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HYDRODYNAMIC INTERACTIONS BETWEEN SHIPS IN SHALLOW WATER

by

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ABSTRACT

A method is developed for predicting the forces and moments acting on a vessel due to the hydrodynamic interactions between vessels in a passing situation. The lateral separation between the vessels is assumed large so that the vessels are in each other's far field. The speed of the interacting vessels is small so that the free surface effects are neglected. The water depth is assumed small so that the ship is close to the bottom. Two cases are investigated, namely, the case when the ship touches the bottom (two dimensional problem), and the case when there is a slight gap beneath the keel (shallow water problem).

The general problem is solved using the method of matched asymptotic expansions. In the far field, the body and vorticity in the unsteady wake region are represented by a distribution of sources and vortices. As typical of slender body theories, the source strength can be directly related to the body geometry. The vorticity must be determined by the solution of a pair of coupled singular integral equations. The integral equations are solved by a successive approximation scheme. The side forces and yaw moments are calculated using the source and vortex strengths.

Numerical results are presented. Due to the lack of experimental data, comparison with experimental results are made for the two dimensional problem only. Qualitative agreements between the numerical and experimental results are indicated.

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I INTRODUCTION

The motion of ships in close proximity and restricted waters involves serious hydrodynamic interactions. Particular cases of interest are ships in a re-fueling maneuver, ships passing each other, a single ship moving parallel to a canal wall or a channel bank, etc. In all these cases, the hydrodynamic interactions are dangerous and collisions may occur. Such dangers become even more serious in shallow water problems because the magnitude of the interaction forces and moments are substantially larger due to additional restrictions on the flow.

Collatz (1963, cf. Ref. [15]) studied the problem of the hydrodynamic interactions between bodies in two dimensional flow. In his theory, no Kutta conditions were imposed on the trailing ends of the moving bodies. This is justified for blunt bodies such as moving cylinders. However, for extended bodies like 8:1 ellipses, Collatz's prediction of the interaction forces are not satisfactory in the cases of both bodies travelling in tandem and on the moving body in the case of one body passing a fixed one. Nevertheless, Collatz's results on the stationary body show good correlation with Oltmann's experiments (1970, cf. Ref. [14]). As later demonstrated by Tuck, (1974, cf. Ref. [8]) this is probably due to the lack of an appropriate Kutta condition on the trailing edges of the moving extended bodies.

Tuck and Newman (1974, cf. Ref. [8]) developed two theories for predicting the hydrodynamic sway force and yaw moment acting on each of two ships when they are moving along parallel paths. For ships in deep water, they developed a three-dimensional slender body theory in which the first approximation for the solution in the neighborhood of each ship is computed as if the other ship were not present. For ships in shallow water, they solve a two dimensional problem by considering the ships simultaneously with Kutta conditions imposed on the trailing edges of the moving ships. They also give a formulation to handle the cross flow beneath the ship's keel and the bottom. They solved the general three dimensional problem but their theory for two dimensional motion was limited to steady state cases in order to avoid the complication of vortex shedding from the trailing edges of the moving ships.

In this work we shall study the problem of hydrodynamic interactions between two ships in relative motions in shallow water. Because of the unsteady nature of the problem, vortex shedding from the trailing edges of the moving ships must be taken into consideration. However, we will assume that the velocity of the moving ship is small and that the interacting bodies are in each other's far field so that wave effects and the details of the flow around each body can be neglected.

If we consider both ships simultaneously we will inevitably arrive at a pair of coupled integral equations whose kernels are functions of time. This causes tremendous computation effort. Instead, a successive approximation scheme is developed, which consists of a system of integral equations with the same time independent kernel.

The two dimensional problem will be studied first in order to develop the successive approximation scheme and test for convergence. Next, the shallow water problem is investigated. In the shallow water problem, the ship is not touching the bottom, so that there is a flow beneath the keel. The flow beneath the keel is treated in a fashion similar to that formulated by Tuck and Newman (1974, cf. Ref. [8]). The far field description is represented by two dimensional singularities distributed along the center lines of the ships plus wake region. The far field can be analyzed using the technique developed in the two dimensional problem. The near field problem is a stream flow past the ship section in a channel. The strength of the singularities will be determined by matching.

In this work, it was found that the unsteady vortex shed from the moving ship provides an important contribution to the sway force acting on the moving ship. The cross flow under the keel was found to be of paramount importance in the shallow water problem. Numerical calculations indicate that a slight gap alters the flow significantly.

This report is divided into two main sections. The first section deals with hydrodynamic interactions between bodies in two dimensional flow. Numerical results for the case of one body passing a fixed one are presented and comparisons with experiments made. In the second section, cross flow beneath the ship's keel is introduced, and numerical results are presented for the case of one ship passing a stationary one in shallow water.

II HYDRODYNAMIC INTERACTIONS BETWEEN BODIES IN TWO DIMENSIONAL FLOW

(1) The case of a moving body passing a fixed one.

The problem considered here is that of a body moving with a constant velocity U passing a stationary body. The two bodies are assumed to be thin and oriented as shown in Figure 1. The fluid is considered as incompressible and inviscid. We shall use a separate coordinate system on each body. Denote the ith body by a Cartesian coordinate (x_i, y_i) such that $x_i=0$, $y_i=0$ represent the mid-body station and center plane, respectively, with x_i measured forward and y_i to starboard. Body 1 is assumed to be fixed and body 2 moving with a constant velocity U.

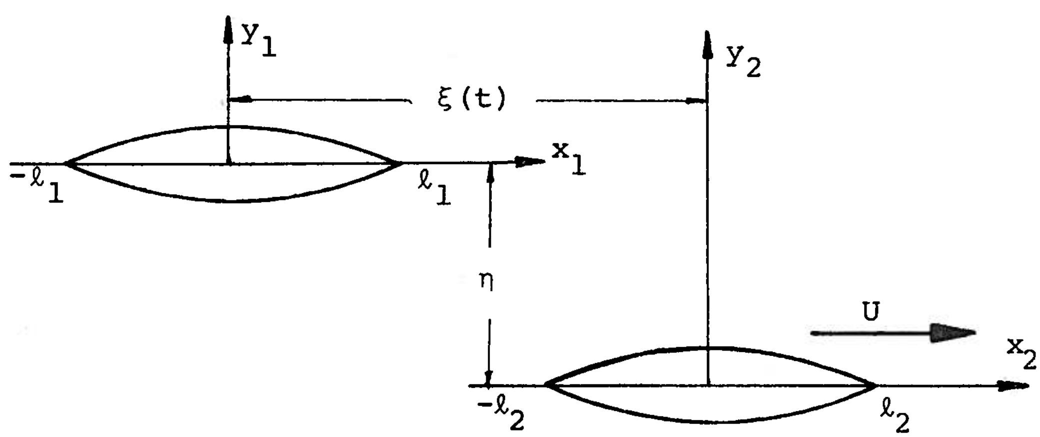


Figure 1. Co-ordinate System

The lateral and longitudinal separations between the bodies are denoted by η and $\xi(t)$, where η is fixed and independent of time, $\xi(t)$ is the "stagger" such that $\xi(0)=0$, i.e.,

$$\xi(t) = x_1 - x_2 = Ut \quad (2.1)$$

For simplicity, we assume the bodies are symmetric about their center-plane, so that

$$y_i = \pm f_i(x_i) \quad , \qquad -\ell_i < x_i < \ell_i \tag{2.2}$$

represents the upper and lower surface of the ith body. Furthermore, $f_i(x_i)$ is assumed to be small compared to ℓ_i and η .

The disturbance velocity potential can be written in a most general form

$$\phi = \frac{1}{2\pi} \sum_{i=1}^{2} \int_{-\ell_i}^{\ell_i} [\sigma_i(\xi,t) \ln\sqrt{(x_i-\xi)^2+y_i^2}$$

$$+ \gamma_i(\xi,t) \tan^{-1} \frac{y_i}{x_i-\xi}] d\xi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{-\ell_2} \gamma_w(\xi,t) \tan^{-1} \frac{y_2}{x_2-\xi} \, d\xi \tag{2.3}$$

Where σ_i is the source strength and γ_i is the vortex strength due to body i, γ_w is the vortex strength of the unsteady wake shedding from the trailing edge of the moving body.

To satisfy the linearized boundary condition on the fixed body, we write:

$$\lim_{y_1 \to \pm 0} \frac{\partial \phi}{\partial y_1} = 0 \qquad -\ell_1 < x_1 < \ell_1 \tag{2.4}$$

Substituting this into (2.3), we obtain

$$0 = \frac{\pm\sigma_1(x_1,t)}{2} + \frac{1}{2\pi} \int_{-\ell_1}^{\ell_1} \frac{\gamma_1(\xi,t)}{x_1-\xi} \, d\xi$$

$$+ \frac{1}{2\pi} \int_{-\ell_2}^{\ell_2} [\sigma_2(\xi,t) \frac{\eta}{(x_2-\xi)^2+\eta^2} + \gamma_2(\xi,t) \frac{x_2-\xi}{(x_2-\xi)^2+\eta^2}] d\xi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{-\ell_2} \gamma_w(\xi,t) \, \frac{x_2-\xi}{(x_2-\xi)^2+\eta^2} \, d\xi$$

(2.5)

where the bar on the integral sign signifies that the integral is taken in a Cauchy's principal-value sense.

From (2.5), it follows that

$$\sigma_1(x_1,t) = 0 \tag{2.6}$$

and that

$$-\!\!\!\!\!\!\int_{-\ell_1}^{\ell_1} \frac{\gamma_1(\xi,t)}{x_1-\xi}\, d\xi + \int_{-\ell_2}^{\ell_2} \gamma_2(\xi,t)\, \frac{x_1-Ut-\xi}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi$$

$$= -\eta \int_{-\ell_2}^{\ell_2} \frac{\sigma_2(\xi,t)\, d\xi}{(x_1-Ut-\xi)^2+\eta^2} - \int_{-\infty}^{-\ell_2} \gamma_w(\xi,t) \frac{x_1-Ut-\xi}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi \tag{2.7}$$

since $x_2=x_1 - Ut$

Similarly, the linearized boundary condition on the moving body gives

$$\lim_{y_2\to\pm 0} \frac{\partial\phi}{\partial y_2} = \mp\, Uf_2'(x_2) \qquad -\ell_2<x_2<\ell_2 \tag{2.8}$$

where the prime denotes differentiation with respect to x_2.

Substituting this and (2.6) into (2.3), we obtain

$$\mp\, Uf_2'(x_2) = \pm\, \frac{\sigma_2(x_2,t)}{2} + \frac{1}{2\pi} -\!\!\!\!\!\!\int_{-\ell_2}^{\ell_2} \frac{\gamma_2(\xi,t)}{x_2-\xi}\, d\xi$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{-\ell_2} \frac{\gamma_w(\xi,t)}{x_2-\xi}\, d\xi + \frac{1}{2\pi} \int_{-\ell_1}^{\ell_1} \gamma_1(\xi,t) \frac{x_1-\xi}{(x_1-\xi)^2+\eta^2}\, d\xi \tag{2.9}$$

It follows that

$$\sigma_2(x_2,t) = -2Uf_2'(x_2) \tag{2.10}$$

and that

$$-\!\!\!\!\!\!\int_{-\ell_2}^{\ell_2} \frac{\gamma_2(\xi,t)}{x_2-\xi}\, d\xi + \int_{-\ell_1}^{\ell_1} \gamma_1(\xi,t)\, \frac{x_2+Ut-\xi}{(x_2+Ut-\xi)^2+\eta^2}\, d\xi$$

$$= - \int_{-\infty}^{-\ell_2} \frac{\gamma_w(\xi,t)}{x_2-\xi}\, d\xi \tag{2.11}$$

by (2.1)

(2.7) and (2.11) are a pair of simultaneous integral equations for the unknowns γ_1 , γ_2. The singular kernel is independent of time, while the others are functions of time. Analytical solutions to such systems are usually impossible to find. It is well known that such a system can be solved by the so-called "time stepping" method. Unfortunately, the time dependent regular kernel makes conventional numerical techniques difficult to apply. The time dependent kernel requires the laborious process of inverting a new coefficient matrix at each time step.

We shall proceed to set up a successive approximation scheme which eliminates the need for solving simultaneously integral equations with time dependent kernels.

Let

$$\gamma_1(\xi,t) = \gamma_1^0(\xi,t) + \gamma_1^1(\xi,t) + \gamma_1^2(\xi,t) + \cdots$$
$$\gamma_2(\xi,t) = \gamma_2^0(\xi,t) + \gamma_2^1(\xi,t) + \gamma_2^2(\xi,t) + \cdots$$
$$\gamma_w(\xi,t) = \gamma_w^0(\xi,t) + \gamma_w^1(\xi,t) + \gamma_w^2(\xi,t) + \cdots \tag{2.12}$$

Putting these into (2.7), (2.11), denoting $\gamma_1^i(\xi,t)$ by γ_1^i, $\gamma_2^i(\xi,t)$ by γ_2^i, etc. and using (2.10), we arrive at

$$\text{(a)} \quad \int_{-\ell_1}^{\ell_1} \frac{\gamma_1^0+\gamma_1^1+\gamma_1^2+\cdots}{x_1-\xi}\, d\xi + \int_{-\ell_2}^{\ell_2} (\gamma_2^0+\gamma_2^1+\gamma_2^2+\cdots)\frac{x_1-Ut-\xi}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi$$

$$= 2U\eta \int_{-\ell_2}^{\ell_2} \frac{f_2'(\xi)}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi - \int_{-\infty}^{-\ell_2} (\gamma_w^0+\gamma_w^1+\gamma_w^2+\cdots)\frac{x_1-Ut-\xi}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi$$

and

(b) $$\int_{-\ell_2}^{\ell_2} \frac{(\gamma_2^0+\gamma_2^1+\gamma_2^2+\ldots)}{x_2-\xi} d\xi + \int_{-\ell_1}^{\ell_1} (\gamma_1^0+\gamma_1^1+\gamma_1^2+\ldots) \frac{(x_2+Ut-\xi)}{(x_2+Ut-\xi)^2+\eta^2} d\xi$$

$$= -\int_{-\infty}^{-\ell_2} \frac{(\gamma_w^0+\gamma_w^1+\gamma_w^2+\ldots)}{x_2-\xi} d\xi$$

(2.13)

(2.13a), (2.13b) can now be rearranged to give the following system

(a) $$\int_{-\ell_1}^{\ell_1} \frac{\gamma_1^0}{x_1-\xi} d\xi = 2U\eta \int_{-\ell_2}^{\ell_2} \frac{f_2'(\xi)}{(x_1-Ut-\xi)^2+\eta^2} d\xi$$

(b) $$\int_{-\ell_2}^{\ell_2} \frac{\gamma_2^0}{x_2-\xi} d\xi = -\int_{-\infty}^{-\ell_2} \frac{\gamma_w^0}{x_2-\xi} d\xi - \int_{-\ell_1}^{\ell_1} \gamma_1^0 \frac{x_2+Ut-\xi}{(x_2+Ut-\xi)^2+\eta^2} d\xi$$

(c) $$\int_{-\ell_1}^{\ell_1} \frac{\gamma_1^1}{x_1-\xi} d\xi = -\int_{-\ell_2}^{\ell_2} \gamma_2^0 \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2} d\xi - \int_{-\infty}^{-\ell_2} \gamma_w^0 \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2} d\xi$$

(d) $$\int_{-\ell_2}^{\ell_2} \frac{\gamma_2^1}{x_2-\xi} d\xi = -\int_{-\infty}^{-\ell_2} \frac{\gamma_w^1}{x_2-\xi} d\xi - \int_{-\ell_1}^{\ell_1} \gamma_1^1 \frac{(x_2+Ut-\xi)}{(x_2+Ut-\xi)^2+\eta^2} d\xi$$

$$\vdots$$

(2.14)

By using the above iteration scheme, we have arrived at a system of integral equations, in which each integral equation contains only one unknown and the kernel is completely independent of time.

Equations (2.14a) and (2.14b) give the first approximations of the scheme. Equations (2.14c) and (2.14d) etc. are the next and higher approximations. Equation (2.14a) is an integral equation for γ_1^0, which is the vorticity generated in the fixed body to counteract the disturbance caused by the moving body. In the first approximation, the moving body is represented as a line distribution of sources causing a disturbance represented by the right hand side of equation (2.14a). Note that the source strength is computed as if the fixed body is not present. As we see in equation (2.14b), γ_1^0 will in turn induce a disturbance on the moving body, causing a distribution of vorticity γ_2^0 along the moving body. Because the problem is unsteady, there is also vorticity in the wake, represented by γ_w^0. γ_2^0 and γ_w^0 will further influence the flow around the fixed body and so on as represented by the second and higher approximations.

From the above system it is obvious that we may consider the hydrodynamic problems for each of the two bodies separately.

(a) The Fixed Body Problem.

From equation (2.13a) of the last section, we obtained the following system of integral equations for the vortex strength γ_1 of the fixed body:

$$\text{(a)} \quad \int_{-\ell_1}^{\ell_1} \frac{\gamma_1^0}{x_1-\xi} d\xi = 2U\eta \int_{-\ell_2}^{\ell_2} \frac{f_2'(\xi) \, d\xi}{(x_1-Ut-\xi)^2+\eta^2}$$

$$\text{(b)} \quad \int_{-\ell_1}^{\ell_1} \frac{\gamma_1^1}{x_1-\xi} d\xi = -\int_{-\ell_2}^{\ell_2} \gamma_2^0 \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2} d\xi - \int_{-\infty}^{-\ell_2} \gamma_w^0 \frac{x_1-Ut-\xi}{(x_1-Ut-\xi)^2+\eta^2} d\xi$$

(c)
$$\int_{-\ell_1}^{\ell_1} \frac{\gamma_1^2}{x_1-\xi}d\xi = -\int_{-\ell_2}^{\ell_2} \gamma_2^1 \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi - \int_{-\infty}^{-\ell_2} \gamma_w^1 \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2}d\xi$$

$$\vdots \qquad (2.15)$$

For the time being, assume the right hand sides of equations (2.15) are known. These equations are singular integral equations of the Cauchy type, and the singular operator is commonly known as a finite Hilbert Transform. Solutions to such equations are not unique, so we must provide more information on the circulation conditions and conditions at infinity. We impose the following conditions on each of the γ_1^i's .

1) There is no net circulation generated on the fixed body.

2) The disturbance must vanish at infinity

The first condition is justified because if the total circulation on the fixed body is other than zero at any instant , Kelvin's theorem requires that there must be vorticity in the fluid. Since there is nothing to convect the vorticity detached from the fixed body, the total circulation must be zero there at all times.

The solution to the integral equation satisfying these conditions is unique, and we may use the inversion formulas (see Appendix A) to write the solution to equation (2.15) directly as

$$\gamma_1^0(x_1,t) = \frac{2U\eta}{\pi^2\sqrt{\ell_1^2-x_1}} \int_{-\ell_1}^{\ell_1} \frac{\sqrt{\ell_1^2-\zeta^2}}{\zeta-x_1}d\zeta \int_{-\ell_2}^{\ell_2} \frac{f_2'(\xi)}{(\zeta-Ut-\xi)^2+\eta^2}\, d\xi \qquad (2.16)$$

and

$$\gamma_1^i(x_1,t) = \frac{2U\eta}{\pi^2\sqrt{\ell_1^2-x_1}} \int_{-\ell_1}^{\ell_1} \frac{\sqrt{\ell_1^2-\zeta^2}}{\zeta-x_1}\, g_i(\zeta,t)\,d\zeta \qquad (2.17)$$

where

$$g_i(x_1,t) = \int_{-\ell_2}^{\ell_2} \gamma_2^{i-1} \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi + \int_{-\infty}^{-\ell_2} \gamma_w^{i-1} \frac{(x_1-Ut-\xi)}{(x_1-Ut-\xi)^2+\eta^2}\, d\xi$$

$$i=1,2,3,\ldots$$

Note that the vortex strength is singular at both ends.

(b) The Moving Body Problem.

From (2.13), we arrived at the following system of integral equations for the vortex strength of the moving body.

$$\int_{-\ell_2}^{\ell_2} \frac{\gamma_2^i(\xi,t)}{x_2-\xi}\, d\xi = -\int_{-\infty}^{-\ell_2} \frac{\gamma_w^i(\xi,t)}{x_2-\xi}\, d\xi - \int_{-\ell_1}^{\ell_1} \gamma_1^i(\xi,t) \frac{x_2+Ut-\xi}{(x_2+Ut-\xi)^2+\eta^2}\, d\xi$$

$$i=0,1,2,\ldots \qquad (2.18)$$

In the above equations, the second term on the right-hand side can be identified as the cross flow induced on the moving body due to the presence of the fixed body; γ_w^i is the unsteady vorticity shed from the trailing edge of the moving body.

At this point, the only known part of the integral equation (2.18) is the second term in the right-hand side. Naturally, we need more conditions in order to obtain the solution. The following conditions still remain to be satisfied.

1) Pressure must be continuous across the wake.
2) The Kutta condition must be satisfied on the trailing edge, i.e., the velocity must be finite there.
3) The total circulation in an ideal fluid must remain constant by Kelvin's Law.

