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RECENT RESEARCH IN PARTIALLY SATURATED OCEAN ACOUSTIC PROCESSES

Anastassios N. Perakis
Efstratios Nikolaidis
Emmanuel Katzouros

THE DEPARTMENT OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING

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Recent Research in Partially Saturated Ocean Acoustic Processes

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Anastassios N. Perakis
Efstratios Nikolaidis
Emmanuel Katzouros

Department of Naval Architecture
and Marine Engineering
College of Engineering
The University of Michigan
Ann Arbor, Michigan 48109

ABSTRACT

An approximate, closed-form expression for the value of the integral encountered in the calculation of the probability density function (PDF) of the envelope of a partially saturated ocean acoustic process is obtained. Other possible derivations for specific cases are presented. Furthermore, an expression of this PDF as a series of modified Bessel functions is presented. Numerical applications show that the closed form expression is always within 1-2% of the exact result, and is obtained at a substantially lower computational effort. The results may also be directly applied to the evaluation of the PDF encountered in the structural reliability analysis of marine diesel engine shafting systems.

Introduction

The scope of this paper is to derive an approximate closed-form expression for an integral of the following form:

$$I = \int_0^{2\pi} \exp\{a\cos\phi + b\sin\phi + c\cos 2\phi + d\sin 2\phi\} d\phi \tag{1}$$

This integral was encountered in the evaluation of the first order probability density function (PDF) of the envelope of the acoustical pressure of a partially saturated process [1], [2], [3]. Let us denote $X(t)$ and $Y(t)$ the cosine and sine components of the envelope. Then [2]:

$$a = \frac{-1}{2(1-\rho_{xy}^2)} \rho_1 \left(\frac{2\rho_{xy}\mu_y}{\sigma_x \sigma_y} - \frac{2\mu_x}{\sigma_x^2}\right) \tag{2}$$

$$b = \frac{-1}{2(1-\rho_{xy}^2)} \rho_1 \left(\frac{2\rho_{xy}\mu_x}{\sigma_x \sigma_y} - \frac{2\mu_y}{\sigma_y^2}\right) \tag{3}$$

$$c = \frac{-1}{2(1-\rho_{xy}^2)} \rho_1^2 \left(\frac{1}{2\sigma_x^2} - \frac{1}{2\sigma_y^2}\right) \tag{4}$$

$$d = \frac{\rho_1^2}{2(1-\rho_{xy}^2)} \frac{\rho_{xy}}{\sigma_x\sigma_y} \tag{5}$$

where μ_x, σ_x^2 are the mean and the variance of the random variable $X(t)$ respectively, μ_y, σ_y^2 are the mean and the variance of the random variable $Y(t)$ respectively, and ρ_{xy} is the correlation coefficient of random variables X and Y.

In general, the quadrature components of the envelope of a narrowband ocean acoustic multipath process are given by

$$X = \sum_{n=1}^{N} \left(r_n \cos\theta_n + N_x^{(n)}\right) \quad ,$$

$$Y = \sum_{n=1}^{N} \left(r_n \sin\theta_n + N_y^{(n)}\right) \quad , \tag{6}$$

where:

N = number of independent paths between source and receiver,

r_n = the amplitude of the nth path

θ_n = phase of the nth path

$N_x^{(n)}$, $N_y^{(n)}$ = zero-mean, uncorrelated Gaussian additive noise for the nth path.

Furthermore, the envelope and the phase of the total signal are defined as:

$$\rho = (X^2 + Y^2)^{1/2}$$

$$\phi = \tan^{-1}(Y/X) \tag{7}$$

At short ranges and low frequencies, or for stable channels, the propagation is said to be unsaturated and the probability density function (PDF) of ρ is Rician [4] and independent of the number of paths. At sufficiently long ranges and/or high frequencies, the propagation is fully saturated, which means that ϕ , the phase of ρ , can be characterized as a random variable uniformly distributed between 0 and 2π , or each path has a phase θ_n that is normally distributed with a standard deviation $> 2\pi$. In this regime, when $N > 4$ and the single path amplitudes r_n are approximately equal, phase random multipath propagation is obtained. It has been found that the envelope $\rho(t)$ of a fully saturated phase random process with additive Gaussian noise obeys a Rayleigh PDF. Moreover, several other statistics and joint PDF's for the phase random process have been obtained, and are presented in a comprehensive summary by Mikhalevsky [5].

In intermediate ranges, where the signal experiences enough perturbations in the channel so that each θ_n can be characterized as a Gaussian random variable but with a standard deviation $<2\pi$, partially saturated propagation is obtained. The frequency/range boundaries between the unsaturated, partially saturated, and fully saturated regimes are dependent upon the ocean dynamics or boundary dynamics of the propagation channel, as well as the magnitude of any relative source-receiver motion.

