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ON THE STEADY TURN OF A SHIP

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ABSTRACT

A wing of zero thickness and small aspect ratio is used as a mathematical model for a ship in steady turn. The shape of the wing is given by the projection of the hull on the vertical plane of symmetry and the aspect-ratio by the relation draft/length of the ship. As the ship moves in a turn, the angle of attack in each section is different; this fact is approximately simulated in the model by adding a camber to the wing and letting the wing follow a straight course.

Near the ship three regions are identified: the region downstream of the section of maximum draft (span), in which the Kutta condition is required to be satisfied, the region near midship, and the region near the bow. In the first region, the condition on the free surface is a rigid wall condition, thus permitting a reflection into the upper space. The acceleration potential concept is used in order to describe the flow in this region. In the region near midship, a reflection into the upper space is also possible. The method of solution is similar to the usual slender body theory for ships. In the region near the bow, where the free surface suffers large deformations, the rigid-wall condition at the free surface is no longer valid. A different problem is formulated here, and Fourier transform approach is used to solve the problem in this region.

The method of matched asymptotic expansions was used to solve the problem. Therefore, the condition at ∞ in each of the problems mentioned in the preceding paragraph is given by the far-field expansion, which describes the flow far from the ship.

For a simple case, when the ship has a constant draft in the bow region, the shape of the free surface near the ship in the bow region has been calculated.

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NOTATION

a	characteristic length in the bow region
$b(x)$	camber function
$c(p)$	auxiliary function
c_1 , c_2	constants arising in the solution of the integral equation
f	function defining a hull geometry
$f(y)$	auxiliary function
F	Froude number
$F(u)$	auxiliary function
F_I	imaginary part of $\phi_z^*(x,y,0)$, $y > -h$
F_R	real part of $\phi_z^*(x,y,0)$, $y > -h$
g	acceleration of gravity
$g(y)$	auxiliary function
$G(u)$	auxiliary function
$G_0(x,y,z)$	function defining the hull surface
$G_1(x,y,z)$	function defining the hull profile
$G_2(x,y,z)$	function defining the free-surface
$h(x)$	function defining the ship draft
H	non-dimensional free-surface elevation
$H(x)$	Heaviside step function
$I_1(x)$	Bessel function
$J_1(x)$	Bessel function
k	variable used in the transformed plane
$K_1(x)$	Bessel function
ℓ	variable used in the transformed plane

L	ship length
$L(x)$	modified Struve function
M_{ij}	homogeneous solution for the usual near-field
p	pressure
$s(p)$	auxiliary function
U	ship speed
u, v	variable
$\vec{V}$	velocity vector
x, y, z	Cartesian coordinates
α	parameter defined as $\alpha = k^2U^2/g$
γ	constant used in defining the order of magnitude of $b(x)$
δ	small parameter
$\delta(x)$	Dirac delta function
ε	slenderness parameter
ζ	variable (real or complex)
η	variable
$\eta(x,z)$	the free surface deformation
μ	Rayleigh viscosity
$\mu(u)$, $\mu_{ij}(u)$	solution of an integral equation
ν	parameter defined as $\nu = g/U^2$
ξ	variable
ρ	water density
$\sigma(x)$	dipole density
$\sigma^*(k) = \sigma^*_I + \sigma^*_R$	Fourier transform of the dipole density
$\phi(x,y,z)$	perturbation velocity potential

$\phi^*(k,y,z)$	Fourier transform of the velocity potential $\phi(x,y,z)$
$\Phi(x,y,z)$	velocity potential
$\Omega(x,y,z)$	acceleration potential

I. INTRODUCTION

In the study of the maneuverability of a surface ship many physical processes are involved. These processes interact among themselves, creating a very complicated picture, which is difficult to describe using any realistic mathematical model.

The presence of the free-surface and its behavior along the ship are certainly the source of the most intriguing and difficult phenomena to be analyzed in the present context, as in many problems of ship hydrodynamics. The simple observation of a ship as it goes through the water is sufficient to show that the deformation of the free-surface does not follow a regular pattern along the ship. In the region near the bow one can observe a very large deformation of the free-surface, and eventually, as the ship speed increases (or the bow becomes blunter), spilling of water occurs, which is more commonly known as "breaking". Downstream of the bow region, the deformation of the free-surface becomes more gentle, as if the effects of gravity were strong compared to the dynamic acceleration of the water particles, to the point of pulling the free-surface to a nearly horizontal plane. In ship maneuverability problems one should not neglect these distinct flow patterns, since, for instance, the turning moment of a ship can be largely affected by these distinct patterns of flow.

In the present work, in an attempt to get a better understanding of the flow around a ship in steady turn, a mathematical model is adopted which has the most relevant features of the real flow and which is simple enough that a boundary value problem can be formulated and solved in a rational way. We replace the ship hull by a wing of small aspect ratio and zero thickness. The aspect ratio is given by the draft/length ratio (the draft is assumed to be small compared to the length) and the shape of the wing by the projection of the ship hull on the vertical plane of symmetry. As the ship moves in a steady turn in calm water, the angle of attack for each section will be different (see Fig. 1). As an approximate description of this fact, a camber is added to the wing model and the wing is assumed to follow a straight path.

In linearized wing theory one can separate the thickness effects from the camber effects. In this thesis only the camber effects are considered. The thickness effect (which is not considered here) would give a zero net lateral force and a symmetric (with respect to the ship plane of symmetry) displacement of the flow.

Because of the model adopted and some assumptions, about order of magnitude, to be made later on, the ship hull will be referred to as a "slender body" or a "slender wing" as well.

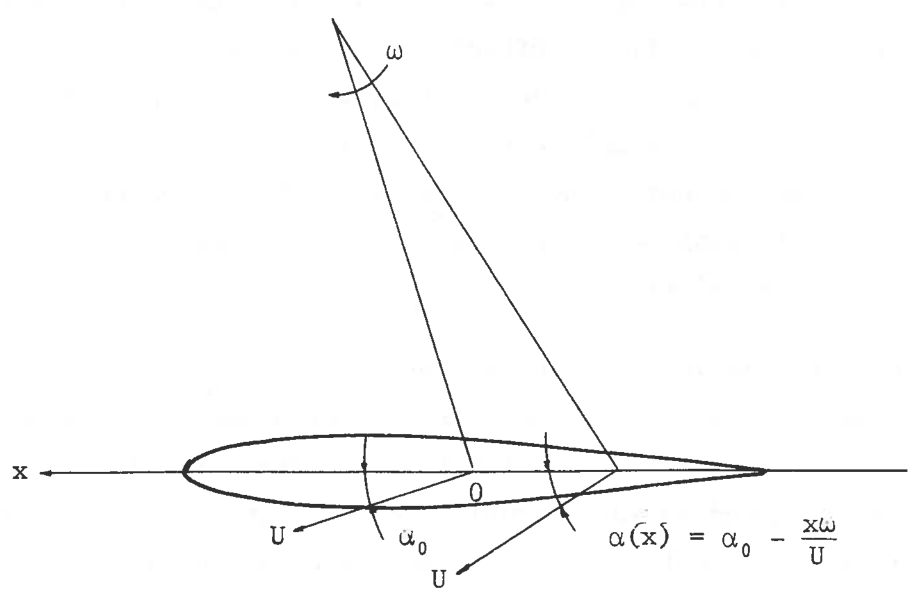

FIGURE 1

THE MATHEMATICAL MODEL

In an earlier work, Fedyayevskiy and Sobolev [2]*used the same type of model in order to find the forces and moments acting on a ship hull in steady turn. They write the integral equation for a lifting surface, where the unknown is the vortex density, then they simplify the equation using the fact that the wing is of small aspect ratio, and they obtain a solution. Unfortunately, their solution gives zero lift generation in the section downstream of the section of maximum span (which corresponds to the maximum draft), which is all right for a flat wing but certainly is not good for a cambered one, as in the present model. Another point that should be observed is that their solution is valid for an infinite fluid region; this would be of no harm if the ordinary slender body theory were to be used, since in this theory the free-surface boundary condition reduces to a rigid wall condition, and a reflection on the upper side is possible, therefore transforming the free-surface problem into an infinite-fluid problem (see details in Chapter II). However, the use of the ordinary slender body theory does not allow us to analyze properly the variation of the flow pattern near the bow.

A different approach is necessary in order to solve the boundary value problem derived from the model, if these effects are of importance. A quick look into the literature related to the present work could provide a guide in how to attack the problem. Jones [4] has studied the problem of a low aspect ratio wing in an infinite fluid region. He used an intuitive "strip approach", and he concluded that "for a low-aspect-ratio wing the development of the lift depends on the expansion of the sections in a downstream direction." Jones' theory says that a decreasing section would require a negative lift and infinite pressure on the trailing edge, thus violating the Kutta condition. A more recent approach to the problem of a low-aspect-ratio wing in an infinite fluid region was given by Wang [15]. He used the method of "matched asymptotic expansions" and he was able to give "a unified approach for treating wing problems of almost all aspect ratios." But yet the question of how

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* Numbers in brackets denote references at the end of the text.

to treat the sections downstream of the section of maximum span, for a cambered wing, was left open (for the subsonic case).

Two exact solutions to the linear problem for wings of special shapes in an infinite fluid region are known. The circular planform wing problem was solved by Kinner [5] and the elliptic by Krienes [6]. Both authors solved the problem in terms of an "acceleration potential" [11] expressed in special coordinate systems. The solutions were given as a series, and it was possible to find solutions corresponding to shock-free entry, which were superposed on the solution for a flat plate at an angle of incidence to give the solution to the problem of a cambered wing with an angle of attack. Kinner's solution was given in terms of Legendre functions, which enables one easily to do a thorough analysis, in particular in the wake region. Krienes' solution is more closely related to the present problem. From his solution it is possible to obtain the results for an elliptic wing of a very low aspect ratio. This was done by Jones, and it showed a good agreement with his "strip approach."

Wu [17] showed how to use the acceleration potential and slender body theory together in analyzing the flow around wings that cannot be treated by the method of Jones or of Fedyayevskiy and Sobolev. Wu's procedure can be used for cambered wings with span decreasing in the downstream direction. (In fact, he develops the method even for time-dependent flows.)

The method of matched asymptotic expansions is used in order to find the solution to the present problem. This method requires a description for the regions far from the ship (the "far field" region). The method of solution for the far field is very similar to the slender-body-theory of ship motions by Ogilvie and Tuck [8]. In the near field, the coordinates are stretched and an analysis of orders of magnitude simplifies the boundary value problem (see Chapter II).

In the sections downstream of the section of maximum span, the sections are decreasing in span and the Kutta condition is required. In this region the concept of "acceleration potential" is used, as developed especially by Wu[17]. In the mid-ship sections the solution is the one given by the ordinary slender body theory. For the region near the bow, where large deformation of the free-surface is present and the flow has characteristics of a high Froude number flow [9], some complications are introduced [10]. The formulation of the problem in this region leads us to a boundary value problem, with a partial differential equation in two dimensions (y and z), but with a condition on the free-surface containing a derivative in the x-direction. A Fourier Transform approach is used to solve this problem. In the bow region the free-surface elevation near the ship is calculated numerically for the simple case of a ship of constant draft.

II. GENERAL FORMULATION

1. Definitions and Assumptions

In order to formulate the general boundary value problem corresponding to the mathematical model presented in the introductory chapter, let us make use of Fig. 2. The above mentioned model corresponds to an idealization of the real flow around a ship (which has some restrictions on its geometrical characteristics, to be mentioned later), in steady turn, in the presence of a free-surface. Let us take a coordinate system fixed on the ship such that the origin is located at the intersection of the bow and the undisturbed free-surface (the ship is assumed not heaving, pitching , or undergoing any other motion except the steady turn, the water is assumed to be calm),

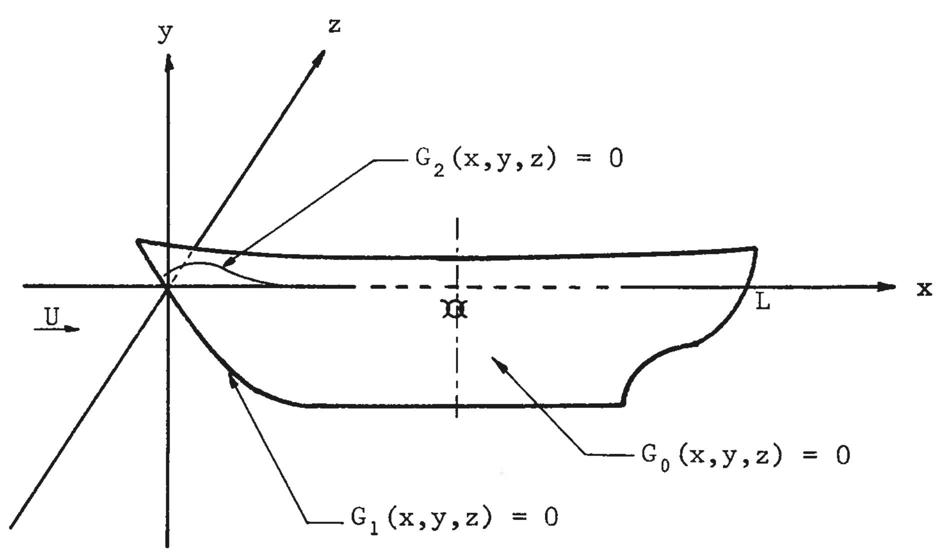


FIGURE 2

THE COORDINATE SYSTEM AND THE BODY DEFINITIONS the x-axis is taken along the ship length, the y-axis vertical upwards and the z-axis horizontal. The section of maximum draft (span) is assumed to be located at mid-ship. In this way, a uniform flow with velocity U is assumed parallel to the positive x-axis.

The ship hull is defined by the following relations:

The hull surface is expressed by

$$G_0(x,y,z) = z - b(x) = 0 \tag{1}$$

The hull contour or ship profile, defined as the hull projection on the vertical plane of symmetry, is

$$G_1(x,y,z) = y - h(x) = 0 \tag{2}$$

The free-surface will be deformed by the movement of the ship. It will be defined by

$$G_2(x,y,z) = y - \eta(x,z) = 0 \tag{3}$$

One should note that, in defining the hull geometry, certain simplifications were made, i.e., it was assumed that the hull surface and contour could be defined by using only two coordinates, x and y or z . It is possible, however, to take more general definitions, i.e. to define the hull surface and contour by three coordinates. This more general case was not followed here since it would introduce some more complications to the treatment without introducing any interesting new feature to the problem.