First, consider the condition on the vortex wake. Denoting (x_2,y_2) by (x,y), define

$$\phi_w^i(x,0,t) = \frac{1}{2}\int_x^{-\ell_2} \gamma_w^i(\xi,t)\,d\xi$$

$$-\infty < x < -\ell_2 \qquad (2.19)$$

From the linearized Bernoulli equation, the pressure must be continuous across the wake region.

$$\frac{P_w{}^i}{\rho} = \left(\frac{\partial \phi_w{}^i}{\partial t} - U\,\frac{\partial \phi_w{}^i}{\partial x}\right)^{\pm} = 0$$

$$y=0,\quad -\infty<x<-\ell_2 \qquad (2.20)$$

where the + and - signs denote that the quantity in the bracket is being evaluated on the upper ..d lower side of the vortex wake.

Upon differentiating with respect to x equation (2.20) and using (2.19), we obtain

$$\left(\frac{\partial}{\partial t} - U\frac{\partial}{\partial x}\right)\gamma_w^i = 0 \qquad -\infty<x<-\ell_2 \qquad (2.21)$$

This is a one dimensional wave equation. The solution to this partial differential equation is

$$\gamma_w^i(x,t) = f^i(x+Ut) \equiv \gamma_w^i(x+Ut)$$

$$-\infty<x<-\ell_2 \qquad (2.22)$$

Physically, this indicates that the vorticity is swept downstream by the flow -U, and remains constant in this reference frame.

Dropping superscripts, each equation of (2.18) takes the form

$$\int_{-\ell_2}^{\ell_2} \frac{\gamma_2(\xi,t)}{x-\xi}d\xi = -\int_{-\infty}^{-\ell_2} \frac{\gamma_w(\xi+Ut)}{x-\xi}d\xi - V(x,t)$$

$$-\ell_2<x<\ell_2 \qquad (2.23)$$

where $V(x,t)$ is a known function.

A particular solution to (2.23) satisfying the Kutta condition can be written as (cf. Appendix A.b):

$$\gamma_2 = \frac{1}{\pi^2}\sqrt{\frac{\ell_2+x}{\ell_2-x}}\left[\pi\int_{-\infty}^{\ell_2}\frac{\gamma_w(\xi+Ut)}{x-\xi}\sqrt{\frac{\xi-\ell_2}{\xi+\ell_2}}\,d\xi - \int_{-\ell_2}^{\ell_2}\sqrt{\frac{\ell_2-\xi}{\ell_2+\xi}}\,\frac{V(\xi,t)}{\xi-x}d\xi\right]$$

$$-\ell_2<x<\ell_2 \qquad (2.24)$$

Invoking Kelvin's theorem that the total circulation taken along a close contour which includes both the body and the wake must be zero leads to the following integral equation for γ_w (cf. Appendix A.b):

$$\int_{-\infty}^{-\ell_2} \sqrt{\frac{\xi-\ell_2}{\xi+\ell_2}}\,\gamma_w(\xi+Ut)\,d\xi = \frac{1}{\pi}\int_{-\ell_2}^{\ell_2} \sqrt{\frac{\ell_2-\xi}{\ell_2+\xi}}\, V(\xi,t)\,d\xi \tag{2.25}$$

Upon a change of variables to $z=\xi+Ut$, we obtain a Volterra integral equation:

$$\int_{-\infty}^{Ut-\ell_2} \sqrt{\frac{z-Ut-\ell_2}{z-Ut+\ell_2}}\quad \gamma_w(z)\,dz = F(t) \tag{2.26}$$

where F(t) is a known function depending only on $\gamma_1^i(x,t)$ from section Ia.

(2) <u>The case of both bodies moving with constant velocities that may differ</u>.

When both of the bodies are in relative motion, unsteady vortex shedding from both trailing edges will occur. Here, we confine our interest to relative motions in which the bodies are travelling along parallel paths.

The "stagger" $\xi(t)$, in equation (2.1) must now be written as

$$\xi(t) = (U_1 - U_2)t \tag{2.27}$$

where U_1 is the velocity of body 1.

The disturbance velocity potential now takes the form

$$\phi = \frac{1}{2\pi}\sum_{i=1}^{2}\Big[\int_{-\ell_i}^{\ell_i}[\sigma_i(\xi,t)\,\ell n\sqrt{(x_i-\xi)^2+y_i^2} + \gamma_i(\xi,t)\tan^{-1}\frac{y_i}{x_i-\xi}]d\xi$$
$$+ \int_{-\infty}^{-\ell_i}\gamma_{wi}(\xi,t)\tan^{-1}\frac{y_i}{x_i-\xi}\,d\xi\Big] \tag{2.28}$$

where γ_{wi} is the unsteady vortex shedding from the trailing edge of body i.

After satisfying the body boundary conditions on each body, and applying the same successive approximation scheme as described in section II.(1), we arrive at the following system of integral equations.

$$\int_{-\ell_i}^{\ell_i} \frac{\gamma_i^0}{x_i-\xi} d\xi = -\int_{-\infty}^{-\ell_i} \frac{\gamma_{wi}^0}{x_i-\xi} d\xi + 2U_j\eta_i \int_{-\ell_j}^{\ell_j} \frac{r_j'(\xi)}{(x_i-Vt-\xi)^2+\eta^2}\, d\xi \tag{2.29}$$

and

$$\int_{-\ell_i}^{\ell_i} \frac{\gamma_i^k}{x_i-\xi} d\xi = -\int_{-\infty}^{-\ell_i} \frac{\gamma_{wi}^k}{x_i-\xi} d\xi - \int_{-\ell_j}^{\ell_j} \gamma_j^{k-1} \frac{x_i-Vt-\xi}{(x_i-Vt-\xi)^2+\eta^2}\, d\xi$$

$$- \int_{-\infty}^{-\ell} \gamma_{wj}^{k-1} \frac{x_i-Vt-\xi}{(x_i-Vt-\xi)^2+\eta^2}\, d\xi \tag{2.30}$$

where: $i,j = 1,2 \quad ; \; i \neq j$

$k = 1,2,3,\ldots$

$V = U_i - U_j$

and $\eta_i = -\eta_j$ is the lateral separation between the bodies.

In the above system of integral equations, the second term on the right hand side of equation (2.29) (the first approximation) is the cross flow on body i induced by a distribution of sources which generate the other body in the absence of body i. γ_i^0 is the vortex strength on body i such that the linearized body boundary conditions are satisfied. γ_{wi}^0 is the unsteady vortex shedding from body i due to the nonuniform flow encountered. Likewise, each equation of (2.30) is an integral equation for the vortex strength (γ_i^k, γ_{wi}^k) required to satisfy the linearized body boundary condition on body i due to a known disturbance (γ_j^{k-1}, γ_{wj}^{k-1}) from the other moving body.

It is obvious that if we let U_i tend to zero, then γ_{wj} vanishes, and the system of equations in section II.(1) is retrieved. Each of the equations in (2.29) and (2.30) takes the same form as equation (2.23), and can be treated in exactly the same manner.

(3) Force and Moment on Each Body.

Once the various sources and vortex strengths are obtained, the pressure on body i can be calculated and hence the force and moment acting on the body can be obtained by integration.

From the Bernoulli equation, the pressure on body i is given by

$$\frac{P}{\rho} = -\left[\frac{\partial \phi}{\partial t} + \frac{1}{2}(\nabla\phi)^2 + C\right] \tag{2.31}$$

where C is a constant.

Note that in the coordinate system fixed on body i, i=1,2 the time derivative must be written as $(\partial/\partial t - U_i \partial/\partial x_i)$.

If we denote the potentials due to the source and vortex distributions on body i and the vortex distribution in the wake of body i by ϕ_i^s, ϕ_i^v, ϕ_i^w, respectively, equation (2.31) can be written

$$\frac{P}{\rho} = -\sum_{j=1}^{2}\left[\frac{\partial}{\partial t}(\phi_j^s+\phi_j^v+\phi_j^w) + \frac{1}{2}\left(\frac{\partial}{\partial x}(\phi_j^s+\phi_j^v+\phi_j^w)\right)^2 + \frac{1}{2}\left(\frac{\partial}{\partial y}(\phi_j^s+\phi_j^v+\phi_j^w)\right)^2\right] + C \tag{2.32}$$

The force acting on body i can thus be obtained by integrating the pressure on the upper and lower surface of the slit $-\ell_i<x_i<\ell_i$, or

$$F_i(t) = \rho\int_{-\ell_i}^{\ell_i}[P]^{\pm}\,dx \tag{2.33}$$

The + and - sign indicates that the contour of integration is taken along $y_i=\pm 0$. Hence, only those terms in (2.32) that are not continuous across $y_i=0$, $-\ell_i<x_i<\ell_i$ contribute to the force on body i. These terms are

$$\frac{\partial}{\partial t}\phi_i^v, \quad \frac{\partial}{\partial x}\phi_i^v \left[\frac{\partial}{\partial x}(\phi_i^s+\phi_i^w+\phi_j^s+\phi_j^v+\phi_j^w)\right]$$

and

$$\frac{\partial}{\partial y}\phi_i^s \left[\frac{\partial}{\partial y}(\phi_i^v+\phi_i^w+\phi_j^s+\phi_j^v+\phi_j^w)\right] \tag{2.34}$$

Here, we consider only the linear terms in the Bernoulli equation.

In a coordinate system fixed on the body i moving with velocity U_i in the positive x direction, the pressure integral appears as

$$F_i(t) = \rho \int_{-\ell_i}^{\ell_i} \left[\frac{\partial \phi_i^v}{\partial t} - U_i \frac{\partial \phi_i^v}{\partial x_i}\right]^{\pm} dx_i \tag{2.35}$$

Dropping subscripts and superscripts, noting that the horizontal velocity component $u(x)$ on the upperside of the boundary surface due to a vortex distribution $\gamma(x)$ is given by

$$u(x) = -\frac{1}{2}\gamma(x) \tag{2.36}$$

and that

$$u = \frac{\partial \phi}{\partial x}, \quad x<\ell \quad ; \quad u=0, \quad x>\ell,$$

we can write

$$\phi(x,t) = \frac{1}{2}\int_x^{\ell} \gamma(\xi,t)\, d\xi, \qquad y=0^+ . \tag{2.37}$$

Thus, the force from equation (2.35) becomes

$$F(t) = \rho \int_{-\ell}^{\ell} \left[\int_x^{\ell} \frac{\partial}{\partial t}\gamma(\xi,t)d\xi + U\,\gamma(x,t)\right] dx \tag{2.38}$$

Using equation (2.24) and integrals from appendix D, after some algebraic manipulations, we arrive at the following (cf. Newman [1]):

$$F(t) = \rho U \int_{-\infty}^{-\ell} \frac{\xi}{\sqrt{\xi^2-\ell^2}} \gamma_w(\xi+Ut)d\xi - \frac{\rho}{\pi} \int_{-\ell}^{\ell} \sqrt{\ell^2-\xi^2} \; \frac{\partial}{\partial t} V(\xi,t)\; d\xi \tag{2.39}$$

Performing a similar analysis for the moment integral

$$M(t) = \rho \int_{-\ell}^{\ell} \left[\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x}\right]^{\pm} xdx \quad , \tag{2.40}$$

we arrive at (cf. Newman [1])

$$M(t) = \frac{\rho U}{2} \int_{-\infty}^{-\ell} \frac{\gamma_w(\xi+Ut)}{\sqrt{\xi^2-\ell^2}} d\xi - \frac{\rho U}{\pi} \int_{-\ell}^{\ell} \sqrt{\ell^2-\xi^2} \; V(\xi,t)\; d\xi$$

$$- \frac{\rho}{2\pi} \int_{-\ell}^{\ell} \xi \sqrt{\ell^2-\xi^2} \; \frac{\partial}{\partial t} V(\xi,t)\; d\xi \tag{2.41}$$

In the above analysis, we observe that once the vorticity in the unsteady wake is determined from the Volterra integral equation (2.26), the force and moment acting on the body can be readily found. The first terms on the right hand sides of equations (2.38) and (2.39) can be regarded as the effect of the unsteady wake on force and moment acting on the body.

In the case of the fixed body, we have $U=0$, therefore, the force and moment is simply given by

$$F(t) = -\frac{\rho}{\pi} \int_{-\ell}^{\ell} \sqrt{\ell^2-\xi^2} \; \frac{\partial}{\partial t} V(\xi,t)\; d\xi \tag{2.42}$$

and

$$M(t) = -\frac{\rho}{2\pi}\int_{-\ell}^{\ell} \xi\sqrt{\ell^2-\xi^2}\ \frac{\partial}{\partial t} V(\xi,t)\ d\xi \tag{2.43}$$

Particularly, since $V(\xi,t)$ is the cross-flow induced on the fixed-body surface by the singularity distributed over the moving body and its wake, we write

$$V(\xi,t) = 2\pi\ \frac{\partial}{\partial y}\ [\phi_2^s+\phi_2^v+\phi_2^w] \tag{2.44}$$

We shall see later a better way to evaluate the contributions from ϕ_2^s and ϕ_2^v to the linearized force on the fixed body is the application of the "Chertock formulas." Meanwhile, let us consider ϕ_2^w , i.e., the effect of the unsteady wake trailing behind the moving body on the fixed one. In the coordinate system on the fixed body,

$$\left.\frac{\partial}{\partial y}\right|_{y=0} = \frac{1}{2\pi}\int_{-\infty}^{-\ell_2} \gamma_w(\xi+Ut)\ \frac{x-Ut-\xi}{(x-Ut-\xi)^2+\eta^2}d\xi \tag{2.45}$$

If we denote the force in (2.42) due to ϕ_2^w by $F_w(t)$, and substitute (2.45) into (2.42), we find, after change of order of integration, that

$$F_w(t) = -\frac{\rho}{\pi}\int_{-\infty}^{-\ell_2} \frac{\partial}{\partial t}\ [\gamma_w(\xi+Ut)\cdot g\ (\xi+Ut)]d\xi \tag{2.46}$$

where

$$g(x+Ut) = \int_{-\ell_1}^{\ell_1} \sqrt{\ell_1^2-\xi^2}\ \frac{x+Ut-\xi}{(x+Ut-\xi)^2+\eta^2}\ d\xi\quad . \tag{2.47}$$

Noting that

$$\frac{\partial}{\partial t}\ f(\xi+Ut) = U\ \frac{\partial f}{\partial \xi}\ (\xi+Ut) \tag{2.48}$$

Upon integration by parts, and using the fact that γ_w and g vanish at ∞, (2.46) yields:

$$F_w(t) = -\rho\ \frac{U}{\pi}\ \gamma_w(-\ell_2+Ut)g(-\ell_2+Ut) \tag{2.49}$$

(4) Numerical results and discussions on two dimensional theory

(a) Numerical results

For the sake of simplicity in computations and experiments to be performed in the future, we have used an idealized "parabolic" representation for the body shape. The two bodies are taken to be identical and assume the following body contour

$$f(x) = \begin{cases} \frac{b}{\ell^2}(\ell^2 - x^2) & |x| < \ell \\ 0 & |x| > \ell \end{cases} \qquad (2.50)$$

where b = half-beam
ℓ = half-length

The length to beam ratio is taken to be $\ell/b=8$.

Here, numerical results are obtained for the case of a body passing a fixed one with constant velocity in a parallel path.

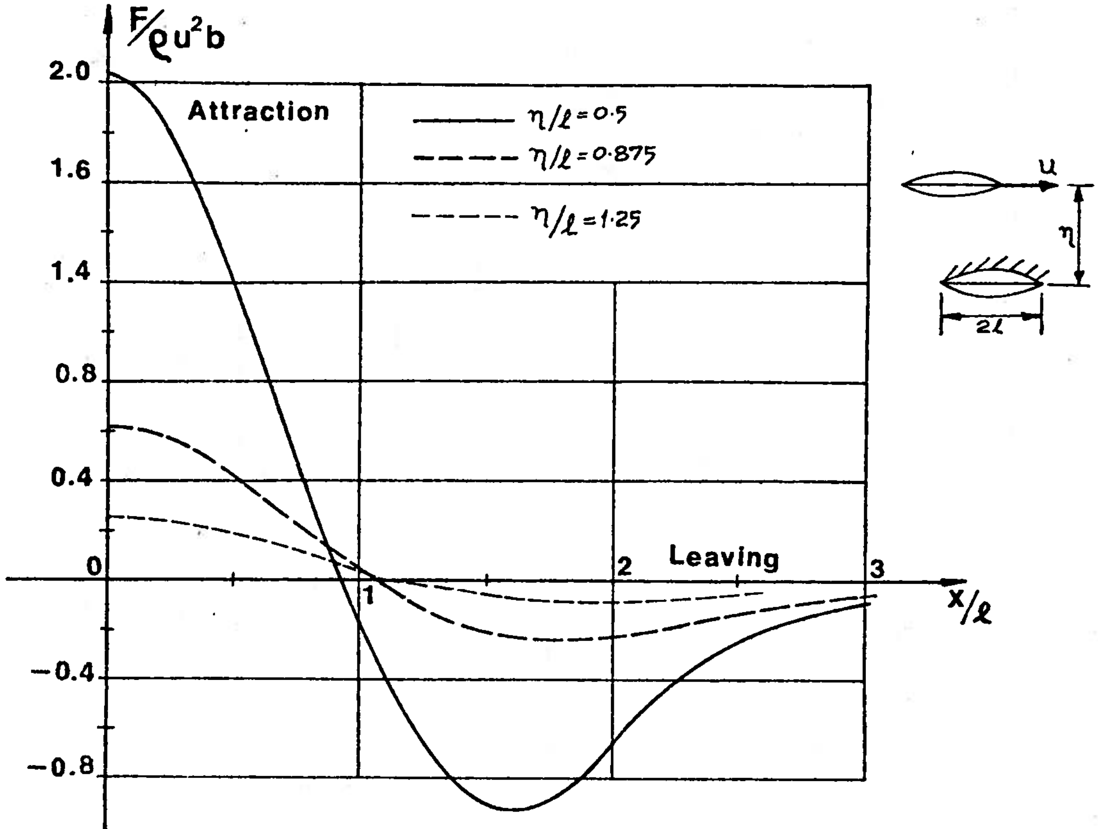
It was found that the successive approximation scheme converged quite fast. For the case of lateral separation $\eta=.625\ell$ (ℓ is the half-length of the bodies), the second approximation (γ_i^1 , $i=1,2$) for the vorticies are within 5% of the first approximation (γ_i^0 , $i=1,2$). Although we expect that the contributions from higher approximations are larger when the lateral separation of the bodies becomes closer.

The force acting on the fixed body due to the influence of the vortex wake trailing behind the moving body (cf. equation (2.49)) reaches its maximum when the moving body begins to leave, i.e. the "stagger" becomes negative, however, this is negligible compared to the force resulting from the vortex distribution along the moving body. Therefore, the force acting on the fixed body, which results dominantly from the first approximation of the vorticities on the moving body, is symmetric about zero "stagger", while the moment is an odd function of the "stagger".

Figures 4, 5 show the forces and moments acting on the fixed body (cf. equation (2.42) and (2.43)) obtained from the first approximation (γ_1^0) for various lateral separations. There is a slight repulsion when the other body is approaching, this is followed by a strong attraction force which reaches its maximum when the moving body is side by side. The force becomes repulsive again when the stern of the moving body passes the bow of the fixed one. As far as the moment is concerned, the stern of the fixed body tends to move away as the other body is approaching. When the bow of the moving body reaches the mid-section of the fixed body, a stronger moment tends to turn the stern of the fixed body towards the moving one. This phenomenon is reversed after the mid sections of both bodies meet. Both the force and moment acting on the fixed body vanish rapidly as the lateral separation between the bodies increases.

The force and moment acting on the moving body differs drastically in nature from that of the fixed body. The vortex wake trailing behind the moving body provides a crucial effect on the force and moment acting on the moving body. As shown in figure 6, without the implementation of the Kutta condition at the trailing edge, there will be no vortex wake and the force acting on the moving body (cf. equation (2.39)) would be symmetrical about zero stagger. However, once a Kutta condition is satisfied at the trailing edge, the vortex shed will continue to influence the force and moment at subsequent time. In this case, as shown in figure 6, the wake effect becomes substantial after the mid-sections of the bodies meet. This is reasonable since the cross flow induced on the moving body reaches the maximum when the bodies are side-by-side. In general, the vortex wake increases the attraction forces acting on the moving body during the period of interaction.

