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ESTIMATION OF STANDARD ERRORS OF THE CHARACTERISTIC ROOTS OF A DYNAMIC ECONOMETRIC MODEL

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1. INTRODUCTION

CONSIDER A LINEAR dynamic system

$$(1)\qquad \sum_{\tau=0}^{m} A_\tau y(t-\tau) + Bx(t) = u(t) \qquad (t = 1, 2, \ldots, T);$$

where $y(t)$ represents the vector of endogenous variables, $x(t)$ the vector of exogenous variables, $u(t)$ the vector of stochastic disturbances, and t the tth period of observation. The matrices $A_\tau\,(\tau = 0, 1, \ldots, m)$ of the structural coefficients are square matrices of order G. It is assumed that the conditions justifying the theorems in [**3**, Ch. 10] are satisfied, and that there are no nonlinear restrictions on the elements of A_τ. The stability of the system is determined by reference to the dominant root of the polynomial equation

$$(2)\qquad \det\left(\sum_{\tau=0}^{m} A_\tau \lambda^{m-\tau}\right) = 0.$$

This equation is the characteristic equation of system (1). The system is stable if the absolute value of the dominant root is less than one. We assume that there are no multiple roots, and that the dominant root is different from zero. If the structural coefficients A_τ in (2) are replaced by their statistical estimates, the roots of the characteristic equation are subject to sampling errors. This was noted by Theil and Boot [**4**] who derived the asymptotic standard error of the dominant root of Klein's Model I, and by Neudecker and van de Panne [**1**] who developed a procedure which could be applied to larger systems as well. The authors of both papers rely upon the availability of an estimated covariance matrix of the derived reduced form coefficients. This matrix is generally not available as a part of simultaneous equation estimation programs but has to be obtained in the form of a first-order approximation from the estimated covariance matrix of the structural coefficients. The purpose of this note is to derive the asymptotic standard errors of the characteristic roots in terms of the asymptotic covariance matrix of the structural coefficients.[2] An estimate of this matrix is available for any system method of estimation such as 3SLS or FIML. We propose a computational procedure which is relatively simple and which can be used for quite large linear models without much difficulty.[3]

2. DETERMINATION OF THE ASYMPTOTIC VARIANCE OF AN ESTIMATED CHARACTERISTIC ROOT

Consider the following implicit function,

$$(3)\qquad \det\left(\sum_{\tau=0}^{m} C_\tau \lambda^{m-\tau}\right) = 0,$$

in the neighborhood of $\lambda = \lambda_1$ and $C_\tau = A_\tau$, where λ_1 is the relevant characteristic root and the elements of C (but not λ) are restricted to real numbers. Note that we can write

$$(4)\qquad \det\left(\sum_{\tau=0}^{m} C_\tau \lambda^{m-\tau}\right) = \sum_{\mu=0}^{M} \alpha_\mu \lambda^\mu,$$

[1] This paper was written while J. Kmenta was on leave from Michigan State University. Both authors wish to thank Manfred Deistler for his helpful comments.

[2] It should be noted that this result is not obtainable by directly generalizing the results of Theil and Boot or of Neudecker and van de Panne. See also Wymer [**5**].

[3] A program written for IBM 360 is available on request.

where α_μ are real and $M = G(m-1)$. Let us set

$$\lambda = \rho(\cos\phi + i\sin\phi). \tag{5}$$

Note that $|\lambda| = \rho$. Our aim is to determine the asymptotic variance of an estimate of ρ, say $\hat{\rho}$, and of an estimate of ϕ, say $\hat{\phi}$. The Taylor expansion of ρ as a function of the parameters in C_τ at the point $\lambda = \lambda_1$ and $C_\tau = A_\tau$ gives

$$\rho - \rho_1 = \sum_{k=1}^{K} \left.\frac{\partial\rho}{\partial c_k}\right|_{c_k=a_k} (c_k - a_k) + \text{terms of higher order in } c_k, \tag{6}$$

where c_k and a_k represent the respective elements of C_τ and A_τ, and ρ_1 corresponds to λ_1. Further, let $\hat{a}$ be a vector of all 3SLS or FIML estimates of the structural coefficients. Because of (6) we can apply the convergence theorem given in [**2**, p. 319] from which it directly follows that the asymptotic distribution of $\sqrt{T}(\hat{\rho} - \rho_1)$ is normal with