Concerning the geometry of the hull, the following assumptions about the orders of magnitude will be made:

$$h(x) = O(\varepsilon)$$

$$b(x) = O(\varepsilon^{1+\gamma}) , \qquad 0 < \gamma < 1/2$$

$$\frac{\partial f(x)}{\partial x} = f'(x) = O[f(x)]$$

where:

$f(x)$ is any quantity describing the hull geometry.

ε is the slenderness parameter, say for example, the aspect ratio as defined in the introduction.

The quantity γ is used in defining the order of magnitude of $b(x)$ because of some problems we will face later, when applying the body boundary condition. Note that the above assumptions are part of what are called "the slender-body-theory assumptions".

Concerning the fluid and the fluid region, the following assumptions will be made:

- the fluid is ideal;
- the fluid region extends to infinity in every direction in the lower half space.

Finally, concerning the flow and the mathematical approach, the assumptions below will be accepted.

- There are two different characteristic lengths besides the ship length, and so it is possible to define two different Froude numbers. Each will have special significance in a specific region.
- The boundary value problem derived from the mathematical model can be linearized.

The existence of different characteristic lengths is suggested by observations and measurements in the towing tank. Experiments [18] show that the flow in the bow region presents characteristics of high

Froude number, much higher than the Froude number defined by using the ship length.

The Froude number based on the ship length is useful in describing the flow over most of the ship. However, in regions near the bow this Froude number based on the ship length is not so useful as a "bow Froude number" based in a local characteristic length. The latter has been used not only in Baba's experiments, but also in other towing tanks for relating data from tests on different scales. As one can expect, this "bow Froude number" must be higher, which implies the existence of a smaller characteristic length in this region; let us denote it by a . It will be assumed that

$$a = O(\varepsilon^{1/2}) \quad .$$

About the assumption made with respect to the linearization of the boundary value problem, one should say that it is rather strong. One can say, for example, that if the angle of attack is not small, separation occurs and our mathematical model breaks down. Or one can say that maneuverability problems are highly non-linear. This is quite true, but the non-linear problem is quite intractable and it is our hope that our linearized problem will lead us to some useful results which can be applied even if the angle of attack is not so small as it is in many airfoil problems.

2. The General Boundary Value Problem

The fluid being ideal and the flow irrotational and steady, the existence of a velocity potential $\Phi(x,y,z)$, is assumed. As the fluid is incompressible, the continuity of the medium is expressed by the Laplace equation:

$$0 = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} \qquad \text{in the fluid region} \qquad (L^i)$$

The condition that the ship hull is a solid non-porous boundary is expressed by the body condition

$$0 = \frac{D}{Dt} G_0(x,y,z) \quad \text{on} \quad G_0(x,y,z) = 0 \qquad (H^i)$$

where the operator D/Dt indicates the substantial derivative, which is given in the steady-state case by

$$\frac{D}{Dt} = [\frac{\partial}{\partial t} + \vec{V} \cdot \nabla]$$

$$= \vec{V} \cdot \nabla$$

where

$$\vec{V} = \nabla\Phi$$

The mathematical model used is a wing and therefore a condition on the trailing edge must be applied, i.e., the Kutta condition, which says that one must have bounded velocity on the trailing edge:

$$|\frac{\partial\Phi}{\partial x}|, \ |\frac{\partial\Phi}{\partial y}|, \ |\frac{\partial\Phi}{\partial z}| \quad \text{bounded on} \quad G_1(x,y,z) = 0, \ x > L/2 \qquad (K^i)$$

On the free-surface, two conditions must be satisfied. The dynamic condition expresses the fact that on the free-surface the pressure is constant and therefore that

$$g\eta + \frac{1}{2}[(\frac{\partial\Phi}{\partial x})^2 + (\frac{\partial\Phi}{\partial y})^2 + (\frac{\partial\Phi}{\partial z})^2] = \text{const. on} \quad y = \eta(x,z) \qquad (A^i)$$

The kinematic condition says that a particle on the free-surface must remain on the free surface. This fact is expressed by

$$0 = \frac{D}{Dt} G_2(x,y,z) \quad \text{on} \quad y = \eta(x,z) \qquad (B^i)$$

Besides the above described conditions, one should not expect to have waves far upstream. This condition is called "radiation condition" and will not be precisely stated now. We will call it condition (R^i).

The velocity potential $\Phi(x,y,z)$ can be written as a sum of the potential for the uniform stream in the positive x direction and a perturbation potential:

$$\Phi(x,y,z) = Ux + \phi(x,y,z) \tag{4}$$

Introducing (4) in the above formulated problem, one has:

$$0 = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \quad \text{in the fluid region} \tag{L}$$

$$0 = Ub' + b'\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial z} \quad \text{on} \quad z = \pm b(x) \tag{H}$$

$$0 = g\eta + U\frac{\partial\phi}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2\right] \quad \text{on} \quad y = \eta(x,z) \tag{A}$$

$$0 = U\frac{\partial\eta}{\partial x} + \frac{\partial\eta}{\partial x}\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y} + \frac{\partial\eta}{\partial z}\frac{\partial\phi}{\partial z} \quad \text{on} \quad y = \eta(x,z) \tag{B}$$

$$\left|\frac{\partial\phi}{\partial x}\right|, \left|\frac{\partial\phi}{\partial y}\right|, \left|\frac{\partial\phi}{\partial z}\right| \quad \text{bounded on} \quad G_1(x,y,z) = 0, \quad x > L/2 \tag{K}$$

Radiation Condition (R)

Note: The following notation will also be used.

$$\phi_x = \frac{\partial\phi}{\partial x}, \quad \phi_y = \frac{\partial\phi}{\partial y}, \quad \text{etc.}$$

f' the x derivative of f, if f is any geometric hull characteristic.

It is next assumed that the perturbation potential can be expressed by an asymptotic expansion

$$\phi(x,y,z) = \sum_{n=1}^{N} \phi_n(x,y,z;\varepsilon) \tag{5}$$

such that

$$\left|\frac{\phi_{n+1}}{\phi_n}\right| = o(1) \quad \text{as} \quad \varepsilon \to 0$$

and therefore one can also write

$$\Phi(x,y,z) = \sum_{n=0}^{N} \phi_n(x,y,z;\varepsilon) \tag{6}$$

where

$$\phi_0(x,y,z) = Ux \tag{6a}$$

In this thesis only the first term in the perturbation velocity potential will be considered and so we will accept the simpler notation,

$$\phi(x,y,z) \sim \phi_1(x,y,z,\varepsilon) \tag{7}$$

Accepting (7) the problem defined by (L),(H),(A),(B),(K), and (R) is also the problem for $\phi_1(x,y,z)$. Later on, this problem will be further simplified after making suitable assumptions.

3. The Method of Matched Asymptotic Expansions

In order to best describe the flow around the ship it is convenient to visualize the flow with different "magnifications" in several regions with individual flow characteristics. This is the usual way employed by those working with the method of matched asymptotic expansions.

Let us first identify the "far-field region". In this region one wants to have an overall picture. For that, one has to be far from the ship (hence the name far-field) and therefore all the details near the ship are lost and cannot be "seen". As will be described later, all that can be "seen" is a flow apparently generated by a line of singularities (in the present case: dipoles).

If one wants to know the details near the ship, some "magnification" has to be used, which is equivalent to stretching the coordinates. One then moves to regions close to the ship - the near field region. There is not a precise boundary between the far and near field, but one can assume roughly that the near-field is characterized by distances of order ε .

In order to stretch the coordinates, let us assume that

$$x = X\delta \; *$$

$$y = Y\varepsilon$$

$$z = Z\varepsilon$$

and

$$\frac{\partial}{\partial X} = O(1)$$

$$\frac{\partial}{\partial Y}, \frac{\partial}{\partial Z} = O(1)$$

Note: For simplification in the algebra involved and for typographic reasons, the above notation will not be used, and the following relations will be understood in the near-field.

$$x = O(\delta)$$

$$y, z = O(\varepsilon)$$

$$\frac{\partial}{\partial x} = O(\delta^{-1}) \quad ; \quad \frac{\partial}{\partial y}, \frac{\partial}{\partial z} = O(\varepsilon^{-1})$$

---

* The reason for using δ and not ε will be clear below. In the present work δ will be different from ε .

In physical terms, this means that in the near field one stretches the coordinates in order to see the details since the gradient of the flow characteristics is very large. In the above relations, assuming $\delta \neq \varepsilon$, we are allowing for two different scales even in the near field, which means two different rates of change. It will also be assumed that $\delta > \varepsilon$, which means that the change in the flow characteristics in the longitudinal direction is smaller than the change in the transverse directions.

If $\delta = 1$, one has the usual slender body theory. Then the above assumptions, together with those made about the orders of magnitude of the hull geometry (see Section II.1) form what are called the slender body assumptions. This set of assumptions seems to be reasonably good for the near-field in general. The region in which it is approximately valid will be called the "usual near-field region".

However, a closer look at the physical picture seems to indicate that δ should be smaller than unity at the end, near the bow. Very close to the bow the usual slender ship theory does not give good results and puts too much emphasis in gravity effects. Formally, if we stretch the longitudinal coordinate in slender ship theory, we are deemphasizing the gravity effects. We will therefore assume that $\delta < 1$. Observations of the flow near the bow show that the aforesaid is true; one sees a sudden longitudinal deformation of the free-surface, to the point that the action of gravity, combined with surface tension, is not strong enough to hold the water particles, and the spilling and wave breaking occur. Of course this is an extreme case beyond our hope to analyze; our concern is to be able to get some results for the case before the breaking starts.

Reported observations and experiments [18] in the towing tank seem to indicate the necessity of using a local Froude number, higher than that usually defined by the ship length, for a best description of the flow. This implies the existence of a local characteristic length, a , much smaller than L . This length is comparable to the distance from the bow to the section in consideration. If an accurate description of the flow characteristics is desired in this region, one must stretch the coordinates, in such a way that the unit length in the stretched coordinate is of the same order of magnitude as the characteristic length. In Section II.1, it was assumed that $a = O(\varepsilon^{1/2})$; therefore, let us take

$$\delta = \varepsilon^{1/2}$$

In the bow region, this assumption seems to be reasonable, since it will lead to a formulation of the problem that admits local waves and high fluid acceleration, not negligible compared to the acceleration of gravity, thus manifesting an upward motion of fluid particles in contrast to what is predicted by the usual slender body theory; i.e., a clamped free surface on the plane $y = 0$, with zero vertical velocity. It should be mentioned that it has been noticed that the usual slender body theory puts much emphasis on the gravity effects. The "bow-near-field" will be identified by using $\delta = \varepsilon^{1/2}$ in the above formulation and using the same assumptions concerning the hull geometry.

For purpose of analysis, it is sufficient to identify these two near-fields. However, it is interesting to note that for $x > L/2$, one extra condition, the Kutta condition is required, i.e., for the regions where the draft (span) decreases as one moves downstream.

Table I on the following page summarizes symbolically what was discussed above.

Table I

SEVERAL REGIONS AND THEIR PROPERTIES

Regions	Far Field	Usual Near Field	Bow Near Field
Dimensions			
x	$O(1)$	$O(1)$	$O(\varepsilon^{1/2})$
y	$O(1)$	$O(\varepsilon)$	$O(\varepsilon)$
z	$O(1)$	$O(\varepsilon)$	$O(\varepsilon)$
Rate of Change			
$\partial/\partial x$	$O(1)$	$O(1)$	$O(\varepsilon^{-1/2})$
$\partial/\partial y$	$O(1)$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1})$
$\partial/\partial z$	$O(1)$	$O(\varepsilon^{-1})$	$O(\varepsilon^{-1})$
Characteristic Length	$L = O(1)$	$H = O(\varepsilon)$	$a = O(\varepsilon^{1/2})$
Froude Number	$\frac{U}{\sqrt{gL}} = O(1)$	$\frac{U}{\sqrt{gH}} = O(\varepsilon^{-1/2})$	$\frac{U}{\sqrt{ga}} = O(\varepsilon^{-1/4})$

III. THE POTENTIAL FUNCTION

1. The Far-Field

The far-field, as identified before, is the region of a distance equal or greater than unity from the ship. At that distance, all the details near the ship are lost and only a disturbance caused by the hull can be seen. A ship turning creates a flow pattern which is hardly symmetric. The asymmetric disturbance caused by the hull can be represented in the far-field description by a line of horizontal dipoles. The source distribution is not considered since the ship is assumed to be of zero thickness. One should not expect to be able to satisfy the body condition nor the Kutta condition by merely using the far-field description, however, the radiation condition (R) must be satisfied.

To these physical arguments, it is possible to give a mathematical formulation. First, the Laplace equation is replaced by the Poisson equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \sigma(x)\delta(y - y_0)\delta'(z) \qquad (8)$$

where

- δ is the Dirac-delta function
- δ' is its derivative
- $y_0<0$ will be set equal to zero at the appropriate time
- $\sigma(x)$ is the dipole density

The free-surface conditions can be linearized to:

$$\begin{aligned} 0 &= g\eta + U\phi_x \\ 0 &= U\eta_x - \phi_y \end{aligned} \qquad \text{on} \quad y = 0\ *$$

---

* The application of the free-surface conditions on the plane $y=0$ implies that these conditions have been transferred from the actual free-surface [16].

and then combined to give:

$$0 = U^2\phi_{xx} + g\phi_y \qquad \text{on } y = 0 \tag{9'}$$

Perhaps the easiest way to satisfy the radiation condition, which excludes the possibility of waves upstream of the ship, is to introduce the concept of the "Rayleigh fictitious viscosity" μ , which is set equal to zero at an appropriate time (see Ogilvie & Tuck[8]). Condition (9') is then modified to:

$$0 = (U\frac{\partial}{\partial x} + \frac{\mu}{2})^2\phi + g\phi_y \qquad \text{on} \quad y = 0 \tag{9}$$

Two alternative solutions to (8) and (9) can be found as follows:

$$\phi(x,y,z) = \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\frac{dk e^{ikx}\sigma^*(k)|k|z\,K_1(|k|\sqrt{y^2+z^2})}{\sqrt{y^2+z^2}}$$

$$- \frac{iU^2 g}{4\pi^2}\int_{-\infty}^{\infty} dk e^{ikx}k^2\sigma^*(k)\lim_{\mu\to 0}\int_{-\infty}^{\infty}\frac{d\ell\,\ell\exp[i\ell z + y\sqrt{k^2+\ell^2}\,]}{\sqrt{k^2+\ell^2}\,[\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - \frac{i\mu}{2})^2]} \tag{10a}$$

and

$$\phi(x,y,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx}\sigma^*(k)\lim_{\mu\to 0}\left[-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\ell\,\ell\exp[i\ell z + y\sqrt{k^2+\ell^2}]}{\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - \frac{i\mu}{2})^2}\right] \tag{10b}$$

where

K_1(---) is the Bessel Function (see [1]).