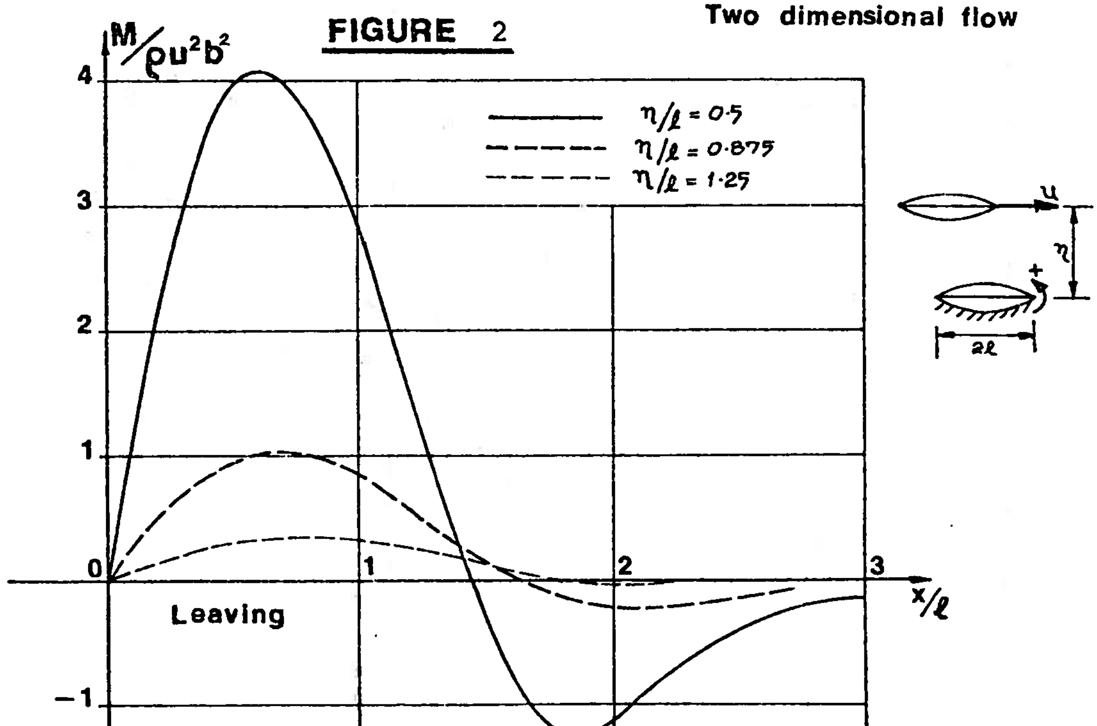
In figures 7 and 8, the total forces and moments acting on the moving body predicted by the calculations are



FORCE ON FIXED BODY AT VARIOUS LATERAL SEPARATION

Two dimensional flow

FIGURE 2



MOMENT ON FIXED BODY AT VARIOUS LATERAL SEPARATION

FIGURE 3

Two dimensional flow

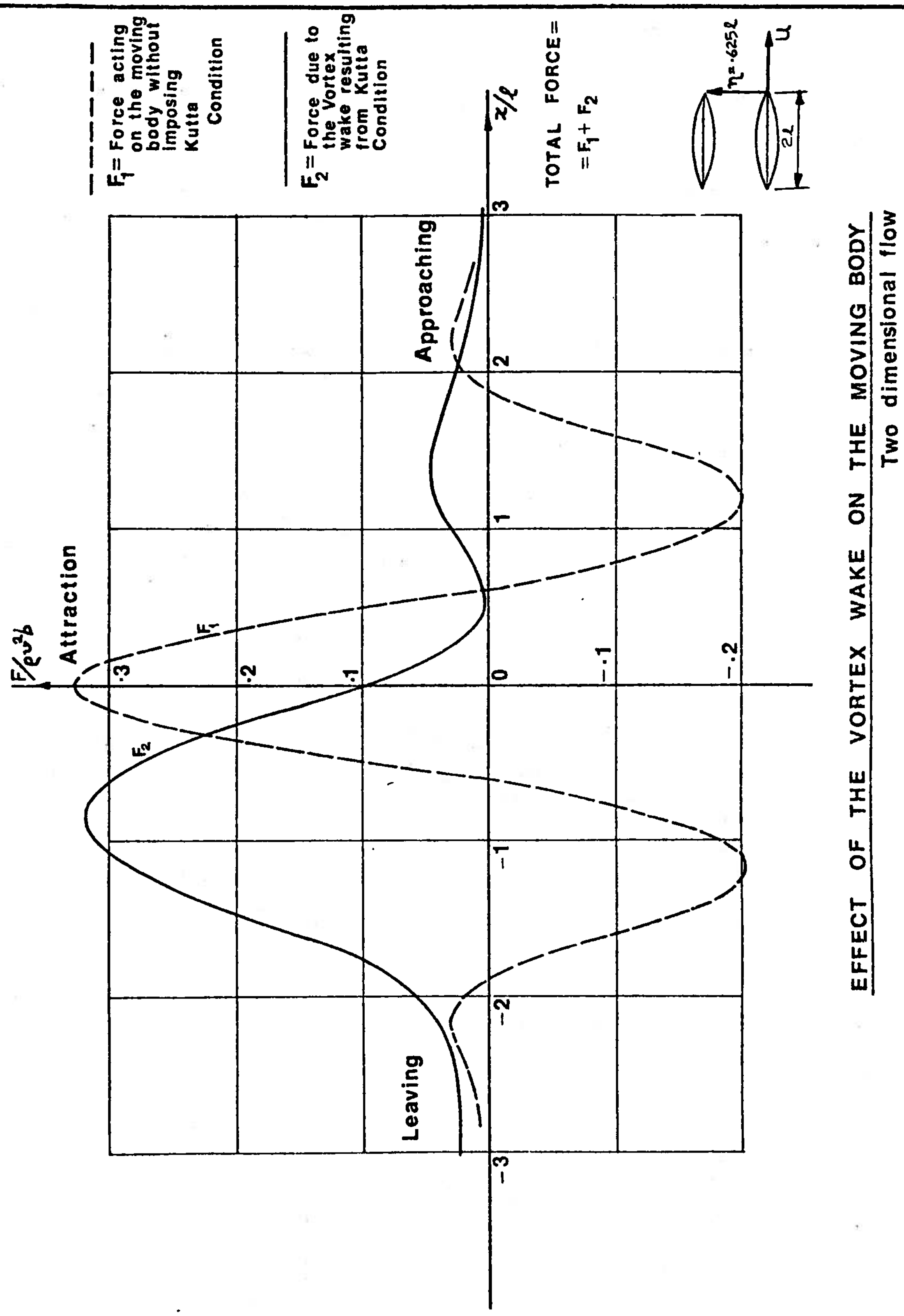
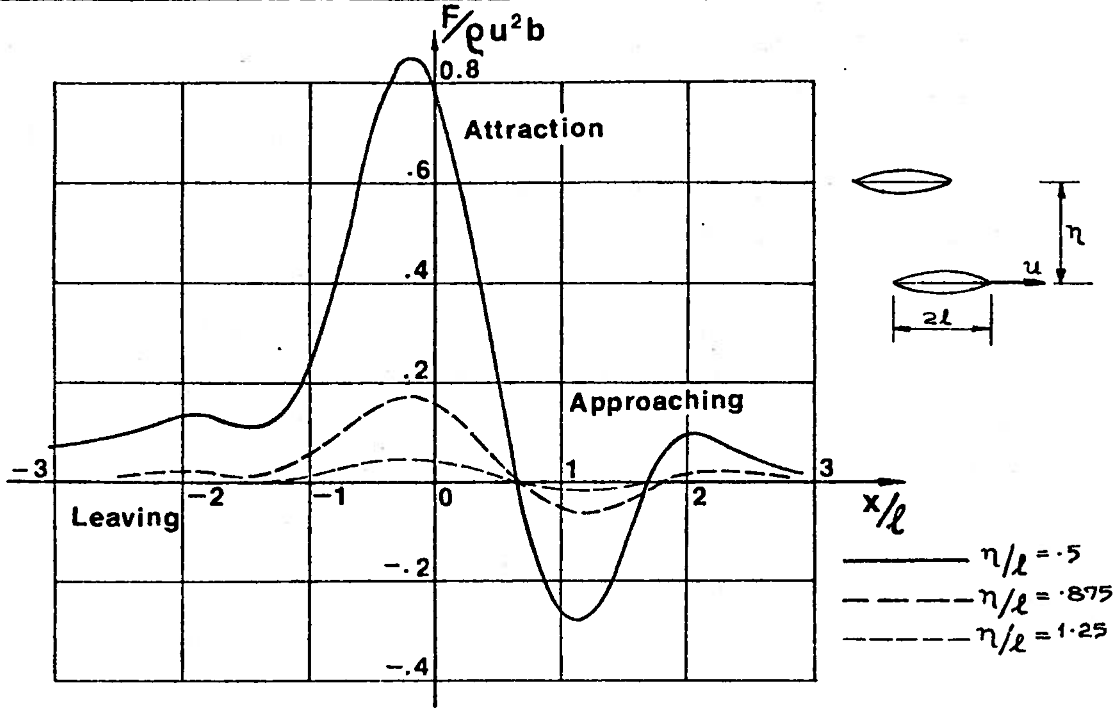

F₁ = Force acting on the moving body without imposing Kutta Condition
F₂ = Force due to the Vortex wake resulting from Kutta Condition
TOTAL FORCE = = F₁ + F₂
η = .625ℓ
U
2ℓ
x/ℓ
F/ρv²b
Attraction
Approaching
Leaving
F₁
F₂
3
2
1
0
−1
−2
−3
·3
·2
·1
−.1
−.2
EFFECT OF THE VORTEX WAKE ON THE MOVING BODY
Two dimensional flow

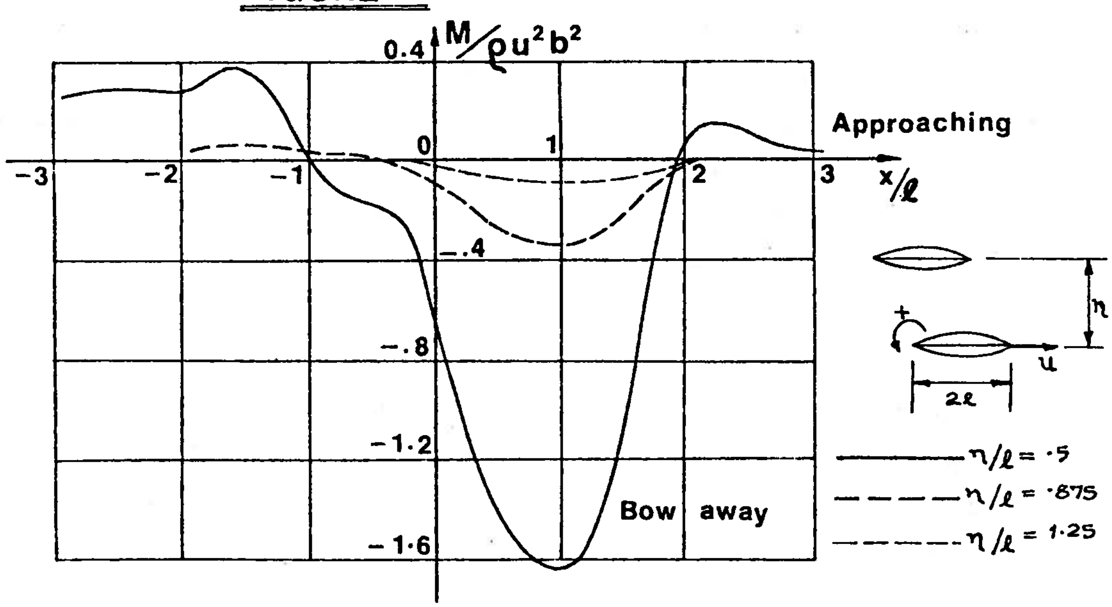
FIGURE 4



FORCE ON MOVING BODY AT VARIOUS LATERAL SEPARATIONS

Two dimensional flow

FIGURE 5



MOMENT ON MOVING BODY AT VARIOUS LATERAL SEPARATIONS

Two dimensional flow

FIGURE 6

plotted against the stagger for various lateral separations. The force is slightly attractive, then becomes repulsive as the bow approaches the stern of the fixed body. As the bow passes, a stronger attraction force occurs and the force maintains attraction over the rest of the interaction period. A slight moment tends to turn the bow towards the stern of the fixed body as the bodies approach one another. As the fixed body is passed, a larger negative moment develops. Finally, the moment again obtains a small positive value as the two bodies separate. As with the fixed body, both forces and moments acting on the moving body diminish quickly as the lateral separation between the bodies is increased.

(b) Comparison with results of Collatz and Oltmann.

Collatz (1963, cf. Ref. [15]) studied the problem of hydrodynamic interactions between bodies in two-dimensional flow. The interacting bodies in his work were elliptical (8:1 ellipses). In the Collatz theory, sources were distributed on the surface of each body. The source strengths were obtained by solving simultaneous equations which result from satisfying the boundary conditions on each body at the same time. In addition, no Kutta conditions, and hence no complication of a vortex wake, were imposed on the moving bodies. The forces and moments acting on the bodies were obtained by integrating the complete Bernoulli equation (including the non-linear terms) along the body surfaces.

Experiments have been performed by Oltmann [16] to check against Collatz's analytical work . In Oltmann's experiments, a long elliptical cylinder was towed parallel to its major axis passed an identical fixed one. Two sets of measurements were taken. The set named "shallow submergence" was taken on a strip near the free surface, another set named "deep submergence" was taken on a strip further down in the fluid. The speed of the moving elliptical cylinder also varied. One set of trails were performed at low speeds (Froude number based on length

$Fn\sim.14-.2$) while the other trails were ran at high speeds (Froude number based on length $Fn\sim.5$). In Oltmann's experiments, measurements were also taken for the steady case of two cylinders moving with the same speed.

In the case of one elliptical body passing another fixed one, if the speed is low ($Fn\sim.14$), Collatz's theoretical predictions for the force and moment (which are even and odd functions of the stagger) acting on the fixed body agreed with Oltmann's experiments extremely well. However, Collatz's prediction of the force and moment acting on the moving body deviated from Oltmann's experiments, particularly after the two bodies were side by side. This is probably due to the lack of a vortex wake trailing behind the moving body. Unfortunately for high speeds ($Fn\sim.5$), the analytical work does not agree with the experiments. One reason pointed out by Oltmann was that the wave and vibration of the towing carriage affected the experiments.

In order to compare the present theory with the above work , we calculate the force acting on the fixed elliptical body. For a lateral separation of $\eta=.625\ell$, the force acting on the fixed body (which is modelled as a flat plate of zero thickness in the present theory) obtained from the first approximation of the scheme is about only 60% everywhere of Collatz's (hence Oltmann's) result. However, this comparison is not valid.

Firstly, for a blunt body like the ellipse, the slender body approximation $\sigma=-2Uf'(x)$ for the source distribution along the center line of the moving body is not valid. The source strength given by the slender body approximation for an ellipse is

$$\sigma(x) = -2U\,\frac{b}{\ell}\,\frac{x}{\sqrt{\ell^2-x^2}} \qquad \text{for} \qquad |x|<\ell \tag{2.51}$$

On the other hand, the exact representation for the source strength corresponding to an ellipse moving parallel to its major axis in an otherwise undisturbed fluid is (cf. Milne-Thomson [3]):

$$\sigma(x) = -2U \frac{b}{(\ell-b)} \frac{x}{\sqrt{c^2-x^2}} \qquad \text{for} \quad |x|<c = \sqrt{\ell^2-b^2} \tag{2.52}$$

A rough estimate between equations (2.51) and (2.52) for the case of $\ell/b=8$ amounts to a difference of about 14%. i.e., Our result will improve by 14% merely by using the exact representation for the source strength.

Secondly, throughout our previous analysis, the thickness (other than assumed to be thin) of the fixed body plays no role in the force. The result would be the same if we replace it by a flat plate. To calculate the forces acting on a fixed body, the application of the "Chertock Formulas" provides a great convenience. For this matter, let ϕ_I represent the flow disturbance (the incident flow) caused by the singularity (source, vortex, etc.) in the absence of the fixed body. The linearized sway force acting on the body can be written as (cf. Ogilvie [2]):

$$F(t) = \rho \oint_{B_1} \left\{ \frac{\partial\psi}{\partial n} \frac{\partial\phi_I}{\partial t} - \psi \frac{\partial}{\partial n} \left(\frac{\partial\phi_I}{\partial t} \right) \right\} d\ell \tag{2.53}$$

where B_1 is the body contour of the fixed body and $\partial/\partial n$ denotes the directional derivative normal to the body surface.

ψ in the above equation is the potential as if the fixed body is moving with unit speed in the y direction, which satisfies the following

$$\begin{aligned} &\nabla^2\psi = 0 && \text{outside } B_1 \\ &(x^2+y^2)\psi < \infty && \text{at infinity} \\ &\frac{\partial\psi}{\partial n} = n_y && \text{on } B_1 \end{aligned} \tag{2.54}$$

where n_y is the y component of the inward unit vector on B_1.

It can be easily shown that if the incident flow is that of a source σ at a position $x=Ut$, $y=-\eta$ moving with velocity U , the force (2.53) is given by:

$$F(t) = \rho U \sigma \frac{\partial \psi}{\partial x} (Ut,-\eta) \tag{2.55}$$

and that of a vortex γ by

$$F(t) = \rho U \gamma \frac{\partial \psi}{\partial y} (Ut,-\eta) \tag{2.56}$$

The sway force on the fixed body due to the singularities distributed on the moving body can then be written as

$$F(t) = \rho U \int_{-\ell_2}^{\ell_2} \left[\sigma_2(\xi) \frac{\partial \psi}{\partial x} (\xi+Ut,-\eta) + \gamma_2(\xi,t) \frac{\partial \psi}{\partial y} (\xi+Ut,-\eta)\right] d\xi \tag{2.57}$$

The unknown ψ in this equation depends only on the shape of the fixed body. Once the body contour is given, the force can be readily determined. For the present case, to evaluate the sway force on an ellipse due to some unsteady disturbances, the ψ problem corresponds to the potential for the ellipse moving parallel to its minor axis with unit velocity. The solution to such problem can be written as (cf. Milne-Thomson [3])

$$\frac{\partial \psi}{\partial x} = \text{Re} \left[i \frac{\ell}{\ell-b} \left(\frac{z}{\sqrt{z^2-\ell^2+b^2}} - 1\right)\right] \tag{2.58}$$

where the quantity inside the bracket is the complex potential, ℓ and b are major and minor axes of the ellipse respectively. If we let $b \to 0$, the ψ problem for the flat plate is obtained, as

$$\frac{\partial \psi}{\partial x} = \text{Re} \left[i \left(\frac{z}{\sqrt{z^2-\ell^2}} - 1\right)\right] \tag{2.59}$$

Referring to equation (2.57), a comparison between equations (2.58) and (2.59) gives a good estimate of the "Thickness effect" of the fixed body. For an ellipse of $\ell/b=8$, the force will be roughly 14% higher than that of a flat plate of the same length.

Finally, the non-linear terms in the Bernoulli equation must be accounted for. The force on the fixed body due to the non-linear terms in the pressure equation (2.34) can be written as

$$F_N(t) = -\rho \int_{-\ell_1}^{\ell_1} \left[\frac{\partial}{\partial x} \phi_1^v \frac{\partial}{\partial x} (\phi_2^s + \phi_2^v + \phi^w) \right]^{\pm} dx$$

$$= -\frac{\rho}{2\pi} \int_{-\ell_1}^{\ell_1} \gamma_1(x,t)\,dx \int_{-\ell_2}^{\ell_2} \frac{\sigma_2(\xi)(x-Ut-\xi) - (\gamma_2(x,t) + \gamma_{w2}(\xi+Ut))\eta}{(x-Ut-\xi)^2 + \eta^2} d\xi \qquad (2.60)$$

The above contribution from the first approximation γ_1^0 will further improve the present results by 8∿10% for various staggers. Therefore, if all these effects are taken into account properly, Collatz's result on the fixed body can be reproduced by the present theory.

With all this in mind, figures 9, 10, 11 and 12 give a comparison between Oltmann's experiments on the forces and moments acting on the elliptical bodies ($\ell/b=8$) and our calculations on the parabolic bodies ($\ell/b=8$) for the low speed cases (Fn∿.14-.2). The behavior of the force and moment acting on the fixed body agree between the theory and experiment as shown in figures 11 and 12. As far as the force acting on the moving body is concerned, the experiment seems to agree with our prediction on the way that the vortex wake will influence the force. This is contrasted to Collatz's prediction of a symmetric force, as indicated in figure 6, where the Kutta condition is not imposed on the moving body. However, an attraction force considerably larger than what we predicted is observed in the experiments when the stern of the moving body leaves the bow of the fixed one. One reason for this may be that the shed vortex is likely to move, while our linear theory predicts once the vortex is shed it remains on the x axis. This may also explain the same phenomenon observed in the moment acting on the moving body, as shown in figure 10.

