Econometrica, Vol. 31, No. 3 (July, 1963)

A MONTE CARLO STUDY OF ALTERNATIVE ESTIMATES OF THE COBB-DOUGLAS PRODUCTION FUNCTION[1]

BY J. KMENTA AND M. E. JOSEPH

A Monte Carlo experiment is carried out to examine the small sample properties of ordinary least squares, indirect least squares, Hoch's, and Klein's estimates of the parameters of the Cobb-Douglas production function. A perfectly competitive model of firms in a single industry is considered in nine situations which differ in the behavior of the disturbances, the variability of inputs, and the position of the average firm. In each case 200 samples of size 20 and 200 samples of size 100 were obtained to approximate the sampling distribution of the various estimators.

1. INTRODUCTION

THE CONDITIONS of profit maximization and the specification of the production function fully determine the equilibrium position of a firm that operates under conditions of perfect competition in the product market, obtains its inputs at fixed prices, and experiences decreasing returns to scale. If all the relationships hold exactly, all firms in the industry will be producing identical quantities of output and will be employing identical quantities of inputs, providing the inputs are freely variable and substitutable.[2] Variations from firm to firm will exist if one or more of the inputs are fixed; in this case, the profit maximizing quantities of output and of inputs will depend on the amount of the fixed input or inputs in each firm. If, however, the production function as well as the profit-maximizing decision equations contain stochastic disturbances, differences in actual outputs and inputs of firms will appear even in the absence of fixed factors of production. In this case a solution of the system of equations for the quantities of variable inputs and of output of any firm shows that each quantity is a function of all disturbances in the system. Consequently, the inputs are not independent of the disturbance in the production function, and single-equation least-squares estimates of the production function parameters based on cross-sectional data will be, in general, biased and inconsistent. This was first noted in a classical article by Marschak and Andrews [5] in 1944. Alternative methods of estimation have since been proposed; these include Klein's, Hoch's, and the indirect least-squares procedure. With the exception of the first, no small sample properties of the suggested estimators have been derived. This paper represents

EDITORS' NOTE: We regret to record that Mr. M. E. Joseph is now deceased and his authorship of the final version of this article is posthumous.

[1] This project was supported by a University of Sydney Research Grant.

[2] A uniform technology throughout the industry is assumed.

an attempt at describing the behavior of these estimators in small samples in a number of specific cases by means of a Monte Carlo experiment.

2. MODELS

2.1. *Common properties.* The industry is envisaged as being perfectly competitive with all firms producing a homogeneous product and employing two homogeneous and substitutable inputs. The price received for the product on the market is 1 unit, and the prices paid for the first and second inputs are 5 and 2 units, respectively. The production function of the ith firm is given by

$$x_{0i} = 0.9796 + 0.5\,x_{1i} + 0.4\,x_{2i} + u_i\,, \tag{1}$$

where x_{0i} is the logarithm[3] of the quantity of output, x_{1i} is the logarithm of the quantity of input 1, x_{2i} is the logarithm of the quantity of input 2, and u_i is the "technical" disturbance which is normally and independently distributed with zero mean and constant variance. In the rest of the paper we shall refer to the coefficient attached to x_1 as a_1, and to that attached to x_2 as a_2. These are the input elasticities of output of Cobb-Douglas fame.

2.2. *Models A, B, C, D and E.*

In the first five models both inputs are variable, and the average firm[4] is optimal. The first-order conditions for profit maximization lead to the following decision equations for the ith firm, after allowing for the presence of disturbances:

$$x_{1i} = -1.000 + x_{0i} + v_{1i}\,, \tag{2}$$

$$x_{2i} = -0.699 + x_{0i} + v_{2i}\,. \tag{3}$$

Here v_{1i} and v_{2i} are the "economic" disturbances expected to be present in the decision process;[5] they are normally and independently distributed with zero means and constant variances.[6] The optimal firm uses 10 units of input 1, 20 units of input 2, and produces 100 units of output. The models are different only with respect to the behavior of the disturbances u_i, v_{1i}, and v_{2i}. The detailed specifications are given in Table I.

[3] All logarithms used in the text of this paper are common logarithms.

[4] "Average" refers to the population mean.

[5] The terms "technical" and "economic" disturbance were coined by Marschak and Andrews.

[6] The condition of independence refers to firms, i.e., $E(v_{1i}v_{1j}) = 0$ and $E(v_{2i}v_{2j}) = 0$ for all $i \neq j$.

TABLE I

SPECIFICATION OF DISTURBANCES IN MODELS A, B, C, D, AND E*

	A	B	C	D	E
$E(u^2)$	.00040	.00020	.00040	.00040	.00020
$E(v_1^2)$	.00040	.00100	.00040	.00040	.00100
$E(v_2^2)$	.00040	.00100	.00040	.00040	.00100
$E(uv_1)$	0	0	0	$.00040(\sqrt{.8})$	.00040
$E(uv_2)$	0	0	0	$.00040(\sqrt{.8})$	.00040
$E(v_1v_2)$	0	0	.00032	.00032	.00080
Correlation coefficient of:					
u and v_1	0	0	0	$+\sqrt{.8}$	$+\sqrt{.8}$
u and v_2	0	0	0	$+\sqrt{.8}$	$+\sqrt{.8}$
v_1 and v_2	0	0	+.8	+.8	+.8

* The relative values of $E(u^2)$, $E(v_1^2)$ and $E(v_2^2)$ conform to some of those selected by Hoch in [2, p. 574].

2.3. *Models F and G.*

Models F and G represent a situation where both inputs are variable but the average firm is not in an optimal position. The decision equations of the ith firm are

$$x_{1i} = -0.97178 + x_{0i} + v_{1i}\,, \tag{4}$$

$$x_{2i} = -0.74133 + x_{0i} + v_{2i}\,. \tag{5}$$

The disturbances v_{1i} and v_{2i} again have zero means and constant variances and are normally and independently distributed. The average firm uses 10 units of input 1, 17.1 units of input 2, and its level of output is 93.7 units. Such a situation could arise in the case where the average firm, after acquiring the desired quantity of input 1, has been prevented from obtaining the optimum quantity of input 2. The distinguishing features of the two models are shown in Table II.