In Ref. [2], the envelope statistics for signals in the partially saturated regime were presented. As the variance of the single path phase goes to zero, or becomes large, the PDF's converge to the unsaturated and fully saturated results respectively.

For the statistics of the amplitude and phase, we only need to consider the correlation matrix of X and Y and the transformations given by Eq. (7). After the numerical calculations have been performed, we see that the joint PDF of ρ and ϕ is:

$$P_{\rho,\phi}(\rho,\phi) = \frac{\rho}{2\pi\sigma_x\sigma_y\left(1-\rho_{xy}^2\right)^{1/2}} \times$$

$$\exp\left[\frac{-1}{2\left(1-\rho_{xy}^2\right)}\left(\frac{(\rho\cos\phi-\mu_x)^2}{\sigma_x^2} - \frac{2\rho_{xy}(\rho\cos\phi-\mu_x)(\rho\sin\phi-\mu_y)}{\sigma_x\,\sigma_y} + \frac{(\rho\sin\phi-\mu_y)^2}{\sigma_y^2}\right)\right] \tag{8}$$

The PDF of the amplitude of the envelope ρ is obtained by integrating the above equation with respect to ϕ from 0 to 2π.

$$p_\rho(\rho) = \frac{\rho}{2\pi\sigma_x\sigma_y\left(1-\rho_{xy}^2\right)^{1/2}} \times \exp\left[\frac{-1}{2\left(1-\rho_{xy}^2\right)}\left(\frac{\rho^2+2\mu_x^2}{2\sigma_x^2} - \frac{2\rho_{xy}\mu_x\mu_y}{\sigma_x\sigma_y} + \frac{\rho^2+2\mu_y^2}{2\sigma_y^2}\right)\right] \times I \tag{9}$$

In the case of the structural reliability studies (see for example [6], [7]) the probability of first excursion failure during a time interval of length T must be calculated. This is achieved by deriving an expression for the probability that random process $\Delta\theta_\ell(t)$ exceeds a threshold $(\Delta\theta_\ell)max$ that has been determined previously. The approach is based on the approximation of the local maxima of $\Delta\theta_\ell(t)$ by the value of the associated envelope process at the time they occur. A discrete time, two-state process is defined with the two states defined by the requirement that the value of $\Delta\theta_\ell(t)$ exceeds or does not exceed the threshold $(\Delta\theta_\ell)_{max}$. The failure probability is calculated assuming that the discrete time process is Markov.

Let t_1, t_2, ... be the times that the local maxima of $\Delta\theta_\ell(t)$ occur. Assuming that the local maxima of $\Delta\theta_\ell(t)$ are approximately equal to the value of the envelope at the time they occur, the upcrossing probability is equal to the probability that at least one of the values of the envelope at times t_1, t_2. ... exceeds the value of the threshold. Since $\Delta\theta_\ell(t)$ is a narrowband process, the time interval between two successive local maxima is approximately constant.

If we assume that $t_{k+1} - t_k = \tau$ = constant for all k=1, ... the expected period between successive maxima is $\tau = 2\pi/\omega_0$, and the expected number of maxima in the time interval T is

$$\nu = T/\tau + 1 \tag{10}$$

Consider a discrete time, two state random process, whose states, denoted by U and P, are defined by the requirements that the envelope of $\Delta\theta_\ell(t)$ exceeds and does not exceed the threshold $(\Delta\theta_\ell)max$, respectively. The time step is constant and equal to τ, with the discrete process taking values at times $0, \tau, \ldots, \nu\tau$. The envelope process $\rho_\ell(t) = \left(X_\ell{}^2(t) + Y_\ell{}^2(t)\right)^{1/2}$ is stationary since $X_\ell(t)$ and $Y_\ell(t)$ are stationary. Hence, the discrete time random process is also stationary. It is assumed that the discrete process is

