Holographic Wilson Loops and Fermions in Consistent Truncations of String Theory and M-Theory

by

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To Javiera, + 1.

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CHAPTER I

Introduction

Our current understanding of elementary particle physics is based on the principles of relativistic quantum field theory and the concept of gauge symmetry. Indeed, three of the fundamental forces of nature, namely, the electromagnetic interaction, the weak force and the strong force, are triumphantly described by a unified quantum field theory, called the Standard Model, that exhibits spontaneously broken $SU(3) \times SU(2) \times U(1)$ gauge invariance. With the advent of the LHC era, new particles will be discovered, and it is reasonable to expect that, at the energy scales being probed, their properties will be explained in a framework not too different from the Standard Model.

Despite their great success, gauge theories are incapable of accommodating gravity, the fourth of the fundamental interactions. A completely satisfactory explanation of quantum gravity still evades modern physics, yet considerable progress has been made with the development of String/M Theory. Originally proposed as an attempt to describe certain features of hadronic physics, String Theory eventually revealed itself as an attractive candidate for the realization of Einstein's dream of a grand unified theory. In remarkable and unexpected ways, it manages to reconcile the principles of quantum mechanics and General Relativity. In modern language, we like to say that String Theory is an ultraviolet complete theory of gravity.

One of the most revolutionary ideas introduced in the framework of quantum gravity is holography. Inspired by black hole thermodynamics, the holographic principle states that a theory of quantum gravity can be represented by a quantum field theory defined in the boundary of the space where the gravitational theory lives. Conversely, given certain assumptions, a quantum field theory can be equivalently described by a gravity dual formulated in a suitable space.

The best understood realization of the holographic principle is the Anti-de Sitter (AdS) / Conformal Field Theory (CFT) correspondence [2, 37, 74, 114, 154], which conjectures the equivalence between String Theory on $AdS_5 \times S^5$ and $d = 4$, $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory. The first is a theory of quantum gravity defined in a ten-dimensional space, while the second is a superconformal quantum field theory that lives in four dimensions and contains no gravitational degrees of freedom. This is a remarkably deep insight given the seemingly very different nature of the two sides of the duality, and it is worth while understanding it in more detail.

A simple test we can run to see if such a correspondence is possible, is to verify that the symmetries of the two theories coincide. Consider first the string theory part of the conjecture. Both, the AdS_5 manifold and the 5-sphere can be defined as embeddings in six-dimensional flat space:

$$\eta_{\mu\nu} x^\mu x^\nu = L^2 \,, \qquad x^\mu \in \mathbb{R}^6 \,, \tag{I.1}$$

where $\eta_{\mu\nu} = \mathrm{diag}\,(-,-,+,+,+,+)$ for AdS_5 and $\eta_{\mu\nu} = \mathrm{diag}\,(1,1,1,1,1,1)$ for S^5. The solution is supported by N units of a self-dual 5-form flux across S^5. In the original construction of [114], N represents the number of D3-branes. The radii of the spaces are determined by the type IIB supergravity equations of motion to be $L^4/\alpha'^2 = 4\pi g_s N$, where g_s is the string coupling constant and α'^2 is the string length. It is clear from this description that $AdS_5 \times S^5$ exhibits an $SO(4,2) \times SO(6)$ isometry. When the supersymmetries of theory are taken into account, this group is augmented to the supergroup $SU(2,2|4)$. String Theory on this background necessarily displays these symmetries. On the other hand, $d = 4$, $\mathcal{N} = 4$ SYM is a superconformal field theory. The conformal group in four dimensions, that is, the group of transformations that leave the Minkowski metric invariant up to an overall factor, is precisely $SO(4,2)$. Being supersymmetric, the theory also possesses and R-symmetry given by $SU(4) \simeq SO(6)$. The full superconformal group is again

$SU(2,2|4)$. Thus, the global symmetries of the two theories are in exact agreement. The duality also identifies the parameters on both sides by $g_{YM}^2 = 4\pi g_s$, where g_{YM} is the coupling constant of the gauge theory. The rank of the gauge group $SU(N)$ is determined by the 5-form flux in the string theory description. Equivalently, we can write

$$\frac{L^4}{\alpha'^2} = \lambda\,, \tag{I.2}$$

with $\lambda \equiv g_{YM}^2 N$ being the 't Hooft coupling. The AdS/CFT correspondence goes far beyond this analysis of global symmetries, boldly stating that the two sides are actually fully equivalent as quantum theories.

Several generalizations of the AdS/CFT correspondence have been proposed, with much of the initial work directed towards finding gravity duals of theories that display confinement and chiral symmetry breaking [102–104, 118, 156]. But perhaps more strikingly, given the string-theoretical origin of the correspondence, holography is proving to be useful in describing some of the most interesting condensed matter and atomic systems, such as superconductors and non-fermi liquids [85, 92, 95, 97, 109]. This line of inquiry is a major driving force in String Theory explorations today.

Traditionally, holography has been implemented in the limit where the number of fields N involved the gauge theory is very large and the 't Hooft coupling λ is kept fixed. On the gravity side, this translates to the fact the string coupling constant g_s is small and non-perturbative effects can be neglected. A further simplification arises when we consider the large λ limit. This can be seen as taking the radius of curvature L large enough so that General Relativity is a good approximation and a consistent quantum gravitational description is not needed. For this reason, the AdS/CFT correspondence and its extensions are particularly powerful at tackling non-perturbative questions about gauge theory dynamics; problems in strongly coupled field theory are mapped to problems in classical gravity. The latter are usually much more tractable than the former. This weak/strong nature of the duality is particularly useful, for it provides us with a robust tool to study a range of issues that are not accessible to standard perturbative techniques in quantum field theory. This is one of the main reasons why the correspondence has received so much attention in the community.

A natural question to ask is whether or not the AdS/CFT correspondence renders predictions that are correct at the quantum level, away from the large N and large λ regimes, where stringy effects are important. An analysis of quantum corrections to classical results is the way to validate our assumptions about the duality.

As we shall describe below, a particularly well-suited framework to study the problem of $1/N$ corrections in the AdS/CFT correspondence is provided by Wilson loop operators. This important class of non-local objects were originally introduced in gauge theories as order parameters to describe confinement in models like QCD. Even in cases that do not display confinement, Wilson loop operators can be used as variables to reformulate the theory in a manifestly gauge-invariant way. Generally speaking, a Wilson loop is non-local operator of the form

$$W_R\,(C) = \mathrm{Tr}_R\, P \exp\left(i\int_C A\right)\,, \tag{I.3}$$

where C is a curve in spacetime and R labels a representation of the gauge group, typically $SU(N)$. Mathematically, $W_R(C)$ is the trace of the holonomy matrix associated to parallel transport in a principal bundle. Physically, the expectation value of this operator measures the phase or effective action of a particle of charge R transported around the loop. In particular, if the path C is chosen as two infinite antiparallel lines, as shown in figure 1.1, the Wilson loop can be understood as the effective quark-antiquark potential in the theory, the behavior of which encodes the possibility of confinement.

Given their prominent role in this context, following the arrival of the AdS/CFT correspondence, the question of how to holographically describe Wilson loops naturally arose. Early investigations [44, 116, 140] revealed that half-BPS operators in $\mathcal{N}=4$ SYM, i.e. the circle and the infinite line, in the fundamental representation of $SU(N)$ have a dual description in terms of a string with an AdS_2 worldsheet that pinches the boundary of AdS_5 along the loop. To cap-

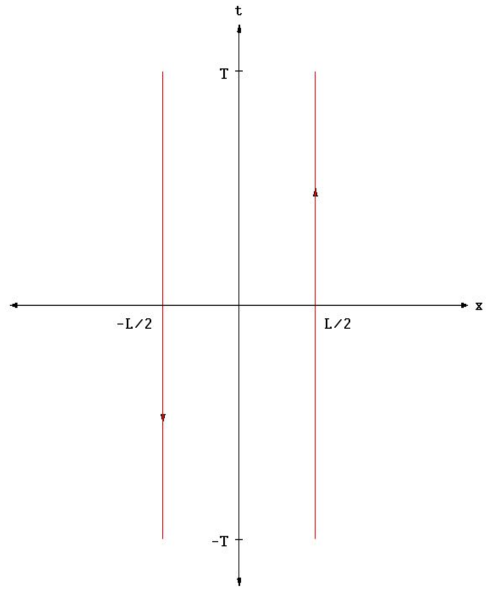

Figure 1.1: Infinite, antiparallel lines, representing the worldlines of a quark-antiquark pair. For $T \to \infty$, the expectation value of the Wilson loop goes like $\langle W \rangle \sim e^{-TV(L,\lambda)}$, where $V(L,\lambda)$ is the effective quark-antiquark potential.

ture more general representations one must consider instead D3 or D5-branes in $AdS_5 \times S^5$ with worldvolume flux [39, 71, 157]. In these cases, the classical supersymmetric solutions correspond to $AdS_2 \times S^2$ and $AdS_2 \times S^4$. One of the main objectives of this thesis is to carry out a systematic study of corrections to the expectation value of half-BPS Wilson loops in $\mathcal{N} = 4$ SYM, thus taking the AdS/CFT correspondence beyond the analysis of ground state solutions. We shall do so by studying the spectrum of excitations of probe D3 and D5-branes dual to operators in the symmetric and antisymmetric representation of $SU(N)$, respectively. The idea is to compute the corrections that arise due to these fluctuations and compare the results to the computation done in the gauge theory description. This task is aided by the fact that, as shown in the seminal work of Erickson-Semenoff-Zarembo and Drukker-Gross [42, 49], the circular Wilson loop is described exactly, to all orders in N and λ, by a Gaussian matrix model.

The other major topic addressed in this thesis is that of consistent truncations of String Theory and M Theory or, more precisely, their low energy limits, type IIB and 11d supergravities, respectively. This is a rich subject on its own, but the main motivation for us stems from holography and its applications to condensed matter systems. Given that the supergravity theories live ten and eleven dimensions, it is of extreme importance to understand exactly how to properly extract realistic lower dimensional physics. A general feature of gauge/gravity dualities is that a d-dimensional field theory is described, at least in the strong coupling regime, by a gravitational theory in $d+1$ dimensions. In order to characterize the relevant three or four-dimensional physics using holography, it is therefore necessary that the gravitational theory live in four or five dimensions.

The process of taking a higher dimensional theory and decomposing it in terms of lower dimensional variables is called compactification. The basic idea is to assume that the total space has the product structure $M \times Y$, where M is the lower dimensional spacetime and Y is some internal manifold, assumed to be compact. One can then decompose the fields generically as

$$\phi(x,y) = \sum_n \phi_n(x) f_n(y) \,, \tag{I.4}$$

where x are coordinates on M, y denote coordinates on Y, and $f_n(y)$ is a complete set of harmonic functions on the internal space. The fields ϕ_n are then interpreted as physical variables in a theory that lives exclusively on M. A prominent example is the compactification of type IIB supergravity on S^5 [100].

In principle, by keeping the full tower of fields, this description is equivalent to the original higher dimensional theory. However, in practice, one would like to truncate the theory and retain only a finite set of fields. In this context, a consistent truncation is defined as reduction of a higher dimensional theory to a lower dimensional one, such that any solution of the later can always be uplifted to a solution of the former. In this thesis we specifically address the reduction of fermionic fields in a recently found class of consistent truncations of type IIB and eleven-dimensional supergravity on so called squashed Sasaki-Einstein manifolds. The mathematical details of these spaces will be left to the appropriate chapters. For now, it will suffice to say that they posses enough structure to allow for the decomposition of the higher dimensional fields in terms of a finite set of functions that are singlets under a certain symmetry group, thus guaranteeing the consistency of the reduction.

A noteworthy property of the class of truncations we focus on is that they retain charged, massive bosonic modes, making them relevant for the study of holographic superconductors. Our work exhibits the couplings of fermionic fields to these modes, allowing for the study of fermion correlators in the presence of condensates. In general, embedding phenomenologically desirable supergravity backgrounds in String/M Theory is of great importance in order to shed light on the existence of UV complete theories dual to condensed matter systems. Moreover, as we have done in this thesis, a "top-down" approach to the construction of such models usually fixes many of the parameters that are introduced by hand in "bottom-up" constructions.

This thesis is organized as follows. In chapter II we tackle the problem of open string fluctuations of the D3-brane configuration dual to the supersymmetric Wilson loop in the symmetric representation of $SU(N)$. In particular, we review the classical solution and find its full spectrum of excitations. By looking at bosonic and fermionic modes explicitly, we show that the spectrum fits nicely into multiplets of $OSp(4^*|4)$, the superalgebra preserved by the solution. We also revisit the case of the fundamental string and provide a general picture for the holographic excitations of Wilson loops in arbitrary representations. Some final remarks and open problems close the chapter.

As a logical continuation, Chapter III addresses in full detail the excitations of a large class of D5-brane solutions that wrap an $S^4 \subset S^5$. We derive the spectrum of excitations, both in the bosonic and fermionic sectors. Specializing to the solution dual to the Wilson loop in the antisymmetric representation of $SU(N)$, its supersymmetric structure is displayed and find agreement with the framework expected from the previous chapter. We then take the analysis of D-brane fluctuations a step further and compute the effective action due to the excitations using heat kernel techniques. We conclude with some comments on future work.

In chapter IV we switch to the subject of dimensional reduction of fermions in eleven-dimensional supergravity. We review the bosonic ansatz in a general class of consistent truncations on 5d squashed Sasaki-Einstein manifolds and motive the reduction ansatz for the gravitino. Following this, we reduce the theory down to four dimensions and obtain the corresponding equations of motion and effective action. A redefinition is fields is necessary to diagonalize the kinetic terms, which we display in detail. In particular, we explicitly display the couplings of the fermionic fields to all the bosonic modes present in the theory. As expected from the general structure of the truncation, we argue that the lower dimensional is properly accommodated in the framework of $d = 4$, $\mathcal{N} = 2$ gauged supergravity. Finally, we discuss some further truncations and examples pertinent to condensed matter applications, and give some final remarks.

Following the same spirit as in chapter IV, we devote V to the study of fermions on consistent truncations of type IIB supergravity on squashed Sasaki-Einstein spaces. The bosonic and fermionic ansatz are discussed in detail to then compute the lower dimensional effective action and equations of motion. As before, the couplings of all modes are written explicitly. In this case, the reduction yields a structure consistent with $d = 5$, $\mathcal{N} = 4$ gauged supergravity. We move on to address further truncations that include, among others, the holographic $3 + 1$ superconductor. The chapter concludes with a summary and possible further studies.

Lastly, the final chapter summarizes the main results achieved in the thesis and explores open questions for new avenues of research.

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[51] A. Faraggi, W. Mück, L. A. Pando Zayas, "One-loop Effective Action of the Holographic Antisymmetric Wilson Loop," submitted to Physical Review D.

[53] A. Faraggi, L. A. Pando Zayas, "The Spectrum of Excitations of Holographic Wilson Loops," JHEP **1105**, 018 (2011).

[9] I. Bah, A. Faraggi, J. I. Jottar, R. G. Leigh, "Fermions and Type IIB Supergravity On Squashed Sasaki-Einstein Manifolds," JHEP **1101**, 100 (2011).

[10] I. Bah, A. Faraggi, J. I. Jottar, R. G. Leigh, L. A. Pando Zayas, "Fermions and $D = 11$ Supergravity On Squashed Sasaki-Einstein Manifolds," JHEP **1102**, 068 (2011).

CHAPTER II

D3-branes and holographic Wilson loops

In the holographic framework, a half BPS Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills in the fundamental, symmetric or antisymmetric representation of $SU(N)$, is best described by a fundamental string, a D3-brane or a D5-brane with fluxes in their worldvolumes, respectively. In this chapter, we derive the spectrum of excitations of such D3-brane in $AdS_5 \times S^5$ explicitly, considering its action in both the bosonic and the fermionic sectors, and demonstrate that it is organized according to short multiplets of the supergroup $OSp(4^*|4)$. We also show that the modes of the fundamental string form an ultra-short multiplet of this supergroup. This way we provide a step towards a unifying picture for the description of holographic excitations of the circular and straight supersymmetric Wilson loops in arbitrary representations.

2.1 Introduction

Wilson loops are important gauge invariant operators in gauge theories. It is possible to reformulate the theory in terms of these nonlocal operators and they also serve as useful order parameters. In the context of the AdS/CFT correspondence Wilson loops were first formulated by Maldacena [117] and Rey-Yee [141]. The prescription was to identify the expectation value of Wilson loops with the action of a fundamental string in the dual supergravity background; Rey-Yee already mentioned the relevance of D-branes with worldvolume fluxes as potential decorating parameters of the Wilson loop.

A particularly important role is played by supersymmetric Wilson loops, most notably, the circular Wilson loop [45]. Its expectation value was conjectured to be computed exactly via a Gaussian matrix model in [43, 50], with a later rigorous proof appearing in [135].

A nontrivial decoration of the circular Wilson loop is the representation of the gauge group. We have by now a good understanding of the fundamental, symmetric and the antisymmetric representations from the holographic point of view [40, 45] and its stringy origin [72, 73]. The work of [40] focused on the D3 which is dual to the Wilson loop in the symmetric representation. By analogy with the giant graviton argument, Yamaguchi developed the case of a D5 brane wrapping S^4 in $AdS_5 \times S^5$ and identified it with the description of a Wilson loop in the antisymmetric representation [158]. Interestingly, using the Gaussian matrix model it was possible to confirm the finer structure of the representations [69, 88, 158, 160].

There is a very interesting characterization depending on the value of k, the number of boxes in the Young tableau of $SU(N)$, relative to N in the large N limit. When k is order one, that is, $k \ll N$, the Wilson loop is effectively described, on the gravity side, by k fundamental strings. For k/N fixed, the most fitting holographic description is that of probe branes with fluxes. Finally, there could also be Wilson loops in representations with $k \sim N^2$; these are almost square Young tableaux. In this case the probe approximation is no longer valid and a fully backreacted supergravity background must be constructed. Such construction has been carried out in some simple cases in [36, 112, 159] and further refined in [70, 131].

Of paramount importance is the computation of quantum corrections to the given expectations values. In a sense, this is the act of taking the AdS/CFT correspondence beyond the comparison of classical ground state configurations. This is the equivalent to high-precision spectroscopy, that is, an analysis of the quantum corrections is the way to validate our assumptions about the correspondence. Indeed, corrections to the circular Wilson loop dual to the fundamental string have been computed in various works [46, 56, 105, 146]. A prescription for computing correlators of Wilson loops with chiral primaries and with another Wilson loop was developed in the early stages of the AdS/CFT correspondence [15]. This prescription was beautifully applied in the case of symmetric and antisymmetric Wilson loops with chiral primaries and was shown to coincide with the calculation from the matrix model in [69].

In this paper our goal is to go beyond the "ground state" analysis of Wilson loops and study the excitations on the gravity side. We study the holographic description of the half BPS Wilson loop in the symmetric representation, that is, a D3 brane configuration in $AdS_5 \times S^5$ whose worldvolume is $AdS_2 \times S^2$. We explicitly compute the bosonic and fermionic fluctuations starting from the action for such a D3 brane as worked out by Martucci and collaborators [120, 121, 123–125]. After this explicit calculation we fit the spectrum of excitations into short multiplets of $OSp(4^*|4)$. Emboldened by our success in the explicit case of the D3 brane, we go on and fit the excitations of the fundamental string and the D5 brane found in the literature into short multiplets of $OSp(4^*|4)$. Thus, using mostly its symmetries, we present the spectrum of excitations of the holographic description of half BPS Wilson loops.

According to the construction in [73], a Wilson loop in an arbitrary representation of $SU(N)$ has a holographic description in terms of coincident D3 branes or, alternatively, coincident D5 branes. A complete analysis of the spectrum then requires dealing with the non-Abelian nature of the corresponding low energy action. The case of a single D3 brane corresponds to a Young tableau with one row.

The chapter is organized as follows. In section 2.2 we review the classical D3 brane configuration. Section 2.3 presents an explicit computation of the spectrum of excitations, both in the bosonic and fermionic sectors. Section 2.4 contains a discussion of the supersymmetric aspects of the spectrum. In section 2.5 we review the status of the excitations of the string and the D5 brane configurations and fit the respective spectra into representations of $OSp(4^*|4)$. We conclude in section with some open problems. We have relegated questions of conventions and more explicit calculations to a series of appendices.

2.2 Review of background geometry and D3-brane configuration

In this section we briefly review the classical D3 brane configuration that describes the BPS Wilson loops we are interested in. It was first introduced in [40]. Throughout the chapter we will work exclusively in Lorentzian signature. See Appendix A for notation and conventions.

2.2.1 $AdS_5 \times S^5$ background

The $AdS_5 \times S^5$ type IIB background is described by a metric and a RR 5-form given by

$$ds^2 = ds^2_{AdS_5} + L^2 d\Omega_5^2\,, \tag{II.1}$$

$$F_5 = -\frac{4}{L}\left(1 + *\right) \mathrm{vol}\left(AdS_5\right)\,. \tag{II.2}$$

Both the AdS_5 space and the 5-sphere have radius L. Since the configuration we study in this thesis is described by a D3-brane with $AdS_2 \times S^2$ worldvolume, it is convenient to introduce coordinates that make this structure manifest. Following [158], we consider a foliation of AdS_5 of the form

$$ds^2_{AdS_5} = L^2\left(\cosh^2(u) ds^2_H + \sinh^2(u) d\Omega_2^2 + du^2\right)\,, \tag{II.3}$$

where ds^2_H is the unit AdS_2 metric. It is clear that the induced geometry on the hypersurface $u = u$ corresponds to $AdS_2 \times S^2$. In these coordinates the volume form reads

$$\mathrm{vol}\left(AdS_5\right) = L^5 \cosh^2(u) \sinh^2(u)\, \mathrm{vol}\left(AdS_2\right) \wedge \mathrm{vol}\left(S^2\right) \wedge du\,. \tag{II.4}$$

Then, the corresponding potential C_4 defined as

$$F_5 = \left(1 + *\right) dC_4\,, \tag{II.5}$$

can be chosen to be

$$C_4 = L^4 g(u)\, \mathrm{vol}\left(AdS_2\right) \wedge \mathrm{vol}\left(S^2\right)\,, \qquad g(u) = \frac{1}{8}\sinh(4u) - \frac{u}{2}. \tag{II.6}$$

This potential is particularly useful since it lies entirely on $AdS_2 \times S^2$. As explained in [40], different gauge choices for C_4 are related by a conformal transformation at the boundary of AdS_5.

2.2.2 Classical D3-brane solution

The bosonic action of a probe D3-brane in the $AdS_5 \times S^5$ background is given by

(II.7) $$S_{D3}^{(B)} = -T_{D3} \int d^4\xi \sqrt{-\det\left(g+\mathcal{F}\right)_{ab}} + T_{D3} \int C_4 \,,$$

where ξ^a, $a = (0,1,2,3)$ are worldvolume coordinates, g_{ab} is the induced metric, and $\mathcal{F} = d\mathcal{A}$ is the field strength of the gauge field living on the brane. The pullback of C_4 onto the worldvolume is implicit in this expression. The tension of a D3-brane is $T_{D3}^{-1} = (2\pi)^3 \alpha'^2 g_s$.

The classical configuration relevant to us is a solution to the equations of motion derived from (II.7). It sits at a fixed point on the 5-sphere while spanning an $AdS_2 \times S^2$ hypersurface in AdS_5 [40]. Recalling the form of the metric (II.3), it is clear that such a D3-brane corresponds to constant u. Furthermore, we only consider an electric flux[1]

(II.8) $$\mathcal{F}_{\alpha\beta} = L^2 \mathcal{F}\,\mathrm{vol}\left(AdS_2\right) \,.$$

It is a simple exercise to check that the equation of motion for u and the flux quantization condition, which is an integral of the equation of motion for the gauge field, are solved by [40]

(II.9) $$\mathcal{F} = \cosh(u) \,,$$

and

(II.10) $$\sinh(u) = \frac{k\sqrt{\lambda}}{4N} \equiv \kappa \,,$$

where $k \in \mathbb{N}$ is the fundamental string charge dissolved on the D3-brane. The point on S^5 where the D3-brane sits is arbitrary.

The $AdS_2 \times S^2$ geometry induced on the D3-brane is

(II.11) $$ds^2_{ind} = L^2 \left(\cosh^2(u) ds^2_{AdS_2} + \sinh^2(u) d\Omega_2^2\right) \,.$$

[1] Introducing a field strength along S^2 would induce a magnetic charge on the D3-brane which is dual to the 't Hooft loop in $\mathcal{N} = 4$ SYM.

Notice that the AdS_2 and S^2 factors have different radii. The solution possesses a $SL(2;\mathbb{R}) \times SO(3) \times SO(5)$ symmetry corresponding to isometries of the worldvolume and rotations of S^5 about a fixed point. It is also shown in [40] that it preserves half of the targetspace supersymmetries, yielding the supergroup $OSp(4^*|4) \subset SU(2,2|4)$. As expected, these coincide with the symmetries preserved by the Wilson loop operator in the dual gauge theory. We will come back to this in section 2.4.

In Euclidean signature, the half-plane (Poincaré) AdS_2 metric can be conformally mapped to the disk by adding the point at infinity. Holographically, the choice of global structure for Euclidean AdS_2 corresponds to selecting either the infinite line or the circular Wilson loop in the gauge theory. This is easily seen by transforming the AdS_5 metric (II.3) to Poincaré coordinates and checking that the embedding pinches the boundary along the appropriate curve. Indeed, for

$$ds^2_{AdS_2} = \frac{1}{r^2}\left(dx^2 + dr^2\right) , \tag{II.12}$$

the transformation $r^2 = \rho^2 + y^2$, $\sinh(u) = \rho/y$ brings the metric (II.3) to the form

$$ds^2_{AdS_5} = \frac{L^2}{y^2}\left(dx^2 + d\rho^2 + \rho^2 d\Omega_2^2 + dy^2\right) . \tag{II.13}$$

We see that the embedding $y = \rho/\kappa$ reaches the boundary $y = 0$ along an infinite line spanned by the coordinate x. Thus, the holographic description of the infinite line Wilson loop is captured by the metric (II.12). In contrast, the circular loop is better described by the disk model of AdS_2,

$$ds^2_H = d\chi^2 + \sinh^2\chi d\psi^2 . \tag{II.14}$$

In this case the coordinate change $\cot\eta = \cosh(u)\sinh\chi$, $\coth\rho = \coth(u)\cosh\chi$ gives

$$ds^2_{AdS_5} = \frac{L^2}{\sin^2\eta}\left(\cos^2\eta d\psi^2 + d\rho^2 + \sinh^2\rho d\Omega_2^2 + d\eta^2\right) . \tag{II.15}$$

This can be put in a more familiar form by writing $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$ and further defining

$$r_1 = \frac{R\cos\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \tag{II.16}$$

$$r_2 = \frac{R\sinh\rho\sin\theta}{\cosh\rho - \sinh\rho\cos\theta}\,, \tag{II.17}$$

$$y = \frac{R\sin\eta}{\cosh\rho - \sinh\rho\cos\theta}\,, \tag{II.18}$$

as was done in [40]. Then,

$$ds^2_{AdS_5} = \frac{L^2}{y^2}\left(dr_1^2 + r_1^2 d\psi + dr_2^2 + r_2^2 d\phi^2 + dy^2\right). \tag{II.19}$$

The D3-brane worldvolume is now described by $\sin\eta = \sinh\rho/\kappa$. As we approach the boundary $y = 0$ ($\eta = 0$) the hypersurface becomes $r_2 = 0$, which corresponds to a circle of radius $r_1 = R$ parameterized by ψ. Notice that the radius of the loop, which appears explicitly in the transformation (II.16), is not present in the metric above as a consequence of scale invariance.

One of the reasons we use the coordinate system (II.3) is that the infinite straight line and circular Wilson loops can be described in a unified way. Notice, however, that in the case of the circular loop, the solution only makes sense in Euclidean signature, since the metric (II.14) does not have a well defined Lorentzian counterpart.

2.3 Open string excitations

In this section we consider fluctuations of the classical D3-brane configuration reviewed above. We construct the quadratic bosonic and fermionic actions and derive the spectrum of excitations.

The study of D-brane fluctuations is, by now, a rather mature subject in the context of the AdS/CFT correspondence. A unifying theme is the fact that some probe branes have worldvolumes containing an AdS_p factor, pointing to the possibility of an effective conformal theory different from the original $\mathcal{N} = 4$ SYM. One of the first works in this direction was provided by [34] in the context of a defect CFT. Perhaps a more widely known example is given by the study of a probe D7-brane whose worldvolume is $AdS_5 \times S^3$ in [106] [101], where the application to $\mathcal{N} = 2$ SYM with fundamental matter was highlighted.

A clean conceptual framework arose in [6,99], where a holographic renormalization description of probe branes was provided. This yielded a recipe for how to read the dimensions of operators dual to modes coming from the defect brane. Interestingly, the fields in this brane are not the $\mathcal{N}=4$ SYM fields which live in the original stack of D3-branes. Due to the crucial role of asymptotic data, it turns out that solutions to the D-brane action are essentially classified as solutions of free fields in $AdS_p \times S^q$, which is the worldvolume of the probe branes. According to [99], the emerging open string modes are effectively described by a scalar in AdS_{d+1} with action

$$S = \frac{1}{2}\int d^{d+1}x\sqrt{g}\left(g^{\alpha\beta}\partial_\alpha\Phi\partial_\beta\Phi + M^2\Phi^2\right). \tag{II.20}$$

This scalar is dual to some gauge-invariant CFT operator with dimension Δ given by $M^2 = \Delta(\Delta - d)$. More importantly for us, the corresponding operator is an operator in the defect theory. A key point in [99] is that in the cases they considered (embedding without worldvolume fluxes) the counterterms for the D-brane action are identical to the counterterms for a free scalar in AdS.

Our work provides a further generalization where the probe brane has flux in its worldvolume. We do not work out the general case similar to [99]; we defer this analysis to the future. It suffices to say that we also find that our open string fluctuations are described by fields in AdS_p albeit with a metric different from the induced metric. Let us elaborate on this point.

Generally speaking, the D-brane fluctuations around a static solution are described by a field theory living on the worldvolume of the brane. In the absence of a background flux $\mathcal{F}_{ab}$ the natural geometry is given by the induced metric, i.e., the pullback of $AdS_5 \times S^5$ to the worldvolume of the brane. As explained in [124], one of the effects of adding a flux is to deform the geometry according to

$$\hat{g}_{ab} = g_{ab} - \mathcal{F}_{ac}g^{cd}\mathcal{F}_{db}\,. \tag{II.21}$$

This deformation is crucial in casting the fermionic part of the action in a canonical form. For the case at hand we readily find,

$$d\hat{s}^2 = L^2 \sinh^2(u) \left(ds^2_{AdS_2} + d\Omega_2^2 \right) \,, \tag{II.22}$$

so that the metric is still given by $AdS_2 \times S^2$ but with equal radii $L \sinh(u)$. In this sense the effect is rather innocuous. Of course, there are other less trivial consequences. In particular, we can infer that changing the radius of AdS_2, from the holographical point of view, amounts to changing the conformal dimension of the dual operators.

2.3.1 Bosonic fluctuations

A general geometric framework to study excitations of extended objects will be developed in the next chapter, when we study fluctuations of probe D5-branes. For now, we will take a pragmatic approach and follow a more naive method to find the spectrum of the D3-brane configuration.

The local symmetries of the complete D-brane action include worldvolume diffeomorphisms and κ-symmetry [124]. Let us comment on the gauge fixing procedure of diffeomorphisms following [46], [124], postponing a discussion of κ-fixing to the next section. Suppose we have a particular embedding $x^m(\xi)$ that solves the Dp-brane equations of motion. The standard (static) gauge condition consists of fixing $x^a(\xi) = \xi^a$ for $p+1$ of the spacetime coordinates. Then, when considering fluctuations δx^m around the solution, we should impose $\delta x^a = 0$ and consider the transverse fluctuations δx^i as physical. We will adopt this static gauge in what follows.

We would like, however, to briefly comment on a more geometrical gauge fixing procedure outlined in [124]. Consider a target space vielbein $E^{\underline{m}} = (E^{\underline{a}}, E^{\underline{i}})$, such that the pull-back of $E^{\underline{a}}$ onto the worldvolume form a vielbein for the induced geometry while the pulled-back $E^{\underline{i}}$ vanish. This explicitly breaks the local Lorentz invariance of the theory to $SO(p+1) \times SO(9-p)$. We can then consider the tangent space fluctuations

$$\chi^{\underline{m}} = E^{\underline{m}}{}_m \delta x^m \tag{II.23}$$

as our worldvolume fields and fix the diffeomorphism invariance by the condition

(II.24) $$\chi^{\underline{a}} = 0.$$

The surviving fields are the transverse modes $\chi^{\underline{i}}$. The choice of AdS_5 coordinates in (II.3) actually makes this method equivalent to choosing the static gauge. However, this gauge fixing condition is better suited to be used in more general coordinates, such as global or Poincaré coordinates in AdS.

Following the above discussion, we choose a gauge such that the coordinates along $AdS_2 \times S^2$ do not fluctuate and expand the remaining bosonic fields as

(II.25) $$u \to u + \delta u\,, \qquad \theta^i \to \theta^i + \delta\theta^i\,, \qquad \mathcal{A} \to \mathcal{A} + a\,,$$

The corresponding tangent space fluctuations are

(II.26) $$\chi^{\underline{4}} = L\delta u, \qquad \chi^{\underline{i}} = Le^{\underline{i}}{}_i\delta\theta^i\,,$$

where $e^{\underline{i}}$ is a vielbein for the unit 5-sphere. For the remaining of the chapter, the index i will denote coordinates on S^5, as we have explicitly written the fluctuation in AdS_5 as $\chi^{\underline{4}}$. Also, all quantities except the fluctuations themselves assume their classical values (II.10) and (II.9).

The quadratic action for the perturbations is obtained by expanding the bosonic action (II.7), reproduce here for convenience,

(II.27) $$S^{(B)}_{D3} = -T_{D3}\int d^4\xi\sqrt{-\det{(g+\mathcal{F})_{ab}}} + T_{D3}\int C_4\,,$$

to second order. To expand the Dirac-Born-Infeld term, we make use the series

(II.28) $$\sqrt{-\det M} \to \sqrt{-\det M}\left\{1 + \frac{1}{2}\text{tr}X + \frac{1}{8}[\text{tr}X]^2 - \frac{1}{4}\text{tr}(X^2) + \mathcal{O}(X^3)\right\}\,,$$

where X denotes the matrix $X = M^{-1}\delta M$, and we have introduced

(II.29) $$M_{ab} = g_{ab} + \mathcal{F}_{ab}\,.$$

A short calculation shows that

$$\delta M_{ab} = \partial_a\chi^{\underline{4}}\partial_b\chi^{\underline{4}} + \delta_{\underline{ij}}\partial_a\chi^{\underline{i}}\partial_b\chi^{\underline{j}} + \frac{1+\coth^2(u)}{L^2}\hat{g}_{ab}\left(\chi^{\underline{4}}\right)^2 + \frac{2\coth(u)}{L}\hat{g}_{ab}\,\chi^{\underline{4}} + f_{ab}\,, \tag{II.30}$$

where $f_{ab} = \partial_a a_b - \partial_b a_a$ and $\hat{g}_{ab}$ is open string metric (II.22), which we write again here,

$$d\hat{s}^2 = L^2\sinh^2(u)\left(ds^2_{AdS_2} + d\Omega_2^2\right)\ . \tag{II.31}$$

Also,

$$\sqrt{-\det M_{ab}} = \coth(u)\sqrt{-\det \hat{g}_{ab}}\,. \tag{II.32}$$

Substituting this in equation (II.28) we obtain

$$\begin{aligned}\sqrt{-\det M_{ab}} \to \sqrt{-\det M_{ab}} + \coth(u)\sqrt{-\det \hat{g}_{ab}}\,\Bigg[&\frac{1}{2}\hat{g}^{ab}\partial_a\chi^{\underline{4}}\partial_b\chi^{\underline{4}} \\ &+\frac{1}{2}\hat{g}^{ab}\delta_{\underline{ij}}\partial_a\chi^{\underline{i}}\partial_b\chi^{\underline{j}} + \frac{4}{L^2}\left(1+\coth^2(u)\right)\left(\chi^{\underline{4}}\right) + \frac{4}{L}\coth(u)\,\chi^{\underline{4}} \\ &+\frac{1}{4}\hat{g}^{ab}\hat{g}^{cd}f_{ac}f_{bd} + \frac{1}{2\cosh(u)}\hat{\epsilon}^{\alpha\beta}f_{\alpha\beta}\Bigg]\ ,\end{aligned} \tag{II.33}$$

where $\hat{\epsilon}_{\alpha\beta}$ is the Levi-Civita tensor associated to the AdS_2 part of the deformed metric (II.31).

Expansion of the Wess-Zumino term in (II.27) is straightforward, as it only requires Taylor-expanding the function $g(u)$ in (II.6). We get

$$C_4 \to C_4 + d^4\xi\ \coth(u)\sqrt{-\det \hat{g}_{ab}}\left[\frac{4}{L^2}\left(1+\coth^2(u)\right)\left(\chi^{\underline{4}}\right)^2 + \frac{4}{L}\coth(u)\,\chi^{\underline{4}}\right]\ . \tag{II.34}$$

Putting everything together, we find that, as expected, the linear term in $\chi^{\underline{4}}$ vanishes. Moreover, the linear term in the gauge field is a total derivative and can be canceled by an appropriate boundary term. Thus, the quadratic action for the bosonic fluctuations reads

$$\begin{aligned}S^{(B,2)}_{D5} = -T_{D3}\coth(u)\int d^4\xi\,\sqrt{-\det \hat{g}_{ab}}\,\Bigg[&\frac{1}{2}\hat{g}^{ab}\left(\partial_a\chi^{\underline{4}}\partial_b\chi^{\underline{4}} + \delta_{\underline{ij}}\partial_a\chi^{\underline{i}}\partial_b\chi^{\underline{j}}\right) \\ &+\frac{1}{4}\hat{g}^{ab}\hat{g}^{cd}f_{ac}f_{bd}\Bigg]\ .\end{aligned} \tag{II.35}$$

The dynamical fields in (II.35) are the scalar $\chi^{\underline{4}}$, the scalars $\chi^{\underline{i}}$ transforming as a **5** under the $SO(5)$ symmetry, and the gauge field a_a. They couple to the deformed geometry (II.31) in the expected manner. We also notice that all the fluctuations turn out to be massless. This was expected for the excitations on S^5, but a non-trivial cancelation between contributions from the DBI and WZ parts of the action occurred for the AdS_5 perturbation $\chi^{\underline{4}}$. As we will see below, we can attribute this to the supersymmetry of the classical solution.

2.3.2 Fermionic fluctuations

We now consider fluctuations of the fermionic degrees of freedom of the D3-brane. In the context of AdS/CFT fermionic excitations have played a relatively secondary role. To our knowledge, there is only one explicit computation of the fermionic spectrum of a D7-brane in $AdS_5 \times S^5$ [101], but without worldvolume fluxes. Some of the classical brane configurations that appear in this thesis have cousins in the context of confining theories where they describe confining k-strings which are bound states of k quarks and k anti-quarks. The study of fluctuations in that context is important for the computation of the Lüscher term. The works [38, 133, 151, 152] presented a study of the fluctuations and found certain universality in the value of the Lüscher term.

The construction of a general quadratic fermionic action was presented in a series of interesting works by Martucci [120, 121, 123–125]. Here we will closely follow the notation and presentation of [124]. For the $AdS_5 \times S^5$ background the action reduces to

$$S^{(F)}_{D3} = \frac{T_{Dp}}{2} \int d^4\xi \, \sqrt{-\det{(g+\mathcal{F})_{ab}}}\, \overline{\Theta}\,(1-\Gamma_{D3})\,\tilde{M}^{ab}\Gamma_b D_a \Theta. \tag{II.36}$$

Here Θ is a doublet of 10d positive chirality Majorana-Weyl spinors, $\overline{\Theta} = i\Theta^\dagger\Gamma^{\underline{0}}$, $\Gamma_a = \partial_a x^m \Gamma_m$ is the pullback of the spacetime Dirac matrices, $\tilde{M}^{ab}$ is the inverse of

$$\tilde{M}_{ab} = g_{ab} + \mathcal{F}_{ab}\tilde{\Gamma}\,, \qquad \tilde{\Gamma} = \Gamma^{11}\otimes\sigma_3\,, \tag{II.37}$$

and Γ_{D3} is a projector ensuring invariance of the action under κ-symmetry. Also, $D_a = \partial_a x^m D_m$

is the pullback of the type IIB covariant derivative, which in our case reads

(II.38) $$D_m = \nabla_m + \frac{1}{16}\not{F}_{(5)}\Gamma_m \otimes (i\sigma_2)\,.$$

As shown in [124], the complete D-brane action is invariant under (linearized) supersymmetry transformations induced by the existence of targetspace Killing spinors. Since the classical embeddings considered here preserve half of the $AdS_5 \times S^5$ supersymmetries, we expect the action for the quadratic fluctuations around these backgrounds to be supersymmetric. Instead of verifying this explicitly, we will show in the next section that the spectrum of excitations falls into multiplets of the appropriate supergroup.

Since the fermionic fields vanish in the classical solution, we can consider Θ in (II.36) as the fluctuation. Moreover, all the bosonic quantities can be evaluated on the background. First, the inverse of the matrix $\tilde{M}_{ab}$ is

(II.39) $$\tilde{M}^{\alpha\beta} = \hat{g}^{\alpha\beta} - \frac{1}{\cosh(u)}\hat{\epsilon}^{\alpha\beta}\tilde{\Gamma}\,, \qquad \tilde{M}^{\mu\nu} = \hat{g}^{\mu\nu}\,,$$

where, as before, $\hat{g}_{ab}$ is given by (II.31). A short calculation reveals that

(II.40) $$\tilde{M}^{\alpha\beta}\Gamma_\beta = \frac{1}{\sinh(u)}e^{R\tilde{\Gamma}}\Gamma^\alpha e^{R\tilde{\Gamma}}\,, \qquad \tilde{M}^{\mu\nu}\Gamma_\nu = e^{R\tilde{\Gamma}}\Gamma^\mu e^{R\tilde{\Gamma}}\,,$$

where we have defined

(II.41) $$R = -\frac{1}{2}\sinh^{-1}\left(\frac{1}{\sinh(u)}\right)\Gamma\,, \qquad \Gamma = \Gamma_{\underline{01}}$$

We recognize $\hat{\Gamma}_\alpha = \sinh(u)\Gamma_\alpha$, $\hat{\Gamma}_\mu = \Gamma_\mu$ as the Dirac matrices associated to the deformed metric (II.31). Thus,

(II.42) $$\tilde{M}^{ab}\Gamma_b D_a = e^{R\tilde{\Gamma}}\left[\hat{\Gamma}^a e^{R\tilde{\Gamma}} D_a e^{-R\tilde{\Gamma}}\right] e^{R\tilde{\Gamma}}\,.$$

Next, by computing the spin connection for the background (II.1), we find that the pullback of the 10d covariant derivative is

$$e^{R\tilde{\Gamma}} D_\alpha e^{-R\tilde{\Gamma}} = \nabla_\alpha + \frac{1}{2L}\hat{\Gamma}_\alpha \Gamma_{\underline{4}} e^{-2R\tilde{\Gamma}} - \frac{\cosh(u)}{2L\sinh(u)}\hat{\Gamma}_\alpha \Gamma^{\underline{01234}} \otimes (i\sigma_2)\ , \tag{II.43}$$

$$e^{R\tilde{\Gamma}} D_\mu e^{-R\tilde{\Gamma}} = \nabla_\mu + \frac{\cosh(u)}{2L\sinh(u)}\hat{\Gamma}_\mu \Gamma_{\underline{4}} - \frac{1}{2L}\hat{\Gamma}_\mu \Gamma^{\underline{01234}} \otimes (i\sigma_2)\, e^{-2R\tilde{\Gamma}}\ , \tag{II.44}$$

where we have taken into account the positive chirality of the spinor on which this operator acts. The terms which come from the extrinsic curvature of the worldvolume and the RR 5-form flux are potential mass terms for the fermionic field Θ. After contracting with $\hat{\Gamma}^a$ one obtains

$$\tilde{M}^{ab}\Gamma_b D_a = e^{R\tilde{\Gamma}}\left[\hat{\Gamma}^a\nabla_a + \frac{\left(1-\Gamma^{(0)}_{D3}\right)}{L\sinh(u)}\Gamma_{\underline{4}}\left(\sinh(u)e^{-2R\tilde{\Gamma}} + \cosh(u)\right)\right]e^{R\tilde{\Gamma}}\ , \tag{II.45}$$

where

$$\Gamma^{(0)}_{D3} = \Gamma_{\underline{01234}} \otimes (i\sigma_2)\ . \tag{II.46}$$

Lastly, we compute the κ-symmetry projector Γ_{D5}. Following the definition in [124], we get

$$\Gamma_{D3} = -\Gamma^{(0)}_{D3}\left(\frac{\cosh(u)}{\sinh(u)} - \frac{1}{\sinh(u)}\Gamma \otimes \sigma_3\right)\ . \tag{II.47}$$

Notice that when acting from the right on the conjugate of a positive chirality spinor we can make the replacement

$$\Gamma_{D3} = -e^{R\tilde{\Gamma}}\Gamma^{(0)}_{D3}e^{-R\tilde{\Gamma}}. \tag{II.48}$$

Because, $\left(\Gamma^{(0)}_{D3}\right)^2 = 1$, it follows that $\left(1+\Gamma^{(0)}_{D3}\right)\left(1-\Gamma^{(0)}_{D3}\right) = 0$. Thus, collecting all the above results, we find that the fermionic action for the D3-brane is given by

$$S^{(F)}_{D3} = \frac{T_{D3}\coth(u)}{2}\int d^4\xi\, \sqrt{-\det \hat{g}_{ab}}\,\overline{\Theta}e^{R\tilde{\Gamma}}\left(1+\Gamma^{(0)}_{D3}\right)\hat{\Gamma}^a\nabla_a e^{R\tilde{\Gamma}}\Theta. \tag{II.49}$$

Naively, we would have expected a mass term coming from the coupling to $\not{F}_5$ in (II.38). However, as we have seen, supersymmetry conspires to precisely cancel this term against the contributions coming from the extrinsic curvature in the 10d spin connection.

The action can be further simplified by redefining $\Theta \to e^{-R\tilde{\Gamma}}\Theta$; the conjugate spinor transforms as $\overline{\Theta} \to \overline{\Theta} e^{-R\tilde{\Gamma}}$. To fix the local κ-symmetry we use the prescription of [124], which is,

$$\tilde{\Gamma}\Theta = \Theta, \qquad \tilde{\Gamma} = \Gamma^{11} \otimes \sigma_3 \tag{II.50}$$

This sets the lower component of Θ to zero. Denoting the upper component also as Θ, the gauge fixed action reads

$$S_{D3}^{(F)} = \frac{T_{D3}\coth(u)}{2} \int d^4\xi \, \sqrt{-\det \hat{g}_{ab}}\, \overline{\Theta}\hat{\Gamma}^a \nabla_a \Theta. \tag{II.51}$$

Now, as is well known from the dimensional reduction of 10d $\mathcal{N} = 1$ SYM down to four dimensions, under $SO(9,1) \to SO(3,1) \times SO(6)$ the Majorana-Weyl spinor Θ is decomposed into a $SO(3,1)$ Weyl spinor θ^A transforming as a $\mathbf{4}$ of $SO(6) \simeq SU(4)$. Further decomposing $SO(6) \to SO(5)$, to accommodate for the symmetries of the Wilson loop, we get a $\mathbf{4}$ of $SO(5) \simeq USp(4)$. This way we obtain the four dimensional fermionic action

$$S_{D3}^{(F)} = \frac{T_{D3}\coth(u)}{2} \int d^4\xi \, \sqrt{-\det \hat{g}_{ab}}\, \overline{\theta}_A \hat{\gamma}^a \nabla_a \theta^A, \tag{II.52}$$

where $\hat{\gamma}_a$ are now 4d Dirac matrices corresponding to the metric $\hat{g}_{ab}$.

2.3.3 Compactification on S^2

Since the D3-brane worldvolume has the product structure $AdS_2 \times S^2$, we can compactify on the sphere. We now display the effective AdS_2 theory in agreement with general expectations of the string theory description. We relegate the details of the calculation to Appendix B. The resulting 2d actions are given by (we omit an overall constant)

$$S_{\chi}^{(2)} = -\frac{1}{2}\int d^2\xi\sqrt{-\det\hat{g}_{\alpha\beta}}\left(\hat{g}^{\alpha\beta}\partial_\alpha\chi_{lm}^{\underline{4}}\partial_\beta\chi_{lm}^{\underline{4}} + \frac{l(l+1)}{L^2\sinh^2(u)}\left(\chi_{lm}^{\underline{4}}\right)^2\right) - \frac{1}{2}\int d^2\xi\sqrt{-\det\hat{g}_{\alpha\beta}}\left(\hat{g}^{\alpha\beta}\delta_{\underline{ij}}\partial_\alpha\chi_{lm}^{\underline{i}}\partial_\beta\chi_{lm}^{\underline{j}} + \frac{l(l+1)}{L^2\sinh^2(u)}\delta_{\underline{ij}}\chi_{lm}^{\underline{i}}\chi_{lm}^{\underline{i}}\right), \tag{II.53}$$

$$S_{a}^{(2)} = -\frac{1}{4}\int d^2\xi\sqrt{-\det\hat{g}_{\alpha\beta}}\left(\hat{g}^{\alpha\beta}\hat{g}^{\gamma\delta}f_{\alpha\gamma}^{lm}f_{\beta\delta}^{lm} + \frac{2l(l+1)}{L^2\sinh^2(u)}\hat{g}^{\alpha\beta}a_\alpha^{lm}a_\beta^{lm}\right) - \frac{1}{2}\int d^2\xi\sqrt{-\det\hat{g}_{\alpha\beta}}\left(\hat{g}^{\alpha\beta}\partial_\alpha a_{lm}\partial_\beta a_{lm} + \frac{l(l+1)}{L^2\sinh^2(u)}\left(a_{lm}\right)^2\right), \tag{II.54}$$

$$S_{\theta}^{(2)} = \int d^2\xi\sqrt{-\det\hat{g}_{\alpha\beta}}\bar{\Theta}_A^{lm}\left(\hat{\nabla\!\!\!/} + \frac{i\left(l+\frac{1}{2}\right)\gamma}{L\sinh(u)}\right)\Theta_{lm}^A, \tag{II.55}$$

where $f_{\alpha\beta}^{lm} = \partial_\alpha a_\beta^{lm} - \partial_\beta a_\alpha^{lm}$. All the geometric quantities appearing above are intrinsically 2-dimensional and are defined in terms of the AdS_2 factor of the deformed metric (II.31). In particular, the Dirac matrices $\gamma_{\underline{\alpha}}$ implicit in the fermionic action are 2×2 matrices. Also, $\gamma = \gamma_{\underline{01}}$.

These expressions follow from the expansion of the 4-dimensional fields in terms of scalar, vector and spinor harmonics on S^2. In the case of θ_{lm}^A, which are 2d Dirac spinors, the quantum number l takes values $l = \frac{1}{2}, \frac{3}{2}, \ldots$, as appropriate for fermions. In all cases the quantum number m ranges from $-l$ to l. Notice that the scalar modes a_{lm}, coming from the gauge field components along the sphere, start at $l = 1$. Also, the $l = 0$ mode of the gauge field is massless so it has no propagating degrees of freedom.

2.4 Supersymmetry

In this section we discuss the symmetries of the BPS Wilson loops and how the spectrum of open string fluctuations fits into representations of the corresponding supergroup.

2.4.1 Symmetries of the Wilson loop

The $\mathcal{N} = 4$ SYM theory has a supersymmetry group given by $SU(2,2|4)$. The bosonic symmetries are $SU(2,2)\times SU(4)$, where $SU(2,2) \simeq SO(4,2)$ is the conformal group in four dimensions and the $SU(4) \simeq SO(6)$ factor acts as an R-symmetry. In the string theory description, these symmetries are realized as isometries of $AdS_5 \times S^5$.

Let us review the subgroup of $SU(2,2|4)$ preserved by the straight line Wilson loop. This is done in detail in [72]. First we recall that a general bosonic Wilson loop operator in a representation R of $SU(N)$ is defined as

(II.56) $$W_R(C) = \mathrm{Tr}_R P \exp\left(i \int_C ds \left(A_\mu \dot{x}^\mu + \phi_I \dot{y}^I\right)\right),$$

where C labels a curve $\big(x^\mu(s), y^I(s)\big)$ in $\mathcal{N}=4$ superspace, I is a vector index of $SO(6)$, and P denotes path ordering along the loop. As shown in [72], in order to preserve supersymmetry, the curve $x^\mu(s)$ must be an infinite timelike line, which we parameterize by $x^\mu(s) = \big(x^0(s), 0, 0, 0\big)$. Supersymmetry also implies that $\dot{y}^I = n^I$, where n^I is a constant unit vector in $\mathbb{R}^6$.

Now, acting on the spacetime coordinates, the generators $(P_\mu, J_{\mu\nu}, D, K_\mu)$ of $SO(4,2)$ read

(II.57) $$\delta x^\mu = a^\mu + w^\mu{}_\nu x^\nu + \lambda x^\mu + b^\mu x^2 - 2b\cdot x x^\mu$$

where $(a^\mu, w^\mu{}_\nu, \lambda, b^\mu)$ are the corresponding transformation parameters. Conserving the form of the loop imposes the conditions

(II.58) $$a^i = w^i{}_0 = b^i = 0$$

Thus, the subgroup preserved by the Wilson loop is generated by (P_0, J_{ij}, D, K_0). The interpretation of these transformations is simple: an infinite line is left invariant by translations along the line, rotations around the line, and dilatations of the coordinates. The operator K_0 generates a special conformal transformation. The generators J_{ij} span the $SU(2) \simeq SO(3)$ algebra while the rest satisfy

(II.59) $$[P_0, K_0] = -2D, \quad [P_0, D] = -P_0, \quad [K_0, D] = K_0.$$

This is the $SU(1,1) \simeq SO(2,1) \simeq SL(2,\mathbb{R})$ algebra. Thus, we see that the infinite line preserves a $SO(4^*) \simeq SL(2,\mathbb{R}) \times SO(3)$ subgroup of $SO(4,2)$. Finally, the choice of a vector n^I breaks the $SO(6)$ R-symmetry down to $SO(5) \simeq USp(4)$.

The infinite line Wilson loop also preserves 16 of the 32 supersymmetries of $SU(2,2|4)$. The original works of [82,128] identified all the possible AdS supergroups and their multiplets. Among the subgroups of $SU(2,2|4)$, the supergroup $OSp\,(4^*|4)$ has $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ as its even subgroup and 16 fermionic generators.

Turning to the holographic description of the BPS Wilson loops, we see that the classical D3-brane solution (II.11) displays a $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ symmetry corresponding to isometries of the worldvolume geometry and rotations of S^5 about a fixed point. It is also shown in [40] that these D3 brane configurations preserve half of the target space supersymmetries. As expected, these coincide with the symmetries preserved by the Wilson loop in the gauge theory dual.

2.4.2 Conformal Dimensions

As we have seen in section 2.3.3, the bosonic open string fluctuations are described by scalar and vector fields in AdS_2, all with masses

$$m_l^2 = l(l+1)/L^2 \sinh^2(u_k) \qquad l = 0^*, 1, \dots \tag{II.60}$$

where * reminds us that some fluctuations do not include the $l = 0$ mode. According to the standard AdS/CFT dictionary, the conformal dimensions of the operators dual to such modes are given by the formulas

$$h_\pm^{scalar} = \frac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2R^2}\right) \qquad h_\pm^{vector} = \frac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2R^2}\right) \tag{II.61}$$

where R is the radius of AdS_{d+1}. In our case $d = 1$ and $R = L\sinh(u_k)$, so

$$h = l + 1 \qquad l = 0^*, 1, \dots \tag{II.62}$$

for all the bosonic fields. Similarly, from the formula

$$h^{spinor} = \frac{d}{2} + |m|R \tag{II.63}$$

we see that the fermionic modes Θ_A^{lm} have

$$h = l + 1 \qquad l = \frac{1}{2}, \frac{3}{2}, \dots \tag{II.64}$$

Notice that the masses depend on the radius of S^2 while the formulas for the conformal dimensions involve the AdS_2 radius. The fact that the perturbations see the deformed metric (II.31) instead of the induced metric (II.11) is crucial to get rational values for h. This is a consequence of the supersymmetry preserved by the D3-brane, as we will see below.

All in all, the spectrum of excitations of the D3 brane is given by a KK tower of fields propagating in AdS_2 labeled by their $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ quantum numbers. This result is summarized in table 2.1. At the lowest level there are six massless and six massive (two triplets of $SO(3)$) bosonic modes. This spectrum is quite different from the expectations based on the calculation using fundamental strings, where the counting was five massless and three massive modes [46, 56, 105, 146].