$$\operatorname{var}\sqrt{T}(\hat{\rho} - \rho_1) = \left(\left.\frac{\partial\rho}{\partial c}\right|_{c=a}\right)' M_{\hat{a}\hat{a}} \left(\left.\frac{\partial\rho}{\partial c}\right|_{c=a}\right), \tag{7}$$

where $M_{\hat{a}\hat{a}}$ is the asymptotic covariance matrix of the estimated structural coefficients $\hat{a}$, and c and a are column vectors of the elements of C_τ and A_τ ($\tau = 0, 1, \ldots, m$). A consistent estimator of (7) is obtained by replacing a by $\hat{a}$ and $M_{\hat{a}\hat{a}}$ by $\hat{M}_{\hat{a}\hat{a}}$.[4] As for the argument ϕ, the asymptotic distribution of $\sqrt{T}(\hat{\phi} - \phi_1)$ is also normal with

$$\operatorname{var}\sqrt{T}(\phi - \phi_1) = \left(\left.\frac{\partial\phi}{\partial c}\right|_{c=a}\right)' M_{\hat{a}\hat{a}} \left(\left.\frac{\partial\phi}{\partial c}\right|_{c=a}\right). \tag{8}$$

If the vector a is subject to the following restriction

$$a = Rb + e, \tag{9}$$

where the matrix R and the vector e are fixed and given, and b is a vector of the free parameters, then (7) will be replaced by

$$\operatorname{var}\sqrt{T}(\hat{\rho} - \rho_1) = \left(\left.\frac{\partial\rho}{\partial c}\right|_{c=a}\right)' R M_{\hat{b}\hat{b}} R' \left(\left.\frac{\partial\rho}{\partial c}\right|_{c=a}\right). \tag{10}$$

Our problem now is to determine $\partial\rho/\partial c$ and $\partial\phi/\partial c$. Substituting for λ from (5) into (4) and denoting the matrix $\Sigma_{\tau=0}^{m} \lambda^{m-\tau} C_\tau$ by $M(C, \lambda)$, we obtain

$$\det M(C, \lambda) = \sum_{\mu=0}^{M} \alpha_\mu \rho^\mu (\cos\mu\phi + i\sin\mu\phi). \tag{11}$$

Further, we define

$$u = \sum_{\mu=0}^{M} \alpha_\mu \rho^\mu \cos\mu\phi, \qquad v = \sum_{\mu=0}^{M} \alpha_\mu \rho^\mu \sin\mu\phi. \tag{12}$$

By reference to (3), we note that in the neighborhood of $\lambda = \lambda_1$ and $C_\tau = A_\tau$ we have to solve the following system:

$$u = 0, \qquad v = 0. \tag{13}$$

Since $\lambda = \lambda_1$ is not a multiple root by assumption, it follows that

$$\left.\frac{\partial}{\partial\lambda}\det M(C, \lambda)\right|_{\lambda=\lambda_1} \neq 0, \tag{14}$$

[4] It may be worth emphasizing that this estimator pertains to the limiting distribution and not to the actual distribution of $\hat{\rho}$ (whose variance may not even exist).

or, because of (11),

$$\sum_{\mu=0}^{M} \alpha_\mu \mu \lambda_1^{\mu-1} \neq 0, \tag{15}$$

which, for $\lambda_1 \neq 0$, is equivalent to

$$\sum_{\mu=0}^{M} \alpha_\mu \mu \lambda_1^{\mu} \neq 0. \tag{16}$$

Let us now set $\lambda_1 = \rho_1(\cos\phi_1 + i\sin\phi_1)$ and rewrite (16) as

$$\sum_{\mu=0}^{M} \alpha_\mu \mu \rho_1^{\mu}(\cos\mu\phi_1 + i\sin\mu\phi_1) \neq 0. \tag{17}$$

Now consider the equations in (12). By taking the first derivatives we obtain

$$\begin{aligned} \frac{\partial u}{\partial \rho} &= \sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu-1}\cos\mu\phi, & \frac{\partial u}{\partial \phi} &= -\sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu}\sin\mu\phi, \\ \frac{\partial v}{\partial \rho} &= \sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu-1}\sin\mu\phi, & \frac{\partial v}{\partial \phi} &= \sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu}\cos\mu\phi, \end{aligned} \tag{18}$$

from which it follows that

$$\det\begin{pmatrix} \dfrac{\partial u}{\partial \rho} & \dfrac{\partial u}{\partial \phi} \\ \dfrac{\partial v}{\partial \rho} & \dfrac{\partial v}{\partial \phi} \end{pmatrix} = \frac{1}{\rho}\left(\sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu}\cos\mu\phi\right)^2 + \frac{1}{\rho}\left(\sum_{\mu=0}^{M} \alpha_\mu \mu \rho^{\mu}\sin\mu\phi\right)^2. \tag{19}$$

