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VALUATION OF CONTINUOUSLY MONITORED DOUBLE BARRIER OPTIONS AND RELATED SECURITIES

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In this paper, we apply Carr's randomization approximation and the operator form of the Wiener-Hopf method to double barrier options in continuous time. Each step in the resulting backward induction algorithm is solved using a simple iterative procedure that reduces the problem of pricing options with two barriers to pricing a sequence of certain perpetual contingent claims with first-touch single barrier features. This procedure admits a clear financial interpretation that can be formulated in the language of embedded options. Our approach results in a fast and accurate pricing method that can be used in a rather wide class of Lévy-driven models including Variance Gamma processes, Normal Inverse Gaussian processes, KoBoL processes, CGMY model, and Kuznetsov's β-class. Our method can be applied to double barrier options with arbitrary bounded terminal payoff functions, which, in particular, allows us to price knock-out double barrier put/call options as well as double-no-touch options.

KEY WORDS: option pricing, double barrier options, double-no-touch options, Lévy processes, Variance Gamma processes, Normal Inverse Gaussian processes, Kuznetsov's β-processes, KoBoL processes, CGMY model, fast Fourier transform, Carr's randomization, Wiener-Hopf factorization, Laplace transform.

1. INTRODUCTION

The problem of finding efficient numerical algorithms for pricing options with one or two barriers has been widely studied in the literature on mathematical finance. In the classical Black–Scholes framework, the problem of pricing continuously monitored double barrier options has been studied by many authors. Among the works devoted to this topic, the earliest of which dates back to 1992, let us mention Kunimoto and Ikeda (1992), Hui (1996), Geman and Yor (1996), Sidenius (1998), Douady (1999), Li (1999), Pelsser (2000), Hui, Lo, and Yuen (2000), Schröder (2000), Geman (2001), Luo (2001), and Kolkiewicz (2002) (the list is by no means complete). For the study of the pricing problem for continuously monitored double barrier options in the double-exponential jump-diffusion model and the relevant background, we refer the reader to Kou (2002), Kou and Wang (2003, 2004), Lipton (2002a,b), Sepp (2004), and Boyarchenko (2006). The

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same methodology can be extended to hyperexponential jump-diffusion models (HEJD) (cf. Boyarchenko 2006; Carr and Crosby 2008). From a broad perspective, the works listed above share one general feature. Using certain probabilistic arguments (such as fluctuation theory for Lévy processes), the authors derive a formula for the Laplace transform of the no-arbitrage price of the option with respect to the time to maturity. It is typically expressed as the sum of an infinite series, with the individual terms being given by explicit analytical formulas. In the Black–Scholes setting, it is furthermore possible to perform explicit Laplace inversion for the individual terms of the series. In the HEJD model, this is no longer the case, but the series itself can be summed explicitly (Boyarchenko 2006), so that the calculation of the option value essentially reduces to a single numerical Laplace inversion. This allows one to develop rather fast pricing algorithms for double barrier options in the Black–Scholes model and in double-exponential jump-diffusion model, a feature which makes these models attractive from the practical viewpoint.

However, empirical evidence shows that other Lévy processes, such as Variance Gamma (VG) processes, Normal Inverse Gaussian (NIG) processes, KoBoL processes, CGMY model (a subclass of KoBoL), often provide a much better fit to the observed stock prices. For stocks of this type, the Black–Scholes model and double-exponential jump-diffusion model may yield inaccurate prices of single and double barrier options when the spot price of the underlying is close to (one of) the barrier(s). The reason is that the value function of the option in the Black–Scholes and HEJD models remains continuously differentiable with respect to the spot price up to and including the barrier(s), while the same property fails for VG, NIG, KoBoL processes, and CGMY model—see Kudryavtsev and Levendorskiĭ (2006), Boyarchenko and Levendorskiĭ (2009a) for more details in the single barrier setting.[1] The accuracy can be improved, to some extent, by considering a natural generalization of the double-exponential jump-diffusion model constructed and studied in Levendorskiĭ (2004) in the context of American options and applied later under the name HEJD models in Jeannin and Pistorius (2007) to price barrier options. The computational advantage of the double-exponential jump-diffusion model and HEJD model stems from the fact that the Wiener-Hopf factorization is almost trivial for these processes (see also Asmussen, Avram, and Pistorius 2004 for the probabilistic background for similar processes, and Lipton 2002a,b, where fluctuation identities and numerical Laplace inversion are used to derive prices of several types of path-dependent options, including single barrier ones). By increasing the number of different jumps that are allowed in the positive and negative directions, one can approximate the jump densities of VG, NIG, and KoBoL processes with the jump densities of HEJDs. The resulting pricing algorithms and numerical examples in the context of options with single barriers can be found in Jeannin and Pistorius (2007). As we noted in Boyarchenko and Levendorskiĭ (2009a), this approach retains the disadvantage that HEJDs have finite jump activity, whereas VG, NIG, and KoBoL processes have infinite jump activity. More importantly, we explained in Boyarchenko and Levendorskiĭ (2009a) that as the number of jumps increases, the computational cost of evaluating the Laplace transform of the value function of the option grows very rapidly, which often necessitates a rather undesirable trade-off between the accuracy and the speed of the calculations.

From the computational viewpoint, the present work continues the tradition of a series of articles including Carr and Madan (1999), Boyarchenko and Levendorskiĭ (2002), Kudryavtsev and Levendorskiĭ (2009), Feng and Linetsky (2008), Boyarchenko and Levendorskiĭ (2009a,b) that demonstrate the high efficiency of numerical methods

[1] The conclusion applies to processes of the recent β-class (Kuznetsov 2010) in the infinite activity case.

based on Fourier transformations with respect to the log-spot price of the underlying asset. We would especially like to single out the important work of Feng and Linetsky (2008), which develops a very fast and accurate method of pricing *discretely monitored* single and double barrier options under a wide class of Lévy-driven models (including NIG and KoBoL processes). Our work can be viewed as the counterpart of Feng and Linetsky (2008) in the setting of continuously monitored options.

One of the key features of our approach, which contributes to its intrinsic accuracy, is that our method does *not* involve replacing the underlying Lévy process of the model with an approximation thereof. The class of Lévy processes to which our method can be applied (see Section 2.6 for a list of examples) includes diffusion processes, HEJDs, VG, NIG, and KoBoL processes (with the CGMY model as a special case), and the class of β-processes (Kuznetsov 2010). At the same time, our algorithm is easy to implement in practice (see Section 4) once an analytic expression for the characteristic exponent is known. The high computational speed is achieved by combining Carr's randomization procedure with the efficient numerical techniques developed in Boyarchenko and Levendorskiĭ (2009a,b).

The rest of this paper is organized as follows. Our approach to the valuation of perpetual double barrier options in Lévy-driven models is explained in Section 2. The key idea is presented in Section 2.2, where we show how the pricing problem for these options can be reduced to a sequence of pricing problems for perpetual first-touch single barrier contingent claims, by means of constructing certain auxiliary embedded options. The remainder of Section 2 is devoted to an explanation of how this sequence of problems can be solved in general, as well as to a description of a wide class of Lévy processes to which our method can be applied, along with a list of concrete examples (Section 2.6). The detailed exposition of the general method for pricing single barrier options using the operator form of the Wiener-Hopf factorization, and improved versions of FFT-technique necessary for an efficient realization of pricing formulas, can be found in Boyarchenko and Levendorskiĭ (2009a,b). In Section 3, we study finite-lived knock-out double barrier options in Lévy-driven models. We use Carr's randomization approximation to compute the no-arbitrage price of such an option, and we reduce each step of the resulting backward induction algorithm to a sequence of calculations involving the expected present value (EPV) operators of the underlying process, which lends our method to an efficient numerical realization. We also derive a formula for the Laplace transform with respect to the maturity date of the value function of a knock-out double barrier option under an arbitrary regular Lévy process of exponential type, which (to the best of our knowledge) is a new result. The shape of our formula is similar to the shapes of the known formulas in the special case of HEJD processes. A detailed algorithm for pricing continuously monitored knock-out double barrier options is presented in Section 4. Numerical examples are presented and discussed in Section 5.

2. PERPETUAL PAYOFFS STREAMS WITH BARRIERS

The seminal works of Carr and Faguet (1996) and Carr (1998) introduced a very useful approximation procedure into mathematical finance, which is nowadays known as *Canadization* or *Carr's randomization* (we prefer the latter term). In a number of different situations, it allows one to develop an efficient algorithm for numerically solving a pricing problem for a finite-lived option by replacing it with a sequence of perpetual option pricing problems. In preparation for using this procedure, we will study pricing problems for perpetual options with single and double barrier features.

2.1. Contingent Claims with Barriers

In this section we consider certain types of perpetual contingent claims on a stock whose price process, $\{S_t\}_{t\geq 0}$, has the form $S_t = S_0 e^{X_t}$, where $S_0 > 0$ and $X = \{X_t\}_{t\geq 0}$ is a one-dimensional Lévy process. (Some examples of Lévy processes used in financial modeling appear in Section 2.6.) We also assume that it is possible to borrow and lend money at a fixed rate of return $q > 0$.

(1) Let $H > 0$ be fixed, let $h = \ln H$, and let $f(x)$ be a bounded measurable[2] function defined on $(-\infty, h)$. These data determine an *up-and-out perpetual stream of payoffs*, whose instantaneous payoff at time $t \geq 0$ (while the stream is active) is equal to $f(\ln S_t)$, and which is abandoned as soon as S_t reaches or exceeds H.
(2) Similarly, if H and h are as above, then a bounded measurable function $f(x)$ defined on $(h, +\infty)$ determines a *down-and-out perpetual stream of payoffs*.
(3) Let H and h be as above, and let $G(x)$ be a bounded measurable function defined on $[h, +\infty)$ (respectively, $(-\infty, h]$). Consider a contingent claim that pays its owner $G(\ln S_t)$ at the first moment when S_t reaches or exceeds H (respectively, reaches or falls below H). It is called a *perpetual up-and-in* (respectively, *down-and-in*) *first-touch contingent claim*.
(4) Finally, fix $0 < H_- < H_+$ and put $h_\pm = \ln H_\pm$. Let $g(x)$ be a bounded measurable function defined on (h_-, h_+), let $G_+(x)$ be a bounded measurable function defined on $[h_+, +\infty)$, and let $G_-(x)$ be a bounded measurable function defined on $(-\infty, h_-]$. Then $g(x)$ determines a *perpetual knock-out double barrier stream of payoffs*, whose instantaneous payoff at time $t \geq 0$ (while the stream is active) is equal to $g(\ln S_t)$, and which is abandoned as soon as S_t leaves the open interval (H_-, H_+). The pair of functions $G_\pm$ determines a *perpetual first-touch double barrier contingent claim*, which pays its owner $G_\pm(\ln S_t)$ at the first moment when S_t leaves the open interval (H_-, H_+), where the subscript "+" or "−" is chosen according to whether $S_t \geq H_+$ or $S_t \leq H_-$.

A pricing method for perpetual streams of types (1)–(2) is recalled in Section 2.3. A pricing method for perpetual contingent claims of type (3) is recalled in Section 2.7. One of the fundamental contributions of the present work is a pricing method for perpetual streams and contingent claims of type (4), which we now develop. Our approach is similar to, and was motivated by, the approach developed by Boyarchenko (2006) in the special case of HEJD models.

2.2. Iterative Procedure for Streams with Double Barriers

In this section, we explain how to value perpetual knock-out double barrier streams of payoffs in terms of perpetual first-touch *single barrier* contingent claims. The arguments we give have a clear financial interpretation, and are valid for an arbitrary process that has the *strong Markov property* (see, e.g., Rogers and Williams 1994, p. 247 or Sato 1999, p. 278).