2d field		4d origin	$SL(2,\mathbb{R})$	$SO(3)$	$SO(5)$	
Bosons	χ^4_{lm}	embedding in AdS_5	$l+1$	l	$\mathbf{1}$	$l \geq 0$
	χ^i_{lm}	embedding in S^5	$l+1$	l	$\mathbf{5}$	$l \geq 0$
	a_μ^{lm}	gauge field along AdS_2	$l+1$	l	$\mathbf{1}$	$l \geq 1$
	a_{lm}	gauge field along S^2	$l+1$	l	$\mathbf{1}$	$l \geq 1$
Fermions	θ_A^{lm}	IIB spinor	$l+1$	l	$\mathbf{4}$	$l \geq \frac{1}{2}$

Table 2.1: KK tower of modes and their transformation properties under $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$. The representations of $SL(2,\mathbb{R})$ are labeled by the $L_0 = h$ eigenvalue of the highest weight state.

2.4.3 Supersymmetry of the spectrum

We are interested in understanding how the excitations we have described can be organized in representations of supersymmetry. Similar fittings of KK modes into AdS supermultiplets have appeared in the context of the AdS/CFT correspondence. In particular, compactifications of supergravity theories on $AdS_2 \times S^2$ have been presented thoroughly in [30, 33, 107, 126]. In these examples the relevant supergroup is $SU(1,1|2)$, which has $SL(2,\mathbb{R}) \times SO(3)$ as its even subgroup. In our case we have an extra $SO(5)$ symmetry which makes $OSp(4^*|4)$ the relevant supergroup.

The spectrum of open string fluctuations should then fall into multiplets of $OSp(4^*|4)$.

Lowest weight representations of the super-group $OSp(2m^*|2n)$ where studied in [81]. In the case of $OSp(4^*|4)$, the so-called doubleton representations can be labeled by a half-integer j and have the following $SO(4^*) \times USp(4) \simeq SL(2,\mathbb{R}) \times SU(2) \times SO(5)$ content (see Appendix C):

$$\mathbf{j} = (j+1, j, \mathbf{5}) \oplus (j+\tfrac{3}{2}, j+\tfrac{1}{2}, \mathbf{4}) \oplus (j+2, j+1, \mathbf{1}) \oplus (j+\tfrac{1}{2}, j-\tfrac{1}{2}, \mathbf{4}) \oplus (j+1, j, \mathbf{1}) \oplus (j, j-1, \mathbf{1}), \tag{II.65}$$

for $j \geq 1$ and

$$\mathbf{0} = (1, 0, \mathbf{5}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}, \mathbf{4}) \oplus (2, 1, \mathbf{1}), \tag{II.66}$$

$$\tfrac{\mathbf{1}}{\mathbf{2}} = (\tfrac{3}{2}, \tfrac{1}{2}, \mathbf{5}) \oplus (2, 1, \mathbf{4}) \oplus (\tfrac{5}{2}, \tfrac{3}{2}, \mathbf{1}) \oplus (1, 0, \mathbf{4}) \oplus (\tfrac{3}{2}, \tfrac{1}{2}, \mathbf{1}). \tag{II.67}$$

for the smallest multiplets.

The guiding principle in identifying the different states in the multiplet is their $SO(5)$ representation. Looking at table 2.1, we first notice that only multiplets with integer j can occur in the spectrum. It is also clear that the scalar excitations in S^5 correspond to the first state in (II.65); these are the only fields that transform as a $\mathbf{5}$ of $SO(5)$. By looking at the first multiplet (II.66) we see that the third state must correspond to a bosonic fluctuation whose excitations start at $j = 1$, i.e. one of the two gauge field fluctuations. Looking at the next multiplet,

$$\mathbf{1} = (2, 1, \mathbf{5}) \oplus (5/2, 3/2, \mathbf{4}) \oplus (3, 2, \mathbf{1}) \oplus (3/2, 1/2, \mathbf{4}) \oplus (2, 1, \mathbf{1}) \oplus (1, 0, \mathbf{1}) \tag{II.68}$$

we realize that the fifth $(2, 1, \mathbf{1})$ and sixth $(1, 0, \mathbf{1})$ states must be identified with the other gauge field fluctuation and the scalar excitation in AdS_5, respectively. In fact, this identification works for any integer j. In table 2.1, this simply amounts to relabeling $l = j - 1$ for ϕ^4_{lm}, $l = j$ for $\phi^{\hat{i}}_{lm}$, $l = j$ for a^{lm}_μ, and $l = j + 1$ for a_{lm}. This implies that the lowest lying modes do not fit in a single multiplet, rather they are split among an entire **0** and part of a **1** multiplet.

Now, recall that the fermionic fluctuations Θ^{lm}_A in table 2.1 are Dirac spinors. By decomposing them into two real spinors and identifying $l = j + \frac{1}{2}$ for one and $l = j - \frac{1}{2}$ for the other, we get the same fermionic content as (II.65).

In summary, we find that the $OSp(4^*|4)$ structure of the spectrum of excitations is

$$\bigoplus_{j\geq 0} \mathbf{j} \tag{II.69}$$

where the multiplets **j** are given by (II.65), (II.66) and (II.67).

2.5 Discussion and conclusions

In this chapter we have tackled the question of excitations of classical configurations that holographically describe the expectation value of supersymmetric Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang-Mills in the symmetric representation of $SU(N)$. Concretely, we considered a probe D3-brane dual to the Wilson loop in the symmetric representation and computed its spectrum of excitations explicitly. Our treatment of the fermionic excitations was exhaustive and we found interesting new properties that were not observed in previous studies. Indeed, most of the analysis of fermions present in the literature has been rather indirect, relying on supersymmetry of the bosonic sector. We found, in particular, that the fermions obtained are massless in the worldvolume of the D3-brane, that is, on $AdS_2 \times S^2$. This fact defies the naive expectation that a background with RR fluxes yields mass terms for the fermionic excitations. We basically witnessed an interesting cancelation between the would be mass term coming from the 5-form flux and the contribution of the extrinsic curvature. The explicit results fit precisely with the structure of supermultiplets of $OSp(4^*|4)$.

As explained in [72, 73, 158], a D3-brane describes the Wilson loop in the symmetric representation of $SU(N)$, while a D5-brane corresponds to the antisymmetric representation. The description in terms of a string captures the fundamental representation of $SU(N)$. After finding such harmonious picture for the D3-brane, we would like to briefly comment on the excitations of the F1 configuration, and see how it fits in the general framework, postponing a full analysis of the D5-brane to the next chapter.

A systematic study of the semi-classical fluctuations of strings dual to the BPS Wilson loops was carried out in [46, 56, 105, 146]. The background solution has an AdS_2 worldsheet embedded in AdS_5. Rotations of the remaining AdS_5 coordinates give an $SO(3)$ symmetry and since the string sits on a fixed point in S^5 the solution also has $SO(5)$ invariance. It turns out that the 3 fluctuations in AdS_5 have $m^2 = 2$ (in units of the AdS_2 radius) while those coming from the 5-sphere are massless. These masses correspond to $SL(2,\mathbb{R})$ quantum numbers $h = 2$ and $h = 1$, respectively. Thus, the bosonic spectrum respects the $SO(3) \times SO(5)$ symmetry of the solution. The fermionic fluctuations are described by eight real degrees of freedom which can be combined into four real 2d fermions. These fields have masses $|m| = 1$ and transform in the fundamental of $SU(2)$ and the $\mathbf{4}$ of $SO(5)$. In terms of their $SL(2,\mathbb{R}) \times SO(3) \times SO(5)$ representations, the complete spectrum of excitations of the fundamental string dual to the BPS Wilson loop is given by

$$(1, 0, \mathbf{5}) \oplus (3/2, 1/2, \mathbf{4}) \oplus (2, 1, \mathbf{1}) \tag{II.70}$$

This is precisely the $\mathbf{j} = \mathbf{0}$ ultra-short multiplet of $OSp(4^*|4)$. Notice that this supersymmetric structure is in agreement with [46], where the authors argued that the fluctuations formed an $\mathcal{N} = 8$ multiplet in two dimensions.

From the field theory perspective, the symmetries of the Wilson loop operator do not depend on the particular representation of the gauge group. We therefore expect that the excitations of the fundamental string and D5-brane dual to the Wilson loop operators fall into representations of the supergroup $OSp(4^*|4)$, as for the D3-brane. Indeed, this is the case. We summarize our findings in table 2.2.

Configuration	Representation	Worldvolume	Isometries	Supergroup
F1	Fundamental	AdS_2	$SL(2,\mathbb{R})$	$OSp(4^*\|4)$
D3	Symmetric	$AdS_2 \times S^2$	$SL(2,\mathbb{R}) \times SO(3)$	
D5	Antisymmetric	$AdS_2 \times S^4$	$SL(2,\mathbb{R}) \times SO(5)$	

Table 2.2: BPS Wilson loops in various representations and their holographic descriptions.

We will finish this section by mentioning a set of problems that we consider worth pursuing and will help to clarify some aspects of this beautiful duality between string theory configurations and expectation values of Wilson loop operators in $\mathcal{N} = 4$ SYM. We list them in increasing speculative order:

- A natural next step for the work presented here is the computation of the one-loop determinants due to the fluctuations. Knowing the one-loop determinant is equivalent to computing the first quantum correction to the expectation value of the half BPS Wilson loops. Clearly, such computation opens the door for comparison with other methods and exact results [135], and will provide further insight into the structure of AdS/CFT in supersymmetric setups. This is a very important computation and we plan to complete it in a separate publication. Let us just advance a few observations of what we glean from our experience here. It seems plausible to achieve a unified treatment of the straight line and the circular Wilson loop, whereby the only difference comes from global aspects of the AdS_2 space where the excitations live. One technical hurdle we anticipate, compared to the fundamental string calculation, is the fact that now we need to include the $SO(3)$ or $SO(5)$ quantum numbers when computing such determinants. Hopefully, the organization into supermultiplets achieved in this paper will serve as a guiding principle.

- Having understood the spectrum of excitations for 1/2 BPS holographic Wilson loops a natural question is whether the 1/4 BPS are amenable to a similar treatment. A clear starting point would be the solutions presented in [41].

- Given the prominent role that the matrix model has played in the context of general representations, it makes sense to expect that results similar to those obtained here could be mirrored in the matrix model side. In particular, it is likely that the computation of the expectation value of the Wilson loops could be organized in terms of excitations that are ultimately classified by $OSp(4^*|4)$.

- One of our original motivations for the study of these configurations was the hope that they might uncover some sort of integrable structure similar to those arising in the context of BMN or spin chains. We did not directly succeeded but hold out some hope that this is possible. We are encouraged by interesting works showing the role of integrability for circular Wilson loops using the fundamental string, even in the context of phase transitions [20, 161].

- Finally, in this paper we did not discuss the interpretation of the field theory dual in any detail. This is a one-dimensional defect CFT and has been quoted in recent works as a model for interesting condensed matter phenomena related to quantum impurity [127, 145]. In such a context, uncovering the precise role of the spectrum of excitations should lead to a deeper understanding of the interactions of the system.

CHAPTER III

D5-branes and holographic Wilson loops

As anticipated in the previous chapter, we now proceed to systematically study the spectrum of excitations and the one-loop determinant of holographic Wilson loop operators in antisymmetric representations of $\mathcal{N}=4$ supersymmetric Yang-Mills theory. Holographically, these operators are described by D5-branes carrying electric flux and wrapping an $S^4 \subset S^5$ in the $AdS_5 \times S^5$ bulk background. We derive the dynamics of both bosonic and fermionic excitations for such D5-branes. In fact, we do this calculation for a more general class of solutions in a $aAdS_5 \times S^5$ background at finite temperature, of which the half-BPS Wilson loop is a particular example. We then show explicitly that it is supersymmetric and calculate the one-loop effective action using heat kernel techniques.

3.1 Introduction

Wilson loop operators play a central role in gauge theories, both as formal variables and as important order parameters. In the context of the AdS/CFT correspondence expectation values of Wilson loops were first formulated by Maldacena [117] and Rey-Yee [141].

One of the most exciting developments early on was the realization that the expectation value of the BPS circular Wilson loop can be computed using a Gaussian matrix model [43, 50]. This conjecture was later rigorously proved in [136]. In a beautiful, now classic work by Gross and Drukker, the matrix model was evaluated and its leading N, large 't Hooft coupling limit was successfully compared with the string theory answer. One of the most intriguing windows opened by this problem is the question of quantum corrections it their entire variety. For example, having an exact field theory answer (Gaussian matrix model) prompted Gross and Drukker to speculate that the exact matrix model result was the key to understanding higher genera on the string theory side. The quantum corrections on the string theory side have been the subject of much investigation starting with earlier efforts in [46, 57] and continuing in more recent works such as [105, 146]. Despite these concerted efforts, it is fair to say that a crisp picture of matching the BPS Wilson loop at the quantum level on both sides of the correspondence has not yet been achieved.

More recently the question of tackling BPS Wilson loops in more general representations has been successfully addressed at leading order. The introduction of general representations gives a new probing parameter, thus expanding the possibilities initiated in the context of the fundamental representation. In the holographic framework, a half BPS Wilson loop in $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory in the fundamental, symmetric or antisymmetric representation of $SU(N)$ is best described by a fundamental string, a D3-brane or a D5-brane with fluxes in their worldvolumes, respectively. Drukker and Fiol computed in [40], using a holographic D3 brane description, the expectation value of a k-winding circular string which, to leading order, coincides with the k-symmetric representation. A more rigorous analysis of the role of the representation was elucidated in [72, 73]. Some progress on the questions of quantum corrections to these configurations immediately followed with a strong emphasis on the field theory side [89, 157, 160]. Developing the gravity side of this correspondence is one of the main motivations for this work. In particular, we derive the spectrum of quantum fluctuations in the bosonic and fermionic sectors for a D5-brane with k units of electric flux in its $AdS_2 \times S^4$ world volume embedded in $AdS_5 \times S^5$. This gravity configuration is the dual of the half BPS Wilson loop in the totally antisymmetric representation of rank k in $\mathcal{N} = 4$ SYM.

Although our main motivation comes from the study of Wilson loops, there is another strong motivation for our study of quantum fluctuations. String theory has heavily relied on the understanding of extended objects in the context of the gauge/gravity correspondence. They have played a key role in interpreting and identifying various hadronic configurations (quarks, baryons, mesons, k-strings). A more general approach on the quantization of these objects is a natural necessity. The long history of failed attempts at quantizing extended objects around flat space might have found its right context. Although largely motivated by holography, it is important by itself that the quantum theory of extended objects in asymptotically AdS world volumes seems to be much better behaved than naively expected. In our simplified setup we are faced with various divergences, but many of them allow for some quite natural interpretations. Although we do not attack the general problem of divergences in a general context, we hope that our analysis could serve as a first step in this more fundamental direction of quantization of extended objects.

In this paper, we systematically study small fluctuations of D5-branes embedded in asymptotically $AdS_5 \times S^5$, with flux in its world volume and wrapping an $S^4 \subset S^5$ [8, 21, 84, 134]. The formalism we develop readily applies to more general backgrounds than just the holographic Wilson loop, including holographic Wilson loop correlators [132, 161] and related finite-temperature configurations [90, 98]. Using this general formalism, we obtain the spectrum of both bosonic and fermionic excitations of D5-branes dual to the half BPS circular Wilson loop. Our analysis is explicit by nature and falls nicely in the group theoretic framework put forward in [52]. We also compute the one-loop effective action using heat kernel techniques.

The chapter is organized as follows. In section 3.2, we introduce the class of D5-brane configurations for which our analysis applies. For completeness, the bulk background geometries and the main features of the D5-brane background configurations are reviewed in sections 3.2.1 and 3.2.2, respectively. Section 3.3 contains the general analysis of the bosonic and fermionic excitations of these D5-branes. The second-order actions for the bosonic and fermionic degrees of freedom are constructed in sections 3.3.1 and 3.3.2, respectively, and their classical field equations are analyzed in sections 3.3.3 and 3.3.4. Sections 3.3.5 and 3.4 deal with the holographic Wilson loop. The spectrum of fluctuations is obtained in section 3.3.5. Section 3.4 presents the calculation of the one-loop effective action using the heat kernel method. We conclude in section 3.5. Technical material pertaining to our notation, the geometry of embeddings and to aspects of the heat kernel method are relegated to a series of appendices.

3.2 Review of background geometry and D5-brane configurations

In this section we will briefly review the bulk background and classical D-brane configurations we are interested in. We will work in Lorentzian signature and switch to Euclidean signature only to discuss functional determinants. We refer the reader to Appendix A for notation and conventions.

3.2.1 Bulk background

We want to study probe D-branes embedded in the following $aAdS_5 \times S^5$ solution of type IIB supergravity:

$$ds^2 = ds^2_{aAdS_5} + L^2 d\Omega_5^2\,, \tag{III.1}$$

$$F_5 = 4L^4\,(1+*)\;\mathrm{vol}\left(S^5\right)\;, \tag{III.2}$$

where

$$ds^2_{aAdS_5} = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}\sum_{i=1}^{3} dx_i^2\,, \qquad f(r) = \frac{r^2}{L^2}\left(1 - \frac{r_+^4}{r^4}\right)\;, \tag{III.3}$$

and vol ( ) denotes the volume form. All the other background fields vanish. Writing the line element on S^5 as

$$d\Omega_5^2 = d\vartheta^2 + \sin^2\vartheta\, d\Omega_4^2\,, \tag{III.4}$$

the potential C_4 corresponding to the 5-form flux $F_5 = dC_4$ is

$$C_4 = \frac{r^4}{L^4} dt\wedge d^3x + L^4 C\left(\vartheta\right)\;\mathrm{vol}\left(S^4\right)\;, \tag{III.5}$$

with

$$C(\vartheta) = \frac{3}{2}\vartheta - \frac{3}{2}\sin\vartheta\cos\vartheta - \sin^3\vartheta\cos\vartheta\,. \tag{III.6}$$

For $L^4 = \lambda\alpha'^2$, where $\lambda \equiv 4\pi g_s N$ is the 't Hooft coupling, the solution (III.1) describes N D3-branes, generically at finite temperature. The black hole horizon radius, r_+, is related to the inverse temperature by

$$r_+ = \frac{\pi L^2}{\beta}\,. \tag{III.7}$$

The zero temperature $AdS_5 \times S^5$ solution is recovered by setting $r_+ = 0$. In this case, we can make the replacement $r \to L^2/z$ to obtain the AdS_5 metric in the standard Poincaré coordinates with boundary at $z = 0$, namely,

$$ds^2_{AdS_5} = \frac{L^2}{z^2}\left(-dt^2 + \sum_{i=1}^{3} dx_i^2 + dz^2\right)\,. \tag{III.8}$$

3.2.2 Classical D5-brane solutions

In the background (III.1), the bosonic part of the D5-brane action is

$$S^{(B)}_{D5} = -T_{D5}\int d^6\xi\sqrt{-\det\left(g+\mathcal{F}\right)_{ab}} + T_{D5}\int C_4 \wedge \mathcal{F}\,, \tag{III.9}$$

where ξ^a, $a = (0,1,2,3,4,5)$ are worldvolume coordinates, g_{ab} is the induced metric, and $\mathcal{F} = d\mathcal{A}$ is the field strength of the gauge field living on the brane. The pullback of C_4 onto the worldvolume is implicit in this expression. The tension of a D5-brane is $T^{-1}_{D5} = (2\pi)^5\,\alpha'^3 g_s$.

The the class of configurations relevant to us are solutions to the equations of motion that follow from (III.9). In this thesis, we consider embeddings such that four of the coordinates ξ^μ, $\mu = (2,3,4,5)$, wrap the $S^4 \subset S^5$ at a constant azimuth angle ϑ, and the remaining two coordinates $\xi^\alpha = (\tau,\sigma)$ span an effective string worldsheet, with induced metric $g_{\alpha\beta}$, in the $aAdS_5$ part of the bulk. By symmetry, the only non-vanishing components of the field strength are (with a slight abuse of notation)

(III.10) $$\mathcal{F}_{\alpha\beta} = \mathcal{F}\epsilon_{\alpha\beta}\,.$$

It follows that

(III.11) $$\det\left(g+\mathcal{F}\right)_{ab} = L^8 \sin^8\vartheta\left(1-\mathcal{F}^2\right)\det g_{\alpha\beta}\,,$$

and the action (III.9) can be written as

(III.12) $$S_{D5}^{(B)} = -\frac{N}{3\pi^2\alpha'}\int d\tau d\sigma\sqrt{-\det g_{\alpha\beta}}\left[\sin^4\vartheta\sqrt{1-\mathcal{F}^2} - C\left(\vartheta\right)\mathcal{F}\right]\,.$$

The prefactor arises from $T_{D5}V_4L^4 = \frac{N}{3\pi^2\alpha'}$, where $V_4 = 8\pi^2/3$ is the volume of the unit S^4.

Quantization of 2-form flux, which is an integral of the equation of motion for $\mathcal{A}_\tau$, and the equation of motion for ϑ are solved by [134] [21]

(III.13) $$\frac{1}{\pi}\left(\vartheta - \sin\vartheta\cos\vartheta\right) = \frac{k}{N}\,,$$

and

(III.14) $$\mathcal{F} = \cos\vartheta\,.$$

Here, $k = 0, 1, \ldots, N$ is the fundamental string charge dissolved on the D5-brane.

One must add to (III.12) appropriate boundary terms [40] [45]

(III.15) $$I_{D5}^{(B)} = -\int d\tau \mathrm{sgn}\left(r'\right)\left(r\pi_r + \mathcal{A}_\tau\pi_{\mathcal{A}}\right)\,,$$

where

(III.16) $$\pi_r = \frac{\partial\mathcal{L}_{D5}}{\partial r'}\,,\qquad \pi_{\mathcal{A}} = \frac{\partial\mathcal{L}_{D5}}{\partial A'_\tau}\,,$$

and the prime denotes a derivative with respect to σ. Putting everything together, one finds that the action of the background D5-brane can be reduced to that of an effective string living in the $aAdS_5$ portion of the 10-dimensional geometry [84]

(III.17) $$S_{D5}^{(B)} + I_{D5}^{(B)} = -\frac{N\sin^3\vartheta}{3\pi^2\alpha'}\left[\int d\tau d\sigma\sqrt{-\det g_{\alpha\beta}} - \int d\tau\,\mathrm{sgn}\left(r'\right) r\frac{\partial\sqrt{-\det g_{\alpha\beta}}}{\partial r'}\right]\,.$$

The D5-brane configurations have an induced metric given by

$$ds^2_{ind} = g_{\alpha\beta} d\xi^\alpha d\xi^\beta + L^2 \sin^2(\vartheta)\, d\Omega_4^2\,, \tag{III.18}$$

and, generically, will preserve only a $SO(5)$ symmetry corresponding to the isometries of S^4. The solution dual to the half-BPS Wilson loop in the antisymmetric representation, however, has an $AdS_2 \times S^4$ worldvolume

$$ds^2_{ind} = L^2 \left(ds^2_{AdS_2} + \sin^2(\vartheta)\, d\Omega_4^2 \right)\,. \tag{III.19}$$

and possesses a $SL(2;\mathbb{R}) \times SO(3) \times SO(5)$ symmetry. The $SL(2;\mathbb{R})$ and $SO(5)$ are realized as isometries of the geometry while the $SO(3)$ corresponds to rotations in the AdS_5 directions transverse to the brane. The solution also preserves half of the targetspace supersymmetries [157], yielding the supergroup $OSp(4^*|4) \subset SU(2,2|4)$. As expected, these coincide with the symmetries preserved by the Wilson loop operator in the dual gauge theory. Notice that the AdS_2 and S^4 factors have different radii.

3.3 Open string excitations

Following the same philosophy as we did for the D3-brane configuration, in this section we consider fluctuations of the bosonic and fermionic degrees of a general class of D5-brane solutions in $AdS_5 \times S^5$. We construct the quadratic action and derive the classical field equations. Thusly, the spectrum of excitations of the half-BPS Wilson loop operators in the anti-symmetric representations of $SU(N)$ is fully derived. Our formalism readily applies to more general backgrounds, including holographic Wilson loop correlators [161] [132] and related finite-temperature configurations [90] [98].

3.3.1 Bosonic fluctuations

Before we show our results, let us define the dynamical variables that parameterize the physical fluctuations. We will make use of well-known geometric relations for embedded manifolds [48], which are reviewed in appendix F. The fields present in (III.9) are the target-space coordinates of the D5-brane x^m and the gauge field components $\mathcal{A}_a$ living on the brane. Both are functions of the worldvolume coordinates ξ^a.

We now recall a few facts from differential geometry that, although known to the reader, we bring to bear explicitly in our calculations. We shall parameterize the fluctuations of x^m around the background coordinates by the generating vector y^m of an exponential map [48]

(III.20) $$x^m \to (\exp_x y)^m = x^m + y^m - \frac{1}{2}\Gamma^m{}_{np} y^n y^p + \mathcal{O}(y^3)\,,$$

thereby obtaining a formulation that is manifestly invariant under bulk diffeomorphisms. Recall that, as familiar from General Relativity, the differences of coordinates are not covariant objects, but vector components are. Here and henceforth, all quantities except the fluctuation variables are evaluated on the background. Locally, the vector components y^m coincide with the Riemann normal coordinates centered at the origin of the exponential map. Riemann normal coordinates are also helpful for performing the calculations, because of a number of simplifying relations that hold at the origin. For example, one can make use of

(III.21) $$\Gamma^m{}_{np} = 0\,, \qquad \Gamma^m{}_{np,q} = -\frac{2}{3}R^m{}_{npq}\,,$$

while the expression for a covariant tensor of rank k is, up to second order in y,

(III.22) $$\begin{aligned} A_{m_1\dots m_k} \to{}& A_{m_1\dots m_k} + A_{m_1\dots m_k;n} y^n \\ &+ \frac{1}{2}\left(A_{m_1\dots m_k;np} + \frac{1}{3}\sum_{l=1}^{k} R^q{}_{npm_l} A_{m_1\dots q\dots m_k}\right) y^n y^p\,. \end{aligned}$$

In the equations that follow, we will implicitly assume the use of a Riemann normal coordinate system. Moreover, we shall drop terms of higher than second order in y. The tangent vectors along the worldvolume (see appendix F), which serve to calculate the pull-back of bulk tensor fields, are given by

(III.23) $$x^m_a \to x^m_a + \nabla_a y^m - \frac{1}{3}R^m{}_{pnq} x^n_a y^p y^q\,.$$

Reparametrization invariance allows us to gauge away the fluctuations that are tangent to the world volume. This leaves us with

(III.24) $$y^m = N^m_{\underline{i}} \chi^{\underline{i}}\,,$$

where the $\chi^{\underline{i}}$ parameterize the fluctuations orthogonal to the worldvolume, or normal fluctuations, and the index $\underline{i}$ runs over all normal directions. The expression above is the natural geometric object related to fluctuations; it has appeared in previous works, for example, [124] and, more explicitly, in [52]. We found it appropriate to provide an explicit account of the origin of this parametrization of the fluctuations. Using the relations summarized in appendix F, this gives rise to

(III.25) $$\nabla_a y^m = -H_{\underline{i}a}{}^b x^m_b \chi^{\underline{i}} + N^m_{\underline{i}} \nabla_a \chi^{\underline{i}}\,,$$

where $H_{\underline{i}a}{}^b$ is the second fundamental form of the background world volume, and ∇_a denotes the covariant derivative including the connections in the normal bundle.

The fluctuations of the gauge field are introduced by

(III.26) $$\mathcal{A}_a \to \mathcal{A}_a + a_a\,.$$

The corresponding fields strength is

(III.27) $$\mathcal{F}_{ab} \to \mathcal{F}_{ab} + f_{ab}\,,$$

where $f_{ab} = \partial_a a_b - \partial_b a_a$.

Following these preliminaries, we now consider fluctuations of the bosonic degrees of freedom of the D5-branes. The goal is to expand the action (III.9), reproduced here for convenience,

(III.28) $$S^{(B)}_{D5} = -T_{D5} \int d^6\xi \sqrt{-\det{(g+\mathcal{F})_{ab}}} + T_{D5} \int C_4 \wedge \mathcal{F}\,,$$

to second order in the fields $\chi^{\underline{i}}$ and a_a. For the Dirac-Born-Infeld term, we make use of the formula

(III.29) $$\sqrt{-\det M} \to \sqrt{-\det M} \left\{ 1 + \frac{1}{2}\mathrm{tr} X + \frac{1}{8}[\mathrm{tr} X]^2 - \frac{1}{4}\mathrm{tr}(X^2) + \mathcal{O}(X^3) \right\}\,,$$

where X denotes the matrix $X = M^{-1}\delta M$, and we have introduced

$$M_{ab} = g_{ab} + \mathcal{F}_{ab}\,. \tag{III.30}$$

Combining (III.22)–(III.25) to obtain the induced metric, we have

$$\delta M_{ab} = -2H_{\underline{i}ab}\chi^{\underline{i}} + f_{ab} + \nabla_a\chi^{\underline{i}}\nabla_b\chi^{\underline{j}}\delta_{\underline{ij}} + \left(H_{\underline{i}a}{}^{c}H_{\underline{j}bc} - R_{mpnq}x^m_a x^n_b N^p_{\underline{i}} N^q_{\underline{j}}\right)\chi^{\underline{i}}\chi^{\underline{j}}\,. \tag{III.31}$$

Substituting (III.31) into (III.29) and using the background relations, one obtains after some calculation

$$\begin{aligned}\sqrt{-\det M_{ab}} \to \frac{\sqrt{-\det \hat{g}_{ab}}}{\sin\vartheta}\Bigg[&\frac{1}{2}\hat{g}^{ab}\left(\delta_{\underline{ij}}\nabla_a\chi^{\underline{i}}\nabla_b\chi^{\underline{j}} + \nabla_a\chi^{\underline{5}}\nabla_b\chi^{\underline{5}}\right) + \frac{1}{4}\hat{g}^{ab}\hat{g}^{cd}f_{ac}f_{bd}\\ &- \frac{1}{2\sin^2\vartheta}\left(H_{\underline{i}\alpha\beta}H_{\underline{j}}{}^{\alpha\beta} + R_{mpnq}g^{\alpha\beta}x^m_\alpha x^n_\beta N^p_{\underline{i}}N^q_{\underline{j}}\right)\chi^{\underline{i}}\chi^{\underline{j}}\\ &+ \frac{2}{L^2\sin^2\vartheta}\left(3\cos^2\vartheta - \sin^2\vartheta\right)\left(\chi^{\underline{5}}\right)^2 + \frac{2\cos^2\vartheta}{L\sin^3\vartheta}\epsilon^{\alpha\beta}f_{\alpha\beta}\chi^{\underline{5}}\\ &+\frac{4\cos\vartheta}{L\sin\vartheta}\chi^{\underline{5}} + \frac{\cos\vartheta}{2\sin^2\vartheta}\epsilon^{\alpha\beta}f_{\alpha\beta} + 1\Bigg]\,,\end{aligned} \tag{III.32}$$

where, as anticipated, $\hat{g}_{ab}$ is the open string metric

$$d\hat{s}^2 = \sin^2\vartheta\left(g_{\alpha\beta}d\xi^\alpha d\xi^\beta + L^2 d\Omega_4^2\right)\,. \tag{III.33}$$

Henceforth, the normal index $\underline{i}$ refers only to the three normal directions within the $aAdS_5$ part of the bulk, as we have indicated explicitly $\chi^{\underline{5}}$ for the normal direction within S^5.

In order to expand the Wess-Zumino term in (III.28), we make use of (III.22), (III.23), (III.27) and the background relations. After some algebra we obtain

$$\begin{aligned}C_4\wedge\mathcal{F} \to d^6\xi\frac{\sqrt{-\det\hat{g}_{ab}}}{\sin\vartheta}\Bigg[&\frac{8\cos^2\vartheta}{L^2\sin^2\vartheta}\left(\chi^{\underline{5}}\right)^2 - \frac{2}{L\sin\vartheta}\epsilon^{\alpha\beta}f_{\alpha\beta}\chi^{\underline{5}}\\ &+\frac{4\cos\vartheta}{L\sin\vartheta}\chi^{\underline{5}} - \frac{C\left(\vartheta\right)}{2\sin^5\vartheta}\epsilon^{\alpha\beta}f_{\alpha\beta} + \frac{C\left(\vartheta\right)\cos\vartheta}{\sin^5\vartheta}\Bigg]\,.\end{aligned} \tag{III.34}$$

Replacing (III.32) and (III.34) in (III.28), the linear terms in $\chi^{\underline{5}}$ are found to cancel as expected for an expansion around a classical solution. The linear term in $f_{\alpha\beta}$ is a total derivative and is canceled by a suitable boundary term. Thus, one ends up with the following quadratic terms in the action,

$$S_{D5}^{(B,2)} = -\frac{T_{D5}}{\sin\vartheta}\int d^6\xi\sqrt{-\det\hat{g}_{ab}}\left[\frac{1}{2}\hat{g}^{ab}\left(\delta_{\underline{ij}}\nabla_a\chi^{\underline{i}}\nabla_b\chi^{\underline{j}} + \nabla_a\chi^{\underline{5}}\nabla_b\chi^{\underline{5}}\right)\right.$$
$$+\frac{1}{4}\hat{g}^{ab}\hat{g}^{cd}f_{ac}f_{bd} - \frac{1}{2\sin^2\vartheta}\left(H_{\underline{i}\alpha\beta}H_{\underline{j}}^{\ \alpha\beta} + R_{mpnq}g^{\alpha\beta}x_\alpha^m x_\beta^n N_{\underline{i}}^p N_{\underline{j}}^q\right)\chi^{\underline{i}}\chi^{\underline{j}}$$
$$\left.-\frac{2}{L^2\sin^2\vartheta}\left(\chi^{\underline{5}}\right)^2 + \frac{2}{L\sin\theta}\hat{\epsilon}^{\alpha\beta}f_{\alpha\beta}\chi^{\underline{5}}\right]. \tag{III.35}$$

The dynamical fields present in (III.35) are the scalar $\chi^{\underline{5}}$, the scalars $\chi^{\underline{i}}$ transforming as a triplet under the $SO(3)$ symmetry of the normal bundle, and the gauge field a_a. They couple to the deformed geometry (III.33) in the expected manner. Given that the effect of deformation is to simply rescale the 2-dimensional factor $g_{\alpha\beta} \to \hat{g}_{\alpha\beta} = \sin^2\vartheta g_{\alpha\beta}$, the spin connection does no change and the covariant derivatives in (III.35) coincide with those computed using $\hat{g}_{ab}$, i.e. $\nabla_a = \hat{\nabla}_a$. Notice also the appearance of the Levi-Civita tensor $\hat{\epsilon}_{\alpha\beta} = \sin^2\vartheta\epsilon_{\alpha\beta}$.

3.3.2 Fermionic fluctuations

We now consider fluctuations of the fermionic degrees of freedom of the D5-branes. This is somewhat easier than the bosonic part, because one just needs the fermionic part of the action, in which all the bosonic fields assume their background values.

The construction of a general quadratic fermionic action was presented in a series of interesting works by Martucci [120, 121, 123–125]. Here we will closely follow the notation and presentation of [124]. For the $aAdS_5 \times S^5$ background the action reduces to

$$S_{D5}^{(F)} = \frac{T_{Dp}}{2}\int d^6\xi\sqrt{-\det\left(g+\mathcal{F}\right)_{ab}}\,\overline{\Theta}\left(1-\Gamma_{D5}\right)\tilde{M}^{ab}\Gamma_b D_a\Theta. \tag{III.36}$$

Here Θ is a doublet of 10d positive chirality Majorana-Weyl spinors, $\overline{\Theta} = i\Theta^\dagger\Gamma^{\underline{0}}$, $\Gamma_a = \partial_a x^m\Gamma_m$ is the pullback of the spacetime Dirac matrices, $\tilde{M}^{ab}$ is the inverse of

$$\tilde{M}_{ab} = g_{ab} + \mathcal{F}_{ab}\tilde{\Gamma}\,, \qquad \tilde{\Gamma} = \Gamma^{11}\otimes\sigma_3\,, \tag{III.37}$$

and Γ_{D5} is a projector ensuring invariance of the action under κ-symmetry. Also, $D_a = \partial_a x^m D_m$ is the pullback of the type IIB covariant derivative, which in our case reads

$$D_m = \nabla_m + \frac{1}{16} \not{F}_{(5)} \Gamma_m \otimes (i\sigma_2) \,. \tag{III.38}$$

As shown in [124], the complete D-brane action is invariant under (linearized) supersymmetry transformations induced by the existence of targetspace Killing spinors. Since the classical embeddings considered here generically do not preserve any of the $AdS_5 \times S^5$ supersymmetries, we do not expect the action for the quadratic fluctuations around these backgrounds to be supersymmetric. For the case of holographic Wilson loop solutions, however, half of the $SU(2,2|4)$ supersymmetries are preserved. Instead of verifying explicitly that the action is supersymmetric, we will show in the next section that the spectrum of excitations falls into multiplets of the appropriate supergroup.

Technically the action in [124] is defined in Lorentzian signature. In section 3.4 we will switch to Euclidean signature to compute the functional determinants due to the D5-brane fluctuations. This raises the problem of imposing the Majorana condition on the 10-dimensional spinors Θ. For our purposes, it will suffice to think of fermions being defined in Lorentzian signature and simply replace $t = ix$ when appropriate.

Now, the inverse of the matrix $\tilde{M}_{ab}$ is found to be

$$\tilde{M}^{\alpha\beta} = \frac{1}{\sin^2\vartheta}\left(g^{\alpha\beta} - \cos\vartheta \epsilon^{\alpha\beta}\tilde{\Gamma}\right) , \qquad \tilde{M}^{\mu\nu} = g^{\mu\nu} \,. \tag{III.39}$$

A short calculation shows that

$$\tilde{M}^{\alpha\beta}\Gamma_\beta = \frac{1}{\sin\vartheta} e^{R\tilde{\Gamma}}\Gamma^\alpha e^{R\tilde{\Gamma}} \,, \qquad \tilde{M}^{\mu\nu}\Gamma_\nu = e^{R\tilde{\Gamma}}\Gamma^\mu e^{R\tilde{\Gamma}} \,, \tag{III.40}$$

where we have defined

$$R = \frac{1}{2}\sinh^{-1}(\cot\vartheta)\,\Gamma \,, \qquad \Gamma = \frac{1}{2}\epsilon_{\alpha\beta}\Gamma^{\alpha\beta} \tag{III.41}$$

Thus,

$$\tilde{M}^{ab}\Gamma_b D_a = e^{R\tilde{\Gamma}}\left[\frac{1}{\sin\vartheta}\Gamma^{\alpha}e^{R\tilde{\Gamma}}D_{\alpha}e^{-R\tilde{\Gamma}} + \Gamma^{\mu}e^{R\tilde{\Gamma}}D_{\mu}e^{-R\tilde{\Gamma}}\right]e^{R\tilde{\Gamma}}\,. \tag{III.42}$$

A derivation of the pullback of the covariant derivative onto the worldvolume is given in Appendix F. Using equation (F.14) and the expression for the RR 5-form flux (III.1) one finds

$$e^{R\tilde{\Gamma}}D_{\alpha}e^{-R\tilde{\Gamma}} = \tilde{\nabla}_{\alpha} - \frac{1}{2}H_{\underline{i}\alpha\beta}\Gamma^{\beta}\Gamma^{\underline{i}}e^{-2R\tilde{\Gamma}} - \frac{1}{4L}\Gamma_{\alpha}\Gamma^{\underline{56789}}\left(1+\Gamma^{11}\right)\otimes(i\sigma_2)\,, \tag{III.43}$$

$$e^{R\tilde{\Gamma}}D_{\mu}e^{-R\tilde{\Gamma}} = \nabla_{\mu} - \frac{1}{2}H_{\underline{5}\mu\nu}\Gamma^{\nu}\Gamma^{\underline{5}} + \frac{1}{4L}\Gamma_{\mu}\Gamma^{\underline{56789}}\left(1+\Gamma^{11}\right)\otimes(i\sigma_2)\,e^{-2R\tilde{\Gamma}}\,, \tag{III.44}$$

where we have abbreviated

$$\tilde{\nabla}_{\alpha} = \nabla_{\alpha} + \frac{1}{4}A_{\underline{ij}\alpha}\Gamma^{\underline{ij}} \tag{III.45}$$

to denote the covariant spinor derivative including the connections in the normal bundle. The extrinsic curvature terms entering this expression are $H^{\underline{5}\mu}{}_{\nu} = (-\cot\vartheta/L)\,\delta^{\mu}{}_{\nu}$ and $H_{\underline{i}\alpha}{}^{\alpha} = 0$, because the 2d part of the background is a minimal surface. Putting these results together yields

$$\begin{aligned}\tilde{M}^{ab}\Gamma_b D_a = {}& e^{R\tilde{\Gamma}}\left[\frac{1}{\sin\vartheta}\Gamma^{\alpha}\tilde{\nabla}_{\alpha} + \Gamma^{\mu}\nabla_{\mu} + \frac{1}{L\sin\vartheta}\Gamma^{\underline{56789}}\otimes(i\sigma_2)\right]e^{R\tilde{\Gamma}} \\ &+ e^{R\tilde{\Gamma}}\left[\frac{2}{L}\left(1+\Gamma^{(0)}_{D5}\right)\cot\vartheta\,\Gamma^{\underline{5}}\right]e^{R\tilde{\Gamma}}\,,\end{aligned} \tag{III.46}$$

where we have replaced $\Gamma^{11} = 1$ since the operator is acting on a positive chirality spinor, and introduced

$$\Gamma^{(0)}_{D5} = \Gamma\Gamma^{\underline{6789}}\otimes\sigma_1\,. \tag{III.47}$$

Notice that $\left(\Gamma^{(0)}_{D5}\right)^2 = 1$.

The final object entering the fermionic action is the projector Γ_{D5}. The general definition can be found in [124]. In our case it reads

$$\Gamma_{D5} = \frac{1}{\sin\vartheta}\Gamma\Gamma^{\underline{6789}}\otimes\sigma_1\left(1+\cot\vartheta\,\gamma\otimes\sigma_3\right)\,, \tag{III.48}$$

and can be rewritten as

$$\Gamma_{D5} = e^{R\tilde{\Gamma}}\Gamma^{(0)}_{D5}e^{-R\tilde{\Gamma}} \tag{III.49}$$

when acting on a conjugate spinor from the right.

Collecting all the formulae, we find that the fermionic action (III.36) becomes

$$\begin{aligned} S^{(F)}_{D5} = \frac{T_{D5}}{2\sin\vartheta}\int d^6\xi\sqrt{-\det\hat{g}_{ab}}\,\overline{\Theta}e^{R\tilde{\Gamma}}\left(1-\Gamma^{(0)}_{D5}\right)\Bigg[&\frac{1}{\sin\vartheta}\Gamma^\alpha\tilde{\nabla}_\alpha + \Gamma^\mu\nabla_\mu \\ &+\frac{1}{L\sin\vartheta}\Gamma^{\underline{56789}}\otimes(i\sigma_2)\Bigg]e^{R\tilde{\Gamma}}\Theta \end{aligned} \tag{III.50}$$

Notice that $\Gamma_\alpha/\sin\vartheta$ can be regarded as the Dirac matrices corresponding to the deformed metric $\hat{g}_{ab}$. This further confirms that the natural worldvolume geometry is given by $\hat{g}_{ab}$ and not the induced metric g_{ab}.

We can further simplify the action by defining a rotated spinor doublet $\Theta' = e^{R\tilde{\Gamma}}\Theta \Leftrightarrow \overline{\Theta}' = \overline{\Theta}e^{R\tilde{\Gamma}}$. To fix κ-symmetry we impose the covariant condition $\tilde{\Gamma}\Theta' = \Theta'$, which sets the lower component of the doublet to zero. The terms that survive this projection are (dropping the primes)

$$S^{(F)}_{D5} = \frac{T_{D5}}{2\sin\vartheta}\int d^6\xi\sqrt{-\det\hat{g}_{ab}}\,\overline{\Theta}\left[\frac{1}{\sin\vartheta}\Gamma^\alpha\tilde{\nabla}_\alpha + \Gamma^\mu\nabla_\mu + \frac{1}{L\sin\vartheta}\Gamma\Gamma^{\underline{5}}\right]\Theta\,, \tag{III.51}$$

where Θ is now a single, 32-component, Majorana-Weyl spinor.

In order to write the action in terms of six-dimensional quantities, as appropriate for a D5-brane, we choose the following representation of the 10d gamma matrices,

$$\Gamma_{\underline{a}} = \gamma_{\underline{a}}\otimes\mathbb{1}_4\,,\qquad \Gamma_{\underline{i}} = \gamma^7\otimes\rho_{\underline{i}}\,,\qquad \Gamma_{\underline{5}} = \gamma^7\otimes\rho_{\underline{5}}\,, \tag{III.52}$$

where $\gamma_{\underline{a}}$ and $\left(\rho_{\underline{i}},\rho_{\underline{5}}\right)$ are $SO(5,1)$ and $SO(4)$ Dirac matrices, respectively, satisfying

$$\{\gamma_{\underline{a}},\gamma_{\underline{b}}\} = 2\eta_{\underline{ab}}\,,\quad \{\rho_{\underline{i}},\rho_{\underline{j}}\} = 2\delta_{\underline{ij}}\,,\quad \{\rho_{\underline{i}},\rho_{\underline{5}}\} = 0\,,\quad \left(\rho_{\underline{5}}\right)^2 = 1\,, \tag{III.53}$$

and $\gamma^7 = \gamma_{\underline{016789}}$ is the $SO(5,1)$ chirality matrix. The 10-d chirality matrix Γ^{11} is then

$$\Gamma^{11} = \gamma^7\otimes\rho^5\,, \tag{III.54}$$

where $\rho^5 = \rho_{\underline{2345}}$. A useful representation of the $SO(4)$ gamma matrices is

$$\rho_{\underline{i}} = \tau_{\underline{i}} \otimes \sigma_2 \,, \qquad \rho_{\underline{5}} = \mathbb{1} \otimes \sigma_1 \,, \tag{III.55}$$

where $\tau^{\underline{i}}$ are Pauli matrices. It follows that

$$\rho^5 = \mathbb{1} \otimes \sigma_3 \,. \tag{III.56}$$

The 10-dimensional spinor Θ can be expanded as

$$\Theta = \sum_{\alpha=\pm} \theta^\alpha \otimes \eta_\alpha \,, \tag{III.57}$$

where, for each $\alpha = \pm$, θ^α is a doublet of $SO(5,1)$ spinors, and η_α are 2-dimensional spinors with $U(1)$ charge α, i.e. $\sigma_2 \eta_\alpha = \alpha \eta_\alpha$. The Weyl condition $\Gamma^{11}\Theta = \Theta$ implies

$$\gamma^7 \theta^\alpha = \alpha \theta^\alpha \,. \tag{III.58}$$

Combining $\theta^\pm$ into a single Dirac spinor doublet

$$\theta = \theta^+ + \theta^- \,, \tag{III.59}$$

the fermionic action reads

$$S^{(F)}_{D5} = \frac{T_{D5}}{2\sin\vartheta} \int d^6\xi \sqrt{-\det \hat{g}_{ab}}\, \overline{\theta} \left[\hat{\gamma}^a \nabla_a + \frac{1}{4} \hat{\gamma}^\alpha \mathcal{A}_{\alpha \underline{ij}} \tau^{ij} + \frac{1}{L \sin\vartheta} \gamma^{\underline{6789}} \right] \theta \,, \tag{III.60}$$

where $\hat{\gamma}_a$ are the 6-dimensional Dirac matrices associated to the deformed metric $\hat{g}_{ab}$. In this expression, the Pauli matrices $\tau^{\underline{ij}} = i\epsilon^{\underline{ijk}}\tau^{\underline{k}}$ act on the doublet structure of θ.

We must point out that it is possible to change the appearance of the "mass" term by performing a chiral rotation $\theta \to e^{i\beta\gamma^7}\theta$. In particular, for $\beta = -\pi/4$ on obtains

$$S^{(F)}_{D5} = \frac{T_{D5}}{2\sin\vartheta} \int d^6\xi \sqrt{-\det \hat{g}_{ab}}\, \overline{\theta} \left[\hat{\gamma}^a \nabla_a + \frac{1}{4} \hat{\gamma}^\alpha \mathcal{A}_{\alpha \underline{ij}} \tau^{ij} - \frac{i}{L \sin\vartheta} \gamma^{\underline{01}} \right] \theta \,, \tag{III.61}$$

In contrast to (III.60), in which the mass term commutes with the 2-d part of the kinetic term and anti-commutes with the 4-d part, in (III.61) it commutes with the 4-d part and anti-commutes with the 4-part. In section 3.4, (III.60) and (III.61) will give rise to two different ways of calculating the heat kernel, with slightly different results.

To conclude the 6-d formulation of the fermionic action, it remains to consider the Majorana condition. To this purpose, we use the intertwiners,

(III.62)
$$B_{(9,1)\pm}\Gamma_{\underline{m}}B^{-1}_{(9,1)\pm} = \pm\Gamma^*_{\underline{m}}, \qquad B^T_{(9,1)\pm} = B_{(9,1)\pm},$$
$$B_{(5,1)\pm}\gamma_{\underline{a}}B^{-1}_{(5,1)\pm} = \pm\gamma^*_{\underline{a}}, \qquad B^T_{(5,1)\pm} = -B_{(5,1)\pm},$$

and

(III.63)
$$B_{(4,0)\pm}\rho_{\underline{i}}B^{-1}_{(4,0)\pm} = \pm\rho^*_{\underline{i}}, \qquad B_{(4,0)\pm}\rho_{\underline{5}}B^{-1}_{(4,0)\pm} = \pm\rho^*_{\underline{5}}, \qquad B^T_{(4,0)\pm} = -B_{(4,0)\pm}.$$

They also satisfy,

(III.64)
$$B_{(9,1)\pm}\Gamma_{11}B^{-1}_{(9,1)\pm} = \Gamma^*_{11}, \quad B_{(5,1)\pm}\gamma^7 B^{-1}_{(5,1)\pm} = \gamma^{7*}, \quad B_{(4,0)\pm}\rho^5 B^{-1}_{(4,0)\pm} = \rho^{5*}.$$

Using the above decomposition, we find that

(III.65)
$$B_{(9,1)\pm} = B_{(5,1)\pm} \otimes B_{(4,0)\pm}$$

It is also easy to see that

(III.66)
$$B_{(4,0)+} = -i\sigma_2 \otimes \mathbb{1}_2\,, \qquad B_{(4,0)-} = -i\sigma_2 \otimes \sigma_3,.$$

Then, writing the doublet θ as

(III.67)
$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix},$$

the Majorana condition $\Theta^* = B_{(9,1)+}\Theta$ becomes the symplectic Majorana condition on the $SO(5,1)$ spinors θ_1 and θ_2, namely,

(III.68)
$$\theta_1^* = B_{(5,1)+}\theta_2\,, \qquad \theta_2^* = -B_{(5,1)+}\theta_1\,.$$

This completes the analysis of the fermionic action.

3.3.3 Equations of motion: bosons

In the computation of the quadratic actions we found the worldvolume fields couple naturally to the open string metric (III.33). To simplify the following analysis, we will rescale the geometry by an overall factor and work with the metric

$$d\hat{s}^2 = g_{\alpha\beta} d\xi^\alpha d\xi^\beta + L^2 d\Omega_4 \,. \tag{III.69}$$

This has the advantage that $\hat{g}_{\alpha\beta} = g_{\alpha\beta}$ so various factors of $\sin\vartheta$ disappear from most expressions.

To derive the bosonic equations of motion, we shall impose the Lorentz gauge

$$\hat{\nabla}_a a^a = 0 \,, \tag{III.70}$$

where $\hat{\nabla}_a$ denotes the covariant derivative with respect to the metric (III.69) and, if acting on fields with indices $\underline{i}$, contains also the appropriate connections for the normal bundle. The condition (III.70) leaves the residual gauge symmetry $a_a \to a_a + \partial_a \lambda$ with $\hat{\nabla}^a \hat{\nabla}_a \lambda = 0$. Taking this into account, the field equations that follow from (III.35) are

$$\left[\delta^{\underline{i}}_{\underline{j}}\, \hat{\nabla}^a \hat{\nabla}_a + H^{\underline{i}}{}_{\alpha\beta} H_{\underline{j}}{}^{\alpha\beta} + R_{mpnq}\, g^{\alpha\beta} x^m_\alpha x^n_\beta N^{\underline{i}p} N^q_{\underline{j}}\right] \chi^{\underline{j}} = 0 \,, \tag{III.71}$$

$$\left(\hat{\nabla}^a \hat{\nabla}_a + \frac{4}{L^2}\right) \chi^{\underline{5}} - \frac{4}{L \sin\vartheta} \epsilon_{\alpha\beta} \nabla^\alpha a^\beta = 0 \,, \tag{III.72}$$

$$\left(\hat{\nabla}^a \hat{\nabla}_a - \frac{1}{2} R_{(2)}\right) a^\alpha - \frac{4 \sin\vartheta}{L} \epsilon^{\alpha\beta} \nabla_\beta \chi^{\underline{5}} = 0 \,, \tag{III.73}$$

$$\left(\hat{\nabla}^a \hat{\nabla}_a - \frac{3}{L^2}\right) a^\mu = 0 \,. \tag{III.74}$$

Here, $R_{(2)}$ denotes the curvature scalar of the 2-d part of the open string metric. So far, the components a_μ and a_α of the gauge fields are not entirely decoupled from each other, because of the gauge condition (III.70). However, we can use the residual gauge freedom to set $\nabla_\alpha a^\alpha = 0$ on-shell. To see this, contract (III.73) with ∇_α, which yields

$$\hat{\nabla}^a \hat{\nabla}_a \nabla_\alpha a^\alpha = 0 \,. \tag{III.75}$$

Thus, for any a^α satisfying (III.75), one can find a residual gauge transformation λ satisfying $\nabla^\alpha\nabla_\alpha\lambda + \nabla_\alpha a^\alpha = 0$ making the fields a_μ and a_α transverse,

$$\nabla_\alpha a^\alpha = \hat{\nabla}_\mu a^\mu = 0\,. \tag{III.76}$$

This still leaves us with the residual gauge transformations satisfying

$$\hat{\nabla}^\mu\hat{\nabla}_\mu\lambda = \nabla^\alpha\nabla_\alpha\lambda = 0\,. \tag{III.77}$$

To continue, we decompose the fields into[1]

$$\chi^{\underline{j}} = \sum_{l=0}^{\infty}\chi_l^{\underline{j}}(\tau,\rho)Y_l(\Omega)\,, \qquad \chi^{\underline{5}} = \sum_{l=0}^{\infty}\chi_l^{\underline{5}}(\tau,\rho)Y_l(\Omega)\,, \tag{III.78}$$

$$a^\alpha = \sum_{l=0}^{\infty}a_l^\alpha(\tau,\rho)Y_l(\Omega)\,, \qquad a^\mu = \sum_{l=0}^{\infty}a_l(\tau,\rho)Y_{l+1}^\mu(\Omega)\,, \tag{III.79}$$

where $Y_l(\Omega)$ and $Y_{l+1}^\mu(\Omega)$ are scalar and transverse vector eigenfunctions of the Laplacian on S^4, respectively. The corresponding eigenvalues and their degeneracies are given by [144]

$$\hat{\nabla}^\mu\hat{\nabla}_\mu Y_l(\Omega) = -\frac{l(l+3)}{L^2}Y_l(\Omega)\,, \qquad D_l(4,0) = \frac{1}{6}(l+1)(l+2)(2l+3)\,, \tag{III.80}$$

$$\hat{\nabla}^\mu\hat{\nabla}_\mu Y_{l+1}^\nu(\Omega) = -\frac{l^2+5l+3}{L^2}Y_{l+1}^\nu(\Omega)\,, \qquad D_{l+1}(4,1) = \frac{1}{2}(l+1)(l+4)(2l+5)\,. \tag{III.81}$$

Substituting (III.78), (III.80) and (III.81) into the field equations (III.71)–(III.74) yields

$$\left[\left(\nabla^\alpha\nabla_\alpha - \frac{l(l+3)}{L^2}\right)\delta^{\underline{i}}_{\underline{j}} + H^{\underline{i}}{}_{\alpha\beta}H_{\underline{j}}{}^{\alpha\beta} + R_{mpnq}\,g^{\alpha\beta}x^m_\alpha x^n_\beta N^{\underline{i}p}N^q_{\underline{j}}\right]\chi_l^{\underline{j}} = 0\,, \tag{III.82}$$

$$\left(\nabla^\alpha\nabla_\alpha - \frac{l(l+3)-4}{L^2}\right)\chi_l^{\underline{5}} - \frac{4}{L\sin\vartheta}\epsilon_{\alpha\beta}\nabla^\alpha a_l^\beta = 0\,, \tag{III.83}$$

$$\left(\nabla^\beta\nabla_\beta - \frac{l(l+3)}{L^2} - \frac{1}{2}R_{(2)}\right)a_l^\alpha - \frac{4\sin\vartheta}{L}\epsilon^{\alpha\beta}\nabla_\beta\chi_l^{\underline{5}} = 0\,, \tag{III.84}$$

$$\left(\nabla^\alpha\nabla_\alpha - \frac{(l+2)(l+3)}{L^2}\right)a_l = 0\,. \tag{III.85}$$

The dynamics of the two components a^α is contained in the field strength $f = \epsilon^{\alpha\beta}\nabla_\alpha a_\beta$. After decomposing f into spherical harmonics on S^4, one can proceed to diagonalize (III.83) and

[1] Notice the index shift for the vector harmonics, which is used to have all sums start from $l = 0$. The sums over other quantum numbers are implicit.