The right-hand side of (19) for $\lambda = \lambda_1$ is different from zero because of (17). Therefore the system (13) has a unique solution for ρ and ϕ in the neighborhood of $C_\tau = A_\tau$. By the implicit function theorem we have

$$\begin{pmatrix} \dfrac{\partial \rho}{\partial c_k} \\ \dfrac{\partial \phi}{\partial c_k} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial \rho} & \dfrac{\partial u}{\partial \phi} \\ \dfrac{\partial v}{\partial \rho} & \dfrac{\partial v}{\partial \phi} \end{pmatrix}^{-1} \begin{pmatrix} \dfrac{\partial u}{\partial c_k} \\ \dfrac{\partial v}{\partial c_k} \end{pmatrix}, \tag{20}$$

where, as before, c_k is an element of C_τ. To obtain $\partial\rho/\partial c_k$ we need to solve (20). Let $c_{\tau hj}$ be an element of C_τ which appears in the hth row and the jth column. Then from (11) we have

$$\begin{aligned} \frac{\partial u}{\partial c_{\tau hj}} + i\frac{\partial v}{\partial c_{\tau hj}} &= \frac{\partial}{\partial c_{\tau hj}} \det M(C,\lambda), \\ &= \lambda^{m-\tau} D_{hj}[M(C,\lambda)], \end{aligned} \tag{21}$$

where $D_{hj}[M(C,\lambda)]$ is the cofactor of the element in the hth row and the jth column of $M(C,\lambda)$. Correspondingly we obtain

$$\begin{aligned} \frac{\partial u}{\partial \rho} + i\frac{\partial v}{\partial \rho} &= \frac{\partial}{\partial \rho}\det M(C,\lambda), \\ &= \sum_h \sum_j D_{hj}[M(C,\lambda)] \sum_{\tau=0}^{m} (m-\tau)\lambda^{m-\tau-1} c_{\tau hj}(\cos\phi + i\sin\phi), \end{aligned} \tag{22}$$

and

$$(23)\qquad \begin{aligned}\frac{\partial u}{\partial \phi} + i\frac{\partial v}{\partial \phi} &= \frac{\partial}{\partial \phi}\det M(C,\lambda),\\ &= \sum_h\sum_j D_{hj}[M(C,\lambda)] \sum_{\tau=0}^{m} (m-\tau)\lambda^{m-\tau-1}c_{\tau hj}i\lambda.\end{aligned}$$

Define

$$(24)\qquad \sum_h\sum_j D_{hj}[M(C,\lambda)] \sum_{\tau=0}^{m} (m-\tau)\lambda^{m-\tau-1}c_{\tau hj} = D_R + iD_I,$$

where D_R and D_I are real numbers. Then (22) and (23) can be written as

$$(25)\qquad \begin{aligned}\frac{\partial u}{\partial \rho} + i\frac{\partial v}{\partial \rho} &= (D_R + iD_I)(\cos\phi + i\sin\phi),\\ &= D_R\cos\phi - D_I\sin\phi + i(D_R\sin\phi + D_I\cos\phi),\end{aligned}$$

$$(26)\qquad \begin{aligned}\frac{\partial u}{\partial \phi} + i\frac{\partial v}{\partial \phi} &= (D_R + iD_I)i\rho(\cos\phi + i\sin\phi),\\ &= -\rho D_R\sin\phi - \rho D_I\cos\phi + i\rho(D_R\cos\phi - D_I\sin\phi).\end{aligned}$$