* indicates the Fourier transform defined as:

$$FT[f(x)] = f^*(k) = \int_{-\infty}^{\infty} dx\, e^{-ikx}f(x) \quad .$$

The above expressions satisfy the 3-D Laplace equation in the fluid region except at the line of singularities, satisfy the "radiation condition", i.e., there are no waves upstream of the ship, and satisfy also the free-surface condition (9').

However, the solutions are singular as one approaches the body, consequently not allowing the application of the body boundary condition as mentioned above. This fact is the reason why one cannot determine the dipole density right now. It is possible to find the behavior of these solutions as one approaches the ship (see Appendix I) and then determine the dipole density, $\sigma(x)$, through a matching process to be presented later on.

2. The Near-Field

As mentioned before, for a detailed description of the flow characteristics near the ship, i.e., in the near-field, it is necessary to stretch the coordinates. It was also mentioned that two near-fields are considered, the "usual near-field" and the "bow near-field".

2.1 The Usual Near-Field

In this region the variation of the flow characteristics in the transverse direction is much larger than the variation observed in the longitudinal direction and so one assumes that differentiation in the longitudinal direction does not change order of magnitude whereas in the transverse directions it does by $O(\varepsilon^{-1})$.

Let us accept then that

$$\text{Froude Number} = O(1)$$

$$\frac{\partial}{\partial x} = O(1)$$

$$\frac{\partial}{\partial y}, \frac{\partial}{\partial z} = O(\varepsilon^{-1})$$

in the regions near the ship, except very close to the bow.

With these assumptions, expressions (L), (H), (A), (B), and (K) take the form (if only the leading terms are considered):

$0 = \phi_{yy} + \phi_{zz}$	in fluid region	(L^i)
$0 = \phi_y$	on $y = 0$	(F^i)
$\phi_z = Ub'(x)$	on $z = \pm 0$, $0 > y > -h$	(H^i)
Kutta condition		(K^i)

About the above formulated problem, it is worth to note that:

a) In conditions (F^i) and (H^i) the boundary condition was transfered from the actual place to the mean position. For the transfering of the boundary condition (H^i) it was necessary to use the assumption that $b(x) = O(\varepsilon^{1+\gamma})$, with $0 < \gamma < 1/2$.

b) The usual radiation condition is missing from this formulation of the boundary value problem in the near-field. A condition at infinity which guarantees the uniqueness of the solution is provided by the matching process.

c) Condition (F^i) is the result of combining conditions (A) and (B) by eliminating the free-surface deformation and derivatives. Only the leading term is considered.

Figure 3 shows a sketch of the problem formulated above for the range $x < L/2$, in which the Kutta condition is not required. The section located at $x = L/2$ is assumed to have the maximum span.

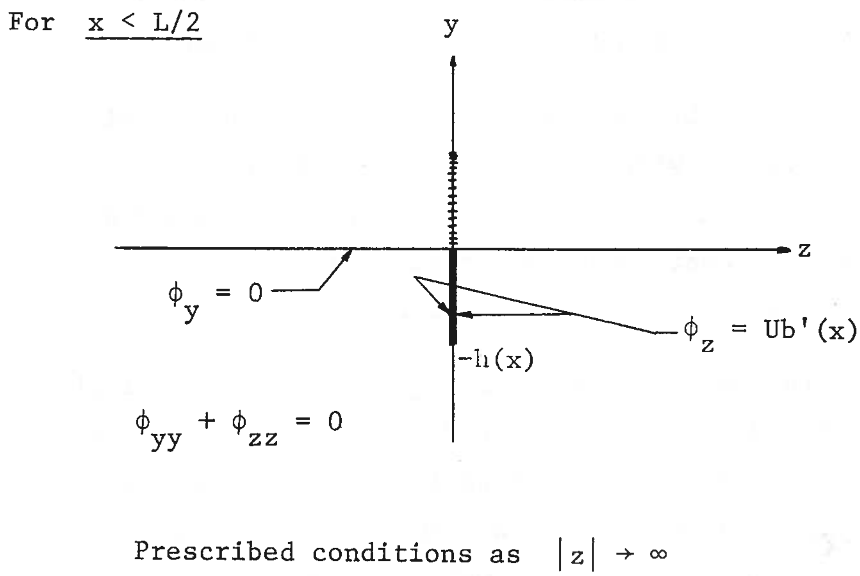


FIGURE 3

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM FOR THE USUAL NEAR-FIELD REGION

If $x < L/2$, i.e., for the sections where the span (draft) is not decreasing (see Chapter II) the Kutta condition is not required and a solution to the boundary value problem can be obtained by using methods of complex variables. Condition (F^i) allows an analytic continuation into the upper half plane, transforming the problem into an infinite-fluid problem. The solution is

$$\phi(x;y,z) = \text{Re}\,\{Ub'(x)[\sqrt{\zeta^2 + h^2} - \zeta]\} + M_{11}(x) \tag{11}$$

where

$\zeta = z + iy$

$M_{11}(x)$ = a solution of the homogeneous problem

The function $M_{11}(x)$ will be determined by the matching process. $M_{11}(x)$ represents an interaction among the sections. This interaction, if significant, can be interpreted partly as the effect of the wake in changing the local angle of attack.

It can be readily oberved that the above solution is not good for the regions where $x > L/2$, since its derivative is unbounded for $\zeta = 0 - ih$. In order to satisfy the Kutta condition, one could proceed mathematically by adding an eigensolution, but it seems that a physical argument could lead us to a simpler way of solving the problem.

Let us first make a transverse cut in the region for which $x>L/2$. What one sees is a distribution of free vortices in the plane of the wing extending from the tip of the section to a point which corresponds to the tip of the section of the maximum span (under the assumption that b is small). Condition (F^i) is still valid and an analytic continuation into the upper half plane is also possible here. A sketch of this is presented in Fig. 4.

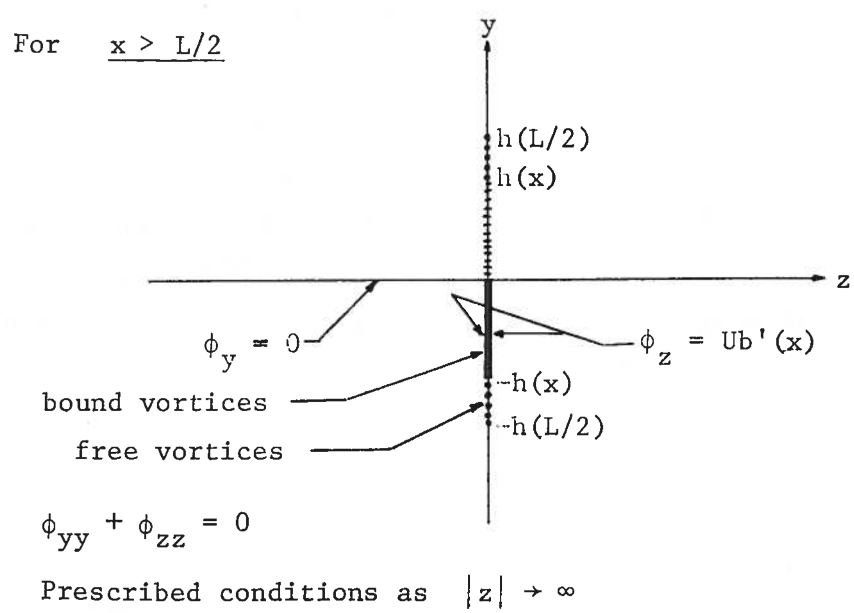


FIGURE 4

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM FOR $x > L/2$

The next thing to be observed is that there is no jump in the pressure across these free-vortices, which form the wake. This fact indicates the physical quantity to deal with; namely, the pressure instead of the velocity potential.

From the Bernoulli equation (neglecting higher order terms), one has that:

$$-\frac{p}{\rho} = U\phi_x \quad ,$$

This suggests the use of the concept of acceleration potential [11], [17]. Following Wu, let us define the acceleration potential function $\Omega(x,y,z)$:

$$\Omega(x,y,z) = U\phi_x(x,y,z)$$

The function so defined satisfies the Laplace equation. The velocity potential can be recovered from the following:

$$\phi(x,y,z) = \phi(L/2,\ y,z) + \frac{1}{U}\int_{L/2}^{x} \Omega(x,y,z)dx \qquad (13)$$

One can express the boundary value problem in terms of the acceleration potential as follows:

$0 = \Omega_{yy} + \Omega_{zz}$	in fluid region	(L^{ii})
$0 = \Omega_y$	on $y = 0$	(F^{ii})
$\Omega_z = U^2b''(x)$	on $z = \pm 0$ $0 > y > -h$	(H^{ii})
$\|\Omega\|$ bounded for $\|z\| < h(L/2)$		(K^{ii})

Observing the formal analogy of this problem with that sketched in Fig. 3, one can write the solution:

$$\Omega(x,y,z) \;=\; \mathrm{Re}\{U^2 b''[\sqrt{\zeta^2 + h^2} - \zeta]\} + M'_{12}(x) \qquad (14)$$

and using the inversion formula (13),

$$\phi(x,y,z) = \phi(L/2,y,z) + \frac{1}{U}\left[\int_{L/2}^{x} dx\ \mathrm{Re}\{U^2 b''[\sqrt{\zeta^2 + h^2} - \zeta]\} + M_{12}(x)\right] \qquad (15)$$

where $M_{12}(x)$ is the solution of the homogeneous problem, and will be determined by the matching process.

Solution (15) satisfies the required condition (K^i) and is taken as the solution for the near-field region, when $x > L/2$.

Let us observe that solutions (11) and (15) do not satisfy the radiation condition in the usual sense. However, something can be said about their behavior as one moves from the ship in the transverse direction. They must have a definite behavior so as to match the far-field solution as presented by equation (10).

2.2 The Bow-Near-Field

In contrast to what is found along almost the entire ship length, the region near the bow presents flow characteristics which are completely different. As discussed in Chapter II, the flow presents characteristics of a high Froude number flow and the rate of change of these characteristics in the longitudinal direction, although smaller, is not negligible, as compared with the rate of change in the transverse direction.

Let us assume in the bow near-field the following:

a. The characteristic length, a , is such that

$$a \;=\; O(\varepsilon^{1/2})$$

and the Froude number, F , is obtained from

$$F^2 = U^2/ag = O(\varepsilon^{-1/2}) \quad .$$

b. The orders of magnitude of the rate of change of the flow characteristics are given by:

$$\frac{\partial}{\partial x} = O(\varepsilon^{-1/2})$$

$$\frac{\partial}{\partial y} , \frac{\partial}{\partial z} = O(\varepsilon^{-1})$$

Introducing the above assumptions in equations (L), (H), (A), and (B), and taking only the leading terms, one has:

$$0 = \phi_{yy} + \phi_{zz} \qquad \text{in fluid region} \qquad (L^{iii})$$

$$0 = U^2\phi_{xx} + g\phi_y \qquad \text{on } y = 0 \qquad (F^{iii})$$

$$\phi_z = Ub'(0) \qquad \text{on } z = \pm 0,\ 0 > y > -h \qquad (H^{iii})$$

In the above formulated problem, several features are worth taking note of:

a) In condition (H^{iii}) and (F^{iii}) the transfering of boundary condition was effected and the latter condition is the result of the elimination of the unknown free-surface elevation (and derivatives) from equations (A) and (B) .

b) In condition (H^{iii}) it was assumed that

$$b'(x) = O(b) \quad .$$

It was also assumed that in the bow near-field the following approximation is valid:

$$b'(x) \simeq b'(0) \quad .$$

c) Condition (F^{iii}) presents the most intriguing feature. One has a Laplace equation in two-dimensions (y and z), (L^{iii}), and a free-surface condition which contains a derivative in the x-direction .

d) As usual in the near-field, the condition at large distances is that the solution matches with the far-field solution.

The presence of the second derivative with respect to x in condition (F^{iii}) does not allow us to use the known methods of a complex variable, as before. By taking the Fourier Transform in the x direction, however, one can eliminate the derivative with respect to x . One then gets a boundary value problem in y and z to solve, having the transform variable k as a parameter.

Taking the x Fourier Transform has its complications. For instance, the body boundary condition says:

$$\phi_z = Ub'(x)$$

and its Fourier Transform is

$$\phi_z^* = \int_{-\infty}^{\infty} dx\ \phi_z(x,y,0)e^{-ikx}$$

i.e., ϕ_z^* contains information from $-\infty$ to $+\infty$. For $x < 0$ the approximation $\phi_z = 0$ on $z = 0$ seems to be reasonable (note that this would not be true if it was an infinite fluid problem). It is also known that in slender ship theory the influence of the flow downstream on the upstream regions is of higher order, therefore one can write approximately,

$$\phi_z^* = \int_{0}^{\infty} dx\ Ub'(0)e^{-ikx}$$

$$= \int_{-\infty}^{\infty} dx\ Ub'(0)e^{-ikx}H(x)$$

where

$H(x)$ is the Heaviside step function.

