Consistent Truncations of IIB Supergravity and a Holographic c-theorem for Higher Derivative Gravity

by

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To my wife.

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CHAPTER I

Introduction

1.1 String theory in general

The Standard Model, or quantum field theory with $SU(3) \times SU(2) \times U(1)$ gauge symmetry plus three generations of quarks and leptons and a spin-0 Higgs boson provides a consistent description of reality up to scales of roughly $10^{-18}m$. It formulates electromagnetic, weak and strong interactions in a uniform framework. Classical gravity, the fourth interaction, is governed by General Relativity and remains to be unified. Attempts to quantize the graviton field quickly lead to non-renormalizable terms due to the power counting of the coupling constant.

The ultra-violet divergence problem of quantum gravity indicates the presence of new physics at short distances beyond the Planck length. One approach is to give up the pointlike structure of particles and incorporate one-dimensional extended objects, namely strings, as the fundamental elements. The spectrum of oscillation modes of strings corresponds to quantized spacetime fields, including the spin-2 graviton. String theory was originally constructed as an attempt to describe the various types of mesons and hadrons discovered in 1960's. Although this description was superseded by quarks and QCD, string theory now has become the most promising theory in quantum gravity and unifying gravity with the standard model.

Curiously, string theory is not consistent in arbitrary dimensions. In flat space string theory, the two-dimensional world-sheet field theory preserves Weyl symmetry.

$$g_{\mu\nu} \to e^{2\omega(x_\mu)} g_{\mu\nu} \tag{I.1}$$

The world-sheet Weyl symmetry is required to guarantee the unitarity and preserve the spacetime Lorentz invariance. In a quantum field theory, the Weyl anomaly for the conformal symmetry is only canceled in 26 dimensional spacetime for the bosonic string and 10 dimensional spacetime for the superstring. String theory is unique because it avoids adjustable parameters or any ad hoc gauge symmetry or discrete symmetry assumptions. String theory also includes spacetime supersymmetry, a symmetry that relates elementary particles of one spin to their superpartners that differ by half spin. Supersymmetry is a successful model in solving the hierarchy problem of the standard model, namely why higgs masses are so much less than the Plank mass. A lot of recent work on string theory has focused on the use of dualities in trying to understand strongly coupled and non-perturbative physics.

1.2 D-branes

In string theory, toroidal compactification is the most basic way to compactify over extra small curved dimensions. T-duality interchanges a small compact internal space with a large one. When we apply the duality to open strings, we have hyperplanes where open strings end, namely D-branes [1]. D-branes themselves are dynamical objects which have mass, tension and carry charges. Different string theories are actually different vacua of a single theory containing general D-brane configurations.

D-branes highly resemble extremal black p-branes [98]. As a family of ten dimensional spacetime solutions for type II supergravity, black p-branes represent p-dimensional extended objects with $9-p$ dimensional transverse space. The black p-branes usually preserve Poincaré symmetry along the directions of branes and spherical symmetry in the

[1] "D" stands for Dirichlet boundary conditions for open strings.

transverse space. The black p-branes look like localized objects with mass and charge for transverse spacetime observers. This is a generalization of the Reissner-Nordström black hole. D-branes natually couple to a C_{p+1} potential [2], which resembles the source of the charge of black p-branes. The tension and mass of D-branes are related to the mass M of the black p-branes. Furthermore, from the horizon structure of Reissner-Nordström solutions, we know there is an upper bound for the supergravity p-brane's charge, $Q \leq M$, to avoid having a naked singularity. This resembles the bound for central charges of spacetime supersymmetry.

To generate a D-brane, T-duality breaks the conservation of momentum in a transverse direction by mirror reflecting the right moving modes. This also breaks half of spacetime supersymmetry. So D-branes are essentially BPS states [3] because exactly half of supersymmetry is broken. D-branes are essentially extremal black p-branes [4]. This type of correspondence is significant because it implies a gauge / gravity dualilty. Consider N coincident D-branes in flat spacetime. Each open string ending on those D-branes has strength g_s, so we have g_sN strength in total. We can understand the open string spacetime as a gauge field in the perturbative region $g_sN \ll 1$. On the black p-brane side, however, the effective curvature is $R^2 \sim g_sN$. So gravity is perturbatively understood at slightly curved spacetime, or $g_sN \gg 1$. The correspondence of D-branes and supergravity p-branes provides a complimentary image of each other at strong coupling or the non-perturbative region.

1.3 Kaluza-Klein reduction

String theory does not exist in arbitrary dimensions. Spacetime unitarity and Lorentz invariance are required to be preserved in the full quantum theory. The world-sheet Weyl

[2] An anti-symmetric $p+1$ potential form. For instance, electro-magnetic potential A_μ is a C_1 potential.

[3] BPS states are massive representations of an extended supersymmetry algebra with mass equal to the supersymmetry central charge. A BPS state breaks exactly half of the supersymmetry.

[4] In the Reissner-Nordström solutions, extremal black holes are the solutions with $Q = M$.

anomaly is cancelled only in 10D spacetime for the superstring. For a realistic theory in 4D, the extra 6D space has to be compactified over a small internal manifold. The procedure of such a compactification is known as Kaluza-Klein reduction. The general procedure is to start from a field in the full space time, and expand it in terms of a complete set of harmonic functions of internal space, so that we get a spectrum of a reduced field only depending on the external spacetime. The simplest model is to reduce a free massless scalar field in flat spacetime in d dimension to $d-1$ dimension.

$$S = \int d^d x \partial_\mu \phi \partial^\mu \phi \tag{I.2}$$

Compactify the d dimensional theory around S^1 over dx^d with radius R:

$$\phi(x^\mu, z) = \sum_{n=-\infty}^{\infty} \phi_n(x^\mu) e^{i\frac{n}{R}z} \tag{I.3}$$

over the complete set $e^{i\frac{n}{R}z}$. So we have a spectrum of reduced scalar fields over x^μ:

$$\phi_n(x^\mu) = \int_0^{2\pi R} \phi(x^\mu, z) e^{-i\frac{n}{R}z} dz \tag{I.4}$$

Because the original scalar field is massless, then:

$$\begin{aligned} \Box\phi(x^\mu, z) &= \sum_{n=-\infty}^{\infty} \left[\Box\phi_n(x^\mu) e^{i\frac{n}{R}z} - \phi_n(x^\mu)(\frac{n}{R})^2 e^{i\frac{n}{R}z}\right] \\ &= \sum_{n=-\infty}^{\infty} \left[m_n^2 - (\frac{n}{R})^2\right] \phi_n e^{i\frac{n}{R}z} = 0 \end{aligned} \tag{I.5}$$

which implies $m_n = \frac{n}{R}$. So we get an infinite tower of massive states except for the zero mode. If we take the limit $R \to 0$, it takes infinite energy to excite the first and higher modes. So the theory reduces to a massless scalar field in a lower dimension.

Now we turn to a more complicated example. We reduce 5D pure gravity over S^1 with radius R.

$$S = \frac{1}{16\pi G_5} \int d^5 x \sqrt{-G} R_5, \quad ds_5^2 = G_{MN} dx^M dx^N \tag{I.6}$$

When we compactify the 5D gravity around T^1 over dx^5 with small radius R, we make the following ansätz where only zero modes are kept, meaning the fields ϕ, $g_{\mu\nu}$ and A_μ are fields of 4D spacetime coordinates only:

$$ds^2 = e^{\frac{1}{\sqrt{3}}\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{-\frac{2}{\sqrt{3}}\phi}(dz + A_\mu dx^\mu)^2, \quad \mu = 0,1,2,3 \tag{I.7}$$

Here $g_{\mu\nu}, \phi, A_\mu$ are fields over all coordinates of dx^μ only, which are defined as:

$$e^{-\frac{2}{\sqrt{3}}\phi} = G_{55}, \quad A_\mu = e^{\frac{2}{\sqrt{3}}\phi} G_{\mu 5}, \quad g_{\mu\nu} = e^{-\frac{1}{\sqrt{3}}\phi} G_{\mu\nu} - e^{-\sqrt{3}\phi} A_\mu A_\nu \tag{I.8}$$

All of these bosonic fields in 4D spacetime are massless due to the Einstein equations.

We get a massless graviton, a massless vector field and a massless dilaton in the lower dimension from a graviton in one higher dimension. Integrating out the action over the internal space S^1, we get:

$$S = \frac{1}{16\pi G_4} \int dx^4 \sqrt{-g} \left[R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4} e^{-\sqrt{3}\phi} F_{\mu\nu} F^{\mu\nu} \right], \quad G_4 = \frac{G_5}{2\pi R} \tag{I.9}$$

In this action, all fields are massless ground states in the Kaluza-Klein tower. We see the massless vector field A_μ is actually a $U(1)$ gauge field. The graviton and dilaton are neutral under this gauge field.

In the above Kaluza-Klein reduction procudure, we have made a hidden assumption that the truncation is consistant, which means all solutions of the truncated theory at lower dimensions are solutions of the full theory. That is, the ground state modes solves the equations of motion of the original and truncated Lagrangian at the same time. This looks trivial in this example, but when we go to more complicated internal spaces and include higher Kaluza-Klein modes, the consistency is not trivial and needs manual inspection. The requirement of consistency sets a strong restriction on the truncation ansatz. In this thesis, we prove the consistency of our truncation ansatz. We also propose some conjectures on what properties of the ansatz would lead to consistency truncations.

Going back to string theory, for the type II superstring [5], we need to compactify over 6D to get a realistic theory. However, the hope that 4D theory resembles the observed world strongly restricts the nature of the compact manifold. Especially the requirement that supersymmetry is preserved restricts us to a special class of manifolds known as Calabi-Yau manifolds. It looks too restrictive to preserve supersymmetry in the 4D theory, since the standard model itself is not supersymmetric. The reason to preserve supersymmetry under compactification is not that we want supersymmetry to persist in the 4D theory, but rather that we do not want to break it at the string scale. Supersymmetry breaking scale is closely related to the higgs masses, which is far below the Planck scale. We preserve supersymmetry at the string scale and break it at a much lower energy level to solve the hierarchy problem. To satisfy the supersymmetry preservation, we require a global spinor as a supersymmetry transformation generator under which the variation of all fermions vanish. Such a global spinor is a Killing spinor over internal space. The existence of such a global Killing spinor requires the internal space to be Ricci-flat and admit a Kähler form, in the absence of fluxes. Conjectured by Calabi and proved by Yau, such manifolds are essentially Kähler manifolds with vanishing first Chern class, or Calabi-Yau manifolds.

People are also interested in compactifying over 5D internal space to yield a 5D anti-de Sitter (AdS) background due to the AdS/CFT correspondence (to be discussed in more details in the next section). It is believed that the low energy supergravity theory in AdS_{d+1} is dual to strongly coupled d-dimensional conformal field theory living on the boundary of AdS. In terms of supersymmetry preservation, the truncation of $AdS_5 \times S^5$ describes $\mathcal{N} = 8$ gauged supergravity [6] in 5D sitting near the horizon of AdS_5, which preserves the maximum 32 supersymmetries. One way to see that is that both AdS_5 and S^5 preserve the maximal

[5] Type II superstring: a theory of closed orinted superstrings. The right-movers and left-movers transform under seperate spacetime supersymmetries. If they have opposite chiralities, it is type IIA superstring. If they have the same chirality, it is type IIB superstring.

[6] Gauged supergravities are supergravity theories with non-abelian gauge fields in the supermultiplet of the graviton. Namely, the gravitino fields are gauged under R symmetry.

symmetry in 5D. Less symmetric internal space could preserve fewer supersymmetries and contain richer structure.

Truncation of type IIB supergravity over $AdS_5 \times S^5$ admits $\mathcal{N} = 8$ supergravity, which further leads to $\mathcal{N} = 4$ super Yang-Mills due to AdS/CFT duality. If we relax the restriction to allow less supersymmetries to be preserved in the compactification, we can truncate type IIB supergravity over $AdS_5 \times M^5$. Since a Calabi-Yau three-fold is required for 6D compactification to preserve supersymmetry, we can think of the compactified 4D theory to be living on the boundary of AdS_5 space. Thus it is natural to write the Calabi-Yau threefold in terms of:

$$ds^2(CY^3) = dr^2 + r^2 ds^2(M^5) \tag{I.10}$$

A Sasaki-Einstein five-manifold M^5 is such a manifold that constructs CY^3 in I.10. That is, the metric cone of M^5 is Ricci-flat and Kähler. On the AdS side, by inserting a stack of D3 branes sitting at the tip of the CY^3 cone, IIB supergravity compactifies on Sasaki-Einstein M^5 to a 5D supergravity. On the CFT side, the dual theory is a $\mathcal{N} = 1$ superconformal field theory (SCFT). The most trivial and maximally symmetric SE_5 is S^5, and the corresponding CY cone is simply $\mathbb{C}^3$. In Sasaki-Einstein geometry, we can always have a killing vector with constant norm [11], which is called the *characteristic* or *Reeb* vector field. So we can in general write a Sasaki-Einetein metric as:

$$ds^2_{SE_5} = (d\psi + A)^2 + ds^2_4(B) \tag{I.11}$$

where $\frac{\partial}{\partial\psi}$ is the killing vector. What is more, the remaining base metric $ds^2_4(B)$ have a Kähler-Einstein structure. The base Kähler-Einstein manifold admits a Kähler $(1,1)$ form J and holomorphic $(2,0)$ form Ω which satisfies $dA = 2J$ and $d\Omega = 3i(d\psi + A)\Omega$. So SE_{2m+1} manifolds are essentially $U(1)$ bundles over KE_{2m} manifolds. More details will be discussed in the next chapter.

The key to obtaining a consistent truncation is expanding the full fields in the right ansatz. Assume the full theory has symmetry group G and the compactified space has symmetry $K \subseteq G$. When we expand the full fields over a complete set of K, we get K singlets ϕ and also non-singlets denoted by ϕ^α which form a representation tower for K. When we make the ansatz that all fields except for singlets vanish, the equations of motion can be solved and the solutions are good for the original full theory. However if we include non-singlet fields, the equations of motion usually lead to more restrictions than modes to be solved. Essentially, under truncation, non-singlet fields are usually sources for higher level modes, leading to an infinite tower of modes, which is usually not realistic for effective theories which can only excite a few low energy level modes. There is generically no guarantee of consistency of an ansatz containing non-singlet modes. Such ansätze need to be checked case by case. The way we check consistency is to write the full equations of motion in terms of lower dimensional fields according to the ansatz first, then decompose each full equation of motion into several equations of motions in the lower dimensional fields. The truncation is consistent if and only if all of these equations of motion of lower dimensional fields are consistent with each other. In the next chapter we will discuss the consistent truncation of type IIB supergravity over a SE_5 manifold up to the second level of Kaluza-Klein reduction. The compactified theory is $\mathcal{N} = 2$ gauged supergravity coupled to a massless $LH + RH$ chiral supermultiplet [7], a massive $LH + RH$ massive gravitino multiplet and a massive vector multiplet. The resulting fields nicely fit into representations of the 4D SCFT [53]:

$$SU(2,2|1) \supset SO(2,4) \times U(1)_q \supset U(1)_E \times SU(2)_L \times SU(2)_R \times U(1)_q \tag{I.12}$$

The full truncated theory corresponds to five-dimensional $\mathcal{N} = 4$ theory. We may truncate

[7] In the superspace notion of supersymmetry, a LH chiral supermultiplet has field contents of a superfield on chiral superspace, a RH supermultiplet has field contents of an superfield on anti-chiral superspace, which is the complex conjugate of chiral superspace.

out the $\mathcal{N} = 2$ massive gravitino multiplet to bring it down to $\mathcal{N} = 2$.

This type of consistent truncation have been of particular interested recently due to the applications of AdS/CFT on condensed matter systems. In one of our further consistent truncations, we demonstrated a supersymmetric completion of a bosonic superconductor theory first demonstrated in [49]. Such truncations implies the possibility of constructing superconductor theories from string theory.

1.4 AdS/CFT duality

The AdS/CFT correspondence by Maldacena [83, 47, 100] has been a popular topic in string theory. The correspondence is a relation between a conformal field theory (CFT) in d dimensions and a gravity theory in $d + 1$ anti de-Sitter (AdS) space. It is a successful realization of the holographic principle, stating that the description of the bulk of spacetime is encoded on its boundary. The first hint of AdS/CFT correspondence is the $SO(2, d)$ symmetry. On the AdS side, it is the maximal symmetric solution with negative curvature for $d+1$ spacetime which could be embeded in flat $d+2$, implying $SO(2, d)$. On the CFT side, the conformal symmetry includes a special conformal transformation vector generator and a dilatation scalar generator, which are integrated into the normal Poincare group to form a larger $SO(2, d)$ group. AdS/CFT is not only a correspondence, but a weak/strong coupling duality, or a gravity/ gauge duality. One of the applications of AdS/CFT duality is to understand the strongly coupled condensed matter system with high critic temperature T_c. In chapter 2, we derive an AdS dual background to a supersymmetric superconductor model as a consistent truncation of IIB supergravity on Sasaki-Einstein manifold.

From a string theory point of view, if we introduce N D3 branes as sources for a five-form in the flat 10D spacetime, we get a compact S^5 and an AdS_5 spacetime solution. On the boundary of AdS_5, we get a 4D super Yang-Mills field theory with $SU(N)$ gauge group. Compactifying over S^5 will preserve all 32 fermionic charges. The super conformal field theory has maximal $\mathcal{N}=4$ supersymmetry which contains 16 fermionic charges. However, for super conformal symmetry, there is another set of 16 fermionic charges. So all 32 fermionic charges are preserved after compactification. The correspondence implies the relation of parameters:

$$\frac{L^4}{\alpha'^2} = 4\pi g_s N = g_{YM}^2 N \tag{I.13}$$

where L is the AdS radius, $\sqrt{\alpha'} = l_s$ is the string length, g_s is string coupling and g_{YM} is the Yang-Mills coupling. In order to implement low energy superstring theory, or supergravity, we require small g_s. On the other hand, we require the AdS background not to be strongly curved to remain within low evergy supergravity, i.e.: $L \gg l_s$, which leads to large 't Hooft coupling $\lambda = g_{YM}^2 N$ on the CFT side. Thus we obtain a weak/strong or gravity/gauge duality theory. On one side we have low energy supergravity $L \gg l_s$, $g_s \ll 1$. On the other side, we have large N and strong λ CFT.

Another point of view is to view both gravity and gauge theory as two decoupled actions. For super Yang-Mills theory, the action is written as $S = S_{bulk} + S_{brane} + S_{int}$, where S_{bulk} represents background 10D supergravity. S_{brane} is defined on the 3+1 world volume which represents the theory near the D3-brane stack and contains super Yang-Mills theory. S_{int} represent the interaction between these two modes. On the AdS side, the p-brane solution is given by [2]:

$$\begin{aligned} ds^2 &= f^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2)^2 + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \\ F_5 &= (1+*)dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge df^{-1} \\ f &= 1 + (\frac{R}{r})^4, \quad R^4 \equiv 4\pi g_s \alpha'^2 N \end{aligned} \tag{I.14}$$

Where F_5 is the field strength coupled to the D3 branes. Denote E_p to be the energy measured by an observer at position r, and E to be the energy measured at infinity. We have: $E = f^{-\frac{1}{4}} E_p$. When we take the limit $r \to 0$ and fix E_p, we get $E \to 0$. This is the theory at low energy from the point of view of the observer at infinity. In this limit $r \ll R$, we have:

$$ds^2 = \frac{r^2}{R^2}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + R^2\frac{dr^2}{r^2} + R^2 d\Omega_5^2 \tag{I.15}$$

which is $AdS_5 \times S^5$. On the other hand, $r \to \infty$ corresponds to the large wave length limit of gravity, which is free gravity. So the theory contains two decoupled low energy regions, $r \to 0$ and $r \to \infty$. The interaction between these two regions is decoupled due to the high energy wall in between. Since $r \to \infty$ is described by low energy supergravity corresponding to S_{bulk} in CFT, the $r \to 0$ theory corresponds to S_{brane}, or $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills CFT.