Markov. The upcrossing probability is given by the equation:

$$P(\text{upcrossing in } [0,T]) = P(\xi_0 > \rho_0) +$$
$$+ P(\xi_1 > \rho_0 , \xi_0 < \rho_0) + P(\xi_2 > \rho_0 , \xi_1 < \rho_0 , \xi_0 < \rho_0) + \dots$$
$$+ P(\xi_{\nu-1} > \rho_0 , \xi_{\nu-2} < \rho_0 , \dots, \xi_0 < \rho_0) , \qquad (11)$$

where ξ_i denotes the ith maximum and ρ_0 equals to the threshold $(\Delta\theta_\ell)$max. With the assumptions that the local maxima are approximately equal to the value of the envelope process at the time they occur and that the time interval between successive maxima is equal to τ eq. (11) yields:

$$P(\text{upcrossing in } [0,T]) = P(U) + P(DU) + \dots + P(D \dots DU) \qquad (12)$$

where $P(U)$ denotes the probability that the value of the two-state Markov process in U and $P(DU)$ denotes the joint probability that the value of the process is D and U on two successive steps. Using the fact that the discrete time process is Markov, eq. (12) yields:

$$P(\text{upcrossing in } [0,T]) = 1 - (1-c)\ (1-b)^{\nu-1} \qquad (13)$$

where c is equal to $P(U)$ and b is equal to the conditional probability that the value of the two-state Markov process is U given that its value in the previous step was D. Coefficients c and d can be obtained as follows:

$$c = \int_{\rho_0}^{\infty} f_\rho(\rho_1)\, d\rho_1 \qquad (14)$$

$$d = \frac{\displaystyle\int_0^{\rho_0}\int_{\rho_0}^{\infty} f_{\rho(t)\rho(t+\tau)}(\rho_1,\rho_2)d\rho_1\, d\rho_2}{\displaystyle\int_0^{\rho_0} f_\rho(\rho_1)\, d\rho_1} \qquad (15)$$

This result has been obtained by Epstein [8].

Since the integrations required for the evaluation of c and d cannot be performed analytically [2], the calculation of the value of the first-order envelope PDF for any value of ρ_1 requires one numerical integration while the calculation of the second-order envelope PDF for any pair ρ_1, ρ_2 requires two, nested numerical integrations.

In these references, the integrals were numerically evaluated using Simpson's formula. Since the computing cost of this evaluation is high, and in the absence of an exact closed-form expression, it is desirable to derive an approximate closed form expression for this integral. This would also provide insight into the limiting behavior of the PDF.

Derivation of the Approximate Closed-Form Expression

Our approach is based on the approximation of the exponent in the neighborhoods of the maxima or minima by second order polynomials having the same roots and maxima. It turns out that the integral obtained after the approximation can be evaluated in closed form.

We use the following abbreviations:

$$F(\phi) = a \cos\phi + b \sin\phi + c \cos 2\phi + d \sin 2\phi \tag{16}$$

$$I = \int_0^{2\pi} \exp F \, d\phi \tag{17}$$

$$F(\phi) = a' \cos(\phi + \delta_1) + b' \cos(2\phi + \delta_2) \tag{18}$$

$$a' = \sqrt{a^2+b^2} \quad , \quad b' = \sqrt{c^2+d^2} \tag{19}$$

$$\delta_1 = \tan^{-1} (-b/a) \quad , \quad \delta_2 = \tan^{-1} (-d/c) \tag{20}$$

First we find the maxima, minima and the roots of the exponent $F(\phi)$. For the stationary points, we set $dF/d\phi = 0$. That gives

$$- a \sin\phi + b \cos\phi - 2c \sin 2\phi + 2d \cos 2\phi = 0$$

If we substitute $\cos\phi = u$, we get:

$$-a\sqrt{1-u^2} + bu - 4c\sqrt{1-u^2}\, u + 2d(2u^2-1) = 0 , \quad \text{or}$$

$$16(d^2+c^2)u^4 + 8\,(bd + ac)u^3 + [a^2+b^2 - 16\,(c^2+d^2)]u^2 +$$

$$+ (-4bd - 8ac)u + 4d^2 - a^2 = 0 \tag{21}$$

For the roots of the exponent, we set $F=0$.

After the substitution $\cos\phi = u$:

$$au + b\sqrt{1-u^2} + c(2u^2-1) + 2du\sqrt{1-u^2} = 0 , \quad \text{or}$$

$$4(c^2+d^2)\,u^4 + 4\,acu^3 + 4bdu^3 + [a^2+b^2 - 4(c^2+d^2)]u^2 -$$

$$- 2\,acu - 4bdu + c^2 - b^2 = 0 \tag{22}$$

Hence, in both cases, we have an algebraic equation of fourth order in u .
We find the roots and the values $\phi_i = \cos^{-1}u_i$, ϕ_i in $[0,2\pi]$.