TABLE II

SPECIFICATION OF DISTURBANCES IN MODELS F AND G

	F	G
$E(u^2)$	0.00040	0.00020
$E(v_1^2)$	0.00040	0.00100
$E(v_2^2)$	0.00040	0.00100
$E(uv_1)$	0.00000	0.00000
$E(uv_2)$	0.00000	0.00000
$E(v_1v_2)$	0.00000	0.00000

TABLE III

PREDETERMINED QUANTITIES OF INPUT 2 IN MODELS H AND I

Quantity of input 2	x_2	Number of firms in samples of size	
		20	100
16.60	1.22	0	1
18.20	1.26	1	6
19.95	1.30	5	24
21.87	1.34	8	38
23.98	1.38	5	24
26.30	1.42	1	6
28.84	1.46	0	1

2.4. *Models H and I.*

In these two models we consider a case where one of the inputs, input 2, is predetermined. The predetermined quantities of input 2 are given in Table III. The decision equation for input 1 of the ith firm is given by equation (2). The predetermined quantities of input 2 are, of course, exact; and the second decision equation containing v_2 disappears from the system. The quantity of input 1 used and of output produced by each firm will depend on the quantity of input 2 as well as on the disturbances u and v_1. The average firm, which uses 21.87 units of input 2, will be employing 10.74 units of input 1 and producing 107.4 units of output.[7] The differences between model H and model I are due to the size of the error variances as shown in Table IV.

TABLE IV

SPECIFICATION OF DISTURBANCES IN MODELS H AND I

	H	I
$E(u^2)$	0.00040	0.00020
$E(v_1^2)$	0.00040	0.00100
$E(uv_1)$	0.00000	0.00000

3. ALTERNATIVE METHODS OF ESTIMATION

3.1. *Hoch's estimates.*

The stimulus to the Monte Carlo study presented here came from Hoch's article [2] on the estimation of parameters of the Cobb-Douglas production

[7] Firms not included in the sample are envisaged as having the same distribution of input 2 as those which are included. A more realistic approach would be to allow the quantities of input 2 to vary from sample to sample. However, the conclusions would not be substantially affected if drawings had been made from the distributions corresponding exactly to the within-sample distributions or their continuous approximations.

function from cross-sectional data. Hoch specified the extent of the single-equation least-squares bias in infinitely large samples and proposed an estimation procedure which removes the bias from the ordinary least squares estimates. In the case where the disturbances are not correlated, i.e., where $E(uv_1) = E(uv_2) = E(v_1 v_2) = 0$, and where both inputs are variable, Hoch's estimates of a_1 and a_2 are given by

$$\hat{\hat{a}}_r = \hat{a}_r \left[1 + \frac{\hat{\hat{S}}_{00}}{\hat{S}_{11}} + \frac{\hat{\hat{S}}_{00}}{\hat{S}_{22}}\right] - \frac{\hat{\hat{S}}_{00}}{\hat{S}_{rr}}, \qquad (r = 1, 2), \tag{6}$$

where $\hat{a}_r$ is the ordinary least squares estimate of a_r, and $\hat{\hat{S}}_{00}$, $\hat{S}_{11}$, and $\hat{S}_{22}$ are the estimates of the error variances $E(u^2)$, $E(v_1^2)$, and $E(v_2^2)$. The estimates of the error variances can be obtained from the sample moments as follows:

$$\hat{S}_{rr} = C_{00} + C_{rr} - 2C_{0r}, \qquad (r = 1, 2),$$

$$\hat{\hat{S}}_{00} = \frac{\hat{S}_{00}}{1 - (\hat{S}_{00}/\hat{S}_{11}) - (\hat{S}_{00}/\hat{S}_{22})},$$

where C_{00} is the sample variance of x_0, C_{rr} is the sample variance of x_r, C_{0r} is the sample covariance of x_0 and x_r, and $\hat{S}_{00} = C_{00} - \hat{a}_1 C_{01} - \hat{a}_2 C_{02}$.

All variances and covariances are represented by their asymptotic values. Hoch suggests using moments of samples of finite size in place of the asymptotic values in empirical investigations. This estimation procedure is, in our case, applicable to models A, B, F, and G.

When input 2 is taken as exogenously determined and the remaining disturbances are not correlated, i.e., $E(uv_1) = 0$, Hoch's estimates become

$$\hat{\hat{a}}_1 = \hat{a}_1 \left[1 + \frac{\hat{\hat{S}}_{00}}{\hat{S}_{11}}\right] - \frac{\hat{\hat{S}}_{00}}{\hat{S}_{11}}, \tag{7a}$$

$$\hat{\hat{a}}_2 = a_2 \left[1 + \frac{\hat{\hat{S}}_{00}}{\hat{S}_{11}}\right]. \tag{7b}$$

$\hat{\hat{S}}_{00}$ is now given by $\hat{S}_{00}/[1 - (\hat{S}_{00}/\hat{S}_{11})]$; other symbols remain unchanged. These estimators were used in our models H and I.

Hoch points out, but does not elaborate, an estimation procedure for the situation where v_1 and v_2 are correlated with each other but not with u, and both inputs are variable. The estimators in this case are

$$\hat{\hat{a}}_r = \hat{a}_r \left[1 + \frac{\hat{\hat{S}}_{00}(\hat{S}_{11} + \hat{S}_{22} - 2\hat{S}_{12})}{\hat{S}_{11}\hat{S}_{22} - \hat{S}_{12}^2}\right] - \frac{\hat{\hat{S}}_{00}(\hat{S}_{pp} - \hat{S}_{12})}{\hat{S}_{11}\hat{S}_{22} - \hat{S}_{12}^2}, \tag{8}$$

$$(r = 1, 2; p = 1, 2; p \neq r).$$

Here $\hat{S}_{12} = C_{00} + C_{12} - C_{01} - C_{02}$, where C_{12} is the sample covariance of x_1 and x_2, and

$$\hat{\hat{S}}_{00} = \frac{\hat{S}_{00}(\hat{S}_{12}^2 - \hat{S}_{11}\hat{S}_{22})}{\hat{S}_{00}(\hat{S}_{11} + \hat{S}_{22} - 2\hat{S}_{12}) + (\hat{S}_{12}^2 - \hat{S}_{11}\hat{S}_{22})};$$

other symbols are defined as in equation (6). These estimators apply to our model C.[8]

When interrelationships exist among the "technical" disturbance u and the "economic" disturbances v_1 and v_2, Hoch's estimation method no longer applies, and cannot, therefore, be used in our models D and E.