As a dual theory, it is essential to have a dictionary which maps physical observables of the two theories. The natural objects for CFT are operators $\mathcal{O}(x)$. The generating function for $\mathcal{O}(x)$ is $< e^{\int d^4x \phi_0(x)\mathcal{O}(x)} >$, where $\phi_0(x)$ corresponds to the source coupled to $\mathcal{O}(x)$. On the AdS side, it is natural to consider a field whose boundary is fixed to $\phi_0(x)$. Thus the string partition function is written as:

$$Z[\phi_0(x)] = \exp\{-S_{SUGRA,Classical}[\phi(x,z)]|\phi(x,0) = \phi_0(x)\} \tag{I.16}$$

Note that we are using the classical action for supergravity. The identification of the CFT generation function and string partition function provides the so-called GKP-Witten relation of AdS/CFT duality [47, 100].

1.5 Application of AdS/CFT for holographic superconductors

One motivation to study the consistent truncation of IIB supergravity on Sasaki-Einstein manifold is to study the superconductors of high critical temperatures T_c. Such superconductors are in the strongly coupling region, which are not well-studied under perturbative field theories. AdS/CFT correspondence provides an applicable approach due to its strong/weak duality feature. To study the d-dimensional strongly coupled superconductor system, we study the corresponding $d+1$ dimensional AdS gravity theory. For instance, to study a 2+1 dimensional superconductor, we start from a four-dimensional AdS gravity with a massive scalar charged under a U(1) gauge symmetry:

$$\mathcal{L}_4 = R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}^2 - |\partial_\mu \phi - iqA_\mu \phi|^2 - m^2|\phi|^2 \tag{I.17}$$

The Hawking temperature is dual to the superconductivity temperature. The scalar is dual to the order parameter which breaks U(1) symmetry and condensates below the critical temperature. We can study the condensation of the charged scalar below critical temperature from the AdS side. If we are able to derive such a superconductor theory from superstring or supergravity theory, we have a UV complete model for the system.

1.6 c-theorem

Conformal invariance requires vanishing trace of the energy-momentum tensor $T_\mu^\mu = 0$. However the conformal anomaly is related to the background spacetime curvature. In 2D CFT, the anomaly form is simple:

$$\langle T_\mu^\mu \rangle = -\frac{c}{12}R \tag{I.18}$$

where R is the 2D Ricci scalar and c is the central charge of the Virasoro algebra, which is directly related to the degrees of freedom of the theory. Zamolodchikov [101] proved a "c-theorem" stating that in 2D field theory, there exists a c-function which monotonically decreases from UV to IR, and whose values at fixed points are identical to the central charges of the conformal field theory. The Zamolodchikov c-theorem has a powerful physics intuition that the degrees of freedom of CFT are removed as the theory flows from UV to IR.

Numerous attempts have been made to generalize the c-theorem to higher dimensions. However, there are multiple candidates for a "c-charge" in higher dimensions. For example, in 4D:

$$\langle T^{\mu}_{\mu} \rangle = \frac{c}{16\pi^2} C^2_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4 \tag{I.19}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the Euler density in 4D. It has been shown that the function a (coefficient for Euler density in corresponding dimension) appears to have a monotonic behavior in the flow toward IR, while the coefficient c has counter-examples. Even though there is no exact physics interpretation of the coefficient a, people still believe it is related to degrees of freedom of the theory. Meanwhile, a holographic proof of the c-theorem is provided based on AdS/CFT duality. The even d-dimensional a-charge is calculated in a holographic manner:

$$a = -\frac{\pi^{-\frac{d}{2}}}{2\kappa^2} \frac{l^{d+1}}{(d/2)!^2} f(AdS) \tag{I.20}$$

where l is the radius of the AdS background, and $f(AdS)$ is simply the on-shell Lagrangian at asymptotic AdS fixed points. When we construct a metric ansatz:

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2_{d-1}) + dr^2 \tag{I.21}$$

it becomes AdS_{d+1} with $A(r) = r/l$ at fixed points. We define an a-function where all l in I.20 is replaced by $A'(r)^{-1}$. Then such an a-function automatically converges to the a-charge at fixed points. We can derive a monotonic flow of $a(r)$ due to the null-energy condition for the matter fields in the bulk gravity theory. The technique is used to derive a c-theorem for gravity with higher derivative corrections. In chapter III, we will prove a c-theorem for higher curvature Lovelock gravity, and discuss the condition for the existence of a c-theorem for arbitrary $f(R)$ and $f(R^{ab}{}_{cd})$ gravity.

1.7 Structure of the thesis

The remainder of this thesis is organized as follows. In chapter II we discuss the construction of consistent supersymmetric truncations of IIB supergravity on five dimensional squashed Sasaki-Einstein manifolds. We first focus on the bosonic sector of gauged $D = 5$, $\mathcal{N} = 2$ supergravity coupled to massive multiplets up to the second Kaluza-Klein level. We derive the effective five-dimensional Lagrangian of the truncated theory and prove the consistency of the truncation. We also discuss some further truncations by setting some supermultiplets to vanish or to dependent on other multiplets. Especially, we present a particular truncation which contains only the LH+RH chiral multiplet, along with the supergravity multiplet. This is essentially the truncation to the lowest non-trivial Kaluza-Klein level. This truncation is particularly interesting as it represents the bosonic sector of a holographic superconductor. We then present the reduction of the fermionic sector of IIB supergravity. We organize the spectrum in terms of AdS supermultiplets, as a supersymmetric completion of the bosonic sector. The supersymmetry variations of the fermions are derived and shown to be consistent with the supermultiplet structure. We construct the Lagrangian for the full truncated theory from the fermionic equations of motion. Finally, we further truncate the fermionic sector to match the holographic superconductor system mentioned above (which is also consistent), as a supersymmetric completion of the bosonic sector.

In chapter III we investigate another broadly discussed application of AdS/CFT correspondence, namely holographic c-theorems. In AdS/CFT, the holographic Weyl anomaly computation relates the a-anomaly coefficient to the properties of the bulk action at the UV fixed point. This universal behavior suggests the possibility of a holographic c-theorem for the a-anomaly under flows to the IR. We prove such a c-theorem for higher curvature Lovelock gravity, where the bulk equations of motion remain second order. We also explore $f(R)$ gravity as a toy model where higher derivatives cannot be avoided. In this case, monoticity of the flow requires an additional condition related to the higher derivative nature of the theory. This is in contrast to the case of $f(R)$ black hole entropy, where the second law follows from application of the full Einstein equations and the null energy condition. We also prove a holographic c-theorem for the a central charge where the bulk is described by a gravitational action built out of an arbitrary function $f(R^{ab}{}_{cd})$ of the Riemann tensor coupled to bulk matter. This theorem holds provided a generalized null energy condition involving both matter and higher curvature gravitational interactions is satisfied. As an example, we consider the case of a curvature-squared action, and find that generically the generalized null energy condition involves not just the bulk matter, but also the sign of R'' where a prime denotes a radial derivative and where R is the bulk scalar curvature.

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CHAPTER II

Consistent truncations of IIB supergravity on squashed Sasaki-Einstein manifold

In this chapter we present a consistent truncation of IIB supergravity on Sasaki-Einstein manifolds. A detailed analysis of the bosonic reduction of IIB is presented, followed by the reduction of the fermionic sector. This chapter is based on work published in [77, 78] in collaboration with James Liu and Phillip Szepietowski.

2.1 Motivations for Studying Massive Truncations of String/M-theory

Recent developments in AdS/CFT have expanded the scope of applications from the realm of strongly coupled relativistic gauge theories to various condensed matter systems whose dynamics are expected to be described by a strongly coupled theory. These include systems with behavior governed by a quantum critical point [60, 56], as well as cold atoms and similar systems exhibiting non-relativistic conformal symmetry [97, 8]. Much current attention is also directed towards holographic descriptions of superfluids and superconductors [48, 54, 59, 55].

The main feature used in the construction of a dual model of superconductivity is the existence of a charged scalar field in the dual AdS background [54, 55]. Turning on temperature and non-zero chemical potential corresponds to working with a charged black hole in AdS. Then, as the temperature is lowered, the charged scalar develops an instability and condenses, so that the black hole develops scalar hair[1]. This condensate breaks the U(1) symmetry, and is a sign of superconductivity (in the case where the U(1) is "weakly gauged" on the boundary).

The basic model dual to a 2+1 dimensional superconductor is simply that of a charged scalar coupled to a Maxwell field and gravity, and may be described by a Lagrangian of the form

$$\mathcal{L}_4 = R + \frac{6}{L^2} - \frac{1}{4}F_{\mu\nu}^2 - |\partial_\mu\psi - iqA_\mu\psi|^2 - m^2|\psi|^2. \tag{II.1}$$

The properties of the system may then be studied for various values of mass m and charge q. While this is a perfectly acceptable framework, a more complete understanding demands that this somewhat phenomenological Lagrangian be embedded in a more complete theory such as string theory, or at least its supergravity limit. For AdS_4 duals of 2+1 dimensional superconductors, this was examined at the linearized level in [27], and embedded into $D = 11$ supergravity at the full non-linear level in [37, 39, 40] for the case $m^2L^2 = -2$ and $q = 2$. Similarly, a IIB supergravity model for an AdS_5 dual to 3+1 dimensional superconductors was constructed in [49] with $m^2L^2 = -3$ and $q = 2$.

The AdS_4 model of [37, 39, 40] and the AdS_5 model of [49] are based on Kaluza-Klein truncations on squashed Sasaki-Einstein manifolds. They both have the unusual feature where the $q = 2$ charged scalar arises from the massive level of the Kaluza-Klein truncation. This appears to go against the standard lore of consistent truncations, where it was thought that truncations keeping only a finite number of massive modes would necessarily be inconsistent. A heuristic argument is that states in the Kaluza-Klein tower carry charges under the internal symmetry, and hence would couple at the non-linear level to source higher and higher states, all the way up the Kaluza-Klein tower. This hints that one way to obtain a consistent truncation is simply to truncate to singlets of the

[1] Recent models have generalized this construction to encompass both p-wave [50, 90] and d-wave [19] condensates.

internal symmetry group, and indeed such a construction is consistent. An example of this is a standard torus reduction, where only zero modes on the torus are kept. On the other hand, sphere reductions to maximal gauged supergravities in $D = 4$, 5 and 7 do not follow this rule, as they are expected to be consistent, even though some of the lower-dimensional fields (such as the non-abelian graviphotons) are charged under the R-symmetry. In fact, the issue of Kaluza-Klein consistency is not yet fully resolved, and often must be treated on a case by base basis. This has led us to explore the squashed Sasaki-Einstein compactifications to see if additional consistent massive truncations may be found.

In addition to embedding holographic models of superconductivity into string theory, several groups have demonstrated the embedding of dual non-relativistic CFT backgrounds into string theory [61, 82, 1]. These geometries where originally constructed from a toy model of a massive vector field coupled to gravity with a negative cosmological constant [97, 8] of the form (given here for a deformation of AdS_5):

$$\mathcal{L}_5 = R + \frac{12}{L^2} - \frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2, \tag{II.2}$$

with mass related to the scaling exponent z according to $m^2L^2 = z(z+2)$. The $z = 2$ and $z = 4$ models ($m^2L^2 = 8$ and $m^2L^2 = 24$, respectively) were subsequently realized within IIB supergravity in terms of consistent truncations retaining a massive vector (along with possibly other fields as well) [61, 82, 1]. These results have further opened up the possibility of obtaining large classes of consistent truncations retaining massive modes of various spin.

2.1.1 Consistent massive truncations of IIB supergravity

For the most part, the massive consistent truncations used in the study of AdS/condensed matter systems have not been supersymmetric[2]. Nevertheless this has motivated us to

[2] The massive truncation given in [37] is supersymmetric, although the connection to a holographic superconductor was done through the non-supersymmetric skew-whiffed case.

investigate the possibility of obtaining new supersymmetric massive truncations of IIB supergravity. In particular, we are mainly interested in reducing IIB supergravity on a Sasaki-Einstein manifold to obtain gauged supergravity in $D = 5$ coupled to possibly massive supermultiplets.

Following the construction of $D = 11$ supergravity [22] and the realization that it admits an $\mathrm{AdS}_4 \times S^7$ vacuum solution [36], it was soon postulated that the Kaluza-Klein reduction on the sphere would give rise to gauged $\mathcal{N} = 8$ supergravity at the "massless" Kaluza-Klein level [33, 31, 32]. This notion was reinforced by a linearized Kaluza-Klein mode analysis demonstrating that the full spectrum of Kaluza-Klein excitations falls into supermultiplets of the $D = 4$, $\mathcal{N} = 8$ superalgebra $\mathrm{OSp}(4|8)$ [10, 93, 15]. However, demonstrating full consistency of the non-linear reduction to gauged $\mathcal{N} = 8$ supergravity has remained elusive. Nevertheless, all indications are that the reduction is consistent [87], and this has in fact been demonstrated for the related case of reducing to $D = 7$ on S^4 [86, 87].

The story is similar for the case of IIB supergravity reduced on S^5. A linearized Kaluza-Klein mode analysis demonstrates that the spectrum of Kaluza-Klein excitations falls into complete supermultiplets of the $D = 5$, $\mathcal{N} = 8$ superalgebra $\mathrm{SU}(2,2|4)$, with the lowest one corresponding to the ordinary $\mathcal{N} = 8$ supergravity multiplet [52, 71]. In this case, only partial results are known about the full non-linear reduction to gauged supergravity, but there is strong evidence for its consistency [70, 23, 81, 24].

More generally, it was conjectured in [89, 30] and [41], that, for any supergravity reduction, it is always possible to consistently truncate to the supermultiplet containing the massless graviton. This is a non-trivial statement, as the truncation must satisfy rather restrictive consistency conditions related to the gauge symmetries generated by the isometries of the internal manifold [29, 62]. This conjecture has recently been shown to be true for Sasaki-Einstein reductions of IIB supergravity on SE_5 [13] and $D = 11$ supergravity on SE_7 [41], yielding minimal $D = 5$, $\mathcal{N} = 2$ and $D = 4$, $\mathcal{N} = 2$ gauged supergravity, respectively (see also [38, 42]).

While states in the same supermultiplet do not necessarily have the same mass in gauged supergravity, the minimal supergravity multiplets, which contain the graviton, gravitino and a graviphoton, are in fact massless. Thus one may suspect that truncations to massless supermultiplets are necessarily consistent. However, it turns out that this is not the case. This was explicitly demonstrated in [62], where, for example, it was shown to be inconsistent to retain the SU(2) $\times$ SU(2) vector multiplets that naturally arise in the compactification of IIB supergravity on $T^{1,1}$.

For many of the above reasons, it has often been a challenge to explore consistent supersymmetric truncations, even at the massless Kaluza-Klein level. However, bosonic truncations retaining massive breathing and squashing modes [12] have been known to be consistent for some time. In this case, consistency is guaranteed by retaining only singlets under the internal symmetry group SU(4) $\times$ U(1) for the squashed S^7 or SU(3) $\times$ U(1) for the squashed S^5. The supersymmetry of background solutions involving the breathing and squashing modes was explored in [76], where it was further conjectured that a supersymmetric consistent truncation could be found that retains the full breathing/squashing supermultiplet.

Although this massive consistent truncation conjecture was made for squashed sphere compactifications, it naturally generalizes to compactification on more general internal spaces, such as Sasaki-Einstein spaces. For $D = 11$ supergravity compactified on a squashed S^7, written as U(1) bundled over CP^3, truncation of the $\mathcal{N} = 8$ Kaluza-Klein spectrum to SU(4) singlets under the decomposition SO(8) $\supset$ SU(4) $\times$ U(1) yields the $\mathcal{N} = 2$ supergravity multiplet[3]

$$n=0: \qquad \mathcal{D}(2,1)_0 = D(3,2)_0 + D(\frac{5}{2},\frac{3}{2})_{-1} + D(\frac{5}{2},\frac{3}{2})_1 + D(2,1)_0, \tag{II.3}$$

at the massless ($n=0$) Kaluza-Klein level. No SU(4) singlets survive at the first ($n=1$) massive Kaluza-Klein level, and the breathing and squashing modes finally make their appearance at the second ($n=2$) Kaluza-Klein level in a massive vector multiplet [76]

$$\begin{aligned} n=2: \qquad \mathcal{D}(4,0)_0 &= D(5,1)_0 + D(\frac{9}{2},\frac{1}{2})_{-1} + D(\frac{9}{2},\frac{1}{2})_1 + D(\frac{11}{2},\frac{1}{2})_{-1} + D(\frac{11}{2},\frac{1}{2})_1 \\ &\quad + D(4,0)_0 + D(5,0)_0 + D(5,0)_{-2} + D(5,0)_2 + D(6,0)_0. \end{aligned} \tag{II.4}$$

Replacing S^7 by SE_7 amounts to replacing CP^3 by an appropriate Kahler-Einstein base B. In this case, the internal isometry is generically reduced from SU(4) $\times$ U(1). Nevertheless, the notion of truncating to SU(4) singlets may simply be replaced by the prescription of truncating to zero modes on the base B. This procedure was in fact done in [37], which constructed the non-linear Kaluza-Klein reduction for all the bosonic fields contained in the above supermultiplets (II.3) and (II.4) and furthermore verified the $\mathcal{N}=2$ supersymmetry.

For the case of IIB supergravity compactified on SE_5, it is straightforward to generalize the squashed S^5 conjecture of [76]. In this case, however, the Kaluza-Klein spectrum is more involved, and is given in Table 2.1. A curious feature shows up here in that an additional LH+RH chiral matter multiplet shows up at the 'massless' Kaluza-Klein level. The $E_0=4$ scalar in this multiplet corresponds to the IIB axi-dilaton, while the additional $E_0=3$ charged scalar is precisely the charged scalar constructed in the holographic model of [49]. At the higher Kaluza-Klein levels, the breathing and squashing mode scalars correspond to the $E_0=8$ and $E_0=6$ scalars in the massive vector multiplet. In addition, consistent truncations involving the $E_0=5$ ($m^2L^2=8$) doublet of vectors in the semi-long LH+RH massive gravitino multiplet and the $E_0=7$ ($m^2L^2=24$) vector in the massive vector

[3] The OSp(4|2) super-representations $\mathcal{D}(E_0,s)_q$ and SO(2,3) representations $D(E_0,s)_q$ are labeled by energy E_0, spin s and U(1) charge q under OSp(4|2) $\supset$ SO(2,3) $\times$ U(1) $\supset$ SO(2) $\times$ SO(3) $\times$ U(1).

n	Multiplet	$SU(2,2\|1)$	$SO(2,4)\times U(1)$
0	supergraviton	$\mathcal{D}(3,\frac{1}{2},\frac{1}{2})_0$	$D(4,1,1)_0+D(3\frac{1}{2},1,\frac{1}{2})_{-1}+D(3\frac{1}{2},\frac{1}{2},1)_1+D(3,\frac{1}{2},\frac{1}{2})_0$
0	LH chiral	$\mathcal{D}(3,0,0)_2$	$D(3\frac{1}{2},\frac{1}{2},0)_1+D(3,0,0)_2+D(4,0,0)_0$
0	RH chiral	$\mathcal{D}(3,0,0)_{-2}$	$D(3\frac{1}{2},0,\frac{1}{2})_{-1}+D(3,0,0)_{-2}+D(4,0,0)_0$
1	LH massive gravitino	$\mathcal{D}(4\frac{1}{2},0,\frac{1}{2})_1$	$D(5\frac{1}{2},\frac{1}{2},1)_1+D(5,\frac{1}{2},\frac{1}{2})_0+D(5,0,1)_2$ $+D(6,0,1)_0+D(4\frac{1}{2},0,\frac{1}{2})_1+D(5\frac{1}{2},0,\frac{1}{2})_{-1}$
1	RH massive gravitino	$\mathcal{D}(4\frac{1}{2},\frac{1}{2},0)_{-1}$	$D(5\frac{1}{2},1,\frac{1}{2})_{-1}+D(5,\frac{1}{2},\frac{1}{2})_0+D(5,1,0)_{-2}$ $+D(6,1,0)_0+D(4\frac{1}{2},\frac{1}{2},0)_{-1}+D(5\frac{1}{2},\frac{1}{2},0)_1$
2	massive vector	$\mathcal{D}(6,0,0)_0$	$D(7,\frac{1}{2},\frac{1}{2})_0+D(6\frac{1}{2},\frac{1}{2},0)_{-1}+D(6\frac{1}{2},0,\frac{1}{2})_1$ $+D(7\frac{1}{2},0,\frac{1}{2})_{-1}+D(7\frac{1}{2},\frac{1}{2},0)_1+D(6,0,0)_0$ $+D(7,0,0)_{-2}+D(7,0,0)_2+D(8,0,0)_0$

Table 2.1: The truncated Kaluza-Klein spectrum of IIB supergravity on squashed S^5 [76], or equivalently on SE_5. Here n denotes the Kaluza-Klein level. The consistent truncation is expected to terminate at level $n=2$ with the breathing mode supermultiplet.

multiplet were constructed in [61, 82, 1] in the context of investigating non-relativistic conformal backgrounds in string theory.