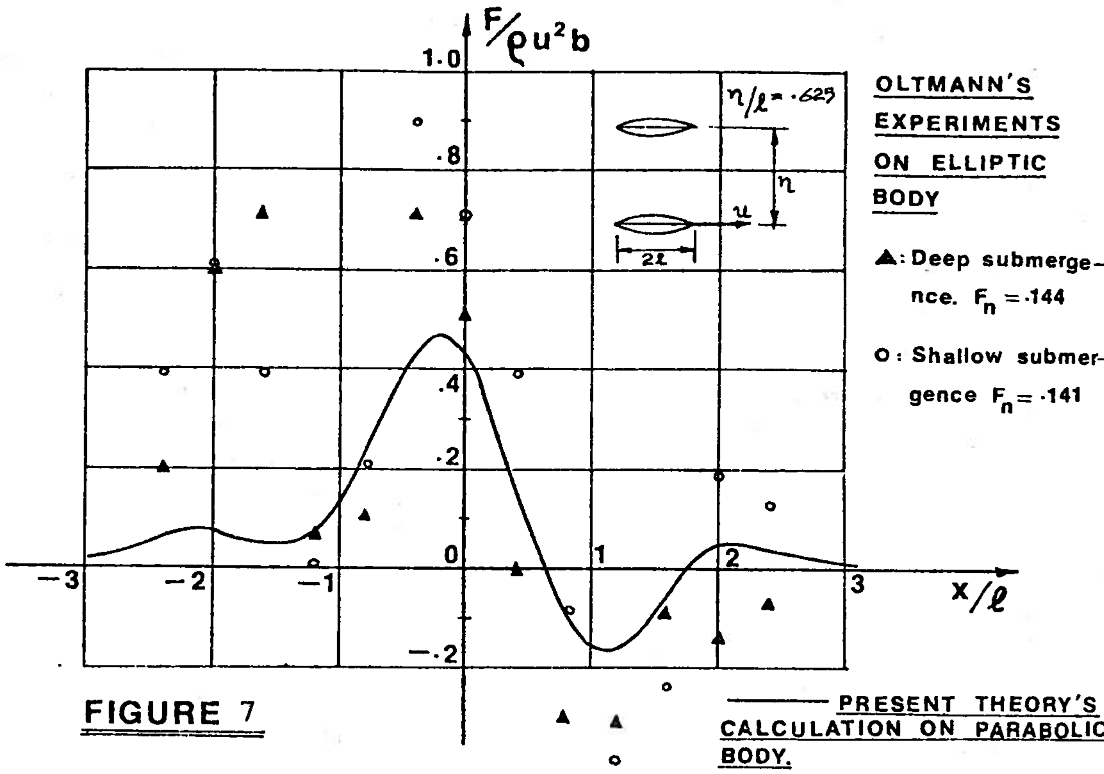


FIGURE 7

FORCES ON MOVING BODY (Two dimensional flow)

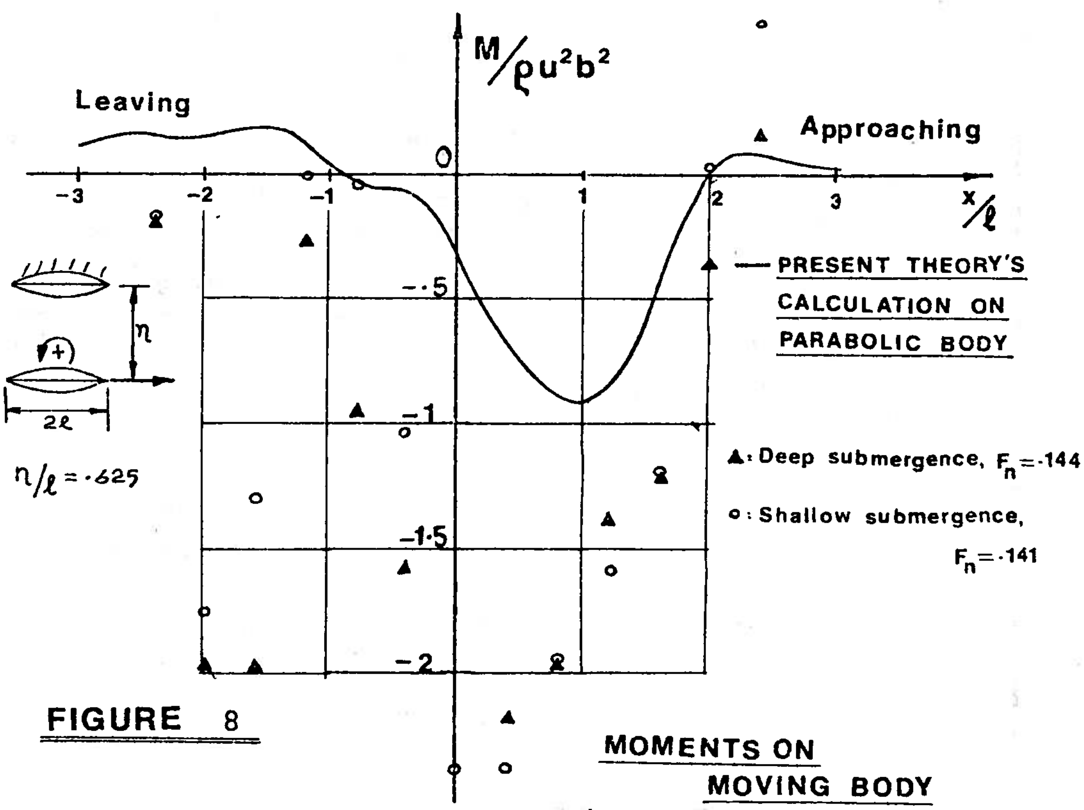
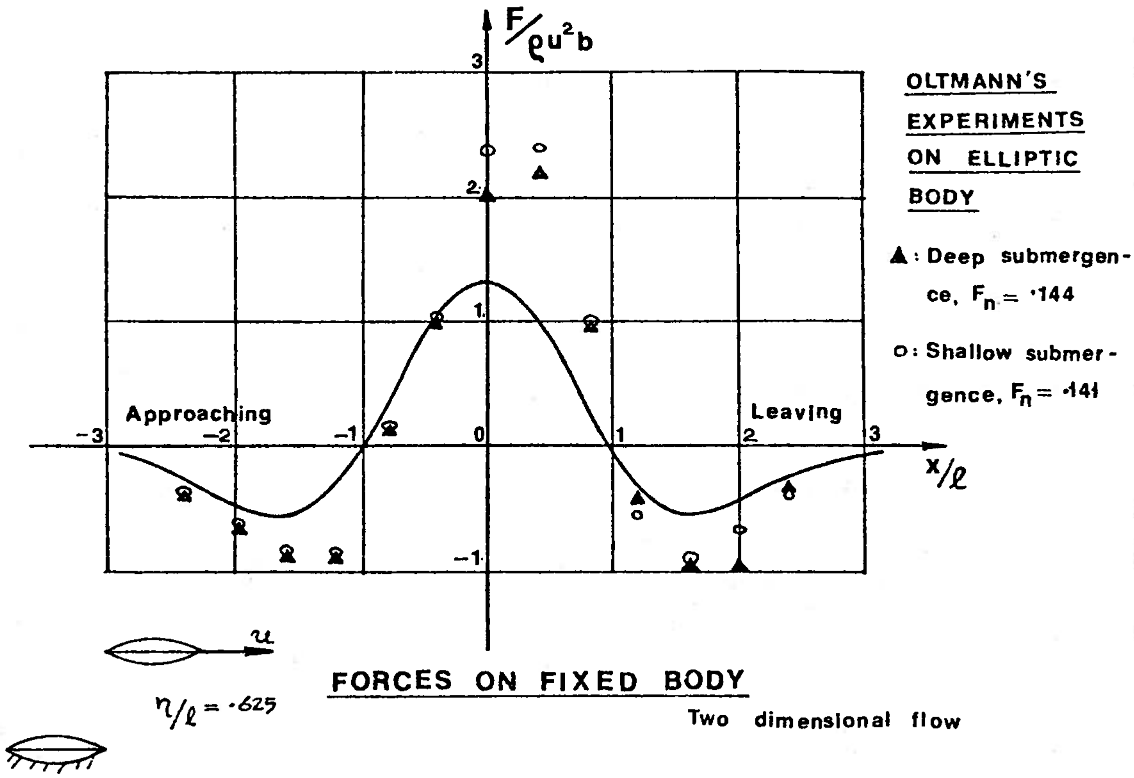


FIGURE 8

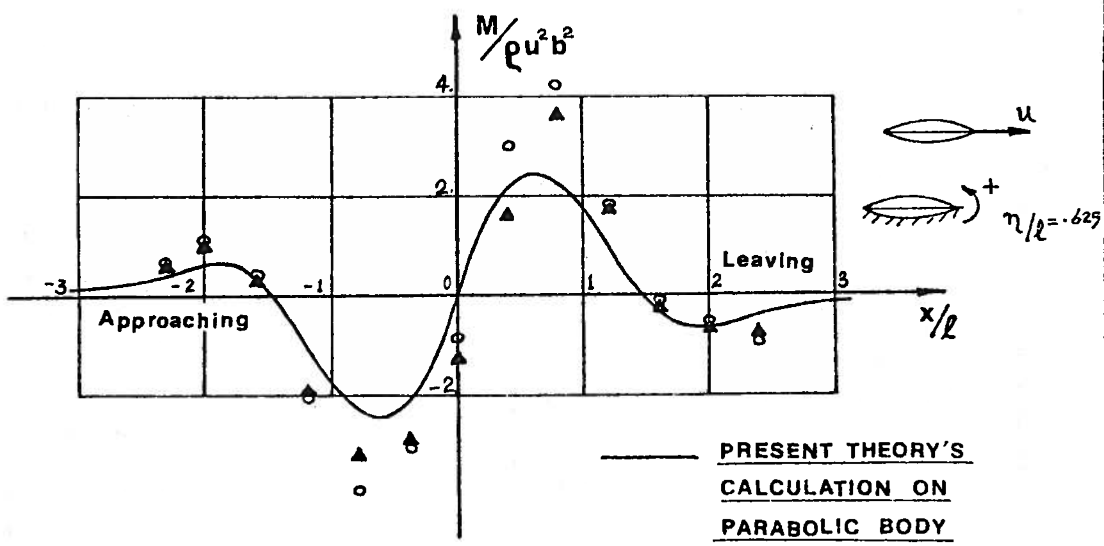
MOMENTS ON MOVING BODY (Two dimensional flow)



FORCES ON FIXED BODY

Two dimensional flow

FIGURE 9



MOMENTS ON FIXED BODY

Two dimensional flow

FIGURE 10

In hydrodynamic interaction problems, the magnitude of the sway force and yaw moment acting on the interacting bodies are directly proportional to the amount of cross flow generated by the moving bodies. For a blunt body (like the ellipse) such a disturbance is bigger than a body of fine shape (like the parabolic body). However, as indicated by the qualitative agreement between our calculations and the experiments, the general behavior of the interacting forces and moments are quite independent of the body shape.

For a fine body with pointed ends defined by two circular arcs, it is possible to obtain an exact representation for the source strength along the center line of the body when it is moving in an otherwise undisturbed fluid (cf. Milne-Thomson [3]). The difference in magnitude of the interaction forces corresponding to the exact and slender body representation for the source strength is substantially smaller than compared to the case for ellipses. This is similar to problems with end effects seen in other slender body problems.

III. HYDRODYNAMIC INTERACTIONS BETWEEN BODIES IN SHALLOW WATER

In order that the two dimensional analysis in the last chapter be applicable to hydrodynamic interactions in shallow water, the ship must be wall sided, and the clearance between the ship's bottom and the bottom of the water must be effectively zero. In addition, the free surface effect must be negligible. This implies that the velocity of the moving ship is small. Under these assumptions, the far field description is truely two dimensional, and the analysis in the last chapter applies.

In most ship problems, the clearance is not effectively zero, but is some small value. The small gap allows a cross flow which substantially alters the forces acting on the ship and must be properly accounted for. In this chapter, we shall investigate the problem in which the ship is not touching the bottom. To account for the cross flow under keel, we shall use the method developed by Tuck and Newman (cf. Ref. [8]). In the outer region, due to the assumption of shallow water and the fact that the details of the flow around the ship can not be seen, the problem reduces to a two dimensional problem in the horizontal plane (x-y plane). The ships can be represented by distributions of singularities along the ship's center line, and the wake consist of a line of vorticity. The strengths of the singularities along the ships can only be determined by matching to the inner region. In the inner region, the usual inner expansion of the Laplace equation reduces the problem to a stream flow past the ship section in a channel in the y-z plane.

The conventional order relations involving the orders O, o will be adopted. Namely, for a small parameter ε

$$f(\varepsilon) = O(g(\varepsilon)) \text{ as } \varepsilon \to 0 \quad \text{if} \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} < \infty$$

and

$$f(\varepsilon) = o(g(\varepsilon)) \text{ as } \varepsilon \to 0 \text{ if } \lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$$

In this case, since the water is shallow, and the ship is slender, we can assume that the draft of the ship T and the depth of the water h are small compared to the length of the ship L. i.e.

$$h/L \ , \ T/L = 0(\varepsilon) \tag{3.1}$$

where ε is a small parameter.

In a co-ordinate system fixed on body $i(i=1,2)$, the linearized free surface boundary condition can be written as

$$\left(\frac{\partial}{\partial t} - U_i \frac{\partial}{\partial x_i}\right)^2 \phi + g\frac{\partial \phi}{\partial z_i} = 0 \tag{3.2}$$

on $z_i=0$

From slender body theory, both the "outer" and "inner" regions defined on this co-ordinate system require the magnitude of the co-ordinate be $x_i=0(1)$ with respect to the ship-length. Hence, the first term in the free surface equation (3.2) is of order $U_i{}^2\phi$. Now, consider the order of magnitude of the second term in equation (3.2). Dropping the subscript i, and expanding ϕ, the total perturbation potential in a Taylor series about the bottom, and using the bottom boundary condition

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{on} \quad z=-h \tag{3.3}$$

and the Laplace equation

$$-\frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) = \nabla^2_{2-D}\phi \tag{3.4}$$

We obtain

$$\phi \simeq \phi(x,y,-h) - \frac{1}{2}(z+h)^2\nabla^2_{2-D}\phi + \ldots \tag{3.5}$$

and

$$\frac{\partial\phi}{\partial z} = -h\ \nabla^2_{2-D}\ \phi + 0(h^2) \qquad \text{on } z=0 \quad . \tag{3.6}$$

Thus, the second term in the free surface equation is of order $gh\phi$.

Therefore, if we assume the conventional Froude number to be

$$\frac{U_i}{\sqrt{gL_i}} = o(\varepsilon^{\frac{1}{2}}) \tag{3.7}$$

the free surface can be replaced by a rigid wall in both the "outer" and "inner" regions.

(1) <u>The far field problem</u>

If the velocities of the moving ships are sufficiently small, the free surface can be replaced by a rigid wall as shown previously. In this case, the far field solution reduces to a two dimensional problem in the x-y plane. The solution is identical to the problem we discussed in the previous chapter. However, since there is a "leakage" at the bottom of each ship, the body boundary conditions described in the two dimensional theory no longer hold. Instead, we have to investigate the cross flow at various cross-sections along the ship hull in order to determine the unknown source and vortex strengths for the "porous" ship.

Performing a similar successive approximation scheme as outlined in Chapter II, the first approximation of the cross flow induced on body i due to body j is a line of sources representing body j moving with velocity U_j in an otherwise undisturbed fluid (as if body i is absent). This will produce a vortex distribution on body i which will in turn cause a disturbance on body j, and so on.

Hence, in a co-ordinate system fixed on body i, the first approximation of the successive approximation scheme for the far field perturbation potential can be written as follows

$$\phi = \frac{1}{2\pi} \int_{-\ell_i}^{\ell_i} [\sigma_i(\xi) \ln\sqrt{(x_i-\xi)^2+y_i^2}$$

$$+ \gamma_i(\xi,t)\tan^{-1}\frac{y_i}{x_i-\xi}]d\xi + \int_{-\infty}^{-\ell_i} \gamma_{iw}(\xi,t)\tan^{-1}\frac{y_i}{x_i-\xi}\, d\xi$$

$$+ \int_{-\ell_j}^{\ell_j} \sigma_j(\xi)\ \ln\sqrt{(x_j-\xi)^2+y_j^2}\ d\xi \qquad (3.8)$$

The last term on the right hand side is the first approximation of the disturbance due to the other moving ship. Second and higher approximations can be written in a similar fashion.

By the virtue of section II(1b), the continuity of the linearized pressure on the vortex wake trailing body i requires

$$\gamma_{wi}(x_i,t) = f(x_i+U_it) \equiv \gamma_{wi}(x_i+U_it) \qquad (3.9)$$

However, the unknowns σ_i , γ_i , etc can now only be determined by matching this equation to the one that governs the "inner" region in a neighborhood near body i. A picture of this far field approximation is shown in Figure 2.

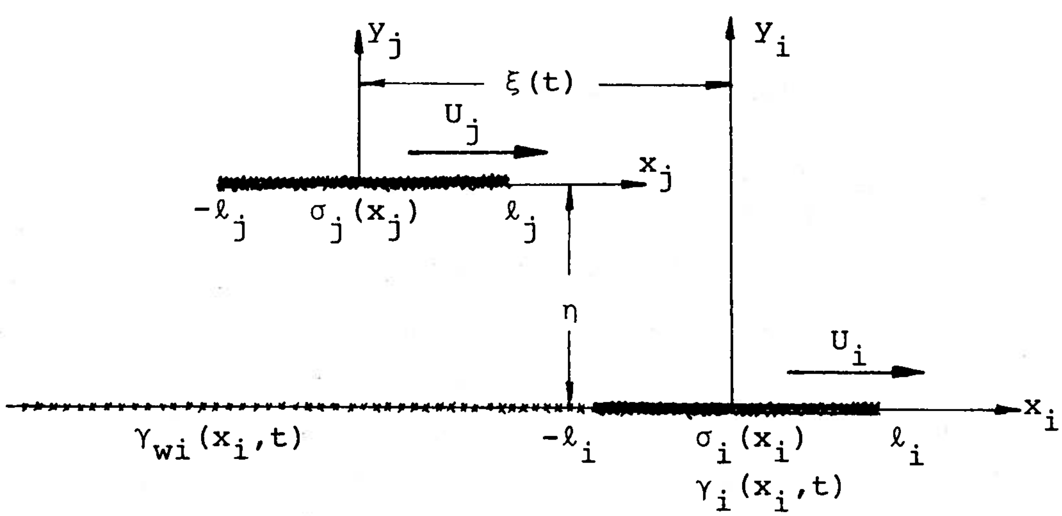


Figure 11. Outer Flow Region (First Approximation)

(2) The near field problem.

Confine our attention to body i. If the ship's hull surface is defined by

$$y_i = \pm F(x_i, z_i) . \qquad (3.10)$$

The hull surface being a stream surface implies that

$$\frac{D}{Dt}(y_i \mp F) = 0 \quad , \quad \text{or}$$

$$\mp \left(\frac{\partial}{\partial t} - U_i \frac{\partial}{\partial x_i}\right) F(x_i, z_i) \mp \frac{\partial \phi}{\partial x_i} \frac{\partial F}{\partial x_i} + \frac{\partial \phi}{\partial y_i} \mp \frac{\partial \phi}{\partial z_i} \frac{\partial F}{\partial z_i} = 0$$

$$\text{on} \quad y_i = \pm F(x_i, z_i) . \qquad (3.11)$$

However, in this region, since the order of magnitude of the co-ordinates are characterized by $y/L = z/L = o(x/L)$. The usual inner expansion of Laplace's equation restricts the flow to the cross-flow plane (i.e. the y-z plane) at each section of constant x, except for a longitudinal velocity component which depends on x only. (cf. Newman [9])

Therefore, the disturbance potentials are typical boundary value problems of the Neumann type, which satisfy the following:

(a) $\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi = 0$

(b) $\frac{\partial \phi}{\partial z} = 0 \quad \text{on} \quad z = -h, 0$

(c) $\frac{\partial \phi}{\partial N} = f(x,z) \quad \text{on} \quad y = \pm F(x,z)$

(d) $\frac{\partial \phi}{\partial y} = \begin{cases} g_1(x) & \text{on} \quad y \to +\infty \\ g_2(x) & \text{on} \quad y \to -\infty \end{cases} \qquad (3.12)$

where for convenience the subscript i has been dropped.

Equation (3.12c) follows from the body boundary condition (3.11) in which N is the two dimensional unit normal to the ship surface given by $y = \pm F(x,z)$, and the function $f(x,z)$ varies according to the order of magnitude of ϕ.

A picture for this type of problem is shown in Figure 3.