2.2.1. *Definition of the Value Function.* We remain in the setup of Section 2.1. In particular, we are given a one-dimensional Lévy process $X = \{X_t\}_{t\geq 0}$ and a real number

[2] Throughout this paper, for a function defined on $\mathbb{R}$, "measurable" will mean "Borel measurable."

$q > 0$, representing the killing rate in our model. We fix $0 < H_- < H_+$, put $h_\pm = \ln H_\pm$, and let $g(x)$ be a bounded measurable function on (h_-, h_+). Let $V_{k.o.}(x; g)$ denote the value function of the associated perpetual knock-out double barrier stream of payoffs. In other words, $V_{k.o.}(x; g)$ is the EPV of this stream, assuming that the initial spot price of the underlying equals $S_0 = e^x$. By definition,

$$V_{k.o.}(x; g) = \mathbb{E}\left[\int_0^{\tau_{h_- - x, h_+ - x}} e^{-qt} g(x + X_t)\, dt\right], \tag{2.1}$$

where τ_{h_-, h_+} is the first entrance time of the process X into the set $(-\infty, h_-] \cup [h_+, +\infty)$, that is, $\tau_{h_-, h_+}(\omega) = \inf\left\{t \geq 0 \,|\, X_t(\omega) \geq h_+ \text{or} X_t(\omega) \leq h_-\right\}$.

REMARK 2.1. In what follows, various contingent claims that we introduce will be implicitly identified with their value functions. The barriers $H_\pm$ are fixed once and for all, so we suppress them from our notation, for the sake of readability.

2.2.2. *Reduction to a First-Touch Double Barrier Claim.* Next, let $G^0(x)$ denote the EPV of the stream of payoffs $\{g(\ln S_t)\}$ that is never abandoned, and, for the notational consistency with many formulas below, introduce $G^0_\pm = G^0$. Any pair of functions $G^0_\pm(x)$ determines a perpetual first-touch double barrier contingent claim with barriers (H_-, H_+) and the payoff $G^0_-(\ln S_t)$, $S_t \leq H_-$, and $G^0_+(\ln S_t)$, $S_t \geq H_+$. Let us denote the value function of this claim by $V_{f.t.}(x; G^0_\pm)$; by definition,

$$\begin{aligned} V_{f.t.}(x; G^0_\pm) = {} & \mathbb{E}\big[e^{-q\tau_{h_- - x}} \mathbb{1}_{\tau_{h_- - x} < \tau_{h_+ - x}} G^0_-(x + X_{\tau_{h_- - x}})\big] \\ & + \mathbb{E}\big[e^{-q\tau_{h_+ - x}} \mathbb{1}_{\tau_{h_+ - x} < \tau_{h_- - x}} G^0_+(x + X_{\tau_{h_+ - x}})\big], \end{aligned} \tag{2.2}$$

where τ_{h_-} (resp., τ_{h_+}) denotes the first entrance time of X into $(-\infty, h_-]$ (resp., $[h_+, +\infty)$). We have $V_{k.o.}(x; g) + V_{f.t.}(x; G^0_\pm) = G^0(x)$ for all $x \in \mathbb{R}$. This follows from Dynkin's formula, but is also clear from the financial viewpoint. Namely, due to the strong Markov property of X, we can interpret $V_{f.t.}(x; G^0_\pm)$ as the EPV of the perpetual stream of payoffs $\{g(\ln S_t)\}$ that becomes *activated* (rather than deactivated) as soon as S_t leaves the open interval (H_-, H_+).

2.2.3. *First Approximation to the Value Function.* Let us try to calculate $V_{f.t.}(x; G^0_\pm)$. Let $G^1_+(x) = V_{d.i.}(x; G^0_-)$ and $G^1_-(x) = V_{u.i.}(x; G^0_+)$ denote[3] the value functions of the perpetual down-and-in and up-and-in first-touch contingent claims, with barriers H_- and H_+, determined by the functions $G^0_-(x)$ and $G^0_+(x)$, respectively. As an initial "approximation" to $V_{f.t.}(x; G^0_\pm)$, we could attempt to use the sum $G^1_+(x) + G^1_-(x)$. However, the portfolio consisting of the contingent claims $G^1_\pm(x)$ is worth more than the contingent claim $V_{f.t.}(x; G^0_\pm)$, because S_t could enter one of the intervals $(0, H_-]$ or $[H_+, +\infty)$, and then enter the other one at a later time.

2.2.4. *Embedded Options.* In order to compensate for the amount by which the sum $G^1_+(x) + G^1_-(x)$ exceeds $V_{f.t.}(x; G^0_\pm)$, we introduce embedded options. Let $G^2_+(x)$ denote the value function of the contingent claim that pays its owner one contingent claim

[3] Note that the subscripts "+" and "−" have been interchanged.

$G^1_-(\ln S_t)$ at the first moment when S_t reaches or falls below H_-. Similarly, let $G^2_-(x)$ denote the value function of the contingent claim that pays its owner one contingent claim $G^1_+(\ln S_t)$ at the first moment when S_t reaches or exceeds H_+. Thus:

$$G^2_+(x) = V_{d.i.}\big(x; V_{u.i.}(\cdot; G^0_+)\big), \quad G^2_-(x) = V_{u.i.}\big(x; V_{d.i.}(\cdot; G^0_-)\big).$$

It is clear that the portfolio consisting of the three claims $V_{f.t.}(x; G^0_\pm)$, $G^2_+(x)$, and $G^2_-(x)$ is worth more than the portfolio consisting of the two claims $G^1_\pm(x)$. The discrepancy between the two portfolios can be measured by another first-touch double barrier contingent claim. Namely, consider the perpetual contingent claim that pays its owner one contingent claim $G^2_\pm(x)$ at the first moment when S_t leaves the open interval (H_-, H_+), where the subscript "+" or "−" is chosen depending on whether $S_t \geq H_+$ or $S_t \leq H_-$. Its value function is $V_{f.t.}(x; G^2_\pm)$, and we have

PROPOSITION 2.2. $V_{f.t.}(x; G^0_\pm) + G^2_+(x) + G^2_-(x) = G^1_+(x) + G^1_-(x) + V_{f.t.}(x; G^2_\pm)$.

The assertion of the proposition is easily verified via a case-by-case inspection of the following mutually exclusive possibilities for the realization of the uncertainty in the future dynamics of the price process $\{S_t\}$:

(i) S_t never leaves the interval (H_-, H_+);
(ii) S_t reaches $[H_+, +\infty)$, but never reaches $(0, H_-]$;
(iii) S_t reaches $(0, H_-]$, but never reaches $[H_+, +\infty)$;
(iv) S_t reaches $[H_+, +\infty)$ before it reaches $(0, H_-]$, then reaches $(0, H_-]$ at a later time, and never returns to $[H_+, +\infty)$ afterwards;
(v) S_t reaches $(0, H_-]$ before it reaches $[H_+, +\infty)$, then reaches $[H_+, +\infty)$ at a later time, and never returns to $(0, H_-]$ afterwards;
(vi) S_t reaches $[H_+, +\infty)$ before it reaches $(0, H_-]$, then reaches $(0, H_-]$ at a later time, and then reaches $[H_+, +\infty)$ again later;
(vii) S_t reaches $(0, H_-]$ before it reaches $[H_+, +\infty)$, then reaches $[H_+, +\infty)$ at a later time, and then reaches $(0, H_-]$ again later.

The verification is straightforward, albeit somewhat tedious, so we skip the details.

2.2.5. *The Valuation Formulas.* We remain in the same setup as before. The formula obtained in Proposition 2.2 can be rewritten as follows:

$$V_{f.t.}(x; G^0_\pm) = G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) + V_{f.t.}(x; G^2_\pm), \tag{2.3}$$

where $G^j_+(x) = V_{d.i.}(x; G^{j-1}_-)$ and $G^j_-(x) = V_{u.i.}(x; G^{j-1}_+)$ for $j = 1, 2$. Using the same notation for $j = 1, 2, \ldots$, and continuing in the same fashion, we obtain a formula[4]

$$V_{f.t.}(x; G^0_\pm) = G^1_+(x) + G^1_-(x) - G^2_+(x) - G^2_-(x) + G^3_+(x) + G^3_-(x) - \cdots, \tag{2.4}$$

where the series on the right-hand side converges absolutely and uniformly on $[h_-, h_+]$. For a proof, it suffices to note that each of the functions in (2.4) is uniformly bounded

[4] We are grateful to the anonymous referee who indicated that decompositions (2.3) and (2.4) are related to the method used by Rogozin (1972) to solve the double exit problem for stable processes, and to several other papers for various classes of Lévy processes—see Kadankova and Veraverbeke (2007) and the bibliography therein.

on $\mathbb{R}$, and, on the strength of the stochastic continuity of a Lévy process, there exists $\epsilon \in (0, 1)$ independent of functions under consideration such that

$$\max_{x \geq h_+} |V_{d.i.}(x; G_-^{j-1})| \leq \epsilon \max_{x \leq h_-} |G_-^{j-1}(x)|, \quad \max_{x \leq h_-} |V_{d.i.}(x; G_+^{j-1})| \leq \epsilon \max_{x \geq h_+} |G_+^{j-1}(x)|.$$

In the Black–Scholes model, once the function $G^0(x)$ is known, the right-hand side of (2.4) can be evaluated explicitly. In the general setting, it can be calculated numerically using the methods outlined in the remainder of this section (the computational aspects are discussed in Section 4).

Finally, formula (2.4) and the discussion in Section 2.2.2 imply the following identity that expresses the value function $V_{k.o.}(x; g)$ of a perpetual knock-out double barrier stream of payoffs as the sum of an absolutely and uniformly convergent series

$$\begin{aligned} V_{k.o.}(x; g) = G^0(x) - G_+^1(x) - G_-^1(x) + G_+^2(x) + G_-^2(x) \\ - G_+^3(x) - G_-^3(x) + G_+^4(x) + G_-^4(x) - \cdots, \end{aligned} \tag{2.5}$$

where $G_+^n(x) = V_{d.i.}(x; G_-^{n-1})$ and $G_-^n(x) = V_{u.i.}(x; G_+^{n-1})$ for all $n \geq 1$.

2.3. Normalized EPV Operators

We remain in the general framework of Section 2.1. In particular, $X = \{X_t\}_{t \geq 0}$ is an arbitrary one-dimensional Lévy process, and $q > 0$ is fixed. The *supremum process* and the *infimum process* of X are the stochastic processes $\bar{X}$ and $\underline{X}$ defined by $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$, $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$. The practical implementation of formulas (2.4) and (2.5) is based on the calculation of the action of the normalized EPV operators $\mathcal{E}_q^{\pm}$ under the processes $\bar{X}$, $\underline{X}$. Their action in the space of measurable bounded functions is defined by

(2.6)
$$(\mathcal{E}_q^+ f)(x) = \mathbb{E}\left[\int_0^\infty q e^{-qt} f(x + \bar{X}_t)\, dt\right], \quad (\mathcal{E}_q^- f)(x) = \mathbb{E}\left[\int_0^\infty q e^{-qt} f(x + \underline{X}_t)\, dt\right].$$

To explain the name "EPV operators," note that each calculates the EPV of a certain stochastic cash flow. For more background on the operators $\mathcal{E}_q^{\pm}$, and efficient numerical realizations, we refer the reader to Boyarchenko and Levendorskiĭ (2009a,b).

For us, the first application of the operators $\mathcal{E}_q^{\pm}$ will be to the problem of pricing perpetual knock-out single barrier streams of payoffs. The formulas we give below were obtained earlier by S.I. Boyarchenko and the second author (see, e.g., Boyarchenko and Levendorskiĭ 2002) under certain additional assumptions, which exclude driftless VG processes. In Boyarchenko and Levendorskiĭ (2009a,b), the reader can find the proof that is valid for arbitrary Lévy processes.