What we have seen so far is that massive consistent truncations of IIB supergravity have been obtained keeping various subsets of the bosonic fields identified in Table 2.1. The goal of the first part of this chapter is to construct a complete non-linear Kaluza-Klein reduction of IIB supergravity on SE_5 retaining all the bosonic fields in the multiplets up to the $n=2$ level. This complements the massive Kaluza-Klein truncation of $D=11$ supergravity [37], and provides another example of a consistent truncation retaining the breathing mode supermultiplet. We proceed in Section 2.2 with the Sasaki-Einstein reduction of IIB supergravity. Then in Section 2.3 we connect the full non-linear reduction with the linearized Kaluza-Klein analysis of [52, 71] and show how the bosonic fields in Table 2.1 are related to the original IIB fields. In Section 2.4 we relate the complete non-linear reduction to previous results by performing additional truncations to a subset of active fields. Finally, we conclude in Section 2.5 with some further speculation on massive consistent truncations of supergravity.

For a discussion of the $\mathcal{N}=4$ nature of the general reduction on Sasaki-Einstein manifolds see [17, 43] which independently worked out the massive consistent truncation of IIB supergravity on SE_5. Also [96] reports related results for a particular truncation of these theories.

2.2 Sasaki-Einstein reduction of IIB supergravity

The bosonic field content of IIB supergravity consists of the NSNS fields (g_{MN}, B_{MN}, ϕ) and the RR potentials (C_0, C_2, C_4). Because of the self-dual field strength $F_5^+ = dC_4$, it is not possible to write down a covariant action. However, we may take a bosonic Lagrangian of the form

$$\mathcal{L}_{\text{IIB}} = R * 1 - \frac{1}{2\tau_2^2} d\tau \wedge * d\bar{\tau} - \frac{1}{2}\mathcal{M}_{ij} F_3^i \wedge * F_3^j - \frac{1}{4}\widetilde{F}_5 \wedge * \widetilde{F}_5 - \frac{1}{4}\epsilon_{ij} C_4 \wedge F_3^i \wedge F_3^j, \tag{II.5}$$

where self-duality $\widetilde{F}_5 = * \widetilde{F}_5$ is to be imposed by hand after deriving the equations of motion.

We have given the Lagrangian in an SL(2,$\mathcal{R}$) invariant form where

$$\tau = C_0 + ie^{-\phi}, \qquad \mathcal{M} = \frac{1}{\tau_2}\begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}, \tag{II.6}$$

and where

$$F_3^i = dB_2^i, \qquad B_2^i = \begin{pmatrix} B_2 \\ C_2 \end{pmatrix}, \qquad \widetilde{F}_5 = dC_4 + \frac{1}{2}\epsilon_{ij} B_2^i \wedge dB_2^j. \tag{II.7}$$

The equations of motion following from (II.5) and the self-duality of $\widetilde{F}_5$ are

$$\begin{aligned} d\widetilde{F}_5 &= \frac{1}{2}\epsilon_{ij} F_3^i \wedge F_3^j, \qquad \widetilde{F}_5 = * \widetilde{F}_5, \\ d(\mathcal{M}_{ij} * F_3^j) &= -\epsilon_{ij} \widetilde{F}_5 \wedge F_3^j, \\ \frac{d * d\tau}{\tau_2} + i\frac{d\tau \wedge * d\tau}{\tau_2^2} &= -\frac{i}{2\tau_2} G_3 \wedge * G_3, \end{aligned} \tag{II.8}$$

and the Einstein equation (in Ricci form)

$$\begin{aligned} R_{MN} &= \frac{1}{2\tau_2^2}\partial_{(M}\tau\partial_{N)}\bar{\tau} + \frac{1}{4}\mathcal{M}_{ij}\left(F_{MPQ}^i F_N^{j\,PQ} - \frac{1}{12} g_{MN} F_{PQR}^i F^{j\,PQR}\right) \\ &\quad + \frac{1}{4 \cdot 4!}\widetilde{F}_{MPQRS}\widetilde{F}_N{}^{PQRS}. \end{aligned} \tag{II.9}$$

In the above we have introduced the complex three-form $G_3 = F_3^2 - \tau F_3^1$. If desired, this allows us to rewrite the three-form equation of motion as

$$d * G = -i\frac{d\tau}{2\tau_2} \wedge *(G_3 + \bar{G}_3) + i\widetilde{F}_5 \wedge G_3. \tag{II.10}$$

2.2.1 The reduction ansatz

Before writing out the reduction ansatz, we note a few key features of Sasaki-Einstein manifolds. A Sasaki-Einstein manifold has a preferred U(1) isometry related to the Reeb vector. This allows us to write the metric as a U(1) fibration over a Kahler-Einstein base B

$$ds^2(SE_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \tag{II.11}$$

where $d\mathcal{A} = 2J$ with J the Kahler form on B. Moreover, B admits an SU(2) structure defined by the (1,1) and (2,0) forms J and Ω satisfying

$$J \wedge \Omega = 0, \qquad \Omega \wedge \bar{\Omega} = 2J \wedge J = 4 *_4 1, \qquad *_4 J = J, \qquad *_4\Omega = \Omega, \tag{II.12}$$

as well as

$$dJ = 0, \qquad d\Omega = 3i(d\psi + \mathcal{A}) \wedge \Omega. \tag{II.13}$$

Note that we are taking the 'unit radius' Einstein condition $R_{ij} = 4g_{ij}$ on the Sasaki-Einstein manifold, which corresponds to $R_{ab} = 6g_{ab}$ on the Kahler-Einstein base.

For the reduction, we write down the most general decomposition of the bosonic IIB fields consistent with the isometries of B. For the metric, we take

$$ds_{10}^2 = e^{2A} ds_5^2 + e^{2B} ds^2(B) + e^{2C}(\eta + A_1)^2, \tag{II.14}$$

where $\eta = d\psi + \mathcal{A}$. Since A_1 gauges the U(1) isometry, it will be related to the $D = 5$ graviphoton. Note, however, that the graviphoton receives additional contributions from the five-form.

The three-form and five-form field strengths can be expanded in a basis of invariant tensors on B. For the three-forms, we work with the potentials

$$B_2^i = b_2^i + b_1^i \wedge (\eta + A_1) + b_0^i \Omega + \bar{b}_0^i \bar{\Omega}. \tag{II.15}$$

The scalars b_0^i are complex, while the remaining fields are real. Note that we do not include a term of the form $\widetilde{b}_0^i J$ in the ansatz, as this field will act simply as a St$\ddot{u}$ckelburg field in the five-dimensional theory. In particular, it does not give rise to any new dynamics in the equations of motion as it can be repackaged as a total derivative plus terms which would simply shift b_2^i and b_1^i,

$$2\widetilde{b}_0^i J = d(\widetilde{b}_0^i \wedge (\eta + A_1)) - d\widetilde{b}_0^i \wedge (\eta + A_1) - \widetilde{b}_0^i F_2. \tag{II.16}$$

Taking $F_3^i = dB_2^i$ gives

$$\begin{aligned} F_3^i &= (db_2^i - b_1^i \wedge F) + db_1^i \wedge (\eta + A_1) - 2b_1^i \wedge J + Db_0^i \wedge \Omega + D\bar{b}_0^i \wedge \bar{\Omega} \\ &\quad + 3ib_0^i \Omega \wedge (\eta + A_1) - 3i\bar{b}_0^i \bar{\Omega} \wedge (\eta + A_1), \end{aligned} \tag{II.17}$$

where D is the U(1) gauge covariant derivative

$$Db_0^i = db_0^i - 3iA_1 b_0^i. \tag{II.18}$$

For convenience, we write this as

$$F_3^i = g_3^i + g_2^i \wedge (\eta + A_1) + g_1^i \wedge J + f_1^i \wedge \Omega + \bar{f}_1^i \wedge \bar{\Omega} + f_0^i \wedge \Omega \wedge (\eta + A_1) + \bar{f}_0^i \wedge \bar{\Omega} \wedge (\eta + A_1), \tag{II.19}$$

where our notation is such that the g^i's are real and the f^i's are complex.

For the self-dual five-form, we take

$$\widetilde{F}_5 = (1+*)[(4+\phi_0)*_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1) + q_2 \wedge \Omega \wedge (\eta + A_1) + \bar{q}_2 \wedge \bar{\Omega} \wedge (\eta + A_1)], \tag{II.20}$$

where $*_4 1$ denotes the volume form on the Kahler-Einstein base B. Note that we have pulled out a constant background component

$$\widetilde{F}_5 = 4(1+*)\mathrm{vol}(SE_5), \tag{II.21}$$

which sets up the Freund-Rubin compactification[4]. The two-forms q_2 are complex, while the other fields are real. For later convenience, we take the explicit 10-dimensional dual in the metric (II.64) to obtain

$$\begin{aligned}\widetilde{F}_5 &= (4+\phi_0) *_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1) + q_2 \wedge \Omega \wedge (\eta + A_1) \\ &\quad + \bar{q}_2 \wedge \bar{\Omega} \wedge (\eta + A_1) + e^{5A-4B-C}(4+\phi_0) * 1 - e^{3A-4B+C} * \mathbb{A}_1 \wedge (\eta + A_1) \\ &\quad + e^{A-C} * p_2 \wedge J + e^{A-C} * q_2 \wedge \Omega + e^{A-C} * \bar{q}_2 \wedge \bar{\Omega}, \end{aligned} \tag{II.22}$$

where $*$ now denotes the Hodge dual in the $D=5$ spacetime.

2.2.2 Reduction of the equations of motion

In order to obtain the reduction, it is now simply a matter of inserting the above decompositions into the IIB equations of motion. The $\widetilde{F}_5$ equation yields

$$\begin{aligned} d(e^{A-C} * p_2) &= 2e^{3A-4B+C} * \mathbb{A}_1 - p_2 \wedge F_2 + \epsilon_{ij} g_1^i \wedge g_3^j, \\ Dq_2 &= 3ie^{A-C} * q_2 + \epsilon_{ij}(f_1^i \wedge g_2^j - f_0^i g_3^j), \end{aligned} \tag{II.23}$$

along with the constraints

$$\begin{aligned} \phi_0 &= -\frac{2i}{3}\epsilon_{ij}(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j), \\ p_2 &= \frac{1}{4}\epsilon_{ij} g_1^i \wedge g_1^j - d[A_1 + \frac{1}{4}\mathbb{A}_1 + \frac{i}{6}\epsilon_{ij}(f_0^i \bar{f}_1^j - \bar{f}_0^i f_1^j)]. \end{aligned} \tag{II.24}$$

The implication of this is that $\widetilde{F}_5$ gives rise to two physical $D=5$ fields, namely a massive vector $\mathbb{A}_1$ and a complex antisymmetric tensor q_2 satisfying an odd-dimensional self-duality equation and with $m^2 = 9$. The mass of $\mathbb{A}_1$ is not directly apparent from (II.23) as it mixes with A_1 from the metric to yield the massless graviphoton as well as a $m^2 = 24$ massive vector.

[4] For simplicity, we have assumed a unit radius ($L=1$) compactification.

The F_3^i equation yields

$$
\begin{aligned}
D(e^{3A+C}\mathcal{M}_{ij} * f_1^j) &= -3ie^{5A-C}\mathcal{M}_{ij} f_0^j * 1 + \epsilon_{ij}[(4+\phi_0)e^{5A-4B-C} f_0^j * 1 - q_2 \wedge g_3^j \\
&\quad + e^{A-C} * q_2 \wedge g_2^j + e^{3A-4B+C} * \mathbb{A}_1 \wedge f_1^j], \\
d(e^{A+4B-C}\mathcal{M}_{ij} * g_2^j) &= \mathcal{M}_{ij}[e^{-A+4B+C} * g_3^j \wedge F + 4e^{3A+C} * g_1^j] \\
&\quad + \epsilon_{ij}[-2e^{A-C} * p_2 \wedge g_1^j - \mathbb{A}_1 \wedge g_3^j - 4e^{A-C}(*q_2 \wedge \bar{f}_1^j + *\bar{q}_2 \wedge f_1^j)], \\
d(e^{-A+4B+C}\mathcal{M}_{ij} * g_3^j) &= \epsilon_{ij}[-(4+\phi_0)g_3^j + \mathbb{A}_1 \wedge g_2^j - 2p_2 \wedge g_1^j - 4(q_2 \wedge \bar{f}_1^j + \bar{q}_2 \wedge f_1^j) \\
&\quad + 4e^{A-C}(\bar{f}_0^j * q_2 + f_0^j * \bar{q}_2)].
\end{aligned}
\tag{II.25}
$$

These correspond to a pair of charged scalars f_0^i, a pair of $m^2 = 8$ massive vectors g_1^i and a pair of massive antisymmetric tensors b_2^i.

The ten-dimensional Einstein equation (II.9) reduces to a five-dimensional Einstein equation, as well as the equations of motion for the breathing and squashing modes B and C and the graviphoton A_1. In particular, in the natural vielbein basis, the frame components of the ten-dimensional Ricci tensor corresponding to the reduction (II.64) are given by

$$
\begin{aligned}
{}^{10}R_{\alpha\beta} &= e^{-2A}[R_{\alpha\beta} - \nabla_\alpha\nabla_\beta(3A+4B+C) - \eta_{\alpha\beta}\partial_\gamma A\partial^\gamma(3A+4B+C) - \eta_{\alpha\beta}\Box A \\
&\quad + 3\partial_\alpha A\partial_\beta A - 4\partial_\alpha B\partial_\beta B - \partial_\alpha C\partial_\beta C + 4(\partial_\alpha A\partial_\beta B + \partial_\alpha B\partial_\beta A) \\
&\quad + (\partial_\alpha A\partial_\beta C + \partial_\alpha C\partial_\beta A)] - \frac{1}{2}e^{2C-4A}F_{\alpha\gamma}F_\beta{}^\gamma, \\
{}^{10}R_{ab} &= \delta_{ab}[6e^{-2B} - 2e^{2C-4B} - e^{-2A}(\Box B + \partial_\gamma B\partial^\gamma(3A+4B+C))], \\
{}^{10}R_{99} &= 4e^{2C-4B} + \frac{1}{4}e^{2C-4A}F_{\gamma\delta}F^{\gamma\delta} - e^{-2A}(\Box C + \partial_\gamma C\partial^\gamma(3A+4B+C)), \\
{}^{10}R_{\alpha 9} &= \frac{1}{2}e^{C-3A}[\nabla^\gamma F_{\alpha\gamma} + F_{\alpha\gamma}\partial^\gamma(A+4B+3C)].
\end{aligned}
\tag{II.26}
$$

The α and β indices correspond to the $D = 5$ spacetime, while a and b correspond to the Kahler-Einstein base B and 9 corresponds to the U(1) fiber direction. The covariant derivatives and frame indices on the right hand side of these quantities are with respect to the $D=5$ metric. In order to reduce to the $D=5$ Einstein frame metric, we now choose $3A+4B+C=0$, or

$$A = -\frac{4}{3}B - \frac{1}{3}C. \tag{II.27}$$

For convenience, we will retain A in the expressions below. However, it is not independent, and should always be thought of as a shorthand for (II.27).

Equating the ten-dimensional Ricci tensor (II.26) to the stress tensor formed out of F_3^i and $\widetilde{F}_5$ of (II.67) and (II.22), we obtain the $D=5$ Einstein equation

$$\begin{aligned}
R_{\alpha\beta} &= \frac{1}{3}\eta_{\alpha\beta}(-24e^{2A-2B} + 4e^{5A+3C} + \frac{1}{2}e^{8A}(4+\phi_0)^2) + \frac{28}{3}\partial_\alpha B\partial_\beta B + \frac{8}{3}\partial_{(\alpha}B\partial_{\beta)}C \\
&\quad + \frac{4}{3}\partial_\alpha C\partial_\beta C + \frac{1}{2\tau_2^2}\partial_{(\alpha}\tau\partial_{\beta)}\bar{\tau} + \frac{1}{2}e^{2C-2A}(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{6}\eta_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}) + \frac{1}{2}e^{-8B}\mathbb{A}_\alpha\mathbb{A}_\beta \\
&\quad + e^{A-C}[(p_{\alpha\gamma}p_\beta{}^\gamma - \frac{1}{6}\eta_{\alpha\beta}p_{\gamma\delta}p^{\gamma\delta}) + 4(q_{(\alpha}{}^\gamma\bar{q}_{\beta)\gamma} - \frac{1}{6}\eta_{\alpha\beta}q_{\gamma\delta}\bar{q}^{\gamma\delta})] \\
&\quad + \mathcal{M}_{ij}[\frac{2}{3}e^{5A-C}\eta_{\alpha\beta}(f_0^i\bar{f}_0^j + \bar{f}_0^i f_0^j) + \frac{1}{2}e^{-2A-2C}(g^i_{\alpha\gamma}g^{j\,\gamma}_\beta - \frac{1}{6}\eta_{\alpha\beta}g^i_{\gamma\delta}g^{j\,\gamma\delta}) \\
&\quad + \frac{1}{4}e^{-4A}(g^i_{\alpha\gamma\delta}g^{j\,\gamma\delta}_\beta - \frac{2}{9}\eta_{\alpha\beta}g^i_{\gamma\delta\epsilon}g^{j\,\gamma\delta\epsilon}) + e^{-4B}(g^i_\alpha g^j_\beta + 2(f^i_\alpha\bar{f}^j_\beta + \bar{f}^i_\alpha f^j_\beta))],
\end{aligned} \tag{II.28}$$

as well as the B, C and A_1 equations of motion

$$\begin{aligned}
d*dB &= [6e^{2A-2B} - 2e^{5A+3C} - \frac{1}{4}e^{8A}(4+\phi_0)^2]*1 - \frac{1}{4}e^{-8B}\mathbb{A}_1\wedge*\mathbb{A}_1 \\
&\quad + \mathcal{M}_{ij}[\frac{1}{8}e^{-2A-2C}g_2^i\wedge*g_2^j + \frac{1}{8}e^{-4A}g_3^i\wedge*g_3^j - \frac{1}{2}e^{5A-C}(f_0^i\bar{f}_0^j + \bar{f}_0^i f_0^j)*1 \\
&\quad - \frac{1}{4}e^{-4B}(g_1^i\wedge*g_1^j + 2(f_1^i\wedge*\bar{f}_1^j + \bar{f}_1^i\wedge*f_1^j))], \\
d*dC &= [4e^{5A+3C} - \frac{1}{4}e^{8A}(4+\phi_0)^2]*1 + \frac{1}{2}e^{2C-2A}F_2\wedge*F_2 + \frac{1}{4}e^{-8B}\mathbb{A}_1\wedge*\mathbb{A}_1 \\
&\quad - \frac{1}{2}e^{A-C}(p_2\wedge*p_2 + 4q_2\wedge*\bar{q}_2) + \mathcal{M}_{ij}[-\frac{3}{8}e^{-2A-2C}g_2^i\wedge*g_2^j \\
&\quad + \frac{1}{8}e^{-4A}g_3^i\wedge*g_3^j - \frac{3}{2}e^{5A-C}(f_0^i\bar{f}_0^j + \bar{f}_0^i f_0^j)*1 \\
&\quad + \frac{1}{4}e^{-4B}(g_1^i\wedge*g_1^j + 2(f_1^i\wedge*\bar{f}_1^j + \bar{f}_1^i\wedge*f_1^j))], \\
d(e^{2C-2A}*F_2) &= (4+\phi_0)e^{-8B}*\mathbb{A}_1 - p_2\wedge p_2 - 4q_2\wedge\bar{q}_2 \\
&\quad + \mathcal{M}_{ij}[4e^{-4B}*(f_0^i\bar{f}_1^j + \bar{f}_0^i f_1^j) + e^{-4A}*g_3^i\wedge g_2^j].
\end{aligned} \tag{II.29}$$

Note that, in order to obtain the $D = 5$ Einstein equation, we had to shift the reduction of ${}^{10}R_{\alpha\beta}$ an appropriate combination of ${}^{10}R_{ab}$ and ${}^{10}R_{99}$ in order to remove the $\eta_{\alpha\beta}\Box A$ component in the first line of (II.26).