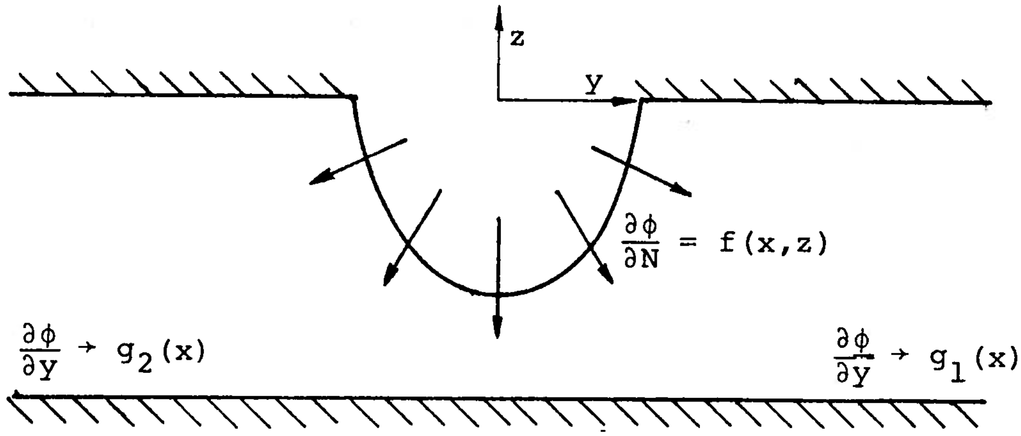


Figure 12. Inner Flow Region

As pointed out by Tuck (Tuck & Newman [8]) the flow in the region immediate next to the body is quite complicated. However, what concerns us most is the quasi-two dimensional flow in the so called "intermediate" region whose lateral distance from the body is large compared to the beam but still small compared to the ship length. In such region, the potential can be written in the following form (cf. Tuck [8] and Beck [6])

$$\phi = v_1(x,t)(y \pm C(x)) \pm v_2(x,t)y + k(x,t) \quad (3.13)$$

The first term on the right hand side of the above equation represents a streaming flow of unknown magnitude $v_1(x,t)$ past the ship cross-section. The second term results from the push-aside flow due to the ship's forward motion, and $k(x,t)$ is just an arbitrary integration constant. The term $C(x)$, which depends only on the shape of the ship cross-section for a given clearance and water depth, is the so-called "blockage" coefficient (cf. Tuck and Taylor [5]). It is the jump in the potential for the two dimensional Neuomann problem of a stream flow past the same body in a channel. $C(x)=0$ denotes the absence of the body, while $C(x)=\infty$ corresponds to complete blockage where the ship "blocks" the channel. As noted by Beck [6], the magnitude of $C(x)$ plays an important role in the matching process.

It is obvious from conservation of mass that (cf. Beck [6]):

$$2v_2(x,t)h = \int_c \frac{\partial\phi}{\partial N}\, d\ell = -US'(x)$$

or

$$v_2(x,t) = \frac{-U}{2h}\frac{ds(x)}{dx} \tag{3.14}$$

where c is the contour of the body cross section and $s(x)$ is the body sectional area.

(3) The Matching Process

Since we have defined the near field of body i to be the region in the neighborhood around the ship, the unsteady vortex wake extending from the stern to infinity must be considered as part of the outer region.

Thus, upon expanding equation (3.8) in a Taylor series about $y=0$, $|x|<\ell$, using equation (2.37) and (2.19), and then taking the limit as $y\to\pm 0$, the inner expansion of the far field solution becomes

$$\begin{aligned}\phi \sim\ & \frac{1}{2\pi}\int_{-\ell_i}^{\ell_i} \sigma(\xi)\,\ell n|x-\xi|\,d\xi \pm \frac{1}{2}\sigma(x)y \\ & \pm \frac{1}{2}\int_{x}^{\ell_i}\gamma(\xi,t)\,d\xi + \frac{y}{2\pi}\int_{-\ell_i}^{\ell_i}\frac{\gamma(\xi,t)}{x-\xi}\,d\xi \\ & + \frac{y}{\pi}\int_{-\infty}^{-\ell_i}\frac{\gamma_w(\xi+Ut)}{x-\xi}\,d\xi \\ & + \int_{-\ell_j}^{\ell_j}\sigma_j(\xi)\,\ell n\sqrt{(x_j-\xi)^2+\eta^2}\,d\xi \\ & + y\int_{-\ell_j}^{\ell_j}\sigma_j(\xi)\frac{\eta}{\sqrt{(x_j-\xi)^2+\eta^2}}\,d\xi + 0(y^2)\end{aligned} \tag{3.15}$$

since $\phi_w(x,0,t)=0$ for $|x|<\ell$.

Matching (3.15) and (3.13) with (3.14) gives the following matching conditions:

$$\text{(a)}\quad \sigma_i(x) = \frac{-U_i}{h}\frac{ds_i(x)}{dx}$$

$$\text{(b)}\quad k(x,t) = \frac{1}{2\pi}\int_{-\ell_i}^{\ell_i}\sigma_i(\xi)\ln|x-\xi|d\xi + \int_{-\ell_j}^{\ell_j}\sigma_j \ln\sqrt{(x_j-\xi)^2+\eta^2}\,d\xi$$

$$\text{(c)}\quad v_1(x,t) = \frac{1}{2\pi}\,⨍_{-\ell_i}^{\ell_i}\frac{\gamma(\xi,t)}{x-\xi}d\xi + \frac{1}{2\pi}\int_{-\infty}^{-\ell_i}\frac{\gamma_w(\xi+Ut)}{x-\xi}\,d\xi$$

$$+ \eta\int_{-\ell_j}^{\ell_j}\sigma_j(\xi)\,\frac{d\xi}{(x_j-\xi)^2+\eta^2}$$

$$\text{(d)}\quad v_1(x,t)\;C(x) = \frac{1}{2}\int_{x}^{\ell_i}\gamma(\xi,t)d\xi \tag{3.16}$$

The same analysis carried out on the other body j will lead to the similar result, particularly we find that

$$\sigma_j = \frac{-U_j}{h}\frac{ds_j(x)}{dx} \tag{3.17}$$

Combining equations (3.16d) and (3.16c), we obtain the following singular integral equation for the unknown vortex distribution γ:

$$⨍_{-\ell_i}^{\ell_i}\frac{\gamma(\xi,t)}{x-\xi}d\xi + \frac{\pi}{C(x)}\int_{x}^{\ell_i}\gamma(\xi,t)\,d\xi$$

$$= -\int_{-\infty}^{-\ell_i}\frac{\gamma_w(\xi+Ut)}{x-\xi}\,d\xi - V(x,t) \tag{3.18}$$

where $V(x,t)$ is a known function. Note that when $C(x)=\infty$, or complete blockage, we retrieve the equation (2.23) for two dimensional flows. The nature of the solution to this integral equation will be discussed in Appendix A.

However, it is well known that the solution to this kind of integral equation is not unique. In fact, there are three unknowns in this equation, namely, γ , γ_w, and an arbitrary function of time $k(t)$ which results from the non-uniqueness of the inverse finite Hilbert Transform. Therefore, this equation must be solved simultaneously with two other equations describing the relations between γ and γ_w. As mentioned in Chapter II, the other conditions we still need to satisfy are Kelvin's theories of constant circulation and the Kutta condition on the trailing edge. Kelvin's theorem merely states that

$$\int_{-\infty}^{-\ell_i} \gamma_w(\xi+Ut)\,d\xi + \int_{-\ell_i}^{\ell_i} \gamma(\xi,t)\,d\xi = 0 \qquad \text{for all } t, \tag{3.19}$$

But, this time, since we can not obtain a closed form for γ as we did in the two dimensional theory, we can not state the Kutta condition as we did previously. For this case the Kutta condition must be stated explicitly (c.f. Appendix B) as:

$$\gamma_w(-\ell_i+Ut) = \gamma(-\ell_i,t) \tag{3.20}$$

i.e., the vortex distribution must be continuous at the trailing edge. The solution satisfying equations (3.18), (3.19), and (3.20) is then unique, and can be determined numerically.

In the case of ship i being fixed, the trailing vortex behind the stern vanishes as we mentioned in II(1a). The vortex distribution along the body will then satisfy the following equation from (3.18):

$$\int_{-\ell_i}^{\ell_i} \frac{\gamma(\xi,t)}{x-\xi}\,d\xi + \frac{\pi}{C(x)} \int_{x}^{\ell_i} \gamma(\xi,t)\,d\xi = V(x,t) \tag{3.21}$$

The appropriate statement of Kelvin's theorem may now be stated as

$$\int_{-\ell_i}^{\ell_i} \gamma(\xi,t)\,d\xi = 0 \qquad \text{for all } t \tag{3.22}$$

This will determine the arbitrary function $k(t)$, thereby making the solution to equations (3.21), and (3.22) unique.

(4) Force and Moment

Once the vortex strength on the "porous" ship is determined, the forces and moments acting on the ship may be obtained by integrating the pressure along the hull.

Neglecting the non-linear terms in the Bernoulli equation, the sway force on body i (i=1,2) can be found by integrating the pressure along the slit $|x|<\ell_i$, $y=\pm 0$. i.e.

$$F(t) = \rho h \int_{-\ell_i}^{\ell_i} \left[\frac{\partial \phi}{\partial t} - U_i \frac{\partial}{x_i} \phi\right]^{\pm} dx_i \qquad (3.23)$$

Since the only jump in the velocity potential is the vortex distribution γ_i, along the body, recall that

$$\phi_i^v = \frac{1}{2} \int_x^{\ell_i} \gamma_i \, d\xi \ ; \ \frac{\partial \phi}{\partial x_i} = \frac{1}{2} \gamma_i \qquad (3.24)$$

$$\text{on } y_i=+0 \quad , \quad x_i<\ell_i$$

Dropping the subscript i, the sway force can be written in terms of γ in the following manner:

$$F(t) = \rho h \left[U \int_{-\ell}^{\ell} \gamma d\xi + \int_{-\ell}^{\ell} d\xi \int_{\xi}^{\ell} \frac{\partial \gamma}{\partial t} dx \right] \qquad (3.25)$$

Noting that the first term in the right hand side is the total circulation on the ship, which is equal in magnitude to the total circulation generated by the unsteady vortex wake, and that upon partial integration of the second term, the sway force may now be written as:

$$F(t) = -\rho U h \int_{-\infty}^{-\ell} \gamma_w d\xi + \rho h \frac{\partial}{\partial t} \int_{-\ell}^{\ell} (\ell+\xi) \gamma d\xi \qquad (3.26)$$

Performing a similar analysis on the yaw moment integral

$$M(t) = \rho h \int_{-\ell}^{\ell} [x(\frac{\partial\phi}{\partial t} - U\frac{\partial\phi}{\partial x})]^{\pm} dx \tag{3.27}$$

We obtain:

$$M(t) = \rho U h \int_{-\ell}^{\ell} \xi\gamma d\xi + \frac{\rho h}{2}\frac{\partial}{\partial t}\int_{-\ell}^{\ell} (\xi^2-\ell^2)\gamma d\xi \tag{3.28}$$

In case ship i is fixed, the sway force and yaw moment are given by the following:

$$F(t) = \rho h \frac{\partial}{\partial t}\int_{-\ell}^{\ell} (\ell+\xi)\gamma d\xi \tag{3.29}$$

and

$$M(t) = \frac{\rho h}{2}\frac{\partial}{\partial t}\int_{-\ell}^{\ell} (\xi^2-\ell^2)\gamma d\xi \tag{3.30}$$

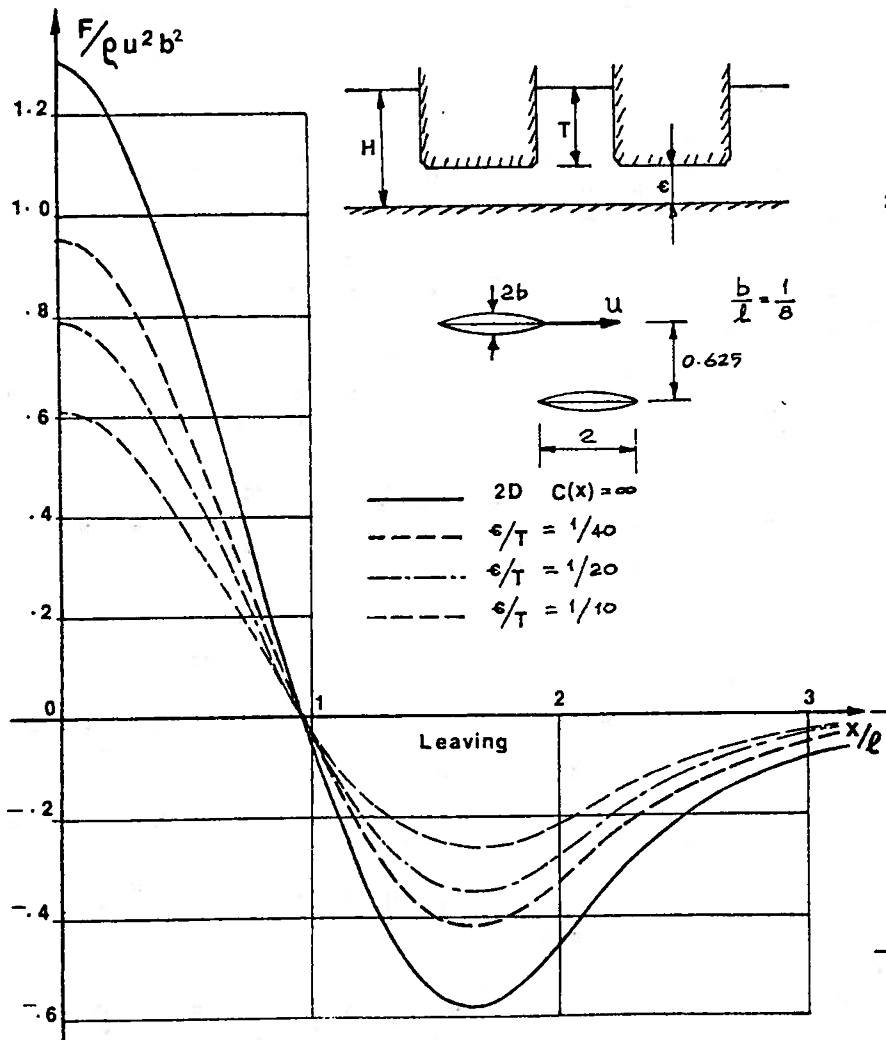
(5) Numerical results and discussions on shallow water theory

For the same reasons stated in the last chapter, our interacting ships are taken to be two idealized identical "parabolic" ships whose water planes are defined by the area enclosed between two parabolic arcs as described by equation (2.58). The parabolic ships are wall sided, and the bottom is flat. The length to beam ratio of each ship is taken to be $\ell/b=8$. Also, the water depth h, to half ship length ratio is taken to be $h/\ell=1/8$.

Numerical results for the case of one ship passing a fixed one with constant velocity in a parallel path will be presented in this section. The case when both ships are moving can be done in a simmilar manner without great difficulty.

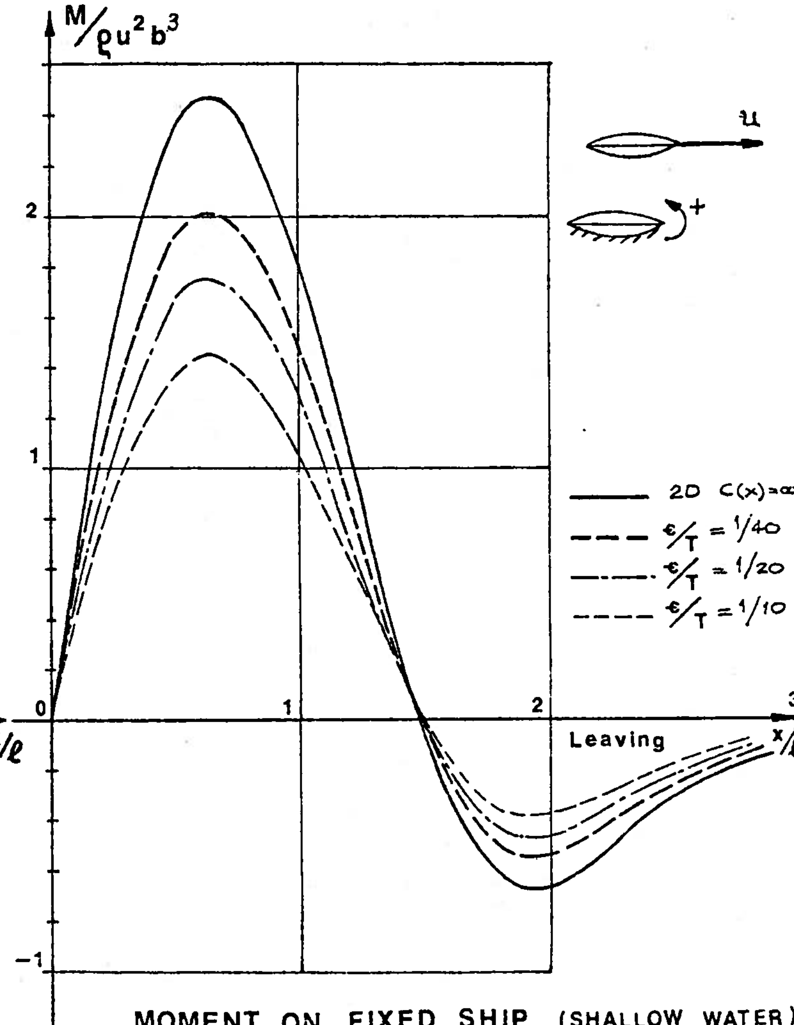
The difficult part in computing the forces and moments acting on each ship is to obtain the appropriate vortex distribution along each ship. The vorticity is found by using a numerical technique in conjunction with the so called "time stepping" method. In the time stepping method, the integral equations (3.18) and (3.21) are solved at each time step using a numerical scheme described in Appendix C. Basically, the ship is divided into a number of segments over which the vorticity assumes a prescribed variation. At each time step, the vorticity along each ship is obtained from satisfying the integral equation at a number of control points so that a system of simultaneous equations result. The vorticity in the wake is found by satisfying the Kutta condition (which requires the vorticity to be continuous from the ship body into the wake) at the trailing edge of the moving ship. The vorticity also assumes a prescribed variation between each time step and stays constant once shed. The details of the numerical scheme are described in Appendix C. Once the vortex distribution is known, the forces and moments can be found by numerical integration and differentiation according to the formulas given in the previous section.

Figures 13 and 14 show the forces and moments acting on the fixed ship for various gap sizes between the ship's bottom and the bottom of the water. It is not surprising that the force and moment obtained from the first approximation of our successive approximation scheme are even and odd functions of the stagger respectively, since the ships (hence the blockage coefficient $C(x)$) are symmetrical about the mid-section. This symmetrical behavior of the force and moment is also found by Newman for the hydrodynamic interaction between ships in deep water. (cf. Ref [8]). In general, if we allow a small clearance beneath the bottom of the ship, the behavior of the force and moment acting on the fixed ship remain unchanged compared to the case of complete blockage ($C(x)=\infty$) corresponding to two dimensional flow. However, the magnitude of the interaction forces and moments change substantially once a small



FORCE ON FIXED BODY (SHALLOW WATER)

FIGURE 13



MOMENT ON FIXED SHIP (SHALLOW WATER)

FIGURE 14

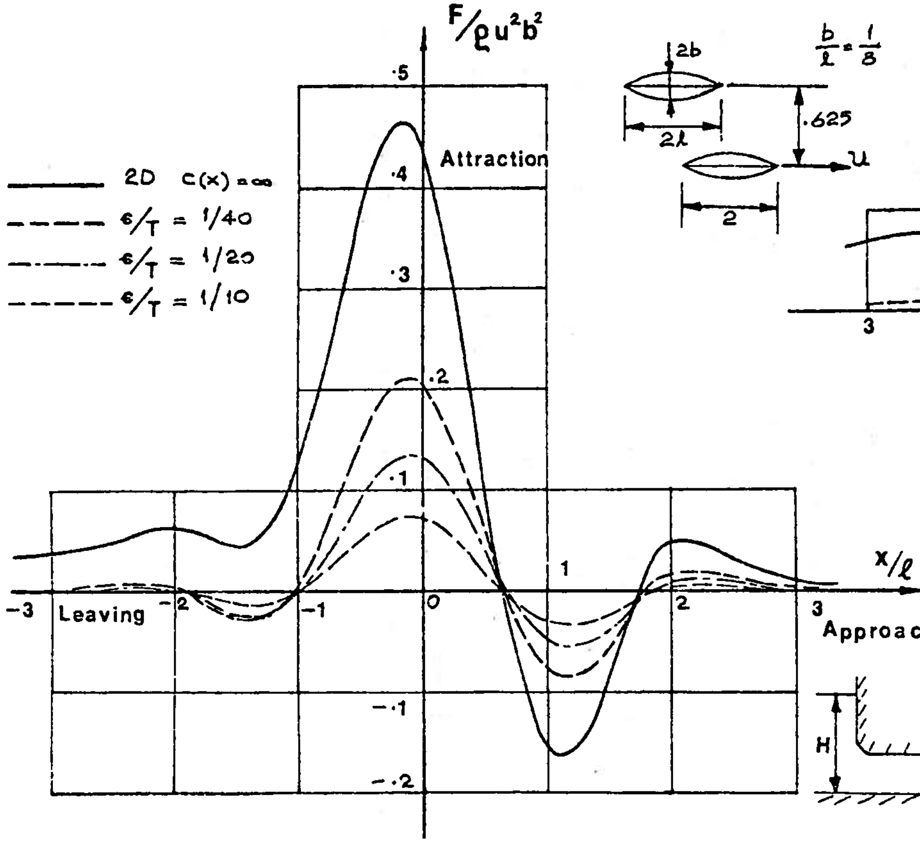
gap is left open beneath the bottom of the ship. The interaction tends to decrease as the gap size increases.