(III.84), which gives rise to the 2-d Klein-Gordon equations

$$\left[\nabla^\alpha\nabla_\alpha - \frac{1}{L^2}(l+3)(l+4)\right]\zeta_l = 0 \qquad \zeta_l = \left[f_l + \frac{\sin\vartheta}{L}(l-1)\chi^{\underline{5}}_l\right], \tag{III.86}$$

$$\left[\nabla^\alpha\nabla_\alpha - \frac{1}{L^2}l(l-1)\right]\eta_l = 0\,, \qquad \eta_l = \left[f_l - \frac{\sin\vartheta}{L}(l+4)\chi^{\underline{5}}_l\right]. \tag{III.87}$$

We should exclude the $l = 0$ case of (III.87), because in this case one can rewrite (III.84) identically as

$$\epsilon^{\alpha\beta}\nabla_\beta\eta_0 = 0\,, \tag{III.88}$$

which implies that this particular mode is not dynamical. A similar result was found in [21]. This matches with the fact that the residual gauge transformation (III.77) is given by a 2-d massless field with $SO(5)$ angular momentum $l = 0$.

To summarize, the classical field equations for the bosonic fluctuations have been reduced to the 2-d field equations (III.82), (III.85), (III.86) and (III.87).

3.3.4 Equations of motion: fermions

Let us now consider the field equations for the fermions. We shall be agnostic about the symplectic Majorana condition (III.68), which can be imposed afterwards. This has the advantage that the following arguments hold also if we switch to Euclidean signature. The Dirac equation following from the action (III.61) is

$$\left[\hat\Gamma^a\tilde\nabla_a - \frac{i}{L}\Gamma^{\underline{01}}\right]\theta = 0\,, \tag{III.89}$$

where now

$$\tilde\nabla_\alpha = \nabla_\alpha + \frac{1}{4}A_{\underline{ij}\alpha}\tau^{\underline{ij}}\,, \qquad \tilde\nabla_\mu = \hat\nabla_\mu \tag{III.90}$$

and $\Gamma_{\underline{a}}$ are $SO(5,1)$ Dirac matrices. Using the $4+2$ decomposition

$$\Gamma^{\underline{\alpha}} = \gamma^{\underline{\alpha}} \otimes \mathbb{1}_4\,, \qquad \Gamma^{\underline{\mu}} = \gamma^{\underline{01}} \otimes \gamma^{\underline{\mu}}\,, \tag{III.91}$$

(III.89) becomes

$$\left[\gamma^{\alpha}\tilde{\nabla}_{\alpha}+\gamma^{\underline{01}}\left(\hat{\gamma}^{\mu}\hat{\nabla}_{\mu}-\frac{i}{L}\right)\right]\theta=0\,. \tag{III.92}$$

Let $\psi_{\mu s}$ be a doublet of 2-d spinors and χ_{ls} a 4-d spinor satisfying the following 2-d and 4-d Dirac equations, respectively,

$$\hat{\gamma}^{\alpha}\tilde{\nabla}_{\alpha}\psi_{\mu s}=s\mu\psi_{\mu s}\,, \qquad (s=\pm 1,\ \mu\geq 0)\,, \tag{III.93}$$

$$\hat{\gamma}^{\mu}\nabla_{\mu}\chi_{ls}=is\frac{l+2}{L}\chi_{ls}\,, \qquad (s=\pm 1,\ l=0,1,2,\ldots)\,. \tag{III.94}$$

The χ_{ls} are just the eigenfunctions of the Dirac operator on the 4-sphere [26]. Then, expanding θ as

$$\theta=\sum_{\mu,l,s,s'}a_{\mu l s s'}\,\chi_{ls'}\otimes\psi_{\mu s}\,, \tag{III.95}$$

and using the property $\gamma^{\underline{01}}\psi_{\mu s}=\psi_{\mu -s}$, (III.92) leads to the following relation for the coefficients,

$$s\mu L a_{\mu l s s'}+i[s'(l+2)-1]a_{\mu l -s s'}=0\,. \tag{III.96}$$

For (III.96) to have a non-trivial solution, it is necessary that

$$\mu L=\begin{cases} l+1 & \text{for } s'=1 \\ l+3 & \text{for } s'=-1\,. \end{cases} \tag{III.97}$$

Summarizing, the classical field equation for the fermionic fluctuations have been reduced to the 2-d Dirac equation (III.93) with the eigenvalues (III.97). Notice, however, that $\psi_{\mu s}$ is a doublet of 2-d Dirac spinors, and $\tilde{\nabla}_{\alpha}$ contains the normal bundle connection term.

3.3.5 Spectrum of operators on half-BPS Wilson loops

The analysis so far has been valid for a general class of D5-brane configurations. As a particular example, we consider the solution dual to a half-BPS Wilson loop in the k-antisymmetric representation of $SU(N)$, and its spectrum of excitations. This solution lives in the zero temperature background (III.8) and has an $AdS_2\subset AdS_5$ worldsheet in addition to wrapping the $S^4\subset S^5$.

Let us determine the geometric quantities needed for the field equations. First, because the bulk is $AdS_5 \times S^5$, the curvature term in (III.82) simply contributes a mass term of $-2/L^2$. Second, as AdS_2 is maximally symmetric, the second fundamental forms $H^{\underline{i}}{}_{\alpha\beta}$ must be proportional to the 2-d induced metric, $g_{\alpha\beta}$. But because they are also traceless (the effective string world-sheet is minimal), we conclude that $H^{\underline{i}}{}_{\alpha\beta} = 0$. Third, an explicit calculation using the formulas in appendix F shows that the $SO(3)$ gauge fields $A_{\underline{ij}\alpha}$ vanish identically.[2] Therefore, the modes of the independent bosonic fields, $\chi_l^{\underline{i}}$, a_l, η_l and ζ_l, satisfy massive Klein-Gordon equations on AdS_2,

(III.98)
$$\left(\nabla^\alpha \nabla_\alpha - m^2\right) \varphi = 0\,.$$

The masses can be read off from (III.82), (III.85), (III.86) and (III.87) and are related to the conformal dimensions of the dual operators by the standard formula

(III.99)
$$h = \frac{1}{2} + \sqrt{\frac{1}{4} + m^2 L^2}\,.$$

For the fermions, the field equation (III.93) is a massive Dirac equation on AdS_2,[3]

(III.100)
$$\left(\gamma^\alpha \nabla_\alpha - m\right) \psi = 0\,,$$

and the (dimensionless) masses mL are given by (III.97). They are related to the conformal dimensions of the dual operators by

(III.101)
$$h = \frac{|m|}{L} + \frac{1}{2}\,.$$

We present our results in table 3.1 in a form similar to table 3 of [52]. The predictions made in that paper are fully confirmed; as expected, the spectrum fits nicely into representations of the supergroup $OSp(4^*|4)$.

[2] This last statement depends, obviously, on the choice of the normal vectors. In general, one gets a pure gauge $A_{\underline{ij}\alpha}$.

[3] Actually, (III.93) is a doublet of Dirac equations, but the (symplectic) Majorana condition that still must be imposed makes it equivalent to a single Dirac equation with an unconstrained spinor.

bosons		
field	m^2L^2	$(h,n)\times(m,l)$
η_l $(l\geq 1)$	$l(l-1)$	$(l,0)\times(0,l)$
ζ_l	$(l+3)(l+4)$	$(l+4,0)\times(0,l)$
a_l	$(l+2)(l+3)$	$(l+3,0)\times(2,l)$
χ_l^i	$(l+1)(l+2)$	$(l+2,1)\times(0,l)$
fermions		
field	mL	$(h,n)\times(m,l)$
ψ_{l+}	$(l+1)$	$(l+\frac{3}{2},\frac{1}{2})\times(1,l)$
ψ_{l+}	$(l+3)$	$(l+\frac{7}{2},\frac{1}{2})\times(1,l)$

Table 3.1: Matching of the bulk fields with multiplets of $OSp(4^*|4)$, cf. table 3 of [52]. The quantum numbers have the following meaning: h is the conformal dimension, $n=0,\frac{1}{2},1$ stand for $SO(3)$ singlets, doublets and triplets, respectively, $m=0,1,2$ for scalar, spinor and vector fields on S^5, respectively, and l is the S^5 angular momentum. In general, $l\geq 0$, except for the field η_l.

3.4 One-loop effective action

Having found the full spectrum of excitations of the half-BPS D5-brane in $AdS_5\times S^5$ dual to the circular Wilson loop, we now proceed to compute the corresponding one-loop effective action using ζ function techniques [91, 153]. Eigenfunctions of the Laplace and Dirac operators in maximally symmetric spaces and their associated heat kernels have been extensively studied [19, 22–26]. We shall follow in spirit the recent calculations of logarithmic corrections to the entropy of black holes in [12, 13], especially with regard to the treatment of zero modes.

We start by providing a general review of the ζ function method, focussing for simplicity on a single massive scalar field and highlighting the scaling properties of the functional determinant. Then, the expansion of the bosonic fields into eigenfunctions on AdS_2 and S^4 is done explicitly, so that we can proceed with the calculation of the bosonic and fermionic heat kernels. At this point we switch to Euclidean signature on the worldvolume.

3.4.1 Computing functional determinants

Let ΔS denote the 1-loop correction to the effective action for a single, real massive scalar field. It is given by

$$e^{-\Delta S} = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int d^d x \sqrt{\det g}\, \phi(-\Box + m^2)\phi} \,. \tag{III.102}$$

where the functional integration measure is defined by

$$1 = \int \mathcal{D}\phi \, e^{-\frac{1}{2}\mu^2 \int d^d x \sqrt{\det g}\, \phi^2} \,. \tag{III.103}$$

The constant μ of dimension inverse length is needed for dimensional reasons, because $[\phi] = L^{1-d/2}$, so that $[S] = 1$. Formally, the functional integral (III.102) is written as a functional determinant

$$e^{-\Delta S} = \left[\mathrm{Det}\left(-\Box + m^2\right)\right]^{-1/2} \,. \tag{III.104}$$

To give an operational definition to these formal expressions, introduce an orthonormal set of eigenstates of $\Box$ satisfying

$$-\Box f_n = \lambda_n f_n \,, \qquad \int d^d x \sqrt{\det g}\, f_n f_m = \delta_{nm} \,. \tag{III.105}$$

If the spectrum of $\Box$ is continuous, the sum is to be understood as an integral with the appropriate spectral measure. In this basis, the field ϕ can be expanded as

$$\phi = \sum_n \phi_n f_n \,. \tag{III.106}$$

Notice the units $[f_n] = L^{-d/2}$ and $[\phi_n] = L$. The integration measure satisfying (III.103) is

$$\mathcal{D}\phi = \prod_n \left(\frac{\mu}{\sqrt{2\pi}} d\phi_n\right) \,, \tag{III.107}$$

and a short calculation shows that (III.102) gives rise to

$$\Delta S = \frac{1}{2}\sum_n \ln \frac{\lambda_n + m^2}{\mu^2} \,. \tag{III.108}$$

In our case, the masses are proportional to $1/L$, where L is the radius of the AdS_2 and S^4 factors. Hence, we can write

(III.109) $$\lambda_n + m^2 = \frac{1}{L^2}\left(\tilde{\lambda}_n + \tilde{m}^2\right)\,,$$

where the $\tilde{\lambda}_n$ are the eigenvalues of the Laplacian $-\tilde{\Box}$ corresponding to $AdS_2 \times S^4$ with unit radius, and $\tilde{m}$ represent dimensionless numbers. Defining the ζ function

(III.110) $$\zeta(s) = \sum_n \left(\tilde{\lambda}_n + \tilde{m}^2\right)^{-s}\,,$$

(III.108) can be expressed as

(III.111) $$\Delta S = -\frac{1}{2}\zeta'(0) - \ln(\mu L)\,\zeta(0) = -\ln\left(L/L_0\right)\,\zeta(0)\,.$$

In the last equation, we have traded the inverse length μ for a renormalization length scale L_0 absorbing also the first term.

In order to study the ζ function, it is convenient to introduce the heat kernel

(III.112) $$K(x,y;t) = \sum_n e^{-\left(\lambda_n + m^2\right)t} f_n(x) f_n(y)\,.$$

Here and henceforth, we have dropped the tilde and implicitly assume unit length $L = 1$. By construction, (III.112) satisfies the heat equation

(III.113) $$\left(\frac{\partial}{\partial t} - \Box + m^2\right) K(x,y;t) = 0\,,$$

with the initial condition $K(x,y;0) = \delta(x,y)$. Setting $x = y$ and integrating over the manifold gives the trace

(III.114) $$Y(t) \equiv \int d^d x\,\sqrt{\det g}\,K(x,x;t) = \sum_n e^{-\left(\lambda_n + m^2\right)t}\,.$$

Then, the ζ function is related to the integrated heat kernel by the Mellin transform,

(III.115) $$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1} Y(t)\,.$$

Notice that since $AdS_2 \times S^4$ is non-compact, the ζ function will diverge; $K(x, x; t)$ is independent of x for a homogeneous space. Thus, $Y(t)$ and $\zeta(s)$ are proportional to the volume of unit $AdS_2 \times S^4$, which must be regularized.

We can separate the integral in (III.115) into

(III.116) $$\zeta(s) = \frac{1}{\Gamma(s)} \left(\int_0^1 dt\, t^{s-1} Y(t) + \int_1^\infty dt\, t^{s-1} Y(t) \right) .$$

The second term converges for any s since $Y(t) \sim e^{-\left(\lambda_0 + m^2\right)t}$ for large t. On the other hand, it can be shown that $Y(t)$ has the asymptotic expansion

(III.117) $$Y(t) \cong \sum_{n=0}^{\infty} a_n t^{(n-d)/2} .$$

as $t \to 0^+$. Substituting this in the first term of (III.116) gives

(III.118) $$\frac{1}{\Gamma(s)} \sum_n \frac{a_n}{s + (n-d)/2} .$$

This shows that $\zeta(s)$ will have poles at $s = d, d-1, \ldots, 1$. The pole at $s = 0$, however, is removed by the gamma function. Inverting (III.115) gives

(III.119) $$Y(t) = \frac{1}{2\pi i} \oint ds\, t^{-s} \Gamma(s) \zeta(s) ,$$

where the integration contour encircles all the poles of the integrand. In particular, (III.119) implies that

(III.120) $$\zeta(0) = a_d .$$

Thus, the problem of computing functional determinants is mapped to the problem of computing the t independent coefficient in the asymptotic expansion of the integrated heat kernel.

The above derivation can be extended to higher spin fields with analogous results. Each field has its own heat kernel and thus its own ζ function. The total effective action is obtained by simply adding the contribution of the integrated heat kernels from all the fields present in the theory. In the case of massless fields, special attention must be paid to possible zero modes of the corresponding kinetic operators as they must be excluded from the definition of the heat kernel. This can be done in an elegant fashion by subtracting from the final heat kernel its value for large t [13]. It turns out, however, that the pieces of the heat kernel that would have to be subtracted have canceled between the contributions from various fields. Moreover, it can be argued that, as far as the logarithmic corrections are concerned, the full heat kernel yields the correct result [13]. For the fluctuations of the D5-brane, a further complication stems from the fact that some modes are coupled and must be diagonalized. We shall deal with these issues at due moment.

3.4.2 Mode decomposition for the bosons

We want to calculate the one-loop effective action for the bosons in the background of the holographic Wilson loop. Let us start with the action (III.35). There are two points we have to address before doing the path integral. First, our fields have physical dimensions $[\chi] = [a] = L$. Thus, to obtain the canonical dimensions used in the last subsection, we must absorb a square root of T_5 into each field.

Second, (III.35) involves the metric $\hat{g}_{ab}$ defined in (III.33), which is $AdS_2 \times S^4$, with both factors of radius $L \sin\vartheta$. The fluctuation fields, however, were defined on the background world volume, which has a "metric" $M_{ab} = g_{ab} + \mathcal{F}_{ab}$, as defined by the Born-Infeld part of the action. This change has an influence on the functional integration measures, which is easily accounted for by a suitable rescaling of the fields. Consider the norms for scalar and vector fields on the background world volume, which are used to define the integration measures,

$$||\chi||^2 = \int d^6\xi \sqrt{\det M_{ab}}\, \chi^2 = \frac{1}{\sin\vartheta} \int d^6\xi \sqrt{\det \hat{g}_{ab}}\, \chi^2 \,, \tag{III.121}$$

and

$$||a||^2 = \int d^6\xi \sqrt{\det M_{ab}}\, M^{ab} a_a a_b = \frac{1}{\sin\vartheta} \int d^6\xi \sqrt{\det \hat{g}_{ab}}\, \hat{g}^{ab} a_a a_b \,. \tag{III.122}$$

The integrals on the right hand sides of (III.121) and (III.122) are the norms that are used to define the integral measures for the path integral on a manifold with metric $\hat{g}_{ab}$. Therefore, in order to write the action (III.35) in terms of integration variables with standard measure and canonical units, we must rescale the fields by

$$\chi \to \sqrt{\frac{\sin\vartheta}{T_{D5}}}\chi\,, \qquad a_a \to \sqrt{\frac{\sin\vartheta}{T_{D5}}}a_a\,. \tag{III.123}$$

Thus, (III.35) gives rise to the Euclidean action

$$\begin{aligned} S^{(B,2)}_{D5,E} = -\int d^6\xi \sqrt{-\det \hat{g}_{ab}} \Bigg[& \frac{1}{2}\delta_{\underline{ij}}\chi^{\underline{i}} \left(\hat{\nabla}_a\hat{\nabla}^a - \frac{2}{L^2\sin^2\vartheta}\right)\chi^{\underline{j}} \\ & + \frac{1}{2}\chi^{\underline{5}}\left(\hat{\nabla}_a\hat{\nabla}^a + \frac{4}{L^2\sin^2\vartheta}\right)\chi^{\underline{5}} - \frac{1}{4}\hat{g}^{ab}\hat{g}^{cd}f_{ac}f_{bd} - \frac{2i}{L\sin\theta}\hat{\epsilon}^{\alpha\beta}f_{\alpha\beta}\chi^{\underline{5}}\Bigg]\,. \end{aligned} \tag{III.124}$$

Notice the i in the last term on the second line, which stems from switching the Levi-Civita tensor to Euclidean signature.

There are two difficulties we have to address in the calculation of the heat kernels. First, there is the gauge invariance, $a_a \to a_a + \partial_a\lambda$. Second, the sector consisting of the gauge field a_α and the scalar $\chi^{\underline{5}}$ must be diagonalized. This problem does not allow us to factorize the heat kernel in a straightforward fashion. Therefore, we choose to do a complete mode expansion of the action into eigenstates on S^4 and AdS_2, which will also allow us to perform the gauge fixing on a state-by-state basis.

Let us start with the mode expansion of the fields appearing in (III.124). Fields that are scalars on S^4 can be decomposed into spherical harmonics, such as

$$\chi^{\underline{5}} = \sum_{l=0}^{\infty} Y_l(\Omega)\,\chi_l^{\underline{5}}(\tau,\sigma)\,. \tag{III.125}$$

We do not explicitly write the sum over the minor angular momentum quantum numbers, which are easily accounted for by remembering the degeneracies $D_l(4,0)$ given in (III.80).

The gauge field a^μ, which is a vector on S^4, decomposes into

$$a^\mu = \sum_{l=1}^{\infty}\left[Y_l^\mu(\Omega)\,a_l(\tau,\sigma) + \sqrt{\frac{L^2}{l(l+3)}}\left(\hat{\nabla}^\mu Y_l(\Omega)\right)b_l(\tau,\sigma)\right]\,. \tag{III.126}$$

In contrast to the expansion of the gauge-fixed classical field, we have to include the longitudinal modes. The square root factor in the second term is necessary in order for the eigenfunctions multiplying the coefficients b_l to be properly normalized.

For the AdS_2 part, we work in the Poincaré metric (D.1). The normalized scalar eigenfunctions of the Laplacian are then given by (D.12), with eigenvalues $-\nabla_\alpha \nabla^\alpha \to \lambda_\nu = (\nu^2 + 1/4)$. Hence, the AdS_2 scalars decompose like

$$\chi^{\underline{5}} = \int_{-\infty}^{\infty} dk \int_{0}^{\infty} d\nu \, f_{(k,\nu)}(x,y)\, \chi^{\underline{5}}_{(k,\nu)}(\Omega) \ . \tag{III.127}$$

For the AdS_2 vector a^α we have to be more careful [13]. Locally, an eigenfunction of the vector Laplacian can be written as $a_\alpha = \lambda^{-1/2}(\nabla_\alpha f_1 + \epsilon_{\alpha\beta}\nabla^\beta f_2)$, where f_1 and f_2 are eigenfunctions of the scalar Laplacian with the same eigenvalue λ. In doing so, we must take care to include a zero mode, which is not a normalizable scalar mode, but which gives rise to a normalizable vector mode. This mode comes from the $\nu = i/2$ case of the scalar eigenfunctions and reads (for unit L)

$$\tilde{f}_k(x,y) = \frac{1}{\sqrt{2\pi|k|}}\, \mathrm{e}^{ikx-|k|y} \ . \tag{III.128}$$

The full expansion of the vector a^α reads, therefore,

$$\begin{aligned} a^\alpha = &\int_{-\infty}^{\infty} dk \int_{0}^{\infty} d\nu \sqrt{\frac{L^2}{\nu^2+1/4}} \left[\left(\nabla^\alpha f_{(k,\nu)}\right) c_{(k,\nu)}(\Omega) + \left(\epsilon^{\alpha\beta}\nabla_\beta f_{(k,\nu)}\right) d_{(k,\nu)}(\Omega)\right] \\ &+ \int_{-\infty}^{\infty} dk \, \left(\nabla^\alpha \tilde{f}_k\right) \tilde{c}_k(\Omega) \ . \end{aligned} \tag{III.129}$$

One can check that the eigenfunctions in front of the mode coefficients $c_{(k,\nu)}$, $d_{(k,\nu)}$ and $\tilde{c}_k$ are orthonormal with respect to the norm $\int d^2x \sqrt{\det g_{\alpha\beta}}\, a^\alpha a_\alpha$. Remember that now $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = +2$, because we are in Euclidean signature.

After doing the mode expansion in (III.124), one obtains

$$
\begin{aligned}
S^{(B,2)}_{D5,E} = \frac{1}{2L^2\sin^2\vartheta} \int\limits_{-\infty}^{\infty} dk \int\limits_{0}^{\infty} d\nu \sum_{l=0(1)}^{\infty} \Bigg\{ & \left[\nu^2 + \frac{1}{4} + l(l+3) + 2\right] \delta_{\underline{ij}} \chi^{\underline{i}}_{l(k,\nu)} \chi^{\underline{j}}_{l(k,\nu)} \\
& + \left[\nu^2 + \frac{1}{4} + l(l+3) - 4\right] \left(\chi^{\underline{5}}_{l(k,\nu)}\right)^2 + 8i\sqrt{\nu^2+\frac{1}{4}}\chi^{\underline{5}}_{l(k,\nu)} d_{l(k,\nu)} + \left[\nu^2 + \frac{1}{4} + l(l+3)\right] d^2_{l(k,\nu)} \\
& + \left[\nu^2 + \left(l+\frac{3}{2}\right)^2\right] a^2_{l(k,\nu)} + \left[\sqrt{l(l+3)}c_{l(k,\nu)} - \sqrt{\nu^2+\frac{1}{4}}b_{l(k,\nu)}\right]^2 \Bigg\}
\end{aligned}
$$

(III.130)

$$
+ \frac{1}{2L^2\sin^2\vartheta} \int\limits_{-\infty}^{\infty} dk \sum_{l=0}^{\infty} l(l+3)\tilde{c}^2_{l,k} \; .
$$

The summation over l starts with 0 for the first two lines, but with 1 for the third line. The last line is the contribution from the special AdS_2 vector modes.

3.4.3 Bosonic heat kernels

Triplet The calculation is simplest for the triplet fields $\chi^{\underline{i}}$. The contribution of each triplet field to the heat kernel is

$$
Y^{\chi^{\underline{i}}}(t) = \mathrm{e}^{-2\bar{t}}\, Y^s_{\widehat{AdS}_2}(\bar{t})\, Y^s_{\widehat{S}^4}(\bar{t}) \; , \tag{III.131}
$$

where $\bar{t} = t/(L\sin\vartheta)^2$, and the heat kernels on $\widehat{AdS}_2$ and $\widehat{S}^4$ (the hats indicate that these are AdS_2 and S^4 of unit radii) are given, respectively, by [13]

$$
Y^s_{\widehat{AdS}_2}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi}\,\mathrm{e}^{-t/4} \int\limits_0^{\infty} d\nu\, \nu \tanh(\pi\nu)\, \mathrm{e}^{-\nu^2 t} \tag{III.132}
$$

and

$$
Y^s_{\widehat{S}^4}(t) = \sum_{l=0}^{\infty} D_l(4,0)\,\mathrm{e}^{-l(l+3)t} = \sum_{l=0}^{\infty} \frac{1}{6}(l+1)(l+2)(2l+3)\,\mathrm{e}^{-l(l+3)t} \; . \tag{III.133}
$$

$V_{\widehat{AdS}_2} = V_{AdS_2}/L^2$ denotes the regulated volume of unit AdS_2. The superscript s on the heat kernels indicates that they are for scalar fields.

Let us rewrite (III.133) by completing the square in the exponent, including the value $l = -1$ in the sum (this does not alter the sum) and shifting the summation index by one. This yields

$$Y^s_{\widehat{S}^4}(t) = -\frac{1}{12}\,\mathrm{e}^{9t/4}\left(1 + 4\frac{\partial}{\partial t}\right)\Sigma^s(t)\;, \tag{III.134}$$

with

$$\Sigma^s(t) = \sum_{l=0}^{\infty}\left(l + \frac{1}{2}\right)\mathrm{e}^{-(l+1/2)^2 t}\;. \tag{III.135}$$

The evaluations of the integral in (III.132) and the infinite sum in (III.135) are carried out in appendix E. Substituting the results into (III.131) we obtain

$$\begin{aligned} Y^{\chi^{\underline{i}}}(t) &= \frac{V_{\widehat{AdS}_2}}{2\pi}\left[-\frac{1}{12}\left(1 + 4\partial_{\bar{t}}\right)\Sigma^s(\bar{t})\right]\left[-\Sigma^s(-\bar{t})\right] \\ &= \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{1}{12\bar{t}^3} - \frac{1}{36\bar{t}^2} - \frac{1}{756} + \cdots\right)\;. \end{aligned} \tag{III.136}$$

Transverse gauge modes Let us integrate over the transverse modes $a_l(k,\nu)$, where $l \geq 1$. From (III.130) we can read off the contribution to the heat kernel

$$Y^a(t) = Y^s_{\widehat{AdS}_2}(\bar{t})\,Y^v_{\widehat{S}^4}(\bar{t})\;, \tag{III.137}$$

where

$$Y^v_{\widehat{S}^4}(t) = \sum_{l=1}^{\infty} D_l(4,1)\,\mathrm{e}^{-(l+1)(l+2)t} = \sum_{l=1}^{\infty}\frac{1}{2}l(l+3)(2l+3)\,\mathrm{e}^{-(l+1)(l+2)t}\;, \tag{III.138}$$

while $Y^s_{\widehat{AdS}_2}(\bar{t})$ is the scalar heat kernel (III.132). The infinite sum in (III.138) can be re-written as

$$Y^v_{\widehat{S}^4}(t) = -\,\mathrm{e}^{t/4}\left(\frac{\partial}{\partial t} + \frac{9}{4}\right)\Sigma^s(t) + 1\;, \tag{III.139}$$

where the 1 can be traced back to a missing $l = 0$ summand after shifting the summation index.

The action (III.124) is invariant under the gauge symmetry $a_\mu \to a_\mu + \partial_\mu\lambda$. Expanding also λ into modes, this translates into

$$\begin{aligned} b_{l(k,\nu)} &\to b_{l(k,\nu)} + \frac{\sqrt{l(l+3)}}{L\sin\vartheta}\lambda_{l(k,\nu)} \qquad (l \geq 1)\;, \\ c_{l(k,\nu)} &\to c_{l(k,\nu)} + \frac{\sqrt{\nu^2 + 1/4}}{L\sin\vartheta}\lambda_{l(k,\nu)} \qquad (l \geq 0)\;. \end{aligned} \tag{III.140}$$

Invariance of (III.130) under (III.140) is immediate upon inspection of the third line in (III.130).

We can now impose a gauge on a mode-by-mode basis. An obvious choice is to fix the coefficients $c_{l(k,\nu)}$, which can be done using Faddeev-Popov. Hence, we must introduce

$$\delta\left(c_{l(k,\nu)}\right)\frac{\sqrt{\nu^2+1/4}}{L\sin\vartheta} \tag{III.141}$$

into the functional integral, where the second factor is the Faddeev-Popov determinant. However, performing the integral over $b_{l(k,\nu)}$, we obtain $\frac{L\sin\vartheta}{\sqrt{\nu^2+1/4}}$, but only for $l\geq 1$. Hence, the net result of gauge fixing, the trivial integration over $c_{l(k,\nu)}$ and the integration over $b_{l(k,\nu)}$ is minus the contribution of an AdS_2 scalar,

$$Y^{gf,b,c}(t) = -Y^{s}_{\widehat{AdS_2}}(\bar{t})\ . \tag{III.142}$$

This compensates the 1 in (III.139). Hence, after gauge fixing, the heat kernel for the vector fields a_μ is

$$\begin{aligned} Y^{a_\mu}(t) = Y^{a}(t) + Y^{gf,b,c}(t) &= \frac{V_{\widehat{AdS_2}}}{2\pi}\left[-\left(\frac{9}{4}+\partial_{\bar{t}}\right)\Sigma^s(\bar{t})\right]\left[-\Sigma^s(-\bar{t})\right] \\ &= \frac{V_{\widehat{AdS_2}}}{2\pi}\left(\frac{1}{4\bar{t}^3}-\frac{7}{12\bar{t}^2}-\frac{19}{1260}+\cdots\right)\ . \end{aligned} \tag{III.143}$$

Mixed sector To integrate over $\chi^5_{l(k,\nu)}$ and $d_{l(k,\nu)}$, we have to deal with the matrix

$$M = \begin{pmatrix} \nu^2+\frac{1}{4}+l(l+3)-4 & 4i\sqrt{\nu^2+\frac{1}{4}} \\ 4i\sqrt{\nu^2+\frac{1}{4}} & \nu^2+\frac{1}{4}+l(l+3) \end{pmatrix}\ . \tag{III.144}$$

Its eigenvalues are

$$(\nu\pm 2i)^2+\frac{1}{4}+l(l+3)+2\ , \tag{III.145}$$

but its determinant can also be written in terms of real factors,

$$\det M = \left[\nu^2+\frac{1}{4}+l(l-1)\right]\left[\nu^2+\frac{1}{4}+(l+3)(l+4)\right]\ . \tag{III.146}$$

The two factors on the right hand side of (III.146) are precisely what one would expect from the classical spectrum.

It is possible to calculate the heat kernel either from the eigenvalues (III.145) or the factors in (III.146). The results of the calculations differ in the scheme dependent divergent terms $1/t^2$ and $1/t$, but we shall perform both calculations, because a similar ambiguity will be encountered for the fermions.

The heat kernel calculation using the eigenvalues (III.145) is similar to the situation encountered in [13], and we shall follow the treatment of that paper. The effect of the mixing between the scalar and the gauge field is a complex shift of the AdS_2 eigenvalue compared to (III.132). Hence, the integrated heat kernel for the $\chi^{\underline{5}}$ and d integration is

(III.147) $$Y_1^{\chi^{\underline{5}},d}(t) = \mathrm{e}^{-2\bar{t}}\, Y^s_{\widehat{S}^4}(\bar{t}) \left[2Y^s_{\widehat{AdS}_2}(\bar{t}) + \delta Y^s_{\widehat{AdS}_2}(\bar{t})\right] ,$$

where

(III.148) $$\delta Y^s_{\widehat{AdS}_2}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi}\, \mathrm{e}^{-t/4} \int\limits_0^\infty d\nu\, \nu \tanh(\pi\nu) \left[\mathrm{e}^{-(\nu-2i)^2 t} + \mathrm{e}^{-(\nu+2i)^2 t} - 2\, \mathrm{e}^{-\nu^2 t}\right] .$$

For the first two terms in the integrand of (III.148), we shift the integration variables to $\nu - 2i$ and $\nu + 2i$, respectively, such as to obtain the same exponent as in the third term. Then, we deform the integral contours such that we have integrals from $-2i$ ($+2i$) to 0 (staying to the right of the imaginary axis) and from 0 to ∞. The latter cancel against the third term in (III.148). Finally, switching the sign of the integration variable in one of the two remaining integrals, they can be combined into

(III.149) $$\delta Y^s_{\widehat{AdS}_2}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi}\, \mathrm{e}^{-t/4} \oint d\nu\, (\nu - 2i) \tanh(\pi\nu)\, \mathrm{e}^{-\nu^2 t} ,$$

where the integration contour circles clockwise around the poles at $\nu = i/2$ and $\nu = 3i/2$. The residue theorem then yields

(III.150) $$\delta Y^s_{\widehat{AdS}_2}(t) = -\frac{V_{\widehat{AdS}_2}}{2\pi} \left(\mathrm{e}^{2t} + 3\right) .$$

Putting everything together, we get

$$\begin{aligned}
Y_1^{\chi^5,d}(t) &= Y^s_{\widehat{S}^4}(\bar t)\left[2\,\mathrm{e}^{-2\bar t}\,Y^s_{\widehat{AdS}_2}(\bar t) - \frac{V_{\widehat{AdS}_2}}{2\pi}\left(3\,\mathrm{e}^{-2\bar t}+1\right)\right]\\
&= \frac{V_{\widehat{AdS}_2}}{2\pi}\left[-\frac{1}{12}\left(1+4\partial_{\bar t}\right)\Sigma^s(\bar t)\right]\left[-2\Sigma^s(-\bar t) - 3\,\mathrm{e}^{\bar t/4} - \mathrm{e}^{9\bar t/4}\right]\\
&= \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{1}{6\bar t^3} - \frac{13}{18\bar t^2} - \frac{1}{3\bar t} - \frac{551}{1890} + \cdots\right)\,. \qquad \text{(III.151)}
\end{aligned}$$

Strictly speaking, we should have removed a zero mode by subtracting the value of the integrated heat kernel at $t=\infty$. One easily finds from (III.133) that $Y^s_{\widehat{S}^4}(\infty)=1$, so the last term in the brackets on the first line of (III.151) contains a zero mode. We shall ignore this for the moment, because the subtraction can be done at the very end [13].

Let us now consider the alternative choice, namely, we perform the calculation using the factors of the determinant (III.146). In this case we get

$$Y_2^{\chi^5,d}(t) = Y^s_{\widehat{AdS}_2}(\bar t)\sum_{l=0}^{\infty} D_l(4,0)\left[\mathrm{e}^{-l(l-1)\bar t} + \mathrm{e}^{-(l+3)(l+4)\bar t}\right]\,. \qquad \text{(III.152)}$$

The infinite sum can be re-written as

$$\sum_{l=0}^{\infty} D_l(4,0)\left[\mathrm{e}^{-l(l-1)t} + \mathrm{e}^{-(l+3)(l+4)t}\right] = \frac{2}{3}\,\mathrm{e}^{t/4}\left(-\partial_t + \frac{47}{4}\right)\Sigma^s(t) + 2\,, \qquad \text{(III.153)}$$

where the 2 stems from extra terms due to shifts of the summation index. Thus, the final result is

$$\begin{aligned}
Y_2^{\chi^5,d}(t) &= \frac{V_{\widehat{AdS}_2}}{2\pi}\left[\frac{2}{3}\left(\frac{47}{4} - \partial_{\bar t}\right)\Sigma^s(\bar t) + 2\,\mathrm{e}^{-\bar t/4}\right]\left[-\Sigma^s(-\bar t)\right]\\
&= \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{1}{6\bar t^3} + \frac{35}{18\bar t^2} + \frac{1}{\bar t} - \frac{551}{1890} + \cdots\right)\,. \qquad \text{(III.154)}
\end{aligned}$$

As anticipated, the results (III.151) and (III.154) differ in the scheme dependent $1/t^2$ and $1/t$ terms. It is worth noting that the relevant terms for the final result, that is, the leading $1/t^3$ terms and the constant terms, are identical in both choices.

Special modes Finally, let us integrate over the special modes $\tilde c$. The AdS_2 part of their heat kernel is obtained from the wave functions (III.128) as

$$K^{\tilde c}_{\widehat{AdS}_2}(t) = \int_{-\infty}^{\infty} dk\,(\nabla^\mu \tilde f_k)(\nabla_\mu \tilde f_k) = \int_{-\infty}^{\infty} dk\,\frac{2k^2y^2}{2\pi|k|}\,\mathrm{e}^{-2|k|y} = \frac{1}{2\pi}\,. \qquad \text{(III.155)}$$

This is independent of t, because the special modes are zero modes on AdS_2.

Thus, the integrated heat kernel for the special AdS vector modes is

$$Y^{\tilde{c}}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi} Y^{s}_{\widehat{S}^4}(\bar{t}) = \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{1}{6\bar{t}^2} + \frac{1}{3t} + \frac{29}{90} + \cdots\right) . \tag{III.156}$$

Note that we have not subtracted the zero mode $l = 0$. However, one can recognize that $Y^{\tilde{c}}(t)$ cancels the third term in the brackets on the first line of (III.151), which contains the zero mode from the $(\chi^{\underline{5}}, d)$ sector, as discussed above. Thus, all bosonic zero modes cancel precisely, and no further subtraction is necessary.

All bosonic modes Let us put together the results for all bosonic fields, (III.136), (III.143), (III.151) [or (III.154)] and (III.156),

$$Y^{bos}(t) = 3Y^{\chi^{\underline{i}}}(t) + Y^{a_\mu}(t) + Y^{\chi^{\underline{5}},d}(t) + Y^{\tilde{c}}(t) . \tag{III.157}$$

Using (III.151) for the mixed sector, we obtain

$$Y^{bos}_1(t) = \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{2}{3\bar{t}^3} - \frac{11}{9\bar{t}^2} + \frac{11}{945} + \cdots\right) , \tag{III.158}$$

while using (III.154) gives rise to

$$Y^{bos}_2(t) = \frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{2}{3\bar{t}^3} + \frac{13}{9\bar{t}^2} + \frac{4}{3\bar{t}} + \frac{11}{945} + \cdots\right) . \tag{III.159}$$

3.4.4 Fermionic heat kernels

We have seen in section 3.3.2 that there are equivalent ways of writing the 6-d fermionic action that are related to each other by chiral rotations. As is well known [61], the fermion integration measure is, in general, not invariant under a chiral rotation in the presence of curvature or gauge fields. To detect whether this is an issue here, let us calculate the fermionic heat kernels corresponding to the actions (III.60) and (III.61), in both cases using a the standard measure for the fermions. We will find that the resulting heat kernels differ in the scheme-dependent $1/t^2$ and $1/t$ terms, but the leading $1/t^3$ term and the constant term are identical, just as we found in the mixed sector of the bosons. This implies that we can safely ignore generic problems with the measure under chiral rotations. Remember that in (III.60) and (III.61) the mass term commutes with either the 2-d or the 4-d part of the kinetic term and anti-commutes with the other. In both cases, the two 6-d spinors in the doublet are not coupled, because $A_{\underline{ij}\alpha} = 0$, and the symplectic Majorana condition (III.68) reduces the doublet to a single independent Dirac spinor, giving rise to a factor of 2 in the action.

Let us start with (III.60). Arguing as for the bosons, we find that θ must be re-scaled like a scalar, so that (III.60) gives rise to

$$S^{(F)}_{D5,E} = \int d^6\xi \sqrt{\det \hat{g}_{ab}}\, \bar{\theta} \left[\hat{\Gamma}^a \nabla_a + \frac{1}{L} \hat{\Gamma}^{\underline{6789}} \right] \theta \,. \tag{III.160}$$

Writing the Dirac operator in the brackets of (III.160) as

$$D = \hat{\Gamma}^\mu \nabla_\mu + \left(\hat{\Gamma}^\alpha \nabla_\alpha + \frac{1}{L} \hat{\Gamma}^{\underline{6789}} \right) \,, \tag{III.161}$$

one can verify that the two terms on the right hand side anti-commute. The 4-d Dirac operator on S^4, $\hat{\Gamma}^\mu \nabla_\mu$, has eigenvalues $\pm i(l+2)/L$ ($l = 0, 1, 2, \ldots$) with degeneracy $D_l(4, \frac{1}{2}) = \frac{2}{3}(l+1)(l+2)(l+3)$ [26]. The 2-d Dirac operator on AdS$_2$, $\hat{\Gamma}^\alpha \nabla_\alpha$ has a continuous spectrum $i\lambda/L$ ($\lambda \geq 0$; the spectral measure can be found in [13, 26]). Taking the square of D, we get

$$D^2 = \left(\hat{\Gamma}^\mu \nabla_\mu \right)^2 + \left(\hat{\Gamma}^\alpha \nabla_\alpha + \frac{1}{L} \hat{\Gamma}^{\underline{6789}} \right)^2 \,. \tag{III.162}$$

Because $\hat{\Gamma}^{\underline{6789}}$ commutes with $\hat{\Gamma}^\alpha \nabla_\alpha$ and has eigenvalues ± 1, we obtain the integrated heat kernel as

$$Y^f_1(t) = -Y^f_{\widehat{S}^4}(\bar{t}) \left[2Y^f_{\widehat{AdS}_2}(\bar{t}) + \delta Y^f_{\widehat{AdS}_2}(\bar{t}) \right] \,, \tag{III.163}$$

where

$$Y^f_{\widehat{S}^4}(t) = -\sum_{l=0}^{\infty} D_l(4, \tfrac{1}{2})\, \mathrm{e}^{-(l+2)^2 t} \,, \tag{III.164}$$

$$Y^f_{\widehat{AdS}_2}(t) = -\frac{V_{\widehat{AdS}_2}}{2\pi} 2 \int_0^\infty d\lambda\, \lambda \coth(\pi\lambda)\, \mathrm{e}^{-\lambda^2 t} \tag{III.165}$$

and

$$\delta Y^f_{\widehat{AdS}_2}(t) = -\frac{V_{\widehat{AdS}_2}}{2\pi} 2 \int_0^\infty d\lambda\, \lambda \coth(\pi\lambda) \left[\mathrm{e}^{-(\lambda+i)^2 t} + \mathrm{e}^{-(\lambda-i)^2 t} - 2\, \mathrm{e}^{-\lambda^2 t}\right] . \tag{III.166}$$

The S^4 part (III.164) is re-written as

$$Y^f_{\widehat{S}^4}(t) = \frac{2}{3} \left(\partial_t + 1\right) \Sigma^f(t) , \tag{III.167}$$

where we have introduced

$$\Sigma^f(t) = \sum_{l=0}^{\infty} l\, \mathrm{e}^{-l^2 t} . \tag{III.168}$$

The explicit evaluation of Σ^f and the integral in (III.165) are relegated to appendix E. The expression (III.166) is obtained along the lines of the first mixed sector calculation in section 3.4.3. One obtains the contour integral

$$\delta Y^f_{\widehat{AdS}_2}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi} 2 \oint_{0^+}^{0^-} d\lambda\, (\lambda - i) \coth(\pi\lambda)\, \mathrm{e}^{-\lambda^2 t} , \tag{III.169}$$

where the contour runs from 0^+ to i along the right of the imaginary axis and back to 0^- along the left. Note that there are no poles inside the contour, and the integrand is regular at $\lambda = i$. However, we cannot close the contour due to the pole at $\lambda = 0$, so that the value of the integral must be defined as the principal value (half of the residue value),

$$\delta Y^f_{\widehat{AdS}_2}(t) = \frac{V_{\widehat{AdS}_2}}{2\pi} 2\pi i \ \mathrm{Res}_{\lambda=0} \left[(\lambda - i) \coth(\pi\lambda)\, \mathrm{e}^{-\lambda^2 t}\right] = \frac{V_{\widehat{AdS}_2}}{2\pi}\, 2 . \tag{III.170}$$

Collecting everything together, we obtain

$$\begin{aligned} Y^f_1(t) &= -\frac{V_{\widehat{AdS}_2}}{2\pi} \left[\frac{2}{3} \left(\partial_{\bar{t}} + 1\right) \Sigma^f(\bar{t})\right] \left[4\Sigma^f(-\bar{t}) + 2\right] \\ &= -\frac{V_{\widehat{AdS}_2}}{2\pi} \left(\frac{2}{3t^3} - \frac{11}{9t^2} + \frac{2}{3t} - \frac{271}{3780} + \cdots\right) . \end{aligned} \tag{III.171}$$

Starting, instead, with the action (III.61), we have the Dirac operator[4]

$$D = \left(\hat{\Gamma}^{\mu}\nabla_{\mu} - \frac{i}{L}\hat{\Gamma}^{\underline{01}}\right) + \hat{\Gamma}^{\alpha}\nabla_{\alpha} \,. \tag{III.172}$$

Analogous arguments as above lead to the integrated heat kernel

$$Y_2^f(t) = Y^f_{\widehat{AdS}_2}(\bar{t}) \sum_{l=0}^{\infty} D_l(4, \tfrac{1}{2}) \left[\mathrm{e}^{-(l+1)^2\bar{t}} + \mathrm{e}^{-(l+3)^2\bar{t}}\right] \,. \tag{III.173}$$

After a short calculation, the infinite sum can be re-written as

$$\sum_{l=0}^{\infty} D_l(4, \tfrac{1}{2}) \left[\mathrm{e}^{-(l+1)^2 t} + \mathrm{e}^{-(l+3)^2 t}\right] = \frac{4}{3}\left(2 - \partial_t\right)\Sigma^f(t) \,. \tag{III.174}$$

Hence, after substituting the results into (III.173), we obtain

$$\begin{aligned} Y_2^f(t) &= -\frac{V_{\widehat{AdS}_2}}{2\pi}\left[\frac{8}{3}\left(\partial_{\bar{t}} - 2\right)\Sigma^f(\bar{t})\right]\Sigma^f(-\bar{t}) \\ &= -\frac{V_{\widehat{AdS}_2}}{2\pi}\left(\frac{2}{3t^3} + \frac{13}{9t^2} - \frac{271}{3780} + \cdots\right) \,. \end{aligned} \tag{III.175}$$

As already anticipated from the results of the mixed sector bosons, the two ways of calculating the heat kernel lead to results that differ in the scheme-dependent $1/t^2$ and $1/t$ terms, but yields identical results for the leading $1/t^3$ and the constant terms.

3.4.5 Combining bosons and fermions

We are now in a position to give the full answer for the heat kernel. As we have two slightly different expressions for the bosons and two for the fermions, there would be four different combinations. One can readily see that the leading $1/t^3$ term cancels in all of them, and the constant term, which is responsible for the scaling, is always the same. We can, however, make the following nice observation, which indicates that supersymmetry does more than just canceling the leading term. It appears natural to combine (III.158) with (III.171), because the heat kernels of the mixed sector bosons and the fermions were calculated with a shift of the eigenvalues on the AdS$_2$ part. Similarly,

[4] In Euclidean signature, the 2-d chirality matrix is $\hat{\Gamma}^{\underline{01}} = i\hat{\Gamma}^{\underline{0}}\hat{\Gamma}^{\underline{1}}$, so that the property $(\hat{\Gamma}^{\underline{01}})^2 = 1$ is maintained.

we should add (III.159) and (III.175), for which the eigenvalue shifts happened on the S^4 part. In these combinations, also the $1/t^2$ terms cancel, and we obtain

$$Y_1(t) = Y_1^s(t) + Y_1^f(t) = \frac{V_{\widehat{AdS_2}}}{2\pi}\left(-\frac{2}{3t} + \frac{1}{12} + \cdots\right) , \tag{III.176}$$

$$Y_2(t) = Y_2^s(t) + Y_2^f(t) = \frac{V_{\widehat{AdS_2}}}{2\pi}\left(\frac{4}{3t} + \frac{1}{12} + \cdots\right) . \tag{III.177}$$

It remains to regularize the infinite volume $V_{\widehat{AdS_2}}$, for which we follow the treatment of [12] complemented with a field theory prescription due to Polyakov [137]. For the circular Wilson loop, it is appropriate to describe unit AdS_2 by the metric

$$ds^2 = d\eta^2 + \sinh^2\eta \; d\phi^2 . \tag{III.178}$$

To regularize the volume we introduce a cut-off η_0, so that the regularized volume of AdS_2 is $2\pi(\cosh\eta_0 - 1)$. In the context of corrections to the entropy of black holes [13] the interpretation of the regularization is as follows. When substituted in the effective action, the term proportional to $\cosh\eta_0$ gives rise, up to a term that vanishes when $\eta_0 \to \infty$, to a divergent contribution $\beta\Delta E$, where $\beta \sim 2\pi\sinh\eta_0$ is the inverse temperature and ΔE is the shift in the ground state energy due to the introduction of the cut-off. This regularization has a simple interpretation on the field theory side as well. In [137], Polyakov studied the evaluation of vacuum expectation values of general Wilson loops and determined a divergent term that is proportional to the length of the contour and can be interpreted as the mass renormalization of the test particle traveling around the contour. Either interpretation leads, for the one-loop correction, to

$$V_{\widehat{AdS_2}} = -2\pi . \tag{III.179}$$

Let us now collect the various pieces and give the final result. Using (III.111), (III.120), (III.176) and (III.179) taking into account also that the appropriate radius of the manifold for canonically normalized fields is $L\sin\vartheta$, as discussed after (III.124), we find for the one-loop effective action

$$\Delta S = \frac{1}{12} \ln \frac{L \sin\vartheta}{L_0} \,. \tag{III.180}$$

3.5 Discussion and conclusions

In this chapter, we have explicitly treated the D5-brane configuration dual to the half-BPS circular Wilson loop in the totally antisymmetric representation. We derived the fluctuations in both, the bosonic and the fermionic sectors. We have also verified that the excitations fall precisely into the expected supermultiplets of $OSp(4^*|4)$. Lastly, we computed the one-loop determinants and provided an answer for the effective action at the one-loop level.

Our work is largely motivated by the applications to the Wilson loops and the potential to take the correspondence beyond the classical ground state by incorporating quantum corrections. This provides a step towards being able to directly compare one-loop corrections from the field theory (Matrix model) and gravity (D-brane) sides. More generally, our work represents a systematic exploration of the various issues that can arise during the quantization of extended objects in the context of the AdS/CFT correspondence. We have encountered and resolved various ambiguities and in the process shed some light on the type of issues that need to be resolved if a coherent quantization of extended objects in curved backgrounds is to be achieved. For example, we hope to have fully clarified the, at times *ad hoc*, process of computing the action for the quadratic fluctuations by explicitly highlighting the differential geometric nature of the fluctuations. We also resolved various technical issues in the computation of the heat kernel for fermions and showed a natural way to determine a scheme. More importantly, at least in our example, we witness that the role of supersymmetry seems to go beyond the expected cancelation of the leading divergence.

There are a few very interesting problems that follow naturally from our work, and we finish by highlighting some of them:

- A natural direction is the calculation and comparison with the matrix model. We hope to report on this interesting issue in an upcoming publication. The task at hand, although conceptually clear, is plagued with many technical issues. Some of these issues are generic to the whole program of comparing expectation values of operators in the field theory and in the gravity dual. We mentioned in the introduction that, even in the apparently simple case of the Wilson loop in the fundamental representation, an agreement has not been found [46, 57, 105, 146]. Hopefully, the extra knob that constitutes the representation might lead to some simplifications.

- In this paper we did not discuss the field theory dual beyond the mere mentioning of the role as half BPS Wilson loops. An important interpretation is provided by the D5-branes as a dual to a one-dimensional defect CFT and has been quoted in recent works as a model for interesting condensed matter phenomena related to quantum impurity models [83, 127, 145]. A similar interpretation of D6-branes as dual descriptions of fermionic impurities in $\mathcal{N}=6$ supersymmetric Chern-Simons-matter theories in $2+1$ dimensions has been advanced in [14]. In such contexts, uncovering the precise role of the spectrum of excitations should lead to a deeper understanding of the interactions of the system.

- More generally, our paper provides a first solid step in the direction of analyzing extended objects at the quantum level. It seems that the analysis of *conformal* branes, that is branes whose world volume contains AdS factors, avoids dealing with the daunting issues encountered in the quantization of extended objects in asymptotically flat spacetimes. We plan to pursue this analysis in the future.

- Recently, Sen and collaborators have studied corrections to the entropy of various black hole configurations using techniques similar to those utilized here. The key technical fact that the near horizon geometry of various black holes contains AdS factors seems to provide a tantalizing playground for our methods. We hope that understanding the quantization of such structures at a deeper level might help clarify difficult issues in black hole physics.

CHAPTER IV

Fermions in consistent truncations of eleven-dimensional supergravity on squashed Sasaki-Einstein manifolds

We now switch gears and abandon the holographic description of Wilson loops to engage with another important area of String Theory, namely, consistent truncations of type IIB supergravity.

In this chapter, we discuss the dimensional reduction of fermionic modes in a recently found class of consistent truncations of $D = 11$ supergravity compactified on squashed seven-dimensional Sasaki-Einstein manifolds. Such reductions are of interest, for example, in that they have $(2+1)$-dimensional holographic duals, and the fermionic content and their interactions with charged scalars are an important aspect of their applications. We derive the lower-dimensional equations of motion for the fermions and exhibit their couplings to the various bosonic modes present in the truncations under consideration, which most notably include charged scalar and form fields. We demonstrate that our results are consistent with the expected supersymmetric structure of the lower dimensional theory, and apply them to a specific example which is relevant to the study of $(2+1)$-dimensional holographic superconductors. This chapter is based on a collaboration with Ibrahima Bah, Juan Ignacio Jottar, Robert Leigh and Leopolodo Pando Zayas, published in [10].

4.1 Introduction

Over the last decade, the gauge/gravity correspondence [3, 75, 115, 155] has generated an unprecedented interest in the construction of new classes of supergravity solutions. The initial efforts were naturally directed at the construction of supergravity backgrounds dual to gauge theories displaying confinement and chiral symmetry breaking [103, 119]. More recently, the search for supergravity backgrounds describing systems that might be relevant for condensed matter physics has considerably expanded our knowledge of classical gravity and supergravity solutions. These include hairy black holes relevant for a holographic description of superfluidity [76, 86, 87], and both extremal and non-extremal solutions with non-relativistic asymptotic symmetry groups (see, for example, [1, 11, 93, 113, 150]).

Since we are usually interested in lower-dimensional physics, the ability to reduce ten or eleven-dimensional supergravity solutions is central. However, only in a few cases can one explicitly construct the full non-linear Kaluza-Klein (KK) spectrum. In the context of eleven-dimensional supergravity, one of the few such examples where the full supersymmetric spectrum of the lower-dimensional theory was worked out at the non-linear level is the reduction of $D = 11$ supergravity on S^4 obtained in [129, 130]. In other cases, the best that can be done is to work with a "consistent truncation" where only a few low-energy modes are taken into account. In this context, by a consistent truncation we mean that any solution of the lower-dimensional effective theory can be uplifted to a solution of the higher dimensional theory. Typically, the intuitive way of thinking about consistent truncations includes the assumption that there is a separation of energy scales that allows one to keep only the "light" fields emerging from the compactification, in such a way that they do not source the tower of "heavy" modes they have decoupled from. Often another principle at work in consistent reductions involves the truncation to chargeless modes when such charges can be defined from the isometries of the compactification manifold; for example, this is the argument behind the consistency of compactifications on tori, where the massless fields carry no charge under the U(1)^n gauge symmetry.

The kind of solutions we are interested in in this paper have as precursors some natural generalizations of Freund-Rubin solutions [60] of the form $AdS_4 \times SE_7$ in $D = 11$ supergravity, where SE_7 denotes a seven-dimensional Sasaki-Einstein manifold. In [66], solutions of $D = 11$ supergravity of this form were shown to have a consistent reduction to minimal $N = 2$ gauged supergravity in four dimensions. Furthermore, a conjecture was put forward in [66], asserting that for any supersymmetric solution of $D = 10$ or $D = 11$ supergravity that consists of a warped product of AdS_{d+1} with a Riemannian manifold M, there is a consistent KK truncation on M resulting in a gauged supergravity theory in $(d+1)$-dimensions.[1] This is a non-trivial statement, since consistent truncations of supergravity theories are hard to come by, even in the cases where the internal manifold is a sphere. While these consistent truncations to massless modes are difficult to construct, the reductions including a finite number of charged (massive) modes were believed to be, in most cases, necessarily inconsistent. In this light, the results of [1, 93, 113] had a quite interesting by-product: while searching for solutions of Type IIB supergravity with non-relativistic asymptotic symmetry groups, consistent five-dimensional truncations including massive bosonic modes were constructed. In particular, massive scalars arise from the breathing and squashing modes in the internal manifold, which is then a "deformed" Sasaki-Einstein space, generalizing the case of breathing and squashing modes on spheres that had been studied in [17,110] (see [18], also). The corresponding truncations including massive modes in $D = 11$ supergravity on squashed SE_7 manifolds were then discussed in [62], and we will use them as the starting point for our work.