The IIB equations of motion thus reduce to (II.23), (II.25), (II.28) and (II.29) as well as the axi-dilaton equation, which we have not written down explicitly, but which will be shown to be consistent below.

2.2.3 The effective five-dimensional Lagrangian

We now wish to construct an effective $D = 5$ Lagrangian which reproduces the above equations of motion. This may be done by noting that the $D = 5$ Einstein equation (II.28) arises naturally from a Lagrangian of the form

$$\begin{aligned}\mathcal{L} \;=\;& R * 1 + (24e^{2A-2B} - 4e^{5A+3C} - \frac{1}{2}e^{8A}(4+\phi_0)^2) * 1 - \frac{28}{3}dB \wedge *dB - \frac{8}{3}dB \wedge *dC \\ & -\frac{4}{3}dC \wedge *dC - \frac{1}{2\tau_2^2}d\tau \wedge *d\bar{\tau} - \frac{1}{2}e^{2C-2A}F_2 \wedge *F_2 - \frac{1}{2}e^{-8B}\mathbb{A}_1 \wedge *\mathbb{A}_1 \\ & -e^{A-C}(p_2 \wedge *p_2 + 4q_2 \wedge *\bar{q}_2) + \mathcal{M}_{ij}[-2e^{5A-C}(f_0^i \bar{f}_0^j + \bar{f}_0^i f_0^j) * 1 \\ & -\frac{1}{2}e^{-2A-2C}g_2^i \wedge *g_2^j - \frac{1}{2}e^{-4A}g_3^i \wedge *g_3^j - e^{-4B}(g_1^i \wedge *g_1^j + 2(f_1^i \wedge *\bar{f}_1^j + \bar{f}_1^i \wedge *f_1^j))] \\ & +\mathcal{L}_{CS}. \end{aligned} \tag{II.30}$$

We have included a Chern-Simons piece $\mathcal{L}_{CS}$ which cannot be determined from the Einstein equation.

It is now possible to verify that (II.30) reproduces all the terms in the equations of motion (II.23), (II.25) and (II.29) involving the metric (*ie* the Hodge *). The remaining terms may be obtained from the addition of the topological piece

$$\begin{aligned}
\mathcal{L}_{CS} &= \frac{2i}{3}(q_2\wedge d\bar{q}_2 - \bar{q}_2\wedge dq_2) - 4A_1\wedge q_2\wedge\bar{q}_2 + 2\epsilon_{ij}b_2^i\wedge db_2^j \\
&\quad +\frac{4i}{3}[(\bar{q}_2 - \frac{i}{6}\epsilon_{ij}\bar{f}_0^i g_2^j)\wedge\epsilon_{kl}(f_1^k\wedge g_2^l - f_0^k g_3^l) - (q_2 + \frac{i}{6}\epsilon_{ij}f_0^i g_2^j)\wedge\epsilon_{kl}(\bar{f}_1^k\wedge g_2^l - \bar{f}_0^k g_3^l)] \\
&\quad -A_1\wedge(p_2 - \frac{1}{4}\epsilon_{ij}g_1^i\wedge g_1^j)\wedge(p_2 - \frac{1}{4}\epsilon_{kl}g_1^k\wedge g_1^l) \\
&\quad -2[\frac{1}{4}\mathbb{A}_1 + \frac{i}{6}\epsilon_{ij}(f_0^i\bar{f}_1^j - \bar{f}_0^i f_1^j)]\wedge\epsilon_{kl}(g_1^k\wedge g_3^l - \frac{1}{4}g_1^k\wedge g_1^l\wedge F_2). \qquad \text{(II.31)}
\end{aligned}$$

Here we recall the definitions

$$f_0^i = 3ib_0^i,\qquad f_1^i = Db_0^i,\qquad g_1^i = -2b_1^i,\qquad g_2^i = db_1^i,\qquad g_3^i = db_2^i - b_1^i\wedge F_2, \qquad \text{(II.32)}$$

implicit in (II.17) and (II.67). Furthermore, ϕ_0 and p_2 are given by (II.24). Note that, while $\mathbb{A}_1$ is massive, and does not have a gauge invariance associated with it, it is natural to make the shift

$$\mathbb{A}_1 \to \mathbb{A}_1' - \frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j - \bar{f}_0^i f_1^j), \qquad \text{(II.33)}$$

so that

$$p_2 = \frac{1}{4}\epsilon_{ij}g_1^i\wedge g_1^j - F_2 - \frac{1}{4}\mathbb{F}_2', \qquad \text{(II.34)}$$

where $\mathbb{F}_2' = d\mathbb{A}_1'$.

We now turn to the axi-dilaton equation obtained from (II.30). Since τ only shows up in the kinetic term and in $\mathcal{M}_{ij}$, we see that the τ equation of motion obtained from the $D=5$ Lagrangian reproduces that obtained from the original IIB Lagrangian. This is because the quantity in the square brackets multiplying $\mathcal{M}_{ij}$ in (II.30) is the straightforward reduction of $-\frac{1}{2}F_3^i\wedge *F_3^j$ in the original IIB Lagrangian (II.5).

2.3 Matching the linearized Kaluza-Klein analysis

The complete $D=5$ Lagrangian, as given by (II.30) and (II.31), is somewhat opaque. Thus in this section, we demonstrate that it in fact contains the fields corresponding to the Kaluza-Klein mass spectrum noted in Table 2.1. To do this, it is sufficient to look at the linearized level. We first note that the effective $D = 5$ fields are the complex scalars (τ, b_0^i), real scalars (B, C), one-form potentials $(A_1, b_1^i, \mathbb{A}_1)$, pair of real two-forms (b_2^i), the complex two-form (q_2), and of course the metric $(g_{\mu\nu})$. The $D = 5$ equations of motion (II.23), (II.25) and (II.29) may be linearized on the matter fields to obtain the set

$$\begin{aligned}
d * db_0^i &= (9\delta_j^i + 12i\mathcal{N}^i{}_j)b_0^j * 1, \\
d * db_1^i &= -8 * b_1^i, \\
d * db_2^i &= -4\mathcal{N}^i{}_j db_2^j, \\
dq_2 &= 3i * q_2, \\
d * F_2 &= 4 * \mathbb{A}_1, \qquad\qquad d * F_2 + \frac{1}{4} d * \mathbb{F}_2 = -2 * \mathbb{A}_1, \\
d * dB &= 4(7B + C) * 1, \qquad d * dC = 16(B + C) * 1.
\end{aligned} \tag{II.35}$$

Here we have introduced

$$\mathcal{N} = \mathcal{M}^{-1}\epsilon = \frac{1}{\tau_2}\begin{pmatrix} -\tau_1 & 1 \\ -|\tau|^2 & \tau_1 \end{pmatrix}, \tag{II.36}$$

with eigenvalues $+i$ and $-i$, corresponding to eigenvectors $\begin{pmatrix} 1 & \tau \end{pmatrix}^T$ and $\begin{pmatrix} 1 & \bar{\tau} \end{pmatrix}^T$, respectively.

The first equation in (II.35) then decomposes into a pair of equations for the complex scalars $b_0^{m^2=-3}$ and $b_0^{m^2=21}$ with masses $m^2 = -3$ and $m^2 = 21$ according to

$$b_0^i = \begin{pmatrix} 1 \\ \tau \end{pmatrix} b_0^{m^2=-3} + \begin{pmatrix} 1 \\ \bar{\tau} \end{pmatrix} b_0^{m^2=21}. \tag{II.37}$$

The second equation is that of an SL(2,$\mathbb{R}$) doublet of real vectors b_1^i with mass $m^2 = 8$. The third equation can be converted to an odd-dimensional self-duality equation [98] $db_2^i = 4\mathcal{N}^i{}_j * b_2^j$, for a doublet of antisymmetric tensors b_2^i with mass $m^2 = 16$. The fourth equation is already in odd-dimensional self-duality form, and shows that the complex antisymmetric tensor q_2 has mass $m^2 = 9$.

The vector equations can be diagonalized

$$d * (F_2 + \frac{1}{6}\mathbb{F}_2) = 0, \qquad d * \mathbb{F}_2 = -24 * \mathbb{A}_1, \tag{II.38}$$

to identify the massless graviphoton $A_1 + \frac{1}{6}\mathbb{A}_1$ and the massive $m^2 = 24$ vector $\mathbb{A}_1$. Finally the B and C equations may be diagonalized to identify the $m^2 = 32$ breathing and $m^2 = 12$ squashing modes

$$d * d\rho = 32\rho * 1, \qquad d * d\sigma = 12\sigma * 1, \tag{II.39}$$

where

$$B = \rho + \frac{1}{2}\sigma, \qquad C = \rho - 2\sigma. \tag{II.40}$$

It is now possible to see how the above linearized modes are organized into $\mathcal{N} = 2$ supermultiplets. As shown in Table 2.1, at the zeroth Kaluza-Klein level, we have the graviton supermultiplet

$$\mathcal{D}(3, \frac{1}{2}, \frac{1}{2})_0 = D(4, 1, 1)_0 + D(3\frac{1}{2}, 1, \frac{1}{2})_{-1} + D(3\frac{1}{2}, \frac{1}{2}, 1)_1 + D(3, \frac{1}{2}, \frac{1}{2})_0, \tag{II.41}$$

with bosonic fields being the graviton $g_{\mu\nu}$ and the massless graviphoton $A_1 + \frac{1}{6}\mathbb{A}_1$. Still at the zeroth level, there is also a LH+RH chiral multiplet

$$\begin{aligned} \mathcal{D}(3,0,0)_2 &= D(3\frac{1}{2}, \frac{1}{2}, 0)_1 + D(3,0,0)_2 + D(4,0,0)_0, \\ \mathcal{D}(3,0,0)_{-2} &= D(3\frac{1}{2}, 0, \frac{1}{2})_{-1} + D(3,0,0)_{-2} + D(4,0,0)_0. \end{aligned} \tag{II.42}$$

The charged $E_0 = 3$ scalar corresponds to the $m^2 = -3$ scalar $b_0^{m^2=-3}$, while the complex $E_0 = 4$ scalar is the axi-dilaton τ.

n	Multiplet	State	Field
0	supergraviton	$D(4,1,1)_0$	$g_{\mu\nu}$
		$D(3,\frac{1}{2},\frac{1}{2})_0$	$A_1 + \frac{1}{6}\mathbb{A}_1$
0	LH+RH chiral	$D(3,0,0)_{\pm 2}$	$b_0^{m^2=-3}$
		$D(4,0,0)_0 + D(4,0,0)_0$	τ
1	LH+RH massive gravitino	$D(5,\frac{1}{2},\frac{1}{2})_0 + D(5,\frac{1}{2},\frac{1}{2})_0$	b_1^i
		$D(5,0,1)_2 + D(5,1,0)_{-2}$	q_2
		$D(6,0,1)_0 + D(6,1,0)_0$	b_2^i
2	massive vector	$D(7,\frac{1}{2},\frac{1}{2})_0$	$\mathbb{A}_1$
		$D(6,0,0)_0$	σ
		$D(7,0,0)_{\pm}2$	$b_0^{m^2=21}$
		$D(8,0,0)_0$	ρ

Table 2.2: Identification of the bosonic states in the Kaluza-Klein spectrum with the linearized modes in the reduction.

At the first Kaluza-Klein level, we have a semi-long LH+RH massive gravitino multiplet

$$\begin{aligned}
\mathcal{D}(4\frac{1}{2},0,\frac{1}{2})_1 &= D(5\frac{1}{2},\frac{1}{2},1)_1 + D(5,\frac{1}{2},\frac{1}{2})_0 + D(5,0,1)_2 + D(6,0,1)_0 \\
&\quad + D(4\frac{1}{2},0,\frac{1}{2})_1 + D(5\frac{1}{2},0,\frac{1}{2})_{-1}, \\
\mathcal{D}(4\frac{1}{2},\frac{1}{2},0)_{-1} &= D(5\frac{1}{2},1,\frac{1}{2})_{-1} + D(5,\frac{1}{2},\frac{1}{2})_0 + D(5,1,0)_{-2} + D(6,1,0)_0 \\
&\quad + D(4\frac{1}{2},\frac{1}{2},0)_{-1} + D(5\frac{1}{2},\frac{1}{2},0)_1.
\end{aligned} \tag{II.43}$$

The bosonic field content is an SL(2,$\mathbb{R}$) doublet of $m^2 = 8$ ($E_0 = 5$) vectors b_1^i, a charged $m^2 = 9$ ($E_0 = 5$) anti-symmetric tensor q_2, and a doublet of $m^2 = 16$ ($E_0 = 6$) anti-symmetric tensors b_2^i.

At the second Kaluza-Klein level, we have a massive vector multiplet

$$\begin{aligned}
\mathcal{D}(6,0,0)_0 &= D(7,\frac{1}{2},\frac{1}{2})_0 + D(6\frac{1}{2},\frac{1}{2},0)_{-1} + D(6\frac{1}{2},0,\frac{1}{2})_1 + D(7\frac{1}{2},0,\frac{1}{2})_{-1} + D(7\frac{1}{2},\frac{1}{2},0)_1 \\
&\quad + D(6,0,0)_0 + D(7,0,0)_{-2} + D(7,0,0)_2 + D(8,0,0)_0.
\end{aligned} \tag{II.44}$$

The massive $E_0 = 7$ vector is the $m^2 = 24$ mode $\mathbb{A}_1$. The real $E_0 = 6$ and $E_0 = 8$ scalars are the $m^2 = 12$ squashing and $m^2 = 32$ breathing modes, σ and ρ, respectively. The charged $E_0 = 7$ scalar is $b_0^{m^2=21}$ with $m^2 = 21$. This identification of the linearized fields with the Kaluza-Klein modes is shown in Table 2.3.

For the case of IIB supergravity on S^5, is interesting to note that these fields lie at the lowest level of the massive trajectories in the Kaluza-Klein mode decomposition of the $D = 10$ fields [52, 71]. We note that the massive Kaluza-Klein tower is built out of scalar, vector and tensor harmonics on S^5, and the lowest harmonics generally have simple behavior on the internal sphere coordinates. For example, the lowest scalar harmonic is the constant mode on the sphere, while the lowest vector harmonics generate the Killing vectors on the sphere. It is presumably the simplicity of the lowest harmonics that allows the truncation to be consistent, even at the non-linear level.

Although the harmonics on SE_5 are more involved (see *e.g.* [18] for the case of $T^{1,1}$), it is clear that the decomposition (II.15) and (II.22) of the $D = 10$ fields in terms of invariant tensors on SE_5 is equivalent to the truncation to the lowest harmonics on the sphere. This appears to be an essential feature guaranteeing the consistency of the massive truncation, and hence we do not expect to be able to keep any additional multiplets in the Kaluza-Klein tower beyond the $n = 2$ level.

2.4 Further truncations

In order to make a connection with previous results on massive consistent truncations of IIB supergravity, we note that the semi-long LH+RH massive gravitino multiplet at the first Kaluza-Klein level may be truncated out by setting

$$b_1^i = 0, \qquad b_2^i = 0, \qquad q_2 = 0. \tag{II.45}$$

It is easy to see that this truncation is consistent, since the respective equations of motion for q_2 in (II.23) and g_2^i and g_3^i in (II.25) are trivially satisfied in this case. The resulting

$D=5$ Lagrangian takes the form

$$\begin{aligned}\mathcal{L} &= R*1+(24e^{2A-2B}-4e^{5A+3C}-\frac{1}{2}e^{8A}(4+\phi_0)^2)*1-\frac{28}{3}dB\wedge *dB-\frac{8}{3}dB\wedge *dC\\
&\quad -\frac{4}{3}dC\wedge *dC-\frac{1}{2\tau_2^2}d\tau\wedge *d\bar\tau-\frac{1}{2}e^{2C-2A}F_2\wedge *F_2-e^{A-C}(F_2+\frac{1}{4}\mathbb{F}_2')\wedge *(F_2+\frac{1}{4}\mathbb{F}_2')\\
&\quad -\frac{1}{2}e^{-8B}[\mathbb{A}_1'-\frac{2i}{3}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j)]\wedge *[\mathbb{A}_1'-\frac{2i}{3}\epsilon_{ij}(f_0^i\bar f_1^j-\bar f_0^i f_1^j)]\\
&\quad -2\mathcal{M}_{ij}[e^{5A-C}(f_0^i\bar f_0^j+\bar f_0^i f_0^j)*1+e^{-4B}(f_1^i\wedge *\bar f_1^j+\bar f_1^i\wedge *f_1^j)]\\
&\quad -A_1\wedge(F_2+\frac{1}{4}\mathbb{F}_2')\wedge(F_2+\frac{1}{4}\mathbb{F}_2'),\end{aligned}\tag{II.46}$$

where

$$f_0^i=3ib_0^i,\qquad f_1^i=Db_0^i,\qquad \phi_0=-\frac{2i}{3}\epsilon_{ij}(f_0^i\bar f_0^j-\bar f_0^i f_0^j).\tag{II.47}$$

A further truncation to the massless $\mathcal{N}=2$ supergravity sector may be obtained by setting

$$b_0^i=0,\qquad B=0,\qquad C=0,\qquad \mathbb{A}_1=0,\tag{II.48}$$

along with taking a constant background for the axi-dilaton, $\tau=\tau_0$. This leaves only $g_{\mu\nu}$ and A_1, and yields the standard Lagrangian for the bosonic fields of minimal gauged supergravity

$$\mathcal{L}=R*1+12g^2*1-\frac{1}{2}F_2\wedge *F_2-\frac{1}{3\sqrt{3}}A_1\wedge F_2\wedge F_2,\tag{II.49}$$

where we have rescaled the graviphoton, $A_1\to\frac{1}{\sqrt3}A_1$, so that it has a canonical kinetic term, and where we have restored the dimensionful gauged supergravity coupling g.