The roots of the fourth order algebraic equation

$$u^4 + \alpha_1 u^3 + \alpha_2 u^2 + \alpha_3 u + \alpha_4 = 0$$

are given by the following [9]:

If y_1 is a real root of the third order equation

$$y^3 - \alpha_2 y^2 + (\alpha_1\alpha_3 - 4\alpha_4)\,y + 4\alpha_2\alpha_4 - \alpha_3^2 - \alpha_1^2\alpha_4 = 0 \tag{23}$$

the four roots are the roots of the quadratic equation:

$$z^2 + \frac{1}{2}\left(\alpha_1 \pm \sqrt{\alpha_1^2-4\alpha_2+4y_1}\right) z + \frac{1}{2}\left(y_1 \pm \sqrt{y_1^2-4\alpha_4}\right) = 0 \tag{24}$$

The roots of the third order equation

$$y^3 + \beta_1 y^2 + \beta_2 y + \beta_3 = 0 \quad \text{are:}$$

$$y_1 = S + T - \frac{1}{3}\beta_1$$

$$y_2 = -\frac{1}{2}(S+T) - \frac{1}{3}\beta_1 + \frac{1}{2}\, i\sqrt{3}\,(S-T) \tag{25}$$

$$y_3 = -\frac{1}{2}(S+T) - \frac{1}{3}\beta_1 - \frac{1}{2} i \sqrt{3}\,(S-T)$$

where: $S = \sqrt[3]{R + \sqrt{Q^3+R^2}}$, $T = \sqrt[3]{R - \sqrt{Q^3+R^2}}$ (26)

and $Q = 1/9\ (3\beta_2 - \beta_1{}^2)$, $R = 1/54\ (9\beta_1\beta_2 - 27\beta_3 - 2\beta_1{}^3)$ (27)

If $(\phi_1, F(\phi_1))$ is a local maximum of the exponent and ϕ_1' is the root following ϕ_1 , the value of the integral

$A_1' = \int_{\phi_1}^{\phi_1'} e^F\, d\phi$ can be approximated by the integral

$$\int_0^{\rho_1'} e^{-\lambda x^2+\mu}\, dx ,$$

where

$$\rho_1' = \phi_1' - \phi_1 \ , \ \mu = F(\phi_1) \ \text{and} \ \lambda = \mu/\rho_1'^2 .$$

Therefore

$$A_1' \simeq \int_0^{\rho_1'} e^{-\lambda x^2+\mu}\, dx = e^\mu \frac{\rho_1'}{2} \sqrt{\frac{\pi}{\mu}}\ \mathrm{erf}(\sqrt{\mu}) \qquad (28)$$

In the same manner, if ϕ_1'' is the root before ϕ_1 the integral

$A_1'' = \int_{\phi_1''}^{\phi_1} e^F\, d\phi$ can be approximated by $e^\mu \frac{\rho_1''}{2} \sqrt{\frac{\pi}{\mu}}\ \mathrm{erf}\ (\sqrt{\mu})$.

Hence

$$A_1 + A_1' + A_1'' = \int_{\phi_1'}^{\phi_1''} e^F\, d\phi \approx e^\mu \frac{\rho_1}{2} \sqrt{\frac{\pi}{\mu}}\ \mathrm{erf}\ (\sqrt{\mu}) \qquad (29)$$

where $\rho_1 = \rho_1' + \rho_1''$ is the distance between the roots.

In the case of local minimum, where we do not expect a significant contribution to the value of the total integral, we use a quadratic approximation of the quantity e^F . If $(\phi_2, F(\phi_2))$ is a local minimum and $\mu = F(\phi_2) < 0$ the integral

$$A_2 = \int_{\phi_2'}^{\phi_2''} e^F\, d\phi$$

where ϕ_2' and ϕ_2'' are the roots of the exponent immediately before and after ϕ_2 respectively, can be approximated by:

$$A_2 = \rho_2\left(\frac{1}{3} + e^{\mu}\right) , \text{ where } \rho_2 = \phi_2'' - \phi_2' \tag{30}$$

Hence, $$I = \sum_{i=1}^{4} A_i \tag{31}$$

where:

$$A_i \cong e^{\mu_i} \frac{\rho_i}{2} \sqrt{\frac{\pi}{\mu_i}} \operatorname{erf}\left(\sqrt{\mu_i}\right) , \; i = 1,2,3,4, \; \mu > 0 \tag{32}$$

and $$Ai \cong \rho_i\left(\frac{1}{3} + e^{\mu_i}\right) , \; i = 1,2,3,4, \; \mu < 0 \tag{33}$$

Note: it is possible to have only 2 roots, ϕ_1 and ϕ_2. In this case we can take $I = A_1 + A_2$. In the case where three stationary points exist between the two roots, we can handle the middle one as if it were another root.

