ESTIMATES OF THE COBB-DOUGLAS PRODUCTION FUNCTION: A REAPPRAISAL

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L'articolo del Prof. J. Kmenta dell'Università del Wisconsin fu inviato al Prof. Fossati, nostro predecessore nella direzione di questa Rivista.

A causa delle circostanze luttuose intervenute, l'articolo del Prof Kmenta ha fatto alcuni giri e ci è pervenuto con notevole ritardo. Questo ritardo, forse, ha nociuto perchè nel frattempo le difficoltose ricerche induttive tendenti a stimare in qualche modo i parametri della Cobb-Douglas hanno fatto alcuni ulteriori passi. Se l'articolo che pubblichiamo avesse effettivamente risentito di tale ritardo, dobbiamo ammettere che ciò fu dovuto a cause di forza maggiore.

Nota del Direttore

The article by Professor J. Kmenta of the University of Wisconsin was sent to Professor Fossati, the late editor of this Review.

Following Professor Fossati's death, the article was passed to and fro and only came to hand after a considerable lapse of time. It may be that this delay has had untoward effects in that the complex research work directed to the evaluation of the Cobb-Douglas parameters has been taken some steps further in the meantime. If the article has in fact suffered as a result of the delay, we feel obliged to state that this was due to circumstances quite beyond our control.

Editor's Note

L'article du Professeur J. Kmenta, de l'Université du Wisconsin, avait été adressé au Professeur Fossati, notre prédécesseur à la direction de cette Revue.

Or, par suite des circonstances douloureuses que l'on sait, cet article, après plusieurs détours, ne nous est parvenu qu'avec un assez long retard. Il se pourrait donc que, du fait de ce retard, certains éléments nouveaux nous aient fait défaut car, entre temps, les difficiles recherches inductives entreprises en vue d'estimer, d'une manière ou d'une autre, les paramètres de Cobb-Douglas ont encore fait quelques pas en avant. Si l'actualité de l'article que nous publions aujourd'hui a eu effectivement à souffrir de ce retard, nous nous en excusons: il s'est agi là, il faut l'avouer, d'un cas de force majeure.

Note du Directeur

Der Artikel von Prof. Kmenta (Wisconsin-Universität) war an den derzeitigen Leiter der Redaktion dieser Zeitschrift, Prof. Fossati, gesandt worden.

Das tragische Ereignis hat es mit sich gebracht, daß dieser Artikel erst auf Umwegen und deshalb reichlich verspätet bei uns eintraf. Das hat ihm vielleicht in gewisser Hinsicht Abbruch getan, denn in der Zwischenzeit sind die komplizierten Induktionsforschungen zur Schätzung der Cobb-Douglas-Parameter ein Stück vorangetrieben worden. Sollte die verzögerte Veröffentlichung des Artikels dessen Wert tatsächlich beeinträchtigt haben, so bitten wir um Verständnis für diesen durch höhere Gewalt herbeigeführten Umstand.

Die Redaktion

I.

In a recent paper published in this journal (1) Professor Walters provides an illuminating discussion on the problem of estimating the parameters of the Cobb-Douglas production function. The three main methods of estimation discussed in the paper are:

1) the method of moments;

2) the method of indirect least squares (regression of the logarithm of output on the logarithm of output per man and on the logarithm of output per unit of capital);

3) the method of factor shares.

The purpose of this note is to point out a significant error in the exposition and application of the method of indirect least squares; to show that — with given specification — this method is identically equivalent to the method of moments; finally, to make some qualifying comments about the method of factor shares.

II.

Let us consider a firm which operates under perfectly competitive conditions on the product market and uses two variable and substitutable inputs obtained at fixed prices. The profit maximizing quantities of output and of inputs are then determined by the following relationships:

$$(1) \qquad y_0 - \alpha_1 y_1 - \alpha_2 y_2 = a_0 + e_0 \, ,$$

$$(2) \qquad y_0 - y_1 = \text{-}\log\,(\alpha_1 p_0/p_1) + e_1 \, ,$$

$$(3) \qquad y_0 - y_2 = \text{-}\log\,(\alpha_2 p_0/p_2) + e_2 \, .$$

Here y_0, y_1 and y_2 represent logarithms of quantities of output and of input 1 and input 2 ; p_0, p_1 and p_2 stand for the respective (fixed) prices, and e_0, e_1 and e_2 are random disturbances. Equation (1) is the Cobb-Douglas production function, and equations (2) and (3) are derived from the first-order conditions of profit maximization.