2.4.1 Truncation to the zeroth Kaluza-Klein level

Beyond the truncation to minimal supergravity discussed above, the first nontrivial truncation involves keeping only the lowest Kaluza-Klein level fields $\{\tau, b_0^{m^2=-3}\}$ dynamical. In what follows we will denote $b_0^{m^2=-3}$ simply as b so that $(b_0^1,b_0^2)=(b,\tau b)$. This truncation is not as simple as setting all other fields to zero, as the equations of motion demand certain constraints to be satisfied. For this case we start with the Lagrangian (B.1), obtained by setting $b_2^i = b_1^i = q_2 = 0$. We then impose the constraints

$$b_0^{m^2=21} = 0, \qquad e^{4B} = e^{-4C} = 1 - 4\tau_2|b|^2, \qquad \mathbb{A}_1 = -4i\tau_2(bD\bar{b} - \bar{b}Db) + 4|b|^2 d\tau_1. \tag{II.50}$$

These in turn imply that

$$\phi_0 = -24\tau_2|b|^2, \qquad p_2 = -dA_1. \tag{II.51}$$

To guarantee consistency, we have to check four constraints from the equations of motion (the B, C, f_0^i, and combined Maxwell Equation). They are all verified to hold identically, and hence the truncation to the supergravity plus the LH+RH chiral multiplet is consistent. The Lagrangian is given by

$$\begin{aligned}\mathcal{L} \;=\;& R*1 + \left[24(1-3\tau_2|b|^2)e^{-4B} - 4e^{-8B} - \frac{1}{2}e^{-8B}(4+\phi_0)^2\right]*1 - 8dB\wedge *dB \\ & -\frac{3}{2}F_2\wedge *F_2 - \frac{1}{2}e^{-8B}\mathbb{A}_1\wedge *\mathbb{A}_1 - 8e^{-4B}\tau_2 Db\wedge *D\bar{b} - 2ie^{-4B}(\bar{b}Db\wedge *d\bar{\tau} - bD\bar{b}\wedge *d\tau) \\ & -\frac{1}{2\tau_2^2}(1+8e^{-4B}\tau_2|b|^2)d\tau\wedge *d\bar{\tau} - A_1\wedge F_2\wedge F_2.\end{aligned} \tag{II.52}$$

This expression can be simplified by defining $\lambda \equiv 4\tau_2|b|^2$, giving

$$\begin{aligned}\mathcal{L} \;=\;& R*1 + \frac{6(2-3\lambda)}{(1-\lambda)^2}*1 - \frac{d\lambda\wedge *d\lambda}{2(1-\lambda)^2} - \frac{(1+\lambda)d\tau\wedge *d\bar{\tau}}{2(1-\lambda)\tau_2^2} - \frac{3}{2}F_2\wedge *F_2 - \frac{\mathbb{A}_1\wedge *\mathbb{A}_1}{2(1-\lambda)^2} \\ & -\frac{8\tau_2 Db\wedge *D\bar{b}}{1-\lambda} - \frac{2i}{1-\lambda}(\bar{b}Db\wedge *d\bar{\tau} - bD\bar{b}\wedge *d\tau) - A_1\wedge F_2\wedge F_2.\end{aligned} \tag{II.53}$$

If we further truncate the model by setting $\tau = ie^{-\phi_0} = ig_s^{-1}$, which is consistent with the equation of motion for τ given in (II.8), this reproduces the model used in [49] to describe a holographic superconductor using a $m^2 = -3$ and $q = 2$ charged scalar. If we denote $b = \sqrt{g_s} f e^{i\theta}$, the truncated Lagrangian reads

$$\begin{aligned}\mathcal{L} \;=\;& R*1 - \frac{3}{2}F_2\wedge *F_2 - A_1\wedge F_2\wedge F_2 \\ & +12\frac{(1-6f^2)}{(1-4f^2)^2}*1 - 8\frac{df\wedge *df}{(1-4f^2)^2} - 8f^2\frac{(d\theta - 3A_1)\wedge *(d\theta - 3A_1)}{(1-4f^2)^2}.\end{aligned} \tag{II.54}$$

A further redefinition $f = \frac{1}{2}\tanh\frac{\eta}{2}$ then reproduces the Lagrangian given in [49].

2.4.2 Truncation to the second Kaluza-Klein level

Starting with the Lagrangian (B.1) with $b_2^i = b_1^i = q_2 = 0$, it is possible to retain the $b_0^{m^2=21}$ scalar by setting $b_0^{m^2=-3} = 0$. In this case, we first let $b_0^2 = \bar{\tau} b_0^1$ and define $b_0^1 = \sqrt{g_s} h e^{i\xi}$, so that (h, ξ) describe the $m^2 = 21$ scalar. Again, the scalar equations of motion lead to constraints, and in particular the first equation in (II.25) yields the equation of motion for h and ξ as well as

$$d(e^{3A+C} * d\tau) + ie^{3A+C}\frac{1}{\tau_2} d\tau \wedge * d\tau = 0. \tag{II.55}$$

This is simply the τ equation of motion without any sources, and the simplest thing to do is to set τ to be constant, $\tau = ie^{-\phi_0} = ig_s^{-1}$. The remaining field content is then $\{g_{\mu\nu}, A_1, \rho, \sigma, b_0^{m^2=21}, \mathbb{A}_1\}$, corresponding to the supergravity multiplet coupled to the massive vector multiplet. It is now straightforward to complete the truncation, and the Lagrangian is given by

$$\begin{aligned}
\mathcal{L} \;=\;& R * 1 + \big(24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho}(1+6h^2)^2\big) * 1 - \frac{40}{3} d\rho \wedge * d\rho - 5 d\sigma \wedge * d\sigma \\
& - \frac{1}{2} e^{\frac{16}{3}\rho - 4\sigma} F_2 \wedge * F_2 - e^{-\frac{8}{3}\rho + 2\sigma}(F_2 + \frac{1}{4}\mathbb{F}_2') \wedge *(F_2 + \frac{1}{4}\mathbb{F}_2') \\
& - \frac{1}{2} e^{-8\rho - 4\sigma}(\mathbb{A}_1' + 8h^2\Gamma) \wedge *(\mathbb{A}_1' + 8h^2\Gamma) - A_1 \wedge (F_2 + \frac{1}{4}\mathbb{F}_2') \wedge (F_2 + \frac{1}{4}\mathbb{F}_2') \\
& - 8\big(e^{-4\rho-2\sigma} dh \wedge * dh + e^{-4\rho-2\sigma} h^2 \Gamma \wedge * \Gamma + e^{-\frac{28}{3}\rho + 2\sigma} h^2 * 1\big),
\end{aligned} \tag{II.56}$$

where we have defined $\Gamma = d\xi - 3A_1$.

We can further truncate this by removing the $m^2 = 21$ scalar (*i.e.* by setting $h = \xi = 0$), giving the Lagrangian

$$\begin{aligned}
\mathcal{L} \;=\;& R * 1 + (24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho}) * 1 - \frac{40}{3} d\rho \wedge * d\rho - 5 d\sigma \wedge * d\sigma \\
& - \frac{1}{2} e^{\frac{16}{3}\rho - 4\sigma} F_2 \wedge * F_2 - e^{-\frac{8}{3}\rho + 2\sigma}(F_2 + \frac{1}{4}\mathbb{F}_2') \wedge *(F_2 + \frac{1}{4}\mathbb{F}_2') - \frac{1}{2} e^{-8\rho - 4\sigma} \mathbb{A}_1' \wedge * \mathbb{A}_1' \\
& - A_1 \wedge (F_2 + \frac{1}{4}\mathbb{F}_2') \wedge (F_2 + \frac{1}{4}\mathbb{F}_2'),
\end{aligned} \tag{II.57}$$

which corresponds to the $m^2 = 24$ massive vector field truncation of [82].

2.4.3 Non-supersymmetric truncations

All the truncations we have listed so far have field content which fills the bosonic sector of AdS_5 supermultiplets and so are presumably supersymmetric truncations. It is also useful to consider truncations which contain dynamical fields belonging to different supermultiplets, without keeping the entire multiplet. In this sense these truncations are not supersymmetric, although they are perfectly consistent truncations and solutions of the ten-dimensional equations of motion. For these truncations, we start with the complete Lagrangian given in (II.30) and (II.31).

Massive vector field

The first non-supersymmetric truncation we will discuss involves keeping the $m^2 = 8$ vector field, b_1^i, and has already been noted in [82]. The field content in this truncation consists of one component of b_1^i (denoted b_1), τ_2, ρ, σ and $g_{\mu\nu}$. Note that the graviphoton is turned off here so that even at the lowest level this cannot be supersymmetric. Furthermore, by keeping only one component of b_1^i, the τ equation of motion demands that we must set $\tau_1 = 0$. With this field content, the $D = 10$ constraints (II.24) are trivially satisfied with $\phi_0 = 0$ and $p_2 = 0$, and the Lagrangian (II.30) becomes [82]

$$\begin{aligned} \mathcal{L} &= R*1 + (24e^{-\frac{16}{3}\rho-\sigma} - 4e^{-\frac{16}{3}\rho-6\sigma} - 8e^{-\frac{40}{3}\rho})*1 - \frac{40}{3}d\rho\wedge *d\rho - 5d\sigma\wedge *d\sigma \\ &\quad -\frac{1}{2\tau_2^2}d\tau_2\wedge *d\tau_2 - \frac{1}{2}\tau_2 e^{\frac{4}{3}\rho+4\sigma}db_1\wedge *db_1 - 4\tau_2 e^{-4\rho-2\sigma}b_1\wedge *b_1. \end{aligned} \tag{II.58}$$

Complex massive anti-symmetric tensor

We can also truncate to theories containing the $m^2 = 9$ complex anti-symmetric tensor field q_2. The field content here is given by, q_2, $\mathbb{A}_1$, B, C, τ, $g_{\mu\nu}$ and A_1. The $D = 10$ constraints become $\phi_0 = 0$ and $p_2 = -dA_1 - \frac{1}{4}d\mathbb{A}_1$. All the other equations of motion are either satisfied by setting the rest of the fields to zero or can be derived from the Lagrangian

$$\begin{aligned}\mathcal{L} &= R*1+(24e^{-\frac{16}{3}\rho-\sigma}-4e^{-\frac{16}{3}\rho-6\sigma}-8e^{-\frac{40}{3}\rho})*1-\frac{40}{3}d\rho\wedge*d\rho-5d\sigma\wedge*d\sigma\\ &\quad -\frac{1}{2}e^{\frac{16}{3}\rho-4\sigma}F_2\wedge*F_2-e^{-\frac{8}{3}\rho+2\sigma}(p_2\wedge*p_2+4q_2\wedge*\bar{q}_2)-\frac{1}{2\tau_2^2}d\tau\wedge*d\bar{\tau}\\ &\quad -\frac{1}{2}e^{-8\rho-4\sigma}\mathbb{A}_1\wedge*\mathbb{A}_1+\frac{2i}{3}(q_2\wedge d\bar{q}_2-\bar{q}_2\wedge dq_2)-A_1\wedge p_2\wedge p_2-4A_1\wedge q_2\wedge\bar{q}_2.\end{aligned} \tag{II.59}$$

Note that it is consistent to further truncate to a constant axi-dilaton $\tau=\tau_0$.

Real massive anti-symmetric tensor

Along similar lines to the case above for a massive vector field, we can set $A_1=0$ and make a truncation including the $m^2=16$ real anti-symmetric tensor doublet b_2^i by keeping only the graviton coupled to b_2^i, τ, ρ and σ. Again, the equations of motion for the other fields are trivially satisfied and the constraints are also trivial $\phi_0=0$ and $p_2=0$. This leaves the Lagrangian

$$\begin{aligned}\mathcal{L} &= R*1+(24e^{-\frac{16}{3}\rho-\sigma}-4e^{-\frac{16}{3}\rho-6\sigma}-8e^{-\frac{40}{3}\rho})*1-\frac{40}{3}d\rho\wedge*d\rho-5d\sigma\wedge*d\sigma\\ &\quad -\frac{1}{2\tau_2^2}d\tau\wedge*d\bar{\tau}-\frac{1}{2}e^{\frac{20}{3}\rho}\mathcal{M}_{ij}db_2^i\wedge*db_2^j+2\epsilon_{ij}b_2^i\wedge*db_2^j.\end{aligned} \tag{II.60}$$

As in the previous truncation it is consistent to further truncate to $\tau=\tau_0$.

2.5 Discussion

In the above, we have examined massive reductions of 10-dimensional IIB supergravity on Sasaki-Einstein manifolds. By utilizing the structure of SE_5, we have given a general decomposition of the IIB fields based on the invariant tensors associated with the internal manifold. The field content obtained in five-dimensions completes the bosonic sector of various AdS_5 supermultiplets, and in particular they fill out the lowest Kaluza-Klein tower up to the breathing mode supermultiplet. This proves, at least at the level of the bosonic fields, the conjecture raised in [76] that a consistent massive truncation may be obtained by truncating to the singlet sector on the Kahler-Einstein base B (which is CP^2 for the squashed S^5) and further restricting to the level of the breathing mode multiplet and below.

As suggested at the end of Section 2.3, it is this truncation to constant modes on the base B that ensures the consistency of the reduction. In a sense, this is a generalization of the old consistency criterion of restricting to singlets of the internal isometry group, except that here restricting to singlets of an appropriate subgroup turned out to be sufficient. For this reason, we believe it is not that the breathing mode is special in itself which allows for a consistent truncation retaining its supermultiplet, but rather that in the examples given here and in [37], the breathing mode superpartners so happen to be the lowest harmonics in their respective Kaluza-Klein towers. It is an unusual feature of Kaluza-Klein compactifications on curved internal spaces that states originating from different levels of the harmonic expansion may combine into a single supermultiplet. Thus, while the breathing mode is always the lowest state in its tower (being a constant mode on the internal space), its superpartners may carry excitations on the internal space. This does not occur for the $\mathcal{N} = 2$ compactification of IIB supergravity on SE_5 (nor does it for $D = 11$ supergravity on SE_7). However, in extended theories, such as IIB supergravity on the round S^5, the superpartners will involve non-trivial harmonics. In particular, the $\mathcal{N} = 8$ superpartners to the breathing mode include a massive spin-2 excitation of the graviton involving the second harmonic (d-waves) on the sphere. Thus we believe it to be unlikely that an $\mathcal{N} = 8$ massive truncation with the breathing mode multiplet will be consistent.

Consistent truncations of the type discussed here have recently been of particular interest in the growing literature on AdS/CFT applications to condensed matter systems. Until recently a strictly phenomenological approach has been taken in this area. In these systems the inclusion of a scalar condensate is required in the gravity theory to source an operator whose expectation value acts as an order parameter describing superconductor/superfluid phase transitions in the strongly coupled system. In the phenomenological approach, the origin of this scalar and its properties have not been of immediate interest; rather the general behavior was determined and many interesting similarities to real condensed matter systems have been noted. However, this approach lacks strong theoretical control in that systems are described by a set of free parameters which can be tuned to provide the property of interest. Recently there has been some work to embed these models in UV complete theories, where the parameters are no longer free but are determined by the underlying features of the theory, such as an origin in string theory. The discussion here has put these reductions into a more general framework and gives further examples of UV complete systems whose duals may have useful applications in the AdS/CMT correspondence.

Given that the fields in these truncations fall into specific supermultiplets it is an obvious and relevant question to discuss their fermionic partners. This would involve reducing the supersymmetry variations and fermion equations in ten-dimensions down to five-dimensions and determining the complete supersymmetric action of these truncations. This is also of interest in terms of AdS/CFT where there has been much interest in describing fermion behavior in condensed matter systems such as the Fermi-liquid theory using the holographic correspondence. In particular, the full supersymmetric action could give us examples of specific interactions studied in these systems coupling scalar condensates to the fermionic excitations [20, 34, 51].

While the consistent truncation procedure in these cases is guaranteed to preserve supersymmetry, until now much of the focus has been on the bosonic sectors. Nevertheless it would be useful to have an explicit realization of the fermion reduction as well. This is especially interesting in light of holographic models of superconductivity in 2+1 [39, 40] and 3+1 [49] dimensions, where electronic properties often involve fermion correlators and not just the bosons. Along these lines, the fermion sector of the reduction of $D = 11$ supergravity on squashed SE_7 was recently constructed in [7]. The procedure is similar to that used in the bosonic reduction. In particular, the eleven-dimensional fermions may be expanded in terms of invariant tensors multiplying Killing spinors. This naturally retains the lowest modes in spinor harmonics in each of the Kaluza-Klein towers, and ensures the overall consistency of the reduction.

In the remaining sections of the chapter, we focus on the $\mathcal{N} = 2$ truncation of IIB supergravity reduced on squashed SE_5, and demonstrate the consistent reduction of the fermion sector, at least to quadratic order in the fermions. As demonstrated in [17, 77, 43, 96], the full bosonic sector of this reduction corresponds to an $\mathcal{N} = 4$ theory. However, by truncating out the $\mathcal{N} = 2$ massive gravitino multiplet, we may bring this down to $\mathcal{N} = 2$. While our main motivation for doing so is to avoid unnecessarily cumbersome expressions related to the massive gravitino sector, we do not see any obstacles to achieving the full reduction if desired. Furthermore, this allows us to highlight some of the features of the reduction from an $\mathcal{N} = 2$ perspective.

Since the reduction of the fermionic sector uses the bosonic reduction as a starting point, we begin with a brief review of the bosons in Section 2.6. We then turn to the reduction of the IIB fermions in Section 2.7 and present the effective five-dimensional theory in Section 2.8. Moreover, as shown in Section 2.9, the resulting $\mathcal{N} = 2$ theory admits a truncation to gauged supergravity coupled to a single hypermultiplet, corresponding to the model of [49] for a holographic superconductor in 3+1 dimensions.

2.6 Review of the bosonic reduction

The reduction of the bosonic sector of IIB supergravity on a squashed Sasaki-Einstein manifold carried out in the previous sections was first done in [17, 77, 43, 96]. From an $\mathcal{N} = 2$ point of view, the resulting theory has on-shell fields corresponding to that of five-dimensional gauged supergravity coupled to a massive hypermultiplet, massive gravitino multiplet and massive vector multiplet [76, 77].

Before turning to the fermions, we review the reduction of the bosonic sector, following the notations and conventions of [77]. For readers' convenience, here we highlight the truncation presented in section 2.4.

Although IIB supergravity does not admit a covariant action, we may take a bosonic Lagrangian of the form

$$\mathcal{L}_{\rm IIB} = R*1 - \frac{1}{2\tau_2^2}d\tau\wedge *d\bar\tau - \frac{1}{2}\mathcal{M}_{ij}F_3^i\wedge *F_3^j - \frac{1}{4}\widetilde F_5\wedge *\widetilde F_5 - \frac{1}{4}\epsilon_{ij}C_4\wedge F_3^i\wedge F_3^j, \tag{II.61}$$

where self-duality $\widetilde F_5 = *\widetilde F_5$ is to be imposed by hand after deriving the equations of motion. Here we have chosen to write the Lagrangian in an SL(2,$\mathbb{R}$) invariant form using

$$\tau = C_0 + ie^{-\phi},\qquad \mathcal{V} = \frac{1}{\sqrt{\tau_2}}\begin{pmatrix}-\tau_1 & 1\\ \tau_2 & 0\end{pmatrix},\qquad \mathcal{M} = \mathcal{V}^T\mathcal{V} = \frac{1}{\tau_2}\begin{pmatrix}|\tau|^2 & -\tau_1\\ -\tau_1 & 1\end{pmatrix}. \tag{II.62}$$

For convenience when coupling to fermions, we also introduce the complexified vielbein $v_i = \mathcal{V}^1{}_i - i\mathcal{V}^2{}_i$, so that

$$v_iF_3^i = \tau_2^{-1/2}(F_3^2 - \tau F_3^1) = \tau_2^{-1/2}G_3, \tag{II.63}$$

where $G_3 = F_3^2 - \tau F_3^1$.

The reduction ansatz follows by taking a metric of the squashed Sasaki-Einstein form

$$ds_{10}^2 = e^{2A}ds_5^2 + e^{2B}ds^2(B) + e^{2C}(\eta + A_1)^2, \tag{II.64}$$

where $d\eta = 2J$ and where we set $3A + 4B + C = 0$ to remain in the Einstein frame. The key to the reduction is to expand the remaining bosonic fields in terms of the invariant forms J and Ω based on the SU(2) structure of the base B and satisfying

$$J\wedge\Omega = 0,\qquad \Omega\wedge\bar\Omega = 2J\wedge J = 4*_4 1,\qquad *_4J = J,\qquad *_4\Omega = \Omega, \tag{II.65}$$

as well as

$$dJ = 0, \qquad d\Omega = 3i(d\psi + \mathcal{A}) \wedge \Omega. \tag{II.66}$$

The bosonic reduction follows by expanding the three-form and five-form field strengths in a basis of invariant tensors on B. Since we will truncate out the massive gravitino multiplet, we set the corresponding bosonic fields to zero. (The complete reduction is given in [77].) In this case, the three-form gives rise to two complex scalars b^i, and is given by

$$F_3^i = f_1^i \wedge \Omega + \bar{f}_1^i \wedge \bar{\Omega} + f_0^i \wedge \Omega \wedge (\eta + A_1) + \bar{f}_0^i \wedge \bar{\Omega} \wedge (\eta + A_1), \tag{II.67}$$

where

$$f_1^i = Db^i, \qquad f_0^i = 3ib^i, \tag{II.68}$$

with D the U(1) gauge covariant derivative

$$Db^i = db^i - 3iA_1 b^i. \tag{II.69}$$

Furthermore, introducing

$$b^i = \begin{pmatrix} 1 \\ \tau \end{pmatrix} b^{m^2=-3} + \begin{pmatrix} 1 \\ \bar{\tau} \end{pmatrix} b^{m^2=21}, \tag{II.70}$$

it is easy to see that

$$v_i f_0^i = 6\sqrt{\tau_2}\, b^{m^2=21}, \qquad \bar{v}_i f_0^i = -6\sqrt{\tau_2}\, b^{m^2=-3}, \tag{II.71}$$

while

$$\begin{aligned} v_i f_1^i &= -2i\sqrt{\tau_2}[Db^{m^2=21} + \frac{i}{2\tau_2}(b^{m^2=-3}d\tau + b^{m^2=21}d\bar{\tau})], \\ \bar{v}_i f_1^i &= 2i\sqrt{\tau_2}[Db^{m^2=-3} - \frac{i}{2\tau_2}(b^{m^2=-3}d\tau + b^{m^2=21}d\bar{\tau})]. \end{aligned} \tag{II.72}$$

These expressions will show up extensively in the fermion reduction below.