While the supergravity truncations we have mentioned above are interesting in their own right, they serve the dual purpose of providing an arena for testing and exploring the ideas of gauge/gravity duality, and in particular its applications to the description of strongly-coupled condensed matter systems. In fact, even though the initial holographic models of superfluids [76, 86, 87] and non-relativistic theories [11, 150] were of a phenomenological ("bottom-up") nature, it soon became apparent that it was desirable to provide a stringy ("top-down") description of these systems. Indeed, a description in terms of ten or eleven-dimensional supergravity backgrounds sheds light on

[1] In the context of holography, the corresponding lower-dimensional modes are dual to the supercurrent multiplet of the d-dimensional dual CFT.

the existence of a consistent UV completion of the lower-dimensional effective bulk theories, while fixing various parameters that appear to be arbitrary in the bottom-up constructions. An important step in this direction was taken in [64, 65], where a $(2+1)$-dimensional holographic superconductor was embedded in M-theory, the relevant feature being the presence of a complex (charged) bulk scalar field supporting the dual field theory condensate for sufficiently low temperatures of the background black hole solution, with the conformal dimensions of the dual operator matching those of the original examples [86, 87]. At the same time, a model for a $(3+1)$-dimensional holographic superconductor embedded in Type IIB string theory was constructed in [77].

Some of the Type IIB truncations have been recently brought into the limelight again, and a more complete and formal treatment of the reduction has been reported. In particular, consistent $N = 4$ truncations of Type IIB supergravity on squashed Sasaki-Einstein manifolds including massive modes have been studied in [27] and [67], while [111] also extended previous truncations to gauged $N = 2$ five-dimensional supergravity to include the full bosonic sector coupled to massive modes up to the second KK level. Similarly, [149] studied holographic aspects of such reductions as well as the properties of solutions of the type $AdS_4 \times \mathbb{R} \times SE_5$. Issues of stability of vacua have been considered in Ref. [16].

It is important to realize that, with the exception of [129, 130], all of the work on consistent truncations that we have mentioned so far discussed the reduction of the bosonic modes only,[2] in the hope that the consistency of the truncation of the fermionic sector is ensured by the supersymmetry of the higher-dimensional theory. In fact, this has been rigorously proven to hold in certain simple cases involving compactifications on a sphere [31, 138]. However, from the point of view of applications to gauge/gravity duality, it is important to know the precise form of the couplings between the various bosonic fields and their fermionic partners, inasmuch as this knowledge would allow one to address relevant questions such as the nature of fermionic correlators in the presence

[2]In some cases (see [18, 66], for example), fermions were considered to the extent that the lower-dimensional solutions preserving supersymmetry were shown to uplift to higher-dimensional solutions which also preserve supersymmetry.

of superconducting condensates, that rely on how the fermionic operators of the dual theory couple to scalars. A related problem involving a superfluid p-wave transition was studied in [4], in the context of (3+1)-dimensional supersymmetric field theories dual to probe $D5$-branes in $AdS^5 \times S^5$. In the case of the $(2+1)$-dimensional field theories which concern us here, some of these issues have been discussed in a bottom-up framework in [54, 78]. We note in particular though that in the presence of scalar excitations, the $d = 4$ gravitino will mix with any other fermions (beyond the linearized approximation). The goal of the present paper is to set the stage for addressing these questions in a more systematic top-down fashion, by explicitly reducing the fermionic sector of the truncations of $D = 11$ supergravity constructed in [62, 64, 65].

This paper is organized as follows. In section 5.2 we briefly review some aspects of the truncations of $D = 11$ supergravity constructed in [62, 64, 65] and the extension of the bosonic ansatz to include the gravitino. In section 4.3 we present our main result: the four-dimensional equations of motion for the fermion modes, and the corresponding effective four-dimensional action functional in terms of diagonal fields. In section 4.4 we reduce the supersymmetry variation of the gravitino, and elucidate the supersymmetric structure of the four-dimensional theory by considering how the fermions fit into the supermultiplets of gauged $N = 2$ supergravity in four dimensions. Thus, we explain how the reduction is embedded in the general scheme of Ref. [5]. In section 5.6 we apply our results to two further truncations of interest: the minimal gauged supergravity theory in four dimensions, and the dual [64, 65] of the $(2 + 1)$-dimensional holographic superconductor. In particular, we briefly discuss the possibility of further truncating the fermionic sector which would be necessary to obtain a simpler theory of fermionic operators coupled to superconducting condensates. We conclude in section 5.7. Various conventions and useful expressions have been collected in the appendices.

4.2 $D = 11$ supergravity on squashed Sasaki-Einstein manifolds

In this section we briefly review the ansatz for the bosonic fields in the consistent truncations of [62, 64, 65], and discuss the extension of this ansatz to include the gravitino.

4.2.1 The bosonic ansatz

The Kaluza-Klein metric ansatz in the truncations of interest is given by [62]

$$ds_{11}^2 = e^{-6U(x)-V(x)} ds_E^2(M) + e^{2U(x)} ds^2(Y) + e^{2V(x)} \big(\eta + A(x)\big)^2 , \qquad \text{(IV.1)}$$

where M is an arbitrary "external" four-dimensional manifold, with coordinates denoted generically by x and four-dimensional Einstein-frame metric $ds_E^2(M)$, and Y is an "internal" six-dimensional Kähler-Einstein manifold (henceforth referred to as "KE base") coordinatized by y and possessing Kähler form J. The one-form A is defined in T^*M and $\eta \equiv d\chi + \mathcal{A}(y)$, where $\mathcal{A}$ is an element of T^*Y satisfying $d\mathcal{A} \equiv \mathcal{F} = 2J$. For a fixed point in the external manifold, the compact coordinate χ parameterizes the fiber of a $U(1)$ bundle over Y, and the seven-dimensional internal manifold spanned by (y, χ) is then a squashed Sasaki-Einstein manifold, with the breathing and squashing modes parameterized by the scalars $U(x)$ and $V(x)$.[3] In addition to the metric, the bosonic content of $D = 11$ supergravity includes a 4-form flux $\hat{F}_4$; the rationale behind the corresponding ansatz is the idea that the consistency of the dimensional reduction is a result of truncating the KK tower to include fields that transform as singlets only under the structure group of the KE base, which in this case corresponds to $SU(3)$. As we will discuss below, this prescription allows for an interesting spectrum in the lower dimensional theory, inasmuch as the $SU(3)$ singlets include fields that are charged under the $U(1)$ isometry generated by ∂_χ. The globally defined Kähler 2-form $J = d\mathcal{A}/2$ and the holomorphic $(3, 0)$-form Σ that define the Kähler and complex structures, respectively, on the KE base Y are $SU(3)$-invariant and can be used in the

[3] In particular, $U - V$ is the squashing mode, describing the squashing of the $U(1)$ fiber with respect to the KE base, while the breathing mode $6U + V$ modifies the overall volume of the internal manifold. When $U = V = 0$, the internal manifold becomes a seven-dimensional Sasaki-Einstein manifold SE_7.

reduction of $\hat{F}_4$ to four dimensions. The $U(1)$-bundle over Y is such that they satisfy[4]

$$\Sigma \wedge \Sigma^* = -\frac{4i}{3} J^3 \,, \qquad \text{and} \qquad d\Sigma = 4i\mathcal{A} \wedge \Sigma \,. \tag{IV.2}$$

More precisely, as will be clear from the discussion to follow below, the relevant charged form Ω on the total space of the bundle that should enter the ansatz for $\hat{F}_4$ is given by

$$\Omega \equiv e^{4i\chi} \Sigma \,, \tag{IV.3}$$

and satisfies

$$d\Omega = 4i\eta \wedge \Omega \,. \tag{IV.4}$$

The ansatz for $\hat{F}_4$ is then [62]

$$\begin{aligned} \hat{F}_4 = & f \,\mathrm{vol}_4 + H_3 \wedge (\eta + A) + H_2 \wedge J + dh \wedge J \wedge (\eta + A) + 2hJ^2 \\ & + \left[X(\eta + A) \wedge \Omega - \frac{i}{4} (dX - 4iAX) \wedge \Omega + \text{c.c.} \right] , \end{aligned} \tag{IV.5}$$

where, as follows from the equations of motion, $f = 6e^{6W}(\epsilon + h^2 + \frac{1}{3}|X|^2)$, with $\epsilon = \pm 1$ and $W(x) \equiv -3U(x) - V(x)/2$, a notation we will use often.[5] All the fields other than (η, J, Ω) are defined on $\Lambda^* T^* M$. The matter fields X and h are scalars, while H_2 and H_3 are 2-form and 3-form field strengths, respectively. In terms of a 1-form potential B_1 and a 2-form potential B_2, the field strengths can be written $H_3 = dB_2$ and $H_2 = dB_1 + 2B_2 + hF$, and it is then easy to verify that the Bianchi identity $d\hat{F}_4 = 0$ is satisfied. As pointed out in [62, 64, 65], when $\epsilon = +1$ the dimensionally reduced theory admits a vacuum solution with vanishing matter fields, which uplifts to an $AdS_4 \times SE_7$ eleven-dimensional solution. On the other hand, by reversing the orientation in the compact manifold (i.e. $\epsilon = -1$) the corresponding vacuum is a "skew-whiffed" $AdS_4 \times SE_7$ solution, which generically does not preserve any supersymmetries, but is nevertheless perturbatively stable [47].

[4]Our conventions for the various form fields are discussed in Appendix A.2.

[5]The normalization of the charged scalar X is related to the one in [62] by $X = \sqrt{3}\chi$. Here, we reserve the notation χ for the fiber coordinate.

4.2.2 The gravitino ansatz

Quite generally, we would like to decompose the gravitino using a separation of variables ansatz of the form

(IV.6) $$\psi_a(x,y,\chi) = \sum_I \psi^I_a(x) \otimes \eta^I(y,\chi)$$

(IV.7) $$\psi_\alpha(x,y,\chi) = \sum_I \lambda^I(x) \otimes \eta^I_\alpha(y,\chi)$$

(IV.8) $$\psi_f(x,y,\chi) = \sum_I \varphi^I(x) \otimes \eta^I_f(y,\chi)\,.$$

The relevant point to understand is how precisely to project to $SU(3)$ singlets, appropriate to the consistent truncation. The first step is to understand how $SU(3)$ acts on the spinors, which is explored fully in Appendix G.

As we have discussed, the seven-dimensional internal space is the total space of a $U(1)$ bundle over a KE base Y. In general, the base is not spin, and therefore spinors do not necessarily exist globally on the base. However, it is always possible to define a $Spin^c$ bundle globally on Y (see [122], for example), and our "spinors" will then be sections of this bundle. The corresponding $U(1)$ generator is proportional to ∂_χ, and hence $\nabla_\alpha - \mathcal{A}_\alpha\partial_\chi$ is the gauge connection on the $Spin^c$ bundle, where ∇_α is the covariant derivative on Y. Of central importance to us in the reduction to $SU(3)$ invariants are the gauge-covariantly-constant spinors, which can be defined on any Kähler manifold [94] and thus satisfy in the present context

(IV.9) $$(\nabla_\alpha - \mathcal{A}_\alpha\partial_\chi)\varepsilon(y,\chi) = 0\,,$$

where

(IV.10) $$\varepsilon(y,\chi) = \varepsilon(y)e^{ie\chi}$$

for fixed "charge" e. Their existence is independent of the metric on the total space of the bundle. Thus, in our discussion, solutions to (V.14) are supposed to exist, and indeed as we will see shortly they must exist in numbers sufficient to give $N = 2$ supersymmetric structure in $d = 4$.

Our next task is to determine the values of the charge e occurring in (V.15). We will do so for a general KE manifold Y of real dimension d_b. Following [68, 139], we start by examining the integrability condition[6]

$$[\nabla_\beta, \nabla_\alpha]\varepsilon = \frac{1}{4}(R_{\delta\gamma})_{\beta\alpha}\Gamma^{\delta\gamma}\varepsilon\,. \tag{IV.11}$$

The key feature is that internal gauge curvature is equal to the Kähler form, $\mathcal{F} = 2J$. Given the assumption (V.15) that $\nabla_\alpha\varepsilon = ie\mathcal{A}_\alpha\varepsilon$, we find

$$[\nabla_\beta, \nabla_\alpha]\varepsilon = -ie\mathcal{F}_{\alpha\beta}\varepsilon = -2ieJ_{\alpha\beta}\,, \tag{IV.12}$$

and hence

$$\frac{1}{4}(R_{\delta\gamma})_{\beta\alpha}J^{\beta\alpha}\Gamma^{\delta\gamma}\varepsilon = -2ieJ_{\alpha\beta}J^{\beta\alpha}\varepsilon = 2ied_b\,\varepsilon\,. \tag{IV.13}$$

Since Y is an Einstein manifold, the Ricci form satisfies

$$Ric = \frac{1}{4}(R_{\delta\gamma})_{\beta\alpha}J^{\beta\alpha}e^\delta \wedge e^\gamma = (d_b+2)J\,, \tag{IV.14}$$

and we then conclude

$$Q\varepsilon \equiv -iJ_{\alpha\beta}\Gamma^{\alpha\beta}\varepsilon = \frac{4ed_b}{d_b+2}\varepsilon\,. \tag{IV.15}$$

In other words, the matrix $Q = -iJ_{\alpha\beta}\Gamma^{\alpha\beta}$ on the left is (up to normalization) the U(1) charge operator.[7] It has maximum eigenvalues $\pm d_b$, and the corresponding spinors have charge

$$e = \pm\frac{d_b+2}{4}\,. \tag{IV.16}$$

These two spinors are charge conjugates of one another, and we will henceforth denote them by $\varepsilon_\pm$. By definition, they satisfy $\not{\mathcal{F}}\varepsilon_\pm = iQ\varepsilon_\pm = \pm id_b\,\varepsilon_\pm$, where $\not{\mathcal{F}} \equiv (1/2)\mathcal{F}_{\alpha\beta}\Gamma^{\alpha\beta}$. As discussed in Appendix G, the spinors with maximal Q-charge are in fact the singlets under the structure group, and we will use them to build the reduction ansatz for the gravitino. In the case at hand $d_b = 6$,

[6] Our Clifford algebra conventions are detailed in Appendix A.2.

[7] This is explored further in Appendix G, in terms of the gravitino states.

the structure group is $SU(3)$, and ε_+ and ε_- transform in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ of Spin(6) $\simeq SU(4)$, respectively, so they have opposite six-dimensional chirality:

$$\gamma_7 \varepsilon_\pm = \pm \varepsilon_\pm \,. \tag{IV.17}$$

Incidentally, we can now understand why it is that $\Omega = e^{4i\chi}\Sigma$ enters the 4-form flux ansatz: defining $\not{\Sigma} = \frac{1}{3!}\Sigma_{\alpha\beta\gamma}\Gamma^{\alpha\beta\gamma}$, we can compute $[Q, \not{\Sigma}\,] = 12\not{\Sigma}$. This means that Σ carries charge $e_\Sigma = 4$. Since the Q charge is realized in the spinors through their χ-dependence, for the holomorphic form we are lead to define $\Omega = e^{4i\chi}\Sigma$, with Σ given by (A.63).

We are now in position to write the reduction ansatz for the gravitino. Taking into account the eleven-dimensional Majorana condition on the gravitino, and dropping all the $SU(3)$ representations other than the singlets, we take

$$\Psi_\alpha(x, y, \chi) = \lambda(x) \otimes \gamma_\alpha\, \varepsilon_+(y) e^{2i\chi} \tag{IV.18}$$

$$\Psi_{\bar{\alpha}}(x, y, \chi) = -\lambda^{\mathbf{c}}(x) \otimes \gamma_{\bar{\alpha}}\, \varepsilon_-(y) e^{-2i\chi} \tag{IV.19}$$

$$\Psi_f(x, y, \chi) = \varphi(x) \otimes \varepsilon_+(y) e^{2i\chi} + \varphi^{\mathbf{c}}(x) \otimes \varepsilon_-(y) e^{-2i\chi} \tag{IV.20}$$

$$\Psi_a(x, y, \chi) = \psi_a(x) \otimes \varepsilon_+(y) e^{2i\chi} + \psi_a^{\mathbf{c}}(x) \otimes \varepsilon_-(y) e^{-2i\chi}\,, \tag{IV.21}$$

where φ, λ and ψ_a are four-dimensional Dirac spinors on M, the superscript $\mathbf{c}$ denotes charge conjugation,[8] and we have used the complex basis introduced in A.2.3 for the KE base directions ($\alpha, \bar{\alpha} = 1, 2, 3$). Notice that all of these modes are annihilated by the gauge-covariant derivative on Y. Equations (IV.18)-(IV.21) provide the starting point for the dimensional reduction of the $D = 11$ supergravity equations of motion down to $d = 4$.

4.3 Four-dimensional equations of motion and effective action

The $D = 11$ equation of motion for the gravitino is

$$\Gamma^{ABC}\hat{D}_B\hat{\Psi}_C + \frac{1}{4}\frac{1}{4!}\left[\Gamma^{ADEFGC}F_{DEFG} + 12\Gamma^{DE}F^{AC}{}_{DE}\right]\hat{\Psi}_C = 0\,. \tag{IV.22}$$

[8] Our charge conjugation conventions are summarized in section A.2.5.

In this paper, we will consider only effects linear in the fermion fields in the equations of motion. Consequently, we will not derive the four-fermion (current-current) couplings that are certainly present in the 4-d Lagrangian. These can be obtained using the same methods that we will develop here, and it would be interesting to do so, as they might be relevant for holographic applications. In Section 4.4, we will show that all of our results fit into the expected $d = 4$ $N = 2$ gauged supergravity, and so the four fermion terms could also be derived by evaluating the known expressions. The spin connection and our conventions for the Clifford algebra and the various form fields can be found in Appendix A.2. Below, we write down the effective four-dimensional equations of motion for the fermion modes λ, φ, ψ_a on M (and their charge conjugates). We then perform a field redefinition in order to write the kinetic terms in diagonal form, and present our main result: the effective four-dimensional action functional for the diagonal fermion fields. The equations of motion that follow from this action have been written explicitly in appendix H.

4.3.1 Reduction of covariant derivatives

We make use of the gravitino ansatz discussed in section 4.2.2 to reduce the eleven-dimensional covariant derivatives. In what follows, we will project the various expressions to the terms proportional to the positive chirality spinor ε_+, and drop the overall factor $e^{2i\chi}$. The $\varepsilon_- e^{-2i\chi}$ contributions are the charge conjugates of the expressions that we will write and thus can be easily resurrected.

Reducing the component in the direction of the fiber, $\Gamma^{fAB}\hat{D}_A\hat{\Psi}_B$, and denoting the resulting expression by $\mathcal{L}_f$, we get

$$
\begin{aligned}
e^W \mathcal{L}_f = & \left[\gamma^{ab} D_a + \frac{1}{2}(\partial^b W) + \gamma^b \not{\partial}(V+3U) + 3ie^{W+V-2U}\gamma_5\gamma^b - \frac{1}{4}e^{V-W}F_{da}\gamma^{ab}\gamma^d\gamma_5\right]\psi_b \\
& + 6\left[\not{D} + \frac{1}{2}\not{\partial}(W+U-V) + \frac{1}{2}e^{V-W}\not{F}\gamma_5 + \frac{3i}{2}e^{W+V-2U}\gamma_5\right]\gamma_5\lambda \\
& + \left(e^{V-W}\not{F} + 6ie^{W+V-2U}\right)\varphi\,, \qquad \text{(IV.23)}
\end{aligned}
$$

where we have defined the four-dimensional gauge-covariant derivative $D_a = \nabla_a - 2iA_a$. Similarly, for the piece coming from the a-component $\Gamma^{aAB}\hat{D}_A\hat{\Psi}_B$, which we denote by $\mathcal{L}^a_{gr}$, after projecting we obtain

$$e^W \mathcal{L}^a_{gr} = \left[\gamma_5\gamma^{abc}D_b + \frac{1}{2}(\partial_b W)\gamma_5\gamma^{abc} - i\left(2e^{W-V} + \frac{3}{2}e^{W+V-2U}\right)\gamma^{ac} - \frac{1}{8}e^{V-W}F_{bd}\gamma^b\gamma^{ac}\gamma^d\right]\psi_c$$
$$+ \left[\gamma^{ab}D_b + \frac{1}{2}(\partial_b V)\gamma^{ab} + \frac{1}{2}\partial^a(W-V) + 3ie^{W+V-2U}\gamma_5\gamma^a + \frac{1}{4}e^{V-W}\gamma_5 F_{bc}\gamma^c\gamma^{ab}\right]\varphi$$
$$+ 6\left[\gamma^{ab}D_b + \frac{1}{2}(\partial_b U)\gamma^{ab} + \frac{1}{2}\partial^a(W-U) + i(2e^{W-V} + e^{W+V-2U})\gamma_5\gamma^a\right.$$

(IV.24)

$$\left. - \frac{1}{8}\gamma_5 e^{V-W}F_{bc}\gamma^b\gamma^a\gamma^c\right]\lambda\,.$$

Finally, for the components in the direction of the KE base, the $SU(3)$-invariants can be extracted by contracting $\Gamma^{\alpha AB}\hat{D}_A\hat{\Psi}_B$ with Γ_α. After projecting, we find

$$e^W\mathcal{L}_b = 6\gamma_5\left[\gamma^{ab}D_a + \frac{1}{2}(\partial^b W) - \frac{1}{2}\gamma^b\not{\partial}(2W-U) + i\left(e^{W+V-2U} + 2e^{W-V}\right)\gamma_5\gamma^b\right.$$
$$\left. + \frac{1}{8}e^{V-W}F_{da}\gamma_5\gamma^a\gamma^b\gamma^d\right]\psi_b$$
$$+ 6\left[-5\not{D} - \frac{5}{2}(\not{\partial}W) + 10ie^{W-V}\gamma_5 + \frac{7}{2}ie^{W+V-2U}\gamma_5 + \frac{5}{4}e^{V-W}\gamma_5\not{F}\right]\lambda$$

(IV.25)
$$+ 3\left[-2\not{D} - \not{\partial}(W+V-U) + 3ie^{W+V-2U}\gamma_5 + e^{V-W}\gamma_5\not{F}\right]\varphi\,.$$

4.3.2 Reduction of fluxes

Having reduced the kinetic terms for the fermion modes, we now turn to the problem of reducing their couplings to the background 4-form flux. More explicitly, we would like to reduce

(IV.26)
$$\frac{1}{4!}\left[\Gamma^{ADEFGC}\hat{F}_{DEFG} + 12\Gamma^{DE}\hat{F}^{AC}{}_{DE}\right]\hat{\Psi}_C$$

by using the ansatz (IV.18)-(IV.21). As we did for the kinetic terms, here we display the expressions obtained by projecting to the terms proportional to the positive chirality spinor ε_+, and drop the overall factor $e^{2i\chi}$.

Evaluating the component of (IV.26) in the direction of the fiber, and denoting the corresponding expression after the projection by $\mathcal{R}_f$, we get

$$e^W \mathcal{R}_f = 3\left[\frac{1}{2} i e^{-W-2U} H_{2\,ab}\gamma^{abc} - \frac{1}{6} e^{-2W-V} H_3^{abc}\gamma_{ab}\gamma_5 - i e^{-2U-V}(\partial^c h)\gamma_5 - 4h e^{W-4U}\gamma^c\right]\psi_c$$
$$+ 6\Big[-i f e^{-3W} + 2i e^{-W-2U}\gamma_5 \not{H}_2 - 4h e^{W-4U}\gamma_5 + i e^{-2U-V}(\not{\partial} h)\Big]\lambda$$

(IV.27)

$$+ 2i e^{-3U}\gamma_5\gamma^{ab}(D_a X)\psi_b^{\mathbf{c}} + 6 e^{-3U}\big[i(\not{D}X) - 4 e^{W-V} X\gamma_5\big]\lambda^{\mathbf{c}}\,.$$

We note that the terms proportional to charge conjugate spinors come about, as explained in the Appendix, because $\not{\Omega}\epsilon_- \sim \epsilon_+$, that is $\not{\Omega}$ is proportional to a "total raising operator" in the Fock basis for gravitino states. We also note that the gauge-covariant derivative D acts on the complex scalar X as $DX = dX - 4iAX$.

Similarly, for the components in the direction of the external manifold, denoted here by $\mathcal{R}^a_{gr}$, we find

$$e^W \mathcal{R}^a_{gr} = \Bigg[3i(\partial_b h)e^{-2U-V}\gamma^{abc} - \frac{3}{2}e^{-W-2U}H_{2\,bd}\epsilon^{abdc} - 12h e^{W-4U}\gamma_5\gamma^{ac}$$
$$+ i f e^{-3W}\gamma^{ac} - e^{-2W-V}H_3{}^{acb}\gamma_b + 3i e^{-W-2U}H_2{}^{ac}\gamma_5\Bigg]\psi_c$$
$$+ 3\left[4h e^{W-4U}\gamma^a - \frac{1}{2} i e^{-W-2U}H_{2\,bc}\gamma^{abc} + \frac{1}{6}e^{-2W-V}H_3{}^{abc}\gamma_5\gamma_{bc} + i(\partial^a h)e^{-2U-V}\gamma_5\right]\varphi$$
$$+ 6\Bigg[2i(\partial_b h)e^{-2U-V}\gamma^{ab}\gamma_5 + \frac{i}{6}\epsilon^{abcd}H_{3\,bcd}e^{-2W-V} + 4h e^{W-4U}\gamma^a$$
$$- i e^{-W-2U}H_{2\,bc}\gamma^{abc} - i e^{-W-2U}H_2{}^{ac}\gamma_c + i(\partial^a h)e^{-2U-V}\gamma_5\Bigg]\lambda$$
$$+ 2e^{-3U}\left[-i(D_b X)\gamma^{abc} + 4X e^{W-V}\gamma_5\gamma^{ac}\right]\psi_c^{\mathbf{c}} + 2i e^{-3U}(D_b X)\gamma_5\gamma^{ab}\varphi^{\mathbf{c}}$$

(IV.28)

$$+ 6e^{-3U}\left[i\gamma_5\gamma^a(\not{D}X) + 4X e^{W-V}\gamma^a\right]\lambda^{\mathbf{c}}\,.$$

Next, let $\mathcal{R}_b$ denote the expression obtained by contracting the components of (IV.26) in the KE base directions with Γ_α and projecting to the ε_+ sector. We then find

$$
\begin{aligned}
e^W \mathcal{R}_b = \Big[& ie^{-2W-V} H_{3;bcd}\epsilon^{abcd}\gamma_5 + 6ie^{-W-2U} H_{2;bc}\gamma_5\gamma^c\gamma^{ab} - 24he^{W-4U}\gamma_5\gamma^a \\
& + 6ie^{-2U-V}(\partial_b h)\left(2\gamma^{ab} - \eta^{ab}\right)\Big]\psi_a \\
& + \Big[-6ife^{-3W}\gamma_5 + 12ie^{-W-2U}\not{H}_2 - 24he^{W-4U} + 6ie^{-2U-V}\gamma_5(\not{\partial}h)\Big]\varphi \\
& + 6\Big[-5ife^{-3W}\gamma_5 + 5e^{-2W-V}\gamma_5\not{H}_3 + 7ie^{-W-2U}\not{H}_2 - 28he^{W-4U} \\
& \quad + 7ie^{-2U-V}\gamma_5(\not{\partial}h)\Big]\lambda + 6e^{-3U}\Big[i(\not{D}X)\gamma^a + 4Xe^{W-V}\gamma_5\gamma^a\Big]\psi_a^{\mathbf{c}} \\
& + 24e^{-3U}\Big[i\gamma_5(\not{D}X) - 4Xe^{W-V}\Big]\lambda^{\mathbf{c}} + 6e^{-3U}\Big[i\gamma_5(\not{D}X) - 4Xe^{W-V}\Big]\varphi^{\mathbf{c}}\,.
\end{aligned}
\tag{IV.29}
$$

Putting the previous results together, we find that the set of equations for the λ, φ and ψ_a modes is given by

$$\mathcal{L}^a_{gr} + \frac{1}{4}\mathcal{R}^a_{gr} = 0 \tag{IV.30}$$

$$\mathcal{L}_f + \frac{1}{4}\mathcal{R}_f = 0 \tag{IV.31}$$

$$\mathcal{L}_b + \frac{1}{4}\mathcal{R}_b = 0\,. \tag{IV.32}$$

These equations can be greatly simplified by a suitable field redefinition which we perform below. For convenience, the resulting equations are written out in full in Appendix H.

4.3.3 Field redefinitions and diagonalization

We now look for a set of fields that produce diagonal kinetic terms for the various modes. The derivative terms in the equations above can be obtained from a Lagrangian density (with respect to the 4-d Einstein measure $d^4x\sqrt{|g|}$) of the form[9]

$$
\begin{aligned}
\mathcal{L}_{kin} = e^W \Big[& \bar{\psi}_a\gamma^{abc}D_b\psi_c + \left(\bar{\varphi} + 6\bar{\lambda}\right)\gamma_5\gamma^{ab}D_a\psi_b + \bar{\psi}_a\gamma_5\gamma^{ab}D_b\left(6\lambda + \varphi\right) \\
& -6\bar{\varphi}\not{D}\lambda - 6\bar{\lambda}\not{D}(5\lambda + \varphi)\Big]\,.
\end{aligned}
\tag{IV.33}
$$

[9] We leave the overall normalization of the Lagrangian unfixed. We note that, as usual, the kinetic terms are real up to a total derivative. In the context of holography, the boundary terms are crucial as they determine the on-shell action. These should be determined separately when necessary.

We can rewrite these terms in diagonal form by means of the following field redefinitions:

$$\zeta_a = e^{W/2}\left[\psi_a - \frac{1}{2}\gamma_5\gamma_a\left(\varphi + 6\lambda\right)\right], \tag{IV.34}$$

$$\eta = e^{W/2}(\varphi + 2\lambda), \tag{IV.35}$$

$$\xi = 6e^{W/2}\lambda\,, \tag{IV.36}$$

so that

(IV.37)

$$\mathcal{L}_{kin} = \bar{\zeta}_a\gamma^{abc}D_b\zeta_c + \frac{3}{2}\bar{\eta}\not{D}\eta + \frac{1}{2}\bar{\xi}\not{D}\xi - \frac{1}{2}\left[\bar{\zeta}_a\gamma^{abc}(\partial_b W)\zeta_c + \frac{3}{2}\bar{\eta}(\not{\partial}W)\eta + \frac{1}{2}\bar{\xi}(\not{\partial}W)\xi\right].$$

The interaction terms are produced by the action of the derivatives on the warping factors involved in the field redefinitions, and they will cancel against similar terms in the interaction Lagrangian. In section 4.4, we will interpret the fields ζ_a, η, ξ in terms of the multiplet content appropriate to the underlying supersymmetry of the $d = 4$ theory. Finally, it is worth noting that given our conventions for charge conjugation (see section A.2.5), the redefinition (IV.34) implies that the corresponding charge conjugate field is given by

$$\zeta_a^{\mathbf{c}} = e^{W/2}\left[\psi_a^{\mathbf{c}} + \frac{1}{2}\gamma_5\gamma_a\left(\varphi^{\mathbf{c}} + 6\lambda^{\mathbf{c}}\right)\right]. \tag{IV.38}$$

4.3.4 Effective $d = 4$ action

By taking appropriate linear combinations of (IV.30)-(IV.32) one can obtain the equations of motion for the diagonal fermion fields (IV.34)-(IV.36). The resulting equations are written explicitly in Appendix H, and can be obtained from the following $d = 4$ action functional:[10]

$$S_F = K\int d^4x\sqrt{-g}\left[\bar{\zeta}_a\gamma^{abc}D_b\zeta_c + \frac{3}{2}\bar{\eta}\not{D}\eta + \frac{1}{2}\bar{\xi}\not{D}\xi + \mathcal{L}^{int}_{\bar{\psi}\psi} + \frac{1}{2}\left(\mathcal{L}^{int}_{\bar{\psi}\psi^{\mathbf{c}}} + \text{c.c.}\right)\right], \tag{IV.39}$$

where K is a normalization constant, " + c.c." denotes the complex conjugate (or, equivalently, the charge conjugate) of $\mathcal{L}^{int}_{\bar{\psi}\psi^{\mathbf{c}}}$, and the interaction pieces $\mathcal{L}^{int}_{\bar{\psi}\psi}$ and $\mathcal{L}^{int}_{\bar{\psi}\psi^{\mathbf{c}}}$ are defined as

[10] In writing the action below, we have performed a chiral rotation of the form $\psi \mapsto e^{i\pi\gamma_5/4}\psi$ in all three fermion fields. This transformation introduces a factor of $i\gamma_5$ in all bilinears of the form $\bar{\psi}\gamma_{a_1}\gamma_{a_2}\dots\gamma_{a_{2k}}\psi$ and $\bar{\psi}\gamma_{a_1}\gamma_{a_2}\dots\gamma_{a_{2k}}\psi^{\mathbf{c}}$, while leaving the rest (e.g. kinetic terms) invariant. This rotation has the virtue of producing standard Dirac mass terms in the truncations we review in section 5.6.

$$\begin{aligned}
\mathcal{L}^{int}_{\bar\psi\psi} = \; & + \frac{3}{4} i(\partial_b h) e^{-2U-V} \bar\zeta_a \gamma_5 \gamma^{abc} \zeta_c + \frac{3}{8} i e^{-2U-V} \bar\eta \gamma_5 (\not\partial h)\eta - \frac{3}{8} i e^{-2U-V} \bar\xi \gamma_5 (\not\partial h)\xi \\
& + \frac{1}{4} e^{-2W-V} H_3{}^{abc} \bar\zeta_a \gamma_5 \gamma_b \zeta_c - \frac{3}{8} e^{-2W-V} \bar\eta \gamma_5 \not H_3 \eta + \frac{3}{8} e^{-2W-V} \bar\xi \gamma_5 \not H_3 \xi \\
& - \frac{i}{4} \bar\zeta_a \left[6\,(\not\partial U) + e^{-2W-V} \gamma_5 \not H_3 \right] \gamma^a \xi + \frac{i}{4} \bar\xi \gamma^a \left[6\,(\not\partial U) - e^{-2W-V} \gamma_5 \not H_3 \right] \zeta_a \\
& - \frac{3}{4} e^{-2U-V} \left[\bar\zeta_a \gamma_5 (\not\partial T) \gamma^a \eta - \bar\eta \gamma_5 \gamma^a (\not\partial T^\dagger) \zeta_a \right] \\
& + \frac{i}{4} \bar\zeta_a \left[-e^{V-W} \left(F + i\gamma_5 * F \right)^{ac} + 3i e^{-W-2U} \gamma_5 (H_2 + i\gamma_5 * H_2)^{ac} \right] \zeta_c \\
& + \frac{3i}{4} e^{V-W} \bar\eta \left(\not F - i\gamma_5 e^{-V-2U} \not H_2 \right) \eta - \frac{i}{8} e^{V-W} \bar\xi \left(\not F + 3i\gamma_5 e^{-V-2U} \not H_2 \right) \xi \\
& + \frac{3}{8} e^{V-W} \left[\bar\zeta_a \left(\not F - i\gamma_5 e^{-V-2U} \not H_2 \right) \gamma^a \eta + \bar\eta \gamma^a \left(\not F - i\gamma_5 e^{-V-2U} \not H_2 \right) \zeta_a \right] \\
& - 3i e^{W-4U} \bar\zeta_a \gamma_5 T^\dagger \gamma^{ac} \zeta_c + 3i e^{W-4U} \bar\eta \gamma_5 T^\dagger \eta + \frac{3}{2} e^{W-4U} \left(\bar\zeta_a \gamma^a \gamma_5 T \eta + \bar\eta T \gamma_5 \gamma^a \zeta_a \right) \\
& - \frac{9i}{2} e^{W-4U} \bar\xi \gamma_5 T \xi - 3i e^{W-4U} (\bar\eta \gamma_5 T \xi + \bar\xi \gamma_5 T \eta) + 3 e^{W-4U} \left(\bar\zeta_a \gamma^a \gamma_5 T \xi + \bar\xi T \gamma_5 \gamma^a \zeta_a \right) \\
& + \frac{1}{4} i \left(\tilde f - 8 e^{W-V} \right) \left(i \bar\zeta_a \gamma^{ac} \zeta_c - 3i \bar\eta \eta + \frac{3}{2} \bar\zeta_a \gamma^a \eta + \frac{3}{2} \bar\eta \gamma^a \zeta_a \right) \\
& + \frac{1}{8} \left(3\tilde f + 8 e^{W-V} \right) \bar\xi \xi + \frac{3}{4} \tilde f \left(\bar\eta \xi + \bar\xi \eta \right) + \frac{1}{4} i \tilde f \left(\bar\xi \gamma^a \zeta_a + \bar\zeta_a \gamma^a \xi \right)
\end{aligned} \tag{IV.40}$$

and[11]

$$\begin{aligned}
\mathcal{L}^{int}_{\bar\psi\psi^{\mathbf{c}}} = e^{-3U} & \left\{ -\frac{i}{2} (D_b X) \bar\zeta_a \gamma_5 \gamma^{abc} \zeta_c^{\mathbf{c}} - \frac{3i}{4} \bar\eta \gamma_5 (\not D X) \eta^{\mathbf{c}} - \frac{1}{4} \bar\zeta_a \gamma_5 (\not D X) \gamma^a \xi^{\mathbf{c}} + \frac{1}{4} \bar\xi \gamma^a \gamma_5 (\not D X) \zeta_a^{\mathbf{c}} \right\} \\
& + X e^{W-V-3U} \left\{ 2i \bar\zeta_a \gamma_5 \gamma^{ac} \zeta_c^{\mathbf{c}} - 6i \bar\eta \gamma_5 \eta^{\mathbf{c}} - \bar\zeta_a \gamma_5 \gamma^a \xi^{\mathbf{c}} + \bar\xi \gamma^a \gamma_5 \zeta_a^{\mathbf{c}} \right. \\
& \left. - 3 \left[\bar\zeta_a \left(\gamma_5 \gamma^a \right) \eta^{\mathbf{c}} + \bar\eta \left(\gamma_5 \gamma^a \right) \zeta_a^{\mathbf{c}} + i \bar\eta \gamma_5 \xi^{\mathbf{c}} + i \bar\xi \gamma_5 \eta^{\mathbf{c}} \right] \right\} ,
\end{aligned} \tag{IV.41}$$

where we have introduced the shorthand

$$\tilde f \equiv f e^{-3W} + 6 e^{W+V-2U} \,, \qquad T \equiv h - i\gamma_5 e^{V+2U} \,. \tag{IV.42}$$

[11] Note that some of the terms written below are actually equal, but we have left them this way to make the $N = 2$ structure of covariant derivatives more manifest. See the next section for details.

We recall that all the fermions have charge ± 2 with respect to the graviphoton, so that $D_a = \nabla_a - 2iA_a$ when acting on ζ, η, ξ, while the complex scalar X has charge -4, i.e. $DX = dX - 4iAX$. It is worth noting that the action (IV.39) is manifestly real (up to total derivatives), and that it can also be obtained by directly reducing the action of $D = 11$ supergravity to the $SU(3)$ singlet sector. In particular, this procedure fixes the normalization constant K in terms of the volume of the KE base Y, the length of the fiber parameterized by χ, the normalization of the internal spinors $\varepsilon_\pm$, and the eleven-dimensional gravitational constant.

4.4 $N = 2$ supersymmetry

To interpret this action further, we consider how the fields fit into supermultiplets of gauged $N = 2$ supergravity in four dimensions, ignoring the possibility of supersymmetry enhancement for special compactifications. Using the same techniques as above, we can reduce the 11-d supersymmetry variations of the fermionic fields.[12] These take the form

$$\delta\Psi_A = \hat{D}_A\Theta + \frac{1}{12}\frac{1}{4!}(\Gamma_A{}^{BCDE} - 8\delta_A^B\Gamma^{CDE})\Theta F_{BCDE}\,. \tag{IV.43}$$

We are interested only in the Grassmann parameters that are $SU(3)$ invariant, and it proves convenient to then write

$$\Theta = e^{W/2}\theta \otimes \varepsilon_+ e^{2i\chi} + e^{W/2}\theta^{\mathbf{c}} \otimes \varepsilon_- e^{-2i\chi}\,. \tag{IV.44}$$

Here, θ is a 4-d Dirac spinor. By making appropriate projections on (IV.43) to terms of definite charge, one obtains the variations of the fields φ, λ, ψ_a. Performing then the change of variables

[12] In what follows we keep only the terms linear in fermions.

(IV.34)-(IV.36), we arrive at the variations

$$\begin{aligned} \delta\eta &= -\frac{1}{4}e^{V-W}\left(\not{F} - ie^{-2U-V}\gamma_5 \not{H}_2\right)\theta + \frac{i}{2}e^{-2U-V}(\not{\partial}T)\theta \\ &\quad -e^{W-4U}T\gamma_5\theta - \frac{1}{4}i\left(\tilde{f} - 8e^{W-V}\right)\theta - 2e^{W-3U-V}X\gamma_5\theta^{\mathbf{c}} \end{aligned} \tag{IV.45}$$

$$\begin{aligned} \delta\xi &= 3\gamma_5(\not{\partial}U)\theta - \frac{1}{2}e^{6U}\not{H}_3\theta - \frac{1}{2}e^{-3U}i(\not{D}X)\theta^{\mathbf{c}} \\ &\quad -\frac{1}{2}i\tilde{f}\theta + 6e^{W-4U}T^\dagger\gamma_5\theta - 2Xe^{W-V-3U}\gamma_5\theta^{\mathbf{c}} \end{aligned} \tag{IV.46}$$

$$\begin{aligned} \delta\zeta_a &= \left(D_a - \frac{3}{4}i(\partial_a h)e^{-2U-V}\gamma_5 + \frac{1}{8}e^{V-W}\gamma_5\left(\not{F} - 3ie^{-V-2U}\gamma_5\not{H}_2\right)\gamma_a\right)\theta \\ &\quad + \left(\frac{1}{8}i\left(\tilde{f} - 8e^{W-V}\right)\gamma_5 + \frac{3}{2}Te^{W-4U}\right)\gamma_a\theta + \frac{1}{8}e^{-2W-V}\gamma_5\left[\gamma_a, \not{H}_3\right]\theta \\ &\quad -Xe^{W-3U-V}\gamma_a\theta^{\mathbf{c}} + \frac{1}{2}e^{-3U}\gamma_5(iD_aX)\theta^{\mathbf{c}}\,. \end{aligned} \tag{IV.47}$$

Now, according to [62], there is a single vector multiplet that contains the scalar $\tau = h + ie^{V+2U}$ (in this notation, $T = \tau P_- + \bar{\tau}P_+$, where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$), and there is universal hypermultiplet containing $\rho = 4e^{6U}$, the pseudoscalar dual to H_3 and X. The gravity multiplet contains the gravitino ζ_a while the vector multiplet and hypermultiplet each contains a Dirac spinor. Examining then the first lines of the variations (IV.45) and (IV.46) written above which contain derivatives of bosonic fields, we can identify the gauginos with η and the hyperinos with ξ.

In the $N = 2$ literature, one usually finds things written in terms of Weyl spinors. For a generic spinor Ψ, we could write

$$\Psi_1 = P_+\Psi, \qquad \Psi_2 = P_+\Psi^{\mathbf{c}} \tag{IV.48}$$

and we then have $\Psi_2^{\mathbf{c}} = P_-\Psi$ and $\Psi_1^{\mathbf{c}} = P_-\Psi^{\mathbf{c}}$. To be specific, let us consider the gaugino variation. It is convenient to first write the charge conjugate equation

$$\begin{aligned} \delta\eta^{\mathbf{c}} &= -\frac{1}{4}e^{V-W}\left(\not{F} - ie^{-2U-V}\gamma_5\not{H}_2\right)\theta^{\mathbf{c}} - \frac{i}{2}e^{-2U-V}(\not{\partial}T)\theta^{\mathbf{c}} \\ &\quad +e^{W-4U}T\gamma_5\theta^{\mathbf{c}} + \frac{1}{4}i\left(\tilde{f} - 8e^{W-V}\right)\theta^{\mathbf{c}} + 2e^{W-3U-V}X^*\gamma_5\theta \end{aligned} \tag{IV.49}$$

and doing the chiral projection, we then obtain

$$\delta\eta_1 = +\frac{i}{2}e^{-2U-V}(\partial\!\!\!/\tau)\theta_1^{\mathbf{c}} - \frac{1}{4}e^{V-W}\left(F\!\!\!/ - ie^{-2U-V}H\!\!\!/_2\right)\theta_1$$

$$\text{(IV.50)} \qquad -e^{W-4U}\bar{\tau}\theta_1 - \frac{1}{4}i\left(\tilde{f} - 8e^{W-V}\right)\theta_1 - 2e^{W-3U-V}X\theta_2$$

$$\delta\eta_2 = -\frac{i}{2}e^{-2U-V}(\partial\!\!\!/\tau)\theta_2^{\mathbf{c}} - \frac{1}{4}e^{V-W}\left(F\!\!\!/ - ie^{-2U-V}H\!\!\!/_2\right)\theta_2$$

$$\text{(IV.51)} \qquad +e^{W-4U}\bar{\tau}\theta_2 + \frac{1}{4}i\left(\tilde{f} - 8e^{W-V}\right)\theta_2 + 2e^{W-3U-V}X^*\theta_1 \,.$$

With a minor change of notation, these expressions can be understood as those that are obtained from working out this specific case of Ref. [5]. (Details of the bosonic sector of this have also recently appeared in Ref. [16]). Indeed, we have worked through the details of deriving the 4-d action using the results of [5]; we will not show this calculation in full here, but just point out the geometric features. The field content is usually presented after dualizing H_2 and H_3 [65][13]

$$\text{(IV.54)} \quad H^{(2)} = \frac{1}{4h^2 + e^{4U+2V}}\left(2h(\tilde{H}^{(2)} + h^2F^{(2)}) - e^{2U+V} * (\tilde{H}^{(2)} + h^2F^{(2)})\right)$$

$$\text{(IV.55)} \quad H^{(3)} = -\frac{1}{4}e^{-12U} * [D\sigma + J_X]$$

where $Da = da + 6(\tilde{B}_1 - \epsilon A_1)$, $\tilde{H}_2 = d\tilde{B}_1$, $J_X = i(X^*DX - DX^*X)$, $\rho = 4e^{6U}$ and $\sigma = 4a$. The hypermultiplet contains the scalars $\{X, \sigma, \rho\}$, while the vector multiplet contains $\tau = h + ie^{V+2U}$. The scalars of the hypermultiplet coordinatize a quaternionic space $\mathcal{HM} \simeq SO(4,1)/SO(4)$ with metric

$$\text{(IV.56)} \qquad ds^2_{\mathcal{H}} = \frac{1}{\rho^2}d\rho^2 + \frac{1}{4\rho^2}\left[d\sigma - i\left(XdX^* - X^*dX\right)\right]^2 + \frac{1}{\rho^2}dXdX^* \,.$$

The vector multiplet scalars coordinatize a special Kähler manifold $\mathcal{SM}$ with Kähler potential

$$\text{(IV.57)} \qquad K_V = -\log\frac{i(\tau - \bar{\tau})^3}{2} \,.$$

[13] It's convenient to note that these imply

$$\text{(IV.52)} \qquad H\!\!\!/_2 = \frac{h+T}{|h+\tau|^2}(\tilde{H\!\!\!/}_2 + h^2F\!\!\!/)$$

$$\text{(IV.53)} \qquad ie^{6U}\gamma_5 H\!\!\!/_3 = \frac{1}{\rho}\left[D\!\!\!/\sigma + J\!\!\!/_X\right]$$

On $\mathcal{SM}$ there is a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \frac{i}{2\pi}\bar{\partial}\partial K_V = \frac{3i}{8\pi}\frac{1}{(Im\tau)^2}$. Each of the fermions is a section of $\mathcal{L}^{1/2}$, with Hermitian connection $\theta = \partial K_V$. In the local coordinates $\tau, \bar{\tau}$, we have $\theta = -\frac{3}{2iIm\tau}d\tau$. Associated naturally to the line bundle is a $U(1)$ bundle with connection $\mathcal{Q} = Im\theta = \frac{3}{2}\frac{dRe\tau}{Im\tau}$. Given $\tau = h + ie^{V+2U}$, this gives $\mathcal{Q} = \frac{3}{2}e^{-V-2U}dh$. The gaugino is also a section of $T\mathcal{SM}$; the Levi-Civita connection on $\mathcal{SM}$ is $\Gamma \equiv \Gamma^\tau{}_\tau = \frac{i}{Im\tau}d\tau = ie^{-V-2U}dh - d(V+2U)$.

Because of the quaternionic structure, $\mathcal{HM}$ possesses three complex structures $\mathcal{J}^\alpha : T\mathcal{HM} \to T\mathcal{HM}$ that satisfy the quaternion algebra $\mathcal{J}^\alpha\mathcal{J}^\beta = -\delta^{\alpha\beta}1 + \epsilon^{\alpha\beta\gamma}\mathcal{J}^\gamma$. Correspondingly, there is a triplet of Kähler forms K_H^α, which we regard as $SU(2)$ Lie algebra valued. Required by $N=2$ supersymmetry, there is a principal $SU(2)$-bundle $\mathcal{SU}$ over $\mathcal{HM}$ with connection such that the hyper-Kähler form is covariantly closed; the curvature of the principal bundle is proportional to the hyper-Kähler form. It follows that the Levi-Civita connection of $\mathcal{HM}$ has holonomy contained in $SU(2) \otimes Sp(2,\mathbb{R})$. The fermions are sections of these bundles as follows:

- gravitino: $\mathcal{L}^{1/2} \times \mathcal{SU}$
- gaugino: $\mathcal{L}^{1/2} \times \mathcal{TSM} \times \mathcal{SU}$
- hyperino: $\mathcal{L}^{1/2} \times \mathcal{THM} \times \mathcal{SU}^{-1}$

In the last line, one means that the hyperino is a section of the vector bundle obtained by deleting the $SU(2)$ part of the holonomy group on $\mathcal{HM}$.

The connections on $\mathcal{SU}$ and $T\mathcal{HM} \times \mathcal{SU}^{-1}$ are evaluated in terms of the hypermultiplet scalars, and one finds the following results, following a translation into Dirac notation. The gravitino covariant derivative reads

$$\mathcal{D}_b\zeta_c = D_b\zeta_c - \frac{3i}{4}e^{-2U-V}(\partial_b h)\gamma_5\zeta_c - \frac{i}{4}e^{6U}(*H_3)_b\zeta_c + \frac{i}{2}e^{-3U}(D_bX)\gamma_5\zeta_c^{\mathbf{c}}\,, \tag{IV.58}$$

which leads to

$$\gamma^{abc}\mathcal{D}_b\zeta_c = \gamma^{abc}D_b\zeta_c + \frac{3i}{4}e^{-2U-V}(\partial_b h)\gamma_5\gamma^{abc}\zeta_c + \frac{1}{4}e^{6U}H_3^{abc}\gamma_5\gamma_b\zeta_c$$
$$- \frac{i}{2}e^{-3U}(D_b X)\gamma_5\gamma^{abc}\zeta_c^{\mathbf{c}}\,. \tag{IV.59}$$

The gaugino covariant derivative is

$$\mathcal{D}_a\eta = D_a\eta - \frac{i}{4}e^{-(2U+V)}(\partial_a h)\gamma_5\eta - \frac{i}{4}e^{6U}(*H_3)_a\eta + \frac{i}{2}e^{-3U}(D_a X)\gamma_5\eta^{\mathbf{c}}\,, \tag{IV.60}$$

giving

$$\not{\mathcal{D}}\,\eta = \not{D}\eta + \frac{i}{4}e^{-(2U+V)}\gamma_5(\not{\partial}h)\eta - \frac{1}{4}e^{6U}\gamma_5\not{H}_3\eta - \frac{i}{2}e^{-3U}\gamma_5(\not{D}X)\eta^{\mathbf{c}}\,. \tag{IV.61}$$

Finally, the hyperino is a section of $T\mathcal{HM}\times\mathcal{SU}^{-1}$. The covariant derivative is then

$$\mathcal{D}_a\xi = D_a\xi + \frac{3i}{4}e^{-(2U+V)}(\partial_a h)\gamma_5\xi + \frac{3i}{4}e^{6U}(*H_3)_a\xi\,. \tag{IV.62}$$

Equivalently,

$$\not{\mathcal{D}}\,\xi = \not{D}\xi - \frac{3i}{4}e^{-(2U+V)}\gamma_5(\not{\partial}h)\xi + \frac{3}{4}e^{6U}\gamma_5\not{H}_3\xi\,. \tag{IV.63}$$

We recognize the pieces of these covariant derivatives in the action given above. Indeed, the action takes the form

$$S_{kin} = K\int d^4x\sqrt{-g}\left[\bar{\zeta}_a\gamma^{abc}\mathcal{D}_b\zeta_c + \frac{3}{2}\bar{\eta}\not{\mathcal{D}}\,\eta + \frac{1}{2}\bar{\xi}\not{\mathcal{D}}\,\xi + \cdots\right]. \tag{IV.64}$$

In comparing to the first few lines of (IV.40) and (IV.41), one can see these covariant derivatives forming. The remaining couplings to F and H_2 and to the scalars can also be derived from the $N=2$ geometric structure, but we will not give further details here.

4.5 Examples

In this section we compare the general effective four-dimensional action to various holographic fermion systems that have been considered in the literature, and look for appropriate further (consistent) truncations of the fermionic sector. We focus mainly on two relevant further truncations, namely, the minimal gauged $N=2$ supergravity theory, and the model of [64, 65], which provided an embedding of the holographic superconductor [86, 87] into M-theory.

4.5.1 Minimal gauged supergravity

As discussed in [62], a possible further truncation entails taking

(IV.65)

$$U=V=W=H_3=h=X=0, \qquad f=6\epsilon\,, \qquad H_2=-\epsilon*F \quad (\text{i.e. } i\gamma_5 \not{H}_2=\epsilon\not{F})\,,$$

which sets all the massive fields to zero, leaving the $N=2$ gravity multiplet only. The corresponding equations for the bosonic fields can be derived from the Einstein-Maxwell action

(IV.66)
$$S_B=K_B\int d^4x\,\sqrt{-g}\,(R-F_{\mu\nu}F^{\mu\nu}+24)\,.$$

The simplest fermionic content that one can consider is a charged massive bulk Dirac fermion minimally coupled to gravity and the gauge field (see for example [108], [55], [28], [78], [79]).

In our context, this truncation has an AdS_4 vacuum solution which uplifts to a supersymmetric $AdS_4\times SE_7$ solution in $D=11$. These solutions are thought of as being dual to three-dimensional SCFTs with $N=2$ supersymmetry (in principle). In this truncation, we note that for $\epsilon=+1$, the variations (IV.45-IV.46) of η and ξ are both zero, and ζ_a decouples from η,ξ. Consequently, it is consistent to set $\eta=\xi=0$ (as we did for their superpartners) in this case, and we then obtain the effective $d=4$ action (IV.39) for the gravity supermultiplet

(IV.67)
$$S=S_B+K\int d^4x\sqrt{-g}\left[\bar{\zeta}_a\gamma^{abc}D_b\zeta_c-i\bar{\zeta}_a\big[(F+i\gamma_5*F)^{ac}+2i\gamma^{ac}\big]\zeta_c\right].$$

We note that this gives the expected couplings between the gravitino and the graviphoton[14] [59], [58] (see [143] also).

If $\epsilon=-1$, supersymmety is broken, and we wish to consider other truncations of the fermionic sector. It appears that there are no non-trivial consistent truncations in this case – if we choose

[14]One can use the identity $F^{bd}\gamma_{[b}\gamma^{ac}\gamma_{d]}=F_{bd}\gamma^{bdac}+2F^{ac}=iF_{bd}\gamma_5\epsilon^{bdac}+2F^{ac}$ to rewrite the coupling of the gravitino to the field-strength in the somewhat more familiar form $\sim F^{bd}\bar{\zeta}_a\gamma_{[b}\gamma^{ac}\gamma_{d]}\zeta_c$.

to set the gravitino to zero for example, its equation of motion gives a constraint on η and ξ that appears to have no non-trivial solutions. To see this, we note the action contains the interaction terms (as usual neglecting 4-fermion couplings)

$$\begin{aligned}\mathcal{L}^{int}_{\bar{\psi}\psi} &= 5\bar{\zeta}_a\gamma^{ac}\zeta_c - \frac{9}{2}i\left(\bar{\zeta}_a\gamma^a\eta + \bar{\eta}\gamma^a\zeta_a\right) - 3i\left(\bar{\zeta}_a\gamma^a\xi + \bar{\xi}\gamma^a\zeta_a\right)\\ &+ \frac{i}{2}\bar{\zeta}_a\left[(F + i\gamma_5 * F)^{ac}\right]\zeta_c + \frac{3}{4}\left[\bar{\zeta}_a \not{F}\gamma^a\eta + \bar{\eta}\gamma^a\not{F}\zeta_a\right]\\ &- 9\bar{\eta}\eta - \frac{7}{2}\bar{\xi}\xi - 3(\bar{\eta}\xi + \bar{\xi}\eta) + \frac{3i}{2}\bar{\eta}\not{F}\eta + \frac{i}{4}\bar{\xi}\not{F}\xi\,.\end{aligned} \tag{IV.68}$$

4.5.2 Fermions coupled to the holographic superconductor

We now consider truncations appropriate to holographic superconductors. We note that the general model contains the charged boson X, of charge twice the charge of the fermion fields. This is one of the basic features of the model considered in [54], which studied charged fermions coupled to the holographic superconductor. It is interesting to see how the couplings used there appear in the top-down model.