According to Lighthill [7], one has

$$\phi^*_z = Ub'(0)[\pi\delta(k) + \frac{1}{ik}]$$

After taking the x Fourier transform, the boundary value problem to solve is

$$0 = \phi^*_{yy} + \phi^*_{zz} \qquad \text{in fluid domain} \qquad (L^{iv})$$

$$0 = \frac{U^2k^2}{g}\phi^* - \phi^*_y \qquad \text{on } y = 0 \qquad (F^{iv})$$

$$\phi^*_z = Ub'(0)[\pi\delta(k) + \frac{1}{ik}] \qquad \text{on } z = \pm 0 \qquad (H^{iv})$$

condition for $|z|\to\infty$ given by the inner expansion of the far-field potential.

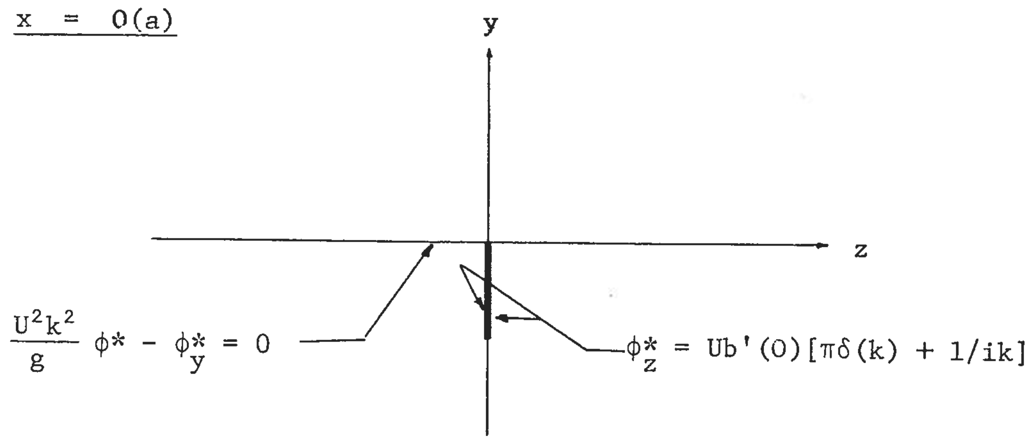
The above problem is sketched in Fig. 5.

Formally, this problem is equivalent to the problem of finding a velocity potential satisfying conditions (L^{iv}), (F^{iv}), (H^{iv}), and the required condition at infinity.

From Appendix I, the inner expansion of the far-field potential in the bow region is obtained, and from that, the behavior of ϕ^* for $z \to \pm\infty$ is

$$\phi^* \sim \phi^*_{+\infty} = [\sigma^*_I(k)\, \alpha e^{\alpha y}\cos\alpha z - \sigma^*_R(k)\alpha e^{\alpha y}\text{sgn}(k)\sin\alpha z] \qquad (16a)$$

$$+i[-\sigma^*_R(k)\, \alpha e^{\alpha y}\cos\alpha z - \sigma^*_I(k)\, \alpha e^{\alpha y}\text{sgn}(k)\sin\alpha z] \quad z \to +\infty$$



$$\phi^*_{yy} + \phi^*_{zz} = 0$$

Prescribed conditions as $|z| \to \infty$

FIGURE 5

TWO-DIMENSIONAL BOUNDARY VALUE PROBLEM
FOR THE BOW REGION

$$\begin{aligned}\phi^* \sim \phi^*_{-\infty} = \; & [-\sigma^*_I(k)\ \alpha e^{\alpha y}\cos\alpha z - \sigma^*_R(k)\ \alpha e^{\alpha y}\mathrm{sgn}(k)\sin\alpha z] \\ & +i[\sigma^*_R(k)\ \alpha e^{\alpha y}\cos\alpha z - \sigma^*_I(k)\ \alpha e^{\alpha y}\mathrm{sgn}(k)\sin\alpha z] \quad z \to -\infty\end{aligned} \tag{16b}$$

where

$$\sigma^*(k) = \sigma^*_R(k) + i\sigma^*_I(k)$$

$$\alpha = \frac{U^2k^2}{g}$$

The elementary solutions $\phi^*_{+\infty}$ and $\phi^*_{-\infty}$ satisfy the required conditions (L^{iv}) and (F^{iv}) and the condition for $z \to \pm\infty$. However, it is not possible to satisfy the body condition using only combinations of these elementary solutions. There is another elementary solution for (L^{iv}) and (F^{iv}).

$$\phi_A^* = e^{-uz}(u\ \cos uy + \alpha\ \ \sin uy); \quad z \geq 0 \tag{17}$$

where

u is any positive real number.

Following Ursell [13], we attempt to write the solution to the proposed problem as

$$\phi_+^* = \phi_{+\infty}^* + \int_0^\infty \{s(p) + ic(p)\} e^{-pz}(p\cos py + \alpha \sin py)\,dp \quad z > 0 \tag{18a}$$

$$\phi_-^* = \phi_{-\infty}^* - \int_0^\infty \{s(p) + ic(p)\} e^{pz}(p\cos py + \alpha \sin py)\,dp \quad z < 0 \tag{18b}$$

where the unknown σ_I^* , σ_R^* , $s(p)$ and $c(p)$ are to be determined by using the body boundary condition.

It should be noted that for $z = 0$

$$\frac{\partial \phi_+^*}{\partial z} = \frac{\partial \phi_-^*}{\partial z}$$

Some other relations for ϕ_+^* , ϕ_-^* and their derivatives on $z = 0$ will be required and stated at the appropriate time.

If one denotes

$$\phi^* = \begin{cases} \phi_+^* & z \geq 0 \\ \phi_-^* & z \leq 0 \end{cases}$$

the velocity potential is recovered by taking the inverse Fourier transform:

$$\phi = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\ e^{ikx} \phi^*$$

The Dipole Density and Functions $c(p)$ and $s(p)$.

Let us denote

$$\frac{\partial\phi^*}{\partial z} = f(y) + ig(y) \qquad z = 0 \tag{19}$$

with

$$f(y) = F_R \qquad \text{for} \qquad 0 > y > -h \tag{20}$$

$$g(y) = F_I \qquad \text{for} \qquad 0 > y > -h \tag{21}$$

$$F_R + iF_I = FT[\phi_z(x;y,0)] \cdot \qquad 0 > y > -h \tag{22}$$

Introducing (18) into (19), and using a Lemma given by Ursell [12],

$$-\alpha\sigma_R^* \,\mathrm{sgn}(k) = 2\int_{-\infty}^{0} dy\, e^{\alpha y} f(y) \tag{23}$$

$$s(p) = -\frac{2}{\pi}\frac{1}{p(p^2+\alpha^2)}\int_{-\infty}^{0} dy\, f(y)(p\cos py + \alpha\sin py) \tag{24}$$

$$-\alpha\sigma_I^* \,\mathrm{sgn}(k) = 2\int_{-\infty}^{0} dy\, e^{\alpha y} g(y) \tag{25}$$

$$c(p) = -\frac{2}{\pi}\frac{1}{p(p^2+\alpha^2)}\int_{-\infty}^{0} dy\, g(y)(p\cos py + \alpha\sin py) \tag{26}$$

The continuity of the fluids requires that for $z = 0$,

$$\frac{\partial\phi_+^*}{\partial y} = \frac{\partial\phi_-^*}{\partial y} \qquad -h > y > -\infty$$

and therefore

$$\phi_+^* = \phi_-^* \qquad z = 0;\ (-h > y > -\infty) \tag{27}$$

since

$$\phi_+^* \text{ and } \phi_-^* \to 0 \quad , \text{ as } \quad y \to -\infty \quad .$$

From (18) and (27) one has

$$\sigma_I^* \alpha e^{\alpha y} = -\int_0^\infty dp\ s(p)(p\cos py + \alpha \sin py) \tag{28}$$

$$\sigma_R^* \alpha e^{\alpha y} = \int_0^\infty dp\ c(p)(p\cos py + \alpha \sin py) \tag{29}$$

Substituting the values of σ_I^* , σ_R^* , $s(p)$, and $c(p)$ one gets

$$-\pi e^{\alpha y}\operatorname{sgn}(k)\int_{-\infty}^0 d\xi e^{\alpha\xi} g(\xi) = \int_{-\infty}^0 du\ f(u) \int_0^\infty \frac{dp(p\cos pu + \alpha \sin pu)(p\cos py + \alpha \sin py)}{p(p^2+\alpha^2)} \tag{30}$$

$$\pi e^{\alpha y}\operatorname{sgn}(k)\int_{-\infty}^0 d\xi e^{\alpha\xi} f(\xi) = \int_{-\infty}^0 du\, g(u) \int_0^\infty \frac{dp(p\cos pu + \alpha \sin pu)(p\cos py + \alpha \sin py)}{p(p^2+\alpha^2)} \tag{31}$$

which is a pair of coupled integral equations. By changing variables as

$$y = -v$$
$$u = -u$$

these equations can be decoupled [12] into

$$\int_0^\infty du[\alpha \int_0^u g(\xi)d\xi + g(u)][\frac{1}{v^2 - u^2}] = 0 \qquad h < v < \infty \tag{32}$$

$$\int_0^\infty du[\alpha \int_0^u f(\xi)d\xi + f(u)][\frac{1}{v^2 - u^2}] = 0 \qquad h < v < \infty \tag{33}$$

Since $f(v)$ and $g(v)$ are known for $0 < v < h$, one can rewrite (32) and (33) into

$$\int_h^\infty du[\alpha\, G(u) + g(u)] \frac{1}{v^2 - u^2} = \lambda_{34} \tag{34}$$

$$\int_h^\infty du[\alpha\, F(u) + f(u)] \frac{1}{v^2 - u^2} = \lambda_{35} \tag{35}$$

where

$$\lambda_{34} = -\int_0^h du[\alpha \int_0^u g(\xi)d\xi + g(u)][\frac{1}{v^2 - u^2}] - \int_h^\infty du[\alpha \int_0^h g(\xi)d\xi][\frac{1}{v^2 - u^2}]$$

$$\lambda_{35} = -\int_0^h du[\alpha \int_0^u f(\xi)d\xi + f(u)][\frac{1}{v^2 - u^2}] - \int_h^\infty du[\alpha \int_0^h f(\xi)d\xi][\frac{1}{v^2 - u^2}]$$

$$F(u) = \int_h^u f(\xi)d\xi$$

$$G(u) = \int_0^u g(\xi)d\xi$$

If one calls

$$\mu_{34}(u) = \alpha\, G(u) + g(u)$$

$$\mu_{35}(u) = \alpha\, F(u) + f(u)$$

equations (34) and (35) fall into the more general type of equations

$$\int_h^\infty \frac{\mu(u)\,du}{v^2-u^2} = \lambda(v) \qquad h < v < \infty$$

which has the following solution [13]

$$\mu(u) = \frac{cu}{\sqrt{u^2-h^2}} + \frac{4}{\pi^2}\frac{u^3}{\sqrt{u^2-h^2}}\int_h^\infty \frac{\lambda(v)\sqrt{v^2-h^2}}{v(v^2-u^2)}\,dv \tag{36}$$

where c is a constant to be determined.

For the present problem two constants c_1 and c_2, arise and can be eliminated by using equations (23) - (26), (28) and (29) and the relations stated below. The Fourier transform of the dipole density and the functions $c(p)$ and $s(p)$ are then determined.

Some relations are stated now, and will be used later on in the next chapter.

$$2\int_h^\infty \begin{bmatrix} g(v) \\ f(v) \end{bmatrix} e^{-\alpha v} dv = \int_h^\infty \begin{bmatrix} \mu_{34}(v) \\ \mu_{35}(v) \end{bmatrix} e^{-\alpha v} dv \tag{37}$$

$$\int_h^\infty \begin{bmatrix} g(v) \\ f(v) \end{bmatrix} (p\cos pv - \alpha \sin pv)\,dv = p\int_h^\infty \left\{\begin{bmatrix} \mu_{34}(v) \\ \mu_{35}(v) \end{bmatrix} - \begin{bmatrix} \mu_{34}(\infty) \\ \mu_{35}(\infty) \end{bmatrix}\right\} \cos pv\,dv - \begin{bmatrix} \mu_{34}(\infty) \\ \mu_{35}(\infty) \end{bmatrix} \sin ph \tag{38}$$

3. The Matching Process

According to Van Dyke [14] the expansions for the far field and the near field potential should obey the following:

"the m term inner expansion of the n term outer expansion must match with the n term outer expansion of the m term inner expansion."

Because of the way the velocity potential is written (see (4)), one can say that ϕ_0 , the first term in the outer or inner expansion, is trivially the uniform flow. It was also stated that only the first term, ϕ_1 , in the perturbation potential ϕ is going to be considered, and therefore we set $\phi = \phi_1$. It is easy to see that in order to satisfy Van Dyke's rule one must have a matching with the first leading term in each expansion, i.e., the leading term in the outer expansion of ϕ_1 (inner) should match the leading term in the inner expansion of ϕ_1 (outer).

3.1 Far Field - Usual Near Field

The expressions for the velocity potential in the far-field are given by equations (10). For the matching it is required to know the behavior as one approaches the ship. From Appendix I, the leading term in the inner expansion of the outer expansion is

$$\phi(x,y,z) \sim \frac{\sigma(x)}{\pi} \cdot \frac{z}{y^2+z^2} \tag{39}$$

The above equation expresses the condition at ∞ that the velocity potential has to satisfy in the usual near field. The velocity potential for the usual near field is expressed by equations (11) and (15) valid for $x < L/2$ and $x > L/2$ respectively (except for regions very near the bow). For the matching process it is required the behavior of these expressions as one moves far away from the ship, i.e., it is required the outer expansion of the inner expansion. From Appendix II, one has the leading term

$$\phi(x,y,z) \sim \frac{z}{y^2+z^2}\left[\frac{Ub'(\xi)h^2(\xi)}{2} + \frac{UH(x-L/2)}{2}\int_{L/2}^{x} du\, b''(u)h^2(u)\right] + M(x) \tag{40}$$

where

$$\xi = \begin{cases} x & \text{if } x < L/2 \\ L/2 & \text{if } x > L/2 \end{cases}$$

$H(x - L/2)$ is the Heaviside step function.