Let $h \in \mathbb{R}$, let f be a bounded measurable function on $(-\infty, h)$, and let us consider a perpetual up-and-out stream of payoffs defined by the function f (cf. Section 2.1(1)). Its value function is given by

$$V_{up\text{-}and\text{-}out}(x; f) = q^{-1} \cdot \left(\mathcal{E}_q^+ \mathbb{1}_{(-\infty, h)} \mathcal{E}_q^- f\right)(x) \tag{2.7}$$

where $\mathbb{1}_{(-\infty,h)}$ denotes the multiplication-by-$\mathbb{1}_{(-\infty,h)}$ operator.

REMARK 2.3. In (2.7), it is more accurate to write $\mathcal{E}_q^- \tilde{f}$ rather than f, where $\tilde{f}$ is a bounded measurable function that coincides with f on $(-\infty, h)$. However, the product $\mathbb{1}_{(-\infty,h)}\mathcal{E}_q^- \tilde{f}$ is independent of the choice of an extension $\tilde{f}$, so we simplify the notation. A similar comment applies to (2.8) and in many other situations considered below.

Similarly, a bounded measurable function f on $(h, +\infty)$ determines a perpetual down-and-out stream of payoffs (cf. Section 2.1(2)). Its value function is given by

$$V_{\textit{down-and-out}}(x; f) = q^{-1} \cdot \big(\mathcal{E}_q^- \mathbb{1}_{(h,+\infty)} \mathcal{E}_q^+ f\big)(x). \tag{2.8}$$

Note that we can either let $h = +\infty$ in (2.7), or let $h = -\infty$ in (2.8), and obtain a formula for the value function of a stream of payoffs that is never abandoned:

$$V_{\text{perpetual}}(x; f) = q^{-1}\big(\mathcal{E}_q^+ \mathcal{E}_q^- f\big)(x) = q^{-1}\big(\mathcal{E}_q^- \mathcal{E}_q^+ f\big)(x). \tag{2.9}$$

In fact, the compositions $\mathcal{E}_q^+\mathcal{E}_q^-$ and $\mathcal{E}_q^-\mathcal{E}_q^+$ are equal to the normalized EPV operator $\mathcal{E}_q$, which is defined by

$$(\mathcal{E}_q f)(x) = \mathbb{E}\left[\int_0^\infty q e^{-qt} f(x + X_t)\, dt\right]. \tag{2.10}$$

This result is one of the forms of the celebrated *Wiener-Hopf factorization formula*, which gave rise to the name "Wiener-Hopf method for pricing barrier options." Once again, we refer to Boyarchenko and Levendorskiĭ (2009a,b) for all the details and additional information.

2.4. How to Calculate the Action of $\mathcal{E}_q^\pm$

In this subsection we explain how the action of the normalized EPV operators $\mathcal{E}_q^\pm$ can be calculated efficiently in practice for a rather wide class of Lévy processes. We also take this opportunity to discuss the conditions under which the numerical methods developed in the present paper can be applied. Several examples that satisfy these conditions and appear frequently in financial modeling are given in Section 2.6.

Let $T_q \sim \operatorname{Exp} q$ be an exponentially distributed random variable with mean q^{-1}, which is independent of the process X. Since the distribution of T_q is $qe^{-qt}\mathbb{1}_{[0,+\infty)}(t)$, we see that $(\mathcal{E}_q^+ f)(x) = \mathbb{E}\big[f(x + \bar{X}_{T_q})\big]$ and $(\mathcal{E}_q^- f)(x) = \mathbb{E}\big[f(x + \underline{X}_{T_q})\big]$ for every bounded measurable function $f(x)$ on $\mathbb{R}$. To rewrite these formulas in a slightly different way, let $p_q^+(dx)$ (respectively, $p_q^-(dx)$) denote the probability distribution of the random variable $\bar{X}_{T_q}$ (respectively, $\underline{X}_{T_q}$). From the definition of $\bar{X}$ and $\underline{X}$, we see that $p_q^+(dx)$ is supported on $[0, +\infty)$ and $p_q^-(dx)$ is supported on $(-\infty, 0]$. We then obtain the following representations of $\mathcal{E}_q^\pm$ as convolution operators:

$$(\mathcal{E}_q^+ f)(x) = \int_0^{+\infty} f(x + y)\, p_q^+(dy), \quad (\mathcal{E}_q^- f)(x) = \int_{-\infty}^0 f(x + y)\, p_q^-(dy). \tag{2.11}$$

In order to calculate the action of $\mathcal{E}_q^\pm$ in practice, it is natural to consider the Fourier transforms of the measures $p_q^\pm(dx)$. Using the normalization of the Fourier transform that is common in probability theory, we define functions $\phi_q^\pm(\xi)$ (the Wiener-Hopf factors) by

$$\phi_q^+(\xi) = \widehat{p}_q^+(\xi) \stackrel{\text{def}}{=} \mathbb{E}\big[e^{i\xi \bar{X}_{T_q}}\big], \quad \phi_q^-(\xi) = \widehat{p}_q^-(\xi) \stackrel{\text{def}}{=} \mathbb{E}\big[e^{i\xi \underline{X}_{T_q}}\big]. \tag{2.12}$$

If the functions $\phi_q^{\pm}(\xi)$ are known, we can (approximately) compute the probability measures $p_q^{\pm}(dx)$ by means of Fourier inversion, which allows us to calculate the action of the operators $\mathcal{E}_q^{\pm}$ using (2.11). The numerical realization of this approach is explained in Section 4.3. In certain cases, such as the Black–Scholes model or double-exponential jump-diffusion model or more general jump-diffusion models (Levendorskiĭ 2004), $\phi_q^{\pm}(\xi)$ are rational functions that are given by explicit formulas. In the recent β-model (Kuznetsov 2010), $\phi_q^{\pm}(\xi)$ can be calculated as infinite products.[5] In general, however, $\phi_q^{\pm}(\xi)$ must be calculated numerically.

To this end, let us recall that every Lévy process $X = \{X_t\}_{t\geq 0}$ has a *characteristic exponent*, which is a continuous function $\psi : \mathbb{R} \to \mathbb{C}$ satisfying $\psi(0) = 0$ and

$$\mathbb{E}\big[e^{i\xi X_t}\big] = e^{-t\psi(\xi)} \quad \forall \xi \in \mathbb{R},\ t \geq 0;$$

and, conversely, the law of a Lévy process is uniquely determined by its characteristic exponent (theorem 7.10, Sato 1999). In this paper, we always work with concrete examples of Lévy processes in terms of their characteristic exponents.

The numerical methods developed in the paper are justified under the following conditions on the characteristic exponent $\psi(\xi)$ and the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$:

(i) There exist $-\infty \leq \lambda_- < 0 < \lambda_+ \leq +\infty$ such that $\psi(\xi)$ admits analytic extension into the open strip $\lambda_- < \mathrm{Im}\xi < \lambda_+$ in the complex plane.[6]

(ii) The following integral formulas for the functions $\phi_q^{\pm}(\xi)$ are valid[7]:

$$\phi_q^{\pm}(\xi) = \exp\left[\pm\frac{1}{2\pi i}\int_{\mathrm{Im}\eta=\omega_{\mp}} \frac{\xi \cdot \ln(1+q^{-1}\psi(\eta))}{\eta(\xi-\eta)}\, d\eta\right], \tag{2.13}$$

where $\lambda_- < \omega_- < 0 < \omega_+ < \lambda_+$ are chosen subject to the condition that $q + \psi(\xi) \notin (-\infty, 0]$ for ξ in the strip $\mathrm{Im}\xi \in [\omega_-, \omega_+]$.

(iii) On each line $\mathrm{Im}\xi = \omega \in (\lambda_-, \lambda_+)$, the derivatives of $\psi(\xi)$ admit estimates

$$|\psi^{(s)}(\xi)| \leq C_s(1+|\xi|)^{\bar{\nu}-s\delta}, \quad s = 1, 2, \ldots, \tag{2.14}$$

where $\bar{\nu} \in \mathbb{R}$ and $\delta > 0$ are independent of s, ω, and ξ, and the constants C_s are independent of ξ (but may depend on ω).

We call a Lévy process *tame* if conditions (i)–(iii) are satisfied. A numerical method of calculating $\phi_q^{\pm}(\xi)$ based on (2.13) is presented in Section 4.2.

2.5. Remarks

a) Typically, one uses explicit formulas for the Wiener-Hopf factors with integration over the real line—see, e.g., Eskin (1981), and then these formulas can be applied

[5] The latter procedure is based on calculation of infinitely many complex roots, and it is not clear if this procedure leads to an efficient computational algorithm for calculation of the Wiener-Hopf factors at many points at once as needed for the algorithm of the present paper.

[6] Then $\psi(\xi) = O(|\xi|^2)$ as $\xi \to \infty$ within every closed sub-strip $\mathrm{Im}\xi \in [\omega_-, \omega_+] \subset (\lambda_-, \lambda_+)$. We are indebted to the anonymous referee for the following simple proof of this statement. On the real line, $\psi(\xi) = O(\xi^2)$ as $\xi \to \pm\infty$ (see Bertoin 1996), and for $a \in (\lambda_-, \lambda_+)$, function $\psi_a(\xi) = \psi(\xi + ia) - \psi(ia)$ is the characteristic exponent of another Lévy process.

[7] From the previous footnote, it is clear that the integral on the right-hand side of (2.13) converges.

when the symbol of an operator (in probability, the characteristic exponent) does not admit analytic continuation. However, this straightforward choice of the line of integration is not computationally efficient if the analytic continuation is possible: if the line of integration is sufficiently far from the real line and the boundary of the strip of analyticity, then the derivatives of the integrand w.r.t. η are small, the derivatives of high order especially. This makes it possible to use a relatively large step $\zeta = \eta_{j+1} - \eta_j$ in numerical integration formulas (for typical parameter values, about 0.5); the integration over the real line, which is possible in our case, and in many other cases, when the singularity of the integrand at $\eta = 0$ is removable, requires a much smaller step, hence, more of CPU time to achieve the same accuracy.

b) One can replace condition (iii) with appropriate bounds on the Lévy density and its derivatives. We prefer (iii) because the verification of conditions (i), (iii) for the model classes is trivial.

We are grateful to the anonymous referee for the suggestion to clarify the following points in more detail.

c) Having in mind applications to Lévy processes of exponential type with not very large $|\lambda_\pm|$, and Carr's randomization method with small time step, hence, large q, we recommend, as a rule of thumb, $\omega_\pm = \lambda_\pm/3$. If q is not large, then this choice will violate the condition in (ii). The integration over a line different from the real line is more efficient in this case as well but one may need to use the residue theorem to justify this reduction, and an additional factor in the formula for $\phi_q^\pm(\xi)$ appears (see section 3.6.3, Boyarchenko and Levendorskiĭ 2002). If $|\lambda_\pm|$ is very large, then the choice $\omega_\pm = \lambda_\pm/3$ will not be efficient. The general idea is to choose a line where the derivatives of the integrand are small.

d) The representation (2.13) is proved for wide classes of Lévy processes of exponential type in theorem 3.2 of Boyarchenko and Levendorskiĭ (2002), with $q + \psi(\eta)$ instead of $1 + q^{-1}\psi(\eta)$ on the RHS.[8] This class contains all the model classes listed below but VG processes and β-model with one of $\gamma_j \le 1$. If the drift μ of a VGP is nonzero, then the proof of theorem 3.3 of Boyarchenko and Levendorskiĭ (2002) can be repeated word by word (VGP being treated as KoBoL of order 0). The resulting representation (3.76) of Boyarchenko and Levendorskiĭ (2002) can be easily transformed into (2.13). Finally, passing to the limit $\mu \to 0$ in both sides of (2.13) (which can be justified), we conclude that it holds for all VGP. The proof for β-model with both of $\gamma_j \le 1$ is similar, but the proof for the cases when only one of $\gamma_j \le 1$ is more technical; we omit it for the sake of brevity.

e) In the case of single barrier options, condition (iii) is unnecessary for both the theoretical study and efficient numerical realization of the pricing algorithm. In the case of double barrier options, it is needed to ensure that $\mathcal{E}_q^\pm$ are continuous operators in spaces of infinitely differentiable functions of exponential decay at infinity, with continuous inverses (for details, see Boyarchenko and Levendorskiĭ 2002). This ensures that the numerical procedure for the inverses $(\mathcal{E}_q^\pm)^{-1}$ used in the paper is sufficiently accurate and stable. For more details, see Remark 2.4.