Refs. [64, 65] considered the following truncation of the bosonic sector

$$\begin{aligned}&h = 0\,, \quad e^{6U} = 1 - \frac{1}{4}|X|^2\,, \quad V = -2U\ (= W), \quad H_2 = *F\,,\\ &H_3 = \frac{i}{4}e^{-12U} * (X^*DX - XDX^*)\,, \quad \epsilon = -1\,, \quad f = 6e^{-12U}\left(-1 + \frac{|X|^2}{3}\right),\end{aligned} \tag{IV.69}$$

where $DX = dX - 4iAX$ as before. As pointed out in [64, 65], in order to set $h = 0$ we need to impose $F \wedge F = 0$ by hand, and thus the truncation (even before considering the fermions) is not consistent. While this restriction allows for black hole solutions carrying electric or magnetic charge only, it excludes solutions of the dyonic type. This theory also has an AdS_4 vacuum solution (with $X = 0$ and $f = -6$), which uplifts to a skew-whiffed $AdS_4 \times SE_7$ solution in $D = 11$. In general, these solutions do not preserve any supersymmetries (an exception being the case where $SE_7 = S^7$).

The $d = 4$ effective action (IV.39) for this truncation is given by

$$S_F = K \int d^4x \sqrt{-g} \left[\bar{\zeta}_a \gamma^{abc} D_b \zeta_c + \frac{3}{2} \bar{\eta} \not{D} \eta + \frac{1}{2} \bar{\xi} \not{D} \xi + \mathcal{L}^{int}_{\bar{\psi}\psi} + \frac{1}{2} \left(\mathcal{L}^{int}_{\bar{\psi}\psi^{\mathbf{c}}} + \text{c.c.} \right) \right], \tag{IV.70}$$

where now

$$\begin{aligned} e^{6U} \mathcal{L}^{int}_{\bar{\psi}\psi} = & \frac{1}{2} \bar{\zeta}_a \left[\left(1 - \frac{|X|^2}{4} \right) i \left(F + i\gamma_5 * F \right)^{ac} - 2 \left(|X|^2 - 5 \right) \gamma^{ac} - \frac{1}{8} \left(X^* \overleftrightarrow{D_b} X \right) \gamma^{bac} \right] \zeta_c \\ & + \frac{3}{4} \bar{\eta} \left[-4 \left(3 - |X|^2 \right) + \frac{1}{8} \left(X^* \overleftrightarrow{\not{D}} X \right) + 2 \left(1 - \frac{|X|^2}{4} \right) i \not{F} \right] \eta \\ & + \frac{3}{8} \bar{\xi} \left[-\frac{4}{3} \left(7 - |X|^2 \right) + \frac{2}{3} \left(1 - \frac{|X|^2}{4} \right) i \not{F} - \frac{1}{4} \left(X^* \overleftrightarrow{\not{D}} X \right) \right] \xi \\ & + \frac{3}{4} \bar{\zeta}_a \left[2i \left(|X|^2 - 3 \right) + \left(1 - \frac{|X|^2}{4} \right) \not{F} \right] \gamma^a \eta + \frac{3}{4} \bar{\eta} \gamma^a \left[2i \left(|X|^2 - 3 \right) + \left(1 - \frac{|X|^2}{4} \right) \not{F} \right] \zeta_a \\ & + \frac{i}{2} \bar{\zeta}_a \left[\left(|X|^2 - 6 \right) + \frac{1}{4} X^* (\not{D} X) \right] \gamma^a \xi + \frac{i}{2} \bar{\xi} \gamma^a \left[\left(|X|^2 - 6 \right) - \frac{1}{4} X (\not{D} X)^* \right] \zeta_a \\ & - \frac{3}{2} \bar{\eta} \left(2 - |X|^2 \right) \xi - \frac{3}{2} \bar{\xi} \left(2 - |X|^2 \right) \eta \end{aligned} \tag{IV.71}$$

and

$$\begin{aligned} e^{3U} \mathcal{L}^{int}_{\bar{\psi}\psi^{\mathbf{c}}} = & \frac{i}{2} \bar{\zeta}_a \gamma_5 \left[-(D_b X) \gamma^{abc} + 4X \gamma^{ac} \right] \zeta^{\mathbf{c}}_c - \frac{3i}{4} \bar{\eta} \gamma_5 \left(\not{D} X + 8X \right) \eta^{\mathbf{c}} \\ & - \frac{1}{4} \bar{\zeta}_a \gamma_5 \left(\not{D} X + 4X \right) \gamma^a \xi^{\mathbf{c}} - \frac{1}{4} \bar{\xi} \gamma_5 \gamma^a \left(\not{D} X + 4X \right) \zeta^{\mathbf{c}}_a \\ & - 3X \left[\bar{\zeta}_a \left(\gamma_5 \gamma^a \right) \eta^{\mathbf{c}} + \bar{\eta} \left(\gamma_5 \gamma^a \right) \zeta^{\mathbf{c}}_a + i \bar{\eta} \gamma_5 \xi^{\mathbf{c}} + i \bar{\xi} \gamma_5 \eta^{\mathbf{c}} \right]. \end{aligned} \tag{IV.72}$$

In order to compare to phenomenologically motivated models, such as the holographic superconductor models, it is instructive to expand in powers of the complex scalar X, it being natural to organize the action by engineering dimension. Since 4-fermi couplings are dimension 6 or higher, we will here keep all terms up to and including dimension five. Doing so we obtain

$$
\begin{aligned}
\mathcal{L}^{int}_{\bar\psi\psi} \simeq\ & \frac{1}{2}i\bar\zeta_a\left[(F+i\gamma_5 * F)^{ac} - 10i\gamma^{ac}\right]\zeta_c + \frac{3}{2}i\bar\eta\left(6i+\not{F}\right)\eta \\
&+\frac{1}{4}i\bar\xi\left(14i+\not{F}\right)\xi + \frac{3}{4}\bar\zeta_a\left(-6i+\not{F}\right)\gamma^a\eta + \frac{3}{4}\bar\eta\gamma^a\left(-6i+\not{F}\right)\zeta_a \\
&-3\left(\bar\eta\xi+\bar\xi\eta+i\bar\zeta_a\gamma^a\xi+i\bar\xi\gamma^a\zeta_a\right) \\
&-\frac{1}{4}i|X|^2\left[i\bar\zeta_a\gamma^{ac}\zeta_c - \frac{3}{2}\left(\bar\zeta_a\gamma^a\eta+\bar\eta\gamma^a\zeta_a\right)+\left(\bar\zeta_a\gamma^a\xi+\bar\xi\gamma^a\zeta_a\right)\right] \\
&+\frac{3}{4}|X|^2\left[\bar\eta\eta-\frac{1}{2}\bar\xi\xi+\left(\bar\eta\xi+\bar\xi\eta\right)\right],
\end{aligned}
\tag{IV.73}
$$

and

$$
\begin{aligned}
\mathcal{L}^{int}_{\bar\psi\psi^{\mathbf{c}}} \simeq\ & \frac{1}{2}i\bar\zeta_a\gamma_5\left[-(D_bX)\gamma^{abc}+4X\gamma^{ac}\right]\zeta^{\mathbf{c}}_c - \frac{3}{4}i\bar\eta\gamma_5\left(\not{D}X+8X\right)\eta^{\mathbf{c}} \\
&-\frac{1}{4}\bar\zeta_a\gamma_5\left(\not{D}X+4X\right)\gamma^a\xi^{\mathbf{c}} - \frac{1}{4}\bar\xi\gamma_5\gamma^a\left(\not{D}X+4X\right)\zeta^{\mathbf{c}}_a \\
&-3X\left(\bar\zeta_a\gamma_5\gamma^a\eta^{\mathbf{c}}+\bar\eta\gamma_5\gamma^a\zeta^{\mathbf{c}}_a+i\bar\eta\gamma_5\xi^{\mathbf{c}}+i\bar\xi\gamma_5\eta^{\mathbf{c}}\right).
\end{aligned}
\tag{IV.74}
$$

Note that we have the same basic couplings as in [54]: we have Majorana couplings between the doubly-charged boson X and spin-$1/2$ fermions. The model is significantly more complicated for several reasons. First, we have kept here several species of spin-$1/2$ fermions, and they are also coupled to the gravitino. An exploration of this model holographically, or a further truncation of the model, would be of interest. We also note that there are generic terms of the form $\bar\psi\gamma_5\not{D}X\psi^{\mathbf{c}}$. These could also be of interest holographically; first in the presence of a boundary chemical potential for A, such a coupling looks similar to the other Majorana coupling near the boundary. But it also would presumably be the most important coupling in non-homogeneous boundary configurations (such as would correspond to spin-wave, nematic order, etc.). We also note that there are generically the "Pauli terms", involving dipole couplings of the fermions to the gauge field strength, which could have important effects in electric or magnetic backgrounds.

It is clear that dropping all of the fermions is a consistent truncation, at least as consistent as the bosonic truncation. It is also apparently possible to keep all of the fermions, although the h

equation of motion will now give a condition including terms non-linear in fermions. It would be interesting to find other truncations of the fermion content. For example, can one reduce, say, to a single species of charged fermion, including the elimination of the gravitino?. If such a truncation exists, it is non-trivial.

4.6 Discussion and conclusions

In this paper, we have explicitly worked out the form of the fermionic action obtained from a consistent truncation of 11-d supergravity on warped Sasaki-Einstein 7-manifolds, which should be thought of as the total space of a $Spin^c$ bundle over a Kähler-Einstein base. The consistent truncation is obtained by restricting to $SU(3)$-invariant excitations. We have checked that the resulting theory is consistent with what is expected from $N = 2$ gauged supergravity in four dimensions, in the case where there is a single vector multiplet and a single hypermultiplet.

This work is relevant to the recent literature on holographic duals of three-dimensional strongly-coupled field theories, particularly to those in which fermions play a central role in the dynamics, such as in superconductors. The theory does contain interesting couplings of the Majorana type, similar to those considered in the literature, as well as some new ones. We have briefly considered several further truncations that are closer to bottom-up models that have been discussed in the literature. Generally, we have found that it is difficult to find truncations of the fermionic sector. In particular, the gravitino is typically coupled to the other fermion fields. As a result, in holographic studies, we expect to see a spin-$3/2$ operator in the dual theory (the boundary supercurrents, in supersymmetric cases), and given appropriate asymptotic bosonic configurations, this operator would mix with other fermionic operators. We have not done an exhaustive job of studying this decoupling problem however, and it would be of interest to do so and to consider a variety of holographic applications.

CHAPTER V

Fermions in consistent truncations of type IIB supergravity on squashed Sasaki-Einstein manifolds

In the same spirit as the previous chapter, here we discuss the dimensional reduction of fermionic modes in a recently found class of consistent truncations of type IIB supergravity compactified on squashed five-dimensional Sasaki-Einstein manifolds. We derive the lower dimensional equations of motion and effective action, and comment on the supersymmetry of the resulting theory, which is consistent with $N = 4$ gauged supergravity in $d = 5$, coupled to two vector multiplets. We compute fermion masses by linearizing around two AdS_5 vacua of the theory: one that breaks $N = 4$ down to $N = 2$ spontaneously, and a second one which preserves no supersymmetries. The truncations under consideration are noteworthy in that they retain massive modes which are charged under a $U(1)$ subgroup of the R-symmetry, a feature that makes them interesting for applications to condensed matter phenomena via gauge/gravity duality. In this light, as an application of our general results we exhibit the coupling of the fermions to the type IIB holographic superconductor, and find a consistent further truncation of the fermion sector that retains a single spin-1/2 mode. This chapter is based on [9], which stemmed from collaboration with Ibrahima Bah, Juan Ignacio Jottar and Robert Leigh.

5.1 Introduction

Recently, consistent truncations of type IIB and 11-d supergravity including massive (charged) modes have sparked a great deal of interest. The relevance of these reductions is two-fold: not only are they novel from the supergravity perspective, but they also constitute an interesting arena to test and extend the ideas of gauge/gravity duality. Indeed, these truncations provide a powerful way of generating solutions of the ten and eleven-dimensional supergravity theories via uplifting of lower dimensional solutions. By definition, this possibility is guaranteed by the consistency of the reduction. Also from a supergravity perspective, the inclusion of massive modes is highly non-trivial; consistent truncations are hard to find, even when truncating to the massless Kaluza-Klein (KK) spectrum. In fact, until not long ago it was widely believed that consistency prevents one from keeping a finite number of massive KK modes. From the gauge/gravity correspondence perspective, in turn, the lower dimensional supergravity theories obtained from these reductions are assumed to possess field theory duals with various amounts of unbroken super(-conformal)symmetry. Strikingly, the inclusion of charged operators on the field theory side, dual to massive bulk fields, opened the door for a stringy ("top-down") modelling of condensed matter phenomena, such as superfluidity and superconductivity and systems with non-relativistic conformal symmetries, via the holographic correspondence. Even though the original work in these directions [11, 76, 86, 87, 150] was based on a phenomenological, "bottom-up" approach, it is clearly advantageous to consider top-down descriptions of these (or similar) systems. Indeed, a description in terms of ten or eleven-dimensional supergravity backgrounds may shed light on the existence of a consistent UV completion of the lower-dimensional effective bulk theories, while possibly fixing various parameters that appear to be arbitrary in the bottom-up constructions.

In this paper we shall be concerned with the consistent truncations of type IIB supergravity on squashed Sasaki-Einstein five-manifolds (SE_5) whose bosonic content was recently considered in [27, 67, 111] (see [149] for related work). These constructions were largely motivated by the results of [113] (see [1, 93] also), which had a quite interesting by-product: while searching for solutions of type IIB supergravity with non-relativistic asymptotic symmetry groups, consistent five-dimensional truncations including massive bosonic modes were constructed. In particular, massive scalars arise from the breathing and squashing modes in the internal manifold, which is then a "deformed" Sasaki-Einstein space, generalizing the case of breathing and squashing modes on spheres that had been studied in [17, 110] (see also [18]). Regarding the internal SE_5 manifold as a $U(1)$ bundle over a Kähler-Einstein (KE) base space of complex dimension two, the guiding principle behind these consistent truncations is to keep modes which are singlets only under the structure group of the KE base. The bosonic sector of the corresponding truncations including massive modes in 11-d supergravity on squashed SE_7 manifolds had been previously discussed in [62], and provided the basis for the embedding of the original holographic AdS_4 superconductors of [86, 87] into M-theory, a connection that was explored in [64, 65]. In our recent work [10] we have extended the consistent truncation of 11-d supergravity on squashed SE_7 to include the fermionic sector, and in particular provided the effective 4-d action describing the coupling of fermion modes to the M-theory holographic superconductor.

At the same time that the work of [64] appeared, the embedding of an asymptotically AdS_5 holographic superconductor into type IIB supergravity was reported in [77]. Continuing with the program we initiated in [10], in the present work we discuss the extension of the consistent truncation of type IIB supergravity on SE_5 discussed in [27, 67, 111] to include the fermionic sector. In particular, as an application of our results we present the effective action describing the coupling of the fermion modes to the holographic superconductor of [77]. Knowing the precise form of said couplings is important from the point of view of the applications of gauge/gravity duality to the description of strongly coupled condensed matter phenomena, insofar as it determines the nature of fermionic correlators in the presence of superconducting condensates, that rely on how the fermionic operators of the dual theory couple to scalars. Hence, we set the stage for the discussion of these and related questions from a top-down perspective. A related problem involving a superfluid p-wave transition was studied in [4], in the context of (3+1)-dimensional supersymmetric field theories dual to probe $D5$-branes in $AdS^5 \times S^5$. In the top-down approach starting from either ten or eleven-dimensional supergravity, inevitably the consistent truncations will include not only spin-1/2 fermions that might be of phenomenological interest but also spin-3/2 fields. One finds that these generally mix together via generalized Yukawa couplings, and this mixing will have implications for correlation functions in the dual field theory. One of our original motivations for the present work as well as [10] was to understand this mixing in more detail and to investigate the existence of "further truncations" which might involve (charged) spin-1/2 fermions alone. As we explain in section 5.6, in the present case we have indeed found such a model, containing a single spin-1/2 field, in the truncation corresponding to the type IIB holographic superconductor.

This paper is organized as follows. In section 5.2 we briefly review some aspects of the truncations of type IIB supergravity constructed in [27, 67, 111] and the extension of the bosonic ansatz to include the fermion modes. In section 5.3 we present our main result: the effective five-dimensional action functional describing the dynamics of the fermions and their couplings to the bosonic fields. We chose to perform this calculation by directly reducing the 10-d equations of motion for the gravitino and dilatino. The resulting action is consistent with 5-d $N = 4$ gauged supergravity, as has been anticipated. In section 5.4 we reduce the supersymmetry variation of the gravitino and dilatino, and comment on the supersymmetric structure of the five-dimensional theory by considering how the fermions fit into the supermultiplets of $N = 4$ gauged supergravity. In principle, a complete mapping to the highly constrained form of $N = 4$ actions could be made, although we do not give all of the details here. The $N = 4$ theory has two vacuum AdS_5 solutions, one with $N = 2$ supersymmetry and one without supersymmetry. In section 5.5 we linearize the fermionic sector in each of these vacua and demonstrate that as expected the gravitini attain masses via the Stückelberg mechanism, which is a useful check on the consistency of our results. In section 5.6 we apply our results to several further truncations of interest: the minimal gauged $N = 2$ supergravity theory in five dimensions, and the dual [77] of the $(3+1)$-dimensional holographic superconductor. We conclude in section 5.7. The details of many of our computations as well as a full accounting of our conventions appear in a series of appendices.

5.2 Type IIB supergravity on squashed Sasaki-Einstein five-manifolds

5.2.1 Bosonic ansatz

In this section we briefly review the ansatz for the bosonic fields in the consistent truncations of [27, 67, 111]. In the following subsection, we will discuss the extension of this ansatz to include the fermionic fields of type IIB supergravity. Here we mostly follow the type IIB conventions of [63,66,67], with slight modifications as we find appropriate. Further details of these conventions can be found in appendix A.3.

The Kaluza-Klein metric ansatz in the truncations of interest is given by [27, 67, 111]

$$ds^2_{10} = e^{2W(x)} ds^2_E(M) + e^{2U(x)} ds^2(KE) + e^{2V(x)} \big(\eta + A_1(x)\big)^2 \,, \tag{V.1}$$

where $W(x) = -\frac{1}{3}(4U(x)+V(x))$. Here, M is an arbitrary "external" five-dimensional manifold, with coordinates denoted generically by x and five-dimensional Einstein-frame metric $ds^2_E(M)$, and KE is an "internal" four-dimensional Kähler-Einstein manifold (henceforth referred to as "KE base") coordinatized by y and possessing Kähler form J. The one-form A_1 is defined in T^*M and $\eta \equiv d\chi + \mathcal{A}(y)$, where $\mathcal{A}$ is an element of T^*KE satisfying $d\mathcal{A} \equiv \mathcal{F} = 2J$. For a fixed point in the external manifold, the compact coordinate χ parameterizes the fiber of a $U(1)$ bundle over KE, and the five-dimensional internal manifold spanned by (y, χ) is then a squashed Sasaki-Einstein manifold, with the breathing and squashing modes parameterized by the scalars $U(x)$ and $V(x)$.[1] In addition to the metric, the bosonic content of type IIB supergravity [96, 148] includes the dilaton Φ, the NSNS 3-form field strength $H_{(3)}$, and the RR field strengths $F_{(1)} \equiv dC_0$, $F_{(3)}$ and $F_{(5)}$,

[1] In particular, $U - V$ is the squashing mode, describing the squashing of the $U(1)$ fiber with respect to the KE base, while the breathing mode $4U + V$ modifies the overall volume of the internal manifold. When $U = V = 0$, the internal manifold becomes a five-dimensional Sasaki-Einstein manifold SE_5.

where C_0 is the axion and $F_{(5)}$ is self-dual. The rationale behind the corresponding ansätze is the idea that the consistency of the dimensional reduction is a result of truncating the KK tower to include fields that transform as singlets only under the structure group of the KE base, which in this case corresponds to $SU(2)$. This prescription allows for an interesting spectrum in the lower dimensional theory, inasmuch as the $SU(2)$ singlets include fields that are charged under the $U(1)$ isometry generated by ∂_χ. The globally defined Kähler 2-form $J = d\mathcal{A}/2$ and the holomorphic $(2,0)$-form $\Sigma_{(2,0)}$ define the Kähler and complex structures, respectively, on the KE base. They are $SU(2)$-invariant and can be used in the reduction of the various fields to five dimensions. The $U(1)$-bundle over KE is such that they satisfy

(V.2) $$\Sigma_{(2,0)} \wedge \Sigma^*_{(2,0)} = 2J^2 \,, \qquad \text{and} \qquad d\Sigma_{(2,0)} = 3i\mathcal{A} \wedge \Sigma_{(2,0)} \,.$$

More precisely, as will be clear from the discussion to follow below, the relevant charged form Ω on the total space of the bundle that should enter the ansatz for the various form fields is given by

(V.3) $$\Omega \equiv e^{3i\chi} \Sigma_{(2,0)} \,,$$

and satisfies

(V.4) $$d\Omega = 3i\eta \wedge \Omega \,.$$

The ansätze for the bosonic fields is then [67]

$$
\begin{aligned}
F_{(5)} &= 4e^{8W+Z}\mathrm{vol}_5^E + e^{4(W+U)} * K_2 \wedge J + K_1 \wedge J \wedge J \\
&\quad + \left[2e^Z J \wedge J - 2e^{-8U} * K_1 + K_2 \wedge J\right] \wedge (\eta + A_1) \\
&\quad + \left[e^{4(W+U)} * L_2 \wedge \Omega + L_2 \wedge \Omega \wedge (\eta + A_1) + \text{c.c.}\right]
\end{aligned} \tag{V.5}
$$

$$
\begin{aligned}
F_{(3)} &= G_3 + G_2 \wedge (\eta + A_1) + G_1 \wedge J + G_0\, J \wedge (\eta + A_1) \\
&\quad + \left[N_1 \wedge \Omega + N_0\, \Omega \wedge (\eta + A_1) + \text{c.c.}\right]
\end{aligned} \tag{V.6}
$$

$$
\begin{aligned}
H_{(3)} &= H_3 + H_2 \wedge (\eta + A_1) + H_1 \wedge J + H_0\, J \wedge (\eta + A_1) \\
&\quad + \left[M_1 \wedge \Omega + M_0\, \Omega \wedge (\eta + A_1) + \text{c.c.}\right]
\end{aligned} \tag{V.7}
$$

$$
C_{(0)} = a \tag{V.8}
$$

$$
\Phi = \phi \tag{V.9}
$$

where vol_5^E and $*$ are the volume form and Hodge dual appropriate to the five-dimensional Einstein-frame metric $ds_E^2(M)$, and $W(x) = -\frac{1}{3}(4U(x) + V(x))$ as before. Several comments are in order. First, all the fields other than (η, J, Ω) are defined on $\Lambda^* T^* M$. Z, a, ϕ, G_0, H_0 are real scalars, and M_0, N_0 are complex scalars. The form fields G_1, G_2, G_3, H_1, H_2, H_3, K_1 and K_2 are real, while M_1, N_1 and L_2 are complex forms. As pointed out in [67], the scalars G_0 and H_0 vanish by virtue of the type IIB Bianchi identities. We also notice that the self-duality of $F_{(5)}$ is automatic in the ansatz (V.5): the first two lines are duals of each other, while the last line is self-dual.

Inserting the ansatz into the type IIB equations of motion and Bianchi identities (Appendix I), one finds that the various fields are related as[2]

$$
\begin{aligned}
H_3 &= dB_2 + \frac{1}{2}(db - 2B_1) \wedge F_2 \\
G_3 &= dC_2 - adB_2 + \frac{1}{2}\left(dc - adb - 2C_1 + 2aB_1\right) \wedge F_2 \\
H_2 &= dB_1 \\
F_2 &= dA_1 \\
G_2 &= dC_1 - adB_1 \\
K_2 &= dE_1 + \frac{1}{2}(db - 2B_1) \wedge (dc - 2C_1) \\
G_1 &= dc - adb - 2C_1 + 2aB_1 \\
H_1 &= db - 2B_1 \\
K_1 &= dh - 2E_1 - 2A_1 + Y^* DX + Y DX^* - X DY^* - X^* DY \\
M_1 &= DY \\
N_1 &= DX - aDY \\
M_0 &= 3iY \\
N_0 &= 3i(X - aY) \\
e^Z &= 1 + 3i(Y^* X - Y X^*), \qquad \text{(V.10)}
\end{aligned}
$$

where $F_2 \equiv dA_1$, X, Y and L_2, M_1, N_1 are complex, and $DY = dY - 3iA_1 Y$, $DX = dX - 3iA_1 X$.

As was explained in detail in [27, 67], the physical scalars parameterize the coset $SO(1,1) \times \big(SO(5,2)/(SO(5) \times SO(2))\big)$, while the structure of the 1-forms and 2-forms is such that a $Heis_3 \times U(1)$ subgroup is gauged.

[2] We have chosen the notation of Ref. [67] apart from replacing their χ, ξ with X, Y, to avoid confusion with the fiber coordinate.

5.2.2 Fermionic ansatz

The fermionic content of type IIB supergravity comprises a positive chirality dilatino and a negative chirality gravitino. Instead of expressing the theory in terms of pairs of Majorana-Weyl fermions, we find it notationally simplest to use complex Weyl spinors. Quite generally, we would like to decompose the gravitino using an ansatz of the form

$$\Psi_a(x,y,\chi) = \sum_I \psi_a^I(x) \otimes \eta^I(y,\chi) \tag{V.11}$$

$$\Psi_\alpha(x,y,\chi) = \sum_I \lambda^I(x) \otimes \eta_\alpha^I(y,\chi) \tag{V.12}$$

$$\Psi_{\mathrm{f}}(x,y,\chi) = \sum_I \varphi^I(x) \otimes \eta_{\mathrm{f}}^I(y,\chi)\,, \tag{V.13}$$

where a, α and f denote the indices in the direction of the external manifold, the KE base, and the fiber, respectively. The projection to singlets under the structure group of the KE base was recently described in great detail for the case of $D = 11$ supergravity compactified on squashed SE_7 manifolds [10]. Since the principles at work in the present case are essentially the same, here we limit ourselves to pointing to a few relevant facts and results. As we have discussed, the five-dimensional internal space is the total space of a $U(1)$ bundle over a KE base. In general, the base is not spin, and therefore spinors do not necessarily exist globally on the base. However, it is always possible to define a $Spin^c$ bundle globally on KE (see [122], for example), and our (c-)spinors will then be sections of this bundle. Indeed, we have seen above that the holomorphic form Ω is also charged under this $U(1)$. The $U(1)$ generator is proportional to ∂_χ, and hence $\nabla_\alpha - \mathcal{A}_\alpha\partial_\chi$ is the gauge connection on the $Spin^c$ bundle, where ∇_α is the covariant derivative on KE. Of central importance to us in the reduction to invariants of the structure group are the gauge-covariantly-constant spinors, which can be defined on any Kähler manifold [94] and satisfy in the present context

$$(\nabla_\alpha - \mathcal{A}_\alpha\partial_\chi)\varepsilon(y,\chi) = 0\,, \tag{V.14}$$

where

$$\varepsilon(y,\chi) = \varepsilon(y)e^{ie\chi} \tag{V.15}$$

for fixed "charge" e. For a KE base of real dimension d_b, these satisfy (see [10], [68] for example)[3]

$$Q\varepsilon \equiv -iJ_{\alpha\beta}\Gamma^{\alpha\beta}\varepsilon = \frac{4ed_b}{d_b+2}\varepsilon\,. \tag{V.16}$$

In other words, the matrix $Q = -iJ_{\alpha\beta}\Gamma^{\alpha\beta}$ on the left is (up to normalization) the U(1) charge operator. It has maximum eigenvalues $\pm d_b$, and the corresponding spinors have charge

$$e = \pm\frac{d_b+2}{4}\,. \tag{V.17}$$

These two spinors are charge conjugates of one another, and we will henceforth denote them by $\varepsilon_\pm$. By definition, they satisfy $\not{\mathcal{F}}\varepsilon_\pm = iQ\varepsilon_\pm = \pm i d_b\,\varepsilon_\pm$, where $\not{\mathcal{F}} \equiv (1/2)\mathcal{F}_{\alpha\beta}\Gamma^{\alpha\beta}$. These spinors with maximal Q-charge are in fact the singlets under the structure group, and they constitute the basic building blocks of the reduction ansatz for the fermions. In the case at hand $d_b = 4$ and the structure group is $SU(2)$; in fact we have an unbroken $SU(2)_L \times U(1)$ subgroup of $Spin(4)$ in which the spinor transforms as $\mathbf{2}_0 \oplus \mathbf{1}_+ \oplus \mathbf{1}_-$. In the complex basis introduced in A.3.3, we find

$$Q_\alpha\varepsilon_\pm = \pm\frac{1}{2}\varepsilon_\pm \quad (\alpha = 1,2) \tag{V.18}$$

and

$$\bar{P}_\alpha\varepsilon_+ = 0, \quad P_\alpha\varepsilon_- = 0\,, \tag{V.19}$$

where $Q_\alpha = \Gamma^{\alpha\bar{\alpha}}$, $P_\alpha = \Gamma^\alpha\Gamma^{\bar{\alpha}}$, and $\bar{P}_\alpha = \Gamma^{\bar{\alpha}}\Gamma^\alpha$. In the Fock state basis, these are $\varepsilon_\pm \leftrightarrow |\pm\frac{1}{2},\pm\frac{1}{2}\rangle$ and the remaining two states form a (charge-zero) doublet. Unlike the two $SU(3)$ singlet spinors that were used to reduce the gravitino in the 11-d case, here the two singlets have the *same* chirality in $4+0$ dimensions, that is $\gamma_{\mathrm{f}}\varepsilon_\pm = \varepsilon_\pm$ (this follows, since $\gamma_{\mathrm{f}} = -\gamma^{1234} = \prod_\alpha 2Q_\alpha$). Similarly,

[3] All of our Clifford algebra and spinor conventions are compiled in Appendix A.3.

for the complex form $\Sigma_{(2,0)}$ we find $[Q, \not{\Sigma}] = 8\not{\Sigma}$, which means that $\Sigma_{(2,0)}$ carries charge $e_\Sigma = 3$ and justifies the definition $\Omega = e^{3i\chi}\Sigma_{(2,0)}$ discussed above.

We are now in position to write the reduction ansatz for the gravitino and dilatino. Dropping all the $SU(2)$ representations other than the singlets, we take

$$\text{(V.20)} \quad \Psi_a(x,y,\chi) = \psi_a^{(+)}(x) \otimes \varepsilon_+(y) e^{\frac{3}{2}i\chi} \otimes u_- + \psi_a^{(-)}(x) \otimes \varepsilon_-(y) e^{-\frac{3}{2}i\chi} \otimes u_-$$

$$\text{(V.21)} \quad \Psi_\alpha(x,y,\chi) = \rho^{(+)}(x) \otimes \gamma_\alpha \varepsilon_+(y) e^{\frac{3}{2}i\chi} \otimes u_-$$

$$\text{(V.22)} \quad \Psi_{\bar{\alpha}}(x,y,\chi) = \rho^{(-)}(x) \otimes \gamma_{\bar{\alpha}} \varepsilon_-(y) e^{-\frac{3}{2}i\chi} \otimes u_-$$

$$\text{(V.23)} \quad \Psi_{\mathrm{f}}(x,y,\chi) = \varphi^{(+)}(x) \otimes \varepsilon_+(y) e^{\frac{3}{2}i\chi} \otimes u_- + \varphi^{(-)}(x) \otimes \varepsilon_-(y) e^{-\frac{3}{2}i\chi} \otimes u_-$$

$$\text{(V.24)} \quad \lambda(x,y,\chi) = \lambda^{(+)}(x) \otimes \varepsilon_+(y) e^{\frac{3}{2}i\chi} \otimes u_+ + \lambda^{(-)}(x) \otimes \varepsilon_-(y) e^{-\frac{3}{2}i\chi} \otimes u_+$$

where $\varphi^{(\pm)}, \rho^{(\pm)}$ and $\psi_a^{(\pm)}$ are $(4+1)$-dimensional spinors on M, the superscript **c** denotes charge conjugation, and we have used the complex basis introduced in A.3.3 for the KE base directions ($\alpha, \bar{\alpha} = 1, 2$). The constant spinors $u_+ = \binom{1}{0}$ and $u_- = \binom{0}{1}$ have been introduced as bookkeeping devices to keep track of the $D = 10$ chiralities. Since our starting spinors were only Weyl in $D = 10$ (as opposed to Majorana-Weyl) there is no relation between, say, $\lambda^{(+)}$ and $\lambda^{(-)}$; they are independent Dirac spinors in $4+1$ dimensions, and the same applies to the rest of the spinors in the ansatz. Although one could write the $(4+1)$-spinors as symplectic Majorana, there is no real benefit to introducing such notation at this point in the discussion. Notice that all of these modes are annihilated by the gauge-covariant derivative on KE. Equations (V.20)-(V.24) provide the starting point for the dimensional reduction of the $D = 10$ equations of motion of type IIB supergravity down to $d = 5$.

According to the charge conjugation conventions in A.3.5, we also find

$$\text{(V.25)}\qquad \Psi^{\mathbf{c}}_a(x,y,\chi) = \psi^{(-)\mathbf{c}}_a(x)\otimes\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_- - \psi^{(+)\mathbf{c}}_a(x)\otimes\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_-$$

$$\text{(V.26)}\qquad (\Psi_\alpha)^{\mathbf{c}}(x,y,\chi) = -\rho^{(+)\mathbf{c}}(x)\otimes\gamma_{\bar\alpha}\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_-$$

$$\text{(V.27)}\qquad (\Psi_{\bar\alpha})^{\mathbf{c}}(x,y,\chi) = \rho^{(-)\mathbf{c}}(x)\otimes\gamma_{\alpha}\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_-$$

$$\text{(V.28)}\qquad \Psi^{\mathbf{c}}_{\mathrm{f}}(x,y,\chi) = \varphi^{(-)\mathbf{c}}(x)\otimes\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_- - \varphi^{(+)\mathbf{c}}(x)\otimes\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_-$$

$$\text{(V.29)}\qquad \lambda^{\mathbf{c}}(x,y,\chi) = -\lambda^{(-)\mathbf{c}}(x)\otimes\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_+ + \lambda^{(+)\mathbf{c}}(x)\otimes\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_+$$

5.3 Five-dimensional equations of motion and effective action

The type IIB fermionic equations of motion to linear order in the fermions are given by (see appendix I for details)

$$\text{(V.30)}\qquad \hat{\not{\mathcal{D}}}\lambda = \frac{i}{8}\not{F}_{(5)}\lambda + \mathcal{O}(\Psi^2)$$

$$\text{(V.31)}\qquad \Gamma^{ABC}\hat{\mathcal{D}}_B\Psi_C = -\frac{1}{8}\not{G}^*\Gamma^A\lambda + \frac{1}{2}\not{P}\Gamma^A\lambda^{\mathbf{c}} + \mathcal{O}(\Psi^3)$$

Here, $\hat{D}$ denotes the flux-dependent supercovariant derivative, which acts as follows:

$$\text{(V.32)}\qquad \hat{\not{\mathcal{D}}}\lambda = \left(\hat{\not{\nabla}} - \frac{3i}{2}\not{Q}\right)\lambda - \frac{1}{4}\Gamma^A\not{G}\Psi_A - \Gamma^A\not{P}\Psi^{\mathbf{c}}_A\,,$$

$$\text{(V.33)}\qquad \hat{\mathcal{D}}_B\Psi_C = \left(\hat{\nabla}_B - \frac{i}{2}Q_B\right)\Psi_C + \frac{i}{16}\not{F}_{(5)}\Gamma_B\Psi_C - \frac{1}{16}S_B\Psi^{\mathbf{c}}_C\,,$$

where $\hat{\nabla}_B$ denotes the ordinary 10-d covariant derivative and we have defined

$$\text{(V.34)}\qquad S_B \equiv \frac{1}{6}\left(\Gamma_B{}^{DEF}G_{DEF} - 9\Gamma^{DE}G_{BDE}\right).$$

As described in Appendix I.1, defining the axion-dilaton $\tau = C_{(0)} + ie^{-\Phi} = a + ie^{-\phi}$ our conventions imply

$$\text{(V.35)}\qquad G = ie^{\Phi/2}\left(\tau dB - dC_{(2)}\right) = -\left(e^{-\phi/2}H_{(3)} + ie^{\phi/2}F_{(3)}\right),$$

and

$$\text{(V.36)}\qquad P = \frac{i}{2}e^{\Phi}d\tau = \frac{d\phi}{2} + \frac{i}{2}e^{\phi}da\,, \qquad Q = -\frac{1}{2}e^{\Phi}dC_{(0)} = -\frac{1}{2}e^{\phi}da\,.$$

It will prove convenient to introduce a compact notation as follows:

(V.37) $$\mathcal{G}_1 = e^{\frac{1}{2}(\phi-4U)}\left(G_1 - ie^{-\phi}H_1\right) \qquad \tilde{\mathcal{G}}_1 = e^{\frac{1}{2}(\phi-4U)}\left(G_1 + ie^{-\phi}H_1\right)$$

(V.38) $$\mathcal{G}_2 = e^{\frac{1}{2}(\phi+4U)}\Sigma\left(G_2 - ie^{-\phi}H_2\right) \qquad \tilde{\mathcal{G}}_2 = e^{\frac{1}{2}(\phi+4U)}\Sigma\left(G_2 + ie^{-\phi}H_2\right)$$

(V.39) $$\mathcal{G}_3 = e^{\frac{1}{2}(\phi+4U)}\Sigma^{-1}\left(G_3 - ie^{-\phi}H_3\right) \qquad \tilde{\mathcal{G}}_3 = e^{\frac{1}{2}(\phi+4U)}\Sigma^{-1}\left(G_3 + ie^{-\phi}H_3\right)$$

(V.40) $$\mathcal{N}_1^{(+)} = e^{\frac{1}{2}(\phi-4U)}\left(N_1 - ie^{-\phi}M_1\right) \qquad \tilde{\mathcal{N}}_1^{(+)} = e^{\frac{1}{2}(\phi-4U)}\left(N_1 + ie^{-\phi}M_1\right)$$

(V.41) $$\mathcal{N}_1^{(-)} = e^{\frac{1}{2}(\phi-4U)}\left(N_1^* - ie^{-\phi}M_1^*\right) \qquad \tilde{\mathcal{N}}_1^{(-)} = e^{\frac{1}{2}(\phi-4U)}\left(N_1^* + ie^{-\phi}M_1^*\right)$$

(V.42) $$\mathcal{N}_0^{(+)} = e^{\frac{1}{2}(\phi-4U)}\Sigma^2\left(N_0 - ie^{-\phi}M_0\right) \qquad \tilde{\mathcal{N}}_0^{(+)} = e^{\frac{1}{2}(\phi-4U)}\Sigma^2\left(N_0 + ie^{-\phi}M_0\right)$$

(V.43) $$\mathcal{N}_0^{(-)} = e^{\frac{1}{2}(\phi-4U)}\Sigma^2\left(N_0^* - ie^{-\phi}M_0^*\right) \qquad \tilde{\mathcal{N}}_0^{(-)} = e^{\frac{1}{2}(\phi-4U)}\Sigma^2\left(N_0^* + ie^{-\phi}M_0^*\right)$$

where the scalar Σ is defined as $\Sigma \equiv e^{2(W+U)} = e^{-\frac{2}{3}(U+V)}$. Its significance will be reviewed later in the paper.

The detailed derivation of the equations of motion is performed in Appendix J, and we will not reproduce them here in the main body of the paper as the expressions are lengthy. Given those equations of motion, we will write an action from which they may be derived. Before doing so, we first consider the kinetic terms and introduce a field redefinition such that the kinetic terms are diagonalized.

5.3.1 Field redefinitions

In order to find the appropriate field redefinitions it is enough to consider the derivative terms, which follow from a Lagrangian density of the form (with respect to the 5-d Einstein frame-measure $d^5x\sqrt{-g_5^E}$)

$$\begin{aligned}L_{kin}^{(\pm)} = e^W\Big[\frac{1}{2}\bar{\lambda}^{(\pm)}\not{D}\lambda^{(\pm)} &+ \bar{\psi}_a^{(\pm)}\left(\gamma^{abc}D_b\psi_c^{(\pm)} - 4i\gamma^{ab}D_b\rho^{(\pm)} - i\gamma^{ab}D_b\varphi^{(\pm)}\right) \\ &- i\bar{\rho}^{(\pm)}\left(4\gamma^{ab}D_a\psi_b^{(\pm)} - 12i\not{D}\rho^{(+)} - 4i\not{D}\varphi^{(\pm)}\right) \\ &+ \bar{\varphi}^{(\pm)}\left(-i\gamma^{ab}D_a\psi_b^{(\pm)} - 4\not{D}\rho^{(\pm)}\right)\Big].\end{aligned} \quad \text{(V.44)}$$

Shifting the gravitino as[4]

$$\psi_a^{(\pm)} = \tilde{\psi}_a^{(\pm)} + \frac{i}{3}\gamma_a \left(\varphi^{(\pm)} + 4\rho^{(\pm)}\right) \quad \Rightarrow \quad \bar{\psi}_a^{(\pm)} = \bar{\tilde{\psi}}_a^{(\pm)} + \frac{i}{3}\left(\bar{\varphi}^{(\pm)} + 4\bar{\rho}^{(\pm)}\right)\gamma_a \,, \tag{V.45}$$

we obtain

$$\begin{aligned} L_{kin}^{(\pm)} = e^W \Big[&\frac{1}{2}\bar{\lambda}^{(\pm)} \not{D}\lambda^{(\pm)} + \bar{\tilde{\psi}}_a^{(\pm)}\gamma^{abc} D_b \tilde{\psi}_c^{(\pm)} + 8\bar{\rho}^{(\pm)}\not{D}\rho^{(\pm)} \\ &+ \frac{4}{3}\left(\bar{\rho}^{(\pm)} + \bar{\varphi}^{(\pm)}\right) \not{D}\left(\rho^{(\pm)} + \varphi^{(\pm)}\right)\Big]. \end{aligned} \tag{V.46}$$

Then we are led to define[5]

$$\tilde{\lambda}^{(\pm)} = e^{W/2}\lambda^{(\pm)} \tag{V.47}$$

$$\zeta_a^{(\pm)} = e^{W/2}\left[\psi_a^{(\pm)} - \frac{i}{3}\gamma_a\left(\varphi^{(\pm)} + 4\rho^{(\pm)}\right)\right] \tag{V.48}$$

$$\xi^{(\pm)} = 4e^{W/2}\rho^{(\pm)} \tag{V.49}$$

$$\eta^{(\pm)} = 2e^{W/2}\left(\rho^{(\pm)} + \varphi^{(\pm)}\right), \tag{V.50}$$

which results in

$$L_{kin}^{(\pm)} = \frac{1}{2}\bar{\tilde{\lambda}}^{(\pm)}\not{D}\tilde{\lambda}^{(\pm)} + \bar{\zeta}_a^{(\pm)}\gamma^{abc}D_b\zeta_c^{(\pm)} + \frac{1}{2}\bar{\xi}^{(\pm)}\not{D}\xi^{(\pm)} + \frac{1}{3}\bar{\eta}^{(\pm)}\not{D}\eta^{(\pm)} \tag{V.51}$$

$$- \frac{1}{2}\left[\bar{\zeta}_a^{(\pm)}\gamma^{abc}\left(\partial_b W\right)\zeta_c^{(\pm)} + \frac{1}{2}\bar{\xi}^{(\pm)}\left(\not{\partial}W\right)\xi^{(\pm)} + \frac{1}{3}\bar{\eta}^{(\pm)}\left(\not{\partial}W\right)\eta^{(\pm)}\right]. \tag{V.52}$$

The W-dependent interaction terms in the second line are produced by the action of the derivatives on the warping factors involved in the field redefinitions, and they will cancel against similar terms in the interaction Lagrangian. We note that the fields we have defined are not canonically normalized. We have done this simply to avoid square-root factors.

The equations of motion written in terms of the fields (V.47)-(V.50) are given explicitly in Appendix J. They follow from an effective $d = 5$ action that we derive below.

[4]To avoid confusion, we note that the notation $\bar{\varphi}^{(\pm)}$ means $(\varphi^{(\pm)})^\dagger\gamma^0$, etc.

[5]One should not confuse the one-form η dual to the Reeb vector field with the fermions $\eta^{(\pm)}$.

5.3.2 Effective action

The equations of motion for the 5d fields (V.47)-(V.50), which are explicitly displayed in appendix J, follow from an effective action functional of the form

$$\begin{aligned}S_{4+1} = K_5 \int d^5x\sqrt{-g_5^E}\Big[&\frac{1}{2}\bar{\tilde{\lambda}}^{(+)}\not{D}\tilde{\lambda}^{(+)} + \bar{\zeta}_a^{(+)}\gamma^{abc}D_b\zeta_c^{(+)} + \frac{1}{2}\bar{\xi}^{(+)}\not{D}\xi^{(+)} + \frac{1}{3}\bar{\eta}^{(+)}\not{D}\eta^{(+)} \\ &+ \frac{1}{2}\bar{\tilde{\lambda}}^{(-)}\not{D}\tilde{\lambda}^{(-)} + \bar{\zeta}_a^{(-)}\gamma^{abc}D_b\zeta_c^{(-)} + \frac{1}{2}\bar{\xi}^{(-)}\not{D}\xi^{(-)} + \frac{1}{3}\bar{\eta}^{(-)}\not{D}\eta^{(-)} \\ &+ \mathcal{L}_{\bar{\psi}\psi}^{(+)} + \mathcal{L}_{\bar{\psi}\psi}^{(-)} + \frac{1}{2}\left(\mathcal{L}_{\bar{\psi}\psi^{\mathbf{c}}}^{(+)} + \mathcal{L}_{\bar{\psi}\psi^{\mathbf{c}}}^{(-)} + \text{ c.c.}\right)\Big]\end{aligned} \tag{V.53}$$

where K_5 is a normalization constant depending on the volume of the KE base, the length of the fiber parameterized by χ, and the normalization of the spinors $\varepsilon_\pm$. Here, $D_a\psi^{(\pm)} = \left(\nabla_a \mp \frac{3i}{2}A_{1a}\right)\psi^{(\pm)}$ for $\psi = \tilde{\lambda}, \psi_a, \eta, \xi$, and the interaction Lagrangians are given by

$$\mathcal{L}_{\bar{\psi}\psi}^{(\pm)} = \mathcal{L}_{mass}^{(\pm)} + \mathcal{L}_1^{(\pm)} + \mathcal{L}_2^{(\pm)} \tag{V.54}$$

where we have defined

$$\begin{aligned}\mathcal{L}_{mass}^{(\pm)} = &\mp\frac{1}{2}\left(e^{-4U}\Sigma^{-1} + \frac{3}{2}\Sigma^2 \pm e^{Z+4W}\right)\bar{\tilde{\lambda}}^{(\pm)}\tilde{\lambda}^{(\pm)} \mp \left(e^{-4U}\Sigma^{-1} + \frac{3}{2}\Sigma^2 \mp e^{Z+4W}\right)\bar{\zeta}_a^{(\pm)}\gamma^{ac}\zeta_c^{(\pm)} \\ &\mp\frac{1}{9}\left(e^{-4U}\Sigma^{-1} - \frac{15}{2}\Sigma^2 \pm 5e^{Z+4W}\right)\bar{\eta}^{(\pm)}\eta^{(\pm)} \pm \frac{3}{2}\left(e^{-4U}\Sigma^{-1} - \frac{1}{2}\Sigma^2 \mp e^{Z+4W}\right)\bar{\xi}^{(\pm)}\xi^{(\pm)} \\ &\pm\frac{1}{3}i\left(e^{-4U}\Sigma^{-1} - 3\Sigma^2 \pm 2e^{Z+4W}\right)\left(\bar{\zeta}_a^{(\pm)}\gamma^a\eta^{(\pm)} + \bar{\eta}^{(\pm)}\gamma^a\zeta_a^{(\pm)}\right) \\ &\mp\frac{2}{3}\left(e^{-4U}\Sigma^{-1} \pm 2e^{Z+4W}\right)\left(\bar{\eta}^{(\pm)}\xi^{(\pm)} + \bar{\xi}^{(\pm)}\eta^{(\pm)}\right) \\ &\mp i\left(e^{-4U}\Sigma^{-1} \mp e^{Z+4W}\right)\left(\bar{\zeta}_a^{(\pm)}\gamma^a\xi^{(\pm)} + \bar{\xi}^{(\pm)}\gamma^a\zeta_a^{(\pm)}\right) \\ &\pm\mathcal{N}_0^{(\pm)}\left[\frac{1}{2}\bar{\tilde{\lambda}}^{(\pm)}\gamma^a\zeta_a^{(\mp)} + \frac{2}{3}i\bar{\tilde{\lambda}}^{(\pm)}\eta^{(\mp)} + \frac{1}{2}i\bar{\tilde{\lambda}}^{(\pm)}\xi^{(\mp)}\right] \\ &\pm\tilde{\mathcal{N}}_0^{(\pm)}\left[\frac{1}{2}\bar{\zeta}_a^{(\pm)}\gamma^a\tilde{\lambda}^{(\mp)} + \frac{2}{3}i\bar{\eta}^{(\pm)}\tilde{\lambda}^{(\mp)} + \frac{1}{2}i\bar{\xi}^{(\pm)}\tilde{\lambda}^{(\mp)}\right]\end{aligned} \tag{V.55}$$

$$\mathcal{L}_1^{(\pm)} = +\frac{1}{8}i\bar{\tilde{\lambda}}^{(\pm)}\left[3e^{\phi}(\partial\!\!\!/ a)+2e^{-4U}K\!\!\!/_1\right]\tilde{\lambda}^{(\pm)}+\frac{1}{4}ie^{-4U}\bar{\zeta}_a^{(\pm)}\left(e^{\phi}\gamma^{abc}(\partial_b a)+2\gamma^{[c}K\!\!\!/_1\gamma^{a]}\right)\zeta_c^{(\pm)}$$
$$+\frac{1}{8}i\bar{\xi}^{(\pm)}\left[e^{\phi}(\partial\!\!\!/ a)+6e^{-4U}K\!\!\!/_1\right]\xi^{(\pm)}+\frac{1}{12}i\bar{\eta}^{(\pm)}\left[e^{\phi}(\partial\!\!\!/ a)-2e^{-4U}K\!\!\!/_1\right]\eta^{(\pm)}$$
$$+\bar{\zeta}_a^{(\pm)}\left(i(\partial\!\!\!/ U)-\frac{1}{2}e^{-4U}K\!\!\!/_1\right)\gamma^a\xi^{(\pm)}+\bar{\xi}^{(\pm)}\gamma^a\left(-i(\partial\!\!\!/ U)-\frac{1}{2}e^{-4U}K\!\!\!/_1\right)\zeta_a^{(\pm)}$$
$$-\frac{1}{2}i\bar{\zeta}_a^{(\pm)}(\Sigma^{-1}\partial\!\!\!/\Sigma)\gamma^a\eta^{(\pm)}+\frac{1}{2}i\bar{\eta}^{(\pm)}\gamma^a(\Sigma^{-1}\partial\!\!\!/\Sigma)\zeta_a^{(\pm)}$$
$$\pm\frac{1}{2}i\bar{\tilde{\lambda}}^{(\pm)}\gamma^a N\!\!\!/_1^{(\pm)}\zeta_a^{(\mp)}\pm\frac{1}{2}i\bar{\zeta}_a^{(\pm)}\tilde{N\!\!\!/}_1^{(\pm)}\gamma^a\tilde{\lambda}^{(\mp)}\pm\frac{1}{2}\bar{\tilde{\lambda}}^{(\pm)}N\!\!\!/_1^{(\pm)}\xi^{(\mp)}\pm\frac{1}{2}\bar{\xi}^{(\pm)}\tilde{N\!\!\!/}_1^{(\pm)}\tilde{\lambda}^{(\mp)}$$
$$\pm\frac{1}{4}i\left(\bar{\tilde{\lambda}}^{(\pm)}G\!\!\!/_1\xi^{(\pm)}+\bar{\xi}^{(\pm)}\tilde{G\!\!\!/}_1\tilde{\lambda}^{(\pm)}\right)\mp\frac{1}{4}\left(\bar{\tilde{\lambda}}^{(\pm)}\gamma^a G\!\!\!/_1\zeta_a^{(\pm)}+\bar{\zeta}_a^{(\pm)}\tilde{G\!\!\!/}_1\gamma^a\tilde{\lambda}^{(\pm)}\right) \tag{V.56}$$

and

$$\mathcal{L}_2^{(\pm)} = +\frac{1}{8}\bar{\tilde{\lambda}}^{(\pm)}\gamma^a\left(iG\!\!\!/_3+G\!\!\!/_2\right)\zeta_a^{(\pm)}+\frac{1}{8}\bar{\zeta}_a^{(\pm)}\left(i\tilde{G\!\!\!/}_3+\tilde{G\!\!\!/}_2\right)\gamma^a\tilde{\lambda}^{(\pm)}$$
$$+\frac{1}{12}i\bar{\tilde{\lambda}}^{(\pm)}\left(iG\!\!\!/_3+G\!\!\!/_2\right)\eta^{(\pm)}+\frac{1}{12}i\bar{\eta}^{(\pm)}\left(i\tilde{G\!\!\!/}_3+\tilde{G\!\!\!/}_2\right)\tilde{\lambda}^{(\pm)}$$
$$+\frac{1}{8}i\bar{\tilde{\lambda}}^{(\pm)}\left(iG\!\!\!/_3-G\!\!\!/_2\right)\xi^{(\pm)}+\frac{1}{8}i\bar{\xi}^{(\pm)}\left(i\tilde{G\!\!\!/}_3-\tilde{G\!\!\!/}_2\right)\tilde{\lambda}^{(\pm)}$$
$$-\frac{1}{4}i\bar{\zeta}_a^{(\pm)}\left(\Sigma^{-2}\gamma^{[c}F\!\!\!/_2\gamma^{a]}\mp2\Sigma\gamma^{[c}K\!\!\!/_2\gamma^{a]}\right)\zeta_c^{(\pm)}\pm\Sigma\bar{\zeta}_a^{(\pm)}\gamma^{[c}L\!\!\!/_2^{(\pm)}\gamma^{a]}\zeta_c^{(\mp)}$$
$$+\frac{1}{6}\bar{\zeta}_a^{(\pm)}\left(\Sigma^{-2}F\!\!\!/_2\pm\Sigma K\!\!\!/_2\right)\gamma^a\eta^{(\pm)}\mp\frac{1}{3}i\Sigma\bar{\zeta}_a^{(\pm)}L\!\!\!/_2^{(\pm)}\gamma^a\eta^{(\mp)}$$
$$+\frac{1}{6}\bar{\eta}^{(\pm)}\gamma^c\left(\Sigma^{-2}F\!\!\!/_2\pm\Sigma K\!\!\!/_2\right)\zeta_c^{(\pm)}\mp\frac{1}{3}i\Sigma\bar{\eta}^{(\pm)}\gamma^c L\!\!\!/_2^{(\pm)}\zeta_c^{(\mp)}$$
$$+\frac{1}{8}i\bar{\tilde{\lambda}}^{(\pm)}\left(\Sigma^{-2}F\!\!\!/_2\pm2\Sigma K\!\!\!/_2\right)\tilde{\lambda}^{(\pm)}\pm\frac{1}{2}\Sigma\bar{\tilde{\lambda}}^{(\pm)}L\!\!\!/_2^{(\pm)}\tilde{\lambda}^{(\mp)}$$
$$+\frac{1}{8}i\bar{\xi}^{(\pm)}\left(\Sigma^{-2}F\!\!\!/_2\mp2\Sigma K\!\!\!/_2\right)\xi^{(\pm)}\mp\frac{1}{2}\Sigma\bar{\xi}^{(\pm)}L\!\!\!/_2^{(\pm)}\xi^{(\mp)}$$
$$-\frac{1}{36}i\bar{\eta}^{(\pm)}\left(5\Sigma^{-2}F\!\!\!/_2\pm2\Sigma K\!\!\!/_2\right)\eta^{(\pm)}\mp\frac{1}{9}\Sigma\bar{\eta}^{(\pm)}L\!\!\!/_2^{(\pm)}\eta^{(\mp)} \tag{V.57}$$

Similarly, the interaction Lagrangian for the coupling to the charge conjugate fields reads

$$
\begin{aligned}
\mathcal{L}^{(\pm)}_{\bar\psi\psi^{\mathbf{c}}} = & \mp \frac{1}{2}\bar{\tilde\lambda}^{(\pm)}\gamma^a \not{P}\,\zeta_a^{(\mp)\mathbf{c}} \pm \frac{1}{2}\bar\zeta_a^{(\pm)}\not{P}\gamma^a\tilde\lambda^{(\mp)\mathbf{c}} \\
& \pm \frac{1}{4}\bar\zeta_a^{(\pm)}\gamma^{[a}\left(-i\not{G}_3 + \not{G}_2 \pm 2\not{G}_1\right)\gamma^{d]}\zeta_d^{(\mp)\mathbf{c}} + \bar\zeta_a^{(\pm)}\left[i\mathcal{N}_{1b}^{(\pm)}\gamma^{abd} - \mathcal{N}_0^{(\pm)}\gamma^{ad}\right]\zeta_d^{(\pm)\mathbf{c}} \\
& \mp \frac{1}{12}i\bar\zeta_a^{(\pm)}\left(i\not{G}_3 - \not{G}_2\right)\gamma^a\eta^{(\mp)\mathbf{c}} - \frac{2}{3}i\mathcal{N}_0^{(\pm)}\bar\zeta_a^{(\pm)}\gamma^a\eta^{(\pm)\mathbf{c}} \\
& \mp \frac{1}{12}i\bar\eta^{(\pm)}\gamma^d\left(i\not{G}_3 - \not{G}_2\right)\zeta_d^{(\mp)\mathbf{c}} - \frac{2}{3}i\mathcal{N}_0^{(\pm)}\bar\eta^{(\pm)}\gamma^d\zeta_d^{(\pm)\mathbf{c}} \\
& \mp \frac{1}{8}i\bar\zeta_a^{(\pm)}\left(i\not{G}_3 + \not{G}_2 \pm 2\not{G}_1\right)\gamma^a\xi^{(\mp)\mathbf{c}} + \frac{1}{2}\bar\zeta_a^{(\pm)}\left(\not{\mathcal{N}}_1^{(\pm)} - i\mathcal{N}_0^{(\pm)}\right)\gamma^a\xi^{(\pm)\mathbf{c}} \\
& \mp \frac{1}{8}i\bar\xi^{(\pm)}\gamma^d\left(i\not{G}_3 + \not{G}_2 \pm 2\not{G}_1\right)\zeta_d^{(\mp)\mathbf{c}} + \frac{1}{2}\bar\xi^{(\pm)}\gamma^d\left(\not{\mathcal{N}}_1^{(\pm)} - i\mathcal{N}_0^{(\pm)}\right)\zeta_d^{(\pm)\mathbf{c}} \\
& \pm \frac{1}{12}\bar\xi^{(\pm)}\left(i\not{G}_3 + \not{G}_2\right)\eta^{(\mp)\mathbf{c}} + \frac{2}{3}\mathcal{N}_0^{(\pm)}\bar\xi^{(\pm)}\eta^{(\pm)\mathbf{c}} \pm \frac{3}{16}\bar\xi^{(\pm)}\not{G}_2\xi^{(\mp)\mathbf{c}} \\
& \mp \frac{1}{36}\bar\eta^{(\pm)}\left(i\not{G}_3 - \not{G}_2 \mp 6\not{G}_1\right)\eta^{(\mp)\mathbf{c}} + \frac{1}{9}i\bar\eta^{(\pm)}\left(3\not{\mathcal{N}}_1^{(\pm)} - 5i\mathcal{N}_0^{(\pm)}\right)\eta^{(\pm)\mathbf{c}} \\
& \pm \frac{1}{12}\bar\eta^{(\pm)}\left(i\not{G}_3 + \not{G}_2\right)\xi^{(\mp)\mathbf{c}} + \frac{2}{3}\mathcal{N}_0^{(\pm)}\bar\eta^{(\pm)}\xi^{(\pm)\mathbf{c}}
\end{aligned} \tag{V.58}
$$

where, in a slight abuse of notation, $\not{P}$ now denotes the 5-d quantity $\not{P} = (1/2)\gamma^b\left(\partial_b\phi + ie^\phi\partial_b a\right)$.