If the matching between (39) and (40) is performed, one has

$$M(x) = 0 \tag{41}$$

$$\frac{\sigma(x)}{\pi} = \frac{Ub'(\xi)h^2(\xi)}{2} + \frac{UH(x - L/2)}{2} \int_{L/2}^{x} du\, b''(u)h^2(u) \tag{42}$$

The fact that $M(x) = 0$ shows that the interaction among sections, to this approximation is of higher order. This can be expected in an infinite fluid problem, but it is not so obvious in a free-surface problem.

3.2 Far Field - Bow Near Field

The matching of the far field with the near field in the bow region was already performed, when in Section III.2.2 the condition on ϕ^* for $z \to \pm\infty$, was taken as expressions (16).

The matching process need not be discussed, therefore, for the bow region.

IV. NUMERICAL RESULTS AND CONCLUSIONS

In this chapter, the free-surface elevation near the ship is calculated for the bow region.

For the simplest case, when the draft is constant, some numerical values are obtained.

It is predicted that the wave elevation is anti-symmetric with respect to the center plane of the ship. In a test with a model, this fact probably cannot be observed since thickness effects (which are not considered here) will have some influence.

1. Free-Surface Elevation for a Ship of Constant Draft

If the ship has a constant draft in the bow region, the calculations are simplified.

The dynamic free-surface condition, if only the leading term is taken, is:

$$0 = g\eta + U\phi_x \qquad \text{on} \quad y = 0$$

Therefore, the free surface elevation near the ship is

$$\eta = -\frac{U}{g}\,\phi_x(x,0,0) \tag{43}$$

or

$$\eta = -\frac{U}{g}\frac{i}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} k\phi^*(x,0,0)$$

Using (18), one obtains

$$\eta(x,\pm 0) = \mp\frac{U}{g}\frac{i}{2\pi}\int_{-\infty}^{\infty} dk e^{ikx} k\{[\sigma_I^*(k)\alpha - i\sigma_R^*(k)\alpha] + \int_0^{\infty}[s(p) + ic(p)]pdp\} \tag{44}$$

The above equation already shows the anti-symmetry in the wave elevation, mentioned before.

The determination of σ_I^* , σ_R^* , $s(p)$ and $c(p)$ proceeds as follows:

From (H^{iv}), (20), (21), and (22)

$$f = F_R(k) \qquad 0 > y > -h$$

$$g = F_I(k) \qquad 0 > y > -h$$

i.e., f and g are independent of y . The right-hand sides of (34) and (35) are:

$$\lambda_{34} = - F_I\{ \frac{\alpha}{2} \log(\frac{v^2}{v^2 - h^2}) + (\frac{1}{2} - \frac{h\alpha}{2})(\frac{1}{v} \log \frac{v + h}{v - h})\}$$

$$\lambda_{35} = - F_R \{\frac{\alpha}{2} \log(\frac{v^2}{v^2 - h^2}) + (\frac{1}{2} - \frac{h\alpha}{2})(\frac{1}{v} \log \frac{v + h}{v - h})\}$$

and the solutions of (34) and (35) are:

$$\mu_{34} = \frac{c_1 u}{\sqrt{u^2 - h^2}} + \frac{2F_I\alpha}{\pi} u \sin^{-1}(\frac{h}{u}) + F_I(1 - \alpha h) \qquad (45)$$

$$\mu_{35} = \frac{c_2 u}{\sqrt{u^2 - h^2}} + \frac{2F_R\alpha}{\pi} u \sin^{-1}(\frac{h}{u}) + F_R(1 - \alpha h) \qquad (46)$$

Using identities (37) and (38) in equations (23) through (26), one gets:

$$- \alpha\sigma_R^* \mathrm{sgn}(k) = \frac{F_R}{\alpha} (1 + \frac{2}{\pi} \int_0^{\alpha h} uK_1(u)du) + c_2 hK_1(\alpha h) \qquad (47)$$

$$- \alpha\sigma_I^* \mathrm{sgn}(k) = \frac{F_I}{\alpha} (1 + \frac{2}{\pi} \int_0^{\alpha h} uK_1(u)du) + c_1 hK_1(\alpha h) \qquad (48)$$

$$s(p) = -\frac{2}{\pi p(p^2+\alpha^2)} \{-\frac{F_R\alpha}{p} \int_0^{ph} uJ_1(u)du - \frac{c_2\pi}{2} phJ_1(ph)\} \qquad (49)$$

$$c(p) = -\frac{2}{\pi p(p^2+\alpha^2)} \{-\frac{F_I\alpha}{p} \int_0^{ph} uJ_1(u)du - \frac{c_1\pi}{2} phJ_1(ph)\} \qquad (50)$$

where J_1 and K_1 are Bessel functions.*

Equations (28) and (29) give us two new relations, which make it possible to determine the constants c_1 and c_2 .

$$\sigma_I^*\alpha = \frac{2F_R}{\pi} \int_0^{\alpha h} uI_1(u)du + \frac{\pi}{2} c_2\alpha hI_1(\alpha h) \qquad (51)$$

$$\sigma_R^*\alpha = -\frac{2F_I}{\pi} \int_0^{\alpha h} uI_1(u)du - \frac{\pi}{2} c_1\alpha hI_1(\alpha h) \qquad (52)$$

where I_1 is a Bessel function.

Equations (47), (48), (51), and (52) give us a system of algebraic equations for determining c_1 , c_2 , σ_I^* , and σ_R^* . Before solving it, a simplification is still possible. From Appendix I, it is assumed that the contribution for the inverse Fourier transform when k is in the neighborhood of zero is negligible. Therefore one can neglect the contribution of

$$F_R = \pi\delta(k)$$

as given by (H^{iv}).

---

* All the special functions are defined according to Abramowitz and Stegun [1].

The constants are then expressed as:

$$c_1 = -\frac{F_I}{h[\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)} \left[\frac{K_1(\alpha h)}{\alpha} \left(1 + \frac{2}{\pi} \int_0^{\alpha h} uK_1(u)du\right) + \frac{2\pi I_1(\alpha h)}{\alpha} \int_0^{\alpha h} uI_1(u)du\right]$$

$$c_2 = -\frac{\pi F_I \operatorname{sgn}(k)}{\alpha h[\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)} [I_1(\alpha h) + L_1(\alpha h)]$$

where L_1 is the modified Struve function.

The desired functions in equation (44) can now be calculated as being:

$$-\alpha\sigma_R^* = \frac{\pi U b'(0)}{\alpha k} \cdot \frac{K_1(\alpha h)}{\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)} [I_1(\alpha h) + L_1(\alpha h)]$$

$$\alpha\sigma_I^* = \frac{\pi^2 U b'(0) \operatorname{sgn}(k)}{\alpha k} \cdot \frac{I_1(\alpha h)}{\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)} [I_1(\alpha h) + L_1(\alpha h)]$$

$$\int_0^\infty s(p)pdp = \frac{Ub'(0)\pi \operatorname{sgn}(k)}{\alpha k} \cdot \frac{\int_0^\infty \frac{pJ_1(ph)}{p^2 + a^2} dp}{\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)} \cdot [I_1(\alpha h) + L_1(\alpha h)]$$

$$\int_0^\infty c(p)pdp = -\frac{2Ub'(0)\alpha}{\pi k} \int_0^\infty \frac{\int_0^{ph} uJ_1(u)du}{p(p^2 + \alpha^2)} + \frac{Ub'(0)}{k[\pi^2 I_1^2(\alpha h) + K_1^2(\alpha h)}$$

$$\cdot \left[\frac{K_1(\alpha h)}{\alpha}\left(1 + \frac{2}{\pi} \int_0^{\alpha h} uK_1(u)du\right) + \frac{2\pi I_1(\alpha h)}{\alpha} \int_0^{\alpha h} uI_1(u)du\right] \int_0^\infty \frac{pJ_1(ph)}{p^2 + \alpha^2} dp$$

For purposes of numerical calculation, the non-dimensional coordinates below are used.

$$X = x\Big/\sqrt{\frac{hU^2}{g}}$$

$$K = k\sqrt{\frac{hU^2}{g}}$$

$$H = \frac{\eta}{\sqrt{\frac{U^2h}{g}}}$$

Figure (6) shows a plot of $H/b'(0)$ versus X . Equation (44) was used for the computations. The numerical calculations were made using the facilities of the University of Michigan Computer Center. Figure (6) shows also an asymptotic estimate of the free-surface elevation for large X . This estimate is obtained from equation (44) by using a theorem of the theory of Fourier transform [7]. For large X , the following asymptotic behavior for $H/b'(0)$ is obtained:

$$\frac{H}{b'(0)} \sim \frac{12}{X^5} \qquad \text{for } X \to \infty \quad .$$

2. Conclusions

The theory presented in this thesis led us to a method for calculating the hydrodynamic reaction acting on a ship hull in maneuver. The effects caused by the presence of the free-surface were taken into account in the present theory. These forces can be calculated by using the Bernoulli equation in order to find the pressure which is then integrated over the hull surface. This method does not give us all the terms in the general equation of motion for a ship in maneuver. As mentioned, only the hydrodynamic reactions for a hull in steady turn can be obtained.

It should also be emphasized that thickness effects were not considered.

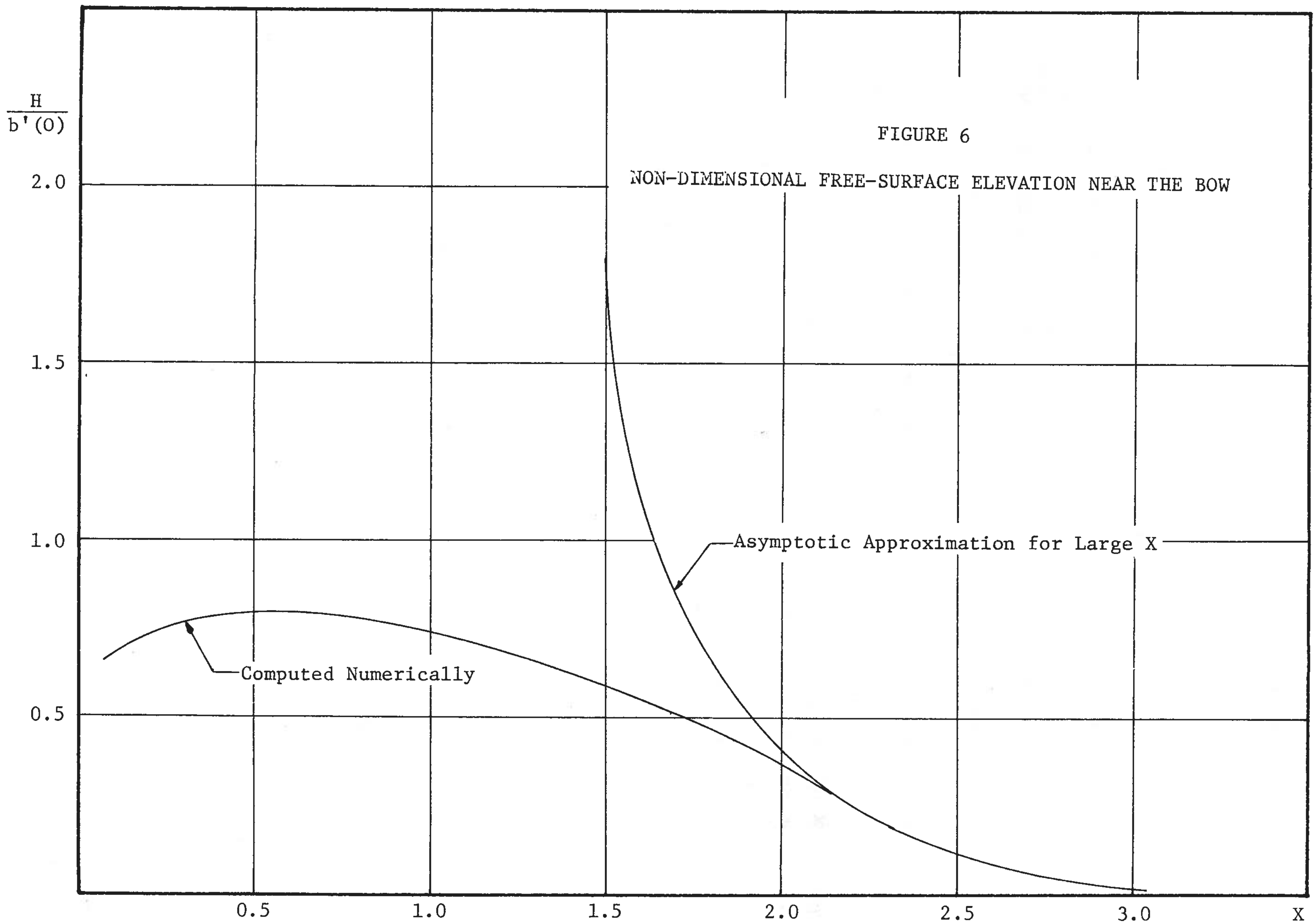


FIGURE 6

NON-DIMENSIONAL FREE-SURFACE ELEVATION NEAR THE BOW

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APPENDIX I

INNER EXPANSION OF THE FAR-FIELD POTENTIAL

1. Inner Expansion of the Far-Field Potential in the Usual Near-Field

In order to find how the far-field potential behaves as one approaches the line of singularities (i.e., the ship) in the region downstream of the bow region, it is convenient to start with the expression (10a) which is:

$$\phi(x,y,z) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{dk e^{ikx} \sigma^*(k)\ |k| z\ K_1(|k| \sqrt{y^2+z^2})}{\sqrt{y^2+z^2}}$$

$$- \frac{iU^2 g}{4\pi^2} \int_{-\infty}^{\infty} dk e^{ikx} k^2 \sigma^*(k) \lim_{\mu\to 0} \int_{-\infty}^{\infty} \frac{\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}] d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - \frac{i\mu}{2})^2]} \qquad \text{(I-1)}$$

Our problem is to find out how this potential behaves as $R = \sqrt{y^2+z^2}$ goes to zero or, more precisely, for $R = O(\varepsilon)$.