[8] If $\omega_\pm \neq 0$, the reduction of the former case to the latter is immediate from the fact that in the case $\psi(\xi) \equiv 1$, the integral in (2.13) equals 0. If the singularity of the integrand is removable, then one can shift the line of integration to the real line and obtain (2.13) with $\omega_\pm = 0$—see Lewis and Mordecki (2008).

2.6. Model Classes of Tame Lévy Processes

(1) A Brownian motion (used in the classical Black–Scholes model) is a tame Lévy process with $\lambda_\pm = \pm\infty$.
(2) In Merton's model (Merton 1976), the underlying log-price process is a Lévy process with the characteristic exponent $\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda\cdot\left(1 - e^{im\xi - \frac{s^2}{2}\xi^2}\right)$, where $\sigma, s, \lambda > 0$ and $\mu, m \in \mathbb{R}$. A process of this kind is also tame with $\lambda_\pm = \pm\infty$.
(3) A HEJD process (Asmussen, Avram, and Pistorius 2004; Levendorskiĭ 2004; Jeannin and Pistorius 2007) has characteristic exponent

$$\psi(\xi) = \frac{\sigma^2}{2}\xi^2 - i\mu\xi + \lambda^+ \cdot \sum_{j=1}^{n^+} \frac{ip_j^+\xi}{i\xi - \alpha_j^+} + \lambda^- \cdot \sum_{k=1}^{n^-} \frac{ip_k^-\xi}{i\xi + \alpha_k^-}, \tag{2.15}$$

where $n^\pm$ are positive integers and $\alpha_j^\pm, \lambda^\pm, p_j^\pm > 0$ satisfy $\sum_{j=1}^{n^\pm} p_j^\pm = 1$. The double-exponential jump-diffusion model can be obtained as a special case of HEJD models by taking $n^+ = n^- = 1$. Conditions (i)–(iii) are satisfied with $(\lambda_-, \lambda_+) = \left(\max\{-\alpha_k^-\}, \min\{\alpha_j^+\}\right)$.
(4) Lévy processes of the *extended Koponen family* (generalizing the class of processes introduced by Koponen 1995) were defined by S.I. Boyarchenko and the second author in Boyarchenko and Levendorskiĭ (2000). Later a subfamily thereof was used in Carr et al. (2002) under the name "CGMY-model," and the full family was used in Boyarchenko and Levendorskiĭ (2002) under the name "KoBoL processes." We use the latter term. The Lévy density of a KoBoL process has the form

$$F(dx) = c_+(-\lambda_-)x^{-\nu_- - 1}e^{\lambda_- x}\mathbb{1}_{(0,+\infty)}dx + c_-\lambda_+ x^{-\nu_+ - 1}e^{\lambda_+ x}\mathbb{1}_{(-\infty,0)}dx, \tag{2.16}$$

where $c_\pm \geq 0$, $\nu_\pm < 2$, and $\lambda_- < 0 < \lambda_+$, see Boyarchenko and Levendorskiĭ (2000, 2002). In the special case where $c_+ = c = c_-$ and $\nu_\pm = \nu \neq 0, 1$ (which corresponds to the CGMY model), the characteristic function of the process is given by

$$\psi(\xi) = -i\mu\xi + c\cdot\Gamma(-\nu)\cdot\left[(-\lambda_-)^\nu - (-\lambda_- - i\xi)^\nu + \lambda_+^\nu - (\lambda_+ + i\xi)^\nu\right]. \tag{2.17}$$

Conditions (i)–(iii) are satisfied with the same parameters $\lambda_\pm$, so there is no conflict of notation.
(5) VG processes were first used in empirical studies of financial markets by Madan and collaborators (Madan and Seneta 1990; Madan and Milne 1991; Madan, Carr, and Chang 1998). The Lévy density of a V.G. process is given by (2.16) with $c_\pm = c > 0$ and $\nu_\pm = 0$. Such a process is also tame with the same $\lambda_\pm$, and the corresponding characteristic exponent is[9]:

$$\psi(\xi) = -i\mu\xi + c\cdot\left[\ln(-\lambda_- - i\xi) - \ln(-\lambda_-) + \ln(\lambda_+ + i\xi) - \ln(\lambda_+)\right]. \tag{2.18}$$

(6) NIG processes were constructed by Barndorff-Nielsen, and were applied to empirical studies of financial markets in Barndorff-Nielsen (1998). The characteristic exponent of a NIG process has the form

$$\psi(\xi) = -i\mu\xi + \delta\cdot\left[\left(\alpha^2 - (\beta + i\xi)^2\right)^{1/2} - (\alpha^2 - \beta^2)^{1/2}\right], \tag{2.19}$$

[9] What we present is not the most common way of writing the formula. Rather, we chose an expression that is equivalent to the standard one and makes the analogy with (2.17) transparent.

where $\alpha > |\beta| > 0, \delta > 0$ and $\mu \in \mathbb{R}$. A natural generalization of NIG defined by (2.19) with $\nu/2, \nu \in (0, 2)$, instead of $1/2$ in the exponents was constructed in Barndorff-Nielsen and Levendorskiĭ (2001). Conditions (i)–(iii) are satisfied with $(\lambda_-, \lambda_+) = (\beta - \alpha, \beta + \alpha)$.

(7) The β-family of Lévy processes constructed in Kuznetsov (2010) is defined by the Lévy density

$$F(dx) = c_1 \frac{e^{-\alpha_1\beta_1 x}}{(1 - e^{-\beta_1 x})^{\gamma_1}} \mathbb{1}_{(0,+\infty)}(x) + c_2 \frac{e^{\alpha_2\beta_2 x}}{(1 - e^{\beta_2 x})^{\gamma_2}} \mathbb{1}_{(-\infty,0)}(x), \tag{2.20}$$

where $c_j \geq 0, \alpha_j, \beta_j > 0$ and $\gamma_j \in (0, 3)$. For any positive integer N, $F(dx)$ can be represented as a sum of finite number of densities of the form (2.16), with decreasing $\nu_\pm$, and not necessarily positive $c_\pm$, plus a density of the form $p_N(x)dx$, where, for each $s = 0, 1, \ldots, p_N^{(s)}(x) = |x|^{N-s}$ as $x \to 0$, and exponentially decays as $x \to +\infty$. Therefore, the straightforward calculations (the same as used to derive the formulas for the characteristic exponent of the processes of the extended Koponen's family in Boyarchenko and Levendorskiĭ (2000, 2002), show that the characteristic exponent of the β model, and its derivatives, admit the same estimates as in the (general) KoBoL model with $\nu_+ = \gamma_2 - 1, \nu_- = \gamma_1 - 1$. Hence, conditions (i), (iii) hold. Moreover, if $\gamma_j > 1$, then β-model is a Lévy process of exponential type and positive order; for these processes, (ii) holds (see theorem 3.2, Boyarchenko and Levendorskiĭ 2002). If both $\gamma_j = 1$, then the proof of (ii) is the same as for VGP model, and if one of $\gamma_j \leq 1$, then a similar proof can be given.

(8) If a nontrivial Brownian motion component is introduced into any of the models (4)–(7), the model becomes a Lévy process of exponential type and order 2, hence, all conditions (i)–(iii) are automatically satisfied.

2.7. First-Touch Single Barrier Claims

We now return to the question of how to calculate the right-hand side of formulas (2.4) and (2.5) in practice. It is clear that the problem of pricing perpetual first-touch contingent claims described in Section 2.1(3) must be solved as an intermediate step. To this end, let us fix $h \in \mathbb{R}$, and let G_+ be a bounded measurable function defined on $[h, +\infty)$. Then G_+ determines a perpetual up-and-in first-touch contingent claim on the underlying $\{S_t = S_0 e^{X_t}\}_{t\geq 0}$. As before, its value function is denoted by $V_{u.i}(x; G_+)$, and, whenever necessary, we may identify G_+ with a convenient extension of G_+ to the whole real line.

In order to calculate $V_{u.i.}(x; G_+)$, let us first suppose that (an extension of) G_+ can be represented as the EPV of a perpetual payoff stream $f(\ln S_t)$ that is never abandoned, where f is a bounded measurable function on the real line. Then, due to the strong Markov property of the process $X = \{X_t\}$, we can interpret $V_{u.i.}(x; G_+)$ as the EPV of the stream of payoffs $f(\ln S_t)$ that is *activated* as soon as S_t reaches or exceeds $H = e^h$. If $V_{u.o.}(x; f)$ denotes the value function of the corresponding perpetual *up-and-out* stream of payoffs, it follows that the sum $V_{u.i.}(x; G_+) + V_{u.o.}(x; f)$ is equal to the EPV of the perpetual stream $f(\ln S_t)$ that is never abandoned.

By construction, the value function of the latter stream is G_+. Furthermore, the function $V_{u.o.}(x; f)$ can be calculated using formula (2.7). It also follows from

Section 2.3 that $G_+ = q^{-1}\mathcal{E}_q^+\mathcal{E}_q^- f$. Combining these facts, we deduce that

$$V_{u.i.}(x; G_+) = q^{-1} \cdot \left(\mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}\mathcal{E}_q^- f\right)(x).$$

Finally, in order to remove the notational dependence on the function f, we note that, thanks to the identity $G_+ = q^{-1}\mathcal{E}_q^+\mathcal{E}_q^- f$, we may (formally) write $q^{-1}\mathcal{E}_q^- f = (\mathcal{E}_q^+)^{-1}G_+$, which leads to the formula

$$V_{u.i.}(x; G_+) = \left(\mathcal{E}_q^+ \mathbb{1}_{[h,+\infty)}(\mathcal{E}_q^+)^{-1} G_+\right)(x). \tag{2.21}$$

REMARK 2.4. It can be shown that $\mathbb{1}_{[h,+\infty)}(\mathcal{E}_q^+)^{-1}G_+$ is independent of the values $G_+(x)$, $x < h$ (see Boyarchenko and Levendorskiĭ 2002 for details). Hence, we can use (2.21) with G_+ defined on a subset of the real line that contains $[h, +\infty)$ identifying G_+ with its sufficiently regular bounded extension (and, whenever convenient, changing the values $G_+(x)$, $x < h$; cf. Remark 2.3). Note that in the case of no-touch options, all functions G, to which (2.21) will be applied, are infinitely differentiable functions, which can be made exponentially decaying after an appropriate change of measure. In the case of calls and puts, some of the functions will have finite smoothness at the strike; however, if the grid is chosen so that the log-strike is on the grid, then this irregularity is not essential because our discretization method is based on piece-wise linear approximation of G_+. Operator $(\mathcal{E}_q^+)^{-1}$ will be represented as a discrete convolution operator with kernel supported on $\mathbb{Z}_+$. Before the discretization, for the case of tame processes, $\mathcal{E}_q^+$ is a PDO with the sufficiently regular symbol so both $\mathcal{E}_q^+$ and its inverse are continuous in the scale of Sobolev spaces of functions exponentially decaying at infinity (see Boyarchenko and Levendorskiĭ 2002 for details). This property can be used to formally justify the discretization approximation used in the paper.