This leads to

$$(27)\qquad \det\begin{pmatrix}\dfrac{\partial u}{\partial \rho} & \dfrac{\partial u}{\partial \phi}\\[2mm] \dfrac{\partial v}{\partial \rho} & \dfrac{\partial v}{\partial \phi}\end{pmatrix} = \rho(D_R^2 + D_I^2),$$

and therefore

$$(28)\qquad \begin{aligned}\begin{pmatrix}\dfrac{\partial u}{\partial \rho} & \dfrac{\partial u}{\partial \phi}\\[2mm] \dfrac{\partial v}{\partial \rho} & \dfrac{\partial v}{\partial \phi}\end{pmatrix}^{-1} &= \frac{1}{\rho(D_R^2 + D_I^2)}\begin{pmatrix}\dfrac{\partial v}{\partial \phi} & -\dfrac{\partial u}{\partial \phi}\\[2mm] -\dfrac{\partial v}{\partial \rho} & \dfrac{\partial u}{\partial \rho}\end{pmatrix},\\ &= \frac{1}{\rho(D_R^2 + D_I^2)}\begin{pmatrix}\rho D_R\cos\phi - \rho D_I\sin\phi & \rho D_R\sin\phi + \rho D_I\cos\phi\\ -D_R\sin\phi - D_I\cos\phi & D_R\cos\phi - D_I\sin\phi\end{pmatrix}.\end{aligned}$$

Further we define

$$(29)\qquad \lambda^{m-\tau}D_{hj}[M(C,\lambda)] = D_{R,\tau hj} + iD_{I,\tau hj}.$$

Then, using (20), (21), and (28) we obtain

$$(30)\qquad \begin{aligned}\frac{\partial \rho}{\partial c_{\tau hj}} &= \frac{1}{D_R^2 + D_I^2}\{(D_R\cos\phi - D_I\sin\phi)D_{R,\tau hj} + (D_R\sin\phi + D_I\cos\phi)D_{I,\tau hj}\},\\ &= \frac{1}{\rho|D_R + iD_I|^2}\ \text{real part of}\ [\lambda(D_R + iD_I)(D_{R,\tau hj} - iD_{I,\tau hj})].\end{aligned}$$

Let us now replace the triple subscript (τhj) by a single subscript k. Note that $h, j = 1, 2, \ldots, G$ and $\tau = 0, 1, \ldots, m$. Further, we note that $1 \leqslant k \leqslant K$, where K is the number of parameters in A_τ $(\tau = 0, 1, \ldots, m)$. Then we can write equation (30) as

$$(31)\qquad \frac{\partial \rho}{\partial c_k} = \frac{1}{D_R^2 + D_I^2}[D_R\cos\phi - D_I\sin\phi)D_{Rk} + (D_R\sin\phi + D_I\cos\phi)D_{Ik}].$$

Similarly, from (20), (21), and (28) we get

$$\text{(32)} \qquad \frac{\partial \phi}{\partial c_k} = \frac{1}{\rho(D_R^2 + D_I^2)}[-(D_R \sin \phi + D_I \cos \phi)D_{Rk} + (D_R \cos \phi - D_I \sin \phi)D_{Ik}].$$

These are the desired results.[5]

The numerical difficulty lies in the calculation of the matrices D_{Rk} and D_{Ik}. To this end we need, according to (29), to determine the matrix $D_{hj}[M(C, \lambda)]$. However, a simple eigenvalue of $M(C, \lambda)$ is, by definition, equal to zero. For such matrices the following relation holds. Let M be a matrix with simple eigenvalue equal to zero. Further, let the corresponding eigenvector be t_1 (i.e., $Mt_1 = 0$), and the eigenvector corresponding to M' be s_1 (i.e., $M's_1 = 0$). Then the adjoint $D[M]$, determined up to the normalization factor μ, is given by

$$\text{(33)} \qquad D[M] = \mu t_1 s_1'.$$

The result in (33) is obtained from the presentation of M in the Jordanian form

$$\text{(34)} \qquad M = T \Lambda T^{-1},$$

where $\Lambda = (\lambda_{ij})$ is a lower triangular matrix with $\lambda_{11} = 0$. We then define $\Lambda(\varepsilon)$ as a matrix which is the same as Λ except for the first element which is given in $\lambda_{11}(\varepsilon) = \varepsilon$, $\varepsilon > 0$. Correspondingly, we define

$$\text{(35)} \qquad M_\varepsilon = T\Lambda(\varepsilon)T^{-1}$$

and note that M_ε is not singular. Then we use the formula

$$\text{(36)} \qquad D[M_\varepsilon] = (\det M_\varepsilon)M_\varepsilon^{-1}$$

and let ε approach zero. This leads to (33). By substitution into (31) the normalization factor μ will cancel out.