Let us first define

$$\phi_A(x,y,z) = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \frac{dk e^{ikx} \sigma^*(k) |k| z\ K_1(|k| \sqrt{y^2+z^2})}{\sqrt{y^2+z^2}} \qquad \text{(I-2)}$$

$$\phi_B(x,y,z) = - \frac{iU^2 g}{4\pi^2} \int_{-\infty}^{\infty} dk e^{ikx} k^2 \sigma^*(k) \lim_{\mu\to 0} \int_{-\infty}^{\infty} \frac{\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}] d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - \frac{i\mu}{2})^2]} \qquad \text{(I-3)}$$

The behavior of ϕ_A for $R = O(\varepsilon)$ is easily found by expanding the Bessel function for small argument [1]. It is:

$$\phi_A(x,y,z) \sim \frac{z}{y^2+z^2} \quad \frac{1}{2\pi^2} \int_{-\infty}^{\infty} dk e^{ikx} \sigma^*(k) + o(\sigma)$$

$$\sim \frac{\sigma(x)}{\pi} \quad \frac{z}{y^2+z^2} + o[\sigma(x)] \qquad \text{(I-4)}$$

The behavior of ϕ_B for $R = O(\varepsilon)$ is now shown to be of negligible magnitude as compared with ϕ_A . The work by Ogilvie & Tuck is then followed. Let

$$I = \lim_{\mu \to 0} \int_{-\infty}^{\infty} \frac{\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}]d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk - \frac{i\mu}{2})^2]} \qquad \text{(I-5)}$$

In the limit, the integrand has singularities given by

$$\sqrt{k^2+\ell_0^2} - (Uk)^2/g = 0$$

or

$$\sqrt{k^2+\ell_0^2} = k^2/\nu \qquad \text{where} \quad \nu = g/U^2$$

$$\ell_0 = \pm \frac{|k|}{\nu} \sqrt{k^2-\nu^2}$$

As $\mu \to 0$ one has a contour integral and it is necessary to determine how to indent it around the singularities. First let us see where the singularities are located in the ℓ-plane .

If $|k| > \nu$, ℓ_0 is real

$|k| < \nu$, ℓ_0 is imaginary

Now the "ficticious viscosity" plays its important role; i.e., it tells us how ℓ_0 approaches the real axis as $\mu \to 0$ and therefore shows how one must indent the contour. Take ℓ_0 as a limiting value as $\mu \to 0$.

$$\ell_0^2 = \frac{1}{g^2}(Uk - \frac{i\mu}{2})^4 - k^2 \qquad \text{as} \quad \mu \to 0$$

$$= k^4/\nu^2 - k^2 - 2iU^3k^3\mu + O(\mu^2) \qquad \text{as} \quad \mu \to 0$$

Therefore, the three following cases are possible.

a) $k > \nu > 0$

If we define $\ell_0^{\pm}$ by

$$\ell_0^{\pm} = \pm\sqrt{\frac{k^4}{\nu^2} - k^2 - i2U^3k^3\mu}$$

we have that:

$$\text{Re } \ell_0^{+} > 0$$

$$\text{Im } \ell_0^{+} < 0$$

and

$$\text{Re } \ell_0^{-} < 0$$

$$\text{Im } \ell_0^{-} > 0$$

As $\mu \to 0$ we have to indent the contour in equation (I-5) as shown in Fig. (I-a).

b) $k < -\nu$

In this case, the analysis is the same and the contour is indented as in Fig. (I-b).

c) $-\nu < k < \nu$

In this case, as $\mu \to 0$, ℓ_0 becomes imaginary and we do not have to indent the contour. (See Fig. I-c.)

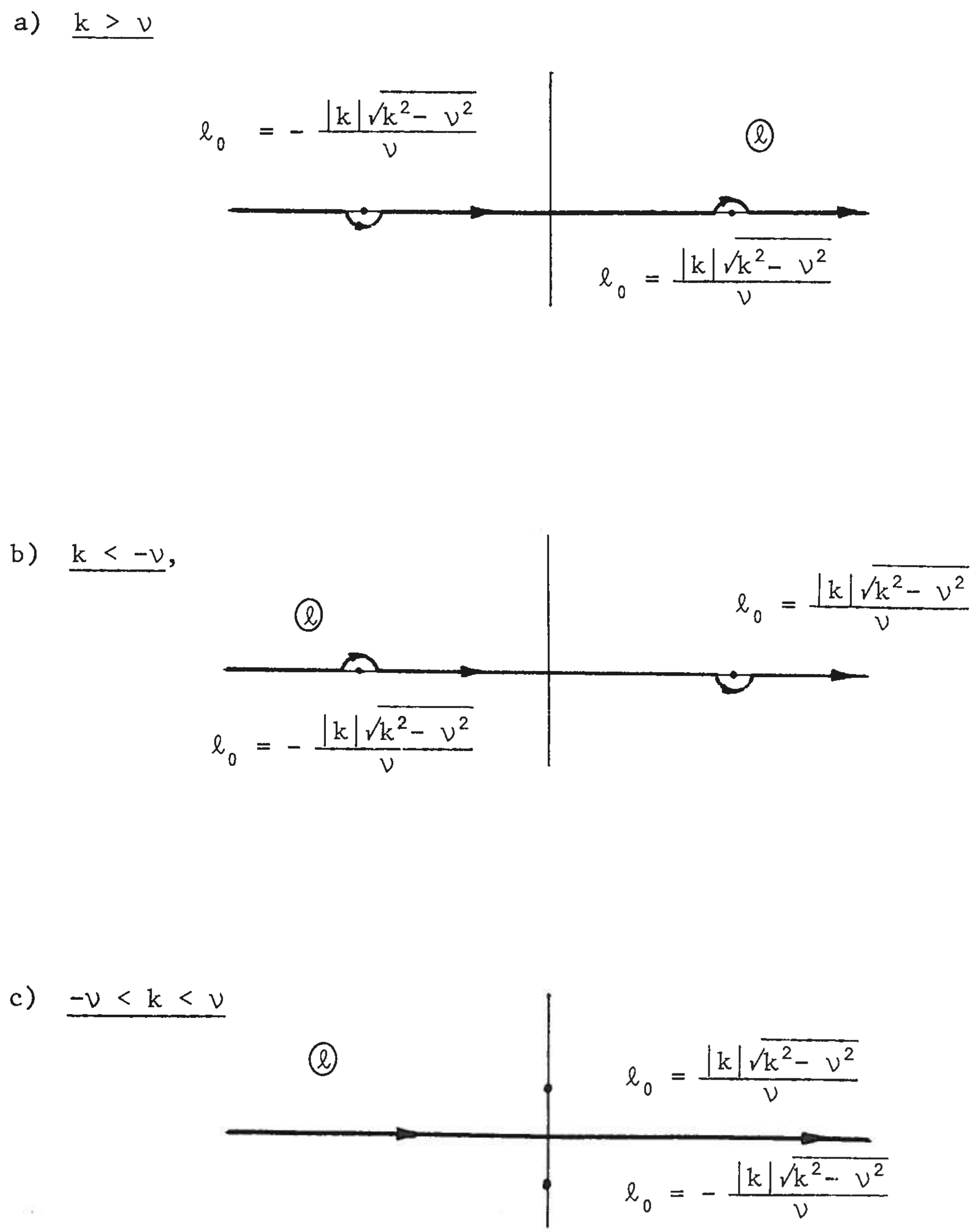


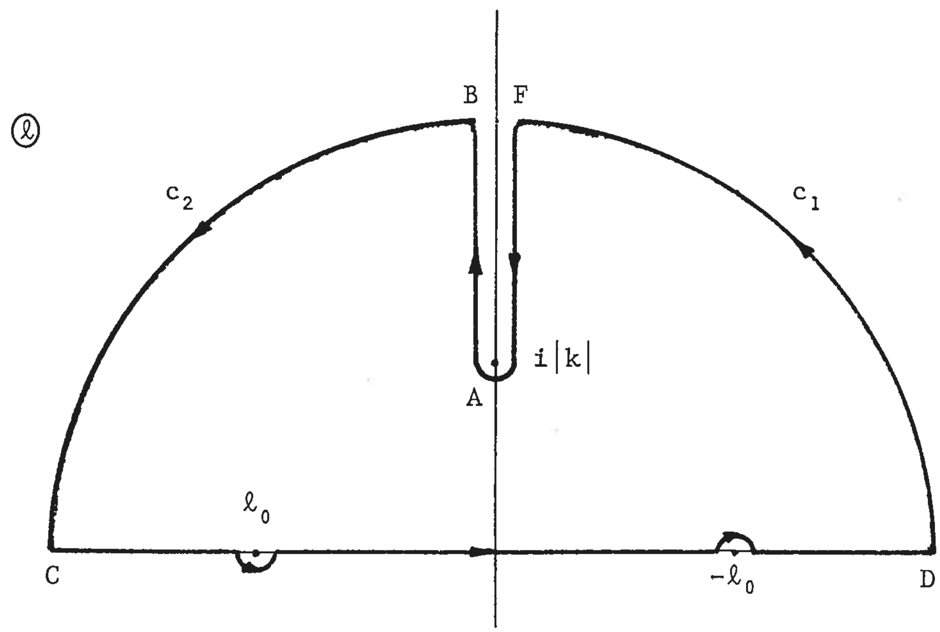
FIGURE I

THE SINGULARITIES OF THE FAR-FIELD POTENTIAL

Under the hypothesis that $z > 0$ let us evaluate the integral I for the three different cases.

a) $k > \nu$

The integrand of I is analytic in the upper-half plane, except on the imaginary axis above $\ell = ik = i|k|$, therefore one can close the contour as shown in Fig. II, and use the residue theorem.



$$\ell_0 = -\frac{|k|\sqrt{k^2-\nu^2}}{\nu}$$

$$\sqrt{k^2+\ell_0^{\,2}} = \frac{k^2}{\nu}$$

FIGURE II

THE CONTOUR OF INTEGRATION FOR $k > \nu$

$$I = 2\pi i \operatorname{Res}(\ell_0) - \int_{FAB} \frac{\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}]d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2} - \frac{1}{g}(Uk)^2]} - \int_{c_1} - \int_{c_2}$$

and, as the points B,C,D,F move to infinity, one has

$$I = 2\pi i \text{ Res } (\ell_0) + \int_{i|k|}^{i\infty} \frac{\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}]d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2}-\frac{1}{g}(Uk)^2]}$$

$$- \int_{i|k|}^{\infty} \frac{\ell \exp[i\ell z - y\sqrt{k^2+\ell^2}]d\ell}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2}+\frac{1}{g}(Uk)^2]} \qquad \text{(I-6)}$$

since the integral along the paths c_1 and c_2 vanish.

The residue of the integrand at ℓ_0 can be calculated as follows:

$$\text{Res } (\ell_0) = \lim_{\ell \to \ell_0} \frac{(\ell - \ell_0)\ell \exp[i\ell z + y\sqrt{k^2+\ell^2}]}{\sqrt{k^2+\ell^2}[\sqrt{k^2+\ell^2}-\frac{1}{g}(Uk)^2]}$$

$$= \exp[-\frac{iz|k|\sqrt{k^2-\nu^2}}{\nu} + y\frac{k^2}{\nu}]$$

The branch contribution can be evaluated as follows:

$$-\int_{FAB} = -\int_{|k|}^{\infty} \frac{\ell \exp[-\ell z + iy\sqrt{\ell^2 - k^2}]\, d\ell}{i\sqrt{\ell^2 - k^2}[i\sqrt{\ell^2 - k^2} - \frac{1}{g}(Uk)^2]}$$

$$+ \int_{|k|}^{\infty} \frac{\ell \exp[-\ell z - iy\sqrt{\ell^2 - k^2}]d\ell}{i\sqrt{\ell^2 - k^2}[i\sqrt{\ell^2 - k^2} + \frac{1}{g}(Uk)^2]}$$

where use of the transformation $\ell = i\ell$ was made. Further,

$$-\int_{FAB} = \int_{|k|}^{\infty} \frac{\ell \exp[-\ell z + iy\sqrt{\ell^2 - k^2}]d\ell}{\sqrt{\ell^2 - k^2}[\sqrt{\ell^2 - k^2} + \frac{1}{g}(Uk)^2]}$$

$$-\int_{|k|}^{\infty} \frac{\ell \exp[-\ell z - iy\sqrt{\ell^2 - k^2}]d\ell}{\sqrt{\ell^2 - k^2}[\sqrt{\ell^2 - k^2} - \frac{1}{g}(Uk)^2]}$$

Call

$$A^{\pm} = \int_{|k|}^{\infty} \frac{\ell \exp[-\ell z \pm iy\sqrt{\ell^2 - k^2}]d\ell}{\sqrt{\ell^2 - k^2}[\sqrt{\ell^2 - k^2} \pm \frac{1}{g}(Uk)^2]}$$

now let $\ell^2 - k^2 = u^2$. Therefore,

$$A^{\pm} = \int_{0}^{\infty} \frac{du \exp[-z\sqrt{u^2 + k^2} \pm iyu]}{u \pm iu_0} \quad , \quad u_0 = k^2/\nu$$

letting y and z approach zero at the same time would lead to a divergent integral. Let us first expand the exponential function with real argument

$$A^{\pm} = \int_{0}^{\infty} \frac{du e^{\pm iyu}}{u \pm iu_0} [1 + z\sqrt{u^2 + k^2} + \ldots.]$$

Taking only the leading term and using the transformation

$$t = \mp iy(y \pm iu_0)$$

one has the contours of integration shown in Fig. III with the following results:

$$A^{\pm} = \mp i\pi e^{-u_0|y|} + Ei(|y|u_0)e^{|y|u_0}$$

where Ei is an exponential integral [1].