The treatment of perpetual down-and-in first-touch contingent claim is entirely analogous. In the situation above, if G_- is a bounded measurable function on $(-\infty, h]$, then, under suitable assumptions, we have the following formula for the value function of the corresponding contingent claim:

$$V_{d.i.}(x; G_-) = \left(\mathcal{E}_q^- \mathbb{1}_{(-\infty,h]}(\mathcal{E}_q^-)^{-1} G_-\right)(x). \tag{2.22}$$

2.8. Perpetual Double Barrier Streams and EPV Operators

We fix a tame one-dimensional Lévy process $X = \{X_t\}_{t\geq 0}$, a real number $q > 0$, barriers $0 < H_- < H_+$, and a bounded measurable function $g(x)$ defined on the interval (h_-, h_+), where $h_\pm = \ln H_\pm$. Let $V_{k.o.}(x; g)$ denote the EPV of the perpetual stream of payoffs $g(\ln S_t)$ that is abandoned as soon as $S_t = S_0 e^{X_t}$ leaves the interval (H_-, H_+), assuming that the killing rate equals q and that the initial spot price of the underlying equals $S_0 = e^x$, where $h_- < x < h_+$ (see (2.1)).

We proved in Section 2.2 that the function $V_{k.o.}(x; g)$ can be calculated using formula (2.5), where the series on the right-hand side converges absolutely and uniformly for all $h_- < x < h_+$. The individual terms of the series can be calculated inductively using the following prescription. First, extend the function g to a bounded measurable function on the whole real line, and calculate[10]

$$G^0(x) = q^{-1}(\mathcal{E}_q^+\mathcal{E}_q^- g)(x) = q^{-1}(\mathcal{E}_q^-\mathcal{E}_q^+ g)(x). \tag{2.23}$$

Next, set $G_\pm^0 = G^0$, and calculate $G_\pm^n$ for $n = 1, 2, 3, \ldots$ using the formulas

$$G_+^n(x) = \left(\mathcal{E}_q^- \mathbb{1}_{(-\infty,h_-]}(\mathcal{E}_q^-)^{-1} G_-^{n-1}\right)(x), \tag{2.24}$$

and

$$G_-^n(x) = \left(\mathcal{E}_q^+ \mathbb{1}_{[h_+,+\infty)}(\mathcal{E}_q^+)^{-1} G_+^{n-1}\right)(x). \tag{2.25}$$

3. FINITE-LIVED DOUBLE BARRIER OPTIONS

3.1. Market Specifications

In this section we consider a model frictionless market consisting of a riskless bond and a risky asset (for instance, a stock). We assume that the riskless rate, $r > 0$, is constant, and let S_t denote the spot price of the underlying at time t. We also assume that $S_t = S_0 e^{X_t}$, where $X = \{X_t\}_{t\geq 0}$ is a Lévy process under a chosen equivalent martingale measure (EMM). We remark that in general, an EMM (also called a "risk-neutral measure") is not unique. We assume that an EMM has been fixed once and for all, and all expectation operators appearing in this section will be with respect to this measure.

Our goal is to study pricing problems for finite-lived knock-out double barrier options on the stock $\{S_t\}$. The types of options we consider are described in Section 3.2, where we also give formulas for their no-arbitrage value functions in terms of certain stochastic expressions. These formulas are unsuitable for practical calculations, and the rest of the section will be devoted to developing more computationally efficient approaches to the valuation of finite-lived double barrier options. We design a fast and accurate procedure for the valuation of finite-lived double barrier options based on Carr's randomization (Carr and Faguet 1996; Carr 1998). Convergence of Carr's randomization procedure for single and double barrier options is proved in Boyarchenko (2008) under a mild assumption, which hold for all model classes of Lévy processes.[11]

3.2. Knock-Out Double Barrier Options

We remain in the setup of Section 3.1. Let us fix two *barriers*, $0 < H_- < H_+$, and define $h_\pm = \ln H_\pm$. We also let $g(x)$ denote a bounded nonnegative measurable function on the interval (h_-, h_+). The *knock-out double barrier option with maturity date $T > 0$, barriers (H_-, H_+) and terminal payoff* $g(x)$ is defined as the contingent claim that expires worthless if the price, S_t, of the underlying leaves the open interval (H_-, H_+) at any time $0 \leq t \leq T$, and pays its owner $g(\ln S_T)$ at time T otherwise.

[10] It is often computationally more efficient to calculate the action of the composition $\mathcal{E}_q^+\mathcal{E}_q^-$ at this step, rather than the action of $\mathcal{E}_q$.

[11] In the case of the double-exponential jump-diffusion model and, more generally, HEJD model, the diffusion component must be nontrivial.

EXAMPLES 3.1.

(1) If $K > 0$ is fixed and $g(x) = (e^x - K)_+$ (or $g(x) = (K - e^x)_+$), we obtain a knock-out double barrier call (put) option with strike price K.
(2) If $g(x) = 1$ for all x, we obtain a double-no-touch (DNT) option (cf. Carr and Crosby 2008).

In general, the no-arbitrage value of a knock-out double barrier option with maturity date $T > 0$, barriers (H_-, H_+) and terminal payoff $g(x)$ is given by the formula

$$V_{k.o.}(x, T; g) = \mathbb{E}\big[e^{-rT}\mathbb{1}_{\{\tau_{h_--x,h_+-x}>T\}}g(x + X_T)\big], \tag{3.1}$$

where $x = \ln S_0$ is the initial log-spot price of the underlying and τ_{h_-,h_+} denotes the first entrance time of the process X into the set $(-\infty, h_-] \cup [h_+, +\infty)$ (cf. Section 2.2.1).

3.3. Carr's Randomization for Double Barrier Options

Let us consider the right-hand side of (3.1). Following Carr (1998), we first replace the deterministic maturity period $[0, T]$ with a random maturity period $[0, T']$, where T' is an exponentially distributed random variable that has mean T and is independent of the process X. By a slight abuse of notation, let us denote the resulting expression by $V_{k.o.}(x, T'; g)$ (even though T' is a random variable, $V_{k.o.}(x, T'; g)$ is still a deterministic quantity). Using the fact that the distribution of T' equals $\frac{1}{T} \cdot e^{-t/T}\mathbb{1}_{[0,+\infty)}(t)\,dt$ and that T' is independent of X, we obtain

$$\begin{aligned}
V_{k.o.}(x, T'; g) &= \mathbb{E}\big[e^{-rT'}\mathbb{1}_{\{\tau_{h_--x,h_+-x}>T'\}}g(x + X_{T'})\big] \\
&= \frac{1}{T} \cdot \mathbb{E}\left[\int_0^\infty e^{-t/T}e^{-rt}\mathbb{1}_{\{\tau_{h_--x,h_+-x}>t\}}g(x + X_t)\,dt\right] \\
&= (1 + rT)^{-1} \cdot (r + T^{-1}) \cdot \mathbb{E}\left[\int_0^{\tau_{h_--x,h_+-x}} e^{-(r+T^{-1})t}g(x + X_t)\,dt\right] \\
&= (1 + rT)^{-1} \cdot (r + T^{-1}) \cdot V_{k.o.}(x; g),
\end{aligned}$$

where $V_{k.o.}(x; g)$ is defined by formula (2.1) with q replaced by $r + T^{-1}$.

In general, of course, one cannot expect $V_{k.o.}(x, T'; g)$ to be a good approximation to $V_{k.o.}(x, T; g)$. Following Carr (1998), we next divide the maturity period of the option into N subintervals, using points $0 = t_0 < t_1 < \ldots < t_N = T$, and we replace each subperiod $[t_s, t_{s+1}]$ with an exponentially distributed random maturity period with mean $\Delta_s = t_{s+1} - t_s$. Moreover, these N random maturity subperiods are assumed to be independent of each other and of the process X. (In Carr 1998, it is assumed that $\Delta_s = T/N$ for all s, but, in principle, we do not have to impose this requirement.)

As in Carr (1998), we can calculate the value function of the claim with this new maturity period using backward induction. Namely, let $V^s(x)$ denote the value function of the option after the first s maturity subperiods. Then, by definition, $V^N(x) = g(x)$, the terminal payoff function. Moreover, for all $0 \le s \le N - 1$, the function $V^s(x)$ can be interpreted as the value function of a knock-out double barrier option with barriers (H_-, H_+), terminal payoff function $V^{s+1}(x)$, and exponentially distributed maturity date with mean Δ_s. Therefore $V^s(x)$ can be calculated using the method we just explained. The resulting algorithm that computes $V^N(x), V^{N-1}(x), \ldots, V^0(x)$ will be referred to as "Carr's randomization for double barrier options."

Let us make this procedure a little more explicit. Further details, as well as the computational aspects of the algorithm sketched below, will be discussed in Section 4.

Carr's randomization algorithm for calculating $V_{k.o.}(x, T; g)$.

1. Choose points $0 = t_0 < t_1 < \cdots < t_N = T$. For each $0 \le s \le N-1$, set $\Delta_s = t_{s+1} - t_s$ and $q_s = r + \Delta_s^{-1}$.
2. Define $V^N(x) = g(x)$, the terminal payoff function of the option.
3. In a cycle with respect to $s = N-1, N-2, \ldots, 0$,

   - first, extend V^{s+1} to a measurable bounded function on the real line and calculate

$$G^{0,s} = \mathcal{E}_{q_s}^{+}\mathcal{E}_{q_s}^{-} V^{s+1}, \quad G_{\pm}^{0,s} = G^{0,s},$$

   - next, for $n = 1, 2, \ldots,$ calculate

$$G_{+}^{n,s} = \mathcal{E}_{q_s}^{-}\mathbb{1}_{(-\infty,h_-]}(\mathcal{E}_{q_s}^{-})^{-1} G_{-}^{n-1,s}, \tag{3.2}$$

$$G_{-}^{n,s} = \mathcal{E}_{q_s}^{+}\mathbb{1}_{[h_+,+\infty)}(\mathcal{E}_{q_s}^{+})^{-1} G_{+}^{n-1,s}, \tag{3.3}$$

   - and then set, for $x \in (h_-, h_+)$,

$$V^s(x) = (1 + r\Delta_s)^{-1} \cdot \big(G^{0,s}(x) - G_{+}^{1,s}(x) - G_{-}^{1,s}(x) + G_{+}^{2,s}(x) + G_{-}^{2,s}(x) \\ - G_{+}^{3,s}(x) - G_{-}^{3,s}(x) + G_{+}^{4,s}(x) + G_{-}^{4,s}(x) - \cdots \big),$$

The function $V^0(x)$ obtained at the last step of this algorithm is the desired *Carr's randomization approximation* to the value function $V_{k.o.}(x, T; g)$ of the original double barrier option with deterministic maturity date T. As the mesh, $\max_s \Delta_s$, of the partition of the maturity period of the option approaches 0, the approximation $V^0(x)$ converges to $V_{k.o.}(x, T; g)$ (Boyarchenko 2008).

REMARK 3.2. In practice, we usually apply Carr's randomization to partitions where $\Delta_s = T/N$ for all $0 \le s \le N-1$, i.e., $t_s = sT/N$ for all s. The reason is that q_s then becomes independent of s, and usually it is computationally much more efficient to calculate the action of the operators $\mathcal{E}_q^{\pm}$ (and their inverses) on many different functions for a fixed value of q, rather than for varying values of q. This feature is responsible for the computational superiority of Carr's randomization over methods based on numerical Laplace inversion.

REMARK 3.3. At each step of the backward induction procedure sketched above, the function V^s is represented as a sum of an infinite series whose terms are expressed in terms of the function V^{s+1} obtained at the previous step. In practice, one must truncate this infinite series to arrive at a finite sum. Numerical examples show that, typically, one only has to keep very few (5–9) terms of the series to force the error of this truncation to become negligible.