The effect of the flow beneath the bottom of the body on the moving ship is quite different from that of the fixed ship. In Figures 15 and 16, the forces and moments acting on the moving ship are plotted against the stagger for various gap sizes. Two interesting phenomena are observed.

Firstly, the magnitude of the interaction force and moment acting on the ship vanishes quite rapidly once a gap appears between the ship's bottom and the bottom of the water. This phenomenon is also indicated by Newman (cf. Ref. [8]). In fact, for interactions in deep water, Newman's theory predicts no force on a passing vessel due to its interaction with a stationary vessel.

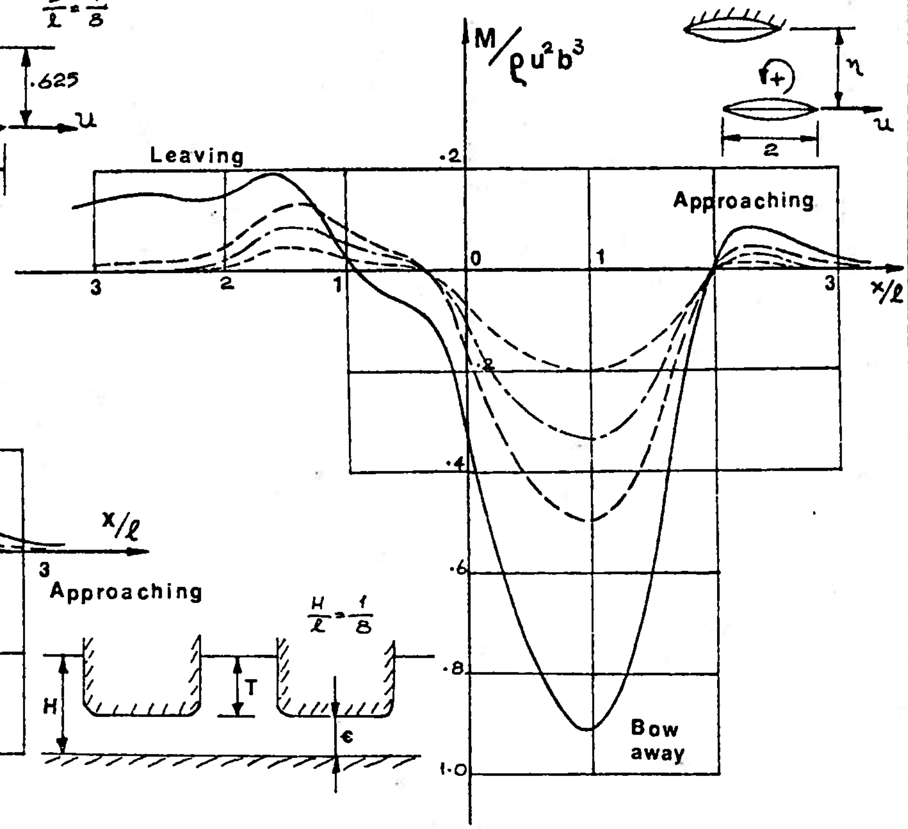
Secondly, the effect of the vortex wake trailing behind the moving ship on the interaction forces and moments diminishes as cross flow is allowed beneath the bottom of the ship. This is indicated by the symmetrical tendency of the curves. As the stern leaves the mid-section of the fixed ship, the theory predicts a repulsion force as contrasted to that of attraction in the two dimensional flow case.

In general, the magnitude of the interaction forces and moments acting on the fixed ship are much bigger than that of the passing ship. This also is true for the two dimensional flow. Unfortunately, for the time being, there is no experimental measurements available for comparison in the case of hydrodynamic interactions in shallow water.



FORCE ON MOVING SHIP (SHALLOW WATER)

FIGURE 15



MOMENT ON MOVING SHIP (SHALLOW WATER)

FIGURE 16

IV SUMMARY AND CONCLUSIONS

The important conclusions that we arrive at in this study probably are:

(1) The importance of the flow beneath the ship in shallow water problems.

(2) The importance of the unsteady vortex wake trailing behind the moving body in two dimensional flow.

(3) The importance of the body shape (end effects) in hydrodynamic interaction problems where slender body approximations are employed.

First, as indicated by the numerical results and the discussions in Appendix A, the temptation to use the two dimensional solution as the first approximation for shallow water problem is erroneous. Once there is a small gap beneath the ship's bottom and the bottom of the water, the flow changes significantly. This is best demonstrated by the force acting on the moving body. As indicated by the calculations, the disturbance caused by the forward motion of the moving ship on the fixed ship will "leak" through the "porous" ship entirely and almost nothing is reflected back once a slight gap is left open. This confirms Newman's (Tuck and Newman [8]) prediction in deep water theory that the force acting on the passing vessel due to its interaction with a stationary vessel is of higher order. It is physically evident that the cross flow induced on a slender body will likely travel beneath the bottom (the distance is of order ε) rather than all the way around the bow and stern (where the distance is of order 1). Similar conclusions have been drawn by Tuck and Taylor [5] and Beck [6].

The sensitivity to the flow under the keel makes both theoretical predictions and experiments for a particular slight gap undependable. Nevertheless, what concerns us most is the trend in the forces and moments as the gap vanishes. Indeed, the introduction of the so-called "blockage coefficient" in shallow water problems does provide a "bridge" between the theory of high aspect ratio (e.g. two dimensional flow in this study) and low aspect ratio (e.g. interactions in shallow water) in the mechanics of fluids.

Secondly, as indicated in figure 6, and Oltmann's experiments, the vortex wake play an important role on the force acting on the moving body in two dimensional flow. Such effects become prominent when the bodies are abeam, (where or when) the unsteady cross flow induced on the moving body reaches a maximum. Although the present theory under predicts the magnitude of such effects (as compared to the experiments), it suggests a qualitative behavior of this important effect. In other maneuvering problems in shallow water, whenever such unsteadiness is encountered, the complication of the unsteady vortex shedding should be taken into account properly.

Finally, in this study, it was found that the shape of the interacting bodies also plays an important role. For a blunt body (like an extended ellipse), the following assumptions usually employed in slender body theory are poor. (1) The source strength is $\sigma(x)=-2Uf'(x)$ along the center line of the body. (2) The body boundary condition is satisfied on the center line of the body.

In hydrodynamic interaction problems, the forces and moments acting on the bodies are directly proportional to the magnitude of the bodies. A body with full ends will produce a bigger disturbance than a fine body moving with the same speed. For blunt bodies, the linear theory generally under predicts the magnitude of the interaction forces due to the above assumptions. However, the linear theory does provide good qualitative prediction on the behavior of the interaction forces and moments.

In this work, we have also developed a successive approximation method to solve a system of coupled simultaneous integral equations whose kernels are functions of time. This scheme is particularly useful when the solution to the system can not be found in an analytic form. In applying the scheme, the system of coupled integral equations breakes down into a system of uncoupled integral equations. The kernel of each integral equation is the same, and does not depend on time. Thus, in defining a numerical scheme to solve the integral equations, the coefficient matrix need be calculated and inverted only once throughout the whole time history. This saves tremendous amounts of computer time.

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APPENDIX A

Solutions to the governing integral equations

a) Solution to airfoil equation for the fixed body in two dimensional flow.

Consider:

$$\int_{-\ell}^{\ell} \frac{\gamma(\xi,t)}{x-\xi}\, d\xi = f(x,t) \tag{A.1}$$

where $f(x,t)$ is an integrable function.

A general solution takes the form: (cf. Tricomi, Ref. [12] pp 178)

$$\gamma(x,t) = \frac{1}{\pi^2\sqrt{\ell^2-x^2}} \int_{-\ell}^{\ell} \frac{\sqrt{\ell^2-\xi^2}}{\xi-x} f(\xi,t)\, d\xi + \frac{c(x)}{\sqrt{\ell^2-x^2}} \tag{A.2}$$

No net circulation on the body $-\ell<x<\ell$ implies

$$\int_{-\ell}^{\ell} \gamma(\xi,t)\, d\xi = 0 \qquad \text{for all } t\,. \tag{A.3}$$

Upon integrating (A.1) from $-\ell$ to ℓ, we obtain

$$c(t) = 0 \qquad \text{for all } t \tag{A.4}$$

b) Solution to airfoil equation for the moving body in two dimensional flow.

Consider:

$$\int_{-\ell}^{\ell} \frac{\gamma(\xi,t)}{x-\xi}\, d\xi = -\int_{-\infty}^{-\ell} \frac{\gamma_w(\xi+Ut)}{x-\xi}\, d\xi - V(x,t) \tag{A.5}$$

A general solution to this equation may also be written as follows (cf. Ref. [12], pp 180)

$$\gamma(x,t) = -\frac{1}{\pi^2}\sqrt{\frac{\ell+x}{\ell-x}} \int_{-\ell}^{\ell} \sqrt{\frac{\ell-x}{\ell+x}}\, \frac{d\xi}{\xi-x} \left[-\int_{-\infty}^{-\ell} \frac{\gamma_w(\zeta+Ut)}{\xi-\zeta}\, d\zeta + V(\xi,t) \right] + \frac{c(x)}{\sqrt{\ell^2-x^2}} \tag{A.6}$$

In order that the Kutta condition be satisfied at $x=-\ell$, we must have

$$c(t)=0 \qquad \text{for all } t.$$

Noting that it is permissible to change order of integration under the Cauchy's Principal Value integral (cf. Ref. [13]) and that (cf. Appendix D):

$$\int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}} \frac{d\xi}{(\xi-x)(\xi-\zeta)} = \frac{\pi}{x-\zeta} \sqrt{\frac{\zeta-\ell}{\zeta+\ell}} \tag{A.7}$$

for $-\ell<x<\ell$, $-\infty<\zeta-\ell$ ·

Equation (A.6) can be rewritten in the following form:

$$\gamma(x,t) = \frac{1}{\pi^2}\sqrt{\frac{\ell+x}{\ell-x}} \left[\pi \int_{-\infty}^{-\ell} \frac{\gamma_w(\zeta+Ut)}{x-\zeta} \sqrt{\frac{\zeta-\ell}{\zeta+\ell}}\, d\zeta \right.$$

$$\left. - \int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}} \frac{V(\xi,t)}{\xi-x} d\xi \right] \tag{A.8}$$

Finally, the total circulation along the body is given by integrating equation (A.8) from $-\ell$ to ℓ. Using the appropriate integral in Appendix D and after some algebraic manipulation, we obtain the following:

$$\Gamma(t) = \int_{-\ell}^{\ell} \gamma(x,t)dx = -\int_{-\infty}^{-\ell} \gamma_w(\zeta+Ut)d\zeta$$

$$+ \int_{-\infty}^{-\ell} \sqrt{\frac{\zeta-\ell}{\zeta+\ell}}\, \gamma_w(\zeta+Ut)d\zeta - \frac{1}{\pi} \int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}}\, V(\xi,t)d\xi \tag{A.9}$$

Since Kelvin's theorem requires

$$\int_{-\ell}^{\ell} \gamma(x,t)dx = -\int_{-\infty}^{-\ell} \gamma_w(\zeta+Ut)d\zeta$$

We obtain the following integral equation for the wake vorticity.

$$\int_{-\infty}^{-\ell} \sqrt{\frac{\zeta-\ell}{\zeta+\ell}}\ \gamma_w(\zeta+Ut)d\zeta - \frac{1}{\pi}\int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}}\ V(\xi,t)d\xi = 0 \qquad (A.10)$$

c) The nature of the solution to the singular integral equation with Cauchy's integral as its dominant part.

In solving the shallow water problem, we arrive at the following systems of integral equations. For the vortex distribution over the moving ship, we have

$$a)\ \left[⨍_{-\ell}^{\ell} \frac{\gamma(\xi,t)}{x-\xi} + \frac{\pi}{C(x)}\int_{x}^{\ell} \gamma(\xi,t)\right]d\xi = -\int_{-\infty}^{-\ell} \frac{\gamma_w(\xi+Ut)}{x-\xi}\, d\xi + V(x,t)$$

$$b)\ \int_{-\infty}^{-\ell} \gamma_w(\xi+Ut)d\xi = -\int_{-\ell}^{\ell} \gamma(\xi,t)d\xi$$

$$c)\ \gamma(-\ell,t) = \gamma_w(-\ell+Ut) \qquad (A.11)$$

For the vortex distribution over the fixed ship, we have

$$a)\ \left[⨍_{-\ell}^{\ell} \frac{\gamma(\xi,t)}{x-\xi} + \frac{\pi}{C(x)}\int_{x}^{\ell} \gamma(\xi,t)\right]d\xi = V(x,t)$$

$$b)\ \int_{-\ell}^{\ell} \gamma(\xi,t)d\xi = 0 \qquad (A.12)$$

Here, we wish to demonstrate that the solutions to the above systems are unique, and possess square root singularity at the proper ends of the ship. This can be done by reducing the above systems into regular Fredholm equations as outlined in Muskhelishvili (cf. Ref. [11]), Chapter 14.

Consider the system (A.11). Transposing the second term on the left hand side of equation (A.11a) to the right hand side and performing a similar algebraic manipulation to those outlined in Appendix A.b, the system can be rewritten in the following form,

$$\gamma(x,t) = \frac{1}{\pi^2}\sqrt{\frac{\ell+x}{\ell-x}}\left[\pi\int_{-\infty}^{-\ell}\frac{\gamma_w(\xi+Ut)}{x-\xi}\sqrt{\frac{\xi-\ell}{\xi+\ell}}\,d\xi + \int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\frac{V(\xi,t)}{\xi-x}d\xi - \pi\int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\frac{d\xi}{(\xi-x)C(\xi)}\int_{\xi}^{\ell}\gamma(\zeta,t)\,d\zeta\right] \tag{A.13}$$

Note that the first term inside the bracket is of order $0(1/\sqrt{\ell+x})$ and the rest are bounded as $x\to-\ell$ (cf. Appendix B). If $\gamma(x,t)$ is integrable over $-\ell\leq x\leq\ell$, then it must possess a square root singularity at $x=\ell$ and bounded at $x=-\ell$. Upon transposing the term involving γ back to the left hand side, equation (A.13) is known as a Fredholm integral equation of the second kind although its kernel is not bounded. However, if we state the unknown γ in a suitable manner, this equation can be reduced to a regular one. Let

$$\gamma'(x,t) = \frac{\gamma(x,t)}{\sqrt{\ell-x}} \tag{A.14}$$

Dropping prime, equation (A.13) can be rewritten as:

$$\gamma(x,t) + \frac{\sqrt{\ell+x}}{\pi}\int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\frac{d\xi}{(\xi-x)C(x)}\int_{\xi}^{\ell}\frac{\gamma(\zeta,t)}{\sqrt{\ell-\zeta}}\,d\zeta$$

$$= \frac{\sqrt{\ell+x}}{\pi^2}\left[\pi\int_{-\infty}^{-\ell}\frac{\gamma_w(\xi+Ut)}{x-\xi}\sqrt{\frac{\zeta-\ell}{\zeta+\ell}}\,d\xi + \int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\frac{V(\xi,t)}{\xi-x}\,d\xi\right] \tag{A.15}$$

Furthermore, replacing the variable ζ by η where

$$\eta = \int_{-\ell}^{\zeta}\frac{d\zeta}{\sqrt{\ell-\zeta}}\ , \quad d\eta = \frac{d\zeta}{\sqrt{\ell-\zeta}} \tag{A.16}$$

Denoting $\gamma(x,t)$ simply by $\gamma(y,t)$, and changing the order of integration in the left hand side of (A.1t) we have

$$\gamma(y,t) + \int_{0}^{2\sqrt{2\ell}} \gamma(\eta,t)\left[\frac{\sqrt{\ell+y}}{\pi}\int_{-\ell}^{-\ell-\eta\sqrt{2\ell}+\eta^2/4} \sqrt{\frac{\ell-\xi}{\ell+\xi}}\,\frac{d\xi}{(\xi-y)C(\xi)}\right]d\eta$$

$$= \frac{\sqrt{\ell+y}}{\pi^2}\left[\pi\int_{-\infty}^{-\ell} \frac{\gamma w(\xi+Ut)}{y-\xi}\sqrt{\frac{\xi-\ell}{\xi+\ell}}\,d\xi + \int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\,\frac{V(\xi,t)}{\xi-y}\,d\xi\right] \tag{A.17}$$

Hence, the kernel is integrable, and all the fundamental Fredholm theorems are applicable to the above equation. In particular, setting the right hand side to zero is physically equivalent to the absence of the other ship, thereby, the homogeneous equation has only the trivial solution. Thus the solution to the Fredholm equation is unique.

On the other hand, a similar analysis performed on the system (A.12) leads to the following:

$$\gamma(x,t) = \frac{1}{\pi^2\sqrt{\ell^2-x^2}} \int_{-\ell}^{\ell} \frac{\sqrt{\ell^2-\xi^2}}{\xi-x}V(\xi,t)d\xi + \int_{-\ell}^{\ell}\frac{\sqrt{\ell^2-\xi^2}}{\xi-x}\,\frac{1}{C(\xi)}\int_{\xi}^{\ell}\gamma(\zeta,t)d\zeta \tag{A.18}$$

Now, γ has square root singularities at both $x=\ell$ and $x=-\ell$. Writing

$$\gamma'(x,t) = \frac{\gamma(x,t)}{\sqrt{\ell^2-x^2}}$$

$$\eta = \int_{-\ell}^{\zeta}\frac{d\zeta}{\sqrt{\ell^2-\zeta^2}} \quad , \quad d\eta = \frac{d\zeta}{\sqrt{\ell^2-\zeta^2}} \tag{A.19}$$

By the same token, equation (A.19) may be written as the following

$$\gamma(y,t) + \int_{0}^{\pi}\gamma(\eta,t)\left[\frac{1}{\pi}\int_{\ell\sin(\eta-\frac{\pi}{2})}^{\ell} \frac{\sqrt{\ell^2-\xi^2}}{\xi-y}\,\frac{d\xi}{C(\xi)}\right]d\eta =$$

$$\frac{1}{\pi^2} \int_{-\ell}^{\ell} \frac{\sqrt{\ell^2-\xi^2}}{\xi-x} \; V(\xi,t)d\xi \quad . \tag{A.20}$$

Again, the argument for equation (A.18) applies here.

Before attempting to define a numerical scheme to solve (A.11) and (A.12), it is worthwhile to investigate how the blockage coefficient $C(x)$ influences the vortex distribution along the ship. For a sufficiently large blockage, i.e., a sufficiently small clearance between the ship's bottom and the bottom of the water, we expect that the flow will not deviate too much from that of the two-dimensional problem.

For illustrative purposes, let us consider the hydrodynamics of the fixed body. In fact, if $C(x)$ in sufficiently large, equation (A.12) can be solved by successive approximation as follows:

(a) $\gamma = \sum_{i=1}^{\infty} \gamma_i$

where

(b) $\displaystyle \int_{-1}^{1} \frac{\gamma_1(\xi,t)}{x-\xi} \, d\xi = V(x,t)$

(c) $\displaystyle \int_{-1}^{1} \frac{\gamma_i(\xi,t)}{x-\xi} \, d\xi = -\frac{\pi}{C(x)} \int_{x}^{1} \gamma_{i-1}(\xi,t)d\xi$

$i = 2,3,\ldots$ (A.21)

The length scale has been non-dimensionalized with respect to ℓ. γ_1 can be viewed as the two dimensional solution, the rest of γ_i can be regarded as perturbations due to the flow under the ship's bottom. A necessary piece of information is for what value of $C(x)$ will the scheme converge?

In the view of equation (A.18), γ_i takes the following form for a fixed t.

$$\gamma_i = \frac{f_i(x)}{\sqrt{1-x^2}} \tag{A.22}$$

Expanding f_i, f_{i+1} in Chebyshev polynomials of the first kind, we write

$$\text{(a)} \quad f_{i+1}(\xi) = \sum_{1}^{N} a_n T_n$$

$$\text{(b)} \quad f_i(\xi) = \sum_{1}^{N} b_n T_n \tag{A.23}$$

Substituting (A.22), (A.23) into equation (A.21c), and applying the proper integrals in Appendix D, we arrive at the following

$$\sum_{1}^{N} a_n U_{n-1}(x) = -\sum_{1}^{N} \frac{bn}{n} \left[\frac{\sin(\cos^{-1}(x))}{C(x)} \right] \tag{A.24}$$

where U_n is the Chebyshev polynomial of the second kind. Multiplying each side of equation (A.24) by $U_n(x)\sqrt{1-x^2}$ and integrating from -1 to 1, we obtain

$$\frac{\pi}{2} a_n = -\sum_{m=1}^{N} \frac{bm}{m} \int_{-1}^{1} \frac{\sin m(\cos^{-1}x)\, U_{n-1}(x)\sqrt{1-x^2}}{C(x)}\, dx \tag{A.25}$$

For a parabolic body, if the asymptotic approximation for $C(x)$ (cf. equation (C.17)) in Appendix C is used, $C(x)$ can be written in the form

$$C(x) = d(1-ex^2) \tag{A.26}$$

where d and e are constants which depend on the body shape, water depth, and the gap size between bottom and hull bottom. Finally, equation (A.25) can be written in the following form

$$a_n = \frac{1}{d\pi} \sum_{m=1}^{N} \frac{bm}{m} \left[\int_{0}^{\pi} \frac{\cos\left(2\left|\frac{m+n}{2}\right|\theta\right)\sin\theta}{1-e\cos^2\theta}\, d\theta - \int_{0}^{\pi} \frac{\cos\left(2\left|\frac{m-n}{2}\right|\theta\right)\sin\theta}{1-e\cos^2\theta} \right] d\theta \tag{A.27}$$

where (m+n)/2, (m-n)/2 must be an integer. A recurrent relation for the integrals inside the bracket is given in Appendix D. Thus, we can readily compare the magnitudes of a_n and b_n . It is found that if the water depth is one-eighth of the length of the parabolic body, in order $|a_n|<|b_n|$, the gap size must be close to one-fortieth of the draft. In other words, the flow changes drastically once a small "leakage" is allowed beneath the bottom of the ship.