(1) WALTERS (1961).

The problem is to estimate the input elasticities α_1 and α_2 from the sample observations of outputs and inputs.

The traditional method of simple least squares applied to equation (1) gives biased and inconsistent estimates because the so-called « independent » variables y_1 and y_2 are not independent of the disturbance term e_0. Equations (2) and (3) show, however, that $(y_0 - y_1)$ and $(y_0 - y_2)$ are independent of e_0 as long as e_0 is independent of e_1 and e_2. We may then form a new equation

$$(4) \qquad y_0 = \gamma_0 + \gamma_1(y_0 - y_1) + \gamma_2(y_0 - y_2) + v$$

and obtain simple least squares estimates of γ_1 and γ_2 which are consistent. Equation (4) can be re-written as

$$(4') \qquad y_0 = \gamma_0/(1 - \gamma_1 - \gamma_2) - [\gamma_1/(1 - \gamma_1 - \gamma_2)]\, y_1 - [\gamma_2/(1 - \gamma_1 - \gamma_2)]\, y_2 + v/(1 - \gamma_1 - \gamma_2)$$

Comparison of (4') with the production function (1) implies that

$$(5) \qquad \alpha_1 = -\gamma_1/(1 - \gamma_1 - \gamma_2) \quad , \quad \alpha_2 = -\gamma_2/(1 - \gamma_1 - \gamma_2)$$

By substituting the least squares estimates of γ_1 and γ_2 we obtain estimates of α_1 and α_2 which are consistent. This is the method of indirect least squares.

An earlier description of the method (2) contained an error which seems to have been inherited by Prof. Walters (3) who states that

$$\gamma_p = \alpha_p/(1 - \alpha_1 - \alpha_2) \qquad (p = 1, 2)$$

whereas it is clear from (5) above that

$$\gamma_p = -\alpha_p/(1 - \alpha_1 - \alpha_2) \qquad (p = 1, 2)\,.$$

The expression for γ_p given by Prof. Walters would be correct only if y_0 were regressed on $(y_1 - y_0)$ and $(y_2 - y_0)$, not on $(y_0 - y_1)$ and $(y_0 - y_2)$ as stated. This makes a substantial difference to the estimated values of α_1 and α_2 since

$$\hat{\alpha}_p \text{ (correct)} = -\hat{\gamma}_p/(1 - \hat{\gamma}_1 - \hat{\gamma}_2) \qquad \text{and}$$

$$\hat{\alpha}_p \text{ (incorrect)} = \hat{\gamma}_p/(1 + \hat{\gamma}_1 + \hat{\gamma}_2) \qquad (p = 1, 2)\,.$$

The existence of the error may explain why the application of the method to the data ot Bronfenbrenner and Douglas (1939) led to the conclusion that « this method was a failure » (4).

(2) See Hoch (1958), p. 572, footnote 11.
(3) Walters, *op. cit.*, p. 132.
(4) *Ibid.*, p. 135.

It should be noted that the consistency property of the indirect least squares estimates of α_1 and α_2 is preserved even if e_1 and e_2 are not independent of each other. However, in this case we may encounter a high degree of relationship between $(y_0 - y_1)$ and $(y_0 - y_2)$ in the sample, resulting in large standard errors of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ in equation (4). That is, we are likely to face the problem of multicollinearity. The method of indirect least squares breaks down completely (in the sense that the estimates become inconsistent) if e_0 is not independent of e_1 and e_2. A difficulty also arises when the sum $(\alpha_1 + \alpha_2)$ is close to unity, i.e., when the returns to scale are close to being constant. In this case $\hat{\gamma}_1$ and $\hat{\gamma}_2$ will tend to be highly unstable in small samples, fluctuating from large positive to large negative values.

III

The problem of estimating α_1 and α_2 may be approached in a different way. By expressing both sides of equations (1) to (3) in terms of variances and covariances we obtain

$$AMA' = U \tag{6}$$

where $A = \begin{bmatrix} 1 & -\alpha_1 & -\alpha_2 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$,

$$M = \begin{bmatrix} \text{Var}(y_0) & & \\ \text{Cov}(y_0,y_1) & \text{Var}(y_1) & \\ \text{Cov}(y_0,y_2) & \text{Cov}(y_1,y_2) & \text{Var}(y_2) \end{bmatrix}$$

and $U = \begin{bmatrix} \text{Var}(e_0) & & \\ \text{Cov}(e_0,e_1) & \text{Var}(e_1) & \\ \text{Cov}(e_0,e_2) & \text{Cov}(e_1,e_2) & \text{Var}(e_2) \end{bmatrix}$.