For the self-dual five-form, we have

$$\widetilde{F}_5 = (1+*)[(4+\phi_0)*_4 1 \wedge (\eta + A_1) + \mathbb{A}_1 \wedge *_4 1 + p_2 \wedge J \wedge (\eta + A_1)], \tag{II.73}$$

where $*_4 1$ denotes the volume form on the Kahler-Einstein base B. The fields ϕ_0 and p_2 are constrained by

$$\begin{aligned} \phi_0 &= -\frac{2i}{3}\epsilon_{ij}(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j), \\ p_2 &= -d[A_1 + \frac{1}{4}\mathbb{A}_1 + \frac{i}{6}\epsilon_{ij}(f_0^i \bar{f}_1^j - \bar{f}_0^i f_1^j)]. \end{aligned} \tag{II.74}$$

Hence the only additional field arising from the five-form is the vector $\mathbb{A}_1$.

Finally, we note that the bosonic field content of this massive truncation is that of gauged supergravity coupled to a hypermultiplet with fields $(\tau, b^{m^2=-3})$ and a massive vector multiplet with fields $(B, C, b^{m^2=21}, \mathbb{A}_1)$. This massive multiplet is actually a vector combined with a hypermultiplet. However, since we are working on shell, one of the scalars has been absorbed into the massive vector. If desired, this scalar may be restored by an appropriate Stueckelberg shift of $\mathbb{A}_1$.

2.7 Reduction of the IIB fermions

We are now prepared to examine the fermionic sector of IIB supergravity [92]. For simplicity in working out the reduction, we follow a Dirac convention throughout. In this case, the fermions consist of a spin-$\frac{3}{2}$ gravitino Ψ_M and a spin-$\frac{1}{2}$ dilatino λ, with opposite chiralities

$$\Gamma_{11}\Psi_M = \Psi_M, \qquad \Gamma_{11}\lambda = -\lambda. \tag{II.75}$$

Our Dirac conventions are detailed in Appendix A.1. In particular, as opposed to [92], we are using a mostly plus metric signature.

In the following we always work to lowest order in the fermions. In this case, the IIB supersymmetry variations on the fermions are given by [92]

$$\begin{aligned}
\delta\lambda &= \frac{i}{2\tau_2}\Gamma^A\partial_A\tau\epsilon^c - \frac{i}{24}\Gamma^{ABC}v_iF^i_{ABC}\epsilon, \\
\delta\Psi_M &= \mathcal{D}_M\epsilon \equiv \left(\nabla_M + \frac{i}{4\tau_2}\partial_M\tau_1 + \frac{i}{16\cdot 5!}\Gamma^{ABCDE}\widetilde{F}_{ABCDE}\Gamma_M\right)\epsilon \\
&\quad + \frac{i}{96}\left(\Gamma_M{}^{ABC} - 9\delta^A_M\Gamma^{BC}\right)v_iF^i_{ABC}\epsilon^c.
\end{aligned} \tag{II.76}$$

The supersymmetry parameter ϵ is chiral with $\Gamma_{11}\epsilon = \epsilon$, and the complexified $SL(2,\mathbb{R})$ vielbein, v_i, was defined above in (II.63). In addition the fermion equations of motion are [92]

$$\begin{aligned}
0 &= \Gamma^M\mathcal{D}_M\lambda - \frac{i}{8\cdot 5!}\Gamma^{MNPQR}F_{MNPQR}\lambda, \\
0 &= \Gamma^{MNP}\mathcal{D}_N\Psi_P + \frac{i}{48}\Gamma^{NPQ}\Gamma^M v_i^* F^{i*}_{NPQ}\lambda - \frac{i}{4\tau_2}\Gamma^N\Gamma^M\partial_N\tau\lambda^c,
\end{aligned} \tag{II.77}$$

where the supercovariant derivative acting on the gravitino is defined in the gravitino variation (II.76). On the other hand, the supercovariant derivative acting on the dilatino takes the form

$$\mathcal{D}_M\lambda = \left(\nabla_M + \frac{3i}{4\tau_2}\partial_M\tau_1\right)\lambda - \frac{i}{2\tau_2}\Gamma^N\partial_N\tau\Psi^c_M + \frac{i}{24}\Gamma^{NPQ}v_iF^i_{NPQ}\Psi_M, \tag{II.78}$$

and is defined so that $\nabla_M\epsilon$ terms drop out of the variation $\mathcal{D}_M\delta\lambda$, as appropriate to supercovariantization.

2.7.1 Killing spinors on SE_5

The starting point of the fermion reduction is the construction of Killing spinors on SE_5. Starting with the undeformed Sasaki-Einstein metric

$$ds^2(SE_5) = ds^2(B) + (d\psi + \mathcal{A})^2, \tag{II.79}$$

the Killing spinor equations then follow from the internal components of the gravitino variation in (II.76) with a constant five-form flux

$$\tilde{F}_5 = 4 *_5 1 + 4 *_4 1 \wedge (d\psi + \mathcal{A}) \tag{II.80}$$

and take the form

$$0 = \delta\Psi_a = \hat{\mathcal{D}}_a\eta \equiv [\hat{\nabla}_a - \mathcal{A}_a\partial_\psi + \frac{1}{2}J_{ab}\tau^b\tau^9 + \frac{i}{2}\tau_a]\eta,$$

$$0 = \delta\Psi_9 = [\partial_\psi - \frac{1}{4}J_{ab}\tau^{ab} + \frac{i}{2}\tau_9]\eta. \tag{II.81}$$

We proceed by assigning a U(1) charge q to the Killing spinor η, so that $\partial_\psi\eta = iq\eta$. Furthermore, since $(J_{ab}\tau^{ab})^2 = -8(1-\tau^9)$, we see that $J_{ab}\tau^{ab}$ has eigenvalues $(4i, -4i, 0, 0)$ with corresponding τ^9 eigenvalues $(-1,-1,1,1)$. The variation $\delta\Psi_9$ then vanishes for the charges $q = (\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$. The $\mathcal{N}=2$ Killing spinor is thus obtained by taking $q = \frac{3}{2}$ and $J_{ab}\tau^{ab}\eta = 4i\eta$.

Having exhausted the content of the $\delta\Psi_9$ equation, we now turn to integrability of $\delta\Psi_a$, which gives the requirement

$$0 = \tau^b[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b]\eta = \tau^b[\delta_{ab}(\tau^9 - 1) - iJ_{ab}(\tau^9 + 2q)]\eta. \tag{II.82}$$

For $q = \frac{3}{2}$ and $\tau^9\eta = -\eta$, this gives the condition $J_{ab}\tau^b\eta = i\tau_a\eta$, which is easily seen to be consistent with the above requirement that $J_{ab}\tau^{ab}\eta = 4i\eta$. After defining $\eta = e^{3i\psi/2}\tilde{\eta}$, we are finally left with the condition

$$[\hat{\nabla}_a - \frac{3i}{2}\mathcal{A}_a]\tilde{\eta} = 0, \tag{II.83}$$

which is solved by taking $\tilde{\eta}$ to be a gauge covariantly constant spinor on the Kahler-Einstein base [44].

To summarize the above, the system (II.81) may be solved to yield a single complex Killing spinor η satisfying

$$\partial_\psi\eta = \frac{3i}{2}\eta, \qquad \tau^9\eta = -\eta, \qquad \tau^b J_{ab}\eta = i\tau_a\eta, \qquad \tau^b\Omega_{ab}\eta = 0. \tag{II.84}$$

The final condition may be obtained by multiplying the penultimate one by Ω_{ca} on both sides and making use of the identity $\Omega_{ca}J_{ab} = -i\Omega_{cb}$, which follows from the relations [43]

$$\Omega_{ac}\Omega^{bc} = 0, \qquad \Omega_{ac}\bar{\Omega}^{bc} = 2\delta_a{}^b - 2iJ_a{}^b. \tag{II.85}$$

The Killing spinor η and its conjugate η^c provide a natural basis of invariant spinors in which to expand the fermions. Furthermore, as discussed in [7], these represent singlets of the $SU(2)$ structure group, thus ensuring consistency of the reduction. Note that η and η^c are related by

$$\tau^b \bar{\Omega}_{ab}\eta = 2\tau_a \eta^c, \tag{II.86}$$

and η^c satisfies the conjugated relations

$$\partial_\psi \eta^c = -\frac{3i}{2}\eta^c, \qquad \tau^9\eta^c = -\eta^c, \qquad \tau^b J_{ab}\eta^c = -i\tau_a\eta^c, \qquad \tau^b\bar{\Omega}_{ab}\eta^c = 0. \tag{II.87}$$

2.7.2 IIB spinor decomposition

We are now in a position to present the fermion decomposition ansatz by expanding the ten-dimensional fermions in terms of η and η^c. Although we will ultimately truncate away the massive gravitino multiplet, we find it instructive to start with the complete ansatz. This allows us to identify which fermions belong in which multiplets, and hence will guide the truncation.

Starting with the IIB dilatino, since it has negative chirality, it may be decomposed as[5]

$$\lambda = e^{-A/2}\lambda \otimes \eta \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{-A/2}\lambda' \otimes \eta^c \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{II.88}$$

The IIB transformation parameter ϵ and gravitino Ψ_A each have positive chirality. Thus we expand the gravitino in ten dimensional flat indices as

$$\begin{aligned}
\Psi_\alpha &= e^{-A/2}\psi_\alpha \otimes \eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-A/2}\psi'_\alpha \otimes \eta^c \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\Psi_a &= e^{-A/2}\psi \otimes \tau_a\eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-A/2}\psi' \otimes \tau_a\eta^c \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\Psi_9 &= e^{-A/2}\psi_9 \otimes \tau_9\eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{-A/2}\psi'_9 \otimes \tau_9\eta^c \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\end{aligned} \tag{II.89}$$

[5]Note that this is a slight abuse of notation, in that λ shows up as both ten-dimensional and five-dimensional fields. The correct interpretation will be obvious from the context.

and the transformation parameter as

$$\epsilon = e^{A/2}\varepsilon \otimes \eta \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{II.90}$$

Note that in all the above we have included relevant warp factors to account for the breathing and squashing modes.

While we have started with a theory with 32 real supercharges, only a quarter of these are preserved in the $\mathrm{AdS}_5 \times SE_5$ background. By focusing on supersymmetries generated by (II.90), we are thus restricting our study to five-dimensional supersymmetry parameterized by a single Dirac spinor. This corresponds to an $\mathcal{N} = 2$ theory, and provides a motivation for us to remove the massive gravitino from subsequent consideration. (If desired, the full spontaneously broken $\mathcal{N} = 4$ symmetry may be obtained by introducing an $\varepsilon \otimes \eta^c$ component in (II.90). However, we will not pursue this here.)

2.7.3 Linearized analysis and the $\mathcal{N} = 2$ supermultiplet structure

Before presenting the fermionic reduction, it is instructive to analyze the linearized equations of motion. Doing so allows us to group the effective five-dimensional fermions into the relevant $\mathcal{N} = 2$ supermultiplets as highlighted in [77]. We start by noting that the five-dimensional fermions consist of the two gravitini ψ_α and ψ'_α, two dilatini λ and λ' and four additional spin-1/2 fields ψ, ψ', ψ_9 and ψ'_9 arising from the internal components of the ten-dimensional gravitino.

In the linearized theory, the equations are greatly simplified and the fermions satisfy free massive Dirac and Rarita-Schwinger equations. The λ and λ' equations are naturally diagonal and the gravitino equations are diagonalized by the following modes,

$$\begin{aligned} \hat{\psi}_\alpha &= \psi_\alpha + \frac{i}{3}\gamma_\alpha \left(4\psi + \psi_9\right), & \psi^{m=11/2} &= 4\psi + \psi_9, \qquad \psi^{m=-9/2} = \psi - \psi_9, \\ \hat{\psi}'_\alpha &= \psi'_\alpha + \frac{i}{10}\left(\gamma_\alpha + 2\nabla_\alpha\right)\left(4\psi' + \psi'_9\right), & \psi'^{m=5/2} &= \psi' - \psi'_9. \end{aligned} \tag{II.91}$$

n	Multiplet	State	Field
0	supergraviton	$D(4,1,1)_0$	$g_{\mu\nu}$
		$D(3\frac{1}{2},1,\frac{1}{2})_{-1}+D(3\frac{1}{2},\frac{1}{2},1)_1$	$\hat{\psi}_\mu$
		$D(3,\frac{1}{2},\frac{1}{2})_0$	$A_1+\frac{1}{6}\mathbb{A}_1$
0	LH+RH chiral	$D(3,0,0)_{\pm 2}$	$b^{m^2=-3}$
		$D(3\frac{1}{2},\frac{1}{2},0)_1+D(3\frac{1}{2},0,\frac{1}{2})_{-1}$	λ'
		$D(4,0,0)_0+D(4,0,0)_0$	τ
1	LH+RH massive gravitino	$D(5\frac{1}{2},\frac{1}{2},1)_1+D(5\frac{1}{2},1,\frac{1}{2})_{-1}$	$\hat{\psi}'_\mu$
		$D(5,\frac{1}{2},\frac{1}{2})_0+D(5,\frac{1}{2},\frac{1}{2})_0$	b_1^i
		$D(5,0,1)_2+D(5,1,0)_{-2}$	q_2
		$D(6,0,1)_0+D(6,1,0)_0$	b_2^i
		$D(4\frac{1}{2},0,\frac{1}{2})_1+D(4\frac{1}{2},\frac{1}{2},0)_{-1}$	$\psi'^{m=5/2}$
		$D(5\frac{1}{2},0,\frac{1}{2})_{-1}+D(5\frac{1}{2},\frac{1}{2},0)_1$	λ
2	massive vector	$D(7,\frac{1}{2},\frac{1}{2})_0$	$\mathbb{A}_1$
		$D(6\frac{1}{2},\frac{1}{2},0)_{-1}+D(6\frac{1}{2},0,\frac{1}{2})_1$	$\psi^{m=-9/2}$
		$D(7\frac{1}{2},0,\frac{1}{2})_{-1}+D(7\frac{1}{2},\frac{1}{2},0)_1$	$\psi^{m=11/2}$
		$D(6,0,0)_0$	σ
		$D(7,0,0)_{\pm 2}$	$b^{m^2=21}$
		$D(8,0,0)_0$	ρ

Table 2.3: Identification of the bosonic and fermionic states in the Kaluza-Klein spectrum with the linearized modes in the reduction.

In all, the linearized modes satisfy,

$$\begin{aligned}
&\gamma^{\mu\alpha\beta}\nabla_\alpha\hat{\psi}_\beta=\frac{3}{2}\gamma^{\mu\alpha}\hat{\psi}_\alpha, &&\gamma^{\mu\alpha\beta}\nabla_\alpha\hat{\psi}'_\beta=-\frac{7}{2}\gamma^{\mu\alpha}\hat{\psi}'_\alpha,\\
&\gamma^\alpha\nabla_\alpha\lambda=\frac{7}{2}\lambda, &&\gamma^\alpha\nabla_\alpha\lambda'=-\frac{3}{2}\lambda',\\
&\gamma^\alpha\nabla_\alpha\psi^{m=11/2}=\frac{11}{2}\psi^{m=11/2}, &&\gamma^\alpha\nabla_\alpha\psi^{m=-9/2}=-\frac{9}{2}\psi^{m=-9/2},\\
&\gamma^\alpha\nabla_\alpha\psi'^{m=5/2}=\frac{5}{2}\psi'^{m=5/2}.
\end{aligned} \tag{II.92}$$

Note that the massive gravitino obtains its mass by absorbing the spin-1/2 combination $4\psi'+\psi'_9$.

As with the fields in the bosonic truncation, we have arrived at a field content which, in the case of the round five-sphere, saturates the lowest harmonic in each of the respective Kaluza-Klein towers as determined in [52, 71]. Noting that, in five dimensions, the relation between the conformal weight Δ and mass m of the fermions is $|m|=\Delta-2$, we can map the fermion fields into $\mathcal{N}=2$ AdS multiplets. First, it is straightforward to see that $\hat{\psi}_\mu$ has $m=3/2$, corresponding to a massless spin-3/2 field in AdS$_5$. Hence it should be identified with the massless gravitino sitting in the supergraviton multiplet. Also at the zeroth Kaluza-Klein level, the LH+RH chiral multiplet contains an $m = 3/2$ fermion which may be identified as λ'. At level $n = 1$, the massive gravitino multiplet has three fermions; one spin-3/2 particle with $m = -7/2$ corresponding to the massive gravitino $\hat{\psi}'_\mu$ and two spin-1/2 particles with $m = 5/2$ corresponding to $\psi'^{m=5/2}$ and $m = 7/2$ corresponding to λ. Finally, at the $n = 2$ Kaluza-Klein level, the massive vector multiplet contains two spin-1/2 particles, $\psi^{m=-9/2}$ and $\psi^{m=11/2}$. These identifications will be further justified by examining the supersymmetry transformations. The complete field content of the supermultiplets is shown in Table 2.3, where the bosonic fields are fully defined in [77].

2.8 The Five-dimensional Theory and $\mathcal{N} = 2$ Supergravity

The linearized analysis above demonstrates that the fields ψ'_α, ψ', ψ'_9 and λ belong to the massive gravitino multiplet. We thus proceed with the $\mathcal{N} = 2$ truncation by setting these to zero

$$\psi'_\alpha = 0, \qquad \psi' = 0, \qquad \psi'_9 = 0, \qquad \lambda = 0. \tag{II.93}$$

It is straightforward to show this this is a consistent truncation, provided the bosonic fields in the massive graviton multiplet are set to zero[6]. Moreover, other than just simplifying the resulting equations, this truncation is natural when explicitly discussing $\mathcal{N} = 2$ supersymmetry as the massive gravitino should really be thought of as descending from a spontaneously broken $\mathcal{N} = 4$ theory.