In the case where the maximum value of the exponent is large, $(\mu > 4)$ - that is usually the case - Laplace's method may be applied [10]. According to Laplace, the major contribution to the value of the integral arises from the immediate vicinity of those points of the interval, at which the exponent assumes its largest value. If $F(\phi)$ has more than one maxima, we may break up the integral into a number of integrals, such that in each integral, $F(\phi)$ reaches its maximum at one of the end-points and at no other point. Accordingly, we shall assume that $F(\phi)$ reaches its maximum at $\phi = \phi_0$, and that $F(\phi) < F(\phi_0)$ in the interval discussed. Since our F is twice continuously differentiable, and $F'(\phi_0) = 0$, $F''(\phi_0) < 0$, we may apply the technique introduced by Laplace. Specifically, a new variable u is defined by the substitution $F(\phi_0) - F(\phi) = u^2$. $F'(\phi)$ will be negative in

$\phi_0 < \phi < \phi_0 + \eta$ for some sufficiently small η. As $F(\phi)$ becomes large:

$$I \cong 2 \int_{\phi_0}^{\phi_0+\eta} e^{F(\phi)} \, d\phi = -4 \int_0^U u \frac{1}{F'(\phi)} \{\exp[F(\phi_0) - u^2]\} \, du \tag{34}$$

where $U = [F(\phi_0) - F(\phi_0+\eta)]^{1/2} > 0$. (35)

Since only the neighbourhood of $u = 0$ matters, we may replace $u/F'(\phi)$ by $-[-2F''(\phi_0)]^{-1/2}$, which is the limit of $u/F'(\phi)$ as $\phi \to \phi_0$, and obtain

$$I \cong \left[\frac{-2}{F''(\phi_0)}\right]^{1/2} 2 \cdot \int_0^u \{\exp[-u^2 + F(\phi_0)]\} \, du \tag{36}$$

By the same argument, we may extend the integration to $u = \infty$ and finally obtain Laplace's result

$$I \cong 2e^{F(\phi_0)} \left[\frac{-\pi}{F''(\phi_0)}\right]^{1/2} \tag{37}$$

In the general case, where we have to deal with two maxima, $\mu_1 = F(\phi_{01})$ and $\mu_2 = F(\phi_{02})$, $\phi \varepsilon [0,2\pi)$, the value of the integral will be:

$$I \cong 2e^{\mu_1} \left[\frac{-2\pi}{F''(\phi_{01})}\right]^{1/2} + 2e^{\mu_2} \left[\frac{-2\pi}{F''(\phi_{02})}\right]^{1/2} \tag{38}$$

<u>Special cases:</u>

(A) If $a^2 + b^2 >> c^2 + d^2$

The exponent $F(\phi)$ can be transformed into the form

$$F(\phi) = a' \cos(\phi+\delta_1) + b' \cos(2\phi + \delta_2)$$

where

$$a' = \sqrt{a^2+b^2}, \quad \delta_1 = \tan^{-1} \frac{b}{a}, \quad b' = \sqrt{c^2+d^2}, \quad \delta_2 = \tan^{-1} \frac{d}{c} .$$

After the substitution $\phi + \delta_1 = \psi$

$$I = \int_0^{2\pi} \exp \{a' \cos\phi + b' \cos(2\phi + \delta)\} \, d\phi =$$

$$= \int_0^{2\pi} \exp F_1(\psi) d\psi \ ,$$

where $F_1(\psi) = a' \cos\psi + b' \cos(2\psi + \delta)$

and $\delta = \delta_2 - 2\delta_1$.

In this case we expect the maximum value of $F_1(\psi)$ to be near $\psi = 0$. For small values of ψ : $a' \cos\psi \cong a' \left(1 - \frac{\psi^2}{2}\right)$

$b' \cos(2\psi + \delta) = b' (\cos 2\psi \cos\delta - \sin 2\psi \sin\delta) \cong$

$\cong \left[(1 - 2\psi^2) \cos\delta - 2\psi \sin\delta\right] b'$

Consequently

$$F_1 \cong a' - \frac{a'}{2} \psi^2 + [\cos\delta - 2\cos\delta \cdot \psi^2 - 2\sin\delta \cdot \psi] b'$$

$$= a' + b'\cos\delta - \psi 2b'\sin\delta - \psi^2 \left(2b'\cos\delta + \frac{a'}{2}\right)$$

To find the value of ψ that maximizes F_1 , set

$\frac{dF_1}{d\psi} = 0$, or equivalently,

$$- 2b'\sin\delta - 4\psi b'\cos\delta + \frac{a'}{2} = 0 \text{ , which yields}$$

$$\psi_0 = \frac{-b'\sin\delta}{2b'\cos\delta + a'/2} \quad (\text{since } a' \gg b' \ , \ |\psi_0| \ll 1) \ . \qquad (39)$$

An approximation to I can now be the quantity $2\pi \ I_0(\mu)$ for $\mu < 4$. (40) where $\mu = F(\phi_0)$ and $I_0(\mu)$ is the modified Bessel function of order 0 . Equation (40) is valid for small values of μ . For large values of μ , we can use equation (37).