If we replace the variances and covariances of M by the appropriate sample values, we can use the six equations of (6) to estimate α_1, α_2 and the variances and covariances of the disturbances. This is a general description of the moments method as formulated by Marschak and Andrews in 1944 [5].

Allowing for symmetry, relationship (6) provides us with six equations to estimate eight parameters (α_1, α_2 and the six variances an covariances of the disturbances). It is obvious, then, that estimation is possible only if further restrictions are introduced. If it is assumed that the disturbance in the production function is

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[5] Actually, MARSCHAK and ANDREWS formulated the problem even more generally by allowing for imperfect competition.

independent of the disturbances in the decision equations, i. e., if Cov (e_0, e_1) = Cov $(e_0, e_2) = 0$, then α_1 and α_2 can be estimated by

$$(7) \qquad \begin{bmatrix} C_{11} - C_{01} & C_{12} - C_{02} \\ C_{12} - C_{01} & C_{22} - C_{02} \end{bmatrix} \begin{bmatrix} \breve{\alpha}_1 \\ \breve{\alpha}_2 \end{bmatrix} = \begin{bmatrix} C_{01} - C_{00} \\ C_{02} - C_{00} \end{bmatrix},$$

where $C_{rs} = \frac{1}{n} \sum_{i=1}^{n} (y_{ri} - \bar{y}_r)(y_{si} - \bar{y}_s)$, $(r, s = 0, 1, 2)$, and $\breve{\alpha}_1$ and $\breve{\alpha}_2$ are estimates of α_1 and α_2. These estimates are consistent and, under certain further assumptions, of maximum likelihood type.

We shall now proceed to show that the moments estimates as specified above are equivalent to the indirect least squares estimates described in section II.

Let $y_0 = x_0$, $(y_0 - y_1) = x_1$, $(y_0 - y_2) = x_2$,

and $M_{rs} = \frac{1}{n} \sum_{i=1}^{n} (x_{ri} - \bar{x}_r)(x_{si} - \bar{x}_s)$, $(r, s = 0, 1, 2)$.

Then

$$\begin{aligned} C_{00} &= M_{00} \\ C_{0p} &= M_{00} - M_{0p} \\ C_{pp} &= M_{00} + M_{pp} - 2M_{0p} \\ C_{12} &= M_{12} + M_{00} - M_{01} - M_{02} \qquad (p = 1, 2). \end{aligned}$$

By substituting into (7) and solving for $\breve{\alpha}_1$ and $\breve{\alpha}_2$ we obtain

$$(8) \qquad \begin{bmatrix} \breve{\alpha}_1 \\ \breve{\alpha}_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -M_{22}M_{01} + M_{12}M_{02} \\ -M_{11}M_{02} + M_{12}M_{01} \end{bmatrix}$$

where

$$D = M_{11}M_{22} - M_{11}M_{02} - M_{22}M_{01} + M_{12}M_{02} + M_{12}M_{01} - M^2_{12}.$$

Now, according to equation (4′) the indirect least squares estimates are

$$(9) \qquad \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix} = \begin{bmatrix} -\hat{\gamma}_1/(1 - \hat{\gamma}_1 - \hat{\gamma}_2) \\ -\hat{\gamma}_2/(1 - \hat{\gamma}_1 - \hat{\gamma}_2) \end{bmatrix},$$

where $\hat{\gamma}_1$ and $\hat{\gamma}_2$ are the simple least squares estimates of γ_1 and γ_2 given by

$$\begin{aligned} \hat{\gamma}_1 &= (M_{22}M_{01} - M_{12}M_{02})/(M_{11}M_{22} - M^2_{12}) \quad \text{and} \\ \hat{\gamma}_2 &= (M_{11}M_{02} - M_{12}M_{01})/(M_{11}M_{22} - M^2_{12}). \end{aligned}$$

Substitution into (9) leads to

$$(10) \qquad \begin{bmatrix} \hat{\alpha}_1 \\ \hat{\alpha}_2 \end{bmatrix} = \frac{1}{D} \begin{bmatrix} -M_{22}M_{01} + M_{12}M_{02} \\ -M_{11}M_{02} + M_{12}M_{01} \end{bmatrix},$$

where D is defined as in (8) above. The right-hand sides of equations (8) and (10) are obviously equal, proving that the two methods of estimation are identically equivalent for all sample sizes.