2.8.1 Supersymmetry Variations

We start with the reduction of the IIB supersymmetry variations given in (II.76). Inserting the fermion ansätze (II.88), (II.89) and (II.90) into the IIB variations, we arrive at

[6] The consistency of this truncation in the bosonic sector has been previously shown in [77, 43, 17].

the following five-dimensional variations[7]

$$\begin{aligned}
\delta\hat{\psi}_\alpha \quad &\equiv \quad \mathcal{D}_\alpha\varepsilon = \Big[D_\alpha + \frac{i}{24}e^{C-A}\left(\gamma_\alpha{}^{\nu\rho} - 4\delta_\alpha{}^\nu\gamma^\rho\right)\left(F_{\nu\rho} - 2e^{-2B-2C}p_{\nu\rho}\right) \\
&\qquad + \frac{1}{12}\gamma_\alpha\left(4e^{A-2B+C} + 6e^{A-C} - (4+\phi_0)e^{A-4B-C}\right)\Big]\varepsilon \\
&\qquad - e^{-2B}\left(v_i f^i_\alpha - \frac{i}{3}e^{A-C}v_i f^i_0\gamma_\alpha\right)\varepsilon^c, \qquad \text{(II.94)}
\end{aligned}$$

$$\begin{aligned}
\delta\psi^{m=11/2} \quad &= \quad \Big[-\frac{i}{2}\gamma^\mu\partial_\mu(4B+C) - \frac{3}{8}e^{-4B}\gamma^\mu\mathbb{A}_\mu + \frac{1}{8}e^{C-A}\gamma^{\mu\nu}\left(F_{\mu\nu} + e^{-2B-2C}p_{\mu\nu}\right) - ie^{A-2B+C} \\
&\qquad - \frac{3i}{2}e^{A-C} + \frac{5i}{8}(4+\phi_0)e^{A-4B-C}\Big]\varepsilon + e^{-2B}\left(\frac{3i}{4}\gamma^\mu v_i f^i_\mu + \frac{7}{4}e^{A-C}v_i f^i_0\right)\varepsilon^c, \qquad \text{(II.95)}
\end{aligned}$$

$$\begin{aligned}
\delta\psi^{m=-9/2} \quad &= \quad \Big[-\frac{i}{2}\gamma^\mu\partial_\mu(B-C) - \frac{1}{4}e^{-4B}\gamma^\mu\mathbb{A}_\mu - \frac{1}{8}e^{C-A}\gamma^{\mu\nu}\left(F_{\mu\nu} + e^{-2B-2C}p_{\mu\nu}\right) \\
&\qquad - \frac{3i}{2}e^{A-2B+C} + \frac{3i}{2}e^{A-C}\Big]\varepsilon + e^{-2B}\left(\frac{i}{2}\gamma^\mu v_i f^i_\mu - \frac{1}{2}e^{A-C}v_i f^i_0\right)\varepsilon^c, \qquad \text{(II.96)}
\end{aligned}$$

$$\delta\lambda' = -\frac{1}{2\tau_2}\gamma^\mu\partial_\mu\tau\varepsilon^c - ie^{-2B}\left(\gamma^\mu v_i\bar{f}^i_\mu - ie^{A-C}v_i\bar{f}^i_0\right)\varepsilon. \qquad \text{(II.97)}$$

The gauge covariant derivative D_α acting on ε is given by $D_\alpha \equiv \nabla_\alpha - \frac{3i}{2}(A_\alpha + \frac{1}{6}e^{-4B}\mathbb{A}_\alpha) + \frac{i}{4\tau_2}\partial_\alpha\tau_1$, where the latter term descends from the traditional charge with respect to the U(1) compensator field, Q_M, in the ten dimensional IIB theory [92]. Furthermore, we have defined the five-dimensional supercovariant derivative $\mathcal{D}_\alpha$ through the gravitino variation in (II.94).

There are several facts worth noting about these expressions. Firstly, we see that these variations fit nicely into the multiplet structure as presented in Table 2.3. In particular, the dilatino variation is built out of τ and $\bar{v}_i f^i$, both of which belong to the LH+RH chiral multiplet, since the latter corresponds to $b^{m^2=-3}$ according to (II.71). On the other hand, $\delta\psi^{m=11/2}$ and $\delta\psi^{m=-9/2}$ contain only terms involving fields from the graviton and massive vector multiplets. [Note that the combination $F_2 + e^{-2B-2C}p_2$ appearing in (II.96) and (II.97) essentially selects the field strength of the massive vector $\mathbb{A}_1$, as can be seen from the definition of p_2 given in (II.74)]. These observations give further justification for the multiplet structure presented in section 2.7.3.

[7]Note that with the Dirac matrix conventions described in the appendix we have $\epsilon^c = i\varepsilon^c \otimes \eta^c \otimes \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

Furthermore, since the breathing mode is $\rho \sim 4B + C$, and the squashing mode is $\sigma \sim B - C$, we can identify $\psi^{m=11/2}$ with the fermionic partner of the breathing mode and $\psi^{m=-9/2}$ as the fermionic partner of the squashing mode as first demonstrated in [76]. Finally, from the gauge covariant derivative, it is evident that the combination $A_\mu + \frac{1}{6}e^{-4B}\mathbb{A}_\mu$ may be identified with the graviphoton, which is consistent with the linearized analysis in [77]. (The combination $F_2 - 2e^{-2B-2C}p_2$ appearing in the gravitino variation is similarly the effective graviphoton field strength.)

The gravitino variation (II.94) is particularly interesting, as we may attempt to read off an $\mathcal{N} = 2$ superpotential from the term proportional to $\gamma_\alpha \varepsilon$

$$W = 2e^{A-2B+C} + 3e^{A-C} - \frac{1}{2}(4 + \phi_0)e^{A-4B-C}. \tag{II.98}$$

Recalling the relations $3A + 4B + C = 0$ and $\phi_0 = -\frac{2i}{3}\epsilon_{ij}\left(f_0^i \bar{f}_0^j - \bar{f}_0^i f_0^j\right)$, we see that the scalar potential can be written as

$$V = 2(\mathcal{G}^{-1})^{ij}\partial_i W \partial_j W - \frac{4}{3}W^2, \tag{II.99}$$

where $(\mathcal{G}^{-1})^{ij}$ is the inverse scalar metric which can be read off from the scalar kinetic terms in the Lagrangian and $\{i, j\}$ run over all scalars in the theory.

To verify (II.99), we made use of the fact that the scalar metric given in [77] is composed of three independent components, pertaining to the independent sets of scalars $\{B, C\}$, $\{b_0^1, b_0^2\}$ and τ, with explicit components

$$(\mathcal{G}^{-1}_{\{B,C\}})^{ij} = \frac{1}{16}\begin{pmatrix} 1 & -1 \\ -1 & 7 \end{pmatrix}, \qquad (\mathcal{G}^{-1}_{\{b_0^1, b_0^2\}})^{ij} = \frac{e^{4B}}{4\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \qquad \mathcal{G}^{-1}_\tau = \tau_2^2. \tag{II.100}$$

Inserting these expressions into (II.99) then exactly reproduces the scalar potential appearing in the bosonic Lagrangian. This is, however, a somewhat surprising relation as the actual gravitino variation (II.94) contains not only the term proportional to the superpotential written above, but another term involving $v_i f_0^i \varepsilon^c$ where $v_i f_0^i$ is proportional to

$b_0^{m^2=21}$, as indicated in (II.71). Based on general $\mathcal{N}=2$ gauged supergravity arguments, this should conceivably also contribute to the scalar potential, but is not taken into account by (II.99).

2.8.2 Equations of Motion

Turning to the equations of motion, the reduction of the dilatino equation is the most straightforward. After a bit of manipulation, we obtain

$$\begin{aligned} 0 \;=\; & \left[\gamma^\mu \mathcal{D}_\mu + \frac{i}{8}\gamma^{\mu\nu}\left(e^{C-A}F_{\mu\nu} - 2e^{-A-2B-C}p_{\mu\nu}\right) - \frac{1}{4}(4+\phi_0)e^{A-4B-C} + e^{A-2B+C} + \frac{3}{2}e^{A-C}\right]\lambda' \\ & -e^{-2B}v_i\left[\frac{4}{5}\gamma^\mu \bar{f}^i_\mu + \frac{28i}{15}\bar{f}^i_0\right]\psi^{m=11/2} - e^{-2B}v_i\left[\frac{4}{5}\gamma^\mu \bar{f}^i_\mu - \frac{4i}{5}\bar{f}^i_0 e^{A-C}\right]\psi^{m=-9/2}, \end{aligned}$$

(II.101)

where the supercovariant derivative acting on the dilatino is defined by

$$\mathcal{D}_\mu\lambda' \equiv D_\mu\lambda' - K(\lambda')\hat{\psi}_\mu = \left[\nabla_\mu + \frac{3i}{4\tau_2}\partial_\mu\tau_1 + \frac{3i}{2}\left(A_\mu + \frac{1}{6}e^{-4B}\mathbb{A}_\mu\right)\right]\lambda' - K(\lambda')\hat{\psi}_\mu. \tag{II.102}$$

The supercovariantization term $K(\lambda')$ acting on $\hat{\psi}_\mu$ is given by the right hand side of the dilatino variation (II.97) with ε replaced by $\hat{\psi}_\mu$ (and similarly ε^c replaced by $\hat{\psi}^c_\mu$).

Starting with the IIB gravitino, we arrive at three equations, corresponding to the α, a, and 9 components. After a fair bit of manipulations, and the appropriate redefinitions given in the first line of (II.91), we obtain the $\psi^{m=11/2}$ and $\psi^{m=-9/2}$ equations

$$
\begin{aligned}
0 \;=\; & \Big[\gamma^\mu \mathcal{D}_\mu + \frac{3i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu - \frac{i}{120}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{11i}{60}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
& + e^A\left(-\frac{17}{12}(4+\phi_0)e^{-4B-C} + \frac{1}{15}e^{-2B+C} - \frac{1}{10}e^{-C}\right)\Big]\psi^{m=11/2} \\
& \Big[\frac{3i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu + \frac{i}{5}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{i}{10}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} + e^A\left(\frac{12}{5}e^{-2B+C} - \frac{12}{5}e^{-C}\right)\Big]\psi^{m=-9/2} \\
& + v_i e^{-2B}\Big[\left(-\frac{2}{5}\gamma^\mu f^i_\mu + \frac{34i}{15}e^{A-C}f^i_0\right)\psi^{c\,m=11/2} + \left(\frac{3}{5}\gamma^\mu f^i_\mu - \frac{7i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2}\Big] \\
& + \bar{v}_i e^{-2B}\left(\frac{3}{4}\gamma^\mu f^i_\mu + \frac{7i}{4}e^{A-C}f^i_0\right)\lambda',
\end{aligned}
$$

(II.103)

$$
\begin{aligned}
0 \;=\; & \Big[\gamma^\mu \mathcal{D}_\mu + \frac{2i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu - \frac{3i}{40}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{3i}{20}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
& + e^A\left(\frac{1}{4}(4+\phi_0)e^{-4B-C} + \frac{13}{5}e^{-2B+C} + \frac{9}{20}e^{-C}\right)\Big]\psi^{m=-9/2} \\
& \Big[\frac{2i}{5}e^{-4B}\gamma^\mu \mathbb{A}_\mu + \frac{2i}{15}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{i}{15}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} + e^A\left(\frac{8}{5}e^{-2B+C} - \frac{8}{5}e^{-C}\right)\Big]\psi^{m=11/2} \\
& + v_i e^{-2B}\Big[\left(\frac{2}{5}\gamma^\mu f^i_\mu - \frac{14i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=11/2} + \left(-\frac{3}{5}\gamma^\mu f^i_\mu - \frac{3i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2}\Big] \\
& + \bar{v}_i e^{-2B}\left(\frac{1}{2}\gamma^\mu f^i_\mu - \frac{i}{2}e^{A-C}f^i_0\right)\lambda'.
\end{aligned}
$$

(II.104)

As in the dilatino case, we have defined the supercovariant derivatives

$$
\begin{aligned}
\mathcal{D}_\mu \psi^{m=11/2} &= \left[\nabla_\mu + \frac{i}{4\tau_2}\partial_\mu\tau_1 - \frac{3i}{2}(A_\mu + \frac{1}{6}e^{-4B}\mathbb{A}_\mu)\right]\psi^{m=11/2} - K(\psi^{m=11/2})\hat{\psi}_\mu, \\
\mathcal{D}_\mu \psi^{m=-9/2} &= \left[\nabla_\mu + \frac{i}{4\tau_2}\partial_\mu\tau_1 - \frac{3i}{2}(A_\mu + \frac{1}{6}e^{-4B}\mathbb{A}_\mu)\right]\psi^{m=-9/2} - K(\psi^{m=-9/2})\hat{\psi}_\mu,
\end{aligned}
$$

(II.105)

with $K(\psi^{m=11/2})$ and $K(\psi^{m=-9/2})$ similarly obtained from the variations (II.96) and (II.97), respectively.

Finally, the gravitino equation takes the form

(II.106)

$$
0 = \gamma^{\mu\nu\rho}\mathcal{D}_\nu\hat{\psi}_\rho - \frac{8}{15}\tilde{K}(\psi^{m=11/2})\gamma^\mu\psi^{m=11/2} - \frac{4}{5}\tilde{K}(\psi^{m=-9/2})\gamma^\mu\psi^{m=-9/2} - \frac{1}{2}\tilde{K}(\lambda')\gamma^\mu\lambda',
$$

where the supercovariant derivative acting on the gravitino is given by the right hand side of the gravitino variation (II.94), and where the $\tilde{K}$ terms are essentially the Dirac conjugates of K. The above equations have the appropriate structure to be obtained from an effective $\mathcal{N}=2$ Lagrangian of the form[8]

$$
\begin{aligned}
e^{-1}\mathcal{L} \;=\;& \bar{\hat{\psi}}_\mu \gamma^{\mu\nu\rho}\mathcal{D}_\nu\hat{\psi}_\rho + \frac{8}{15}\bar{\psi}^{m=11/2}\gamma^\mu D_\mu \psi^{m=11/2} + \frac{4}{5}\bar{\psi}^{m=-9/2}\gamma^\mu D_\mu\psi^{m=-9/2} + \frac{1}{2}\bar{\lambda}'\gamma^\mu D_\mu \lambda' \\
& + \left[\bar{\hat{\psi}}_\mu\left(-\frac{8}{15}\tilde{K}(\psi^{m=11/2})\gamma^\mu\psi^{m=11/2} - \frac{4}{5}\tilde{K}(\psi^{m=-9/2})\gamma^\mu\psi^{m=-9/2} - \frac{1}{2}\tilde{K}(\lambda')\gamma^\mu\lambda'\right) + \text{h.c.}\right] \\
& + \cdots .
\end{aligned}
\tag{II.107}
$$

The full fermionic Lagrangian (to quadratic order in the fermions) is given in Appendix B.1.

Although we have worked only to quadratic order in the fermions, it is clear from the nature of the invariant spinors η and η^c that higher spinor harmonics would not be excited by this subset of states. Thus, if desired, the consistent truncation may be extended to the four-fermi terms as well. However, we expect this to be quite tedious and not particularly worth pursuing.

2.9 A supersymmetric holographic superconductor

In this final section we demonstrate the consistency of a particularly interesting truncation to the lowest Kaluza-Klein level, namely the supersymmetric completion of the bosonic truncation first demonstrated in [49]. As we demonstrate, this is a fully consistent truncation, so long as we keep all fields in the graviton and LH+RH chiral multiplets. However, it is a nontrivial truncation, in that it is not consistent to naively set the other fields in the above reduction to zero. Instead, the “backreaction” on the truncated fields must be taken into account, effectively setting these modes equal to something depending on the

[8] Note that some care must be taken when considering the conjugate spinor terms. Nevertheless, the various conjugate terms do assemble themselves properly into a consistent effective fermionic Lagrangian. This is one place where a more conventional symplectic-Majorana approach would allow the manipulations to be more transparent.

dynamical fields. Due to this backreaction on the non dynamical fields, the resulting Lagrangian is nonlinear and so describes a non-trivial coupling of $\mathcal{N}=2$ supergravity with a single hypermultiplet.

In the bosonic sector the truncation amounts to keeping only $\{\tau, b^{m^2=-3}\}$ and the graviton and graviphoton dynamical. In what follows, we will denote $b^{m^2=-3}$ simply as b so that $(b_0^1, b_0^2) = (b, \tau b)$. This requires the following constraints on the other terms in the reduction [17, 77]

$$b^{m^2=21}=0, \qquad e^{4B}=e^{-4C}=1-4\tau_2|b|^2, \qquad \mathbb{A}_1=-4i\tau_2(bD\bar{b}-\bar{b}Db)+4|b|^2 d\tau_1, \tag{II.108}$$

and

$$\phi_0=-24\tau_2|b|^2, \qquad p_2=-dA_1. \tag{II.109}$$

For the fermions, by analyzing the supersymmetry transformations of the spin-$\frac{1}{2}$ fields in this truncation, it is evident that if we set

$$\psi=-\psi_9=-\frac{i}{2}b\tau_2^{1/2}e^{-2B}\lambda', \tag{II.110}$$

the resulting system will be consistent with the supersymmetry transformations. It turns out that under this identification the fermion equations of motion also degenerate into a single expression, resulting in a theory containing only λ' and $\hat{\psi}_\mu$ in the fermionic sector.

Moving directly to the Lagrangian, we write this as a sum of bosonic and fermionic contributions $\mathcal{L}=\mathcal{L}_b+\mathcal{L}_f$, where

$$\begin{aligned}\mathcal{L}_b \;=\;& R*1+\frac{6(2-3\chi)}{(1-\chi)^2}*1-\frac{d\chi\wedge *d\chi}{2(1-\chi)^2}-\frac{(1+\chi)d\tau\wedge *d\bar{\tau}}{2(1-\chi)\tau_2^2}-\frac{3}{2}F_2\wedge *F_2-\frac{\mathbb{A}_1\wedge *\mathbb{A}_1}{2(1-\chi)^2}\\ &-\frac{8\tau_2 Db\wedge *D\bar{b}}{1-\chi}-\frac{2i}{1-\chi}(\bar{b}Db\wedge *d\bar{\tau}-bD\bar{b}\wedge *d\tau)-A_1\wedge F_2\wedge F_2,\end{aligned} \tag{II.111}$$

and

$$
\begin{aligned}
e^{-1}\mathcal{L}_f \;=\;& \bar{\hat{\psi}}_\alpha \gamma^{\alpha\beta\sigma} D_\beta \hat{\psi}_\sigma + \frac{3i}{8}\bar{\hat{\psi}}_\alpha \left(\gamma^{\alpha\beta\rho\sigma} + 2g^{\alpha\beta}g^{\rho\sigma}\right) F_{\beta\rho}\hat{\psi}_\sigma + \frac{1}{2}\bar{\tilde{\lambda}}\gamma^\alpha D_\alpha \tilde{\lambda} + \frac{3i}{16}\bar{\tilde{\lambda}}\gamma^{\mu\nu}F_{\mu\nu}\tilde{\lambda} \\
&+\frac{1}{2}e^{-4B}\left(3\tau_2(b\bar{D_\mu b} - \bar{b}D_\mu b)\bar{\tilde{\lambda}}\gamma^\mu\tilde{\lambda} + \frac{3}{2}(1+8\tau_2|b|^2)\bar{\tilde{\lambda}}\tilde{\lambda}\right) \\
&+e^{-4B}\left(-\frac{3}{2}\bar{\hat{\psi}}_\alpha\gamma^{\alpha\sigma}\hat{\psi}_\sigma + \tau_2(\bar{b}D_\beta b - b\bar{D_\beta b})\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\hat{\psi}_\sigma\right) \\
&+\tau_2^{1/2}e^{-4B}\left(D_\mu b\bar{\hat{\psi}}_\alpha\gamma^\mu\gamma^\alpha\tilde{\lambda} + 3b\bar{\hat{\psi}}_\alpha\gamma^\alpha\tilde{\lambda} + h.c.\right) \\
&+\frac{e^{-2B}}{\tau_2^{1/2}}\left(-b\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\partial_\beta\tau\hat{\psi}^c_\sigma + \tau_2^{1/2}\bar{\hat{\psi}}_\alpha\gamma^\mu\partial_\mu\tau\gamma^\alpha\tilde{\lambda}^c + h.c.\right),
\end{aligned}
\tag{II.112}
$$

where we have defined $\tilde{\lambda} \equiv e^{-2B}\lambda'$, $\chi = \tau_2|b|^2$ and we have redefined the gauge covariant derivative acting on b as $D_\mu b = \left(\partial_\mu - 3iA_\mu - \frac{i}{2\tau_2}\partial_\mu\tau_1\right)b$, and similarly for $\tilde{\lambda}$ and $\hat{\psi}_\alpha$.

This truncation is of interest for many of the condensed matter applications of the AdS/CFT correspondence involving the coupling of a charged scalar and fermion. In particular the original motivation for the bosonic truncation was in describing a superconducting phase transition using holographic methods within a controlled system, *i.e,* one which is derived directly from a UV complete theory. This truncation hence completes the story by demonstrating the embedding into a fully supersymmetric theory. It would be interesting to consider the dynamics of this theory, and whether there is a supersymmetric superconducting phase transition. Note however that this analysis would be complicated by the presence of the gravitino. After all, it is not consistent to simply set the gravitino field defined here to be zero. Since the gravitino couples to the supercurrent, this suggests that the holographic superconductor model of [49] in fact has an underlying (although spontaneously broken) supersymmetry.