4. EXPLICIT PRICING ALGORITHM

4.1. General Remarks

The method we will use to implement the pricing algorithm outlined in Section 3.3 uses the same technical tools as the methods we employed in Boyarchenko and Levendorskiĭ (2009a,b). In particular, in Section 4.2 we use the integral formulas (2.13) to calculate (approximately) the values of the Wiener-Hopf factors $\phi_q^{\pm}(\xi)$ (defined by (2.12)) on a suitable grid of points in $\mathbb{R}$, and in Section 4.3 we use these values to realize the normalized EPV operators $\mathcal{E}_q^{\pm}$ as certain (discretized) convolution operators. The inverse operators $(\mathcal{E}_q^{\pm})^{-1}$ that appear in the formulas of Section 3.3 will be realized numerically by explicitly calculating the inverses of the discretized forms of $\mathcal{E}_q^{\pm}$ in Section 4.4.

However, there exists a difference between the method of this paper and the methods of Boyarchenko and Levendorskiĭ (2009a,b) that is important from both the conceptual and the technical viewpoints. Namely, when one is dealing with an option that has a single barrier H, it suffices to work with a single uniformly spaced grid of points on the real line (it was referred to in Boyarchenko and Levendorskiĭ (2009a,b) as "*the* x-grid"). For example, if the option is a down-and-out barrier option, the x-grid has the form $h, h+\Delta, h+2\Delta, \dots, h+(M-1)\Delta$, where $h = \ln H$ and $\Delta > 0$. At each step in the backward induction procedure based on Carr's randomization for single barrier options, one only works with the arrays of values of various auxiliary functions at the points of this fixed x-grid.

On the other hand, a glance at the formulas of Section 3.3 will convince the reader that a single x-grid will not suffice for a numerical implementation of our pricing algorithm for *double barrier* options. Instead, we must work with five x-grids: the "main" one, which begins at $h_- = \ln H_-$ and ends at $h_+ = \ln H_+$ (this is the grid of points at which the values of the function $V_{k.o.}(x, T; g)$ will be calculated), and four longer "auxiliary" ones, which extend to the left and to the right of the points h_+ and h_-.

4.2. Calculation of the Wiener-Hopf Factors

In order to be able to implement the numerical realization of the operators $\mathcal{E}_q^{\pm}$ described in Section 4.3, we must first calculate the values of the functions $\phi_q^{\pm}(\xi)$ that appear in (2.12). Apart from a few special cases (such as the HEJDs Asmussen, Avram, and Pistorius 2004; Jeannin and Pistorius 2007; Boyarchenko 2006; Carr and Crosby 2008), no explicit formulas for $\phi_q^{\pm}(\xi)$ are known. Instead, one must use the integral formulas (2.13), and apply the procedures explained in Boyarchenko and Levendorskiĭ (2009a,b).

4.3. Two Numerical Realizations of the Operators $\mathcal{E}_q^{\pm}$

We begin by recalling the numerical realizations of the operators $\mathcal{E}_q^{\pm}$ that were employed in Boyarchenko and Levendorskiĭ (2009a,b). Formulas (2.11) represent $\mathcal{E}_q^{\pm}$ as operators of convolution with certain probability measures $p_q^{\pm}(dx)$ supported on the half-axes $[0, +\infty)$ and $(-\infty, 0]$, respectively. Away from 0, each of them is represented by an (infinitely differentiable and exponentially decaying) function $g_q^{\pm}(x)$, which can be found via Fourier inversion:

$$g_q^{\pm}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x}(\phi_q^{\pm}(\xi) - \phi_q^{\pm}(\infty))\, d\xi, \quad \pm x > 0, \tag{4.1}$$

where constants $\phi_q^{\pm}(\infty)$ can be easily calculated using Boyarchenko and Levendorskiĭ (2002, 3.76). With the notation of Section 2.4, we have $\phi_q^+(\infty) = p_q^+(\{0\}) = \mathbb{P}[\bar{X}_{T_q} = 0]$ and $\phi_q^-(\infty) = p_q^-(\{0\}) = \mathbb{P}[\underline{X}_{T_q} = 0]$. It follows that the action of the operators $\mathcal{E}_q^+$ and $\mathcal{E}_q^-$ on a bounded measurable function $f(x)$ can be written in the following way:

$$(\mathcal{E}_q^+ f)(x) = p_q^+(\{0\}) \cdot f(x) + \int_0^{+\infty} f(x+y) g_q^+(y)\, dy \tag{4.2}$$

and

$$(\mathcal{E}_q^- f)(x) = p_q^-(\{0\}) \cdot f(x) + \int_{-\infty}^{0} f(x+y) g_q^-(y)\, dy. \tag{4.3}$$

Here, $p_q^{\pm}(\{0\})$ are constants responsible for the *scalar components* of the operators $\mathcal{E}_q^{\pm}$ (the remaining summands in (4.2)–(4.3) are their *integral components*).

REMARK 4.1. It can happen in practice that $p_q^+(\{0\})$ or $p_q^-(\{0\})$ is nonzero. For instance, suppose X is a V.G. process, or a KoBoL process of order $\nu < 1$, with nonzero drift, $\mu \neq 0$. If $\mu > 0$, then $p_q^-(\{0\}) \neq 0$, and if $\mu < 0$, then $p_q^+(\{0\}) \neq 0$.

4.3.1. *First Numerical Realization of* $\mathcal{E}_q^+$. Let us consider a uniformly spaced grid of points $\vec{x} = (x_j)_{j=1}^M$ on the real line, where $x_j = x_1 + (j-1)\Delta$ for all $1 \le j \le M$, and $\Delta > 0$ is fixed. Given a function $f(x)$ whose values at the points of $\vec{x}$ are known, we would like to calculate approximately the values of $(\mathcal{E}_q^+ f)(x)$ at the same points.

To this end, we use the *enhanced* realization of convolution operators,[12] following the methods developed in Boyarchenko and Levendorskiĭ (2009a,b). Specifically, we approximate $f(x)$ with a piecewise linear function on the interval $[x_1, x_M]$ using the approximations

$$f(x) \approx f_j + \Delta^{-1} \cdot (f_{j+1} - f_j) \cdot (x - x_j), \quad x_j \le x \le x_{j+1}, \tag{4.4}$$

and we approximate $f(x)$ by 0 outside of $[x_1, x_M]$. As we saw in section 3.6.2 of Boyarchenko and Levendorskiĭ (2009a), this leads to the following approximation of the values of the function $(\mathcal{E}_q^+ f)(x)$:

$$(\mathcal{E}_q^+ f)(x_k) \approx -d_k^+ \cdot f_M + \sum_{j=k}^{M} c_{k-j}^+ \cdot f_j \quad (1 \le k \le M), \tag{4.5}$$

where $f_j = f(x_j)$ for $1 \le j \le M$,

$$d_k^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-M)\Delta\xi} \cdot (\phi_q^+(\xi) - \phi_q^+(\infty)) \cdot \frac{e^{-i\xi\Delta} + i\xi\Delta - 1}{(i\xi\Delta)^2}\, d\xi$$

for $1 \le k \le M$,

$$c_\ell^+ = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot (\phi_q^+(\xi) - \phi_q^+(\infty)) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2}\, d\xi$$

for $1 - M \le \ell \le -1$, and c_0^+ is equal to the constant $p_q^+(\{0\})$ appearing in (4.2). Probabilistic considerations suggest that in order to improve the accuracy of the approximation

[12] We remark that this idea goes back to A. Eydeland (Eydeland 1994; Eydeland and Mahoney 2001).

(4.5), the discretized form of the operator $\mathcal{E}_q^+$ should also act as an "expectation-type" operator. This means that once the coefficients c_ℓ^+ were found for $\ell \neq 0$ using the formulas above, one should set

$$c_0^+ = 1 - \sum_{1-M \leq \ell \leq -1} c_\ell^+ . \tag{4.6}$$

This observation, which is related to the possibility that the scalar component of $\mathcal{E}_q^+$ may be nonzero (cf. Remark 4.1), becomes especially relevant when one must apply the operator $\mathcal{E}_q^+$ multiple times in the course of a given calculation (as in Section 3.3). In such a situation, if (4.6) does not hold, the errors of the approximation (4.5) will necessarily accumulate over the course of the computation, which may lead to significant errors of the final result produced by the algorithm.

4.3.2. *First Numerical Realization of* $\mathcal{E}_q^-$. The enhanced numerical realization of $\mathcal{E}_q^-$ is obtained similarly by Boyarchenko and Levendorskiĭ (2009a, section 3.6.3). We let $\vec{x}$ and $f(x)$ be as above, and consider the same piecewise linear approximation to $f(x)$ as in Section 4.3.1. It leads to the following approximation of the values of the function $(\mathcal{E}_q^- f)(x)$:

$$(\mathcal{E}_q^- f)(x_k) \approx -d_k^- \cdot f_1 + \sum_{j=1}^{k} c_{k-j}^- \cdot f_j \quad (1 \leq k \leq M), \tag{4.7}$$

where $f_j = f(x_j)$,

$$d_k^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i(k-1)\Delta\xi} \cdot (\phi_q^-(\xi) - \phi_q^-(\infty)) \cdot \frac{e^{i\xi\Delta} - i\xi\Delta - 1}{(i\xi\Delta)^2} \, d\xi,$$

for $1 \leq k \leq M$,

$$c_\ell^- = \frac{\Delta}{2\pi} \int_{-\infty}^{\infty} e^{i\ell\Delta\xi} \cdot (\phi_q^-(\xi) - \phi_q^-(\infty)) \cdot \frac{e^{i\xi\Delta} + e^{-i\xi\Delta} - 2}{(i\xi\Delta)^2} \, d\xi$$

for $1 \leq \ell \leq M-1$, and c_0^- is the coefficient $p_q^-(\{0\})$ appearing in (4.3). As in Section 4.3.1, once the coefficients c_ℓ^- for $\ell \neq 0$ are found, we set

$$c_0^- = 1 - \sum_{1 \leq \ell \leq M-1} c_\ell^- . \tag{4.8}$$

4.3.3. *Second Numerical Realization of* $\mathcal{E}_q^+$. The numerical realization of the operator $\mathcal{E}_q^+$ described in Section 4.3.1 alone does not suffice for implementing the backward induction algorithm of Section 3.3. For example, the auxiliary function

$$f_+^{n-1,s} = \mathbb{1}_{[h_+,+\infty)} (\mathcal{E}_{q_s}^+)^{-1} G_+^{n-1,s} \tag{4.9}$$

is supported on $[h_+, +\infty)$ and its values are calculated at the points of a grid $x_j = h_+ + (j-1)\Delta$, $j = 1, 2, \ldots, M$; however, we would like to calculate the values $G_-^{n,s}(x) = (\mathcal{E}_{q_s}^+ f)(x)$ at points $x < h_+$.

To this end, we extend $f(x)$ by 0 for $x < h_+$, and we calculate the values of $(\mathcal{E}_q^+ f)(x)$ on the new[13] grid $(x_{j+1-M})_{j=1}^M$, where we extend the meaning of the notation x_j by setting $x_j = x_1 + (j-1)\Delta$ for all integers j.

The approximate formulas for the values of $(\mathcal{E}_q^+ f)(x)$ on the new grid can be easily obtained from the formulas of Section 4.3.1, so we simply present the final result:

$$(\mathcal{E}_q^+ f)(x_{k+1-M}) \approx \sum_{j=1}^{k} c_{k-j+1-M}^+ \cdot f_j \quad (1 \le k \le M), \tag{4.10}$$

where $f_j = f(x_j)$ and the coefficients c_ℓ^+ are calculated as in Section 4.3.1.