The estimators which are based on correct assumptions about the nature of the interrelationship among the disturbances will be referred to as "proper." However, in real life, the research worker is never certain about the nature of interrelationship among the disturbances in the population and usually relies on an intelligent guess. Now, it is quite conceivable that an incorrect guess is made, and an estimation procedure which was designed for a different state of the world is applied. If this happens, the question of the consequences of a wrong decision becomes interesting. When applying Hoch's estimation method, one could, for instance, assume no correlation among the disturbances, although, in fact, some correlation exists. The consequences of such an error of judgement have been traced by applying Hoch's estimation procedure designed for the case of uncorrelated disturbances (i.e., estimation formulas applicable to models A, B, F, and G) to the situation where the "economic" disturbances are correlated with each other but not with the "technical" disturbance (model C), and to the situation where all disturbances are correlated with each other (models D and E). The estimators based on this kind of incorrect assumption will be called "estimators for uncorrelated errors" and abbreviated by "EUE." Another error of judgement could be that of applying the estimation procedure designed to deal with a situation where the "economic" disturbances are correlated with each other but not with the "technical" disturbance (i.e., estimation formulas applicable to model C) to the case where all disturbances are correlated (models D and E). The use of such an incorrect assumption leads to estimators which we term "estimators for partly correlated errors" and abbreviate by "EPCE."

[8] A printing error in the text of Hoch's article resulted in an incorrect expression for C_{00} in his equation (4.1) on page 575 of [2]. C_{00} should equal

$$D^2[\sum_p \sum_q a_p a_q S_{pq} + S_{00} + 2 \sum_p a_p S_{0p}] .$$

3.2. *Klein's estimates.*

The conditions of profit maximization require that, for each input, the value of the marginal product should be equal to the price of input. If it is assumed that any departure from this equality on the part of individual firms is due only to the operation of the "economic" disturbances, the input elasticities of the Cobb-Douglas function can be estimated by

$$\log \tilde{a}_r = \frac{1}{n} \sum_{i=1}^{n} \log \frac{P_{ri} X_{ri}}{P_{0i} X_{0i}}, \qquad (r = 1, 2). \tag{9}$$

Here P_{0i} and P_{ri} stand for prices, and X_{0i} and X_{ri} for quantities of output and of rth input of the ith firm. In perfect competition the prices are, of course, the same for each firm. This estimation procedure was suggested by Klein.[9] In the price situation applicable to our models, Klein's estimates are given by

$$\log \tilde{a}_1 = 0.699 + \bar{x}_1 - \bar{x}_0, \tag{10a}$$

$$\log \tilde{a}_2 = 0.301 + \bar{x}_2 - \bar{x}_0, \tag{10b}$$

where small letters represent logarithms of the quantity of output and of inputs as before.[10] The procedure gives, under the stated assumptions, best linear unbiased and maximum likelihood estimates of $\log a_1$ and $\log a_2$.

The fundamental difference between Klein's and Hoch's estimation methods lies in the assumption about the position of the average firm. Klein's estimates are based on the assumption that the average firm is optimal, whereas Hoch's estimates allow for the existence of a parametric restraint on the profit maximizing efforts of the average firm. In consequence, Klein's estimates become biased and inconsistent when the average firm is not optimal, or when one of the inputs is predetermined.[11] In the latter case the bias appears only with respect to the coefficient attached to the predetermined input.

3.3. *Indirect least squares estimates.*[12]

The logarithmic form of the Cobb-Douglas production function, extended by the inclusion of the "technical" disturbance, is

$$x_{0i} = a_0 + a_1 x_{1i} + a_2 x_{2i} + u_i. \tag{11}$$

[9] Klein [**3**, pp. 193–196].

[10] In our models we envisage that prices do not vary from firm to firm, and that they are ascertainable without errors of measurement.

[11] Unless, of course, the average firm possesses a profit maximizing optimum quantity of the fixed input.

[12] This method was suggested by H. Theil; see Hoch [**2**, p. 572, footnote 11]. Hoch's exposition of the method contains an error: the variable z_q should be defined as $(x_q - x_0)$ and not as $(x_0 - x_q)$; otherwise the conclusion reached does not follow.

By deducting $x_{0t}(a_1 + a_2)$ from both sides of (11) and dividing through by $(1 - a_1 - a_2)$, we obtain

$$x_{0t} = b_0 + b_1 z_{1t} + b_2 z_{2t} + e_t \,, \tag{12}$$

where

$$z_{rt} = x_{rt} - x_{0t} \,,$$
$$b_r = a_r/(1 - a_1 - a_2) \qquad (r = 1, 2) \,,$$

and

$$e_t = u_t/(1 - a_1 - a_2) \,.$$

Since the profit maximizing decision equations lead to

$$x_{rt} - x_{0t} = \text{constant} + v_{rt} \,,\text{[13]} \qquad (r = 1, 2) \,,$$

where v_{rt} is the "economic" disturbance, least squares estimates of b_r based on equation (12) will be consistent as long as $E(uv_r) = 0$. From these we can obtain consistent estimates of a_r.

If input 2 is fixed, we have only one decision equation, and the appropriate substitution leads to

$$x_{0t} = b_0^* + b_1^* z_{1t} + b_2^* x_{2t} + e_t^* \,, \tag{13}$$

where

$$b_r^* = a_r/(1 - a_1) \qquad (r = 1, 2) \,,$$
$$z_{1t} = x_{1t} - x_{0t} \,,$$

and

$$e_t^* = u_t/(1 - a_1) \,.$$

Here again, least squares estimates of b_r^* based on equation (13) will lead to consistent estimates of a_r providing $E(uv_1) = 0$.

The indirect least squares estimates, like Hoch's, are not restricted by the assumption that the average firm lies in an optimal position. If the profit maximizing efforts of the average firm are subjected to parametric restraints $(x_{rt} - x_{0t})$, will still be a linear function of v_{rt}, and the consistency of the estimates of a_r will be preserved.

4. ASYMPTOTIC VALUES OF THE ESTIMATORS

Asymptotic values of each of the estimators in each of the models specified in Section 2 above are presented in Table V. The abbreviation O.L.S. refers

[13] The constant may include a parametric restraint on the profit maximizing efforts of the firm if applicable.