(B) If $a^2 + b^2 \ll c^2 + d^2$

Let $\psi = \phi + \delta_2/2$, and $\delta = \delta_1 - \frac{1}{2}\delta_2$

The exponential then becomes:

$F_2(\psi) = a'\cos(\psi+\delta) + b'\cos 2\psi$

Again, the maximum value of $F_2(\psi)$ will be near $\psi=0$. Another local maximum will be near $\psi = \pi$.

The following approximations are used:

$a' \cos(\psi+\delta) = a' (\cos\psi \cos\delta - \sin\psi \sin\delta) \simeq$

$\simeq \left[\left(1- \psi^2/2\right) \cos\delta - \psi \sin\delta\right]a'$, and

$b' \cos 2\psi \simeq b' (1-2\psi^2)$

Consequently,

$F_2 \simeq a' \cos\delta - a' \psi^2/2 \cdot \cos\delta - a'\psi \sin\delta + b' - 2b'\psi^2$

To find the value of ψ that maximizes F_2, set

$dF_2/d\psi = 0$, or

$- a' \sin\delta - 2\psi \left(a/2 + 2b'\right) = 0$

Hence $\psi_{01} = \dfrac{-a'\sin\delta}{a'+4b'}$. (41)

Now define $\mu_1 = F(\phi_0)$, $\phi_0 = \psi_0 - \delta_2/2$

For the second maximum we have

$$\psi_{02} = \frac{-a'\sin(\delta+\pi)}{a'+4b'} = \frac{a'\sin\delta}{a'+4b'} = - \psi_{01} \quad (42)$$

and $\mu_2 = F(\psi_{02})$.

An approximation to I can now be the quantity

$$\pi I_0(\mu_1) + \pi I_0(\mu_2) , \quad (43)$$

where I_0 is the modified Bessel function of order 0 for small values of μ_1 , μ_2 . For large values of μ_1 , μ_2 (>4) we can use equation (38).

(C) If parameters a, b, c, d are all small (<<1) , an approximation can be obtained by expanding the exponent, integrating term by term and dropping the higher order terms:

$$e^F = 1 + \frac{F}{1!} + \frac{F^2}{2!} + \ldots,\ \text{hence}$$

$$I = \int_0^{2\pi} e^F\, d\phi \cong \int_0^{2\pi} (1 + F + \frac{1}{2} F^2)\, d\phi =$$

$$= \int_0^{2\pi} d\phi + a \int_0^{2\pi} \cos\phi d\phi + b \int_0^{2\pi} \sin\phi d\phi + c \int_0^{2\pi} \cos 2\phi d\phi + d \int_0^{2\pi} \sin 2\phi d\phi +$$

$$+ a^2 \int_0^{2\pi} \cos^2\phi d\phi + b^2 \int_0^{2\pi} \sin^2\phi d\phi + c^2 \int_0^{2\pi} \cos^2 2\phi d\phi + d^2 \int_0^{2\pi} \sin^2 2\phi d\phi +$$

$$+ 2ab \int_0^{2\pi} \cos\phi \sin\phi\, d\phi + 2cd \int_0^{2\pi} \cos 2\phi \sin 2\phi\, d\phi + 2ac \int_0^{2\pi} \cos\phi \cos 2\phi\, d\phi +$$

$$+ 2bc \int_0^{2\pi} \sin\phi \cos 2\phi\, d\phi + 2ad \int_0^{2\pi} \cos\phi \sin 2\phi\, d\phi + 2bd \int_0^{2\pi} \sin\phi \sin 2\phi\, d\phi =$$

$$= 2\pi + (a^2 + b^2 + c^2 + d^2) \frac{\pi}{2} \tag{44}$$

Derivation of a Bessel Series Approximation

The first order PDF can be evaluated in terms of an infinite power series of modified Bessel functions. The evaluation is based on the expansion of the cosine terms of the exponential into power series as follows:

$$a'\cos(\phi+\delta_1) = \sum_{n=0}^{\infty} \varepsilon_n (-1)^n I_n(a')\cos(n\phi+n\delta_1) \quad \text{and} \tag{45}$$

$$b'\cos(2\phi+\delta_2) = \sum_{m=0}^{\infty} \varepsilon_m (-1)^m I_m(b')\cos(2m\phi+m\delta_2) \tag{46}$$

where

$\varepsilon_{n(m)} = 1$ for $n(m) = 0$ and

$\varepsilon_{n(m)} = 2$ for $n(m) \geq 1$.