4.3.4. *Second Numerical Realization of $\mathcal{E}_q^-$.* Similarly to Section 4.3.3, we now consider the following problem. In the situation of Section 4.3.2, let us suppose that we know the values of $f(x)$ on the grid $\vec{x} = (x_j)_{j=1}^M$, where $x_M = h_-$. We extend f by 0 on $(h_-, +\infty)$, and calculate the values of $(\mathcal{E}_q^- f)(x)$ on the new[14] grid $(x_{j+M-1})_{j=1}^M$. This leads to the following approximation:

$$(\mathcal{E}_q^- f)(x_{k+M-1}) \approx \sum_{\ell=k}^{M} c_\ell^- \cdot f_{M+k-\ell} \quad (1 \le k \le M), \tag{4.11}$$

where $f_j = f(x_j)$ and the coefficients c_ℓ^- are calculated as in Section 4.3.1.

4.4. Calculation of the Inverses of the Discretized Versions of $\mathcal{E}_q^\pm$

As the last ingredient in the algorithm presented below, we consider the problem of calculating numerically the auxiliary functions of the form (4.9), which entails inverting the operators $\mathcal{E}_q^\pm$ in a suitable sense. We discovered that this problem has a solution that is both convenient and computationally efficient. Namely, we invert the *discretized* forms of the operators $\mathcal{E}_q^\pm$ that were described in Sections 4.3.1–4.3.2 above.

4.4.1. *The Inverse of the Discretized Form of $\mathcal{E}_q^+$.* The right-hand side of (4.5) can be viewed as an operator acting on the M-dimensional space of vectors $\vec{f} = (f_j)_{j=1}^M$ via

$$\mathcal{E}_{q,disc}^+ \vec{f} = c_0^+ \cdot \vec{f} + \mathcal{E}_{q,sub}^+ \vec{f}$$

where

$$(\mathcal{E}_{q,sub}^+ \vec{f})_k \stackrel{\text{def}}{=} -d_k^+ \cdot f_M + \sum_{j=k+1}^{M} c_{k-j}^+ \cdot f_j \quad (1 \le k \le M).$$

Numerical experiments show that in the situations that arise in practice, the scalar component of $\mathcal{E}_{q,disc}^+$, defined by the coefficient c_0^+, dominates the remaining term $\mathcal{E}_{q,sub}^+$,

[13] In other words, the new grid was obtained by shifting the old grid to the left, so that the endpoint of the new grid coincides with the initial point of the old grid.

[14] In other words, the new grid was obtained by shifting the old grid to the right, so that the initial point of the new grid coincides with the endpoint of the old grid.

which implies that the inverse of $\mathcal{E}^{+}_{q,disc}$ can be accurately computed. A direct calculation shows that the inverse is an operator of a similar type:

$$(\mathcal{E}^{+}_{q,disc})^{-1}(\vec{f})_k = \sum_{j=k}^{M} a^{+}_{k-j} \cdot (f_j + b^{+}_j f_M) \quad (1 \le k \le M), \tag{4.12}$$

where $b^{+}_j = d^{+}_j/(c^{+}_0 - d^{+}_1)$, $1 \le j \le M$, and the coefficients $a^{+}_{-\ell}$, for $0 \le \ell \le M-1$, can be found inductively using

$$a^{+}_0 = (c^{+}_0)^{-1}, \quad a^{+}_{-\ell} = -(c^{+}_0)^{-1} \cdot \sum_{j=1}^{\ell} c^{+}_{-j} a^{+}_{j-\ell} \quad (1 \le \ell \le M-1). \tag{4.13}$$

REMARK 4.2. We do not know of a way of calculating the coefficients $a^{+}_{-\ell}$ that is more efficient than doing it one step at a time, using (4.13). However, for the values of M that occur in practice, this calculation only takes a small fraction of a second.

4.4.2. *The Inverse of the Discretized Form of* $\mathcal{E}^{-}_q$. The right-hand side of (4.7) can be viewed as an operator acting on the M-dimensional space of vectors $\vec{f} = (f_j)^{M}_{j=1}$ via

$$(\mathcal{E}^{-}_{q,disc}\vec{f})_k = -d^{-}_k \cdot f_1 + \sum_{j=1}^{k} c^{-}_{k-j} \cdot f_j \quad (1 \le k \le M).$$

The obvious analogue of the comment appearing before formula (4.12) applies here as well, so the inverse of $\mathcal{E}^{-}_{q,disc}$ can be accurately calculated by means of the formula

$$(\mathcal{E}^{-}_{q,disc})^{-1}(\vec{f})_k = \sum_{j=1}^{k} a^{-}_{k-j} \cdot (f_j + b^{-}_j f_1) \quad (1 \le k \le M), \tag{4.14}$$

where $b^{-}_j = c^{-}_j/(c^{-}_0 - d^{-}_1)$, $1 \le j \le M$, and the coefficients a^{-}_{ℓ}, for $0 \le \ell \le M-1$, can be found inductively using

$$a^{-}_0 = (c^{-}_0)^{-1}, \quad a^{-}_{\ell} = -(c^{-}_0)^{-1} \cdot \sum_{j=1}^{\ell} c^{-}_j a^{-}_{\ell-j} \quad (1 \le \ell \le M-1). \tag{4.15}$$

5. NUMERICAL EXAMPLES

The calculations based on the algorithm of Section 4, the results of which are presented below, were performed in MATLAB© 7.3.0 (R2006b), on a PC with characteristics Intel® Core™ 2 Duo T7200 (2.00GHz, 4MB L2 Cache, 667MHz FSB), under the Genuine Windows® XP Professional operating system.

We assume that under a chosen EMM, the log-spot price, $X_t = \ln S_t$, of the underlying follows a KoBoL process (see Section 2.6(4)) with parameters $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$. (These parameters are taken from the examples that appear in Kudryavtsev and Levendorskiĭ 2009; Boyarchenko and Levendorskiĭ 2009a,b.) As in Boyarchenko

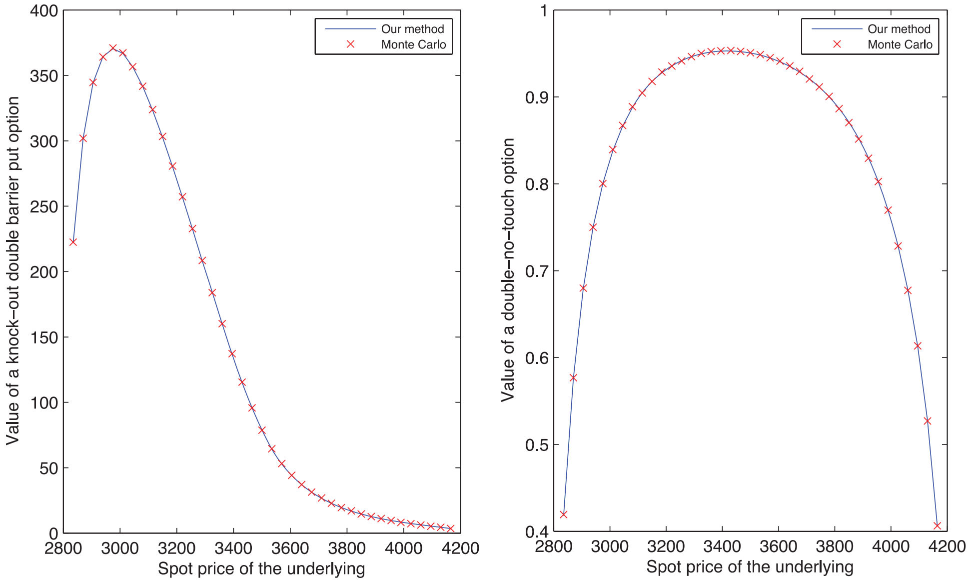

FIGURE 5.1. Prices of a knock-out double barrier put option (left panel) and of a double-no-touch option (right panel) in the KoBoL model. Solid lines represent the results obtained using the algorithm of Section 4. Crosses represent the results obtained using Monte Carlo simulations. *KoBoL parameters:* $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx -0.0423$. *Option parameters:* $K = 3{,}500$, $H_- = 2{,}800$, $H_+ = 4{,}200$, $r = 0.03$, $T = 0.1$. *Algorithm parameters:* $n = 812$ (number of points on the "main" x-grid), $\Delta = \frac{\ln H_+ - \ln H_-}{n-1} \approx 0.005$, $M = 4{,}096$, $M_2 = 4$, $M_3 = 16$, $\zeta_1 \approx 0.767$, $m = 8$ (for the calculation of the Wiener-Hopf factors), $N = 80$ (number of time steps), $\epsilon = 10^{-7}$ (error tolerance for the iterative procedure).

and Levendorskiĭ (2009a), we assume that the riskless rate is $r = 0.03$, which allows us to find the remaining parameter, $\mu \approx -0.0423$, from the EMM condition $\psi(-i) + r = 0$. For this market, we used the algorithm of Section 4 to compute the prices of a knock-out double barrier put option on the stock $S_t = e^{X_t}$, with strike price $K = 3,500$, lower barrier $H_- = 2,800$, upper barrier $H_+ = 4,200$, and maturity date $T = 0.1$ years. We also computed the prices of a double-no-touch option with the same parameters. We then compared our results with the results obtained by the Monte Carlo method. The results of our calculations are represented graphically in Figure 5.1, and are also recorded in Table 5.1 that appears after the list of references.

The auxiliary parameters of our algorithm are specified in the captions to the figure and to the table. The calculation of all the prices took a total of 14 seconds for each of the two types of options. The iterative procedure used at each step of Carr's randomization converged after just two iterations. For the calculations based on Monte Carlo simulations, we used 500,000 trajectories, with 20,000 time steps per year, i.e., 2,000 steps along each trajectory.

We observe that for a knock-out double barrier put option, the agreement between the price calculated using our algorithm and the Monte Carlo price is quite good. The discrepancy does not exceed 0.35% in the in-the-money region for the option, and mostly remains under 1% in the out-of-the-money region (with three exception of size less than

TABLE 5.1
Prices of a Knock-Out Double Barrier Put Option and of a Double-no-Touch Option in the KoBoL Model

Spot price	Knock-out double barrier put			Double-no-touch option		
	Our price	MC price	MC error	Our price	MC price	MC error
81%	221.7694	222.5448	0.0035	0.4179	0.4192	0.0029
82%	301.8436	301.8782	0.0001	0.5767	0.5768	0.0001
83%	344.3249	344.6685	0.0010	0.6792	0.6799	0.0011
84%	364.5144	364.1193	−0.0011	0.7498	0.7500	0.0003
85%	370.4507	370.9815	0.0014	0.8005	0.8004	−0.0002
86%	366.8240	367.2755	0.0012	0.8380	0.8394	0.0017
87%	356.5682	356.6483	0.0002	0.8663	0.8670	0.0009
88%	341.6168	341.7309	0.0003	0.8879	0.8887	0.0009
89%	323.3053	323.8835	0.0018	0.9047	0.9046	−0.0001
90%	302.5926	303.3371	0.0025	0.9177	0.9179	0.0002
91%	280.1974	280.7147	0.0018	0.9279	0.9287	0.0009
92%	256.6810	257.1714	0.0019	0.9358	0.9355	−0.0003
93%	232.5051	232.8017	0.0013	0.9418	0.9415	−0.0003
94%	208.0710	208.4755	0.0019	0.9463	0.9463	0.0000
95%	183.7520	183.8869	0.0007	0.9495	0.9501	0.0006
96%	159.9202	160.1593	0.0015	0.9516	0.9523	0.0006
97%	136.9744	137.2299	0.0019	0.9527	0.9531	0.0004
98%	115.3690	115.5238	0.0013	0.9529	0.9534	0.0005
99%	95.6461	95.7481	0.0011	0.9522	0.9524	0.0003
100%	78.4371	78.6724	0.0030	0.9506	0.9508	0.0002
101%	64.2703	64.5565	0.0045	0.9482	0.9487	0.0006
102%	53.0395	53.2834	0.0046	0.9448	0.9451	0.0003
103%	44.1569	44.2211	0.0015	0.9405	0.9413	0.0009
104%	37.0615	37.1324	0.0019	0.9351	0.9357	0.0006
105%	31.3298	31.3603	0.0010	0.9286	0.9294	0.0009
106%	26.6506	26.9603	0.0116	0.9207	0.9207	0.0000
107%	22.7940	22.7653	−0.0013	0.9113	0.9115	0.0002
108%	19.5879	19.5361	−0.0026	0.9001	0.9005	0.0004
109%	16.9017	17.0190	0.0069	0.8868	0.8865	−0.0003
110%	14.6345	14.5777	−0.0039	0.8710	0.8703	−0.0007
111%	12.7073	12.6848	−0.0018	0.8521	0.8515	−0.0007
112%	11.0571	11.1604	0.0093	0.8296	0.8295	−0.0001
113%	9.6328	9.5769	−0.0058	0.8023	0.8026	0.0003
114%	8.3919	8.2587	−0.0159	0.7693	0.7696	0.0005
115%	7.2973	7.1954	−0.0140	0.7286	0.7285	−0.0002
116%	6.3145	6.2712	−0.0069	0.6778	0.6773	−0.0008
117%	5.4057	5.2579	−0.0273	0.6130	0.6134	0.0007
118%	4.5180	4.5381	0.0045	0.5274	0.5270	−0.0007
119%	3.5337	3.5406	0.0020	0.4062	0.4063	0.0003