IV

The factor shares method of estimation utilizes the fact that, under the specifications of the perfectly competitive equilibrium, α_1 and α_2 are equal to the proportion of the total value of output spent on input 1 and input 2 respectively. If random disturbances are present, as in our equation (1) to (3), the equality will be true only for firms with zero (logarithmic) disturbances. Since it is generally reasonable to assume that $E(e_0) = E(e_1) = E(e_2) = 0$, the equality holds for the average firm [6].

Estimates of α_1 and α_2 are then simply obtained by estimating the quantities of output and of inputs of the average firm from sample means. The relationships are given by equations (2) and (3) and the estimates are

$$\text{(11)} \qquad \log \hat{\alpha}_r = \log(p_r/p_0) + \bar{y}_1 - \bar{y}_0$$

or

$$\text{(11')} \qquad \log \hat{\alpha}_r = \frac{1}{n} \sum_{i=1}^{n} \log (Y_{ri}p_{ri}/Y_{0i}p_{0i}) , \qquad (r = 1, 2)$$

where Y_0, Y_1 and Y_2 stand for quantities, and p_0, p_1 and p_2 for prices, of output and of the two inputs. This method of estimation was suggested by Klein in 1953 and leads to best linear unbiased estimates of $\log \alpha_1$ and $\log \alpha_2$.

The important assumption underlying the method of factor shares, and embodied in the model as described by equations (1) to (3), is that the average firm is exactly in the position of profit maximizing equilibrium. In other words, maximum profit is attained only by the average firm (or firms) while the profit of the remaining firms falls short of the maximum value. Thus the method of factor shares cannot be used to test for the (average) efficiency of allocation of resources in an industry since this is assumed *a priori*. The assumption is not necessary for the method of indirect least squares (and not, of course, for the equivalent method of moments) since these estimates retain their desirable properties even if the profit maximizing efforts of the firms are subjected to parametric restraints. In this case equations (2) and (3) may be changed to

$$\text{(2')} \qquad y_0 - y_1 = - \log (\alpha_1 p_0 R_1/p_1) + e_1 ,$$

$$\text{(3')} \qquad y_0 - y_2 = - \log (\alpha_2 p_0 R_2/p_2) + e_2 ,$$

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(6) « Average » firm is one which produces average (logarithmic) quantity of output and employs average (logarithmic) quantities of inputs.

where R_1 and R_2 are constants representing the profit maximizing restrictions. If there are no restrictions, then $R_1 = R_2 = 1$. The presence of R_1 and R_2, even when they differ from unity, does not affect the properties of the indirect least squares estimates. But if R_1 or R_2 are not equal to one, estimates based on the method of factor shares will be biased and inconsistent. The size of the bias would then depend on the extent to which R_1 or R_2 deviate from unity, i. e., on the distance of the average firm from the optimal position. This point was emphasized by Hoch in 1958 who developed an estimation procedure which allows for the presence of parametric restraints as shown in equations (2') and (3') above. It is possible to show that Hoch's estimation method, in its generalized form, is in fact equivalent to the method of indirect least squares and of moments.

V

The foregoing discussion served to clarify some of the issues involved in estimating the parameters of the Cobb-Douglas production function from cross-sectional data. It appears that, in general, we have a choice between the method of indirect least squares (or the equivalent method of moments) and the method of factor shares, but each method has its limitation. For the method of indirect least squares to give consistent results, it is necessary that the disturbance in the production function is independent of the disturbances in the decision equations; that is, that there is no relationship between « technical » and « economic » efficiency in individual firms. The method of factor shares, on the other hand, depends in a crucial way on the condition that the average firm is one which makes maximum profit. This method thus cannot be used at all to test for the efficiency of allocation of resources. Further, information about the small sample properties of various estimates (7) shows that the indirect least squares estimates tend to have a considerably larger variance than the factor shares estimates. In other words, in the absence of other information about the industry, the choice of an estimation procedure depends on the use to which the estimates are to be put. If the research worker is interested in testing the efficiency of allocation of resources in the industry, the method of factor shares is inappropriate. If the purpose of the research is, for instance, a comparison of marginal productivity in various regions or industries, either of the methods can be used; the choice will depend on the relative importance attached to consistency as compared to variability.

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(7) See KMENTA and JOSEPH (1963).

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