In the case where λ is real, equation (31) will specialize to

$$\text{(37)} \qquad \frac{\partial \rho}{\partial c_k} = \frac{D_{Rk}}{D_R},$$

where, according to (24), (29), and (33),

$$D_{Rk} = \lambda_1^{m-\tau}(t_1 s_1')_{hj} \qquad \text{and} \qquad D_R = \sum_h \sum_j (t_1 s_1')_{hj}.$$

3. AN OUTLINE OF A FLOW CHART

To illuminate the computational procedure involved in estimating the standard errors of the characteristic roots, we present an outline of the flow chart that was used as the basis for the program which we have developed.

(i) Read in the matrices $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_m$ and the covariance matrix

$$\hat{M}_{\hat{a}\hat{a}} = \{\operatorname{cov}(\hat{a}_{\tau hj}, \hat{a}_{\tau' h' j'})\}.$$

(ii) Select the relevant root, say λ_1, of the polynomial equation

$$\det\left(\sum_{\tau=0}^{m} \lambda^{m-\tau} A_\tau\right) = 0.$$

[5] Some caution should be exercised in using the estimated standard error of $\hat{\rho}$ for testing the hypothesis of stability since a good deal of the distribution theory for simultaneous equation estimators depends on the assumption of stability in the first place. Strictly speaking, the hypothesis of stability can be rejected when $\hat{\rho}$ is significantly greater than one, but it is not confirmed when $\hat{\rho}$ is smaller than one. (We are grateful to Franklin Fisher for drawing our attention to this point.)

(iii) Construct the matrix

$$\hat{M} = \sum_{\tau=0}^{m} \lambda_1^{m-\tau}\hat{A}_\tau.$$

This matrix will be, in general, complex.

(iv) Calculate the cofactors of all elements of $\hat{M}$ and set up the adjoint of $\hat{M}$, denoted by D. Note that the procedure here is as follows. We wish to solve the linear homogeneous system $\hat{M}t_1 = 0$; t_1 is a $G \times 1$ vector defined in connection with equation (33). Without a loss of generality, the last component of t_1 is normalized to 1. Let $\hat{M}_1$ be a nonsingular matrix obtained from $\hat{M}$ by deleting the last row and the last column. Further, let $\tilde{t}_1$ be a $(G-1) \times 1$ vector obtained from t_1 by deleting the last component. Then instead of

$$\hat{M}t_1 = 0$$

we have the system

$$\hat{M}_1\tilde{t}_1 = -\hat{m}_G,$$

where $\hat{m}_G$ represents the last column of $\hat{M}$ without the last component. In the same way we solve the system

$$\hat{M}'s_1 = 0,$$

where s_1 is a vector defined in connection with equation (33). Finally, we obtain the matrix D as the product

$$D = t_1 s_1'.$$

(v) Calculate the sum

$$\sum_h \sum_j D_{hj}(\hat{M}) \sum_{\tau=0}^{m} (m-\tau)\lambda^{m-\tau-1}\hat{a}_{\tau hj} = D_R + iD_I,$$

where $D_{hj}(\hat{M})$ stands for the element in the jth row and hth column of D.

(vi) Calculate, for all $\tau = 0, 1, \ldots, m$, and all $h, j = 1, 2, \ldots, G$,

$$\lambda_1^{m-\tau}D_{hj}(\hat{M})$$

and write it as

$$D_{Rk} + iD_{Ik},$$

where k represents the triple subscript (τhj).

(vii) For all k calculate

$$\left.\frac{\partial\rho_1}{\partial c_k}\right|_{c_k=a_k} = \frac{1}{D_R^2 + D_I^2}[(D_R\cos\phi_1 - D_I\sin\phi_1)D_{Rk} + (D_R\sin\phi_1 + D_I\cos\phi_1)D_{Ik}].$$

(viii) Finally, obtain the estimated variance

$$\text{est. var}(\hat{\rho}_1) = \sum_k \sum_k \left(\left.\frac{\partial\rho_1}{\partial c_k}\right|_{c_k=\hat{a}_k}\right)\left(\left.\frac{\partial\rho_1}{\partial c_{k'}}\right|_{c_{k'}=\hat{a}_{k'}}\right)\text{cov}(\hat{a}_k, \hat{a}_{k'}).$$

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