APPENDIX B

The appropriate Kutta condition for the moving ship in shallow water.

In the case of two dimensional flow, the Kutta condition can be stated in such a way that the velocity must be bounded at the trailing edge of the moving body.

However, in the case of the interaction in shallow water, we must state the Kutta condition explicitely in order to define a numerical scheme to solve equation (3.22). In other words, the value of the vorticity at the trailing edge must be related to that of the vortex wake immediately behind the trailing edge. In this appendix, we wish to show that the vorticity is continuous from the body through the trailing edge into the vortex wake.

As indicated in Appendix A, the vorticity along the moving ship can be written in the following equivalent form (c.f. equation (A.13))

$$\gamma(x,t) = \frac{1}{\pi^2}\sqrt{\frac{\ell+x}{\ell-x}}\left[-\pi\int_{-\infty}^{-\ell}\sqrt{\frac{\ell-\xi}{-\xi-\ell}}\,\frac{\gamma_w(\xi+Ut)}{\xi-x}d\xi + \int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\,\frac{V(\xi,t)}{\xi-x}d\xi\right.$$

$$\left. -\pi\int_{-\ell}^{\ell}\sqrt{\frac{\ell-\xi}{\ell+\xi}}\,\frac{d\xi}{\xi-x}\,\frac{1}{C(\xi)}\int_{\xi}^{\ell}\gamma(\zeta,t)\,d\zeta\right] \tag{B.1}$$

Consider the first term inside the bracket. By adding and subtracting $\gamma_w(-\ell+Ut)$, we may write

$$\int_{-\infty}^{-\ell}\sqrt{\frac{\ell-\xi}{-\ell-\xi}}\,\frac{\gamma_w(\xi+Ut)}{\xi-x}d\xi = \sqrt{2\ell}\,\gamma_w(-\ell+Ut)\int_{-\infty}^{-\ell}\frac{d\xi}{(\xi-x)\sqrt{-\ell-\xi}}$$

$$+\int_{-\infty}^{-\ell}\frac{\gamma_w(\xi+Ut)\sqrt{\ell-\xi}-\sqrt{2\ell}\,\gamma_w(-\ell+Ut)}{(\xi-x)\sqrt{-\ell-\xi}}\,d\xi \tag{B.2}$$

Note that the second term on the right hand side of the above equation is bounded as $x\to-\ell$, and the integral

$$\int_{-\infty}^{-\ell} \frac{d\xi}{(\xi-x)\sqrt{-\ell-\xi}} = \frac{-2}{\sqrt{x+\ell}} \int_0^{\infty} \frac{d\zeta}{\zeta^2+1} = \frac{-\pi}{\sqrt{x+\ell}} \tag{B.3}$$

by setting

$$\zeta = \sqrt{\frac{-\ell-\xi}{\ell+x}}$$

A similar manipulation operated on the rest of the terms inside the bracket of equation (B.1) leads to the following:

$$\int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}} \frac{V(\xi,t)}{\xi-x} d\xi = \sqrt{2\ell}\, V(-\ell,t) \int_{-\ell}^{\ell} \frac{d\xi}{(\xi-x)\sqrt{\ell+\xi}} + \begin{pmatrix} O(1) \\ \text{as } x \to -\ell \end{pmatrix}$$

and

$$\int_{-\ell}^{\ell} \sqrt{\frac{\ell-\xi}{\ell+\xi}} \frac{d\xi}{\xi-x} \frac{1}{C(\xi)} \int_{\xi}^{\ell} \gamma(\zeta,t)d\zeta$$

$$= \frac{\sqrt{2\ell}}{C(-\ell)} \int_{-\ell}^{\ell} \gamma(\zeta,t)d\zeta \int_{-\ell}^{\ell} \frac{d\xi}{(\xi-x)\sqrt{\ell+\xi}} + \begin{pmatrix} O(1) \\ \text{as } x \to -\ell \end{pmatrix} . \tag{B.4}$$

Since

$$\int_{-\ell}^{\ell} \frac{d\xi}{(\xi-x)\sqrt{\ell+\xi}} = -\frac{1}{\sqrt{\ell+x}} \ln\left(\frac{\sqrt{2\ell}+\sqrt{\ell+x}}{\sqrt{2\ell}-\sqrt{\ell+x}}\right) < \infty$$

$$\text{as} \quad x \to -\ell \tag{B.5}$$

Then, taking the limit as $x \to -\ell$, equation (B.1) can be written as:

$$\lim_{x \to -\ell} \gamma(x,t) = \frac{1}{\pi^2} \sqrt{\frac{\ell+x}{\ell-x}} \left[\frac{\pi^2\sqrt{2\ell}}{\sqrt{x+\ell}} \gamma_w(-\ell+Ut)+O(1)\right] \tag{B.6}$$

Thus, the following expression for the Kutta condition is obtained

$$\gamma(-\ell,t) = \gamma_w(-\ell+Ut) \tag{B.7}$$

APPENDIX C

Numerical procedures

In this appendix, we will set up the numerical scheme to solve the Volterra integral equation (2.26) for the two dimensional flow and the singular integral equation (3.18) which results from allowing flow beneath the bottom of the ship. The sway force and yaw moment acting on each of the bodies can be evaluated numerically once the appropriate vortex strength along the body and the unsteady wake is known.

(a) Numerical solution to the Volterra integral equation

If we non-dimensionalize the length scale with respect to the half-ship length ℓ, equation (2.26) can be written in the form:

$$\int_{-\infty}^{-1} \sqrt{\frac{\xi-1}{\xi+1}}\, \gamma_w(\xi+\tau)\,d\xi = F(\tau;\delta) \tag{C.1}$$

where $\tau=Ut/\ell$ is the non-dimensional "stagger" and $\delta=\eta/\ell$ the lateral separation between the bodies. $F(\tau,\delta)$ is a known function related to the cross flow along the body when the other body is (τ,δ) apart. Let the unknown wake $\gamma_w(\xi+\tau)= f(\zeta)$, then upon change of integration variable, the following Volterra integral equation is obtained:

$$\int_{-\infty}^{T} \sqrt{\frac{\zeta-T-2}{\zeta-T}}\, f(\zeta)\,d\zeta = F(T+1)$$

$$\text{where} \quad T=\tau-1 \tag{C.2}$$

The basic method of solution of this integral equation will be to divide the unsteady wake into a number of "time steps" with the control points located at both ends of each time step.

For this numerical scheme, the integral equation can be rewritten as

$$\sum_{i=2}^{j} \int_{T_{i-1}}^{T_i} f(\zeta)K(T_j;\zeta)d\zeta = F(T_j+1) \qquad (C.3)$$

where

$$K(T_j;\zeta) = \sqrt{\frac{\zeta-T_j-2}{\zeta-T_j}}$$

In defining a numerical scheme, to solve (C.3), we must assume a "starting" step T_1 at which f is vanishing small. Indeed, when the ships are more than three ship-lengths apart, the cross flow $F(t)$ calculated is practically zero. Thus we set:

$$f(T_1) = 0 \quad \text{at} \quad T_1=-11. \qquad (C.4)$$

In the computer program.

Assume the wake vorticity distribution $f(t)$ varies linearly during each time step, the "shape" function for $f(T)$ in each time step may be written as

$$f(T) = \frac{1}{(T_i-T_{i-1})} [(T_i-T)f_{i-1}+(T-T_{i-1})f_i]$$

for

$$T_{i-1} \leq T \leq T_i \quad , \quad i=1,2,3,... \qquad (C.5)$$

This is illustrated in Figure C1.

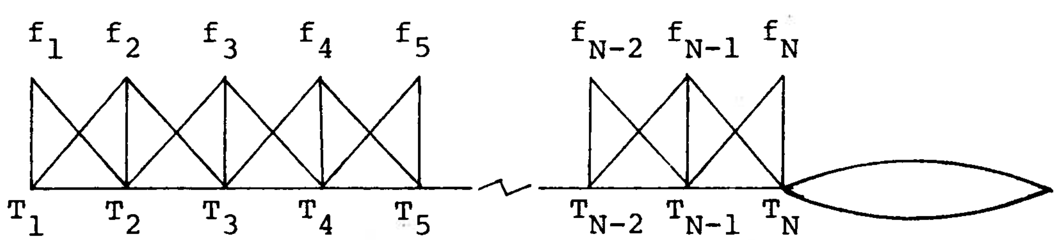

Figure C1 Shape Function for the Volterra Integral Equation

The integral equation (C.3) can now be replaced by the following system.

$$f_j A_j + \sum_{i=1}^{j-1} f_i K_{ij} = F(T_j + 1)$$

$$j=1,2,3,\ldots \qquad (C.6)$$

where

$$f_1 = 0$$

$$T_1 = -11$$

$$A_j = \frac{1}{T_j - T_{j-1}} \int_{T_{j-1}}^{T_j} K(\zeta;T_j)\ (\zeta - T_{j-1})\, d\zeta$$

$$K_{ij} = \frac{1}{(T_i - T_{i-1})} \int_{T_{i-1}}^{T_i} K(\zeta;T_j)(\zeta - T_{i-1})\, d\zeta$$

$$+ \frac{1}{T_{i+1} - T_i} \int_{T_i}^{T_{i+1}} K(\zeta;T_j)(T_{i+1} - \zeta)\, d\zeta$$

Furthermore, A_j and K_{ij} can be integrated analytically using the appropriate relations in Appendix D. Thus the wake vorticity can be completely determined at each time step and there is no matrix inversion involved in obtaining the unknown f_i's of the system (C.6).

In particular, for the case of a moving body past a fixed one, the known function $F(T)$ corresponding to the first approximation γ_w^0 for the wake of the moving body (cf. (2.25)) takes the form

$$F(\tau) = \frac{1}{\pi} \int_{-1}^{1} \sqrt{\frac{1-\xi}{1+\xi}}\, V(\xi,\tau)\, d\xi \qquad (C.7)$$

where

$$V(\xi,\tau) = \frac{U\delta}{\pi^3} \int_{-1}^{1} \frac{\xi+\tau-x}{(\xi+\tau-x)^2+\delta^2} \frac{dx}{\sqrt{1-x^2}} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x-y}\, dy \int_{-1}^{1} \frac{G'(z)}{(y-\tau-z)^2+\delta^2}\, dz$$

and $G(z)=(b/\ell)(1-z^2)$ is the non-dimensional body contour of the identical parabolic bodies.

Interchanging the orders of integration and using the relations of Appendix D to carry out two of the integrations analytically, $F(\tau)$ can be reduced to a double integral of the following form:

$$F(\tau) = \int_{-1}^{1} \frac{\alpha(x,\tau)}{\sqrt{1-x^2}}\, dx \int_{-1}^{1} \beta(x,\lambda,\tau)\, d\lambda \qquad \text{(C.8)}$$

This can be integrated numerically using Simpson's quadrature on the dummy variable λ and Chebyshev quadrature on the remaining integral.

To determine the forces and moments acting on the body, the terms involving the cross flow $V(x,t)$ along the body in equations (2.39) and (2.41) are treated in a similar manner. In actual calculation, an eight-point Simpson's quadrature and an eight-point Chebyshev quadrature give three significant figures (compared to a forty-point Simpson's and a forty-point Chebyshev's quadrature). On the other hand, the terms involving the wake vorticity $\gamma_w(x+\tau)$ in equations (2.39) and (2.41) are more complicated. However, this can be done in a similar way as we treated the kernel in equation (C.1) before, and will not be discussed here again.

The computer program for finding the wake vorticity from the Volterra integral equation (C.2) has been checked in two ways. First, we set the unknown $f(x)$ of equation (C.2) to be a linear function which is zero when $T\leq-11$ and increases linearly for $T>-11$. The function, obtained by integrating the left hand side of (C.2), is used as the input function for the computer program. The difference between the computed result and the assumed linear function is within 0.01% everywhere for a time step of length .2 . Secondly, for the case of a body passing a fixed one, the following lengths for various time steps are used:

length of each time step:

Δ for $|T+1|<3$.

2Δ for $3\leq|T+1|<6$.

4Δ for $|T+1|\geq 6$.

The solution is found to be unsensitive to the length of each time step, the difference between the solutions corresponding to $\Delta=.2$ and $\Delta=.05$ is less than 1%.

(b) Numerical solutions to the Singular Integral equations with Cauchy's integral as the dominant part.

Here, we will set up a numerical scheme to solve the integral equation (3.18) subjected to conditions given by equation (3.19) and (3.20).

The unknown vorticity in equation (3.18) consists of two parts that are different in nature. Namely, those along the ship and those in the wake, which can be treated separately. The time history of the interaction is divided into finite number of time steps. At a given time, the ship is divided into a finite number of segments with a control point chosen properly at each segment. This yields a system of N unknown vorticities satisfied at N-1 control points with an extra condition supplied by the Kutta condition (3.20) and the circulation relation (3.19). If we non-dimensionalize the length scale with respect to the half-ship length ℓ, such a system can be written during the Mth time step as:

$$\text{(a)}\quad \sum_{j=2}^{N}\int_{\xi_{j-1}}^{\xi_j}\gamma(\xi)K_1(x;\xi)d\xi=-\sum_{i=2}^{M}\int_{T_{i-1}}^{T_i}\gamma_w(\xi+\tau)K_2(x;\xi)d\xi+V(x,\tau)$$

$$\text{(b)}\quad \sum_{i=2}^{M}\int_{T_{i-1}}^{T_i}\gamma_w(\xi+\tau)d\xi=-\sum_{j=2}^{N}\int_{\xi_{j-1}}^{\xi_j}\gamma(\xi)d\xi \ . \qquad \text{(C.9)}$$

and

$$\gamma_w(-1+\tau) = \gamma(-1)$$

where

$\tau=\frac{Ut}{\ell}$, is the non-dimensional stagger.

$\Delta T=(T_i-T_{i-1})$, is the distance travelled by the ship over the ith step.

$K_1(x;\xi) = \frac{1}{x-\xi} + \frac{\pi}{C(x)} H(x-\xi)$; H is the heaviside step function

$K_2(x;\xi) = \frac{1}{x-\xi}$

In defining a numerical scheme for the above system, two points we must remember. First, we must pick a "starting time" at which the vortex distributions are negligible. Second, the singular nature of the vortex distribution at the leading edge must be considered.

We will assume the vortex wake equals zero at T_1 and varies linearly over each "time step" ΔT. Remember that the vortex wake remains constant once it is shed. At a given time, we will also assume that the vortex distribution varies linearly over each segment except at the leading edge. At the leading edge, a square root singularity is anticipated. (cf. Appendix A) Therefore, a built-in square root singularity is constructed in that segment. $\gamma(x)$ can now be written as follows:

For the segment belonging to the leading edge:

$$\text{(a)} \quad \gamma(x) = \frac{(x_N-x)\gamma_{N-1}+(x-x_{N-1})\gamma_N}{\sqrt{(x_N-x_{N-1})(x_N-x)}} \qquad x_N<x<x_{N-1}$$

For all the remaining segments:

$$\text{(b)} \quad \gamma(x) = \frac{(x_i-x)\gamma_{i-1}+(x-x_{i-1})\gamma_i}{(x_i-x_{i-1})} \qquad x_{i-1}<x<x_i \qquad \text{(C.10)}$$

It has been shown that (cf. Ref. [9]) for the two dimensional airfoil equation, the numerical result does not depend much on the location of the control point as long as the above shape functions are employed. For convenience, we place the control points at the middle of each segment. The scheme is illustrated in Figure C2.

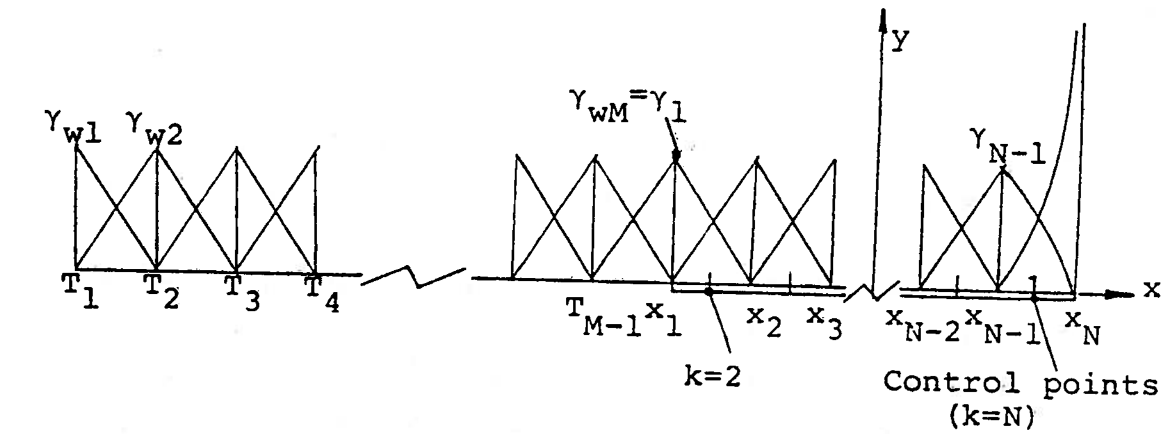

Figure C2 Shape Functions for the system C.9.

Substituting equations (C.5) and (C.10) into the system (C.9), we obtain the following linear algebraic system of N equations and N unknowns of γ_i for the Mth time step.

(a) $$\sum_{i=1}^{N} \gamma_i B_{ik} = \sum_{i=2}^{M} \gamma_{wi} C_{ik} + V_k \qquad k=2,3,\ldots N$$

(b) $$\sum_{i=1}^{N} \gamma_i D_i = -\sum_{i=2}^{M} \gamma_{wi} E_i$$

where

$$B_{1,k} = \int_{-1-(T_M-T_{M-1})}^{-1} \frac{K_2(x;x_k)\,[x+1+(T_M-T_{M-1})]}{T_M - T_{M-1}}\,dx \qquad i=1.$$

$$+ \int_{x_1}^{x_2} \frac{K_1(x;x_k)\,(x_2-x)}{x_2-x_1}\,dx$$

$$B_{i,k} = \int_{x_{i-1}}^{x_i} \frac{K_1(x;x_k)(x-x_{i-1})}{x_i-x_{i-1}}\,dx + \int_{x_i}^{x_{i+1}} \frac{K_1(x;x_k)(x_{i+1}-x)}{x_{i+1}-x_i}\,dx$$

$i=2,3,4\ldots N-2$

$$B_{N-1,k} = \int_{x_{N-2}}^{x_{N-1}} \frac{K_1(x;x_k)(x-x_{N-1})}{x_{N-1}-x_{N-2}}\,dx + \int_{x_{N-1}}^{x_N} \frac{K_1(x;x_k)(x_N-x)}{\sqrt{(x_N-X_{N-1})(x_N-x)}}\,dx$$

$i=N-1$

$$B_{N,k} = \int_{x_{N-1}}^{x_N} \frac{K_1(x;x_k)(x-x_{N-1})}{\sqrt{(x_N-x_{N-1})(x_N-x)}}\,dx \qquad i=N$$

$$C_{i,k} = \int_{-1-(T_M-T_{i-1})}^{-1-(T_M-T_i)} \frac{K_1(x;x_k)(x-x_{N-1})}{T_i-T_{i-1}}\,dx$$

$$- \int_{-1-(T_M-T_i)}^{-1-(T_M-T_{i+1})} \frac{K_2(x;x_k)(1+T_M-T_{i+1}+x)}{T_{i+1}-T_i}\,dx$$

$$D_1 = \frac{T_M-T_{M-1}}{2} + \frac{x_2-x_1}{2}$$

$$D_i = \frac{x_{i+1}-x_{i-1}}{2} \qquad i = 2,3,\ldots N-1$$

$$D_{N-1} = \frac{x_{N-1}-x_{N-2}}{2} + \int_{x_{N-1}}^{x_N} \frac{x_N-x}{x_N-x_{N-1}}\,dx$$

$$D_N = \int_{x_{N-1}}^{x_N} \frac{x_N - x_{N-1}}{\sqrt{(x_N - x_{N-1})(x_N - x)}} \, dx$$

$$E_i = \frac{T_{i+1} - T_{i-1}}{2}$$

and

$$V_k = V(x_k, \tau) \tag{C.11}$$

Note that in the terms $B_{1,k}$ and D_1, we have used the Kutta condition

$$\gamma_1 = \gamma_{wM} \tag{C.12}$$

to obtain the first part. All the integrals in (C.11) except the term V_k can be integrated analytically using the appropriate formulas in appendix D. The term V_k is the resultant cross flow due to the various singularities distributed over the other ship.

