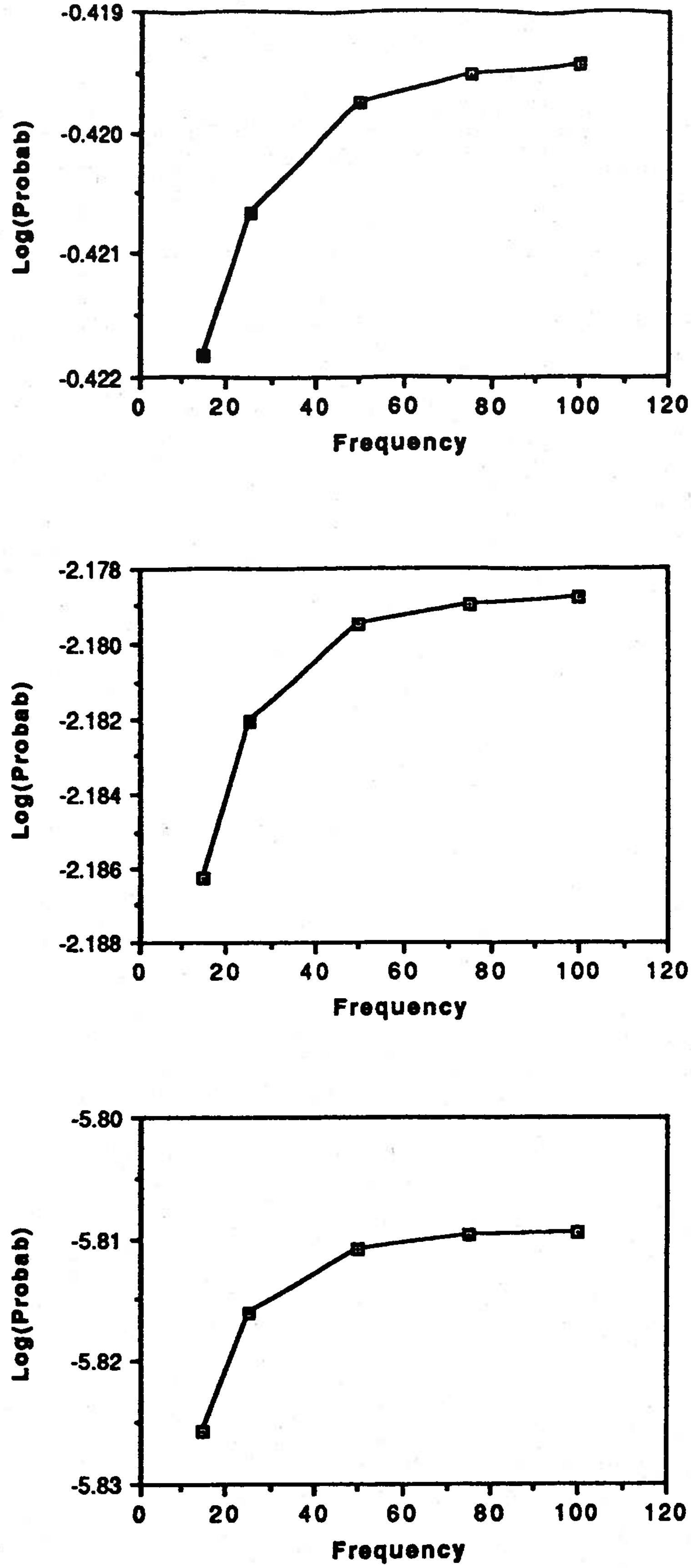


Figure 5.1
$h(x)=x \exp(-x^2)$



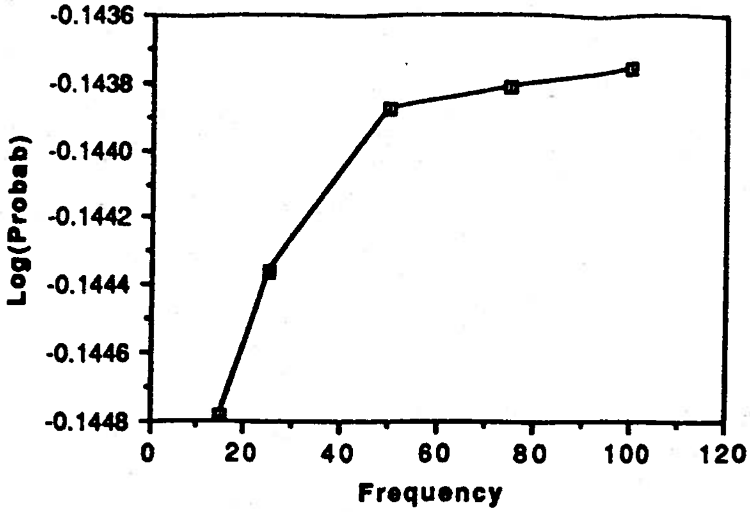
ρ_o=20.0 ksi

ρ_o=25.0 ksi

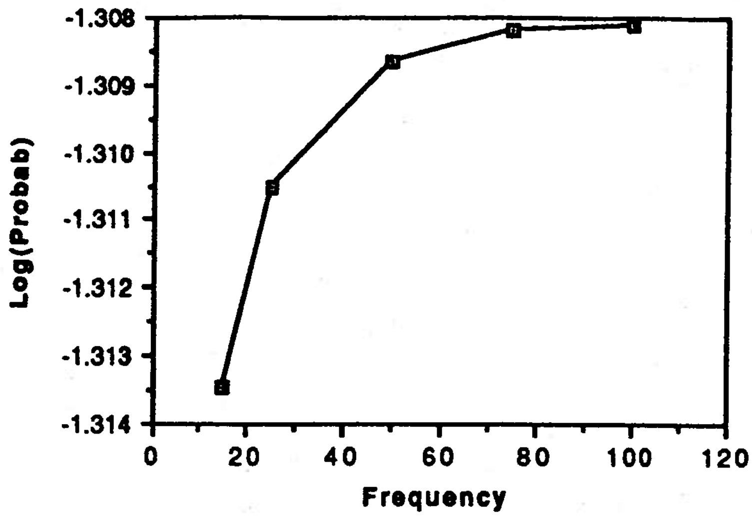
ρ_o=32.5 ksi

Figure 5.2(a)

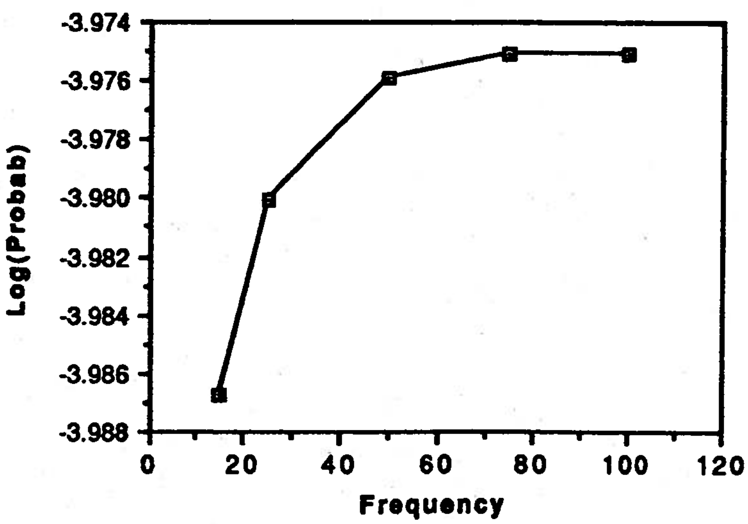
Logarithm of the probability of upcrossing threshold ρ_o (in ksi) versus frequency ω (in rad/sec), for σ_X=σ_Y=25.0 ksi.



ρ_o=20.0 ksi



ρ_o=25.0 ksi



ρ_o=32.5 ksi

Figure 5.2(b)

Logarithm of the probability of upcrossing threshold ρ_o (in ksi) versus frequency ω (in rad/sec), for σ_X=15.0 ksi and σ_Y=35.0 ksi.

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