TABLE V*

ASYMPTOTIC VALUES OF ESTIMATES OF $a_1 = 0.5$

Model	O.L.S.	Hoch's			Klein's	I.L.S.
		Proper**	EUE**	EPCE**		
A	.50000	.50000	—	—	.50000	.50000
B	.50000	.50000	—	—	.50000	.50000
C	.50000	.50000	.50000	—	.50000	.50000
D	.50000	—	.50000	.50000	.50000	.50000
E	.50000	—	.50000	.50000	.50000	.50000
F	.50000	.50000	—	—	.53362	.50000
G	.50000	.50000	—	—	.53362	.50000
H	.75000	.50000	—	—	.50000	.50000
I	.58333	.50000	—	—	.50000	.50000
ASYMPTOTIC VALUES OF ESTIMATES OF $a_2 = 0.4$						
A	.46667	.40000	—	—	.40000	.40000
B	.42857	.40000	—	—	.40000	.40000
C	.45263	.40000	−.40000	—	.40000	.40000
D	.45135	—	.44855	.44967	.40000	.44984
E	.43158	—	.43011	.43077	.40000	.43077
F	.46667	.40000	—	—	.36279	.40000
G	.42857	.40000	—	—	.36279	.40000
H	.20000	.40000	—	—	.40720	.40000
I	.33333	.40000	—	—	.40720	.40000

* Since the computations for this table have been carried out on a desk calculator, the accuracy of the figures is subject to errors due to rounding.
** For explanation of the terms see the final paragraph of Section 3.1.

to "ordinary least squares" and I.L.S. to "indirect least squares" estimates.

It appears that the ordinary least squares estimates are generally inconsistent. The apparent consistency of ordinary least squares estimates of a_1 in models A through G is due to the fact that $a_1 = 0.5$ and that the variance of v_1 is equal to the variance of v_2 in our models. The particularity of this case can readily be seen from equations (6) and (8) and, by contrast, from equation (7a).[14] By using a value of a_1 which leads to consistent least squares estimates, we hoped to provide a "control" of the Monte Carlo process. The degree of inconsistency of the ordinary least squares estimates of a_2 is higher in models where all the disturbances have equal variances than in models where the "technical" disturbance has a relatively smaller variance compared to the variances of the "economic" disturbances.

Hoch's estimation procedure applied to models for which it was designed renders estimates which are always consistent. Inconsistency appears when

[14] By substituting the asymptotic values for Hoch's estimate of a_1 and for the error variances.

incorrect assumptions are made about the nature of interrelationship of the disturbances.[15] Here again, the degree of inconsistency is lower when the variance of u is small relative to the variances of v_1 and v_2.

Klein's estimates are consistent except in models where the average firm is not optimal or when estimating coefficients attached to fixed inputs. Inconsistency in Klein's estimates, when it appears, is unaffected by the relative sizes of the error variances.

Indirect least squares estimates are consistent except in models in which the "technical" disturbance is correlated with the "economic" disturbances.[16] In the latter case the degree of inconsistency is somewhat lower if $E(u^2)$ is small relative to $E(v_1^2)$ and $E(v_2^2)$.

5. DESIGN OF THE MONTE CARLO EXPERIMENT

The Monte Carlo results for each of the models were obtained in two stages. The first stage consisted of generating the random disturbances and calculating the corresponding values of the variables x_0, x_1, and x_2. Samples of size 20 and 100 were then obtained.[17] The values of the various estimators of the parameters and of the error variances were then computed and printed.[18] A signal was built into the program to indicate whenever Hoch's condition $C_{00} \leqslant C_{rs}$ $(r = 0, 1, 2;\ s = 1, 2)$ was violated. The whole process was repeated 200 times for each model. In the second stage a summary of the empirical sampling distribution was produced. This gave, for each of the estimates of a_1 and a_2, a frequency distribution, sum, sum of squares, and sum of cubes. The sums only of the estimates of the error variances and covariances were produced. All this information was also given for those samples where sample moments did not conform to Hoch's condition specified above.[19]

[15] The inconsistency appears only in the estimates of a_2 for reasons explained above.

[16] Again, the inconsistency appears in our models only with respect to estimates of a_2.

[17] Samples of size 100 were built from a completely new set of disturbances so that there is no overlapping with samples of size 20.

[18] The estimates of the error variances and covariances were based on sample moments defined by

$$C_{rs} = \frac{1}{n} \sum_{i=1}^{n} (x_{ri} - \bar{x}_r)(x_{si} - \bar{x}_s)\,, \qquad (r, s = 0, 1, 2)\,.$$

That is, the sums were divided by the number of sample observations rather than by the number of degrees of freedom. Thus, our estimates of the error variances contain a downward bias. However, the effect on Hoch's estimates of the parameters of the production function is negligible.

[19] It appeared that the characteristics of the sampling distributions were not noticeably affected by leaving out those samples in which the condition was not fulfilled.

The entire operation was carried out on SILLIAC, the electronic computer at the University of Sydney. It took approximately 40 minutes of machine time to produce 200 samples of size 20 and 70 minutes for samples of size 100. For each 200 samples of whatever size, about 20 minutes were required for the output of the results for individual samples. This could have been avoided since the second stage of the work could have proceeded without it, but obtaining the individual results was considered worth the sacrifice of machine time.

6. NUMERICAL RESULTS AND THEIR EVALUATION

The results of the computations appear in Tables VI and VII. Table VI summarizes the main characteristics of the empirical sampling distributions of the various estimators of the production function parameters in each of the models. It gives, for each estimation method and for each sample size, the mean and the second and third moment about the mean, calculated from the 200 samples obtained. In addition, we show the estimated bias, which is simply the difference between the mean of the 200 sample estimates and the true value of the parameter, and we also give the second moment about the true value. The latter is sometimes called the "mean squared error" and has been calculated as a sum of the second moment about the mean and the squared bias. Table VII shows the means of the estimates of the variances and co-variances of u, v_1, and v_2 based on the 200 samples. These are compared with the asymptotic values which are denoted as estimates from "sample size ∞." This table is of interest in connection with Hoch's estimation procedure in which the estimated error variances are used.

The main question which we wish to answer from our results is concerned with the ranking of the individual estimators according to their performance in the experiment.[20] It seems reasonable to prefer an estimator which has the smallest bias and, at the same time, the smallest variance. Such estimators will be given the highest rank. The difficulty arises when comparing estimators whose ranking according to the size of the bias differs from that according to the size of the variance. Some authors suggested the use of the mean squared error (i.e., the second moment about the true value) as an appropriate criterion in these cases.[21] However, such a criterion implies fixed valuations of unbiasedness and efficiency which may not apply in all circumstances. At the same time, unfavourable extremes in either direction

[20] Generalizations from the particular situations examined in the study involve a critical assessment of the realism and likelihood of these situations. This, in turn, depends upon the field of application and the technical and economic characteristics of the micro-units.