For the case of a ship passing a fixed one, the vortex distribution along the fixed body is given by equation (3.21) subjected to the circulation relation (3.22). However, here, we must allow square root singularities at both ends. (cf. Appendix A). In addition to the shape functions (C.10), the segment belonging to the trailing edge must be replaced by:

$$\gamma(x) = \frac{(x_2 - x)\gamma_1 + (x - x_1)\gamma_2}{\sqrt{(x_2 - x_1)(x - x_1)}} \qquad -1 < x < x_2 \tag{C.13}$$

The integral equation (3.21) and (3.22) can be rewritten as

$$\text{(a)} \quad \sum_{i=1}^{N} \gamma_i B_{ik} = V_k \qquad k = 2, 3, \ldots N$$

$$\text{(b)} \quad \sum_{i=1}^{N} \gamma_i D_i = 0$$

where B_{ik} and D_i are given in (C.11) except the following

$$B_{1,k} = \int_{x_1}^{x_2} \frac{K_1(x;x_k)(x_2-x)}{\sqrt{(x_2-x_1)(x-x_1)}} dx$$

$$B_{2,k} = \int_{x_1}^{x_2} \frac{K_1(x;x_k)(x-x_1)}{\sqrt{(x_2-x_1)(x-x_1)}} + \int_{x_2}^{x_3} \frac{K_1(x;x_k)(x_3-x)}{x_3-x_2} dx$$

$$D_1 = \int_{x_1}^{x_2} \frac{(x_2-x)}{\sqrt{(x_2-x)(x-x_1)}} dx$$

and

$$D_2 = \frac{x_3-x_2}{2} + \int_{x_1}^{x_2} \frac{x-x_1}{x_2-x_1} dx \qquad \text{(C.14)}$$

In the above system, the term V_k for the first approximation in our successive scheme takes the form:

$$V_k = \frac{U\eta}{h} \int_{-1}^{1} \frac{s'(\xi) \ d\xi}{(x_k-\xi-\tau)^2+\delta^2} \qquad \text{(C.15)}$$

where $s'(\xi)$ is the change in sectional area of the passing ship.

As far as the first approximation of the cross flow induced on the moving ship is concerned, the term V_k of (C.11) is given by

$$V_k = -\int_{-1}^{1} \gamma_f(\xi,\tau) \frac{x_k+\tau-\xi}{(x_k+\tau-\xi)^2+\delta^2} d\xi \qquad \text{(C.16)}$$

where γ_f is the vortex distribution over the fixed body which satisfies (C.14) and (C.15).

In solving the systems (C.11) and (C.14), the blockage coefficient C(x) along the length of each ship must be given. In general, finding the exact value of C(x), which involves solving the two-dimensional problem of a unit flow passing the given ship section, requires a tremendous amount of work. Fortunately, the following asymptotic approximation for C(x) at any section is found by Taylor (cf. Ref. [10])

$$C(x) \sim \frac{B(x)}{2\varepsilon} + \frac{2h}{\pi} - \frac{B(x)}{2} - \frac{2h}{\pi} \ln 4\varepsilon \qquad (C.17)$$

B(x) = full beam of section

T(x) = draft of section

h = water depth

$\varepsilon = (h-T(x))/h$ = normalized gap size between bottom and hull bottom

In the computer program, a NxN coefficient matrix A is formed for the unknowns γ_N of the system (C.11). The elements in the last row of the matrix are given by D_i in equation (C.11b), while the rest are given by B_{ik} in (C.11a). The LU-decomposition of the matrix A is first obtained by using Gaussian elimination with partial pivoting. The solution to the system of linear equations is then obtained by back-substitution in the LU-decomposition of A. We note that only the elements in the first column of A depend on the size of the time step ΔT, if ΔT is kept constant, the process of LU-decomposition of the matrix A need only be carried out once.

The same procedure is followed in solving the system (C.14), except that the coefficient matrix is completely independent of time.

The computer program has been checked in many ways. The results of the limiting case of C(x) tends to infinity match with that of the two dimensional flow (cf. equations (C.6)). Both the number of segments along the ship and the number of time steps in the course have been varied to check convergence of the numerical integration methods. Although the results are quite sensitive to the value of the blockage coefficient C(x) , yet for a given C(x), the numerical scheme is stable.

APPENDIX D

Table of integrals and series

Here we list some of the most frequently used integrals and series in this text.

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\xi}{\xi-x}(1-\xi^2)^{-1/2} = \begin{cases} 0 & |x|<1 \\ (x^2-1)^{-1/2} & x<-1 \end{cases} \tag{D.1}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\xi}{\xi-x}(1-\xi^2)^{1/2} = \begin{cases} -x & |x|<1 \\ -x-(x^2-1)^{1/2} & x<-1 \end{cases} \tag{D.2}$$

$$\int_{-1}^{1}\frac{d\xi}{\sqrt{1+\xi}(\xi-x)} = \frac{1}{\sqrt{1+x}}\ln\frac{\sqrt{2}-\sqrt{1+x}}{\sqrt{2}+\sqrt{1+x}} \qquad |x|<1 \tag{D.3}$$

$$\int_{y}^{1}\frac{d\xi}{\xi-x}(1-\xi^2)^{1/2} = \frac{\pi x}{2} + \sqrt{1-y^2} - x\sin^{-1}y +$$

$$\sqrt{1-x^2}\ \ln\left[\frac{|x-y|}{1+\sqrt{(1-x^2)(1-y^2)}-xy}\right]$$

$$|x|\ ,\ |y|<1, \qquad x\neq y \tag{D.4}$$

$$\frac{1}{\pi}\int_{-1}^{1}\frac{d\xi}{(\xi-x)}T_n(\xi)(1-\xi^2)^{-1/2} = U_{n-1}(x) \tag{D.5}$$

$$\int_{x}^{1}\frac{T_n(\xi)}{(1-\xi^2)^{1/2}}d\xi = \frac{\sin(n\ (\cos^{-1}x))}{n} \tag{D.6}$$

where T_n, U_n are the Chebyshev polynomials of the first and second kind respectively

$$\frac{1}{\pi}\int_{-1}^{1}\frac{(a\xi+b)\qquad d\xi}{(\xi^2+2c\xi+d)\sqrt{1-\xi^2}} = \text{sgn}\ (c)\ \frac{a(s-d-1)+2cb}{[2s^2(s+2c^2-d-1)]^{1/2}}$$

where $\qquad s = [(d+1)^2-4c^2]^{1/2}$

and $\qquad c^2<d \qquad$ (D.7)

For small c

$$\frac{1}{\pi}\int_{-1}^{1}\frac{a\xi+b}{\xi^2+2c\xi+d}\frac{d\xi}{\sqrt{1-\xi^2}} \sim (b-\frac{ac}{1+d})/[d(d+1)]^{1/2}$$

$$\int\sqrt{\frac{y+2-x}{y-x}}\,dx = G + H$$

$$\int x\sqrt{\frac{y+2-x}{y-x}}\,dx = \left(\frac{y+x-1}{2}\right)G + (y+\frac{1}{2})H \qquad (D.8)$$

where

$$G = -[(y-x)(y-x+2)]^{\frac{1}{2}}$$

$$H = \ell n(y-x+1+G)$$

$$\beta a_{n+1} - 2(2-\beta)a_n + \beta a_{n-1} = \frac{8}{(2n+1)(2n-1)}$$

where

$$a_n = \int_0^{\pi}\frac{\sin\theta\cos 2n\theta}{1-\beta\cos^2\theta}d\theta \qquad (D.9)$$

APPENDIX E

Hydrodynamic interactions for two dimensional unsteady motions in unbounded region

Consider a disturbance (body), traveling in the presence of a stationary object S in ideal fluid. The pressure in the fluid is given by the Bernoulli equation:

$$-\frac{P(x,y,t)}{\rho} = \frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} \tag{E.1}$$

where $\Phi(x,y,t)$ is the velocity potential.

The form acting on the body S may be written as:

$$\vec{F} = -\rho \int_S \vec{n} \left(\frac{\partial \Phi}{\partial t} + \frac{(\nabla\Phi)^2}{2} \right) d\ell \tag{E.2}$$

where $\vec{n}$ is the inward normal on S.

The force consists of two parts which can be treated in different ways. Denote the part that is given by the time derivative of Φ by F_t and the other by F_s as follows:

$$\vec{F}_t = -\rho \int_S \vec{n} \frac{\partial \Phi}{\partial t} d\ell \tag{E.3}$$

and

$$\vec{F}_s = -\rho \int_S \vec{n} \frac{(\nabla\Phi)^2}{2} d\ell \tag{E.4}$$

Furthermore, denote the singularities interior to S by Φ_I and exterior by Φ_O, such that

$$\Phi = \Phi_I + \Phi_O \tag{E.5}$$

(E.4) can be re-written in the following form:

$$F_{sx} - iF_{sy} = i\frac{\rho}{2} \int \left(\frac{dw}{dz}\right)^2 dz \tag{E.6}$$

where $w=w_O+w_I$ is the complex potential, and F_{sx}, F_{sy} are the x and y component of the force $\vec{F}_s$.

If the singularities are multipoles, the force can be further simplifed to

$$F_{sx} - iF_{sy} = \frac{i\rho}{2} \left\{ \sum_j 2\pi i \ \mathrm{Res}\left[\left(\frac{dw}{dz}\right)^2 , \zeta_j\right] \right\} \qquad (E.7)$$

by the virtue of the residue theorem, where ζ_j are the locations of the singularities inside S. A further simplification leads to the following:

$$F_{sx} - iF_{sy} = 2\pi\rho \sum_j \frac{\alpha_j}{(n-1)} \frac{d^{n+1}w_o}{dz^{n+1}} \Bigg|_{z=\zeta_j} \qquad (E.8)$$

where α_j is the strength of the singularity at ζ_j, and $n+1$ is the order of the pole at ζ_j.

A similar analysis on the moment acting on S leads to

$$M_S = \mathrm{Re}\left[+2\pi\rho \sum_j \frac{\alpha_j}{(n-1)} \frac{d^n}{dz^n} \left(z \frac{dw_o}{dz}\right) \Bigg|_{z=\zeta_j}\right] \qquad (E.9)$$

As far as the force $\vec{F}_t$ is concerned, the application of Chertock Formulas provides great convenience. Define an auxiliary function ψ as follows:

$$\nabla^2\psi = 0 \qquad \text{outside S}$$
$$(x^2+y^2)\cdot\psi<\infty \qquad \text{at infinity}$$
$$\frac{\partial\psi}{\partial n} = n_k \qquad \text{on S} \qquad (E.10)$$

Where n_k is the k-th component of $\vec{n}$. Now, the k-th component of $\vec{F}_t$ can be written as follows (cf. Ogilvie [2])

$$F_{tk} = -\sum_j \oint_{S_{\varepsilon j}} \left\{\psi \frac{\partial}{\partial n}\left(\frac{\partial\Phi_o}{\partial t}\right) - \frac{\partial\psi}{\partial n}\frac{\partial\Phi_o}{\partial t}\right\} d\ell_j \qquad (E.11)$$

where $S_{\varepsilon j}$ is a circle of infinitesimal radius ε enclosing the singularity at ζ_j outside S. To calculate the moment, we only need to put

$$\frac{\partial\psi}{\partial n} = xn_y - yn_x$$

in (E.10). Notice that when the moving body is sufficiently far away from the fixed one, Φ_O can be approximated by an incident flow (which represents the moving body in the absence of the fixed one). However, when the bodies are in each other's proximity, Φ_O , Φ_I must be determined simultaneously.

As an illustrative example, consider the force acting on a fixed cylinder due to another cylinder passing in a straight path as shown in Figure E1 .

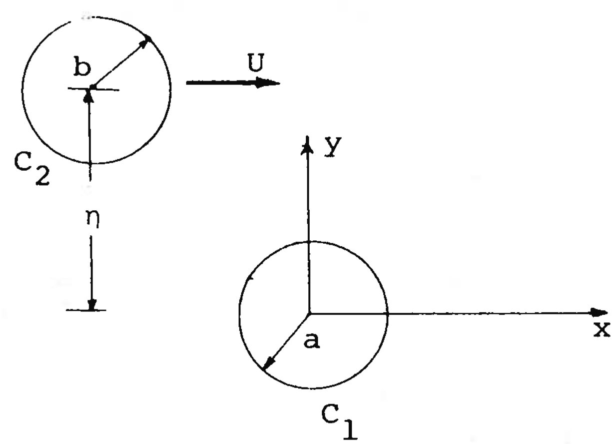

Figure E1.

Axis System for Two Cylinder

The complex potential can be represented by a system of dipoles and images within the two cylinders in the following way

$$W = \sum_{j=1}^{\infty} \frac{\mu_{1j} e^{i(2\alpha-\pi)}}{z-\zeta_{1j}} + \sum_{j=0}^{\infty} \frac{\mu_{2j}}{z-\zeta_{2j}} \tag{E.12}$$

where the subscript 1 denotes the fixed cylinder C_1 and 2 denotes the moving cylinder C_2. μ_{20} is the strength of the dipole representing C_2 in the absence of C_1 . μ_{1j} is the image of μ_{2j-1} in C_1 , μ_{2j} is the image of μ_{1j} in C_2. ζ_{1j} are interior to C_1 while ζ_{2j} are exterior.

The unknowns in (E.12) are given by the following relations (cf. Ref. [3])

$$\alpha = \text{Arg}\ (\zeta_{20})$$

$$|\zeta_{ij}| \cdot |\zeta_{2j-1}| = a^2$$

$$|\zeta_{20}-\zeta_{1j}|\cdot|\zeta_{20}-\zeta_{2j}| = b^2$$

$$\mu_{20} = Ub^2$$

$$\mu_{1j} = \mu_{2j-1}\frac{a^2}{|\zeta_{2j}|^2}$$

$$\mu_{2j} = \mu_{1j}\frac{b^2}{|\zeta_{20}-\zeta_{1j}|^2} \tag{E.13}$$

where ζ_{kj} = complex position of the kj^{th} image

$\zeta_{kj} = \xi_{kj}(t) + i\ \eta_{kj} \qquad k=1,2$

It is obvious that the series in(E.12) converge provide that

$$\zeta > a + b \ . \tag{E.14}$$

From (E.8), we have

$$F_{\ddot{s}x} - iF_{sy} = 2\pi\rho \sum_{j=1}^{\infty} \mu_{1j} \left(\sum_{k=0}^{\infty} \frac{2\mu_{2k}}{(\zeta_{1j}-\zeta_{2k})^3} \right) \tag{E.15}$$

Solution to (E.10) for the body C_1 with $\frac{\partial\psi}{\partial n} = n_x$ takes the following form.

$$\psi = -\frac{a^2x}{x^2+y^2} \tag{E.16}$$

While for $\frac{\partial\psi}{\partial n} = n_y$, we have

$$\psi = -\frac{a^2y}{x^2+y^2} \tag{E.17}$$

Furthermore, if Φ_o in (E.11) is a series of dipoles, (E.11) can be written as

$$F_{tk} = 2\pi\rho \sum_j \frac{\partial}{\partial t} (\vec{\mu}_j\cdot\vec{\nabla})\ \psi\ (\zeta_j) \tag{E.18}$$

where the dipole strength $\vec{\mu}_j$ is written in vector form to take care of the direction of the axis.

Finally, putting

$$\Phi_o = \sum_{j=0}^{\infty} \frac{\mu_{2j}}{z-\zeta_{2j}} \ , \tag{E.19}$$

we have, from (E.16)

$$F_{tx} = 2\pi\rho a^2 \sum_{j=0}^{\infty} \frac{\partial}{\partial t} \left[\mu_{2j} \frac{\xi_{2j}^{\,2} - \eta^2}{(\xi_{2j}^{\,2} + \eta^2)^2} \right] \tag{E.20}$$

and from (E.17)

$$F_{ty} = 4\pi\rho a^2 \eta \sum_{j=0}^{\infty} \frac{\partial}{\partial t} \left[\mu_{2j} \frac{\xi_{2j}}{(\xi_{2j}^{\,2} + \eta^2)^2} \right] \tag{E.21}$$

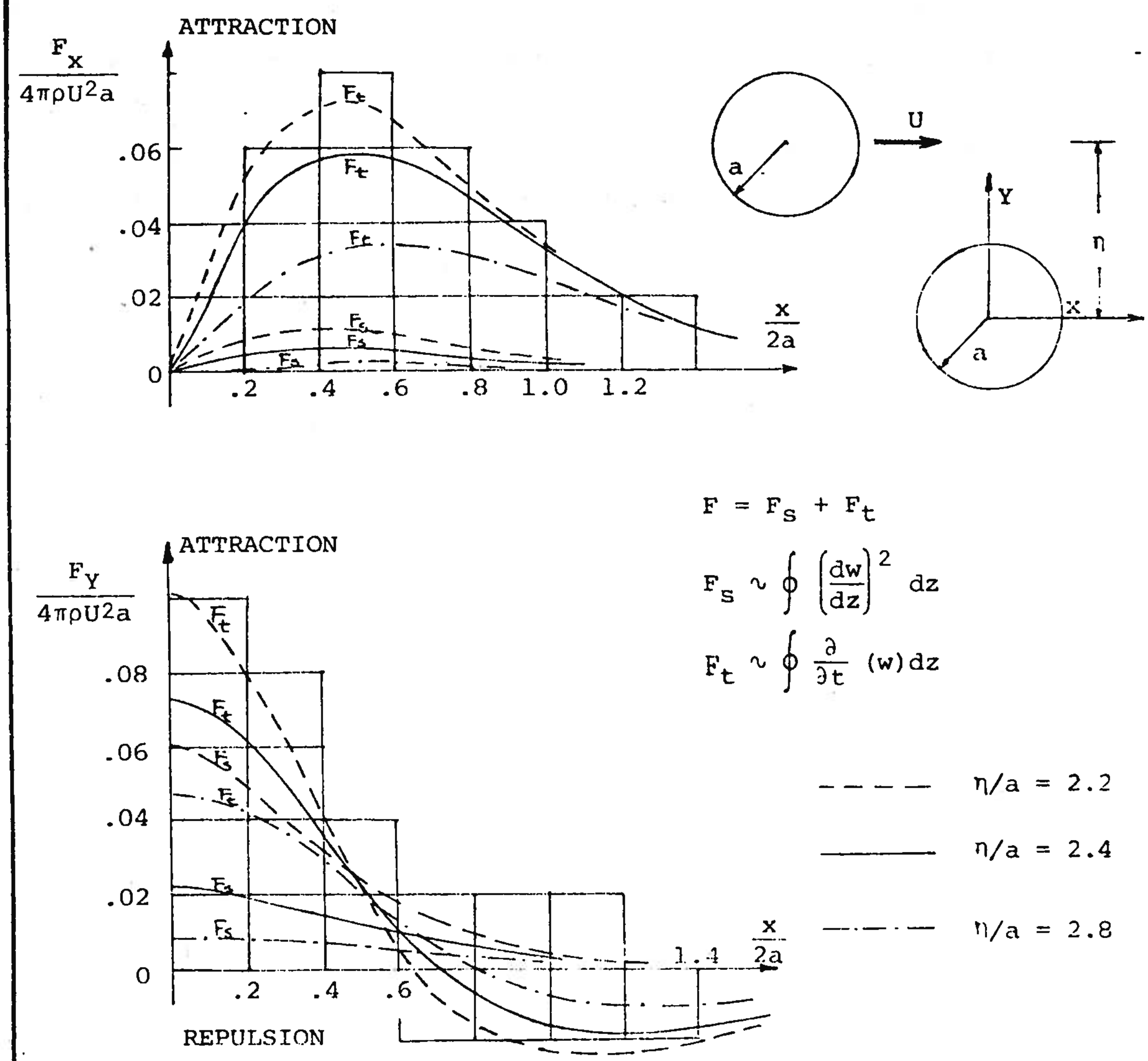

Figure E2.
FORCES ACTING ON FIXED CYLINDER

The forces acting on the fix cylinder have been computed using equations (E.15), (E.20) and (E.21). The results from these computations are shown in Figure E2. Both the horizontal and vertical forces are considered positive if they are attractive (i.e. directed towards the moving cylinder). Only the results for positive values of x are shown since they are the same for negative x. From the figure we can see clearly that the magnitude of the interaction forces increases significantly as the separation distance (η) decreases. The F_t component of the interaction force is the largest, which implies that the $\partial\Phi/\partial t$ term dominates in Bernoulli's equation. However, as the separation distance decreases the F_S component becomes more and more important. In linear theory the F_S component is usually neglected since it depends on the square of the velocity. The results of Figure E2 seem to indicate that this approximation becomes less accurate as the separation distance decreases.