Plugging (45) and (46) in (17):

$$I = \int_0^{2} \sum_{n=0} \sum_{m=0}^{\infty} \varepsilon_n \varepsilon_m \ (-1)^{m+n} \ I_n(a')I_m(b')\cos(n\phi+n\delta_1) .$$

$$\cos(2m\phi+m\delta_2)d\phi \ , \quad \text{or}$$

$$I = \frac{1}{2} \sum_{n_1=1}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \int_0^{2\pi} \varepsilon_m \varepsilon_n (-1)^{m+n} \ I_n(a')I_m(b')$$

$$\cos\big((2m + (-1)^{n_1} n)\phi + (m\delta_2 + (-1)^{n_1} n\delta_1)\big)d\phi \qquad (47)$$

The only terms of the sum in (47) that are not zero are for $n=2m$ and $n_1=1$. Dropping all the terms that are zero eq. (47) yields:

$$I = \pi I_0(a') \ I_0(b') + \pi \sum_{m=1}^{\infty} \varepsilon_m \ \varepsilon_{2m} \ (-1) \ I_m(a') \ I_{2m}(b')$$

$$\cos m(\delta_2 - 2\delta_1) \qquad (48)$$

In most cases, 3 or 4 terms is the sum of eq. (47) are enough to obtain a reasonable estimate of I. However, the evaluation of the modified Bessel functions requires a considerable amount of computational effort.

Examples

1. General case: Quantities $a^2 + b^2$ and $c^2 + d^2$ are of the same order of magnitude. Two numerical examples will be considered:

a) $a = b = c = d = 1$.

Equation (22) becomes:

$8u^4 + 8u^3 - 6u^2 - 6u = 0$, or

$u \ (4u^2 - 3) \ (u + 1) = 0$, whose roots are:

$u_1 = 0$, $u_2 = -1$, $u_3 = \sqrt{3/4}$, $u_4 = -\sqrt{3/4}$

Since $u = \cos\phi$, the corresponding values of ϕ are:

$\phi_1 = \frac{\pi}{2} = 1.5708$, $\phi_2 = \pi = 3.1416$, $\phi_3 = 3.6652$, $\phi_4 = 5.7596$

Equation (21) becomes:

$$32\,u^4 + 16\,u^3 - 30\,u^2 - 12\,u + 3 = 0 \;\rightarrow$$

$$u^4 + 0.5\,u^3 - 0.9375\,u^2 - 0.3750\,u + 0.0938 = 0$$

The third order equation:

$$y^3 - \alpha_2 y^2 + (\alpha_1\alpha_3 - 4\alpha_4)\,y + (4\alpha_2\alpha_4 - \alpha_3^2 - \alpha_1^2\alpha_4) = 0$$

becomes:

$$y^3 + 0.9375\,y^2 - 0.5625y - 0.5156 = 0$$

Hence:

$$Q = \frac{3\beta_2 - \beta_1^2}{9} = -\,0.2852$$

$$R = \frac{9\beta_1\beta_2 - 27\beta_3 - 2\beta_1^3}{54} = 0.1394$$

$$P = Q^3 + R^2 = -\,0.0038\;,\text{ hence}$$

$$\sqrt{D} = i\sqrt{-P} = i\;0.0613$$

$$R + \sqrt{D} = 0.1394 + i\;0.0613$$

$$S = \sqrt[3]{0.1394 + i\;0.0613} =$$

$$= \sqrt[3]{0.1523\quad(\cos 0.4141 + i\sin 0.4141)} =$$

$$= 0.5340\;(\cos 0.1380 + i\sin 0.1380) =$$

$$= 0.5289 + i\;0.0735$$

$$T = \sqrt[3]{0.1394 - i\;0.0613} = 0.5289 - i\;0.0735$$

Thus: $y_1 = S + T - \frac{1}{3}\beta_1 = 0.7454$.