It is worth noticing that this action can be also obtained by direct dimensional reduction of the following $D = 10$ action:

$$
\begin{aligned}
S_{9+1} = K_{10}\int d^{10}x\sqrt{-g_{10}}\Big[& \frac{1}{2}\bar\lambda\left(\hat{\not{\nabla}} - \frac{3i}{2}\not{Q} - \frac{i}{8}\not{F}_{(5)}\right)\lambda + \frac{1}{8}\left(\bar\Psi_A\not{G}^*\Gamma^A\lambda - \bar\lambda\Gamma^A\not{G}\Psi_A\right) \\
& - \frac{1}{4}\left(\bar\lambda\Gamma^A\not{P}\Psi_A^{\mathbf{c}} + \bar\Psi_A\not{P}\Gamma^A\lambda^{\mathbf{c}} + \frac{1}{8}\bar\Psi_A\Gamma^{ABC}S_B\Psi_C^{\mathbf{c}} + \text{ c.c.}\right) \\
& + \bar\Psi_A\Gamma^{ABC}\left(\hat\nabla_B - \frac{i}{2}Q_B + \frac{i}{16}\not{F}_{(5)}\Gamma_B\right)\Psi_C\Big],
\end{aligned} \tag{V.59}
$$

from which the 10-d fermionic equations of motion can be derived. As usual in the context of AdS/CFT, the bulk action would have to be supplemented by appropriate boundary terms in order to compute correlation functions of the dual field theory operators holographically.

5.4 $N = 4$ supersymmetry

It is expected that the Lagrangian we have derived has $N = 4$ $d = 5$ supersymmetry, and we will provide evidence that that is the case. We expect to find the gravity multiplet (containing the graviton, the scalar Σ and vectors) and a pair of vector multiplets (containing the rest of the scalars and vectors). Let us consider the supersymmetry variations of the 10-d theory. These are

$$\delta\lambda = \not{P}\varepsilon^{\mathbf{c}} + \frac{1}{4}\not{G}\varepsilon \tag{V.60}$$

$$\delta\Psi_A = \hat{\nabla}_A\varepsilon - \frac{1}{2}iQ_A\varepsilon + \frac{i}{16}\not{F}_{(5)}\Gamma_A\varepsilon - \frac{1}{16}S_A\varepsilon^{\mathbf{c}} \tag{V.61}$$

where

$$S_A = \frac{1}{6}\left(\Gamma_A{}^{DEF}G_{DEF} - 9\Gamma^{DE}G_{ADE}\right) = \Gamma_A\not{G} - 2G_{ADE}\Gamma^{DE} \tag{V.62}$$

as before. Given the consistent truncation (assuming throughout that the SE_5 is not S^5), the variational parameters must also be $SU(2)$ singlets:

$$\varepsilon = e^{W/2}\theta^{(+)}(x)\otimes\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_- + e^{W/2}\theta^{(-)}(x)\otimes\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_- \tag{V.63}$$

$$\varepsilon^{\mathbf{c}} = e^{W/2}\theta^{(-)\mathbf{c}}(x)\otimes\varepsilon_+(y)e^{\frac{3}{2}i\chi}\otimes u_- - e^{W/2}\theta^{(+)\mathbf{c}}(x)\otimes\varepsilon_-(y)e^{-\frac{3}{2}i\chi}\otimes u_- \,. \tag{V.64}$$

The evaluation of the variations proceeds much as the calculations leading to the equations of motion, and we find

$$\delta\tilde{\lambda}^{(\pm)} = \pm\not{P}\theta^{(\mp)\mathbf{c}} - \frac{1}{4}\Big(i\not{G}_3 + \not{G}_2 \mp 2\not{G}_1\Big)\theta^{(\pm)} \mp i\left(\not{N}_1^{(\pm)} - i\mathcal{N}_0^{(\pm)}\right)\theta^{(\mp)} \tag{V.65}$$

$$\begin{aligned}\delta\xi^{(\pm)} &= \Big[2i(\not{\partial}U) + e^{-4U}\not{K}_1 - 2ie^{Z+4W} \pm 2ie^{-4U}\Sigma^{-1}\Big]\theta^{(\pm)} \\ &\quad \mp\frac{1}{4}\Big(\not{G}_3 - i\not{G}_2 \mp 2i\not{G}_1\Big)\theta^{(\mp)\mathbf{c}} - \left(\not{N}_1^{(\pm)} - i\mathcal{N}_0^{(\pm)}\right)\theta^{(\pm)\mathbf{c}}\end{aligned} \tag{V.66}$$

$$\begin{aligned}\delta\eta^{(\pm)} &= \left[-\frac{3}{2}i(\Sigma^{-1}\not{\partial}\Sigma) - \frac{1}{2}\Sigma^{-2}\not{F}_2 \mp \frac{1}{2}\Sigma\not{K}_2 \mp ie^{-4U}\Sigma^{-1} \pm 3i\Sigma^2 - 2ie^{Z+4W}\right]\theta^{(\pm)} \\ &\quad \pm i\Sigma\not{L}_2^{(\pm)}\theta^{(\mp)} \mp \frac{1}{4}\Big(\not{G}_3 + i\not{G}_2\Big)\theta^{(\mp)\mathbf{c}} + 2i\mathcal{N}_0^{(\pm)}\theta^{(\pm)\mathbf{c}}\end{aligned} \tag{V.67}$$

$$\begin{aligned}\delta\zeta_a^{(\pm)} &= \left[\nabla_a \mp \frac{3}{2}iA_a + \frac{1}{4}ie^{\phi}\partial_a a - \frac{1}{2}ie^{-4U}K_{1a}\right]\theta^{(\pm)} + \gamma_a\left(\pm\frac{1}{3}e^{-4U}\Sigma^{-1} \pm \frac{1}{2}\Sigma^2 - \frac{1}{3}e^{Z+4W}\right)\theta^{(\pm)} \\ &\quad + \frac{1}{8}i\Sigma^{-2}\left(\not{F}_2\gamma_a - \frac{1}{3}\gamma_a\not{F}_2\right)\theta^{(\pm)} \mp \frac{1}{4}i\Sigma\left(\not{K}_2\gamma_a - \frac{1}{3}\gamma_a\not{K}_2\right)\theta^{(\pm)} \\ &\quad \mp\frac{1}{8}\left[i\left(\not{G}_3\gamma_a - \frac{1}{3}\gamma_a\not{G}_3\right) - \left(\not{G}_2\gamma_a - \frac{1}{3}\gamma_a\not{G}_2\right) \mp 4\mathcal{G}_{1a}\right]\theta^{(\mp)\mathbf{c}} \\ &\quad \mp\frac{1}{2}\Sigma\left(\not{L}_2^{(\pm)}\gamma_a - \frac{1}{3}\gamma_a\not{L}_2^{(\pm)}\right)\theta^{(\mp)} + \left(i\mathcal{N}_{1a}^{(\pm)} + \frac{1}{3}\mathcal{N}_0^{(\pm)}\gamma_a\right)\theta^{(\pm)\mathbf{c}}\,.\end{aligned} \tag{V.68}$$

Consulting for example [32, 147], one sees immediately that it is $\delta\eta^{(\pm)}$ that contains $\Sigma^{-1}\partial\!\!\!/\Sigma$, and thus we deduce that it is $\eta^{(\pm)}$ that sits in the $N=4$ gravity multiplet. These could be assembled into four symplectic-Majorana spinors, forming the $\mathbf{4}$ of $USp(4) \sim SO(5)$. The remaining fermions $\xi^{(\pm)}, \tilde{\lambda}^{(\pm)}$ can then be arranged into an $SO(2)$ doublet of $USp(4)$ quartets, appropriate to the pair of vector multiplets.

5.5 Linearized analysis

5.5.1 The supersymmetric vacuum solution

It has been shown that the $N=4$ possesses a supersymmetric vacuum with $N=2$ supersymmetry. To see the details of the Stückelberg mechanism at work, we linearize the fermions around the vacuum, in which all of the fluxes are zero and the scalars take the values $U=V=X=Y=Z=0$. Around this vacuum, the supersymmetry variations reduce to

$$\delta\eta^{(+)} = \delta\xi^{(+)} = \delta\tilde{\lambda}^{(+)} = 0 \tag{V.69}$$

$$\delta\zeta_a^{(+)} = D_a\theta^{(+)} + \frac{1}{2}\gamma_a\theta^{(+)} \tag{V.70}$$

$$\delta\eta^{(-)} = \delta\xi^{(-)} = -4i\theta^{(-)} \tag{V.71}$$

$$\delta\lambda^{(-)} = 0 \tag{V.72}$$

$$\delta\zeta_a^{(-)} = D_a\theta^{(-)} - \frac{7}{6}\gamma_a\theta^{(-)}\,. \tag{V.73}$$

These correspond to unbroken $N=2$ supersymmetry parametrized by $\theta^{(+)}$, while the supersymmetry given by $\theta^{(-)}$ is broken. In our somewhat unusual normalizations of the fermions, as given in (V.51), we can deduce that the Goldstino is proportional to $g = \frac{1}{10}\left(\eta^{(-)} + \frac{3}{2}\xi^{(-)}\right)$ (orthogonal to the invariant mode $\frac{1}{10}\left(\eta^{(-)}-\xi^{(-)}\right)$). The kinetic terms in this vacuum then take the form

$$S_{svac}=\frac{1}{2}\left(\bar{\tilde{\lambda}}^{(+)}\not{D}\,\tilde{\lambda}^{(+)}-\frac{7}{2}\bar{\tilde{\lambda}}^{(+)}\tilde{\lambda}^{(+)}\right)+\frac{1}{2}\left(\bar{\tilde{\lambda}}^{(-)}\not{D}\,\tilde{\lambda}^{(-)}+\frac{3}{2}\bar{\tilde{\lambda}}^{(-)}\tilde{\lambda}^{(-)}\right)$$
$$+\frac{2}{15}\left(\bar{\kappa}_1^{(+)}\not{D}\,\kappa_1^{(+)}-\frac{11}{2}\bar{\kappa}_1^{(+)}\kappa_1^{(+)}\right)+\frac{1}{5}\left(\bar{\kappa}_2^{(+)}\not{D}\,\kappa_2^{(+)}+\frac{9}{2}\bar{\kappa}_2^{(+)}\kappa_2^{(+)}\right)+20\left(\bar{h}\not{D}\,h-\frac{5}{2}\bar{h}h\right)$$
$$+\bar{\zeta}_a^{(-)}\gamma^{abc}D_b\zeta_c^{(-)}+\frac{7}{2}\bar{\zeta}_a^{(-)}\gamma^{ac}\zeta_c^{(-)}+\left(\frac{40}{3}i\bar{\zeta}_a^{(-)}\gamma^a g+\text{c.c.}\right)-\frac{700}{9}\bar{g}g+\frac{40}{3}\bar{g}\not{D}\,g$$

(V.74)

$$+\bar{\zeta}_a^{(+)}\gamma^{abc}D_b\zeta_c^{(+)}-\frac{3}{2}\bar{\zeta}_a^{(+)}\gamma^{ac}\zeta_c^{(+)}\,,$$

where $\kappa_{1,2}^{(+)}$ are linear combinations of $\eta^{(+)},\xi^{(+)}$. Since the geometry is AdS_5, the fourth line represents a "massless" gravitino, while, defining the invariant combination $\Psi_a=\zeta_a^{(-)}+\frac{7}{6}i\gamma_a g-iD_a g$, the third line becomes

$$\bar{\Psi}_a\gamma^{abc}D_b\Psi_c+\frac{7}{2}\bar{\Psi}_a\gamma^{ab}\Psi_b\,, \tag{V.75}$$

the action of a massive gravitino. This is the Proca/Stückelberg mechanism. We see then that we have fermion modes of mass $\{\frac{11}{2},\frac{7}{2},\frac{5}{2},\frac{3}{2},-\frac{3}{2},-\frac{7}{2},-\frac{9}{2}\}$ which correspond to the fermionic modes of unitary irreps of $SU(2,2|1)$ and which also coincide with the lowest rungs of the KK towers of the sphere compactification [100]. The corresponding features in the bosonic spectrum were noted in [27, 67]. Specifically, in the language of Ref. [80], the $p=2$ sector contains $\zeta_a^{(+)},\tilde{\lambda}^{(-)}$, $p=3$ contains $\zeta_a^{(-)},\tilde{\lambda}^{(+)},\eta^{(-)},\xi^{(-)}$ and $p=4$ contains $\eta^{(+)},\xi^{(+)}$.

5.5.2 The Romans AdS_5 vacuum

The non-supersymmetric AdS vacuum [80, 142] of the theory has radius $\sqrt{8/9}$, and vevs

$$e^{4U}=e^{-4V}=\frac{2}{3}\,,\qquad Y=\frac{e^{i\theta}}{\sqrt{12}}e^{\phi/2}\qquad X=(a+ie^{-\phi})Y\,, \tag{V.76}$$

where θ is an arbitrary constant phase. The axion a and dilaton ϕ are arbitrary [27, 67]. For the various quantities appearing in the effective action we have

$$\mathcal{G}_i=\tilde{\mathcal{G}}_i=\mathcal{N}_1^{(\pm)}=\tilde{\mathcal{N}}_1^{(\pm)}=\mathcal{N}_0^{(+)}=\tilde{\mathcal{N}}_0^{(-)}=K_1=K_2=L_2=0\,, \tag{V.77}$$

where $i = 1, 2, 3$, and

$$e^{-4W} = \frac{2}{3}\,, \quad \Sigma = 1\,, \quad e^{Z} = \frac{1}{2}\,, \quad P = 0\,, \quad \left(\mathcal{N}_0^{(-)}\right)^* = \tilde{\mathcal{N}}_0^{(+)} = -\frac{3}{\sqrt{2}}e^{i\theta}\,. \tag{V.78}$$

We then find

$$\begin{aligned} \mathcal{L}^{(+)}_{mass} = & -\frac{15}{8}\bar{\tilde{\lambda}}^{(+)}\tilde{\lambda}^{(+)} - \frac{9}{4}\bar{\zeta}^{(+)}_a\gamma^{ac}\zeta^{(+)}_c + \frac{1}{4}\bar{\eta}^{(+)}\eta^{(+)} + \frac{3}{8}\bar{\xi}^{(+)}\xi^{(+)} \\ & - 2\left(\bar{\eta}^{(+)}\xi^{(+)} + \bar{\xi}^{(+)}\eta^{(+)}\right) - \frac{3}{4}i\left(\bar{\zeta}^{(+)}_a\gamma^a\xi^{(+)} + \bar{\xi}^{(+)}\gamma^a\zeta^{(+)}_a\right) \\ & - \frac{3}{\sqrt{2}}e^{i\theta}\left(\frac{1}{2}\bar{\zeta}^{(+)}_a\gamma^a\tilde{\lambda}^{(-)} + \frac{2}{3}i\bar{\eta}^{(+)}\tilde{\lambda}^{(-)} + \frac{1}{2}i\bar{\xi}^{(+)}\tilde{\lambda}^{(-)}\right) \end{aligned} \tag{V.79}$$

$$\begin{aligned} \mathcal{L}^{(-)}_{mass} = & \frac{9}{8}\bar{\tilde{\lambda}}^{(-)}\tilde{\lambda}^{(-)} + \frac{15}{4}\bar{\zeta}^{(-)}_a\gamma^{ac}\zeta^{(-)}_c - \frac{13}{12}\bar{\eta}^{(-)}\eta^{(-)} - \frac{21}{8}\bar{\xi}^{(-)}\xi^{(-)} \\ & + i\left(\bar{\zeta}^{(-)}_a\gamma^a\eta^{(-)} + \bar{\eta}^{(-)}\gamma^a\zeta^{(-)}_a\right) + \frac{9}{4}i\left(\bar{\zeta}^{(-)}_a\gamma^a\xi^{(-)} + \bar{\xi}^{(-)}\gamma^a\zeta^{(-)}_a\right) \\ & + \frac{3}{\sqrt{2}}e^{-i\theta}\left(\frac{1}{2}\bar{\tilde{\lambda}}^{(-)}\gamma^a\zeta^{(+)}_a + \frac{2}{3}i\bar{\tilde{\lambda}}^{(-)}\eta^{(+)} + \frac{1}{2}i\bar{\tilde{\lambda}}^{(-)}\xi^{(+)}\right) \end{aligned} \tag{V.80}$$

$$\begin{aligned} \mathcal{L}^{(-)}_{\bar{\psi}\psi^{\mathbf{c}}} = \frac{3}{\sqrt{2}}e^{-i\theta}\Big(& \bar{\zeta}^{(-)}_a\gamma^{ad}\zeta^{(-)\mathbf{c}}_d - \frac{5}{9}\bar{\eta}^{(-)}\eta^{(-)\mathbf{c}} + \frac{2}{3}i\bar{\zeta}^{(-)}_a\gamma^a\eta^{(-)\mathbf{c}} + \frac{2}{3}i\bar{\eta}^{(-)}\gamma^d\zeta^{(-)\mathbf{c}}_d \\ & + \frac{i}{2}\bar{\zeta}^{(-)}_a\gamma^a\xi^{(-)\mathbf{c}} + \frac{i}{2}\bar{\xi}^{(-)}\gamma^d\zeta^{(-)\mathbf{c}}_d - \frac{2}{3}\bar{\xi}^{(-)}\eta^{(-)\mathbf{c}} - \frac{2}{3}\bar{\eta}^{(-)}\xi^{(-)\mathbf{c}}\Big) \end{aligned} \tag{V.81}$$

and

$$\mathcal{L}^{(\pm)}_1 = \mathcal{L}^{(\pm)}_2 = \mathcal{L}^{(+)}_{\bar{\psi}\psi^{\mathbf{c}}} = 0\,. \tag{V.82}$$

We see by inspection that indeed both gravitinos are massive. For example, $\zeta^{(+)}_a$ eats the goldstino proportional to $g^{(+)} = \frac{3}{2}i\xi^{(+)} - \mathcal{N}_0^{(-)*}\tilde{\lambda}^{(-)}$, while the Goldstino eaten by $\zeta^{(-)}_a$ is a linear combination of $\xi^{(-)}, \eta^{(-)}$ and their conjugates.

5.6 Examples

As an application of our general result (V.53), in this section we discuss the coupling of the fermions to some further bosonic truncations of interest, including the minimal gauged $N = 2$ supergravity theory in $d = 5$, and the holographic AdS_5 superconductor of [77].

5.6.1 Minimal $N = 2$ gauged supergravity in five dimensions

Perhaps the simplest further truncation one could consider that retains fermion modes entails taking $U = V = Z = K_1 = L_2 = G_i = H_i = M_q = N_q = 0$ ($i = 1, 2, 3$ and $q = 0, 1$) and $K_2 = -F_2$. It is then consistent to set $\tilde{\lambda}^{(\pm)} = \eta^{(\pm)} = \xi^{(\pm)} = 0$ together with $\zeta_a^{(-)} = 0$. This gives the right fermion content of minimal $N = 2$ gauged supergravity in $d = 5$, which is one Dirac gravitino ($\zeta_a^{(+)}$ in our notation), with an action given by

$$S_{4+1} = K_5 \int d^5x \sqrt{-g_5^E} \left[\bar{\zeta}_a^{(+)} \gamma^{abc} D_b \zeta_c^{(+)} + \mathcal{L}^{(+)}_{\bar{\psi}\psi} \right] \tag{V.83}$$

where

$$\mathcal{L}^{(+)}_{\bar{\psi}\psi} = -\frac{3}{2} \bar{\zeta}_a^{(+)} \gamma^{ac} \zeta_c^{(+)} - \frac{3}{4} i \bar{\zeta}_a^{(+)} \gamma^{[c} \not{F}_2 \gamma^{a]} \zeta_c^{(+)} \,, \tag{V.84}$$

and $D_a = \nabla_a - (3i/2) A_{1a}$ as before.

5.6.2 No $p = 3$ sector

A possible further truncation of the bosonic sector considered in [67] entails taking $G_i = H_i = L_2 = 0$ ($i = 1, 2, 3$). In the notation of [80], this corresponds to eliminating the bosonic fields belonging to the $p = 3$ sector. By studying the equations of motion provided in appendix J we find that the fermion modes split into two decoupled sectors, as depicted in figure 5.1. It is therefore consistent to set the modes in either of these sectors to zero.

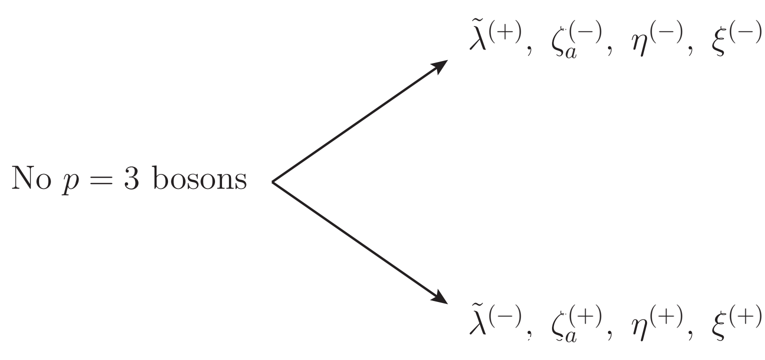

Figure 5.1: Decoupling of the fermion modes in the futher truncation obtained by eliminating the bosons in the "$p = 3$ sector".

We note the first set of fermion fields are all in the $p = 3$ sector, while the second set are in $p = 2, 4$. It seems reasonable therefore to suggest that the latter truncation corresponds to an $N = 2$ gauged supergravity theory coupled to a vector multiplet and two hypermultiplets (this was suggested in [27, 67] in the context of the bosonic sector.) The former truncation would apparently be non-supersymmetric.

5.6.3 Type IIB holographic superconductor

As discussed in [27, 67], the type IIB holographic superconductor of [77] can be obtained by truncating out the bosons of the $p = 3$ sector as discussed above, and further setting $a = \phi = h = 0$ and $X = iY$, $K_2 = -F_2$, $e^{4U} = e^{-4V} = 1 - 4|Y|^2$, which implies $\tilde{E}_1 = 0$ and

(V.85) $$e^Z = 1 - 6|Y|^2\,, \qquad K_1 = 2i\,(Y^* DY - Y DY^*) \equiv 2iY^* \overleftrightarrow{D} Y\,.$$

In terms of the variables we have defined, this truncation implies

(V.86) $$\mathcal{G}_i = \tilde{\mathcal{G}}_i = \mathcal{N}_q^{(+)} = \tilde{\mathcal{N}}_q^{(-)} = 0$$

($i = 1, 2, 3$ and $q = 0, 1$) together with

(V.87) $$\begin{aligned} \mathcal{N}_1^{(-)} &= -2ie^{-2U} DY^*\,, & \qquad \mathcal{N}_0^{(-)} &= -6e^{-2U} Y^*\,, \\ \tilde{\mathcal{N}}_1^{(+)} &= 2ie^{-2U} DY\,, & \qquad \tilde{\mathcal{N}}_0^{(+)} &= -6e^{-2U} Y\,, \end{aligned}$$

and

(V.88) $$P = 0\,, \qquad \Sigma = 1\,, \qquad e^{-4W} = 1 - 4|Y|^2\,.$$

By analyzing the equations of motion given in appendix J, we find that in this case there is a further decoupling of the fermion modes with respect to the no $p = 3$ sector truncation discussed above. As depicted in figure 5.2, the $\tilde{\lambda}^{(+)}$ mode now decouples from $\zeta_a^{(-)}$, $\eta^{(-)}$, $\xi^{(-)}$ as well, resulting in three fermion sectors, which can then be set to zero independently.

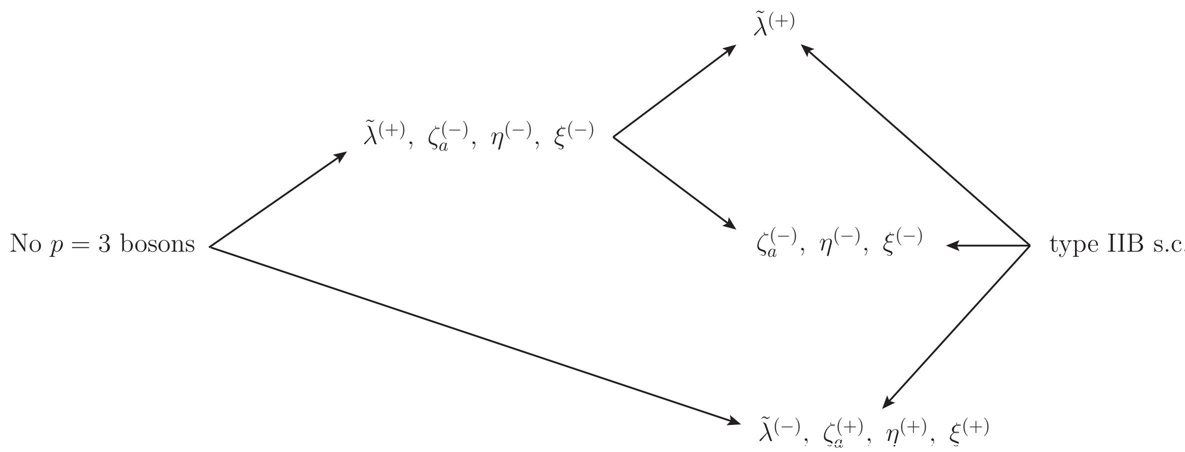

Figure 5.2: Further decoupling of fermion modes in the type IIB holographic superconductor truncation.

A single spin-1/2 fermion

The simplest scenario corresponds of course to keeping the $\tilde{\lambda}^{(+)}$ mode only, for which the effective action (V.53) reduces to

(V.89)
$$S_{4+1} = K_5 \int d^5x \sqrt{-g_5^E} \left[\frac{1}{2} \bar{\tilde{\lambda}}^{(+)} \not{D} \tilde{\lambda}^{(+)} + \mathcal{L}^{(+)}_{\bar{\psi}\psi} \right]$$

with

(V.90)
$$\mathcal{L}^{(+)}_{\bar{\psi}\psi} = -\frac{1}{2} \bar{\tilde{\lambda}}^{(+)} \left(\frac{3}{2} + \frac{1}{4} i \not{F}_2 + \frac{2 - 6|Y|^2 + Y^* \overleftrightarrow{\not{D}}\, Y}{1 - 4|Y|^2} \right) \tilde{\lambda}^{(+)} \,,$$

where we recall that $DY = dY - 3iA_1 Y$, and $\not{D}\tilde{\lambda}^{(+)} = \left(\not{\nabla} - \frac{3i}{2}\not{A}_1\right)\tilde{\lambda}^{(+)}$. As pointed out in [67], we can make contact with the notation of [77] by setting $A_1 = (2/3)A$ and $Y = (1/2)e^{i\theta}\tanh(\eta/2)$. Notice that $\tilde{\lambda}^{(+)}$ only couples derivatively to the phase of the charged scalar Y. The model (V.89) is particularly well suited for an exploration of fermion correlators via holography, inasmuch as the presence of a single spin-1/2 field makes the application of all the standard gauge/gravity duality techniques possible. Naturally, such a program becomes more involved in the presence of mixing between the gravitino and the spin-1/2 fields.

Retaining half of the fermionic degrees of freedom

For the $\tilde{\lambda}^{(-)}, \zeta_a^{(+)}, \xi^{(+)}, \eta^{(+)}$ sector we find that (V.53) reads

$$S_{4+1} = K_5 \int d^5x \sqrt{-g_5^E} \Big[\frac{1}{2} \bar{\tilde{\lambda}}^{(-)} \not{D} \tilde{\lambda}^{(-)} + \bar{\zeta}_a^{(+)} \gamma^{abc} D_b \zeta_c^{(+)} + \frac{1}{3} \bar{\eta}^{(+)} \not{D} \eta^{(+)} + \frac{1}{2} \bar{\xi}^{(+)} \not{D} \xi^{(+)} + \mathcal{L}_{\bar{\psi}\psi} \Big] \tag{V.91}$$

with

$$\begin{aligned}
\mathcal{L}_{\bar{\psi}\psi} = {}& \frac{3}{8} i \bar{\tilde{\lambda}}^{(-)} \not{F}_2 \tilde{\lambda}^{(-)} + 3 \left(e^{-4U} |Y|^2 + \frac{1}{4} \right) \bar{\tilde{\lambda}}^{(-)} \tilde{\lambda}^{(-)} - \frac{1}{2} e^{-4U} \bar{\tilde{\lambda}}^{(-)} \Big(Y^* \overleftrightarrow{\not{D}} Y \Big) \tilde{\lambda}^{(-)} \\
& - \frac{3}{4} i \bar{\zeta}_a^{(+)} \gamma^{[c} \not{F}_2 \gamma^{a]} \zeta_c^{(+)} - 3 \left(2e^{-4U} |Y|^2 + \frac{1}{2} \right) \bar{\zeta}_a^{(+)} \gamma^{ac} \zeta_c^{(+)} - e^{-4U} \bar{\zeta}_a^{(+)} \gamma^{[c} \Big(Y^* \overleftrightarrow{\not{D}} Y \Big) \gamma^{a]} \zeta_c^{(+)} \\
& - \frac{i}{12} \bar{\eta}^{(+)} \not{F}_2 \eta^{(+)} + \frac{1}{6} e^{-4U} \bar{\eta}^{(+)} \left(1 + 2Y^* \overleftrightarrow{\not{D}} Y \right) \eta^{(+)} \\
& + \frac{3}{8} i \bar{\xi}^{(+)} \not{F}_2 \xi^{(+)} + \frac{3}{4} e^{-4U} \bar{\xi}^{(+)} \left(3 - 2Y^* \overleftrightarrow{\not{D}} Y \right) \xi^{(+)} - 3 \bar{\xi}^{(+)} \xi^{(+)} \\
& - e^{-2U} \bar{\tilde{\lambda}}^{(-)} \gamma^a \Big(\not{D} Y^* - 3Y^* \Big) \zeta_a^{(+)} - e^{-2U} \bar{\zeta}_a^{(+)} \Big(\not{D} Y + 3e^{-2U} Y \Big) \gamma^a \tilde{\lambda}^{(-)} \\
& + 2ie^{-4U} \bar{\xi}^{(+)} \gamma^a \Big(Y \not{D} Y^* - 3|Y|^2 \Big) \zeta_a^{(+)} - 2ie^{-4U} \bar{\zeta}_a^{(+)} \Big(Y^* \not{D} Y + 3|Y|^2 \Big) \gamma^a \xi^{(+)} \\
& + ie^{-2U} \bar{\tilde{\lambda}}^{(-)} \Big(\not{D} Y^* + 3Y^* \Big) \xi^{(+)} + ie^{-2U} \bar{\xi}^{(+)} \Big(\not{D} Y - 3Y \Big) \tilde{\lambda}^{(-)} \\
& - 4ie^{-2U} \left(Y \bar{\eta}^{(+)} \tilde{\lambda}^{(-)} - Y^* \bar{\tilde{\lambda}}^{(-)} \eta^{(+)} \right) - 2 \left(\bar{\xi}^{(+)} \eta^{(+)} + \bar{\eta}^{(+)} \xi^{(+)} \right),
\end{aligned} \tag{V.92}$$

where we recall that $e^{4U} = 1 - 4|Y|^2$. We note the presence of a variety of couplings between the fermions and the charged scalar, as well as Pauli couplings.

The $\zeta_a^{(-)}$, $\eta^{(-)}$, $\xi^{(-)}$ sector

For the remaining decoupled sector containing the $\zeta_a^{(-)}$, $\eta^{(-)}$, $\xi^{(-)}$ modes we find

$$S_{4+1} = K_5 \int d^5x \sqrt{-g_5^E} \Big[\bar{\zeta}_a^{(-)} \gamma^{abc} D_b \zeta_c^{(-)} + \frac{1}{3} \bar{\eta}^{(-)} \not{D} \eta^{(-)} + \frac{1}{2} \bar{\xi}^{(-)} \not{D} \xi^{(-)} + \mathcal{L}_{\bar{\psi}\psi} + \frac{1}{2} \left(\mathcal{L}^{(-)}_{\bar{\psi}\psi^{\mathrm{c}}} + \text{c.c.} \right) \Big] \tag{V.93}$$

where now

$$
\begin{aligned}
\mathcal{L}_{\bar{\psi}\psi} = e^{-4U}\Bigg[& \left(\frac{7}{2} - 12|Y|^2\right)\bar{\zeta}_a^{(-)}\gamma^{ac}\zeta_c^{(-)} + \frac{1}{9}\left(-\frac{23}{2} + 60|Y|^2\right)\bar{\eta}^{(-)}\eta^{(-)} \\
& - \frac{3}{2}\left(\frac{3}{2} - 4|Y|^2\right)\bar{\xi}^{(-)}\xi^{(-)} + \frac{2}{3}\left(-1 + 12|Y|^2\right)\left(\bar{\eta}^{(-)}\xi^{(-)} + \bar{\xi}^{(-)}\eta^{(-)}\right) \\
& + \frac{4}{3}i\left(1 - 6|Y|^2\right)\left(\bar{\zeta}_a^{(-)}\gamma^a\eta^{(-)} + \bar{\eta}^{(-)}\gamma^a\zeta_a^{(-)}\right) \\
& + 2i\left(1 - 3|Y|^2\right)\left(\bar{\zeta}_a^{(-)}\gamma^a\xi^{(-)} + \bar{\xi}^{(-)}\gamma^a\zeta_a^{(-)}\right) \\
& - \bar{\zeta}_a^{(-)}\gamma^{[c}Y^*\overleftrightarrow{\not{D}}Y\gamma^{a]}\zeta_c^{(-)} - \frac{3}{2}\bar{\xi}^{(-)}Y^*\overleftrightarrow{\not{D}}Y\xi^{(-)} + \frac{1}{3}\bar{\eta}^{(-)}Y^*\overleftrightarrow{\not{D}}Y\eta^{(-)} \\
& - 2i\bar{\zeta}_a^{(-)}Y^*\not{D}Y\gamma^a\xi^{(-)} + 2i\bar{\xi}^{(-)}\gamma^aY\not{D}Y^*\zeta_a^{(-)}\Bigg] \\
& + \frac{1}{4}i\bar{\zeta}_a^{(-)}\gamma^{[c}\not{F}_2\gamma^{a]}\zeta_c^{(-)} - \frac{1}{8}i\bar{\xi}^{(-)}\not{F}_2\xi^{(-)} - \frac{7}{36}i\bar{\eta}^{(-)}\not{F}_2\eta^{(-)} \\
& + \frac{1}{3}\bar{\zeta}_a^{(-)}\not{F}_2\gamma^a\eta^{(-)} + \frac{1}{3}\bar{\eta}^{(-)}\gamma^c\not{F}_2\zeta_c^{(-)}
\end{aligned}
\tag{V.94}
$$

and

$$
\begin{aligned}
\mathcal{L}^{(-)}_{\bar{\psi}\psi^{\mathbf{c}}} = e^{-2U}\Bigg[& 2\bar{\zeta}_a^{(-)}\left(\gamma^{abd}D_bY^* + 3\gamma^{ad}Y^*\right)\zeta_d^{(-)\mathbf{c}} + 4iY^*\left(\bar{\zeta}_a^{(-)}\gamma^a\eta^{(-)\mathbf{c}} + \bar{\eta}^{(-)}\gamma^a\zeta_a^{(-)\mathbf{c}}\right) \\
& - i\bar{\zeta}_a^{(-)}\left(\not{D}Y^* - 3Y^*\right)\gamma^a\xi^{(-)\mathbf{c}} - i\bar{\xi}^{(-)}\gamma^d\left(\not{D}Y^* - 3Y^*\right)\zeta_d^{(-)\mathbf{c}} \\
& - 4Y^*\left(\bar{\xi}^{(-)}\eta^{(-)\mathbf{c}} + \bar{\eta}^{(-)}\xi^{(-)\mathbf{c}}\right) + \frac{2}{3}\bar{\eta}^{(-)}\left(\not{D}Y^* - 5Y^*\right)\eta^{(-)\mathbf{c}}\Bigg].
\end{aligned}
\tag{V.95}
$$

The models (V.89), (V.91) and (V.93) display a variety of couplings between the fermions and the charged scalar, the fermions and their charge conjugates, and Pauli couplings as well. From the gauge/gravity duality point of view, these couplings might be of phenomenological interest and give rise to features that have not been observed so far in the simpler non-interacting fermion models in the literature. The exploration of these directions in the context of AdS/CFT will be pursued elsewhere.

5.7 Discussion and conclusions

Continuing with the program initiated in [10], where we performed the reduction of the fermionic sector in the consistent truncations of $D = 11$ supergravity on squashed Sasaki-Einstein sevenmanifolds [62], in the present paper we have considered the reduction of fermions in the recently found consistent truncations of type IIB supergravity on squashed Sasaki-Einstein five-manifolds [27, 67, 111]. A common denominator of these KK reductions is that they consistently retain charged (massive) scalar and p-form fields. This feature not only establishes them as relevant from a supergravity perspective, but it also makes them particulary suitable for the description of various phenomena, such as superfluidity and superconductivity, by means of holographic techniques.

In particular, as an application of our results we have discussed the coupling of fermions to the $(4+1)$-dimensional type IIB holographic superconductor of [77], which complements our previous result for the coupling of fermions to the $(3+1)$-dimensional M-theory holographic superconductor constructed in [64]. It is interesting to note the differences between these two effective theories. For example, the coupling of the fermions to their charge conjugates (i.e. Majorana-like couplings) was found to play a central role in the $(3 + 1)$-model of [10]. Although such couplings are still present in the general truncation discussed in the present work, they are absent in the further truncation corresponding to the holographic $(4 + 1)$-dimensional superconductor. More importantly, while a simple further truncation of the fermion sector that could result in a more manageable system well suited for holographic applications eluded us in our previous work, in the present scenario we have found a very simple model (c.f. (V.89)) describing a single spin-1/2 Dirac fermion interacting with the charged scalar that has been shown to condense for low enough temperatures of a corresponding black hole solution of the bosonic field equations [77]. It would be interesting to apply our results to the holographic computation of fermion correlators in the presence of these superconducting condensates. Similarly, our results can be used to explore fermion correlators in other situations as well.

CHAPTER VI

Final remarks

The AdS/CFT correspondence is one of the most remarkable results in the theoretical physics and has been a major research engine for over a decade. In this thesis, we explored the possibility of testing the duality beyond the large N and large λ limits by focusing on the holographic description of supersymmetric Wilson loops. In particular, we found the spectra of excitations of D3 and D5-branes dual to these operators and showed how they fit into representations of $OSp(4^*|4)$. This way we provided a step towards a unifying picture for the description of holographic excitations of supersymmetric Wilson loops in arbitrary representations. In the case of the D5-brane, we took the next step and computed the effective action due to the brane fluctuations. The remaining task is to consider corrections on the gauge theory side and compare these with our predictions.

At the same time, motivated by the study of holographic condensed matter systems, we developed the details of the dimensional reduction of fermions in consistent truncations of type IIB and eleven-dimensional supergravity on squashed Sasaki-Einstein spaces. Such reductions are of interest, for example, in that they have $3 + 1$ and $(2 + 1)$-dimensional holographic duals, and the fermionic content and their interactions with charged scalars are an important aspect of their applications. We derived the lower dimensional equations of motion and the corresponding effective action, and were able to show how the theory fits into the framework of gauged supergravities. Our results for the couplings to various bosonic modes allow for the computation of fermion correlators in backgrounds that are dual to superconductors.

String Theory has relied heavily on the understanding of D-branes to make statements about its non-perturbative properties. This is how, for example, the web of dualities uncovered in the 1990's emerged, providing us with the unified picture of M Theory that we have today. Moreover, the AdS/CFT correspondence originated from a novel way of thinking about these extended objects. D-branes also led to the unveiling of the microscopic origin of the thermodynamics of certain black holes. Despite their prominent role in our modern grasp of fundamental physics, we have been unable to formulate a consistent approach to the quantization of D-branes. We hope that the semi-classical understanding of D-brane configurations dual to Wilson loops developed in this thesis will shed some light on a few of the issues that can arise when pursuing the quantization of extended objects.

APPENDICES

APPENDIX A

Notation, conventions and useful formulae

A.1 Chapters (II) and (III)

Here we summarize the conventions used throughout chapters (II) and (III).

Ten-dimensional curved coordinates x^m are labeled by Latin indices from the middle of the alphabet $m, n, \ldots = (0, \ldots, 9)$. The values $(0, 1, 2, 3, 4)$ and $(5, 6, 7, 8, 9)$ correspond to $aAdS_5$ and S^5, respectively. In both chapters, worldvolume coordinates are denoted by $a, b, \ldots$.

In chapter II, Latin indices $i, j, \ldots = (5, 6, 7, 8, 9)$ are used to label coordinates on S^5, which, together with x^4, are orthogonal to the D3-brane. Greek indices from the beginning of the alphabet $\alpha, \beta, \ldots = (0, 1)$ denote coordinates on the AdS_2 part of the worldvolume, while $\mu, \nu, \ldots = 2, 3$ label directions along S^2. All corresponding flat indices are underlined.

In chapter III, the indices $i, j, \ldots = (2, 3, 4, 5)$ represent directions transverse to the D5-branes. If the value $i = 5$, which corresponds to the azimuthal direction in S^5, is written explicitly, the range should only include $i, j, \ldots = (2, 3, 4)$; we hope this is clear from context. Greek indices $\alpha, \beta, \ldots = (0, 1)$ are used for the coordinates of the effective string embedded in the $aAdS_5$ part of the background geometry, whereas Greek indices from the middle of the alphabet $\mu, \nu, \ldots = (6, 7, 8, 9)$ denote the coordinates of the S^4 part of the D5-brane world volume. The corresponding flat indices are underlined. In contrast to [124], the Levi-Civita symbols $\epsilon_{a_1 \ldots a_n}$ are *tensors*, i.e., they include the appropriate factors of $\sqrt{|\det g|}$. With the exception of section 3.4, we assume

Lorentzian signature for the 2-d part of the world sheet, which implies $\epsilon_{\alpha\beta}\epsilon^{\alpha\beta} = -2$.

A.2 **Chapter** (IV)

In this Appendix we introduce the various conventions used in chapter IV of the thesis, and collect some useful results.

A.2.1 Conventions for forms and Hodge duality

We normalize all the (real) form fields according to

$$\begin{aligned}\omega &= \omega_{a_1\dots a_p}\, e^{a_1}\otimes e^{a_2}\cdots\otimes e^{a_p}\\ &= \frac{1}{p!}\omega_{a_1\dots a_p}\, e^{a_1}\wedge\cdots\wedge e^{a_p}\,.\end{aligned} \tag{A.1}$$

In d spacetime dimensions, the Hodge dual acts on the basis of forms as

$$*(e^{a_1}\wedge\cdots\wedge e^{a_p}) = \frac{1}{(d-p)!}\epsilon_{b_1\dots b_{d-p}}{}^{a_1\dots a_p}\, e^{b_1}\wedge\cdots\wedge e^{b_{d-p}}\,, \tag{A.2}$$

where $\epsilon_{b_1\dots b_{d-p}a_1\dots a_p}$ are the components of the Levi-Civita *tensor*. Equivalently, for the components of the Hodge dual $*\omega$ of a p-form ω we have

$$(*\omega)_{a_1\dots a_{d-p}} = \frac{1}{p!}\epsilon_{a_1\dots a_{d-p}}{}^{b_1\dots b_p}\omega_{b_1\dots b_p}\,. \tag{A.3}$$

In the $(3+1)$-dimensional external manifold M we adopt the convention $\epsilon_{0123} = +1$ for the components of the Levi-Civita tensor in the orthonormal frame.

A.2.2 Elfbein and spin connection

As discussed in section 5.2, the Kaluza-Klein metric ansatz of [62] is given by

$$ds^2_{11} = e^{2W(x)}ds^2_E(M) + e^{2U(x)}ds^2(Y) + e^{2V(x)}\big(d\chi + \mathcal{A}(y) + A(x)\big)^2\,, \tag{A.4}$$

where $W(x) = -3U(x) - V(x)/2$ as in the body of the paper. We now introduce the eleven-dimensional orthonormal frame $\hat{e}^M$. Denoting by $a, b, \dots$ the tangent indices to M, by $\alpha, \beta, \dots$

the tangent indices to the KE base Y, and by f the index associated with the U(1) fiber direction χ, our choice of elfbein reads

$$\hat{e}^a = e^W e^a \tag{A.5}$$

$$\hat{e}^\alpha = e^U e^\alpha \tag{A.6}$$

$$\hat{e}^f = e^V \Big(d\chi + \mathcal{A}(y) + A(x) \Big), \tag{A.7}$$

where e^a and e^α are orthonormal frames for M and Y, respectively. The dual basis is then

$$\hat{e}_a = e^{-W} \big(e_a - A_a \partial_\chi \big) \tag{A.8}$$

$$\hat{e}_\alpha = e^{-U} \big(e_\alpha - \mathcal{A}_\alpha \partial_\chi \big) \tag{A.9}$$

$$\hat{e}_f = e^{-V} \partial_\chi \,. \tag{A.10}$$

Denoting by $\omega^a{}_b$ the spin connection associated with $ds^2(M)$ and by $\omega^\alpha{}_\beta$ the spin connection appropriate to $ds^2(Y)$, for the eleven-dimensional spin connection $\hat{\omega}^M_N$ we find

$$\hat{\omega}^\alpha_a = e^{U-W} (\partial_a U) e^\alpha \tag{A.11}$$

$$\hat{\omega}^f_a = e^{V-W} \left[\frac{1}{2} F_{ab} e^b + (\partial_a V) \big(d\chi + \mathcal{A} + A \big) \right] \tag{A.12}$$

$$\hat{\omega}^f_\alpha = e^{V-U} \frac{1}{2} \mathcal{F}_{\alpha\beta} e^\beta \tag{A.13}$$

$$\hat{\omega}^a_b = \omega^a_b - 2\eta^{ac} \partial_{[c} W \eta_{b]d} e^d - \frac{1}{2} e^{2(V-W)} F^a_b \big(d\chi + \mathcal{A} + A \big) \tag{A.14}$$

$$\hat{\omega}^\alpha_\beta = \omega^\alpha_\beta - \frac{1}{2} e^{2(V-U)} \mathcal{F}^\alpha_\beta \big(d\chi + \mathcal{A} + A \big), \tag{A.15}$$

where η_{ab} is the flat metric in $(3+1)$ dimensions, $F \equiv dA$ and $\mathcal{F} \equiv d\mathcal{A} = 2J$, J being the Kähler form on Y.

A.2.3 Fluxes

The ansatz (IV.5) for the 4-form flux $\hat{F}_4$, reproduced here for convenience, is [62]

$$\begin{aligned} \hat{F}_4 = {} & f\, \mathrm{vol}_4 + H_3 \wedge (\eta + A) + H_2 \wedge J + dh \wedge J \wedge (\eta + A) + 2hJ^2 \\ & + \left[X(\eta + A) \wedge \Omega - \frac{i}{4} \left(dX - 4iAX \right) \wedge \Omega + \text{c.c.} \right]. \end{aligned} \tag{A.16}$$

We will often use a complex basis on T^*Y. If y denote real coordinates on Y, we define $z^1 \equiv \frac{1}{2}(y^1 + iy^2)$, $z^{\bar{1}} \equiv \frac{1}{2}\left(y^1 - iy^2\right)$, and similarly for $z^2, z^{\bar{2}}, z^3, z^{\bar{3}}$. With this normalization, the Kähler form J and the holomorphic (3,0)-form Σ are given by

$$J = 2i \sum_{\alpha=1,2,3} e^\alpha \wedge e^{\bar{\alpha}} \tag{A.17}$$

$$\Sigma = \frac{8}{3!}\epsilon_{\alpha\beta\gamma}\, e^\alpha \wedge e^\beta \wedge e^\gamma \,, \tag{A.18}$$

where we have chosen $\epsilon_{123} = +1$. Similarly, the forms on the external manifold can be written

$$\mathrm{vol}_4 = \frac{1}{4!}\epsilon_{abcd}\, e^a \wedge e^b \wedge e^c \wedge e^d \tag{A.19}$$

$$H_2 = \frac{1}{2!} H_{2\,ab}\, e^a \wedge e^b \tag{A.20}$$

$$H_3 = \frac{1}{3!} H_{3\,abc}\, e^a \wedge e^b \wedge e^c \,. \tag{A.21}$$

The components of $\hat{F}_4$ with respect to the eleven-dimensional frame $\hat{e}^M$ are then (in the real basis for T^*Y)

$$\hat{F}_{abcf} = e^{-3W-V} H_{3\,abc} \tag{A.22}$$

$$\hat{F}_{a\alpha\beta f} = e^{-W-2U-V}(\partial_a h) J_{\alpha\beta} \tag{A.23}$$

$$\hat{F}_{f\alpha\beta\gamma} = X e^{-3U-V}\Omega_{\alpha\beta\gamma} + \text{c.c.} \tag{A.24}$$

$$\hat{F}_{abcd} = f e^{-4W}\epsilon_{abcd} \tag{A.25}$$

$$\hat{F}_{ab\alpha\beta} = e^{-2W-2U} J_{\alpha\beta} H_{2\,ab} \tag{A.26}$$

$$\hat{F}_{\alpha\beta\gamma\delta} = 4h e^{-4U}(J_{\alpha\beta}J_{\gamma\delta} - J_{\alpha\gamma}J_{\beta\delta} + J_{\alpha\delta}J_{\beta\gamma}) \tag{A.27}$$

$$\hat{F}_{a\alpha\beta\gamma} = -\frac{i}{4}(D_a X) e^{-3U-W}\Omega_{\alpha\beta\gamma} + \text{c.c.} \tag{A.28}$$

A.2.4 Clifford algebra

We choose the following basis for the $D = 11$ Clifford algebra:

(A.29) $$\Gamma^a = \gamma^a \otimes \mathbb{1}_8$$

(A.30) $$\Gamma^\alpha = \gamma_5 \otimes \gamma^\alpha$$

(A.31) $$\Gamma_f = \gamma_5 \otimes \gamma_7$$

where the $\{\gamma^a\}$ are a basis for $C\ell(3,1)$ with $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and the $\{\gamma^\alpha\}$ are a basis for $C\ell(6)$ with $\gamma_7 = i\prod_\alpha \gamma^\alpha$. These dimensions are such that we can define Majorana spinors in each case. In $D = 11$, we take Γ^0 to be anti-Hermitian and the rest Hermitian. This means that γ^0 is anti-Hermitian, while $\gamma^a (a \neq 0)$, γ_5, γ_7 and γ^α are Hermitian. We also have $\gamma_5^2 = 1$ and $\gamma_7^2 = 1$. In the standard basis, the $\{\gamma^a, \gamma_5\}$ are 4×4 matrices while the $\{\gamma^\alpha, \gamma_7\}$ are 8×8 matrices. It will also be convenient to define

(A.32) $$\Gamma_7 = \prod_\alpha \Gamma^\alpha = \mathbb{1}_4 \otimes \gamma_7$$

(A.33) $$\Gamma_5 = \prod_a \Gamma^a = \gamma_5 \otimes \mathbb{1}_8\,.$$

Some useful identities involving the $C\ell(3,1)$ gamma matrices include

(A.34) $$\epsilon_{abcd} = -i\gamma_5\gamma_{abcd}\,, \quad \epsilon_{abcd}\gamma^a = i\gamma_5\gamma_{bcd}\,, \quad \epsilon_{abcd}\gamma^{cd} = 2i\gamma_5\gamma_{ab}\,, \quad \epsilon_{abcd}\gamma^{bcd} = 6i\gamma_5\gamma_a\,.$$

A.2.5 Charge conjugation conventions

In $d = 4$ dimensions with signature $(-,+,+,+)$ we can define unitary intertwiners B_4 and C_4 (the charge conjugation matrix), unique up to a phase, satisfying

(A.35) $$B_4\gamma_a B_4^\dagger = \gamma_a^* \qquad B_4^T = B_4$$

(A.36) $$B_4\gamma_5 B_4^\dagger = -\gamma_5^* \qquad B_4^* B_4 = \mathbb{1}\,,$$

and

$$C_4\gamma_a C_4^\dagger = -\gamma_a^T \qquad\qquad C_4^T = -C_4 \tag{A.37}$$

$$C_4\gamma_5 C_4^\dagger = \gamma_5^T \qquad\qquad C_4 = B_4^T\gamma_0 = B_4\gamma_0\,. \tag{A.38}$$

If ψ is any spinor, its charge conjugate $\psi^{\mathbf{c}}$ is then defined as

$$\psi^{\mathbf{c}} = B_4^{-1}\psi^* = B_4^\dagger\psi^* = \gamma_0 C_4^\dagger\psi^*\,. \tag{A.39}$$

In (3+1) dimensions one can define Majorana spinors. By definition, a spinor ψ is Majorana if $\psi = \psi^{\mathbf{c}}$. Notice that in (3+1) dimensions this condition relates opposite chirality spinors. Similarly, we can define the charge conjugates of a spinor Ψ in (10+1) dimensions and a spinor η in 7 Euclidean dimensions as

$$\Psi^{\mathbf{c}} = B_{11}^{-1}\Psi^*\,, \qquad \text{where} \qquad B_{11}\Gamma_M B_{11}^{-1} = \Gamma_M^*\,, \tag{A.40}$$

$$\eta^{\mathbf{c}} = B_7^{-1}\eta^*\,, \qquad \text{where} \qquad B_7\gamma_\alpha B_7^{-1} = -\gamma_\alpha^*\,. \tag{A.41}$$

Defining $\psi^{\mathbf{c}}$ in the (3+1)-dimensional space M by using the intertwiner B_4 defined above, (as opposed to using an intertwiner B_{4-} satisfying $B_{4-}\gamma_a B_{4-}^\dagger = -\gamma_a^*$ and $B_{4-}^T = -B_{4-}$), ensures that the charge conjugation operation acts uniformly in all the 11 directions, with

$$B_{11} = B_4 \otimes B_7\,. \tag{A.42}$$

A.3 Chapter V

In this Appendix we introduce the various conventions used in chapter V, and collect some useful results.