[21] See, e.g., C. F. Christ in [**1**, p. 840].

TABLE VI

POINT ESTIMATES OF $a_1 = 0.5$

		Sample Size	Mean	Bias	2nd Moment About Mean	2nd Moment About True Value	3rd Moment About Mean
Model A	O.L.S.	20	.50019	.00019	.01089	.01089	−.00011
		100	.50122	.00122	.00168	.00168	.00000
	Hoch's	20	.44101	−.05899	.54542	.54890	−2.36826
		100	.50757	.00757	.01711	.01716	.01124
	Klein's	20	.50047	.00047	.00003	.00003	−.00001
		100	.50024	.00024	.00001	.00001	.00000
	I.L.S.	20	.50452	.00452	.05802	.05804	−.01153
		100	.49749	−.00251	.00559	.00560	−.00007
Model B	O.L.S.	20	.49943	−.00057	.00446	.00446	.00004
		100	.50197	.00197	.00074	.00074	.00000
	Hoch's	20	.49583	−.00417	.00779	.00781	−.00001
		100	.50203	.00203	.00084	.00084	.00001
	Klein's	20	.50093	.00093	.00007	.00007	.00000
		100	.50021	.00021	.00002	.00002	.00000
	I.L.S.	20	.49664	−.00336	.00719	.00720	.00003
		100	.50186	.00186	.00084	.00084	−.00001
Model C	O.L.S.	20	.47711	−.02289	.07139	.07191	−.00414
		100	.48941	−.01059	.01296	.01307	.00071
	Hoch's-proper*	20	.48919	−.01081	.22922	.22934	−.06114
		100	.48756	−.01244	.02471	.02486	−.00012
	Hoch's-EUE*	20	.12467	−.37533	17.73167	17.87254	−248.39269
		100	.83289	.33289	4.35147	4.46229	31.06847
	Klein's	20	.49972	−.00028	.00003	.00003	−.00001
		100	.49992	−.00008	.00001	.00001	−.00001
	I.L.S.	20	.48949	−.01051	.23050	.23061	−.05858
		100	.48763	−.01237	.02499	.02514	−.00015
Model D	O.L.S.	20	.50367	.00367	.00449	.00450	.00014
		100	.49915	−.00085	.00071	.00071	.00000
	Hoch's-EUE*	20	.50386	.00386	.00497	.00498	.00015
		100	.49908	−.00092	.00079	.00079	.00000
	Hoch's-EPCE*	20	.50387	.00387	.00478	.00479	.00014
		100	.49940	−.00060	.00074	.00074	.00000
	Klein's	20	.49989	−.00011	.00002	.00002	.00000
		100	.50007	.00007	.00001	.00001	.00000
	I.L.S.	20	.50394	.00394	.00478	.00479	0.0015
		100	.49949	−.00051	.00073	.00073	0.0000

TABLE VI (cont.)
POINT ESTIMATES OF $a_1 = 0.5$

		Sample Size	Mean	Bias	2nd Moment About Mean	2nd Moment About True Value	3rd Moment About Mean
Model E	O.L.S.	20	.49647	−.00353	.00157	.00158	.00000
		100	.49892	−.00108	.00025	.00025	.00000
	Hoch's-EUE*	20	.49641	−.00359	.00163	.00164	.00000
		100	.49843	−.00157	.00026	.00026	.00001
	Hoch's-EPCE*	20	.49657	−.00343	.00158	.00159	.00001
		100	.49919	−.00081	.00026	.00026	.00000
	Klein's	20	.49994	−.00006	.00008	.00008	.00000
		100	.50022	.00022	.00002	.00002	.00000
	I.L.S.	20	.49658	−.00342	.00159	.00160	−.00001
		100	.49910	−.00090	.00026	.00026	.00000
Model F	O.L.S.	20	.49406	−.00594	.00767	.00771	.00287
		100	.49970	−.00030	.00177	.00177	.00002
	Hoch's	20	.51466	.01466	.42812	.42833	2.66812
		100	.49351	−.00649	.00611	.00615	−.00025
	Klein's	20	.53478	.03478	.00003	.00125	.00000
		100	.53348	.03348	.00001	.00113	.00000
	I.L.S.	20	.50042	.00042	.10085	.10085	.09960
		100	.49604	−.00396	.00516	.00518	.00001
Model G	O.L.S.	20	.50190	.00190	.00489	.00489	.00019
		100	.49894	−.00106	.00077	.00077	.00001
	Hoch's	20	.49688	−.00312	.00840	.00841	.00029
		100	.49788	−.00212	.00111	.00111	−.00001
	Klein's	20	.53427	.03427	.00007	.00124	−.00001
		100	.53309	.03309	.00001	.00110	.00000
	I.L.S.	20	.49553	−.00447	.00830	.00832	.00007
		100	.49774	−.00226	.00109	.00109	−.00001
Model H	O.L.S.	20	.74739	.24739	.00357	.06477	.00001
		100	.74937	.24937	.00067	.06285	−.00002
	Hoch's	20	.50230	.00230	.02268	.02268	−.01309
		100	.49127	−.00873	.00394	.00402	−.00022
	Klein's	20	.50024	.00024	.00003	.00003	.00000
		100	.49991	−.00009	.00001	.00001	.00000
	I.L.S.	20	.47084	−.02916	.03756	.03841	−.03744
		100	.48814	−.01186	.00398	.00412	−.00023

TABLE VI (cont.)
POINT ESTIMATES OF $a_1 = 0.5$

		Sample Size	Mean	Bias	2nd Moment About		3rd Moment About Mean
					Mean	True Value	
Model I	O.L.S.	20	.58272	.08272	.00292	.00976	.00006
		100	.58351	.08351	.00033	.00730	−.00001
	Hoch's	20	.49730	−.00270	.00394	.00395	−.00022
		100	.50076	.00076	.00050	.00050	.00000
	Klein's	20	.50185	.00185	.00007	.00007	−.00001
		100	.49977	−.00023	.00001	.00001	.00001
	I.L.S.	20	.48865	−.01135	.00547	.00560	−.00126
		100	.50031	.00031	.00050	.00050	.00001

* For explanation of the terms see the final paragraph of Section 3.1.

are likely to render a particular estimator useless regardless of its other desirable properties. Thus, an estimator which is unbiased but displays a very high degree of variability, or one which has a small variance but is heavily biased, is of a very limited use. The definition of "usefulness" and consequently the limits to the acceptable magnitude of the bias and of the variance will, of course, depend upon the particular function imposed on the estimator, and the answer could perhaps best be found with the help of decision theory.