Note: The first column contains the spot price as a percentage of 3,500. The errors reported in columns 4 and 7 are the relative errors. If V_{MC} denotes the Monte Carlo price of an option, and V denotes the price obtained using our algorithm, the relative error is defined as $(V_{MC} - V)/V$.
KoBoL parameters: $\nu = 0.5$, $c = 1$, $\lambda_+ = 9$, $\lambda_- = -8$, $\mu \approx -0.0423$.
Option parameters: $K = 3{,}500$ (for the double barrier put), $H_- = 2{,}800$, $H_+ = 4{,}200$, $r = 0.03$, $T = 0.1$.
Algorithm parameters: $n = 812$ (number of points on the "main" x-grid), $\Delta = \frac{\ln H_+ - \ln H_-}{n-1} \approx 0.005$, $M = 4{,}096$, $M_2 = 4$, $M_3 = 16$, $\zeta_1 \approx 0.767$, $m = 8$ (for the calculation of the Wiener-Hopf factors), $N = 80$ (number of time steps), $\epsilon = 10^{-7}$ (error tolerance for the iterative procedure).

1.6%, and one about 2.7%). For a double-no-touch option, the agreement is even better: the discrepancy does not exceed 0.2% with the only exception 0.29%, and mostly remains under 0.1%, with only a couple of exceptions.

REFERENCES

ASMUSSEN, S., F. AVRAM, and M. PISTORIUS (2004): Russian and American Put Options under Exponential Phase-type Lévy Models, *Stoch. Process. Appl.* 109(1), 79–111.

BARNDORFF-NIELSEN, O. (1998): Processes of Normal Inverse Gaussian Type, *Finance Stoch.* 2, 41–68.

BARNDORFF-NIELSEN, O., and S. LEVENDORSKIĬ (2001): Feller Processes of Normal Inverse Gaussian Type, *Quantit. Finance* 1, 318–331.

BERTOIN, J. (1996): *Lévy Processes*, Cambridge Tracts in Mathematics, Vol. 121, Cambridge, UK: Cambridge University Press.

BOYARCHENKO, M. (2008): Carr's Randomization for Finite-Lived Barrier Options: Proof of Convergence, Working Paper. Available at SSRN: http://papers.ssrn.com/abstract=1275666.

BOYARCHENKO, M., and S. LEVENDORSKIĬ (2009a): Prices and Sensitivities of Barrier and First-touch Digital Options in Lévy-driven Models, *Int. J. Theor. Appl. Finance* 12(8), 1125–1170.

BOYARCHENKO, M., and S. LEVENDORSKIĬ (2009b): Refined and Enhanced Fast Fourier Transform Techniques, with an Application to the Pricing of Barrier Options, Working Paper. Available at SSRN: http://papers.ssrn.com/abstract=1142833.

BOYARCHENKO, S. (2006): Two-point Boundary Problems and Perpetual American Strangles in Jump-diffusion Models, Working Paper. Available at SSRN: http://ssrn.com/abstract=896260.

BOYARCHENKO, S., and S. LEVENDORSKIĬ (2000): Option Pricing for Truncated Lévy Processes, *Int. J. Theor. Appl. Finance* 3(3), 549–552.

BOYARCHENKO, S., and S. LEVENDORSKIĬ (2002): *Non-Gaussian Merton-Black-Scholes Theory*, Adv. Ser. Stat. Sci. Appl. Probab., Vol. 9, River Edge, NJ: World Scientific Publishing Co.

CARR, P. (1998): Randomization and the American Put, *Rev. Finan. Stud.* 11(3), 597–626.

CARR, P., and J. CROSBY (2008): A Class of Lévy Process Models with Almost Exact Calibration to Both Barrier and Vanilla FX Options, Working Paper. Available at: http://www.john-crosby.co.uk/.

CARR, P., and D. FAGUET (1996): Valuing Finite-Lived Options as Perpetual, Working Paper. Available at SSRN: http://ssrn.com/abstract=706 or DOI: http://dx.doi.org/10.2139/ssrn.706.

CARR, P., H. GEMAN, D. MADAN, and M. YOR (2002): The Fine Structure of Asset Returns: An Empirical Investigation, *J. Business* 75, 305–332.

CARR, P., and D. MADAN (1999): Option Valuation Using the Fast Fourier Transform, *J. Computat. Finance* 2(4), 61–73.

DOUADY, R. (1999): Closed Form Formulas for Exotic Options and Their Lifetime Distribution, *Int. J. Theor. Appl. Finance* 2(1), 17–42.

ESKIN, G. (1981): *Boundary Value Problems for Elliptic Pseudodifferential Equations*, Transl. Math. Monogr., Vol. 9, Providence, RI: American Mathematical Society.

EYDELAND, A. (1994): A Fast Algorithm for Computing Integrals in Function Spaces: Financial Applications, *Computat. Econ.* 7, 277–285.

EYDELAND, A., and D. MAHONEY (2001): The Grid Model for Derivative Pricing, Mirant Technical Report.

FENG, L., and V. LINETSKY (2008): Pricing Discretely Monitored Barrier Options and Defaultable Bonds in Lévy Process Models: A Fast Hilbert Transform Approach, *Math. Finance* 18(3), 337–384.

GEMAN, H. (2001): Time Changes, Laplace Transforms and Path-dependent Options, *Computat. Economics* 17(1), 81–92.

GEMAN, H., and M. YOR (1996): Pricing and Hedging Double-barrier Options: A Probabilistic Approach, *Math. Finance* 6(4), 365–378.

HUI, C. (1996): One-touch Double Barrier Binary Option Values, *Appl. Finan. Econ.* 6(4), 343–346.

HUI, C., C. LO, and P. YUEN (2000): Comment on "Pricing Double Barrier Options Using Laplace Transforms" by Antoon Pelsser, *Finance Stoch.* 4(1), 105–107.

JEANNIN, M., and M. PISTORIUS (2007): A Transform Approach to Calculate Prices and Greeks of Barrier Options Driven by a Class of Lévy Processes, King's College London and Nomura Models and Methodology Group Preprint. Available at: http://arxiv.org/abs/0812.3128.

KADANKOVA, T., and N. VERAVERBEKE (2007): On Several Two-boundary Problems for a Particular Class of Lévy Processes, *J. Theor. Prob.* 20(4), 1073.

KOLKIEWICZ, A. (2002): Pricing and Hedging More General Double-barrier Options, *J. Computat. Finance* 5(3), 1–26.

KOPONEN, I. (1995): Analytic Approach to the Problem of Convergence of Truncated Lévy Flights towards the Gaussian Stochastic Process, *Phys. Rev. E* 52, 1197–1199.

KOU, S. (2002): A Jump-diffusion Model for Option Pricing, *Manage. Sci.* 48(8), 1086–1101.

KOU, S., and H. WANG (2003): First Passage Times of a Jump Diffusion Process, *Adv. Appl. Prob.* 35(2), 504–531.

KOU, S., and H. WANG (2004): Option Pricing under a Double Exponential Jump Diffusion Model, *Manage. Sci.* 50(9), 1178–1192.

KUDRYAVTSEV, O., and S. LEVENDORSKIĬ (2006): Pricing of First Touch Digitals under Normal Inverse Gaussian Processes, *Int. J. Theor. Appl. Finance* 9(6), 915–950.

KUDRYAVTSEV, O., and S. LEVENDORSKIĬ (2009): Fast and Accurate Pricing of Barrier Options under Lévy Processes, *Finance Stoch.* 13(4), 531–562.

KUNIMOTO, N., and M. IKEDA (1992): Pricing Options with Curved Boundaries, *Math. Finance* 2(4), 275–298.

KUZNETSOV, A. (2010): Wiener-Hopf Factorization and Distribution of Extrema for a Family of Lévy Processes, *Ann. Appl. Prob.* 20(5), 1801–1830.

LEVENDORSKIĬ, S. (2004): Pricing of the American Put under Lévy Processes, *Int. J. Theor. Appl. Finance* 7(3), 303–335.

LEWIS, A., and E. MORDECKI (2008): Wiener-Hopf Factorization and Distribution of Extrema for a Family of Lévy Processes, *Ann. Appl. Prob.* 45, 118–134.

LI, A. (1999): The Pricing of Double Barrier Options and Their Variations, *Adv. Futures Options Res.* 10, 17–41.

LIPTON, A. (2002a): Assets with Jumps, *Risk*, 149–153.

LIPTON, A. (2002b): Path-dependent Options on Assets with Jumps, 5th Columbia-Jaffe Conference. Available at http://www.math.columbia.edu/~lrb/columbia2002.pdf.

LUO, L. (2001): Various Types of Double-barrier Options, *J. Computat. Finance* 4(3), 125–138.

MADAN, D., P. CARR, and E. CHANG (1998): The Variance Gamma Process and Option Pricing, *Eur. Finance Rev.* 2, 79–105.

MADAN, D., and F. MILNE (1991): Option Pricing with V.G. Martingale Components, *Math. Finance* 1(4), 39–55.

MADAN, D., and E. SENETA (1990): The Variance Gamma (V.G.) Model for Share Market Returns, *J. Business* 63, 511–524.

MERTON, R. (1976): Option Pricing When Underlying Stock Returns Are Discontinuous, *J. Finan. Econ.* 3, 125–144.

PELSSER, A. (2000): Pricing Double Barrier Options Using Laplace Transforms, *Finance Stoch.* 4(1), 95–104.

ROGERS, L., and D. WILLIAMS (1994): *Diffusions, Markov Processes, and Martingales*. Vol. 1: Foundations, 2nd ed., Chichester: John Wiley & Sons, Ltd.

ROGOZIN, B. (1972): The Distribution of the First Hit for Stable and Asymptotically Stable Walks on an Interval, *Theor. Probab. Appl.* 17, 332–338.

SATO, K. (1999): *Lévy Processes and Infinitely Divisible Distributions*, Cambridge Stud. Adv. Math., Vol. 68, Cambridge: Cambridge University Press.

SCHRÖDER, M. (2000): On the Valuation of Double-barrier Options: Computational Aspects, *J. Computat. Finance* 3(4), 5–33.

SEPP, A. (2004): Analytical Pricing of Double-barrier Options under a Double-exponential Jump Diffusion Process: Applications of Laplace Transform, *Int. J. Theor. Appl. Finance* 7(2), 151–175.

SIDENIUS, J. (1998): Double Barrier Options: Valuation by Path Counting, *J. Comput. Finance* 1(3), 63–79.