A.3.1 Conventions for forms and Hodge duality

We normalize all the form fields according to

$$\begin{aligned}\omega &= \omega_{a_1\ldots a_p}\, e^{a_1}\otimes e^{a_2}\cdots\otimes e^{a_p}\\ &= \frac{1}{p!}\omega_{a_1\ldots a_p}\, e^{a_1}\wedge\cdots\wedge e^{a_p}\,.\end{aligned} \tag{A.43}$$

Similarly, all the slashed p-forms are defined with the normalization

$$\not{\omega} = \frac{1}{p!}\gamma^{a_1\ldots a_p}\omega_{a_1\ldots a_p}\,. \tag{A.44}$$

In d spacetime dimensions, the Hodge dual acts on the basis of forms as

$$*(e^{a_1}\wedge\cdots\wedge e^{a_p}) = \frac{1}{(d-p)!}\epsilon_{b_1\ldots b_{d-p}}{}^{a_1\ldots a_p}\, e^{b_1}\wedge\cdots\wedge e^{b_{d-p}}\,, \tag{A.45}$$

where $\epsilon_{b_1\ldots b_{d-p}a_1\ldots a_p}$ are the components of the Levi-Civita *tensor*. Equivalently, for the components of the Hodge dual $*\omega$ of a p-form ω we have

$$(*\omega)_{a_1\ldots a_{d-p}} = \frac{1}{p!}\epsilon_{a_1\ldots a_{d-p}}{}^{b_1\ldots b_p}\omega_{b_1\ldots b_p}\,. \tag{A.46}$$

In the $(4+1)$-dimensional external manifold M we adopt the convention $\epsilon_{01234} = +1$ for the components of the Levi-Civita tensor in the orthonormal frame.

A.3.2 Zehnbein and spin connection

As discussed in section 5.2, the Kaluza-Klein metric ansatz of [27], [67], [111], [149] is given by

$$ds^2_{10} = e^{2W(x)}ds^2_E(M) + e^{2U(x)}ds^2(KE) + e^{2V(x)}\big(d\chi + \mathcal{A}(y) + A_1(x)\big)^2\,, \tag{A.47}$$

where $W(x) = -\frac{1}{3}\left(4U(x) + V(x)\right)$ as in the body of the paper. We now introduce the ten-dimensional orthonormal frame $\hat{e}^M$. Denoting by $a, b, \ldots$ the tangent indices to M, by $\alpha, \beta, \ldots$ the tangent indices to the Kähler-Einstein base KE, and by f the index associated with the U(1) fiber direction χ, our choice of zehnbein reads

$$\hat{e}^a = e^W e^a \tag{A.48}$$

$$\hat{e}^\alpha = e^U e^\alpha \tag{A.49}$$

$$\hat{e}^{\mathrm{f}} = e^V\Big(d\chi + \mathcal{A}(y) + A_1(x)\Big), \tag{A.50}$$

where e^a and e^α are orthonormal frames for M and KE, respectively. The dual basis is then

$$\hat{e}_a = e^{-W}\big(e_a - A_{1a}\partial_\chi\big) \tag{A.51}$$

$$\hat{e}_\alpha = e^{-U}\big(e_\alpha - \mathcal{A}_\alpha\partial_\chi\big) \tag{A.52}$$

$$\hat{e}_{\rm f} = e^{-V}\partial_\chi\,. \tag{A.53}$$

Denoting by ${\omega^a}_b$ the spin connection associated with $ds^2_E(M)$ and by ${\omega^\alpha}_\beta$ the spin connection appropriate to $ds^2(KE)$, for the ten-dimensional spin connection $\hat{\omega}^M_N$ we find

$$\hat{\omega}^{\alpha}_{\ a} = e^{U-W}(\partial_a U)e^\alpha \tag{A.54}$$

$$\hat{\omega}^{\rm f}_{\ a} = e^{V-W}\left[\frac{1}{2}F_{2\,ab}e^b + (\partial_a V)\big(d\chi + \mathcal{A} + A_1\big)\right] \tag{A.55}$$

$$\hat{\omega}^{\rm f}_{\ \alpha} = e^{V-U}\frac{1}{2}\mathcal{F}_{\alpha\beta}e^\beta \tag{A.56}$$

$$\hat{\omega}^{a}_{\ b} = \omega^a_{\ b} - 2\eta^{ac}\partial_{[c}W\eta_{b]d}e^d - \frac{1}{2}e^{2(V-W)}F_{2\ b}^{a}\big(d\chi + \mathcal{A} + A_1\big) \tag{A.57}$$

$$\hat{\omega}^{\alpha}_{\beta} = \omega^\alpha_\beta - \frac{1}{2}e^{2(V-U)}\mathcal{F}^\alpha_\beta\big(d\chi + \mathcal{A} + A_1\big), \tag{A.58}$$

where η_{ab} is the flat metric in $(4+1)$ dimensions, $F_2 \equiv dA_1$ and $\mathcal{F} \equiv d\mathcal{A} = 2J$, J being the Kähler form on KE.

A.3.3 Fluxes

The ansätze for the form fields fields, reproduced here for convenience, is as presented in Ref. [67]

$$\begin{aligned} F_{(5)} &= 4e^{8W+Z}\mathrm{vol}_5^E + e^{4(W+U)} * K_2 \wedge J + K_1 \wedge J \wedge J \\ &+ \left[2e^{Z} J \wedge J - 2e^{-8U} * K_1 + K_2 \wedge J\right] \wedge (\eta + A_1) \\ &+ \left[e^{4(W+U)} * L_2 \wedge \Omega + L_2 \wedge \Omega \wedge (\eta + A_1) + \text{c.c.}\right] \end{aligned} \tag{A.59}$$

$$\begin{aligned} F_{(3)} &= G_3 + G_2 \wedge (\eta + A_1) + G_1 \wedge J + G_0\, J \wedge (\eta + A_1) \\ &+ \left[N_1 \wedge \Omega + N_0\, \Omega \wedge (\eta + A_1) + \text{c.c.}\right] \end{aligned} \tag{A.60}$$

$$\begin{aligned} H_{(3)} &= H_3 + H_2 \wedge (\eta + A_1) + H_1 \wedge J + H_0\, J \wedge (\eta + A_1) \\ &+ \left[M_1 \wedge \Omega + M_0\, \Omega \wedge (\eta + A_1) + \text{c.c.}\right] \end{aligned} \tag{A.61}$$

As pointed out in the body of the paper, notice that we have $G_0 = H_0 = 0$ by virtue of the type IIB Bianchi identities. We will often use a complex basis on T^*KE. If y denote real coordinates on KE, we define $z^1 \equiv \frac{1}{2}(y^1 + iy^2)$, $z^{\bar{1}} \equiv \frac{1}{2}\left(y^1 - iy^2\right)$, and similarly for $z^2, z^{\bar{2}}$. With this normalization, the Kähler form J and the holomorphic (2,0)-form $\Sigma_{(2,0)}$ are given by

$$J = 2i \sum_{\alpha=1,2} e^{\alpha} \wedge e^{\bar{\alpha}} \tag{A.62}$$

$$\Sigma_{(2,0)} = \frac{2^2}{2!} \epsilon_{\alpha\beta}\, e^{\alpha} \wedge e^{\beta}\,, \tag{A.63}$$

where we have chosen $\epsilon_{12} = +1$. The components of $F_{(5)}$ with respect to the ten-dimensional frame $\hat{e}^M$ are then (in the real basis for T^*KE)

$$F_{(5)abcde} = 4e^{Z+3W}\epsilon_{abcde} \tag{A.64}$$

$$F_{(5)abcd\mathrm{f}} = -2e^{-4U-W}\epsilon_{abcd}{}^{e}K_{1;e} \tag{A.65}$$

$$F_{(5)a\alpha\beta\gamma\delta} = 6e^{-4U-W}K_{1;a}J_{[\alpha\beta}J_{\gamma\delta]} \tag{A.66}$$

$$F_{(5)\alpha\beta\gamma\delta\mathrm{f}} = 12e^{Z+3W}J_{[\alpha\beta}J_{\gamma\delta]} \tag{A.67}$$

$$F_{(5)abc\alpha\beta} = \frac{1}{2}e^{W+2U}\epsilon_{abc}{}^{de}\left(K_{2;de}J_{\alpha\beta} + L_{2;de}\Omega_{\alpha\beta} + L^*_{2;de}\bar{\Omega}_{\alpha\beta}\right) \tag{A.68}$$

$$F_{(5)ab\alpha\beta\mathrm{f}} = e^{W+2U}\left(K_{2;ab}J_{\alpha\beta} + L_{2;ab}\Omega_{\alpha\beta} + L^*_{2;ab}\bar{\Omega}_{\alpha\beta}\right). \tag{A.69}$$

Similarly for the components of $F_{(3)}$ with respect to the ten-dimensional frame we find

$$F_{(3)abc} = e^{-3W}G_{3\,abc} \tag{A.70}$$

$$F_{(3)ab\mathrm{f}} = e^{-2W-V}G_{2\,ab} \tag{A.71}$$

$$F_{(3)a\alpha\beta} = e^{-W-2U}\left[G_{1\,a}J_{\alpha\beta} + (N_{1\,a}\Omega_{\alpha\beta} + \text{ c.c.})\right] \tag{A.72}$$

$$F_{(3)\alpha\beta\mathrm{f}} = e^{-2U-V}\left[G_0J_{\alpha\beta} + (N_0\Omega_{\alpha\beta} + \text{ c.c.})\right], \tag{A.73}$$

with an analogous expression for $H_{(3)}$.

A.3.4 Clifford algebra

We choose the following basis for the $D = 10$ Clifford algebra:

$$\Gamma^a = \gamma^a \otimes \mathbb{1}_4 \otimes \sigma_1 \tag{A.74}$$

$$\Gamma^\alpha = \mathbb{1}_4 \otimes \gamma^\alpha \otimes \sigma_2 \tag{A.75}$$

$$\Gamma_{\mathrm{f}} = \mathbb{1}_4 \otimes \gamma_{\mathrm{f}} \otimes \sigma_2\,, \tag{A.76}$$

where $a = 0, 1, ..., 4$, $\alpha = 1, ..., 4$, whence[1]

$$\Gamma^{ab} = \gamma^{ab} \otimes \mathbb{1}_4 \otimes \mathbb{1}_2 \tag{A.77}$$

$$\Gamma^{\alpha\beta} = \mathbb{1}_4 \otimes \gamma^{\alpha\beta} \otimes \mathbb{1}_2 \tag{A.78}$$

$$\Gamma_{11} = -\Gamma^0\Gamma^1 ... \Gamma^9 = \mathbb{1}_4 \otimes \mathbb{1}_4 \otimes \sigma_3 \,. \tag{A.79}$$

The γ^a generate $C\ell(4,1)$ while the γ^α generate $C\ell(4,0)$. We have $\gamma^{01234} = -i\mathbb{1}_4$ in $C\ell(4,1)$ and $\gamma_{\mathrm{f}} = -\gamma^1\gamma^2\gamma^3\gamma^4$ in $C\ell(4,0)$.

Notice that $\gamma^{abcde} = i\epsilon_5^{abcde}$. Some useful identities involving the $C\ell(4,1)$ gamma matrices are then

$$\epsilon_{abcde}\gamma^{abcde} = -i5!\,, \qquad \epsilon^e{}_{abcd}\gamma^{abcd} = -i4!\gamma^e\,, \tag{A.80}$$

$$\epsilon^{de}{}_{abc}\gamma^{abc} = +i3!\gamma^{de}\,, \qquad \epsilon^{cde}{}_{ab}\gamma^{ab} = +i2!\gamma^{cde}\,. \tag{A.81}$$

It is also useful to notice that the Kähler form on KE satisfies

$$J_{\alpha\beta}J_{\gamma\delta}\epsilon^{\alpha\beta\gamma\delta} = 8\,, \qquad J_{\alpha\beta}J_{\gamma\delta}\gamma^{\alpha\beta\gamma\delta} = -8\gamma_{\mathrm{f}}\,, \qquad J_{\alpha\beta}J_{\gamma\delta}\gamma^{\beta\gamma\delta} = -2\gamma_\alpha\gamma_{\mathrm{f}}\,. \tag{A.82}$$

A.3.5 Charge conjugation conventions

In $d = 5$ dimensions with signature $(-,+,+,+,+)$ we can define unitary intertwiners $B_{4,1}$ and $C_{4,1}$ (the charge conjugation matrix), unique up to a phase, satisfying

$$B_{4,1}\gamma^a B_{4,1}^{-1} = -\gamma^{a*}\,, \qquad B_{4,1}^T = -B_{4,1}\,, \qquad B_{4,1}^* B_{4,1} = -\mathbb{1}\,, \tag{A.83}$$

and

$$C_{4,1}\gamma_a C_{4,1}^{-1} = \gamma_a^T\,, \qquad C_{4,1}^T = -C_{4,1}\,, \qquad C_{4,1} = B_{4,1}^T\gamma_0 = -B_{4,1}\gamma_0\,. \tag{A.84}$$

If ψ is any spinor in $(4+1)$ dimensions, its charge conjugate $\psi^{\mathbf{c}}$ is then defined as

$$\psi^{\mathbf{c}} = B_{4,1}^{-1}\psi^* = B_{4,1}^\dagger\psi^* = -\gamma_0 C_{4,1}^\dagger\psi^*\,. \tag{A.85}$$

[1] We take $\gamma^4 = i\gamma^{0123}$ in $C\ell(4,1)$. There is of course the opposite sign choice, leading to an inequivalent irrep of $C\ell(4,1)$.

In (4+1) dimensions it is not possible to define Majorana spinors satisfying $\psi^{\mathbf{c}} = \psi$. It is possible, however, to define *symplectic Majorana* spinors. These satisfy $\psi_i^{\mathbf{c}} = \Omega_{ij}\psi_j$, where Ω_{ij} is the USp(4)-invariant symplectic form. This fact becomes particulary relevant when dealing with $N = 4$ supergravity in $d = 5$ dimensions, inasmuch as the symplectic Majorana spinors allow to make the action of the R-symmetry manifest.

In analogy with (A.85), we can define the charge conjugates of a spinor Ψ in (9+1) dimensions and a spinor ε in 5 Euclidean dimensions as

$$\Psi^{\mathbf{c}} = B_{9,1}^{-1}\Psi^* , \qquad \text{where} \qquad B_{9,1}\Gamma_M B_{9,1}^{-1} = \Gamma_M^* , \qquad B_{9,1}^T = B_{9,1} \tag{A.86}$$

$$\varepsilon^{\mathbf{c}} = B_5^{-1}\varepsilon^* , \qquad \text{where} \qquad B_5\gamma_\alpha B_5^{-1} = \gamma_\alpha^* , \qquad B_5^T = -B_5 , \tag{A.87}$$

where B_5 and $B_{9,1}$ are the corresponding unitary intertwiners. We then find

$$B_{9,1} = B_{4,1} \otimes B_5 \otimes \sigma_3 . \tag{A.88}$$

Notice that B_5 is unitary and antisymmetric, and therefore for a spinor ε in five Euclidean dimensions we have $(\varepsilon^{\mathbf{c}})^{\mathbf{c}} = -\varepsilon$. In particular, in terms of the gauge-covariantly constant spinors $\varepsilon_\pm$ introduced in section 5.2, we have that defining ε_- as the charge conjugate of ε_+, this is $e^{-\frac{3i}{2}\chi}\varepsilon_- \equiv \left(e^{\frac{3i}{2}\chi}\varepsilon_+\right)^{\mathbf{c}}$, implies that $\left(e^{-\frac{3i}{2}\chi}\varepsilon_-\right)^{\mathbf{c}} = -e^{\frac{3i}{2}\chi}\varepsilon_+$. We also define the unitary intertwiner $C_{9,1}$ (the charge-conjugation matrix) in $(9+1)$ dimensions, which satisfies

$$C_{9,1}\Gamma_M C_{9,1}^{-1} = -\Gamma_M^T \qquad\qquad C_{9,1} = B_{9,1}^T\Gamma_0 = B_{9,1}\Gamma_0 . \tag{A.89}$$

Notice that defining $\Psi^{\mathbf{c}}$ in the (9+1)-dimensional space by using the intertwiner $B_{9,1}$ introduced above (as opposed to using an intertwiner $B_{9,1}^-$ satisfying $B_{9,1}^-\Gamma_M B_{9,1}^{-\dagger} = -\Gamma_M^*$) allows one to choose a basis, if so desired, where the charge conjugation operation in $D = 10$ reduces to complex conjugation. In this basis all the $C\ell(9,1)$ gamma-matrices are real, with $B_{9,1} = \mathbb{1}$ and a corresponding (9+1) charge-conjugation matrix $C_{9,1} = B_{9,1}^T\Gamma_0 = \Gamma_0$.

APPENDIX B

Compactification on S^2

In this appendix we discuss the details of the compactification on S^2. Deviating from the main text, we will use indices $\alpha, \beta, \ldots$ to denote all coordinates on $AdS_2 \times S^2$, with $\mu, \nu, \ldots$ and $i, j, \ldots$ labeling the directions along AdS_2 and S^2, respectively. For a sphere with radius R we use the properly normalized real spherical harmonics

$$\hat{g}^{ij}\nabla_i\nabla_j Y^{lm} = -\frac{l(l+1)}{R^2}Y^{lm}, \qquad R^2 \oint d\Omega_2\, Y^{lm}Y^{l'm'} = \delta^{ll'}\delta^{mm'}. \tag{B.1}$$

Recall that the deformed $AdS_2 \times S^2$ metric is

$$d\hat{s}^2 = R^2\left(ds_H^2 + d\Omega_2^2\right)\,, \tag{B.2}$$

with $R = L\sinh(u)$.

B.1 Scalars

A scalar field on $AdS_2 \times S^2$ can be expanded as

$$\phi(\sigma^0, \sigma^1, \theta, \phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \phi_{lm}(\sigma^0, \sigma^1)Y^{lm}(\theta, \phi). \tag{B.3}$$

Then, the reduction of the action is

$$\int d^4\sigma\sqrt{|\hat{g}|}\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int d^2\sigma\sqrt{|\hat{g}|}\left(\hat{g}^{\mu\nu}\partial_\mu\phi_{lm}\partial_\nu\phi_{lm} + \frac{l(l+1)}{R^2}\left(\phi_{lm}\right)^2\right). \tag{B.4}$$

On the right hand side, in a slight abuse of notation, we have used $\hat{g}$ for the determinant of the AdS_2 factor of the metric.

B.2 Gauge Field

In order to expand the gauge field we define the following vector fields on S^2:

(B.5)
$$Y_i^{lm}(\theta,\phi) = \frac{R}{\sqrt{l(l+1)}}\partial_i Y^{lm}(\theta,\phi), \qquad \hat{Y}_i^{lm}(\theta,\phi) = \epsilon_i^{\ j} Y_j^{lm}(\theta,\phi), \qquad l \geq 1.$$

Here ϵ is the covariantly constant tensor. They satisfy

(B.6)
$$\nabla^i Y_i^{lm} = -\frac{\sqrt{l(l+1)}}{R} Y^{lm}, \quad \epsilon^{ij}\partial_i Y_j^{lm} = 0, \quad \nabla^i \hat{Y}_i^{lm} = 0, \quad \epsilon^{ij}\partial_i \hat{Y}_j^{lm} = \frac{\sqrt{l(l+1)}}{R} Y^{lm},$$

as well as,

(B.7)
$$\begin{aligned} R^2 \oint d\Omega\, g^{ij} Y_i^{lm} Y_j^{l'm'} &= \delta^{ll'}\delta^{mm'}, \\ R^2 \oint d\Omega\, g^{ij} \hat{Y}_i^{lm} \hat{Y}_j^{l'm'} &= \delta^{ll'}\delta^{mm'}, \\ R^2 \oint d\Omega\, g^{ij} Y_i^{lm} \hat{Y}_j^{l'm'} &= 0. \end{aligned}$$

Moreover, they form a complete set of vector fields on S^2. Thus, we can decompose a gauge field on $AdS_2 \times S^2$ as

(B.8)
$$a_\mu(\sigma^0,\sigma^1,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_\mu^{lm}(\sigma^0,\sigma^1) Y^{lm}(\theta,\phi),$$

(B.9)
$$a_i(\sigma^0,\sigma^1,\theta,\phi) = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \Big(a_{lm}(\sigma^0,\sigma^1)\hat{Y}_i^{lm}(\theta,\phi) + b_{lm}(\sigma^0,\sigma^1) Y_i^{lm}(\theta,\phi)\Big).$$

Notice that the expansion for the components a_i starts at $l = 1$.

Now, under a gauge transformation

(B.10)
$$a'_\alpha = a_\alpha - \partial_\alpha \Lambda,$$

with

(B.11)
$$\Lambda(\sigma^0,\sigma^1,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} c_{lm}(\sigma^0,\sigma^1) Y^{lm}(\theta,\phi),$$

we have

$$a'_\mu = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left(a_\mu^{lm} - \partial_\mu c_{lm} \right) Y^{lm}, \tag{B.12}$$

$$a'_i = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \left(a_{lm} \hat{Y}_i^{lm} + \left(b_{lm} - \frac{\sqrt{l(l+1)}}{R} c_{lm} \right) Y_i^{lm} \right). \tag{B.13}$$

By choosing $c_{lm} = R/\sqrt{l(l+1)} b_{lm}$ we can gauge fix $b'_{lm} = 0$ and consider the following ansatz for the gauge field:

$$a_\mu(\sigma^0, \sigma^1, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_\mu^{lm}(\sigma^0, \sigma^1) Y^{lm}(\theta, \phi), \tag{B.14}$$

$$a_i(\sigma^0, \sigma^1, \theta, \phi) = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} a_{lm}(\sigma^0, \sigma^1) \hat{Y}_i^{lm}(\theta, \phi). \tag{B.15}$$

The residual gauge symmetry is

$$a_\mu^{00\prime} = a_\mu^{00} - \partial_\mu c_{00}. \tag{B.16}$$

Substituting this in the action we get

(B.17)

$$\int d^4\sigma \sqrt{|\hat{g}|} \hat{g}^{\alpha\gamma} \hat{g}^{\beta\delta} F_{\alpha\beta} F_{\gamma\delta} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \int d^2\sigma \sqrt{|\hat{g}|} \left[\hat{g}^{\mu\nu} \hat{g}^{\rho\sigma} f_{\mu\rho}^{lm} f_{\nu\sigma}^{lm} + \frac{2l(l+1)}{R^2} \hat{g}^{\mu\nu} a_\mu^{lm} a_\nu^{lm} \right.$$
$$\left. + 2\hat{g}^{\mu\nu} \partial_\mu a_{lm} \partial_\nu a_{lm} + \frac{2l(l+1)}{R^2} \left(a_{lm} \right)^2 \right],$$

where

$$f_{\mu\nu}^{lm} = \partial_\mu a_\nu^{lm} - \partial_\nu a_\mu^{lm}. \tag{B.18}$$

B.3 Fermions

For the expansion of fermionic fields we introduce the eigenspinors of the Dirac operator on the two-sphere which satisfy (see [30, 33, 126])

$$\hat{\gamma}^i \hat{\nabla}_i \chi_{lm}^\pm = \pm i \mu_l \chi_{lm}^\pm, \qquad \mu_l = \frac{\left(l + \frac{1}{2}\right)}{R}, \tag{B.19}$$

with $l = \frac{1}{2}, \frac{3}{2}, \ldots$ and $m = -l, -l+1, \ldots, l-1, l$. They form a complete set for spinor fields on S^2 and satisfy the orthonormality relations

$$R^2 \oint d\Omega\, \chi^{s\dagger}_{lm} \chi^{s'}_{l'm'} = \delta^{ss'} \delta_{ll'} \delta_{mm'}. \tag{B.20}$$

The relative phase can be chosen so that

$$\gamma_{\underline{23}} \chi^{\pm}_{lm} = \pm i \chi^{\mp}_{lm}. \tag{B.21}$$

Spinor fields on $AdS_2 \times S^2$ can be decomposed as

$$\psi(\sigma^0, \sigma^1, \theta, \phi) = \sum_{l=\frac{1}{2}}^{\infty} \sum_{m=-l}^{l} \left(\psi^+_{lm}(\sigma^0, \sigma^1) \otimes \chi^+_{lm}(\theta, \phi) + \psi^-_{lm}(\sigma^0, \sigma^1) \otimes \chi^-_{lm}(\theta, \phi) \right). \tag{B.22}$$

Using the gamma matrix representation

$$\Gamma_{\underline{\mu}} = \gamma_{\underline{\mu}} \otimes \mathbb{1}, \qquad \Gamma_{\underline{i}} = \gamma \otimes \gamma_{\underline{i}}, \tag{B.23}$$

where $\gamma = \gamma_{\underline{01}}$, we find

(B.24)

$$\hat{\Gamma}^{\alpha} \hat{\nabla}_{\alpha} \psi = \sum_{l=\frac{1}{2}}^{\infty} \sum_{m=-l}^{l} \left(\hat{\gamma}^{\mu} \hat{\nabla}_{\mu} + i \mu_l \gamma \right) \psi^+_{lm} \otimes \chi^+_{lm} + \sum_{l=\frac{1}{2}}^{\infty} \sum_{m=-l}^{l} \left(\hat{\gamma}^{\mu} \hat{\nabla}_{\mu} - i \mu_l \gamma \right) \psi^-_{lm} \otimes \chi^-_{lm}.$$

The fermionic action then reads

$$\begin{aligned} \int d^4\sigma \sqrt{|\hat{g}|} \bar{\psi} \hat{\Gamma}^{\alpha} \hat{\nabla}_{\alpha} \psi = & \sum_{l=\frac{1}{2}}^{\infty} \sum_{m=-l}^{l} \int d^2\sigma \sqrt{|\hat{g}|} \bar{\psi}^+_{lm} \left(\hat{\gamma}^{\mu} \hat{\nabla}_{\mu} + i \mu_l \gamma \right) \psi^+_{lm} \\ & + \sum_{l=\frac{1}{2}}^{\infty} \sum_{m=-l}^{l} \int d^2\sigma \sqrt{|\hat{g}|} \bar{\psi}^-_{lm} \left(\hat{\gamma}^{\mu} \hat{\nabla}_{\mu} - i \mu_l \gamma \right) \psi^-_{lm}. \end{aligned} \tag{B.25}$$

In this paper we deal with four-dimensional Weyl spinors. Using the above decomposition, the Weyl condition implies

$$\gamma \psi^{\pm}_{lm} = \mp i \psi^{\mp}_{lm}. \tag{B.26}$$

With this we can eliminate ψ_{lm}^{-} in favor of ψ_{lm}^{+} or vise-versa. Dropping the superscript we get

$$\int d^4\sigma\sqrt{|\hat{g}|}\bar{\psi}\hat{\gamma}^{\alpha}\hat{\nabla}_{\alpha}\psi = 2\sum_{l=\frac{1}{2}}^{\infty}\sum_{m=-l}^{l}\int d^2\sigma\sqrt{|\hat{g}|}\,\bar{\psi}_{lm}\left(\hat{\gamma}^{\mu}\hat{\nabla}_{\mu} + i\mu_l\gamma\right)\psi_{lm}, \tag{B.27}$$

where ψ_{lm} are unconstrained 2d Dirac spinors.

APPENDIX C

The $OSp\,(4^*|4)$ Algebra

In this section we briefly review the representations of $OSp(4^*|4)$ relevant in this paper. We closely follow [81].

The $OSp\,(2m^*|2n) \supset SO(2m^*) \times USp(2n)$ algebra has a Jordan structure with respect to its maximum subalgebra $g^0 = U(m|n) \supset U(m) \times U(n)$, this is,

$$g = g^{-1} \oplus g^0 \oplus g^{+1}, \qquad [g^m, g^n] \subseteq g^{m+n} \tag{C.1}$$

with $g^m = 0$ for $|m| > 1$. This decomposition is at the heart of the representation theory of this superalgebra. Denoting $A_{AB} \in g^{-1}$, $M^A{}_B \in g^0$, $A^{AB} \in g^{+1}$, the commutation relations of $OSp\,(2m^*|2n)$ read

$$[M^A{}_B, M^C{}_D] = \delta^C{}_B M^A{}_D - (-1)^{(\deg A+\deg B)(\deg C+\deg D)}\,\delta^A{}_D M^C{}_B, \tag{C.2}$$

$$[M^A{}_B, A_{CD}] = -\delta^A{}_C A_{BD} - \delta^A{}_D A_{CB}, \tag{C.3}$$

$$[M^A{}_B, A^{CD}] = \delta^C{}_B A^{AD} + \delta^D{}_B A_{CA}, \tag{C.4}$$

$$[A_{AB}, A^{CD}] = \delta^C{}_B M^D{}_A + \text{permutations}. \tag{C.5}$$

The index $A = (i, \mu)$, $i = 1, \ldots, m$, $\mu = 1, \ldots, n$ is in the fundamental of $U(m|n)$. Also, $\deg(i) = -\deg(\mu) = 1$.

This algebra can be realized by introducing f pairs of super oscillators $\xi_A(r)$ and $\eta_A(r)$, $r =$

$1, \ldots, f$,

$$\xi_A(r) = \begin{pmatrix} a_i(r) \\ \\ \alpha_\mu(r) \end{pmatrix}, \qquad \eta_A(r) = \begin{pmatrix} b_i(r) \\ \\ \beta_\mu(r) \end{pmatrix}, \tag{C.6}$$

where a and b are bosonic and α and β are fermionic. The $OSp\,(2m^*|2n)$ generators are then

$$A_{AB} = \xi_A \cdot \eta_B - \eta_A \cdot \xi_B, \tag{C.7}$$

$$A^{AB} = \eta^B \cdot \xi^A - \xi^B \cdot \eta^A, \tag{C.8}$$

$$M^A{}_B = \xi^A \cdot \xi_B + (-1)^{(\deg A)(\deg B)} \eta_B \cdot \eta^A. \tag{C.9}$$

The dot product means sum over r.

In the case of $m = n = 2$ the bosonic bilinears

$$A_{ij} = a_i \cdot b_j - b_i \cdot a_j, \quad A^{ij} = b^j \cdot a^i - a^j \cdot b^i, \quad M^i{}_j = a^i \cdot a_j + b_j \cdot b^i, \tag{C.10}$$

generate $SO(4^*) \simeq SL(2,\mathbb{R}) \times SU(2)$. Indeed, defining

$$B^- = \frac{1}{2}\epsilon^{ij} A_{ij}, \quad B^+ = \frac{1}{2}\epsilon_{ij} A^{ij}, \quad B^0 = \frac{1}{2} M^i{}_i, \quad I^i{}_j = M^i{}_j - \frac{1}{2}\delta^i{}_j M^k{}_k. \tag{C.11}$$

it is easy to see that B generate $SL(2,\mathbb{R})$ while I generate $SU(2)$. The fermionic bilinears

$$A_{\mu\nu} = \alpha_\mu \cdot \beta_\nu - \beta_\mu \cdot \alpha_\nu, \quad A^{\mu\nu} = \beta^\nu \cdot \alpha^\mu - \alpha^\nu \cdot \beta^\mu, \quad M^\mu{}_\nu = \alpha^\mu \cdot \alpha_\nu + \beta_\nu \cdot \beta^\mu, \tag{C.12}$$

span $USp(4) \simeq SO(5)$.

Representations of $OSp(4^*|4)$ are formed by taking a state $|\Omega\rangle$ that transforms irreducibly under $U(m|n)$ and is annihilated by g^{-1},

$$A_{AB}|\Omega\rangle = 0, \tag{C.13}$$

and acting on it with A^{AB}. Such a state can, in turn, be built from the oscillator vacuum $|0\rangle$ by acting with $\xi^A = \xi_A^\dagger$ and $\eta^A = \eta_A^\dagger$. Thus, $|\Omega\rangle$ is characterized by a Young tableau of $U(m|n)$. In general,

(C.14) $$B^0|0\rangle = f|0\rangle$$

(C.15) $$I^i{}_j|0\rangle = 0$$

so $|0\rangle$ has $SL(2,\mathbb{R}) \times SU(2)$ quantum numbers $(h,l) = (f,0)$.

Let us work out the ultrashort multiplet obtained by starting with $|0\rangle$ for $f = 1$. In this case, $|0\rangle$ has quantum numbers $(h,l) = (1,0)$. We have the following $SO^*(4) \times USp(4)$ lowest weight states in the representation:

(C.16) $$|0\rangle,\ A^{i\mu}|0>,\ A^{i\mu}A^{j\nu}|0>.$$

The even generators A^{ij} (or equivalently B^+) and $A^{\mu\nu}$ act within a given irrep of $SO^*(4)$ and $USp(4)$, respectively. We can then compute the $SL(2,\mathbb{R}) \times SU(2) \times SO(5)$ quantum numbers of the states above. We find,

(C.17) $$\mathbf{0} = (1,0,\mathbf{5}) \oplus (3/2,1/2,\mathbf{4}) \oplus (2,1,\mathbf{1})$$

More general doubleton ($f = 1$) representations of $OSp(4^*|4)$ have $SL(2,\mathbb{R}) \times SU(2) \times SO(5)$ content,

(C.18) $$\mathbf{j} = (j+1,j,\mathbf{5}) \oplus (j+3/2,j+1/2,\mathbf{4}) \oplus (j+2,j+1,\mathbf{1}) \\ \oplus (j+1/2,j-1/2,\mathbf{4}) \oplus (j+1,j,\mathbf{1}) \oplus (j,j-1,\mathbf{1}),$$

for $j > 1/2$ and

(C.19) $$\frac{\mathbf{1}}{\mathbf{2}} = (3/2,1/2,\mathbf{5}) \oplus (2,1,\mathbf{4}) \oplus (5/2,3/2,\mathbf{1}) \oplus (1,0,\mathbf{4}) \oplus (3/2,1/2,\mathbf{1}),$$

when $j = 1/2$. These are obtained by starting with the vacuum $\xi^{A_1}\xi^{A_2}\cdots\xi^{A_{2j}}|0\rangle$, which has $SL(2,\mathbb{R}) \times SU(2)$ quantum numbers $(j+1,j)$.

APPENDIX D

Scalar heat kernel on AdS_2

Here we show an explicit derivation of the scalar heat kernel on AdS_2 using Poincaré coordinates and verify that it coincides with the calculation done in global coordinates [13].

We begin by finding the eigenfunctions and eigenvalues. Consider the AdS_2 metric in Poincaré coordinates

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\ . \tag{D.1}$$

The Laplacian reads

$$\Box = y^2 \left(\partial_x^2 + \partial_y^2\right)\ . \tag{D.2}$$

Assuming a dependence of the form e^{ikx}, the spectral problem becomes

$$-y^2 \left(\partial_y^2 - k^2\right) \phi_{(k,\nu)}(y) = \left(\nu^2 + \frac{1}{4}\right) \phi_{(k,\nu)}(y)\ , \tag{D.3}$$

where we have written the eigenvalues as $\nu^2 + 1/4$.

The two independent solutions to this equation are

$$\phi^{(1)}_{(k,\nu)}(y) = \sqrt{y} L_{i\nu}\left(|k| y\right) \qquad \text{and} \qquad \phi^{(2)}_{(k,\nu)}(y) = \sqrt{y} K_{i\nu}\left(|k| y\right) \tag{D.4}$$

where

$$L_\mu(z) = \frac{i\pi}{2} \frac{I_{-\mu}(z) + I_\mu(z)}{\sin(\mu\pi)} \tag{D.5}$$

$$K_\mu(z) = \frac{\pi}{2} \frac{I_{-\mu}(z) - I_\mu(z)}{\sin(\mu\pi)} \tag{D.6}$$

and I_α is the modified Bessel function of the first kind. Of course, K_α is the usual modified Bessel function of the second kind. It is better to consider L_α and K_α (as opposed to I_α and K_α) as independent solutions since they are both real when the order is imaginary and the argument real.

If ν is purely imaginary, both solutions fail to be square integrable. For real ν, the asymptotic behavior as $y \to 0^+$ is

$$L_{i\nu}(y) = \sqrt{\frac{\pi}{\nu \sinh(\nu\pi)}} \left[\nu \cos\left(\nu \ln(y/2) - c_\nu\right) + O(y^2)\right] , \tag{D.7}$$

$$K_{i\nu}(y) = -\sqrt{\frac{\pi}{\nu \sinh(\nu\pi)}} \left[\nu \sin\left(\nu \ln(y/2) - c_\nu\right) + O(y^2)\right] , \tag{D.8}$$

where c_ν is a constant, and

$$L_{i\nu}(y) = \frac{1}{\sinh(\nu\pi)} \sqrt{\frac{\pi}{2y}} e^{y} \left[1 + O\left(\frac{1}{y}\right)\right] , \tag{D.9}$$

$$K_{i\nu}(y) = \sqrt{\frac{\pi}{2y}} e^{-y} \left[1 + O\left(\frac{1}{y}\right)\right] , \tag{D.10}$$

when $y \to \infty$. From this we see that both solutions vanish as we approach the boundary $y = 0$, but only $\phi^{(2)}_{(k,\nu)}$ vanishes as $y \to \infty$. In other words, only $\phi^{(2)}_{(k,\nu)}$ is square integrable.

The relation (see Kontorovich-Lebedev transform)

$$\int_0^\infty dy\, \frac{K_{i\mu}(y) K_{i\nu}(y)}{y} = \frac{\pi^2}{2\mu \sinh(\pi\mu)} \delta(\mu - \nu) , \tag{D.11}$$

sets the normalization of the eigenfunctions as

$$f_{(k,\nu)}(x,y) = \frac{1}{\sqrt{\pi^3}} \sqrt{\nu \sinh(\pi\nu)} e^{ikx} \sqrt{y} K_{i\nu}\left(|k| y\right) , \tag{D.12}$$

where $k \in \mathbb{R}$ and $\nu \geq 0$.

Now, the diagonal heat kernel is

$$K\left((x,y),(x,y);t)\right) = \int dk d\nu\, e^{\left(\nu^2 + \frac{1}{4}\right)t} f^*_{(k,\nu)}(x,y) f_{(k,\nu)}(x,y) \tag{D.13}$$

Using the above eigenfunctions this is

$$K\left((x,y),(x,y);t)\right) = \frac{1}{\pi^3} \int_0^\infty d\nu\, e^{-\left(\nu^2 + \frac{1}{4}\right)t} \nu \sinh(\nu\pi) \int_{-\infty}^\infty dk\, y K_{i\nu}(|k| y)^2 \tag{D.14}$$

$$= \frac{2}{\pi^3} \int_0^\infty d\nu\, e^{-\left(\nu^2 + \frac{1}{4}\right)t} \nu \sinh(\nu\pi) \int_0^\infty dk\, K_{i\nu}(k)^2 \tag{D.15}$$

This does not depend on y, as expected. The norm of the modified Bessel function is

$$\int_0^\infty dx\, K_{i\nu}(x)^2 = \frac{\pi}{4}\Gamma\left(\frac{1}{2}+i\nu\right)\Gamma\left(\frac{1}{2}-i\nu\right) \tag{D.16}$$

$$= \frac{\pi^2}{4\cosh(\pi\nu)} \tag{D.17}$$

Therefore,

$$K\left((x,y),(x,y);t)\right) = \frac{1}{2\pi R^2}\int_0^\infty d\nu\, e^{-\left(\nu^2+\frac{1}{4}\right)t}\nu\tanh(\nu\pi) \tag{D.18}$$

This is the same expression one gets when working with global coordinates on the disk.

APPENDIX E

Integrals and infinite sums

We will perform here the evaluation of the integrals and infinite sums needed for the heat kernel calculations of bosons and fermions in section 3.4.

For the bosons, let us start with the infinite sum (III.135),

$$\Sigma^s(t) = \sum_{l=0}^{\infty} \left(l + \frac{1}{2} \right) \mathrm{e}^{-(l+1/2)^2 t} \ . \tag{E.1}$$

Converting the sum into a contour integral that picks up suitable poles, as outlined in [13], one obtains

$$\Sigma^s(t) = \mathrm{Im} \int_0^{\mathrm{e}^{i\kappa}\infty} d\nu \, \nu \tan(\pi\nu) \, \mathrm{e}^{-\nu^2 t} \ . \tag{E.2}$$

Here, $0 < \kappa \ll 1$, so that $\mathrm{Im}\,\nu > 0$ in the integrand. Now, we write $\tan(\pi\nu) = i \tanh(-i\pi\nu)$ and expand the $\tanh$ as

$$\tanh(\pi\nu) = 1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \, \mathrm{e}^{-2\pi\nu k} \tag{E.3}$$

to obtain

$$\tan(\pi\nu) = i \left[1 - 2 \sum_{k=1}^{\infty} (-1)^{k+1} \, \mathrm{e}^{2\pi i\nu k} \right] \ . \tag{E.4}$$

The integral in (E.2) can be done exactly for the first term of the expansion, while in the remaining terms we expand $\mathrm{e}^{-\nu^2 t}$ as a power series in t, integrate and perform the summation over k. The result is [cf. (2.18) of [13]]

$$\begin{aligned}\Sigma^s(t) &= \frac{1}{2t} + 2\sum_{n=0}^{\infty} \frac{(2n+1)!}{n!(2\pi)^{2n+2}} t^n \left(1 - 2^{-2n-1}\right) \zeta(2n+2) \\ &= \frac{1}{2t} + \frac{1}{2}\sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \left(1 - 2^{1-2n}\right) |B_{2n}| \\ &= \frac{1}{2t} + \frac{1}{24} + \frac{7}{960} t + \frac{31}{16128} t^2 + \mathcal{O}(t^3)\ . \end{aligned} \tag{E.5}$$

On the first line, $\zeta(s)$ denotes the Riemann zeta function, which we expressed in terms of the Bernoulli numbers B_{2n} on the second line.

Consider now the integral in (III.132). In analogy with the calculation above, we expand the $\tanh$ using (E.3) such that the leading term is captured by the integral over the first term of the expansion. For the remaining terms, expand $\mathrm{e}^{-\nu^2 t}$ as a power series in t, integrate and perform the summation over k. The result is [cf. (2.15) of [13]]

$$\int\limits_0^\infty d\nu\, \nu \tanh(\pi\nu)\, \mathrm{e}^{-\nu^2 t} = \frac{1}{2t} - \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-t)^{n-1}}{n!} \left(1 - 2^{1-2n}\right) |B_{2n}| = -\Sigma^s(-t)\ . \tag{E.6}$$

Again, we have expressed the Riemann zeta functions in terms of Bernouuli numbers, and the last equality results from a direct comparison with the second line of (E.5).

Similar calculations must be done for the fermion contributions. Consider the infinite sum (III.168)

$$\Sigma^f(t) = \sum_{l=0}^{\infty} l\, \mathrm{e}^{-l^2 t}\ . \tag{E.7}$$

Converting the sum into a contour integral, one obtains

$$\Sigma^f(t) = -\operatorname{Im} \int\limits_0^{\mathrm{e}^{i\kappa}\infty} d\nu\, \nu \cot(\pi\nu)\, \mathrm{e}^{-\nu^2 t}\ . \tag{E.8}$$

Write $\cot(\pi\nu) = -i \coth(-i\pi\nu)$ and expand the $\coth$ as

$$\coth(\pi\nu) = 1 + 2\sum_{k=1}^{\infty} \mathrm{e}^{-2\pi\nu k} \tag{E.9}$$

to obtain

$$\cot(\pi\nu) = -i \left[1 + 2 \sum_{k=1}^{\infty} \mathrm{e}^{2\pi i \nu k} \right] . \tag{E.10}$$

Continuing as for $\Sigma^s(t)$, we obtain [cf. (3.3.16) of [13]]

$$\begin{aligned} \Sigma^f(t) &= \frac{1}{2t} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{t^{n-1}}{n!} \, |B_{2n}| \\ &= \frac{1}{2t} - \frac{1}{12} - \frac{1}{120} t - \frac{1}{504} t^2 + \mathcal{O}(t^3) . \end{aligned} \tag{E.11}$$

Finally, an analogous calculation for the integral in (III.165) yields

$$\int_0^{\infty} d\nu \, \nu \coth(\pi\nu) \, \mathrm{e}^{-\nu^2 t} = \frac{1}{2t} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{(-t)^{n-1}}{n!} \, |B_{2n}| = -\Sigma^f(-t) . \tag{E.12}$$

APPENDIX F

Geometry of embedded manifolds

To describe the embedding of a Dp-brane worldvolume in the bulk, we shall use the structure equations of embedded manifolds [48]. We shall denote with Latin indices $m, n, \dots$ the curved bulk coordinates and with Latin indices $a, b, \dots$ the worldvolume coordinates. Latin indices i, j are used for the directions normal to the worldvolume. The corresponding flat indices are underlined.

A d-dimensional Riemannian manifold $\mathbb{M}$ embedded in a $\tilde{d}$-dimensional Riemannian manifold $\tilde{\mathbb{M}}$ ($d < \tilde{d}$) is described by $\tilde{d}$ differentiable functions x^m ($m = 1, \dots, \tilde{d}$) of d variables ξ^a ($a = 1, \dots, d$). The ξ^a are coordinates on $\mathbb{M}$ (the worldvolume), whereas $x^m(\xi)$ specify the location in $\tilde{\mathbb{M}}$ (the bulk). The tangent vectors to the world volume are given by $x^m_a(\xi) \equiv \partial_a x^m(\xi)$. They provide the pull-back of any bulk quantity onto the world volume. For example, the induced metric is

$$g_{ab} = x^m_a x^n_b g_{mn}\,. \tag{F.1}$$

In addition, there are $d_\perp = \tilde{d} - d$ normal vectors $N^m_{\underline{i}}$, $\underline{i} = 1, \dots, d_\perp$. Together with the x^m_a, they satisfy the orthogonality and completeness relations

$$N^m_{\underline{i}} x^n_a g_{mn} = 0 \qquad N^m_{\underline{i}} N^n_{\underline{j}} g_{mn} = \delta_{\underline{i}\underline{j}}\,, \qquad g^{ab} x^m_a x^n_b + \delta^{\underline{i}\underline{j}} N^m_{\underline{i}} N^n_{\underline{j}} = g^{mn}\,. \tag{F.2}$$

We shall adopt a covariant notation raising and lowering indices with the appropriate metric tensors. The freedom of choice of the normal vectors gives rise to a group $O(n)$ of local rotations of the normal frame.

The geometric structure of the embedding is determined, in addition to the intrinsic geometric quantities, by the second fundamental form $H^{\underline{i}}{}_{ab}$, which describes the extrinsic curvature, and the gauge connection in the normal bundle, $A^{\underline{ij}}{}_{a} = -A^{\underline{ji}}{}_{a}$. They are determined by the equations of Gauss and Weingarten, respectively,

(F.3) $$\nabla_a x^m_b \equiv \partial_a x^m_b + \Gamma^m{}_{np} x^n_a x^p_b - \Gamma^c{}_{ab} x^m_c = H^{\underline{i}}{}_{ab} N^m_{\underline{i}} \,,$$

(F.4) $$\nabla_a N^m_{\underline{i}} \equiv \partial_a N^m_{\underline{i}} + \Gamma^m{}_{np} x^n_a N^p_{\underline{i}} - A^{\underline{j}}{}_{\underline{i}a} N^m_{\underline{j}} = -H_{\underline{i}a}{}^b x^m_b \,.$$

As is evident here, by using the appropriate connections, ∇_a denotes the covariant derivative with respect to all indices. The integrability conditions of the differential equations (F.3) and (F.4) are the equations of Gauss, Codazzi and Ricci, which are, respectively,

(F.5) $$R_{mnpq} x^m_a x^n_b x^p_c x^q_d = R_{abcd} + H^{\underline{i}}{}_{ad} H_{\underline{i}bc} - H^{\underline{i}}{}_{ac} H_{\underline{i}bd} \,,$$

(F.6) $$R_{mnpq} x^m_a x^n_b N^p_{\underline{i}} x^q_c = \nabla_a H_{\underline{i}bc} - \nabla_b H_{\underline{i}ac} \,,$$

(F.7) $$R_{mnpq} x^m_a x^n_b N^p_{\underline{i}} N^q_{\underline{j}} = F_{\underline{ij}ab} - H_{\underline{i}a}{}^c H_{\underline{j}cb} + H_{\underline{i}b}{}^c H_{\underline{j}ca} \,,$$

where $F_{\underline{ij}ab}$ is the field strength in the normal bundle,

(F.8) $$F_{\underline{ij}ab} = \partial_a A_{\underline{ij}b} - \partial_b A_{\underline{ij}a} + A_{\underline{ik}a} A^{\underline{k}}{}_{\underline{j}b} - A_{\underline{ik}b} A^{\underline{k}}{}_{\underline{j}a} \,.$$

As mentioned before, the covariant derivative in (F.6) contains also the connections $A^{\underline{j}}{}_{\underline{i}a}$.

Let us derive the expression for the pull-back of the spinor bulk covariant derivative on the world volume of the brane, which is needed in sections 2.3.2 and 3.3.2,

(F.9) $$x^m_a \nabla_m = x^m_a \left(\partial_m + \frac{1}{4} \omega_m{}^{\underline{np}} \Gamma_{\underline{np}} \right) .$$

The bulk spin connections can be obtained by

(F.10) $$\omega_m{}^{\underline{np}} = -e^{\underline{p}}_q \left(\partial_m e^{q\underline{n}} + \Gamma^q{}_{mp} e^{p\underline{n}} \right) \,,$$

and similarly for the world volume spin connections.

Let us pick a local frame adapted to the embedding,

$$e^m_{\underline{n}} = \begin{cases} x^m_a e^a_{\underline{a}} & \text{for } \underline{n} = \underline{a}, \\ N^m_{\underline{i}} & \text{for } \underline{n} = \underline{i}. \end{cases} \tag{F.11}$$

Then, using (F.3) and (F.4), it is straightforward to show that

$$x^m_a \left(\partial_m e^{q\underline{n}} + \Gamma^q{}_{mp} e^{p\underline{n}}\right) = \begin{cases} H^{\underline{i}}_{ab} N^q_{\underline{i}} e^b_{\underline{a}} + \omega_{a\underline{b}\underline{a}} e^{b\underline{b}} x^q_b & \text{for } \underline{n} = \underline{a}, \\ -H_{\underline{i}a}{}^b x^q_b + A^{\underline{j}}{}_{\underline{i}a} N^q_{\underline{j}} & \text{for } \underline{n} = \underline{i}. \end{cases} \tag{F.12}$$

Hence, one finds for the pull-back of the bulk spin connections

$$x^m_a \omega_{m\underline{a}\underline{b}} = \omega_{a\underline{a}\underline{b}}\,, \qquad x^m_a \omega_{m\underline{a}\underline{i}} = -H_{\underline{i}ab} e^b_{\underline{a}}\,, \qquad x^m_a \omega_{m\underline{i}\underline{j}} = A_{\underline{i}\underline{j}a}\ . \tag{F.13}$$

Consequently, (F.9) becomes

$$x^m_a \nabla_m = \nabla_a - \frac{1}{2} H_{\underline{i}ab} \Gamma^b \Gamma^{\underline{i}} + \frac{1}{4} A_{\underline{i}\underline{j}a} \Gamma^{\underline{i}\underline{j}}\,. \tag{F.14}$$

APPENDIX G

More on $SU(3)$ singlets

The crucial feature of the truncations we are examining is that we retain only singlets under the structure group of the KE base. To further understand the structure in play in the reduction of the fermionic degrees of freedom, we consider the corresponding problem on gravitino *states*.

In the complex basis, the Γ matrices act as raising and lowering operators on the states. The raising operators transform as a $\mathbf{3}$ of $SU(3)$ and the lowering operators as a $\bar{\mathbf{3}}$. Using complex notation, we write $\Gamma^1 = \frac{1}{2}\left[\Gamma^1 + i\Gamma^2\right]$, etc. where the matrix on the left-hand side is understood to be defined in the complex basis and those on the right are in the real basis. We then see that Γ^α and $\Gamma^{\bar{\alpha}}$ satisfy Heisenberg algebras, and we can associate Fock spaces to each pair. Then, $P_1 = \Gamma^1\Gamma^{\bar{1}}$ is a projector, and we are led to define the set of projection operators (we are using complex indices, so $\alpha = 1, 2, 3$)

(G.1) $$P_\alpha = \Gamma^\alpha\Gamma^{\bar{\alpha}}, \quad \bar{P}_\alpha = \Gamma^{\bar{\alpha}}\Gamma^\alpha \quad \text{(no sum)}$$

and "charge" operators [1]

(G.2) $$Q_\alpha = \Gamma^{\alpha\bar{\alpha}} \ \text{(no sum)}$$

Since a spinor can be thought of in the corresponding Fock space representation as $|\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\rangle$, with the $\pm\frac{1}{2}$ being eigenvalues of Q_α, the $SU(3)$ singlets are those spinors that satisfy

(G.3) $$Q_\alpha\varepsilon_\pm = \pm\frac{1}{2}\varepsilon_\pm, \quad \forall\ \alpha$$

[1] Note that Γ^1, Γ^2, Q_1 can be identified as the generators J_x, J_y, J_z of the spin-$1/2$ representation of an $SU(2)$ subgroup, and similarly for Γ^3, Γ^4, Q_2, etc.

The six other states are in non-trivial representations of $SU(3)$. Note that $\Gamma_7 = \prod_\alpha 2Q_\alpha$, so the positive (negative) chirality spinor has an even (odd) number of minus signs, and Γ_7 is the "volume form" (the product of all the signs). The (c-)spinors are in the $\mathbf{4} + \bar{\mathbf{4}}$ of Spin$(6) \simeq SU(4)$, with the two conjugate representations corresponding to the two chiral spinors. We can now appreciate the significance of the operator Q that we encountered in section 5.2: it is (up to normalization) the "total charge operator" $Q = 2\sum_\alpha 2Q_\alpha$. It is clear that it is the $SU(3)$ singlets that have maximum charge $Q = \pm 6$, where the sign is correlated with the chirality. The other spinor states are in $\mathbf{3}$ and $\bar{\mathbf{3}}$ and have Q-charges ∓ 2. We then find that the ordinary spinor consists of $\{|\mathbf{1}, 6\rangle_+, \{|\mathbf{3}, -2\rangle_+, \{|\bar{\mathbf{3}}, 2\rangle_-, \{|\mathbf{1}, -6\rangle_-\}$, where the subscript on the ket indicates the γ_7-chirality. In the weight language, the $|\mathbf{1}, 6\rangle_+$ corresponds to $|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$ and the $|\mathbf{1}, -6\rangle_-$ corresponds to $|-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle$, and it is clear from the construction that they are related by charge conjugation.

As described in the body of the paper, we focus on the $SU(3)$ singlet spinors $\varepsilon_\pm$, and consequently discard all but the internal spinors

$$\varepsilon(y, \chi) = \varepsilon_\pm(y) e^{\pm 2i\chi} = \varepsilon_\pm(y) e^{\pm 2i\chi}\,. \tag{G.4}$$

Notice that $\varepsilon_\pm$ are not only γ_7-chiral, but they satisfy the projections

$$\bar{P}_\alpha \varepsilon_+ = 0, \quad P_\alpha \varepsilon_- = 0, \quad \forall \alpha \tag{G.5}$$

Finally, the gravitino states can be thought of as the spin-1/2 spinor tensored with $|\mathbf{3}, 4\rangle \oplus |\bar{\mathbf{3}}, -4\rangle$ (i.e. the representations corresponding to the raising/lowering operators). Thus, the gravitino states transform as $\{|\mathbf{3}, 10\rangle, |\mathbf{1}, 6\rangle, |\mathbf{8}, 6\rangle, |\bar{\mathbf{3}}, 2\rangle, |\mathbf{6}, 2\rangle, |\mathbf{3}, -2\rangle\}$ and their conjugates. This totals 48 states, which is the right counting.