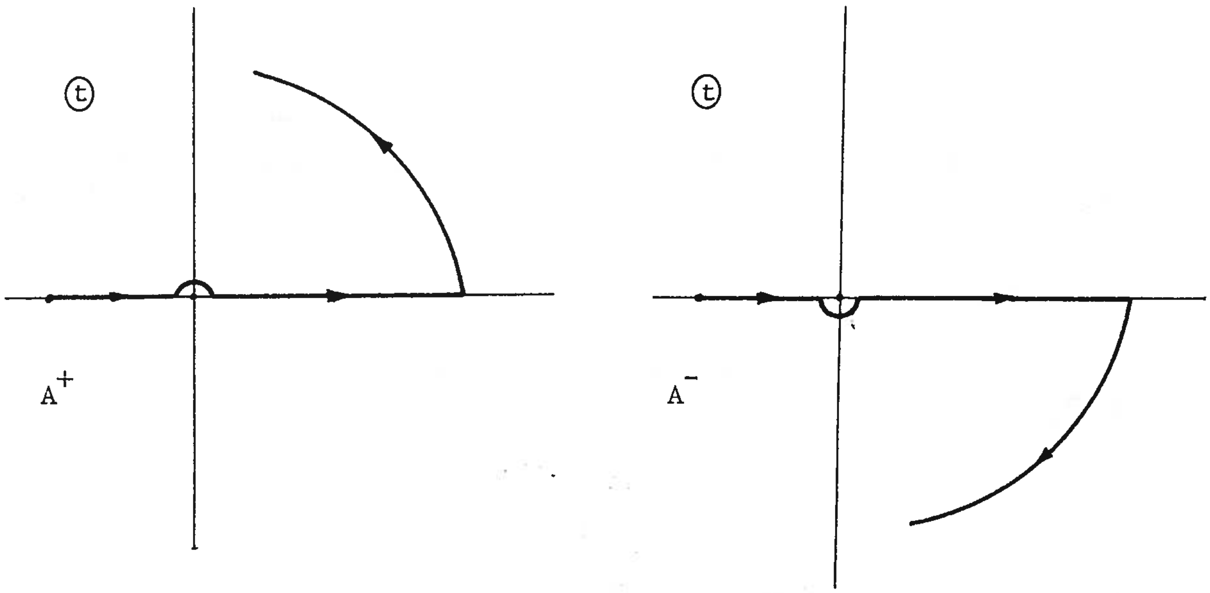


FIGURE III

PATH OF INTEGRATION IN THE t-PLANE

Therefore, one has

$$-\int_{FAB} \sim -2\pi i\, e^{|y|k^2/\nu} \tag{I-8}$$

and expanding the exponential function in the residue as well as in the branch contribution for small arguments

$$I = 2\pi i[1 + (\frac{yk^2}{\nu} + \frac{i|k|\sqrt{k^2 - \nu^2}\,z}{\nu}) + \ldots\ldots]$$

$$-2\pi i[1 + \frac{yk^2}{\nu} + \ldots\ldots]$$

$$I \sim 2\pi i\left[\frac{i|k|\sqrt{k^2 - \nu^2}\ z}{\nu} + O(\varepsilon^2)\right] \tag{I-9}$$

Noticing that the leading term is $O(\varepsilon)$, one concludes that the inner expansion of ϕ_B (see I-3) is negligible compared to ϕ_A (see I-4).

b) $k < -\nu$

The same general approach is used. The contour is closed as shown in Fig. IV.

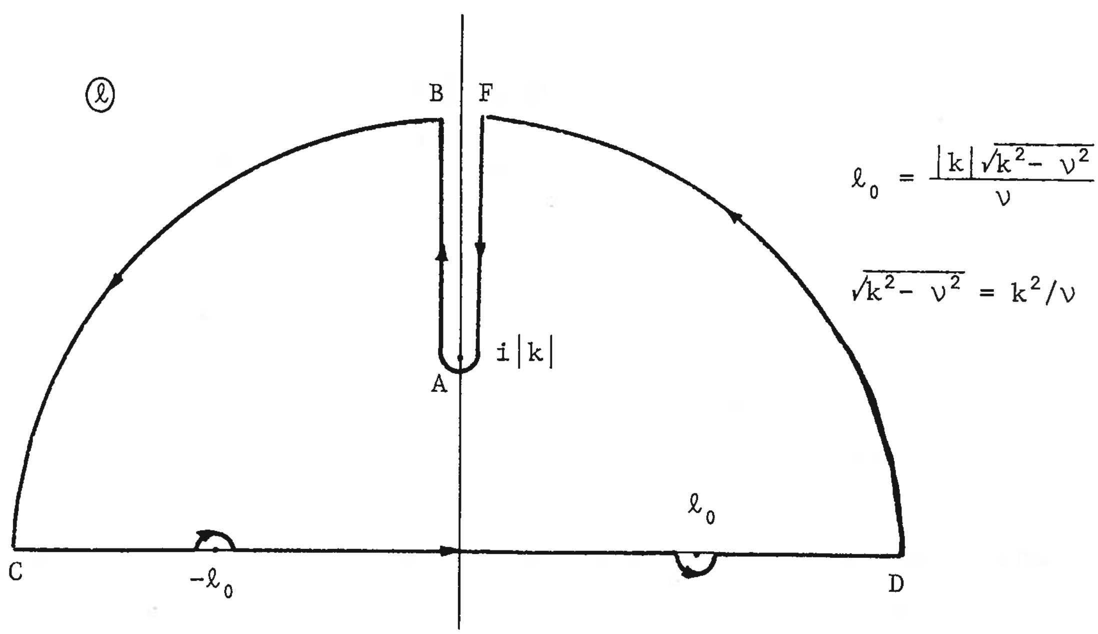


FIGURE IV

THE CONTOUR OF INTEGRATION FOR $k < -\nu$

The residue is:

$$\mathrm{Res}(\ell_0) = \exp\left[\frac{iz|k|\sqrt{k^2-\nu^2}}{\nu} + y\,\frac{k^2}{\nu}\right] \tag{I-10}$$

and the same estimate for the branch contribution as in Case a) is valid, leading to the same argument about the order of magnitudes of the inner expansions of ϕ_A and ϕ_B .

c) $-\nu < k < k$

Again the same general approach is used. The contour, however, must be closed as in Fig. V.

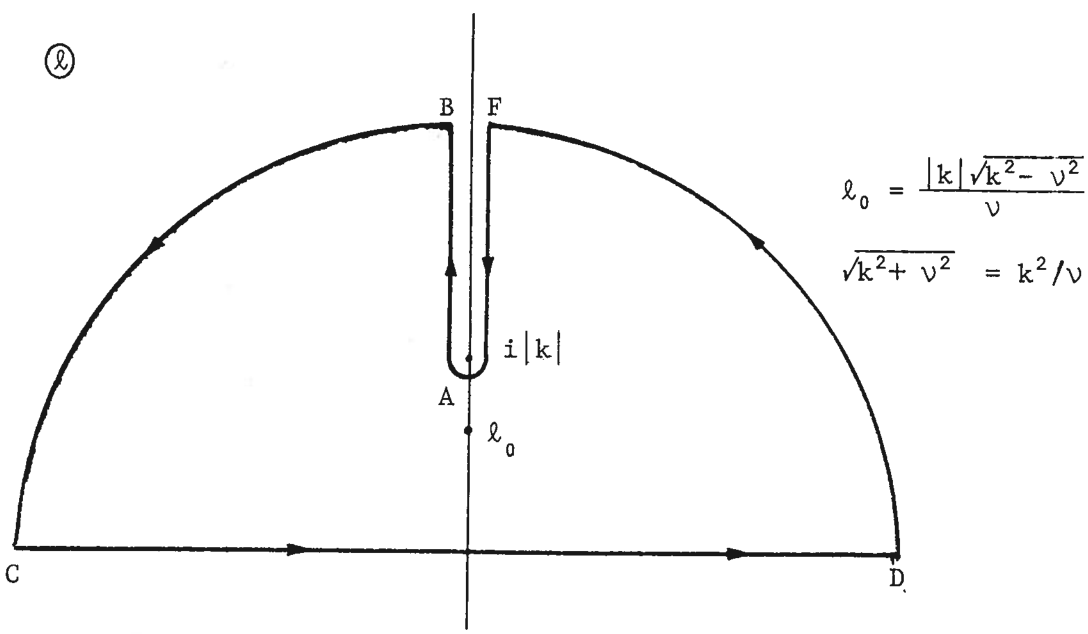


FIGURE V

THE CONTOUR OF INTEGRATION FOR $-\nu < k < \nu$

Now the singularity is located on the imaginary axis, but below the point $\ell = i|k|$.

The residue is:

$$\mathrm{Res}(\ell_0) = \exp[-\frac{iz|k|\sqrt{\nu^2 - k^2}}{\nu} + y\frac{k^2}{\nu}] \tag{I-11}$$

and the same estimate for the orders of magnitude of the inner expansion of ϕ_A and ϕ_B can be reached.

If now the assumption that $z < 0$ is made,the same result holds true.

The inner expansion(considering only the leading term) of (I-1) is therefore given by the leading term of ϕ_A , i.e.,

$$\phi(x,y,z) \sim \frac{\sigma(x)}{\pi} \frac{z}{y^2 + z^2} + o[\sigma(x)] \tag{I-12}$$

2. Inner Expansion of the Far-Field Potential in the Bow Region

For the matching of the far-field with the bow near-field, it is required to have the inner expansion in the bow region of the outer expansion; in other words, the behavior of the far-field potential in the region near the bow. In the main text the bow region was characterized by distances which are $O(\varepsilon^{1/2})$ in the longitudinal direction, and distances which are $O(\varepsilon)$ in the transverse direction. In order to find the required behavior of the velocity potential in the bow region, we first restrict ourselves to a region where the distances are $O(\varepsilon^{1/2})$ from the point of intersection of the bow and the undisturbed free surface, in all directions. Later, we will restrict even more the region of interest by letting the distance $O(\varepsilon)$ in the transverse direction from the ship.

Let us introduce the non-dimensional coordinates *

$$x = \xi a$$

$$y = \eta a$$

$$z = \zeta a$$

where a is a length which is $O(\varepsilon^{1/2})$.

For the present purpose it is convenient to start working with the expression (10b) of the far-field velocity potential, given in the main text. We can rewrite expression (10b) in non-dimensional coordinates as follows:

$$\phi(\xi,\eta,\zeta) = -\frac{1}{4\pi^2}\int_{-\infty}^{\infty} dk e^{ik\xi}\sigma^*(k) \lim_{\mu\to 0}\int_{-\infty}^{\infty} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{\ell^2+k^2}]d\ell}{\sqrt{k^2+\ell^2} - (\frac{1}{ga})(Uk - \frac{i\mu}{2})^2} \tag{I-13}$$

As we did in the previous section of this Appendix, let us define

$$I = \lim_{\mu\to 0}\int_{-\infty}^{\infty} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{k^2+\ell^2}]d\ell}{\sqrt{k^2+\ell^2} - \frac{1}{ga}(Uk - \frac{i\mu}{2})^2} \tag{I-14}$$

As far as the singularities of this expression are concerned, we can note a formal analogy with the expression (I-5). We can then identify three cases and the path of integration has to be indented as shown in Fig. VI. This figure also shows how we close the contour in order to use the residue theorem.

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* ξ, η, ζ, should cause no confusion with the main text, since they are used as non-dimensional coordinates only in this Appendix.

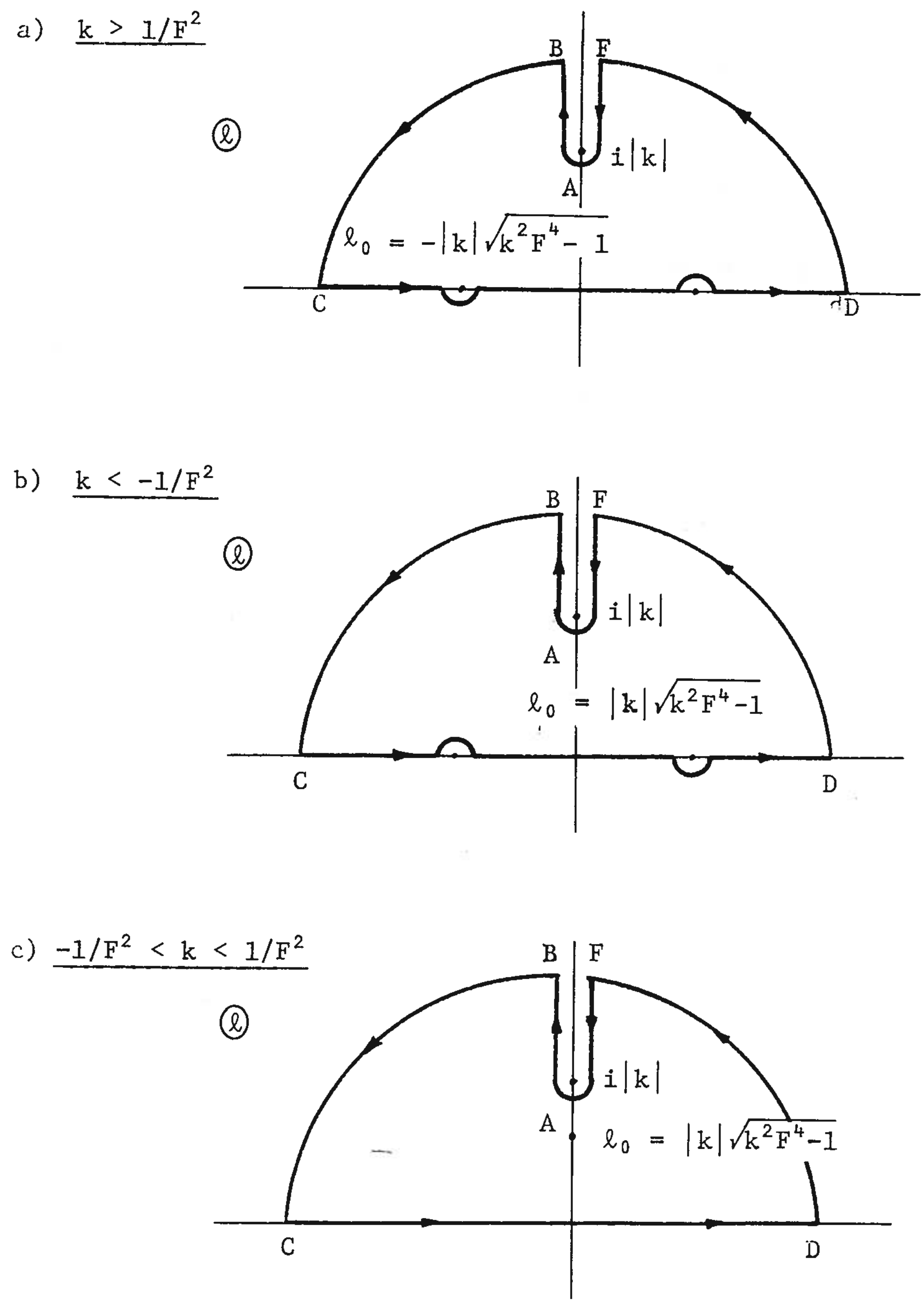

FIGURE VI

THE CONTOUR OF INTEGRATION FOR THE POTENTIAL

IN THE BOW REGION

Under the assumption that $\zeta > 0$ let us consider three cases and then, if the square of the Froude number is

$$F^2 = \frac{U^2}{ga} = O(\varepsilon^{-1/2})$$

as defined in the main text, we have:

a) $k > \dfrac{1}{F^2}$

The contour of integration is shown in Fig. VI a) . By the residue theorem we have:

$$I = 2\pi i \,\mathrm{Res}(\ell_0) - \int_{FAB} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{k^2 + \ell^2}]d\ell}{\sqrt{k^2+\ell^2} - \frac{1}{ga}(Uk)^2} \tag{I-15}$$

if we let the points B, C and D move to infinity.