On the criterion of smallest bias and smallest variance, and considering estimates of both parameters simultaneously, our results indicate that Klein's estimates are best[22] in models A, B, C, D, E, and H with respect to both sample sizes, and in model I with respect to samples of size 20. In models F, G (both sample sizes), and I (samples of size 100), Klein's estimates showed the smallest variance but not the smallest bias. These results are not surprising in view of the assumptions underlying the derivation of Klein's estimates and considering the construction of the models. In all the models in which Klein's estimates were best, with the exception of models H and I, the average firm was in the position of a profit maximizing equilibrium with respect to both inputs. In models H and I, one of the inputs was predetermined, but the average firm was only at a slight distance from the optimum position. In models F and G the average firm was not optimal by virtue of a restraint on input 2,[23] and Klein's estimates became biased.

The criterion of smallest bias and smallest variance does not lead to an

[22] Minor differences in the estimated bias in samples of size 20 have been disregarded.

[23] In models F and G the distances between the mean and the optimum (logarithmic) quantity of input 2 were 0.288 and 0.267 of the standard deviation of x_2 respectively.

TABLE VI (cont.)
POINT ESTIMATES OF $a_2 = 0.4$

		Sample Size	Mean	Bias	2nd Moment About Mean	2nd Moment About True Value	3rd Moment About Mean
Model A	O.L.S.	20	.46678	.06678	.01112	.01558	.00019
		100	.46588	.06588	.00172	.00606	.00000
	Hoch's	20	.45821	.05821	.61623	.61962	3.02213
		100	.38525	−.01475	.01194	.01216	−.00253
	Klein's	20	.40053	.00053	.00002	.00002	−.00001
		100	.40004	.00004	.00000	.00000	.00000
	I.L.S.	20	.36672	−.03328	.07091	.07202	−.01957
		100	.40003	.00003	.00569	.00569	−.00001
Model B	O.L.S.	20	.42917	.02917	.00448	.00533	−.00003
		100	.42624	.02624	.00077	.00146	−.00001
	Hoch's	20	.40141	.00141	.00750	.00750	−.00032
		100	.39767	−.00233	.00091	.00091	.00000
	Klein's	20	.39986	−.00014	.00003	.00003	−.00001
		100	.40011	.00011	.00001	.00001	.00000
	I.L.S.	20	.39989	−.00011	.00700	.00700	−.00017
		100	.39760	−.00240	.00088	.00089	.00000
Model C	O.L.S.	20	.47634	.07634	.07087	.07670	.00379
		100	.46326	.06326	.01299	.01699	−.00073
	Hoch's-proper*	20	.39732	−.00268	.23640	.23641	−.00708
		100	.41257	.01257	.02491	.02507	−.00006
	Hoch's-EUE*	20	.74172	.34172	16.93982	17.05659	113.26060
		100	−.04410	−.44410	4.20123	4.39845	−12.40962
	Klein's	20	.39992	−.00008	.00001	.00001	.00000
		100	.40002	.00002	.00000	.00000	.00000
	I.L.S.	20	.39688	−.00312	.23858	.23859	−.01375
		100	.41216	.01216	.02521	.02536	−.00003
Model D	O.L.S.	20	.44764	.04764	.00450	.00677	−.00015
		100	.45224	.05224	.00069	.00342	.00000
	Hoch's-EUE*	20	.44473	.04473	.00497	.00697	−.00018
		100	.44952	.04952	.00078	.00323	.00000
	Hoch's-EPCE*	20	.44585	.04585	.00478	.00688	−.00017
		100	.45063	.05063	.00073	.00329	.00000
	Klein's	20	.39976	−.00024	.00002	.00002	.00000
		100	.39998	−.00002	.00000	.00000	.00000
	I.L.S.	20	.44579	.04579	.00478	.00688	−.00017
		100	.45038	.05038	.00074	.00332	.00000

TABLE VI (cont.)
POINT ESTIMATES of $a_2 = 0.4$

		Sample Size	Mean	Bias	2nd Moment About Mean	2nd Moment About True Value	3rd Moment About Mean
Model E	O.L.S.	20	.43514	.03514	.00156	.00279	−.00001
		100	.43264	.03264	.00025	.00131	.00000
	Hoch's-EUE*	20	.43374	.03374	.00162	.00276	−.00002
		100	.43033	.03033	.00027	.00119	.00000
	Hoch's-EPCE*	20	.43418	.03418	.00158	.00275	−.00002
		100	.43158	.03158	.00026	.00126	.00000
	Klein's	20	.39973	−.00027	.00005	.00005	−.00001
		100	.39993	−.00007	.00001	.00001	.00000
	I.L.S.	20	.43417	.03417	.00158	.00275	−.00002
		100	.43164	.03164	.00027	.00127	.00000
Model F	O.L.S.	20	.47536	.07536	.01137	.01707	−.00044
		100	.46702	.06702	.00180	.00629	−.00004
	Hoch's	20	.39479	−.00521	.22874	.22877	−.68406
		100	.40508	.00508	.00581	.00584	−.00015
	Klein's	20	.36308	−.03692	.00001	.00137	.00000
		100	.36283	−.03717	.00000	.00138	.00000
	I.L.S.	20	.38215	−.01785	.12710	.12742	−.19739
		100	.40128	.00128	.00545	.00545	−.00004
Model G	O.L.S.	20	.42731	.02731	.00486	.00561	−.00017
		100	.42954	.02954	.00077	.00164	.00000
	Hoch's	20	.40075	.00075	.00887	.00887	−.00070
		100	.40149	.00149	.00121	.00121	.00000
	Klein's	20	.36300	−.03700	.00003	.00140	.00000
		100	.36292	−.03708	.00001	.00138	.00000
	I.L.S.	20	.40165	.00165	.00817	.00817	−.00040
		100	.40162	.00162	.00118	.00118	.00000
Model H	O.L.S.	20	.19856	−.20144	.00898	.04956	.00026
		100	.20104	−.19896	.00166	.04124	.00000
	Hoch's	20	.38557	−.01443	.03639	.03660	.00474
		100	.41023	.01023	.00779	.00790	.00032
	Klein's	20	.40650	.00650	.00007	.00011	.00000
		100	.40694	.00694	.00001	.00006	.00000
	I.L.S.	20	.43110	.03110	.05131	.05228	.04993
		100	.40947	.00947	.00459	.00468	.00012