APPENDIX H

$d = 4$ equations of motion

Here we explicitly collect the equations of motion for the diagonal fermion fields ζ_a, η and ξ. To this end we define the following linear combinations

(H.1) $\mathcal{L}_\zeta^a \equiv e^{\frac{3W}{2}} \gamma_5 \mathcal{L}_{gr}^a$ $\qquad$ $\mathcal{R}_\zeta^a \equiv e^{\frac{3W}{2}} \gamma_5 \mathcal{R}_{gr}^a$

(H.2) $\mathcal{L}_\eta \equiv e^{\frac{3W}{2}} \left(\frac{2}{3} \gamma_5 \mathcal{L}_f + \frac{1}{3} \gamma_a \mathcal{L}_{gr}^a \right)$ $\qquad$ $\mathcal{R}_\eta \equiv e^{\frac{3W}{2}} \left(\frac{2}{3} \gamma_5 \mathcal{R}_f + \frac{1}{3} \gamma_a \mathcal{R}_{gr}^a \right)$

(H.3) $\mathcal{L}_\xi \equiv \frac{2}{3} e^{\frac{3W}{2}} \left(\frac{1}{2} \mathcal{L}_b - \gamma_5 \mathcal{L}_f + \gamma_a \mathcal{L}_{gr}^a \right)$ $\qquad$ $\mathcal{R}_\xi \equiv \frac{2}{3} e^{\frac{3W}{2}} \left(\frac{1}{2} \mathcal{R}_b - \gamma_5 \mathcal{R}_f + \gamma_a \mathcal{R}_{gr}^a \right),$

where $\mathcal{L}_f, \mathcal{L}_{gr}^a, \mathcal{L}_b$ and $\mathcal{R}_f, \mathcal{R}_{gr}^a, \mathcal{R}_b$ are given in section 4.3. After performing the chiral rotation of the fermion fields described in section 4.3, the equations of motion then read

$$
\begin{aligned}
\text{(H.4)}\quad 0 &= \mathcal{L}^a_\zeta + \frac{1}{4}\mathcal{R}^a_\zeta \\
&= \gamma^{abc}D_b\zeta_c + \frac{1}{4}\Big[-ie^{V-W}\left(F + i\gamma_5 * F\right)^{ac} - 12ie^{W-4U}\gamma_5(h + i\gamma_5 e^{V+2U})\gamma^{ac} \\
&\qquad + 3i(\partial_b h)e^{-2U-V}\gamma_5\gamma^{abc} - 3e^{-W-2U}\gamma_5\left(H_2 + i\gamma_5 * H_2\right)^{ac} \\
&\qquad - \left(fe^{-3W} + 6e^{W+V-2U} - 8e^{W-V}\right)\gamma^{ac} + e^{-2W-V}{H_3}^{abc}\gamma_5\gamma_b\Big]\zeta_c \\
&\quad + \frac{3}{8}\Big[i\left(fe^{-3W} + 6e^{W+V-2U} - 8e^{W-V}\right) + e^{V-W}\left(\not{F} - i\gamma_5 e^{-V-2U}\not{H}_2\right) \\
&\qquad - 4e^{W-4U}\gamma_5\left(h + i\gamma_5 e^{V+2U}\right) - 2e^{-2U-V}\gamma_5\not{\partial}\left(h - i\gamma_5 e^{V+2U}\right)\Big]\gamma^a\eta \\
&\quad + \frac{1}{4}\Big[i\left(fe^{-3W} + 6e^{W+V-2U}\right) - 12e^{W-4U}\gamma_5\left(h + i\gamma_5 e^{V+2U}\right) - 6i\left(\not{\partial}U\right) \\
&\qquad - ie^{-2W-V}\gamma_5\not{H}_3\Big]\gamma^a\xi + \frac{i}{2}\gamma_5\left[-(D_bX)e^{-3U}\gamma^{abc} + 4Xe^{W-3U-V}\gamma^{ac}\right]\zeta^{\mathbf{c}}_c \\
\text{(H.5)}\quad &\quad - \frac{1}{4}\gamma_5\Big[e^{-3U}(\not{D}X)\gamma^a + 4Xe^{W-3U-V}\gamma^a\Big]\xi^{\mathbf{c}} - 3Xe^{W-3U-V}\gamma_5\gamma^a\eta^{\mathbf{c}}\,,
\end{aligned}
$$

$$
\begin{aligned}
0 &= \mathcal{L}_\eta + \frac{1}{4}\mathcal{R}_\eta \\
&= \not{D}\eta + \Big[\frac{1}{2}\left(fe^{-3W} + 6e^{W+V-2U} - 8e^{W-V}\right) - \frac{1}{4}e^{-2W-V}\gamma_5\not{H}_3 + \frac{i}{4}e^{-2U-V}\gamma_5(\not{\partial}h) \\
&\qquad + \frac{i}{2}e^{V-W}\left(\not{F} - i\gamma_5 e^{-V-2U}\not{H}_2\right) + 2ie^{W-4U}\gamma_5\left(h + i\gamma_5 e^{V+2U}\right)\Big]\eta \\
&\quad + \frac{1}{4}\Big[i\left(fe^{-3W} + 6e^{W+V-2U} - 8e^{W-V}\right) + 4e^{W-4U}\gamma_5\left(h - i\gamma_5 e^{V+2U}\right)\Big]\gamma^b\zeta_b \\
&\quad + \frac{1}{4}\gamma^b\Big[e^{V-W}\left(\not{F} - i\gamma_5 e^{-V-2U}\not{H}_2\right) - 2e^{-2U-V}\gamma_5\not{\partial}\left(h + i\gamma_5 e^{V+2U}\right)\Big]\zeta_b \\
&\quad + \frac{1}{2}\Big[\left(fe^{-3W} + 6e^{W+V-2U}\right) - 4i\gamma_5 e^{W-4U}\left(h - i\gamma_5 e^{V+2U}\right)\Big]\xi \\
&\quad - \frac{i}{2}\gamma_5\left[e^{-3U}(\not{D}X) + 8Xe^{W-3U-V}\right]\eta^{\mathbf{c}} + \left(-2ie^{W-3U-V}X\right)\gamma_5\xi^{\mathbf{c}} \\
\text{(H.6)}\quad &\quad + \left(-2Xe^{W-3U-V}\gamma_5\gamma^c\right)\zeta^{\mathbf{c}}_c\,,
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \mathcal{L}_\xi + \frac{1}{4}\mathcal{R}_\xi \\
&= \not{D}\xi + \frac{3}{4}\Bigg[\frac{8}{3}e^{W-V} + \left(fe^{-3W} + 6e^{W+V-2U}\right) - \frac{i}{3}e^{V-W}\left(\not{F} + 3i\gamma_5 e^{-V-2U}\not{H}_2\right) \\
&\qquad\qquad + e^{-2W-V}\gamma_5\not{H}_3 - 12ie^{W-4U}\gamma_5\left(h - i\gamma_5 e^{V+2U}\right) - ie^{-2U-V}\gamma_5(\not{\partial}h)\Bigg]\xi \\
&\quad + \frac{1}{2}\Bigg[i\gamma_5\gamma^a e^{-2W-V}\not{H}_3 + 6i\gamma^a\left(\not{\partial}U\right) + i\gamma^a\left(fe^{-3W} + 6e^{W+V-2U}\right) \\
&\qquad + 12e^{W-4U}\gamma_5\left(h - i\gamma_5 e^{V+2U}\right)\gamma^a\Bigg]\zeta_a \\
&\quad + \frac{3}{2}\Bigg[\left(fe^{-3W} + 6e^{W+V-2U}\right) - 4ie^{W-4U}\gamma_5\left(h - i\gamma_5 e^{V+2U}\right)\Bigg]\eta \\
&\quad - \frac{e^{-3U}}{2}\gamma_5\gamma^a\Big[(\not{D}X) + 4Xe^{W-V}\Big]\zeta_a^{\mathbf{c}} - 6iXe^{W-3U-V}\gamma_5\eta^{\mathbf{c}}\,.
\end{aligned}
\tag{H.7}
$$

We recall that all the fermions have charge ± 2 with respect to the graviphoton, so that $D_a = \nabla_a - 2iA_a$ when acting on ζ, η, ξ, while the complex scalar X has charge -4, i.e. $DX = dX - 4iAX$. Naturally, the equations of motion for the charge conjugate fields $\zeta_a^{\mathbf{c}}, \eta^{\mathbf{c}}, \xi^{\mathbf{c}}$ can be obtained by taking the complex conjugate of the equations above and using the rules given in section A.2.5. Alternatively, the above equations can be obtained directly by taking functional derivatives of the effective action (IV.39).

APPENDIX I

Type IIB supergravity

In this appendix we briefly review the field content and equations of motion of type IIB supergravity [96, 148]. We follow the conventions of [67], [63], [66] closely, and adapt our fermionic conventions accordingly.

I.1 Bosonic content and equations of motion

In the $SU(1,1)$ language of [148], the bosonic content of type IIB supergravity includes the metric, a complex scalar B, "composite" complex 1-forms P and Q (that can be written in terms of B), a complex 3-form G, and a real self dual five-form $F_{(5)}$. The corresponding equations of motion read (to linear order in the fermions)

$$D * P = -\frac{1}{4} G \wedge *G \tag{I.1}$$

$$D * G = P \wedge *G^* - iG \wedge F_{(5)} \tag{I.2}$$

$$\begin{aligned} R_{MN} &= P_M P_N^* + P_N P_M^* + \frac{1}{96} F_{(5)MP_1P_2P_3P_4} F_{(5)N}{}^{P_1P_2P_3P_4} \\ &+ \frac{1}{8}\left(G_M{}^{P_1P_2} G^*_{NP_1P_2} + G_N{}^{P_1P_2} G^*_{MP_1P_2} - \frac{1}{6} g_{MN} G^{P_1P_2P_3} G^*_{P_1P_2P_3} \right) \end{aligned} \tag{I.3}$$

together with the self-duality condition $*F_{(5)} = F_{(5)}$. Similarly, the Bianchi identities read

$$dF_{(5)} - \frac{i}{2} G \wedge G^* = 0 \tag{I.4}$$

$$DG + P \wedge G^* = 0 \tag{I.5}$$

$$DP = 0\,. \tag{I.6}$$

In this language there is a manifest local $U(1)$ invariance and Q is the corresponding gauge field, with field-strength $dQ = -iP \wedge P^*$. Similarly, G has charge 1 and P has charge 2 under the $U(1)$, so $D * G \equiv d * G - iQ \wedge *G$ and $D * P \equiv d * P - 2iQ \wedge *P$. Notice that Einstein's equation (I.3) has been rewritten by using the trace condition $R = 2P^R P^*_R + \frac{1}{24} G^{P_1P_2P_3} G^*_{P_1P_2P_3}$.

In the body of the paper we have worked in the $SL(2,\mathbb{R})$ language which is more familiar to string theorists. The translation between the two formalisms involves a gauge-transformation and field-redefinitions.[1] Here we just quote the result that links this formalism with the fields used in the rest of the paper. Writing the axion-dilaton τ and the NSNS and RR 3-forms $H_{(3)}$ and $F_{(3)}$ as

$$\tau \equiv C_{(0)} + ie^{-\Phi}\,, \qquad F_{(3)} = dC_{(2)} - C_{(0)} dB_{(2)}\,, \qquad H_{(3)} = dB_{(2)}\,, \tag{I.7}$$

for the 3-form G we have[2] [63]

$$G = ie^{\Phi/2}\left(\tau dB - dC_{(2)}\right) = -\left(e^{-\Phi/2} H_{(3)} + ie^{\Phi/2} F_{(3)}\right), \tag{I.8}$$

and similarly

$$P = \frac{i}{2} e^{\Phi} d\tau\,, \qquad Q = -\frac{1}{2} e^{\Phi} dC_{(0)}\,. \tag{I.9}$$

In terms of these fields, the equations of motion (I.1)-(I.3) become [67] (to linear order in the

[1] The gauge transformation has the form $P \to e^{2i\theta} P$, $Q \to Q + d\theta$, $G \to e^{\frac{i}{2}\theta} G$, where θ is a τ-dependent phase. These phases are then absorbed by a redefinition of the fermions. More details can be found in [35] [29], for example.

[2] Note that our forms $F_{(3)}$ and G are related to the traditional string theory forms $F_{(3)st} = dC_{(2)}$ and $G_{st} = F_{(3)st} - \tau H_{(3)}$ by $F_{(3)} = F_{(3)st} - C_{(0)} H_{(3)}$ and $G = -iG_{st}/\sqrt{\mathrm{Im}\tau}$. It's not our fault.

fermions)

$$0 = d(e^{\Phi} * F_{(3)}) - F_{(5)} \wedge H_{(3)} \tag{I.10}$$

$$0 = d(e^{2\Phi} * F_{(1)}) + e^{\Phi} H_{(3)} \wedge *F_{(3)} \tag{I.11}$$

$$0 = d(e^{-\Phi} * H_{(3)}) - e^{\Phi} F_{(1)} \wedge *F_{(3)} - F_{(3)} \wedge F_{(5)} \tag{I.12}$$

$$0 = d * d\Phi - e^{2\Phi} F_{(1)} \wedge *F_{(1)} + \frac{1}{2} e^{-\Phi} H_{(3)} \wedge *H_{(3)} - \frac{1}{2} e^{\Phi} F_{(3)} \wedge *F_{(3)} \tag{I.13}$$

$$\begin{aligned} R_{MN} &= \frac{1}{2} e^{2\Phi} \nabla_M C_{(0)} \nabla_N C_{(0)} + \frac{1}{2} \nabla_M \Phi \nabla_N \Phi + \frac{1}{96} F_{MP_1P_2P_3P_4} F_N{}^{P_1P_2P_3P_4} \\ &\quad + \frac{1}{4} e^{-\Phi} \left(H_M{}^{P_1P_2} H_{NP_1P_2} - \frac{1}{12} g_{MN} H^{P_1P_2P_3} H_{P_1P_2P_3} \right) \\ &\quad + \frac{1}{4} e^{\Phi} \left(F_M{}^{P_1P_2} F_{NP_1P_2} - \frac{1}{12} g_{MN} F^{P_1P_2P_3} F_{P_1P_2P_3} \right) \end{aligned} \tag{I.14}$$

while the Bianchi identities (I.4)-(I.6) now read

$$dF_{(5)} + F_{(3)} \wedge H_{(3)} = 0 \tag{I.15}$$

$$dF_{(3)} + F_{(1)} \wedge H_{(3)} = 0 \tag{I.16}$$

$$dF_{(1)} = 0 \tag{I.17}$$

$$dH_{(3)} = 0\,. \tag{I.18}$$

These identities are solved by writing $F_{(5)} = dC_{(4)} - C_{(2)} \wedge H_{(3)}$, $F_{(1)} = dC_{(0)}$, together with $H_{(3)} = dB_{(2)}$ and $F_{(3)} = dC_{(2)} - C_{(0)} dB_{(2)}$ as in (I.7).

I.2 Fermionic content and equations of motion

Our conventions for the type IIB fermionic sector are based on those of [35], [7], with slight modifications needed to conform with our bosonic conventions. The type IIB fermionic content consists of a chiral dilatino λ and a chiral gravitino Ψ, with equations of motion given by (to linear order in the fermions)

$$\hat{\not{D}}\lambda = \frac{i}{8}\not{F}_{(5)}\lambda + \mathcal{O}(\Psi^2) \tag{I.19}$$

$$\Gamma^{ABC}\hat{D}_B\Psi_C = -\frac{1}{8}\not{G}^*\Gamma^A\lambda + \frac{1}{2}\not{P}\Gamma^A\lambda^{\mathbf{c}} + \mathcal{O}(\Psi^3) \tag{I.20}$$

Here, $\hat{D}$ denotes the flux-dependent supercovariant derivative, which acts as follows:

$$\hat{\not{D}}\lambda = \left(\hat{\not{\nabla}} - \frac{3i}{2}\not{Q}\right)\lambda - \frac{1}{4}\Gamma^A\not{G}\Psi_A - \Gamma^A\not{P}\Psi_A^{\mathbf{c}} \tag{I.21}$$

$$\hat{D}_B\Psi_C = \left(\hat{\nabla}_B - \frac{i}{2}Q_B\right)\Psi_C + \frac{i}{16}\not{F}_{(5)}\Gamma_B\Psi_C - \frac{1}{16}S_B\Psi_C^{\mathbf{c}}\,, \tag{I.22}$$

where $\hat{\nabla}_B$ denotes the ordinary 10-d spinor covariant derivative and we have defined

$$S_B \equiv \frac{1}{6}\left(\Gamma_B{}^{DEF}G_{DEF} - 9\Gamma^{DE}G_{BDE}\right). \tag{I.23}$$

The gravitino and dilatino have opposite chirality in $d = 10$, and we choose $\Gamma_{11}\Psi_A = -\Psi_A$, $\Gamma_{11}\lambda = +\lambda$. Since $F_{(5)}$ is self-dual, our conventions then imply $\not{F}_{(5)} = -\Gamma_{11}\not{F}_{(5)}$. Thus, for any spinor ε satisfying $\Gamma_{11}\varepsilon = -\varepsilon$ we have $\not{F}_{(5)}\varepsilon = 0$ and $\not{F}_{(5)}\Gamma_A\varepsilon = \{\not{F}_{(5)}, \Gamma_A\}\varepsilon = \frac{1}{12}F_{(5)ACDEF}\Gamma^{CDEF}\varepsilon$. The corresponding SUSY variations of the fermions read

$$\delta\lambda = \not{P}\varepsilon^{\mathbf{c}} + \frac{1}{4}\not{G}\varepsilon \tag{I.24}$$

$$\delta\Psi_A = \left(\hat{\nabla}_A - \frac{i}{2}Q_A\right)\varepsilon + \frac{i}{16}\not{F}_{(5)}\Gamma_A\varepsilon - \frac{1}{16}S_A\varepsilon^{\mathbf{c}}\,. \tag{I.25}$$

APPENDIX J

$d = 5$ equations of motion

In this appendix we present the dimensional reduction of the fermionic equations of motion in full detail, and rewrite them in final form in terms of the fields (V.47)-(V.50) which possess diagonal kinetic terms in the effective action. In the calculations below we encounter a number of expressions involving $\varepsilon_\pm$ that need evaluation. We collect them here:

$$\not{J}\varepsilon_+ = \frac{1}{2}iQ\varepsilon_+ = 2i\varepsilon_+ \qquad \not{J}\varepsilon_- = \frac{1}{2}iQ\varepsilon_- = -2i\varepsilon_- \tag{J.1}$$

$$\not{\Omega}\varepsilon_- e^{-\frac{3}{2}i\chi} = 4\varepsilon_+ e^{\frac{3}{2}i\chi} \qquad \overline{\not{\Omega}}\varepsilon_+ e^{\frac{3}{2}i\chi} = -4\varepsilon_- e^{-\frac{3}{2}i\chi} \tag{J.2}$$

$$\gamma^\alpha\gamma_\alpha\varepsilon_+ = 4\varepsilon_+ \qquad \gamma^\alpha\not{J}\gamma_\alpha\varepsilon_+ = \gamma^{\bar\alpha}\not{\Omega}\gamma_{\bar\alpha}\varepsilon_- = 0\,. \tag{J.3}$$

J.1 Reduction of the dilatino equation of motion

We begin by performing the reduction of the $D = 10$ equation of motion for the dilatino, as given in (I.19).

J.1.1 Derivative operator

We first reduce the 10-d derivative operator $\hat\nabla_A - (3i/2)Q_A$ acting on the dilatino. Defining

$$e^W\left(\hat{\not{\nabla}} - \frac{3i}{2}\not{Q}\right)\lambda \equiv \mathcal{L}^+_\lambda \otimes \varepsilon_+ e^{\frac{3i}{2}\chi} \otimes u_- + \mathcal{L}^-_\lambda \otimes \varepsilon_- e^{-\frac{3i}{2}\chi} \otimes u_- \tag{J.4}$$

we find

$$\mathcal{L}^\pm_\lambda = \left(\not{D} + \frac{1}{2}\not{\partial}W + \frac{3}{4}ie^\phi(\not{\partial}a)\right)\lambda^{(\pm)} + \frac{1}{4}i\Sigma^{-2}\not{F}_2\lambda^{(\pm)} \mp \left(e^{-4U}\Sigma^{-1} + \frac{3}{2}\Sigma^2\right)\lambda^{(\pm)}\,, \tag{J.5}$$

where $\not{D}\lambda^{(\pm)} = \left(\not{\nabla} \mp \frac{3}{2}i\not{A}_1\right)\lambda^{(\pm)}$ is the gauge-covariant five-dimensional connection acting on $\lambda^{(\pm)}$.

J.1.2 Couplings

We now reduce the various terms involving the couplings of the dilatino, including the flux-dependent terms in the supercovariant derivative. Defining

$$e^W\left(\frac{i}{8}\not{F}_{(5)}\lambda + \frac{1}{4}\Gamma^A\not{G}\Psi_A\right) \equiv \mathcal{R}^+_{1\lambda}\otimes\varepsilon_+ e^{\frac{3i}{2}\chi}\otimes u_- + \mathcal{R}^-_{1\lambda}\otimes\varepsilon_- e^{-\frac{3i}{2}\chi}\otimes u_- \tag{J.6}$$

we find

$$\begin{aligned}\mathcal{R}^{(\pm)}_{1\lambda} = {}& e^{Z+4W}\lambda^{(\pm)} - \frac{1}{2}ie^{-4U}\not{K}_1\lambda^{(\pm)} \mp \frac{1}{2}i\Sigma\not{K}_2\lambda^{(\pm)} \mp \Sigma\not{L}^{(\pm)}_2\lambda^{(\mp)}\\ & - \frac{1}{4}i\gamma^a\not{G}_3\psi^{(\pm)}_a - \frac{1}{4}\gamma^a\not{G}_2\psi^{(\pm)}_a + \frac{1}{4}\not{G}_3\left(\varphi^{(\pm)} + 4\rho^{(\pm)}\right) - \frac{1}{4}i\not{G}_2\left(\varphi^{(\pm)} - 4\rho^{(\pm)}\right)\\ & \pm\frac{1}{2}\gamma^a\not{G}_1\psi^{(\pm)}_a \pm \frac{1}{2}i\not{G}_1\varphi^{(\pm)} \mp i\gamma^a\not{N}^{(\pm)}_1\psi^{(\mp)}_a \pm \not{N}^{(\pm)}_1\varphi^{(\mp)}\\ & \mp\gamma^a\mathcal{N}^{(\pm)}_0\psi^{(\mp)}_a \mp i\mathcal{N}^{(\pm)}_0\varphi^{(\mp)}\,,\end{aligned} \tag{J.7}$$

where we have introduced the notation $\not{L}^{(+)}_2 = (1/2!)L_{2\,ab}\gamma^{ab}$ and $\not{L}^{(-)}_2 = (1/2!)L^*_{2\,ab}\gamma^{ab}$. Similarly, defining

$$e^W\Gamma^A\not{P}\Psi^{\mathbf{c}}_A \equiv \mathcal{R}^+_{2\lambda}\otimes\varepsilon_+ e^{\frac{3i}{2}\chi}\otimes u_- + \mathcal{R}^-_{2\lambda}\otimes\varepsilon_- e^{-\frac{3i}{2}\chi}\otimes u_- \tag{J.8}$$

we obtain

$$\mathcal{R}^{(\pm)}_{2\lambda} = \pm\not{P}\psi^{(\mp)\mathbf{c}}_a \pm i\not{P}\left(4\rho^{(\mp)\mathbf{c}} + \varphi^{(\mp)\mathbf{c}}\right). \tag{J.9}$$

where, in a slight abuse of notation, $\not{P} = (1/2)\left(\not{\partial}\phi + ie^{\phi}\not{\partial}a\right)$ when appearing in 5-d equations. In terms of the quantities computed above, the 10-d dilatino equation reduces to two equations for the five-dimensional fields, given by

$$\mathcal{L}^{(\pm)}_\lambda - \mathcal{R}^{(\pm)}_{1\lambda} - \mathcal{R}^{(\pm)}_{2\lambda} = 0\,. \tag{J.10}$$

J.2 Reduction of the gravitino equation of motion

We now reduce the equation of motion for the $D = 10$ gravitino, as given in (I.20).

J.2.1 Derivative operator

Here we define

$$e^W\Gamma^{aBC}\left(\hat\nabla_B-\frac{i}{2}Q_B\right)\Psi_C=\mathcal{L}^{(+)a}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+ +\mathcal{L}^{(-)a}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.11}$$

$$e^W\tilde\sigma_2\Gamma_\alpha\Gamma^{\alpha BC}\left(\hat\nabla_B-\frac{i}{2}Q_B\right)\Psi_C=\mathcal{L}^{(+)}_{base}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+ +\mathcal{L}^{(-)}_{base}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.12}$$

$$e^W\Gamma^{\mathrm{f}BC}\left(\hat\nabla_B-\frac{i}{2}Q_B\right)\Psi_C=\mathcal{L}^{(+)}_{\mathrm{f}}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+ +\mathcal{L}^{(-)}_{\mathrm{f}}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.13}$$

where $\tilde\sigma_2\equiv\mathbb{1}_4\otimes\mathbb{1}_4\otimes\sigma_2$. Then, for the components of the derivative operator in the external manifold directions we find

$$\begin{aligned}\mathcal{L}^{(\pm)a}=&\gamma^{abc}\left(D_b+\frac{1}{2}\partial_bW+\frac{1}{4}ie^{\phi}(\partial_ba)\right)\psi_c^{(\pm)}\\&-\frac{1}{4}i\Sigma^{-2}\gamma^{[c}\not{F}_2\gamma^{a]}\psi_c^{(\pm)}\mp\left(\Sigma^{-1}e^{-4U}+\frac{3}{2}\Sigma^2\right)\gamma^{ab}\psi_b^{(\pm)}\\&-4i\gamma^{ab}\left[D_b+\frac{1}{2}\partial_bW+\frac{1}{4}ie^{\phi}(\partial_ba)\right]\rho^{(\pm)}-i(\Sigma^{-1}\not{\partial}\Sigma)\gamma^a\rho^{(\pm)}+4i(\not{\partial}U)\gamma^a\rho^{(\pm)}\\&\pm2i\left(3\Sigma^2+\Sigma^{-1}e^{-4U}\right)\gamma^a\rho^{(\pm)}-\frac{1}{2}\Sigma^{-2}F_{2\,bd}\gamma^b\gamma^a\gamma^d\rho^{(\pm)}\\&-i\gamma^{ab}\left[D_b+\frac{1}{2}\partial_bW+\frac{1}{4}ie^{\phi}(\partial_ba)\right]\varphi^{(\pm)}-i(\Sigma^{-1}\not{\partial}\Sigma)\gamma^a\varphi^{(\pm)}\\&\pm2i\Sigma^{-1}e^{-4U}\gamma^a\varphi^{(\pm)}+\frac{1}{4}\Sigma^{-2}F_{2\,bc}\gamma^c\gamma^{ab}\varphi^{(\pm)}\,.\end{aligned}\tag{J.14}$$

Similarly, the components in the direction of the KE base yield

$$\begin{aligned}\mathcal{L}^{(\pm)}_{base}=&-4i\gamma^{ab}\left[D_a+\frac{1}{2}\partial_aW+\frac{1}{4}ie^{\phi}(\partial_aa)\right]\psi_b^{(\pm)}+i\gamma^b(\Sigma^{-1}\not{\partial}\Sigma)\psi_b^{(\pm)}-4i\gamma^b(\not{\partial}U)\psi_b^{(\pm)}\\&+\frac{1}{2}\Sigma^{-2}F_{2\,da}\gamma^a\gamma^b\gamma^d\psi_b^{(\pm)}\pm2i\left(\Sigma^{-1}e^{-4U}+3\Sigma^2\right)\gamma^b\psi_b^{(\pm)}\\&-12\left[\not{D}+\frac{1}{2}(\not{\partial}W)+\frac{1}{4}ie^{\phi}(\not{\partial}a)\right]\rho^{(\pm)}\pm2\left(2\Sigma^{-1}e^{-4U}+9\Sigma^2\right)\rho^{(\pm)}-3i\Sigma^{-2}\not{F}_2\rho^{(\pm)}\\&-4\left[\not{D}+\frac{1}{2}\not{\partial}W+\frac{1}{4}ie^{\phi}(\not{\partial}a)-\frac{3}{4}(\Sigma^{-1}\not{\partial}\Sigma)-(\not{\partial}U)\right]\varphi^{(\pm)}\\&\pm2\Sigma^{-1}e^{-4U}\varphi^{(\pm)}-2i\Sigma^{-2}\not{F}_2\varphi^{(\pm)}\,.\end{aligned}\tag{J.15}$$

Finally, for the fiber component of the derivative operator we obtain

$$
\begin{aligned}
\mathcal{L}_{\mathrm{f}}^{(\pm)} =& - i\gamma^{ab}\left[D_a + \frac{1}{2}\partial_a W + \frac{1}{4}ie^{\phi}(\partial_a a)\right]\psi_b^{(\pm)} + i\gamma^b(\Sigma^{-1}\not{\partial}\Sigma)\psi_b^{(\pm)} \\
& \pm 2i\Sigma^{-1}e^{-4U}\gamma^b\psi_b^{(\pm)} + \frac{1}{4}\Sigma^{-2}F_{2\,da}\gamma^{ab}\gamma^d\psi_b^{(\pm)} \\
& - 4\left[\not{D} + \frac{1}{2}\not{\partial}W + \frac{1}{4}ie^{\phi}(\not{\partial}a) + \frac{3}{4}(\Sigma^{-1}\not{\partial}\Sigma) + \not{\partial}U\right]\rho^{(\pm)} \\
& \pm 2\Sigma^{-1}e^{-4U}\left(2\varphi^{(\pm)} + \rho^{(\pm)}\right) - i\not{F}_2\Sigma^{-2}\left(\varphi^{(\pm)} + 2\rho^{(\pm)}\right).
\end{aligned}
\tag{J.16}
$$

J.2.2 Couplings

Next, define

$$
\begin{aligned}
e^W\left(-\frac{1}{8}\not{G}^*\Gamma^a\lambda - \frac{i}{16}\Gamma^{aBC}\not{F}_{(5)}\Gamma_B\Psi_C\right) =& \mathcal{R}_1^{(+)a}\otimes\varepsilon_+ e^{\frac{3i}{2}\chi}\otimes u_+ \\
& + \mathcal{R}_1^{(-)a}\otimes\varepsilon_- e^{-\frac{3i}{2}\chi}\otimes u_+
\end{aligned}
\tag{J.17}
$$

$$
\begin{aligned}
e^W\left(-\frac{1}{8}\tilde{\sigma}_2\Gamma_\alpha\not{G}^*\Gamma^\alpha\lambda - \frac{i}{16}\tilde{\sigma}_2\Gamma_\alpha\Gamma^{\alpha BC}\not{F}_{(5)}\Gamma_B\Psi_C\right) =& \mathcal{R}_{1\,base}^{(+)}\otimes\varepsilon_+ e^{\frac{3i}{2}\chi}\otimes u_+ \\
& + \mathcal{R}_{1\,base}^{(-)}\otimes\varepsilon_- e^{-\frac{3i}{2}\chi}\otimes u_+
\end{aligned}
\tag{J.18}
$$

$$
\begin{aligned}
e^W\left(-\frac{1}{8}\not{G}^*\Gamma^{\mathrm{f}}\lambda - \frac{i}{16}\Gamma^{\mathrm{f}BC}\not{F}_{(5)}\Gamma_B\Psi_C\right) =& \mathcal{R}_{1\,\mathrm{f}}^{(+)}\otimes\varepsilon_+ e^{\frac{3i}{2}\chi}\otimes u_+ \\
& + \mathcal{R}_{1\,\mathrm{f}}^{(-)}\otimes\varepsilon_- e^{-\frac{3i}{2}\chi}\otimes u_+\,.
\end{aligned}
\tag{J.19}
$$

We find

$$\mathcal{R}_1^{(\pm)a} = \left(-\frac{1}{8}i\tilde{\mathcal{G}}_3 \pm \frac{1}{4}\tilde{\mathcal{G}}_1 - \frac{1}{8}\tilde{\mathcal{G}}_2\right)\gamma^a\lambda^{(\pm)} \mp \left(\frac{1}{2}i\tilde{\mathcal{N}}_1^{(\pm)} + \frac{1}{2}\tilde{\mathcal{N}}_0^{(\pm)}\right)\gamma^a\lambda^{(\mp)}$$
$$+ e^{Z+4W}\gamma^{ba}\psi_b^{(\pm)} - \frac{1}{2}ie^{-4U}\gamma^{[b}\not{K}_1\gamma^{a]}\psi_b^{(\pm)} + e^{-4U}\{\not{K}_1,\gamma^a\}\rho^{(\pm)} - \frac{1}{4}e^{-4U}[\not{K}_1,\gamma^a]\varphi^{(\pm)}$$
$$\mp \frac{1}{2}i\Sigma\gamma^{[b}\not{K}_2\gamma^{a]}\psi_b^{(\pm)} \mp \Sigma\gamma^{[b}\not{L}_2^{(\pm)}\gamma^{a]}\psi_b^{(\mp)} - \frac{1}{4}\Sigma\left(\pm[\not{K}_2,\gamma^a]\varphi^{(\pm)} \mp 2i[\not{L}_2^{(\pm)},\gamma^a]\varphi^{(\mp)}\right)$$
$$+ \Sigma\gamma^a\left(\pm\not{K}_2\rho^{(\pm)} \mp 2i\not{L}_2^{(\pm)}\rho^{(\mp)}\right) \tag{J.20}$$

$$\mathcal{R}_{1\,base}^{(\pm)} = \left(\frac{1}{2}\tilde{\mathcal{G}}_3 + \frac{i}{2}\tilde{\mathcal{G}}_2\right)\lambda^{(\pm)} + e^{-4U}\{\gamma^b,\not{K}_1\}\psi_b^{(\pm)} - 6ie^{-4U}\not{K}_1\rho^{(\pm)} + 4e^{Z+4W}(\varphi^{(\pm)} + 3\rho^{(\pm)})$$
$$- \Sigma\left[\pm i\not{K}_2\left(i\gamma^a\psi_a^{(\pm)} + \varphi^{(\pm)} + 2\rho^{(\pm)}\right) \pm 2\not{L}_2^{(\pm)}\left(i\gamma^a\psi_a^{(\mp)} + \varphi^{(\mp)} + 2\rho^{(\mp)}\right)\right] \tag{J.21}$$

$$\mathcal{R}_{1\,\mathrm{f}}^{(\pm)} = \left(\frac{1}{8}\tilde{\mathcal{G}}_3 \pm \frac{1}{4}i\tilde{\mathcal{G}}_1 - \frac{1}{8}i\tilde{\mathcal{G}}_2\right)\lambda^{(\pm)} \pm \left(\frac{1}{2}\tilde{\mathcal{N}}_1^{(\pm)} - \frac{1}{2}i\tilde{\mathcal{N}}_0^{(\pm)}\right)\lambda^{(\mp)}$$
$$- \frac{1}{4}e^{-4U}[\gamma^b,\not{K}_1]\psi_b^{(\pm)} \mp \frac{1}{4}\Sigma[\gamma^b,\not{K}_2]\psi_b^{(\pm)} \pm \frac{1}{2}i\Sigma[\gamma^b,\not{L}_2^{(\pm)}]\psi_b^{(\mp)}$$
$$+ 4e^{Z+4W}\rho^{(\pm)} \mp i\Sigma\not{K}_2\rho^{(\pm)} \mp 2\Sigma\not{L}_2^{(\pm)}\rho^{(\mp)}\,. \tag{J.22}$$

We now reduce the couplings to the charge conjugate spinors in the gravitino equation. We write

$$\frac{1}{2}e^W\not{P}\Gamma^a\lambda^{\mathbf{c}} + \frac{1}{16}e^W\Gamma^{aBC}S_B\Psi_C^{\mathbf{c}} = \mathcal{R}_2^{(+)a}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+$$
$$+ \mathcal{R}_2^{(-)a}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.23}$$
$$\frac{1}{2}e^W\tilde{\sigma}_2\Gamma_\alpha\not{P}\Gamma^\alpha\lambda^{\mathbf{c}} + \frac{1}{16}e^W\tilde{\sigma}_2\Gamma_\alpha\Gamma^{\alpha BC}S_B\Psi_C^{\mathbf{c}} = \mathcal{R}_{2\,base}^{(+)}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+$$
$$+ \mathcal{R}_{2\,base}^{(-)}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.24}$$
$$\frac{1}{2}e^W\not{P}\Gamma^{\mathrm{f}}\lambda^{\mathbf{c}} + \frac{1}{16}e^W\Gamma^{\mathrm{f}BC}S_B\Psi_C^{\mathbf{c}} = \mathcal{R}_{2\,\mathrm{f}}^{(+)}\otimes\varepsilon_+e^{\frac{3i}{2}\chi}\otimes u_+$$
$$+ \mathcal{R}_{2\,\mathrm{f}}^{(-)}\otimes\varepsilon_-e^{-\frac{3i}{2}\chi}\otimes u_+ \tag{J.25}$$

obtaining

$$\begin{aligned}
\mathcal{R}_2^{(\pm)a} = & \pm \frac{1}{2} \not{P} \gamma^a \lambda^{(\pm)\mathbf{c}} \pm \frac{1}{8} i \mathcal{G}_3{}^{ebc} \left(\delta_e^d \gamma^a \gamma_{bc} - \delta_e^a \gamma^d \gamma_{bc} - \frac{1}{3} \gamma^{ad} \gamma_{ebc} \right) \psi_d^{(\mp)\mathbf{c}} \\
& \pm \frac{1}{4} \mathcal{G}_2{}^{eb} \left(\delta_e^d \gamma^a \gamma_b - \delta_e^a \gamma^d \gamma_b - \frac{1}{2} \gamma^{ad} \gamma_{eb} \right) \psi_d^{(\mp)\mathbf{c}} \\
& - \frac{1}{2} \mathcal{G}_{1e} \gamma^{eda} \psi_d^{(\mp)\mathbf{c}} - i \mathcal{N}_{1e}^{(\pm)} \gamma^{eda} \psi_d^{(\pm)\mathbf{c}} + \mathcal{N}_0^{(\pm)} \gamma^{ab} \psi_b^{(\pm)\mathbf{c}} \\
& \pm \frac{1}{24} \mathcal{G}_{3ebc} \gamma^{aebc} \left(\varphi^{(\mp)\mathbf{c}} + 4 \rho^{(\mp)\mathbf{c}} \right) \mp \frac{1}{8} i \mathcal{G}_{2eb} \gamma^{aeb} \left(\varphi^{(\mp)\mathbf{c}} - 4 \rho^{(\mp)\mathbf{c}} \right) \\
& \pm \frac{1}{4} i \gamma^a \not{\mathcal{G}}_2 \varphi^{(\mp)\mathbf{c}} - \frac{1}{2} i \mathcal{G}_{1b} \gamma^{ba} \varphi^{(\mp)\mathbf{c}} + i \gamma^a \not{\mathcal{G}}_1 \rho^{(\mp)\mathbf{c}} \\
& + \mathcal{N}_{1b}^{(\pm)} \gamma^{ba} \varphi^{(\pm)\mathbf{c}} - 2 \gamma^a \not{\mathcal{N}}_1^{(\pm)} \rho^{(\pm)\mathbf{c}} - 2 i e \gamma^a \mathcal{N}_0^{(\pm)} \rho^{(\pm)\mathbf{c}} ,
\end{aligned} \tag{J.26}$$

$$\begin{aligned}
\mathcal{R}_{2\,base}^{(\pm)} = & \pm 2 i \not{P} \lambda^{(\pm)\mathbf{c}} + 2 i \not{\mathcal{N}}_1^{(\pm)} \left(\varphi^{(\pm)\mathbf{c}} + 2 \rho^{(\pm)\mathbf{c}} \right) - 2 \mathcal{N}_0^{(\pm)} \left(\varphi^{(\pm)\mathbf{c}} + 2 \rho^{(\pm)\mathbf{c}} \right) \\
& + \not{\mathcal{G}}_1 \left(\varphi^{(\mp)\mathbf{c}} + 2 \rho^{(\mp)\mathbf{c}} \right) \mp i \not{\mathcal{G}}_3 \left(\varphi^{(\mp)\mathbf{c}} + 3 \rho^{(\mp)\mathbf{c}} \right) \\
& - 2 i \mathcal{N}_0^{(\pm)} \gamma^d \psi_d^{(\pm)\mathbf{c}} - 2 \not{\mathcal{N}}_1^{(\pm)} \gamma^d \psi_d^{(\pm)\mathbf{c}} + i \not{\mathcal{G}}_1 \gamma^d \psi_d^{(\mp)\mathbf{c}} \\
& \mp \frac{1}{6} \mathcal{G}_{3ebc} \gamma^{debc} \psi_d^{(\mp)\mathbf{c}} \pm \frac{1}{2} i \mathcal{G}_{2eb} \gamma^{deb} \psi_d^{(\mp)\mathbf{c}}
\end{aligned} \tag{J.27}$$

and

$$\begin{aligned}
\mathcal{R}_{2\,\mathrm{f}}^{(\pm)} = & \pm i \frac{1}{2} \not{P} \lambda^{(\pm)\mathbf{c}} \mp \frac{1}{24} \mathcal{G}_{3ebc} \gamma^{debc} \psi_d^{(\mp)\mathbf{c}} \mp \frac{1}{4} i \mathcal{G}_2{}^{db} \gamma_b \psi_d^{(\mp)\mathbf{c}} + \frac{1}{2} i \mathcal{G}_{1e} \gamma^{ed} \psi_d^{(\mp)\mathbf{c}} \\
& - \mathcal{N}_{1e}^{(\pm)} \gamma^{ed} \psi_d^{(\pm)\mathbf{c}} \mp i \not{\mathcal{G}}_3 \rho^{(\mp)\mathbf{c}} + \not{\mathcal{G}}_1 \rho^{(\mp)\mathbf{c}} + 2 i \not{\mathcal{N}}_1^{(\pm)} \rho^{(\pm)\mathbf{c}} - 2 \mathcal{N}_0^{(\pm)} \rho^{(\pm)\mathbf{c}} .
\end{aligned} \tag{J.28}$$

In terms of the quantities computed above, the 10-d gravitino equation reduces to the following set of equations for the five-dimensional fields:

$$0 = \mathcal{L}^{(\pm)a} - \mathcal{R}_1^{(\pm)a} - \mathcal{R}_2^{(\pm)a} \tag{J.29}$$

$$0 = \mathcal{L}_{base}^{(\pm)} - \mathcal{R}_{1\,base}^{(\pm)} - \mathcal{R}_{2\,base}^{(\pm)} \tag{J.30}$$

$$0 = \mathcal{L}_{\mathrm{f}}^{(\pm)} - \mathcal{R}_{1\,\mathrm{f}}^{(\pm)} - \mathcal{R}_{2\,\mathrm{f}}^{(\pm)} . \tag{J.31}$$

Instead of working with the equations of motion given in this form, it is convenient to rewrite them in terms of the fields (V.47)-(V.50) whose kinetic terms are diagonal. We do so below.

J.3 Equations of motion in terms of diagonal fields

The $d = 5$ equations of motion for the diagonal fields (V.47)-(V.50) are given by

$$0 = \mathcal{L}^{(\pm)}_{\tilde{\lambda}} - \mathcal{R}^{(\pm)}_{1\tilde{\lambda}} - \mathcal{R}^{(\pm)}_{2\tilde{\lambda}} \tag{J.32}$$

$$0 = \mathcal{L}^{(\pm)a}_{\zeta} - \mathcal{R}^{(\pm)a}_{1\,\zeta} - \mathcal{R}^{(\pm)a}_{2\,\zeta} \tag{J.33}$$

$$0 = \mathcal{L}^{(\pm)}_{\eta} - \mathcal{R}^{(\pm)}_{1\,\eta} - \mathcal{R}^{(\pm)}_{2\,\eta} \tag{J.34}$$

$$0 = \mathcal{L}^{(\pm)}_{\xi} - \mathcal{R}^{(\pm)}_{1\,\xi} - \mathcal{R}^{(\pm)}_{2\,\xi} \tag{J.35}$$

Here,

$$\mathcal{L}^{(\pm)}_{\tilde{\lambda}} = e^{W/2}\mathcal{L}^{(\pm)}_{\lambda} \tag{J.36}$$

$$= \not{D}\tilde{\lambda}^{(\pm)} + \frac{1}{4}i\Sigma^{-2}\not{F}_2\tilde{\lambda}^{(\pm)} \mp \left(e^{-4U}\Sigma^{-1} + \frac{3}{2}\Sigma^2\right)\tilde{\lambda}^{(\pm)} + \frac{3}{4}ie^{\phi}(\not{\partial}a)\tilde{\lambda}^{(\pm)} \tag{J.37}$$

where now $\not{D}\tilde{\lambda}^{(\pm)} = \left(\not{\nabla} \mp \frac{3i}{2}\not{A}\right)\tilde{\lambda}^{(\pm)}$ and

$$\mathcal{R}^{(\pm)}_{1\tilde{\lambda}} = e^{W/2}\mathcal{R}^{(\pm)}_{1\lambda} \tag{J.38}$$

$$\begin{aligned} &= \left(e^{Z+4W} - \frac{1}{2}ie^{-4U}\not{K}_1 \mp \frac{1}{2}i\Sigma\not{K}_2\right)\tilde{\lambda}^{(\pm)} \mp \Sigma\not{L}^{(\pm)}_2\tilde{\lambda}^{(\mp)} \\ &\quad + \left(-\frac{1}{4}i\gamma^a\not{G}_3 - \frac{1}{4}\gamma^a\not{G}_2 \pm \frac{1}{2}\gamma^a\not{G}_1\right)\zeta^{(\pm)}_a \mp \left(i\gamma^a\not{N}^{(\pm)}_1 + \gamma^a\not{N}^{(\pm)}_0\right)\zeta^{(\mp)}_a \\ &\quad + \left(\frac{1}{6}\not{G}_3 - \frac{1}{6}i\not{G}_2\right)\eta^{(\pm)} \mp \frac{4}{3}i\mathcal{N}^{(\pm)}_0\eta^{(\mp)} + \frac{1}{4}\left(\not{G}_3 + i\not{G}_2 \mp 2i\not{G}_1\right)\xi^{(\pm)} \\ &\quad \mp \left(\not{N}^{(\pm)}_1 + i\mathcal{N}^{(\pm)}_0\right)\xi^{(\mp)} \end{aligned} \tag{J.39}$$

Similarly,

$$\mathcal{R}^{(\pm)}_{2\tilde{\lambda}} = e^{W/2}\mathcal{R}^{(\pm)}_{2\lambda} = \pm\gamma^a\not{P}\,\zeta^{(\mp)\mathbf{c}}_a\,. \tag{J.40}$$

In the same way, for the $\zeta_a^{(\pm)}$ equation of motion we find

$$
\begin{aligned}
\mathcal{L}_\zeta^{(\pm)a} &= e^{W/2}\mathcal{L}^{(\pm)a} \\
&= \gamma^{abc}\left[D_b + \frac{1}{4}ie^{\phi}(\partial_b a)\right]\zeta_c^{(\pm)} \mp \left(e^{-4U}\Sigma^{-1} + \frac{3}{2}\Sigma^2\right)\gamma^{ac}\zeta_c^{(\pm)} \\
&\quad - \frac{1}{4}i\Sigma^{-2}\gamma^{[c}\not{F}_2\gamma^{a]}\zeta_c^{(\pm)} + \left[i(\not{\partial}U)\gamma^a \mp ie^{-4U}\Sigma^{-1}\gamma^a\right]\xi^{(\pm)} \\
&\quad - \frac{1}{2}i(\Sigma^{-1}\not{\partial}\Sigma)\gamma^a\eta^{(\pm)} + \frac{1}{6}\Sigma^{-2}\not{F}_2\gamma^a\eta^{(\pm)} \pm \frac{i}{3}\left(e^{-4U}\Sigma^{-1} - 3\Sigma^2\right)\gamma^a\eta^{(\pm)}
\end{aligned}
\tag{J.41--J.42}
$$

$$
\begin{aligned}
\mathcal{R}_{1\,\zeta}^{(\pm)a} &= e^{W/2}\mathcal{R}_1^{(\pm)a} \\
&= \left(-\frac{1}{8}i\tilde{\not{\mathcal{G}}}_3 - \frac{1}{8}\tilde{\not{\mathcal{G}}}_2 \pm \frac{1}{4}\tilde{\not{\mathcal{G}}}_1\right)\gamma^a\tilde{\lambda}^{(\pm)} \mp \left(\frac{1}{2}i\tilde{\not{\mathcal{N}}}_1^{(\pm)} + \frac{1}{2}\tilde{\mathcal{N}}_0^{(\pm)}\right)\gamma^a\tilde{\lambda}^{(\mp)} \\
&\quad + \left(e^{Z+4W}\gamma^{ca} - \frac{1}{2}ie^{-4U}\gamma^{[c}\not{K}_1\gamma^{a]} \mp \frac{1}{2}i\Sigma\gamma^{[c}\not{K}_2\gamma^{a]}\right)\zeta_c^{(\pm)} \mp \Sigma\gamma^{[c}\not{L}_2^{(\pm)}\gamma^{a]}\zeta_c^{(\mp)} \\
&\quad + \left(-ie^{Z+4W} + \frac{1}{2}e^{-4U}\not{K}_1\right)\gamma^a\xi^{(\pm)} + \left(-\frac{2i}{3}e^{Z+4W} \mp \frac{1}{6}\Sigma\not{K}_2\right)\gamma^a\eta^{(\pm)} \\
&\quad \pm \frac{1}{3}i\Sigma\not{L}_2^{(\pm)}\gamma^a\eta^{(\mp)}
\end{aligned}
\tag{J.43--J.44}
$$

and

$$
\begin{aligned}
\mathcal{R}_{2\,\zeta}^{(\pm)a} &= e^{W/2}\mathcal{R}_2^{(\pm)a} \\
&= \mp\frac{1}{2}\not{P}\gamma^a\tilde{\lambda}^{(\mp)\mathbf{c}} \pm \frac{1}{8}i\mathcal{G}_3{}^{ebc}\left[\frac{1}{3}\gamma^{da}\gamma_{ebc} + (\delta_e^d\gamma^a - \delta_e^a\gamma^d)\gamma_{bc}\right]\zeta_d^{(\mp)\mathbf{c}} \\
&\quad \pm \left(\frac{1}{8}\mathcal{G}_2{}^{eb}\gamma_e\gamma^{da}\gamma_b \mp \frac{1}{2}\mathcal{G}_{1b}\gamma^{abd}\right)\zeta_d^{(\mp)\mathbf{c}} + \left(-i\mathcal{N}_{1b}^{(\pm)}\gamma^{abd} + \mathcal{N}_0^{(\pm)}\gamma^{ad}\right)\zeta_d^{(\pm)\mathbf{c}} \\
&\quad \mp \left(\frac{1}{12}\not{\mathcal{G}}_3 + \frac{i}{12}\not{\mathcal{G}}_2\right)\gamma^a\eta^{(\mp)\mathbf{c}} + \frac{2}{3}i\mathcal{N}_0^{(\pm)}\gamma^a\eta^{(\pm)\mathbf{c}} \\
&\quad \mp \frac{1}{8}\left(\not{\mathcal{G}}_3 - i\not{\mathcal{G}}_2 \mp 2i\not{\mathcal{G}}_1\right)\gamma^a\xi^{(\mp)\mathbf{c}} - \frac{1}{2}\left(\not{\mathcal{N}}_1^{(\pm)} - i\mathcal{N}_0^{(\pm)}\right)\gamma^a\xi^{(\pm)\mathbf{c}}.
\end{aligned}
\tag{J.45--J.46}
$$

For the $\eta^{(\pm)}$ equation of motion we have

$$\mathcal{L}_{\eta}^{(\pm)} = e^{W/2}\left(\mathcal{L}_{\mathrm{f}}^{(\pm)} + \frac{i}{3}\gamma_a \mathcal{L}^{(\pm)a}\right) \tag{J.47}$$

$$= \frac{2}{3}\left[\not{D} + \frac{1}{4}ie^{\phi}(\not{\partial}a)\right]\eta^{(\pm)} + \left(-\frac{5}{18}i\Sigma^{-2}\not{F}_2 \mp \frac{2}{9}e^{-4U}\Sigma^{-1} \pm \frac{5}{3}\Sigma^2\right)\eta^{(\pm)}$$

$$+ \left[\frac{1}{3}\Sigma^{-2}\gamma^c\not{F}_2 + i\gamma^c(\Sigma^{-1}\not{\partial}\Sigma) \pm \frac{2}{3}ie^{-4U}\Sigma^{-1}\gamma^c \mp 2i\Sigma^2\gamma^c\right]\zeta_c^{(\pm)}$$

$$\mp \frac{4}{3}e^{-4U}\Sigma^{-1}\xi^{(\pm)} \tag{J.48}$$

$$\mathcal{R}_{1\eta}^{(\pm)} = e^{W/2}\left(\mathcal{R}_{1\,\mathrm{f}}^{\pm} + \frac{i}{3}\gamma_a \mathcal{R}_1^{(\pm)a}\right) \tag{J.49}$$

$$= \left(\frac{1}{6}\tilde{\not{G}}_3 - \frac{1}{6}i\tilde{\not{G}}_2\right)\tilde{\lambda}^{(\pm)} \mp \frac{4}{3}i\tilde{\mathcal{N}}_0^{(\pm)}\tilde{\lambda}^{(\mp)}$$

$$+ \left(-\frac{4}{3}ie^{Z+4W}\gamma^c \mp \frac{1}{3}\Sigma\gamma^c\not{K}_2\right)\zeta_c^{(\pm)} \pm \frac{2}{3}i\Sigma\gamma^c\not{L}_2^{(\pm)}\zeta_c^{(\mp)} + \frac{8}{3}e^{Z+4W}\xi^{(\pm)}$$

$$+ \left(\frac{10}{9}e^{Z+4W} + \frac{1}{3}ie^{-4U}\not{K}_1 \pm \frac{1}{9}i\Sigma\not{K}_2\right)\eta^{(\pm)} \pm \frac{2}{9}\Sigma\not{L}_2^{(\pm)}\eta^{(\mp)} \tag{J.50}$$

and

$$\mathcal{R}_{2\eta}^{(\pm)} = e^{W/2}\left(\mathcal{R}_{2\,\mathrm{f}}^{(\pm)} + \frac{i}{3}\gamma_a \mathcal{R}_2^{(\pm)a}\right) \tag{J.51}$$

$$= \mp \frac{1}{6}\gamma^d\left(\not{G}_3 + i\not{G}_2\right)\zeta_d^{(\mp)\mathbf{c}} + \frac{4}{3}i\mathcal{N}_0^{(\pm)}\gamma^d\zeta_d^{(\pm)\mathbf{c}}$$

$$\pm \frac{1}{18}\left(i\not{G}_3 - \not{G}_2 \mp 6\not{G}_1\right)\eta^{(\mp)\mathbf{c}} - \left(\frac{10}{9}\mathcal{N}_0^{(\pm)} + \frac{2i}{3}\not{\mathcal{N}}_1^{(\pm)}\right)\eta^{(\pm)\mathbf{c}}$$

$$\mp \frac{1}{6}\left(i\not{G}_3 + \not{G}_2\right)\xi^{(\mp)\mathbf{c}} - \frac{4}{3}\mathcal{N}_0^{(\pm)}\xi^{(\pm)\mathbf{c}}\,. \tag{J.52}$$

Finally, for the $\xi^{(\pm)}$ equation of motion we have

$$\mathcal{L}_{\xi}^{(\pm)} = e^{W/2}\left(i\gamma_a\mathcal{L}^{(\pm)a} + \mathcal{L}^{(\pm)}_{base} - \mathcal{L}^{(\pm)}_{\mathrm{f}}\right) \tag{J.53}$$

$$= 2\left[\not{D} + \frac{1}{4}ie^{\phi}(\not{\partial}a)\right]\xi^{(\pm)} + \frac{1}{2}i\Sigma^{-2}\not{F}_2\xi^{(\pm)} \pm 3\left(2e^{-4U}\Sigma^{-1} - \Sigma^2\right)\xi^{(\pm)}$$

$$+\left[\mp 4ie^{-4U}\Sigma^{-1}\gamma^c - 4i\gamma^c(\not{\partial}U)\right]\zeta_c^{(\pm)} \mp \frac{8}{3}e^{-4U}\Sigma^{-1}\eta^{(\pm)} \tag{J.54}$$

$$\mathcal{R}_{1\xi}^{(\pm)} = e^{W/2}\left(i\gamma_a\mathcal{R}_1^{(\pm)a} + \mathcal{R}^{(\pm)}_{1\,base} - \mathcal{R}^{(\pm)}_{1\,\mathrm{f}}\right) \tag{J.55}$$

$$= \left(\frac{1}{2}\tilde{\not{\mathcal{G}}}_3 + \frac{1}{2}i\tilde{\not{\mathcal{G}}}_2 \mp i\not{\mathcal{G}}_1\right)\tilde{\lambda}^{(\pm)} \mp \left(2\tilde{\not{\mathcal{N}}}_1^{(\pm)} + 2i\tilde{\mathcal{N}}_0^{(\pm)}\right)\tilde{\lambda}^{(\mp)}$$

$$+\left(-4ie^{Z+4W}\gamma^c + 2e^{-4U}\gamma^c\not{K}_1\right)\zeta_c^{(\pm)} + \frac{16}{3}e^{Z+4W}\eta^{(\pm)}$$

$$+\left(6e^{Z+4W} - 3ie^{-4U}\not{K}_1 \pm i\Sigma\not{K}_2\right)\xi^{(\pm)} \pm 2\Sigma\not{L}_2^{(\pm)}\xi^{(\mp)} \tag{J.56}$$

and

$$\mathcal{R}_{2\xi}^{(\pm)} = e^{W/2}\left(i\gamma_a\mathcal{R}_2^{(\pm)a} + \mathcal{R}^{(\pm)}_{2\,base} - \mathcal{R}^{(\pm)}_{2\,\mathrm{f}}\right) \tag{J.57}$$

$$= \mp\left(\frac{1}{2}\gamma^d\not{\mathcal{G}}_3 - \frac{1}{2}i\gamma^d\not{\mathcal{G}}_2 \mp i\gamma^d\not{\mathcal{G}}_1\right)\zeta_d^{(\mp)\mathbf{c}} + \left(2i\mathcal{N}_0^{(\pm)}\gamma^d - 2\gamma^d\not{\mathcal{N}}_1^{(\pm)}\right)\zeta_d^{(\pm)\mathbf{c}}$$

$$\mp\left(\frac{1}{3}i\not{\mathcal{G}}_3 + \frac{1}{3}\not{\mathcal{G}}_2\right)\eta^{(\mp)\mathbf{c}} - \frac{8}{3}\mathcal{N}_0^{(\pm)}\eta^{(\pm)\mathbf{c}} \mp \frac{3}{4}\not{\mathcal{G}}_2\xi^{(\mp)\mathbf{c}}\,. \tag{J.58}$$

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