The residue is:

$$\mathrm{Res}\ (\ell_0) = k^2F^2\exp[-i\zeta|k|F^2\sqrt{k^2 - F^2} + \eta k^2F^2] \tag{I-16}$$

The branch-cut contribution is:

$$-\int_{FAB} = -\int_{|k|}^{\infty} \frac{\ell \exp[-\ell\zeta + i\eta\sqrt{\ell^2 - k^2}\,]d\ell}{i\sqrt{\ell^2 - k^2} - \frac{U^2k^2}{ga}} + \int_{|k|}^{\infty} \frac{\ell \exp[-\ell\zeta - i\eta\sqrt{\ell^2 - k^2}]d\ell}{-i\sqrt{\ell^2 - k^2} - \frac{U^2k^2}{ga}}$$

Now we want to show that the branch-out contribution is negligible as compared to the residue contribution.

Let us first observe the following inequalities

$$\left| \int_{|k|}^{\infty} \frac{\ell \exp[-\ell\zeta \pm i\eta\sqrt{\ell^2 - k^2}]d\ell}{\pm\sqrt{\ell^2 - k^2} - \frac{U^2k^2}{ga}} \right| \leqslant \int_{|k|}^{\infty} \frac{\ell e^{-\ell\zeta}\, d\ell}{\sqrt{|\ell^2 - k^2| + k^4F^4}} \leqslant \int_{|k|}^{\infty} \frac{\ell e^{-\ell\zeta}\, d\ell}{k^2F^2}$$

If now we restrict ourselves to the bow region, y and z are $O(\varepsilon)$. This means that η and ζ are $O(\varepsilon^{1/2})$ since in the spherical region defined by distances $O(\varepsilon^{1/2})$ from the bow η and ζ were $O(1)$ [see the beginning of this Section]. With these orders of magnitude for η and ζ and remembering that $F^2 = O(\varepsilon^{-1/2})$ we can neglect the branch-out contribution as compared to the residue contribution.

Therefore, we write:

$$I \sim 2\pi i\{k^2F^2\exp[-i\zeta kF^2\sqrt{k^2- F^2} + \eta k^2F^2]\} \cdot \qquad \text{(I-17)}$$

b) $\underline{\underline{k < -1/F^2}}$

The contour of integration is shown in Fig. VI b). The arguments are the same as in the previous case and we can write

$$I \sim 2\pi i\{k^2F^2\exp[i\zeta|k|F^2\sqrt{k^2- F^2} + \eta k^2F^2]\} \qquad \text{(I-18)}$$

c) $\underline{\underline{-1/F^2 < k < 1/F^2}}$

We now want to show that in equation (I-13) the contribution of the integral in k, for $k = O(\varepsilon^{1/2})$ is negligible.

From the formulation of the boundary value problem in the main text, we know that

$$\phi(x,y,z) = O(\varepsilon^{5/2}) \quad *$$

and that

$$\sigma^*_R(k) = -\sigma^*_R(k) = O(\varepsilon^2) \qquad \text{if} \quad k = O(\varepsilon^{1/2})$$

$$\sigma^*_I(k) = O(\varepsilon^4) \qquad \text{if} \quad k = O(\varepsilon^{1/2})$$

---

* For simplicity we are assuming $\gamma = 1/2$, see page 8.

Let us denote

$$I_0 = \int_{-m}^{m} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{\ell^2 + k^2}]d\ell}{\sqrt{\ell^2 + k^2} - \frac{U^2k^2}{ga}}$$

such that $\ell < |k| = O(\varepsilon^{1/2})$ if $-m < \ell < m$.

Then the following estimate is valid in the near field:

$$|I_0| < 2\zeta \int_{0}^{m} \frac{\ell^2 d\ell}{|k| - \frac{U^2k^2}{ga}} = O(\varepsilon^{3/2})$$

If we denote

$$I_1 = \int_{m}^{\infty} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{\ell^2 + k^2}]d\ell}{\sqrt{\ell^2 + k^2} - \frac{U^2k^2}{ga}}$$

we have that in the near field

$$|I_1| \leq \int_{m}^{\infty} \frac{d\ell e^{\ell\eta}\ell}{\ell(\frac{m - F^2k^2}{m})} = - \frac{e^{\eta m}}{m - F^2k^2} = O(\varepsilon^{-1/2})$$

If we denote

$$I_2 = \int_{-\infty}^{-m} \frac{\ell \exp[i\ell\zeta + \eta\sqrt{\ell^2 + k^2}]d\ell}{\sqrt{\ell^2 + k^2} - \frac{U^2k^2}{ga}}$$

in the near field the following is valid:

$$|I_2| \leq O(\varepsilon^{-1/2})$$

by using the same arguments as used in I_1 .

Now we can use all these facts to show that $\phi^0(x,y,z) = O(\varepsilon^3)$ is negligible as compared to $\phi(x,y,z)$, where

$$\phi^0(x,y,z) = -\frac{1}{4\pi^2}\int_{-N}^{N} dke^{ikx}\sigma^*(k)I \quad , \quad N = O(\varepsilon^{1/2})$$

For that, let:

$$\phi^0 = \int_{-N}^{N} dke^{ikx}I\sigma_R^*(k) + i\int_{-N}^{N} dke^{ikx}I\sigma_I^*(k)$$

The second integral is $O(\varepsilon^4)$ and therefore,

$$\phi^0 \sim \int_{-N}^{N} dk\ \cos kx\ \sigma_R^* I_0 + \int_{-N}^{N} dk\ \cos kx\ \sigma_R^*(I_1 + I_2)$$

$$+i\int_{-N}^{N} dk\ \sin kx\ \sigma_R^* I_0 + \int_{-N}^{N} dk\ \sin kx\ \sigma_R^*(I_1 + I_2)$$

$$+ O(\varepsilon^4)$$

The first integral is $O(\varepsilon^4)$ and the third is $O(\varepsilon^5)$ and therefore neglected. The second integral has an odd integrand and therefore is equal to zero.

For $x = O(1)$ and $k = O(\varepsilon^{1/2})$ the sine function can be expanded in series and the fourth integral is $O(\varepsilon^3)$.

This shows that

$$\phi^0 = O(\varepsilon^3)$$

and can be neglected as compared to $\phi(x,y,z)$.

We can now use equations (I-17) and (I-18) to find the required behavior of the potential, by introducing them into equation (I-13). As we are interested only in the leading term, we expand the square root in equations (I-17) and (I-18), taking only the leading term, then combining in a single expression:

$$I \sim 2\pi i\{k^2F^2\exp[k^2F^2\eta - ik|k|F^2\zeta]\} \qquad \text{(I-19)}$$

To derive (I-19) we made the assumption that $\zeta > 0$. If $\zeta < 0$, we can use the same arguments; however, the paths of integration are not those of Fig. VI, but we must close the contour in the lower half-plane. The result is:

$$I \sim -2\pi i\{k^2F^2\exp[k^2F^2\eta - ik|k|F^2\zeta]\} \qquad \text{(I-20)}$$

Introducing (I-19) and (I-20) in equation (I-13) we have

$$\phi(\xi,\eta,\zeta) \sim -\frac{\mathrm{sgn}(\zeta)}{2\pi}\int_{-\infty}^{\infty} dk\, ie^{ik\xi}\sigma^*(k)k^2F^2\exp[k^2F^2\eta - iF^2k|k|\ |\zeta|]$$

In order to use this result in the main text, it is convenient to express it in dimensional coordinates x, y and z . One then gets:

$$\phi(x,y,z) \sim \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ikx}\sigma^*(k,y,z) \qquad \text{(I-21)}$$

where

$$\phi^*(k,y,z) = -\,\mathrm{sgn}(z)i\sigma^*(k)\,\frac{k^2U^2}{g}\exp\left[\frac{k^2U^2}{g}y - \frac{iU^2k|k||z|}{g}\right]$$

We call (I-21) the leading term of the inner expansion of the outer expansion.

APPENDIX II

OUTER EXPANSION OF NEAR-FIELD POTENTIAL

1. Outer Expansion of the Usual Near-Field

In this appendix the behavior of the expression for the velocity potential is found as one moves away from the body. From the text, it is found that one has two different expressions for the potential in the usual near-field. One which is valid in the region near the ship for $x < L/2$ except very close to the bow [equation (11)] and the other which is valid for $x > L/2$ [equation (15)].

Let us start with equation (11):

$$\begin{aligned} \phi(x;y,z) &= \mathrm{Re}\{Ub'[\sqrt{\zeta^2 + h^2} - \zeta]\} + M_{11}(x) \\ &= \mathrm{Re}\left[Ub'\zeta\left(\sqrt{1 + \frac{h^2}{\zeta^2}} - 1\right)\right] + M_{11}(x) \end{aligned} \qquad (11)$$

and, as one moves far away from the ship

$$h/\zeta = O(\varepsilon)$$

Therefore, expanding the square root, we have

$$\phi(x;y,z) = \mathrm{Re}\{Ub'\zeta[1 + \frac{h^2}{2\zeta^2} - \frac{h^4}{8\zeta^4} + \ldots\ldots -1]\} + M_{11}(x)$$

$$\phi(x;y,z) \sim \mathrm{Re}\{\frac{Ub'h^2}{2\zeta} + \ldots\ldots\ldots\} + M_{11}(x)$$

and

$$\phi(x;y,z) \sim \frac{Ub'h^2}{2} \frac{z}{y^2 + z^2} + M_{11}(x) + \ldots\ldots\ldots \qquad \text{(II-1)}$$

This is the leading term of the outer expansion of the near-field potential expressed by (11).

The outer expansion of the near-field potential for $x > L/2$ is obtained as follows:

$$\phi(x;y,z) = \phi(L/2,y,z) + \frac{1}{U}\int_{L/2}^{x} dx\,\mathrm{Re}\{U^2 b''[\sqrt{\zeta^2+h^2} - \zeta] + M_{12}(x) \tag{15}$$

The first term can be considered as the potential (11) for $x \to L/2$. Therefore, the outer expansion is:

$$\phi(L/2;y,z) \sim \frac{Ub'h^2}{2}\,\frac{z}{z^2+y^2} \qquad \text{where} \quad b' = b'(x = L/2), \quad h = h(x = L/2)$$

In order to find the outer expansion of the integral term, one can use the same arguments as before, and

$$\frac{1}{U}\int_{L/2}^{x} dx\,\mathrm{Re}\{U^2 b''[\sqrt{\zeta^2+h^2} - \zeta]\} \sim \frac{\mathrm{Re}}{U}\int_{L/2}^{x} dx\left[\frac{U^2 b'' h^2}{2}\,\frac{1}{\zeta} + \ldots\ldots\right]$$

$$\sim \frac{U}{2}\int_{L/2}^{x} dx\; b''h^2\,\frac{z}{y^2+z^2}$$

$$\sim \frac{z}{y^2+z^2}\,\frac{U}{2}\int_{L/2}^{x} dx\; b''h^2$$

The outer expansion of (15) is:

$$\phi(x;y,z) \sim \frac{z}{y^2+z^2}\left[\frac{Ub'h^2}{2} + \frac{U}{2}\int_{L/2}^{x} dx\; b''h^2\right] + M_{12}(x) \tag{II-2}$$

which is the leading term in the outer expansion of the potential (15).

One can combine (II-1) and (II-2) by giving the general expression for the outer expansion of the usual near-field.

$$\phi(x;y,z) \sim \frac{z}{y^2 + z^2} \quad \frac{Ub'(\xi)h^2(\xi)}{2} + \frac{UH(x - L/2)}{2} \int_{L/2}^{x} du b''h^2 \} + M(x) \tag{II-3}$$

where:

$$\xi = \begin{cases} x & \text{for } x < L/2 \\ L/2 & \text{for } x > L/2 \end{cases}$$

$H(x - L/2)$ = the Heaviside function.

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13. ABSTRACT
A wing of zero thickness and small aspect ratio is used as a mathematical model for a ship in steady turn. The shape of the wing is given by the projection of the hull on the vertical plane of symmetry and the aspect-ratio by the relation draft/length of the ship. As the ship moves in a turn, the angle of attack in each section is different; this fact is approximately simulated in the model by adding a camber to the wing and letting the wing follow a straight course.

Near the ship three regions are identified: the region downstream of the section of maximum draft (span), in which the Kutta condition is required to be satisfied, the region near midship, and the region near the bow. In the first region, the condition on the free surface is a rigid wall condition, thus permitting a reflection into the upper space. The acceleration potential concept is used in order to describe the flow in this region. In the region near midship, a reflection into the upper space is also possible. The method of solution is similar to the usual slender-body theory for ships. In the region near the bow, where the free surface suffers large deformations, the rigid-wall condition at the free surface is no longer valid. A different problem is formulated here, and Fourier transform approach is used to solve the problem in this region.

The method of matched asymptotic expansions was used to solve the problem. Therefore, the condition at infinity in each of the problems mentioned above is given by the far-field expansion, which describes the flow far from the ship.

For a simple case, when the ship has a constant draft in the bow region, the shape of the free surface near the ship in the bow region has been calculated.

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