TABLE VI (cont.)
POINT ESTIMATES OF $a_2 = 0.4$

		Sample Size	Mean	Bias	2nd Moment About		3rd Moment About Mean
					Mean	True Value	
Model I	O.L.S.	20	.34582	−.05418	.00838	.01131	.00013
		100	.33200	−.06800	.00177	.00639	−.00001
	Hoch's	20	.41582	.01582	.01166	.01191	.00033
		100	.39899	−.00101	.00171	.00171	−.00001
	Klein's	20	.40571	.00571	.00007	.00010	.00000
		100	.40716	.00716	.00002	.00007	.00000
	I.L.S.	20	.42681	.02681	.01173	.01245	.00251
		100	.39639	−.00361	.00150	.00151	.00000

* For explanation of the terms see the final paragraph of Section 3.1.

unambiguous ranking of the estimators in models F, G (both sample sizes), and I (samples of size 100). Ordinary least squares estimates have a smaller variance than Hoch's and indirect least squares estimates, but they are heavily biased in estimating a_2 in models F and G, and in estimating both parameters in model I. In model F both Hoch's and indirect least squares estimates exhibit a small bias, but I.L.S. estimates have a smaller variance, particularly in samples of size 20. In models G (both sample sizes) and I (samples of size 100) only minor differences in the sampling distributions of Hoch's and I.L.S. estimates can be observed.

The obtained sampling distributions of individual estimators have displayed special characteristics which are worth noting. These will be discussed separately with respect to each of the estimators.

The main characteristic of ordinary least squares estimates in all models where both inputs were variable (i.e., models A through G) was the pronounced upward bias of estimates of a_2 and the lack of any substantial bias of estimates of a_1. In cases where one of the inputs was predetermined (i.e., models H and I), O.L.S. estimates of both a_1 and a_2 were biased. The observed bias was always greater in models in which the variances of the disturbances were equal than in models where $E(u^2)$ was small relative to $E(v_1^2)$ and $E(v_2^2)$. The variance of O.L.S. estimates appeared to be always smaller or approximately equal to that of Hoch's and indirect least squares estimates. In general, the main results of the O.L.S. estimation procedure conformed to the expectations which could be formed on the basis of asymptotic values of Table V. The lack of bias in estimates of a_1 in models A through G is to be explained by the particularity of the case when the true value of the parameter is 0.5. It is to be noted that the asymptotic bias of O.L.S. estimates of

TABLE VII

POINT ESTIMATES OF ERROR VARIANCES

Model	Error Variance*	Sample Size	Mean of Sample Estimates
A	$E(v_1^2)$	20	.000383
		100	.000391
		∞	.000400
	$E(v_2^2)$	20	.000379
		100	.000394
		∞	.000400
	$E(u^2)$-proper	20	.000346
		100	.000447
		∞	.000400
	SSE/n	20	.000117
		100	.000131
		∞	.000133
B	$E(v_1^2)$	20	.000938
		100	.000984
		∞	.001000
	$E(v_2^2)$	20	.000933
		100	.000996
		∞	.001000
	$E(u^2)$-proper	20	.000192
		100	.000195
		∞	.000200
	SSE/n	20	.000126
		100	.000138
		∞	.000143
C	$E(v_1^2)$	20	.000374
		100	.000400
		∞	.000400
	$E(v_2^2)$	20	.000367
		100	.000398
		∞	.000400
	$E(u^2)$-proper	20	.000519
		100	.000396
		∞	.000400
	$E(u^2)$-EUE	20	.001277
		100	−.000294
		∞	.003600
	$E(v_1v_2)$-proper	20	.000290
		100	.000319
		∞	.000320
	SSE/n	20	.000160
		100	.000184
		∞	.000190

TABLE VII (cont.)
POINT ESTIMATES OF ERROR VARIANCES

Model	Error Variance*	Sample Size	Mean of Sample Estimates
D	$E(v_1^2)$	20	.000384
		100	.000398
		∞	.000400
	$E(v_2^2)$	20	.000385
		100	.000396
		∞	.000400
	$E(u^2)$-EUE	20	.000010
		100	.000011
		∞	.000011
	$E(u^2)$ EPCE	20	.000009
		100	.000010
		∞	.000012
	$E(v_1v_2)$-EPCE	20	.000306
		100	.000319
		∞	.000320
	SSE/n	20	.000009
		100	.000009
		∞	.000011
E	$E(v_1^2)$	20	.000954
		100	.000989
		∞	.001000
	$E(v_2^2)$	20	.000939
		100	.000975
		∞	.001000
	$E(u^2)$-EUE	20	.000009
		100	.000011
		∞	.000011
	$E(u^2)$-EPCE	20	.000009
		100	.000010
		∞	.000011
	$E(v_1v_2)$ -EPCE	20	.000753
		100	.000784
		∞	.000800
	SSE/n	20	.000009
		100	.000010
		∞	.000010
F	$E(v_1^2)$	20	.000377
		100	.000395
		∞	.000400
	$E(v_2^2)$	20	.000383
		100	.000397
		∞	.000400

TABLE VII (cont.)
POINT ESTIMATES OF ERROR VARIANCES

Model	Error Variance*	Sample Size	Mean of Sample Estimates
F (cont.)	$E(u^2)$-proper	20	.000601
		100	.000411
		∞	.000400
	SSE/n	20	.000112
		100	.000128
		∞	.000133
G	$E(v_1^2)$	20	.000913
		100	.000994
		∞	.001000
	$E(v_2^2)$	20	.000955
		100	.000993
		∞	.001000
	$E(u^2)$-proper	20	.000190
		100	.000200
		∞	.000200
	SSE/n	20	.000123
		100	.000140
		∞	.000143
H	$E(v_1^2)$	20	.000379
		100	.000401
		∞	.000400
	$E(u^2)$-proper	20	.000400
		100	.000414
		∞	.000400
	SSE/n	20	.000165
		100	.000198
		∞	.000200
I	$E(v_1^2)$	20	.000951
		100	.000993
		∞	.001000
	$E(u^2)$-proper	20	.000179
		100	.000195
		∞	.000200
	SSE/n	20	.000143
		100	.000161
		∞	.000167