The quadratic equation

$$z^2 + \frac{1}{2}\left\{\alpha_1 \pm \sqrt{\alpha_1^2 - 4\alpha_2 + 4y_1}\right\} z + \frac{1}{2}\left\{y_1 \pm \sqrt{y_1^2 - 4\alpha_4}\right\} = 0$$

becomes:

(1) $Z^2 + 1.5711\ Z + 0.5851 = 0$

with roots: $Z_1 = -\ 0.6008$, $Z_2 = -\ 0.9644$

and (2) $Z^2 - 1.0711\ Z + 0.1602 = 0$

with roots: $Z_3 = 0.1798$, $Z_4 = 0.8914$

Since $\phi_i = \cos^{-1} u_i$, $\phi_i \varepsilon [0,2\pi]$ we have:

$\phi_1 = 0.4704$, $\phi_2 = 2.2228$, $\phi_3 = 3.4092$, $\phi_4 = 4.8932$

and $\mu_1 = F(\phi_1) = 2.7418$, $\mu_2 = F(\phi_2) = -\ 1.0402$

$\mu_3 = F(\phi_3) = 0.1414$, $\mu_4 = F(\phi_4) = -\ 2.0930$

Since the values of μ_i are small (<4) , we use formulas (29) and (30):

$$A_1 = e^{\mu_1} \frac{\rho_1}{2} \sqrt{\frac{\pi}{\mu_1}}\ \mathrm{erf}(\sqrt{\mu_1}) =$$

$$= 15.5149\ \frac{2.0944}{2} \sqrt{\frac{3.1416}{2.7418}}\ \mathrm{erf}\ (1.6558) = 15.6523$$

$$A_2 = \rho_2 \left(\frac{1}{3} + e^{\mu_2}\right) = 1.5708\ (0.3333 + 0.3574) = 1.0850$$

$A_3 = 0.3979$, $A_4 = 0.9563$, and

$I \simeq \sum A_i = 18.0915$

The exact value of I , calculated numerically, is 17.7622, and hence the difference is only - 1.854% .

b) $a = b = c = d = 5$. Equation (21) is the same as in (a) .

Hence $\phi_{01} = 0.4704$ and $\mu_1 = 13.7090$

$F''(\phi_{01}) = -\ 34.6665$.

$\phi_{02} = 3.4092$ and $\mu_2 = 0.7070$

$F''(\phi_{02}) = -\ 21.2595$

Using formula (38) we get $I \sim 382{,}719$.

The exact value of I is 387,820, and hence the error is + 1.315%

2. Special case A: $a = 10$, $b = 1$, $c = 1$, $d = 1$

$a' = \sqrt{a^2+b^2} = 10.0499$, $\delta_1 = \tan^{-1}(-b/d) = -0.0997$

$b' = \sqrt{c^2+d^2} = 1.4142$, $\delta_2 = \tan^{-1}(-d/c) = -0.7854$

$\delta = \delta_2 - 2\delta_1 = -0.5860$

$$\psi_0 = \frac{-b'\sin\delta}{2b'\cos\delta + a'/2} = 0.1060\,, \quad \phi_0 = \psi_0 - \delta_1 = 0.2057$$

$\mu = F(\phi_0) = 11.3099$, $e^\mu = 81{,}685$

$F''(\phi_0) = 15.2591$

Hence $I \cong \cdot e^\mu \cdot \sqrt{\dfrac{2\pi}{F''(\phi_0)}} = 52{,}416$

The exact value is $I = 53{,}361$, and the error $+\ 1.771\%$

3. Special case B: $a^2 + b^2 \ll c^2 + d^2$

Say: $a = 1$, $b = 1$, $c = 1$, $d = 10$

$a' = \sqrt{a^2+b^2} = 1.4142$, $\delta_1 = \tan^{-1}(-b/a) = -0.7854$

$b' = \sqrt{c^2+d^2} = 10.0499$, $\delta_2 = \tan^{-1}(-d/c) = -1.4711$,

$$\delta = \delta_1 - \frac{1}{2}\delta_2 = -0.0498$$

For the first maximum we use eq. (41)

$$\psi_{01} = \frac{-a'\sin\delta}{a'+4b'} = 0.0017\,, \text{ or } \quad \phi_{01} = \psi_{01} - \frac{1}{2}\delta_2 = 0.7373$$

$\mu_1 = 11.4624$, $F''(\phi_{01}) = -41.6118$

For the second maximum

$$\psi_{02} = -\psi_{01} = -0.0017\,, \text{ or } \quad \phi_{02} = \psi_{02} - \frac{1}{2}\delta_2 + \pi = 3.8755$$

$\mu_2 = 8.6371$, $F''(\phi_{02}) = -38.7869$

Using formula (38) we get: $I \sim 39{,}212$.

The exact value of I is 39,702 and hence the error is 1.233%

Further Research

An approximate closed form expression for the joint PDF of the envelope in two distinct points in time is a desirable future research direction. Laplace's method has been applied to integrals depending on two large variables by Fulks [11] and Thomsen [12] and to double and multiple integrals by Hsu [13], [14], [15] and Rooney [16]. If the stationary points of the exponent $F(\phi_1, \phi_2)$ can be analytically located, an approximate closed form expression may be derived.

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