* For explanation of the term "proper," EUE, and EPCE see the final paragraph of Section 3.1. SSE/n represents the mean of the residual sum of squares of the least squares regression.

a_2 would have been even greater had the true value been smaller than 0.4 as chosen by us.[24]

Hoch's estimates, applied to situations for which they were properly designed, exhibited a great degree of instability in samples of size 20 in models A and F. These models were specified by having variable inputs and uncorrelated disturbances with equal variances, and differ only with respect to the position of the average firm. The instability of Hoch's estimates was largely due to the high degree of variation of $\hat{\hat{S}}_{00}$, the estimate of $E(u^2)$. A particularly disturbing feature was the negative value of $\hat{\hat{S}}_{00}$ which occurred in 4.0 per cent of all samples of size 20 in model A, and in 4.5 per cent of all samples of size 20 in model F. Since a negative value of an estimate of a variance is clearly unacceptable, Hoch's estimation procedure breaks down in these cases.[25] Hoch's estimates improved noticeably when applied to samples of size 100 of models A and F, but their bias and variance were still larger than that of indirect least squares estimates. In the remaining models (i.e., B, C, G, H, and I) Hoch's estimates appeared to be more satisfactory; in models B, C, and G their sampling distributions were found to be quite similar to those of indirect least squares estimates. In general, Hoch's estimates tended to show a smaller bias and variance in larger samples and in models in which the variance of the "technical" disturbance was small compared to the variances of the "economic" disturbances. Thus it may be possible to use sample estimates of error variances as a rough indication of the degree of stability of Hoch's estimates.

When Hoch's estimates designed for models with uncorrelated disturbances are applied to model C where, in fact, the "economic" disturbances were positively correlated (i.e., estimates labelled EUE), it is found that the estimates contain a large bias and are extremely unstable. If the sample size is increased, the bias is still very large but the variance is somewhat reduced. This instability arises mostly from what at first appears to be an arithmetical accident but should be considered an integral hazard of the Hoch type of estimator. It may occur in all cases where this estimator is used, "properly" or "improperly." When the asymptotic values of both numerator and denominator in the various expressions for $\hat{\hat{S}}_{00}$ are small, the sample values of $\hat{\hat{S}}_{00}$ are extremely erratic.

In models D and E all disturbances are positively correlated, therefore estimates of the Hoch type cannot be derived unless incorrect assumptions about the interrelationship of the disturbances are made. When this is done,

[24] See Hoch [**2**, Table I, p. 574].

[25] Hoch pointed out that all asymptotic moments have to be greater than or equal to C_{00} but mentioned that this may not apply to moments of finite samples. However, $\hat{\hat{S}}_{00}$ was found to be negative even in samples in which this condition held.

the results indicate that it makes very little difference whether it is assumed that there is no correlation at all (estimates EUE) or that correlation exists only between the two "economic" disturbances (estimates EPCE). In either case, the estimates of a_2 have a large bias compared to estimates of a_1. The variances of both types of estimators (i.e., EUE and EPCE) are approximately the same, and they are larger in model D than in model E. It is also interesting to compare the performance of the EUE type estimators in models C and D since these models differ only by virtue of the fact that in C the "economic" disturbances are uncorrelated with the "technical" disturbance, whereas in D they are positively correlated. It was found that the estimators have a substantially smaller bias and variance in model D than in model C. Furthermore, if Hoch's estimation procedure designed for the situation of model C is applied to model D (i.e., estimators EPCE), the variance of the estimators is considerably smaller than that of the "proper" estimators in model C. Thus, we come to the interesting conclusion that a violation of the assumption of independence between the "technical" and the "economic" disturbances in fact reduces the variance of Hoch's estimates.[26]

The main characteristics of Klein's estimates were already discussed earlier in this section. Klein's estimates have by far the smallest variance, but they become biased when the average firm is not in an optimal position. The bias, when it appears, depends on the distance of the average from the optimum quantities of output and of inputs.

The performance of indirect least squares estimates can be conveniently described by comparison with Hoch's estimates. I.L.S. estimates performed better in models A and F, particularly in samples of size 20, since Hoch's estimates tended to be unstable. On the other hand, Hoch's estimates appeared to have a smaller bias and variance in samples of size 20 in models H and I. The latter seems to suggest that I.L.S. estimates are not well suited for small sample estimation in models with a predetermined input. In all other cases, the differences between the estimated sampling distributions of the two types of estimates were found to be small or negligible. In models D and E, in which I.L.S. estimates are inconsistent and for which proper Hoch's estimates do not exist, I.L.S. estimates tended to behave very much like Hoch's EUE and EPCE estimates.

In general, our results indicate that no single estimation procedure is satisfactory in all circumstances. Ordinary least squares estimates tend to have an upward bias; Klein's estimates, though highly efficient, are biased in the absence of effective profit maximization; and Hoch's, and to some

[26] This conclusion has to be confined to the case of positive correlation of the disturbances since the case of negative correlation has not been examined.

extent indirect least squares estimates, can be highly unstable in small samples. A choice of an estimation method has to depend on the specific field of application and on the knowledge of the technical and economic characteristics of the industry.

University of Sydney, and
University of New South Wales

REFERENCES

[1] CHRIST, CARL F.: "Simultaneous Equation Estimation: Any Verdict Yet?," *Econometrica*, Vol. 28, October, 1960, pp. 835–845.

[2] HOCH, IRVING: "Simultaneous Equation Bias in the Context of the Cobb-Douglas Production Function," *Econometrica*, Vol. 26, October, 1958, pp. 566–578.

[3] KLEIN, L. R.: *A Textbook of Econometrics*, New York: Row, Peterson and Company, 1953.

[4] KONIJN, H. S.: "Estimation of an Average Production Function from Surveys," *Economic Record*, Vol. 35, April, 1959, pp. 118–125.

[5] MARSCHAK, J., AND W. J. ANDREWS: "Random Simultaneous Equations and the Theory of Production," *Econometrica*, Vol. 12, July-October, 1944, pp. 143–205.

[6] NAGAR, A. L.: "A Monte Carlo Study of Alternative Simultaneous Equation Estimators," *Econometrica*, Vol. 28, July, 1960, pp. 573–590.

[7] WAGNER, H.: "A Monte Carlo Study of Simultaneous Linear Structural Equations," *Econometrica*, Vol. 26, January, 1958, pp. 117–133.