While the truncation first presented in [49] did not include the axi-dilaton, as in the bosonic case, it is consistent to fix τ as well. This simplifies the Lagrangian to be

$$\begin{aligned} e^{-1}\mathcal{L} &= R - \frac{3}{4}F_{\mu\nu}F^{\mu\nu} - e^{-1}A_1 \wedge F_2 \wedge F_2 \\ &\quad +12\frac{(1-6f^2)}{(1-4f^2)^2} - 8\frac{\partial_\mu f \partial^\mu f}{(1-4f^2)^2} - 8f^2\frac{(\partial_\mu\theta - 3A_\mu)(\partial^\mu\theta - 3A^\mu)}{(1-4f^2)^2} \\ &\quad +\bar{\hat{\psi}}_\alpha \gamma^{\alpha\beta\sigma} D_\beta \hat{\psi}_\sigma + \frac{1}{2}\bar{\tilde{\lambda}}\gamma^\alpha D_\alpha \tilde{\lambda} + \frac{3i}{8}\bar{\hat{\psi}}_\alpha \left(\gamma^{\alpha\beta\rho\sigma} + 2g^{\alpha\beta}g^{\rho\sigma}\right) F_{\beta\rho}\hat{\psi}_\sigma + \frac{3i}{16}\bar{\tilde{\lambda}}\gamma^{\mu\nu}F_{\mu\nu}\tilde{\lambda} \\ &\quad +\frac{1}{1-4f^2}\left(\frac{3}{4}(1+8f^2)\bar{\tilde{\lambda}}\tilde{\lambda} - \frac{3}{2}\bar{\hat{\psi}}_\alpha \gamma^{\alpha\sigma}\hat{\psi}_\sigma - if^2(\partial_\mu\theta - 3A_\mu)\left(3\bar{\tilde{\lambda}}\gamma^\mu\tilde{\lambda} + 2\bar{\hat{\psi}}_\alpha\gamma^{\alpha\beta\sigma}\hat{\psi}_\sigma\right)\right) \\ &\quad +\left(\frac{e^{i\theta}}{1-4f^2}\left((\partial_\mu f + if(\partial_\mu\theta - 3A_\mu))\,\bar{\hat{\psi}}_\alpha\gamma^\mu\gamma^\alpha\tilde{\lambda} + 3f\bar{\hat{\psi}}_\alpha\gamma^\alpha\tilde{\lambda}\right) + h.c.\right), \end{aligned}$$

(II.113)

where we have defined $b = \sqrt{g_s} f e^{i\theta}$ and $\tau = i g_s^{-1}$.

Finally, it is worth noting that although this theory involves a charged scalar coupled to the fermion $\tilde{\lambda}$, it lacks the Majorana coupling $\phi\lambda\lambda$ that has been of recent interest in studies involving gapped fermions in the bosonic condensate [20, 34, 51]. While this coupling is allowed by charge conservation, the explicit reduction shows that it is not present. More generally, examination of Table 2.3 demonstrates that the $b^{m^2=21}$ scalar in the massive vector multiplet may have such a coupling, and in fact the equations of motion (II.103) and (II.104) show that it is exists for both $\psi^{m=11/2}$ and $\psi^{m=-9/2}$. It would be curious to see if this $b^{m^2=21}$ scalar may play a role in novel models of holographic superconductors.

CHAPTER III

Holographic c-theorems for higher derivative gravity

In this chapter we prove a holographic c-theorem for higher curvature Lovelock gravity, where the bulk equations of motion remain second order. We also explore gravitational action built out of arbitrary functions $f(R)$ and $f(R^{ab}{}_{cd})$ coupled to bulk matter. In both cases, monoticity of the flows require additional conditions related to the higher derivative nature of the theory. This chapter is based on [75, 79] in collaboration with James Liu and Wafic Sabra.

3.1 Motivation for a holographic c-theorem for higher derivative gravity

Central charges in conformal field theories can often be thought of as a proxy for the number of degrees of freedom exhibited by the theory. In this context, the Zamolodchikov c-theorem [101] is a powerful result for two-dimensional conformal field theories. It states that there exists a c-function which is monotonically decreasing along flows from the UV to IR, and which is equal to the central charge at the fixed points of the flow. This is a direct indication that UV degrees of freedom of the CFT are removed as the theory flows to the IR.

While two dimensional conformal field theories are rather special, there have been numerous attempts to generalize the c-theorem to higher dimensions. However, one obstacle that needs to be surmounted in doing so is the realization that there may be multiple candidates for a satisfactory c-function. For example, in four dimensions, the Weyl anomaly has the well known form

$$\langle T^\mu_\mu \rangle = \frac{c}{16\pi^2} C^2_{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} E_4, \tag{III.1}$$

where $E_4 = R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2$ is the four-dimensional Euler density. In this context, Cardy demonstrated that, while the c anomaly coefficient may not be the proper object to investigate, the a coefficient appears to have the desired monotonicity property along flows [14]. This has subsequently been confirmed in various situations [88, 68, 6, 5], and further investigated in the context of a-maximization [65, 74, 73, 9].

While the above investigations have been carried out in a field theory context, AdS/CFT allows the possibility of a holographic version of the c-theorem. At large N, the a and c anomalies are equal, and can be computed holographically; for $\mathcal{N} = 4$ super-Yang-Mills, the result is simply $a = c = N^2/4$ [58]. This result may be extended to renormalization group flows, which correspond to radial flow in the bulk dual. A c-theorem can then be proven at leading order in large N by examining the flow equations for domain wall solutions interpolating between the UV and IR [3, 45, 35, 91].

Myers and Sinha have extended the leading order holographic c-theorem to the case where the bulk theory may contain higher order curvature terms, corresponding to moving away from the leading behavior in AdS/CFT [94, 84, 95, 85]. In particular, they considered the addition of a Gauss-Bonnet and 'quasi-topological' curvature-cubed term to the bulk action. By generalizing the holographic a-anomaly, computed for a general bulk action in [64], they constructed an appropriate a-function that is monotonic along radial flow. A key element of this holographic c-theorem rests on the fact that the equations of motion contain no higher than second derivatives of the metric when expanded on the AdS background. Gauss-Bonnet gravity manifestly satisfies this requirement, as does the quasi-topological theory, which was explicitly constructed to have this property.

Here we extend the investigation of Myers and Sinha in two directions. Firstly, we prove a holographic c-theorem for the case of Lovelock bulk theories. These theories generalize Gauss-Bonnet gravity in $d+1$ dimensions by the addition of d' dimensional Euler densities where $d' < d$. While this result is mostly formal, in the sense that higher Lovelock terms are only present in theories in dimensions too large for practical applications, it nevertheless suggests that the holographic c-theorem extends to arbitrary orders in the bulk curvature, so long as higher derivative terms in the equations of motion are able to be controlled.

Secondly, we address what happens when the bulk action is not restricted to second order equations of motion by examining $f(R)$ gravity as a toy model. In this case, we find that the constructed a-function may deviate from monotonicity by a term that is explicitly of higher derivative order. We suggest that this term, which may lead to a violation of the holographic c-theorem, is related to the additional ghost modes of the theory. From an AdS/CFT point of view, this would correspond to a breakdown of unitarity in the dual field theory.

3.2 A holographic c-theorem for Lovelock gravity

While higher-curvature gravitational actions generically lead to higher derivative equations of motion and ensuing pathologies such as ghosts, a special class of higher-curvature actions may be constructed that nevertheless give rise to second order equations of motion for the metric. These are the Lovelock actions, which are constructed out of the $(d+1)$-dimensional continuation of lower dimensional Euler densities [80]

$$S = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{-g} \sum_m \alpha_m L^{(m)} + S_{\text{matter}}. \tag{III.2}$$

Here the m-th Lovelock term is the Euler invariant in $2m$ dimensions

$$L^{(m)} = \frac{1}{2^m} \delta^{a_1 b_1 \cdots a_m b_m}_{c_1 d_1 \cdots c_m d_m} R_{a_1 b_1}{}^{c_1 d_1} \cdots R_{a_m b_m}{}^{c_m d_m}. \tag{III.3}$$

In particular, $L^{(0)}=1$ is a cosmological constant, $L^{(1)}=R$ is the ordinary Einstein-Hilbert term and $L^{(2)}=R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2$ is the Gauss-Bonnet invariant. The equation of motion following from (III.2) is simply $G_{ab}=\kappa^2 T_{ab}$, where the generalized Einstein tensor is given by $G_{ab}=\sum_m \alpha_m G^{(m)}_{ab}$, with

$$G^{e\,(m)}_{f}=-\frac{1}{2^{m+1}}\delta^{ea_1b_1\cdots a_mb_m}_{fc_1d_1\cdots c_md_m}R_{a_1b_1}{}^{c_1d_1}\cdots R_{a_mb_m}{}^{c_md_m}. \tag{III.4}$$

With a suitable choice of the cosmological constant, we take the bulk Lovelock action (III.2) to be dual to a d-dimensional CFT. As demonstrated in [64], the d-dimensional type A trace anomaly (*i.e.* the term proportional to the Euler characteristic) is universal in holographic renormalization, and its coefficient may be expressed as

$$a_{\mathrm{UV}}=-\frac{\pi^{d/2}}{2\kappa^2}\frac{\ell^{d+1}}{(d/2)!^2}f(\mathrm{AdS}), \tag{III.5}$$

where

$$f(\mathrm{AdS})=\sum_m \alpha_m L^{(m)}\Big|_{\mathrm{AdS}} \tag{III.6}$$

is the on-shell Lagrangian evaluated on the asymptotic AdS background with radius ℓ.

In order to construct a suitable a-function, we need to promote the a_{UV} central charge (III.5) at the UV fixed point into a function $a(r)$ of the radial flow. Here, for simplicity, we consider a radial slicing of the bulk space into flat slices of the form

$$ds^2=e^{2A(r)}(-dt^2+d\vec{x}^2_{d-1})+dr^2, \tag{III.7}$$

with curvature components

$$R_{\mu\nu\rho\sigma}=-A'^2(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\sigma}g_{\nu\rho}),\qquad R_{\mu r\nu r}=-(A''+A'^2)g_{\mu\nu}. \tag{III.8}$$

This allows us to evaluate the individual Lovelock terms

$$L^{(m)}=(-A'^2)^m\frac{(d+1)!}{(d+1-2m)!}-2mA''(-A'^2)^{m-1}\frac{d!}{(d+1-2m)!}. \tag{III.9}$$

The second term is unimportant at AdS fixed points since we have

$$A \sim r/\ell, \qquad A' \sim 1/\ell, \qquad A'' \sim 0, \qquad \text{as} \qquad r \to \infty. \tag{III.10}$$

As a result, the on-shell value of $L^{(m)}$ takes the form

$$L^{(m)}\Big|_{\mathrm{AdS}} = (-1)^m \frac{(d+1)!}{(d+1-2m)!\ell^{2m}}, \tag{III.11}$$

so that the a_{UV} central charge (III.5) may be expressed as

$$a_{\mathrm{UV}} = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \sum_m \alpha_m (-1)^m \frac{(d+1)!\ell^{d+1-2m}}{(d+1-2m)!}. \tag{III.12}$$

The extension of $a_{\mathrm{UV}} \to a(r)$ into the bulk is by no means unique. As a first attempt to do so, we make the substitution

$$\ell \to \ell_{\mathrm{eff}}(r) \equiv \frac{1}{A'(r)}, \tag{III.13}$$

so that

$$a_0(r) = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \sum_m \alpha_m (-1)^m \frac{(d+1)!}{(d+1-2m)!(A')^{d+1-2m}}. \tag{III.14}$$

This satisfies the requirement that $a(r)$ reproduces the central charge at fixed points of the flow. In addition, it incorporates the metric function at the first derivative level, so that

$$a_0'(r) = -\frac{\pi^{d/2}}{2\kappa^2 (d/2)!^2} \frac{A''}{(A')^d} \sum_m \alpha_m (-1)^{m+1} \frac{(d+1)!(A')^{2m-2}}{(d-2m)!} \tag{III.15}$$

is linear in A''. Following [35, 84], we aim to demonstrate that $a_0'(r)$ is monotonic by appealing to the bulk equations of motion. To do so, we first compute the generalized Einstein tensor components (III.4) using the curvature components (III.8) for the bulk metric. The resulting two independent components are

$$\begin{aligned} G_t^{t\,(m)} &= -\frac{d!}{2(d-2m)!}(-A'^2)^m + \frac{m(d-1)!}{(d-2m)!}A''(-A'^2)^{m-1}, \\ G_r^{r\,(m)} &= -\frac{d!}{2(d-2m)!}(-A'^2)^m, \end{aligned} \tag{III.16}$$

so that

$$G_t^t - G_r^r = A'' \sum_m m\alpha_m(-1)^{m+1}\frac{(d-1)!(A')^{2m-2}}{(d-2m)!}. \tag{III.17}$$

Comparison with (III.15) shows that, while the form of $a_0'(r)$ is suggestive, it nevertheless does not match with the difference $G_t^t - G_r^r$. However, as indicated above, $a_0(r)$ is not necessarily unique, and using the Einstein equation as a guide, we now construct a modified a-function which is monotonic.

To proceed, we first note that, at AdS fixed points where the bulk matter sector contributes vanishing vacuum energy, the background satisfies the vacuum Einstein equation

$$0 = G_r^r\Big|_{\text{AdS}} = \sum_m \alpha_m(-1)^{m+1}\frac{d!(A')^{2m}}{2(d-2m)!}\Big|_{\text{AdS}}. \tag{III.18}$$

(Recall that the cosmological constant term is included in the gravitational sector through α_0.) This allows us to add a vanishing on-shell contribution to (III.6), so that

$$f(\text{AdS}) = 2G_r^r + \sum_m \alpha_m L^{(m)}\Big|_{\text{AdS}} = \sum_m 2m\alpha_m(-1)^m\frac{d!}{(d+1-2m)!\ell^{2m}}. \tag{III.19}$$

In this case, we are led to the definition

$$a(r) = -\frac{\pi^{d/2}}{\kappa^2(d/2)!^2}\sum_m m\alpha_m(-1)^m\frac{d!}{(d+1-2m)!(A')^{d+1-2m}}. \tag{III.20}$$

Note that the shift removes the cosmological constant term α_0 from the definition of $a(r)$, and matches what is done in constructing a suitable a-function in the leading two-derivative gravity. This a-function can now be seen to satisfy

$$a'(r) = -\frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{G_t^t - G_r^r}{(A')^d} = -\frac{d\pi^{d/2}}{(d/2)!^2}\frac{T_t^t - T_r^r}{(A')^d} \geq 0, \tag{III.21}$$

where the inequality corresponds to the null energy condition. Therefore we have found an appropriate extension of the a central charge which is indeed monotonic along flows from the UV to the IR. This result extends the observation of [84, 85] that a general holographic c-theorem may be obtained in the presence of higher order corrections, provided the (linearized) equations of motion remain second order.

3.3 $f(R)$ gravity

In proving the holographic c-theorem for Lovelock gravity, we have constructed the $a(r)$ function (III.20) entirely out of the first derivative of the metric function, A'. This ensures that $a(r)$ reproduces the a central charge at fixed points of the flow where $A' \sim 1/\ell$. However, this construction also guarantees that $a'(r)$ is linear in the second derivative A'' so that it may be connected to the equations of motion. This connection suggests that having second order equations is an essential aspect of obtaining the c-theorem. On the other hand, bulk AdS duals are often considered to be effective theories where higher derivative corrections naturally arise (*e.g.* in the string α' expansion). Thus it is important to address whether any holographic c-theorem could hold in such higher derivative theories as well.

Here we take one step towards a fully general investigation by considering the case of $f(R)$ gravity. Such theories have been considered from a cosmological point of view, and are closely related to Brans-Dicke theory (see *e.g.* [26]). However, here we are mainly interested in $f(R)$ gravity as a toy model exhibiting higher derivative equations of motion. The action is given by

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}f(R) + S_{\text{matter}}, \tag{III.22}$$

where $f(R)$ is a fixed but arbitrary function of the scalar curvature R. The resulting equation of motion is

$$G_{ab} \equiv FR_{ab} - \frac{1}{2}fg_{ab} + (g_{ab}\Box - \nabla_a\nabla_b)F = \kappa^2 T_{ab}, \tag{III.23}$$

where

$$F(R) = \frac{df(R)}{dR}. \tag{III.24}$$

Since $F(R)$ is second order in derivatives, the equation of motion is in general fourth order.

In order to construct an appropriate a-function for $f(R)$ gravity, we follow [35, 84] and explore the difference in the Einstein equation components $G^t_t - G^r_r$. To proceed, we take the same metric (III.7) as used above, and compute the Ricci components

$$R_{\mu\nu} = -(A'' + dA'^2)g_{\mu\nu}, \qquad R_{rr} = -d(A'' + A'^2), \qquad R = -d(2A'' + (d+1)A'^2). \tag{III.25}$$

In this case, the Einstein equation (III.23) splits into

$$\begin{aligned} G^\mu_\nu = -[F(A'' + dA'^2) + \frac{1}{2}f - (d-1)A'F' - F'']\delta^\mu_\nu &= \kappa^2 T^\mu_\nu, \\ G^r_r = -F(dA'' + dA'^2) - \frac{1}{2}f + dA'F' &= \kappa^2 T^r_r. \end{aligned} \tag{III.26}$$

Taking the difference of G^t_t and G^r_r gives

$$G^t_t - G^r_r = (d-1)A''F - A'F' + F'' = \kappa^2(T^t_t - T^r_r). \tag{III.27}$$

Note that F is a function of R, which is in turn a function of A according to (III.25). Thus the higher derivatives of A are encoded in the $-A'F' + F''$ terms in this equation.

Our aim is to construct a suitable $a(r)$ which reproduces the a-anomaly at the UV boundary and which is subject to a flow governed by (III.27). We start with the anomaly coefficient itself, given by (III.5)

$$a_{\mathrm{UV}} = -\frac{\pi^{d/2}}{2\kappa^2}\frac{\ell^{d+1}}{(d/2)!^2}f(\mathrm{AdS}), \tag{III.28}$$

where this time $f(\mathrm{AdS})$ is simply the on-shell value of $f(R)$ at the asymptotic AdS fixed point

$$f(\mathrm{AdS}) = f(-d(d+1)\ell^{-2}). \tag{III.29}$$

As we saw above, the extension of a_{UV} to the interior is not unique. A straightforward choice would be to replace $f(\mathrm{AdS})$ by $f(R)$, so that

$$a_0(r) = -\frac{\pi^{d/2}}{2\kappa^2(d/2)!^2}\frac{f(R)}{(A')^{d+1}}. \tag{III.30}$$

This is similar to our choice of $a_0(r)$ in (III.14), although here we allow $f(R)$ to contain A'' through the dependence on the curvature scalar. However, differentiation of $a_0(r)$ with respect to r does not give any obvious correspondence with the difference (III.27). Thus we seek an improvement to $a_0(r)$, just as we did for the Lovelock case.

Since we do not want to destroy the matching of the a-function with the actual a-anomaly at AdS critical points, we can adjust $a_0(r)$ by at most functions which vanish at such points. A natural possibility for such a function is to take the equations of motion at a critical point. In this case, the stress tensor T_{ab} vanishes, and furthermore the functions f and F become constant. The rr equation of motion (III.26) then simplifies to

$$G^r_r\Big|_{\mathrm{AdS}} = \left[-dA'^2F - \frac{1}{2}f\right]_{\mathrm{AdS}} = 0, \tag{III.31}$$

where $A' = 1/\ell$. This suggests that we shift $f(R)$ in (III.30) by $2G^r_r$, just as we did in the Lovelock case. The resulting a-function then takes the form

$$a(r) = \frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{F(R)}{(A')^{d-1}}, \tag{III.32}$$

where R is given in (III.25). Note that $F(R)$ in the numerator of this expression is essentially the derivative of the Lagrangian with respect to R (or equivalently with respect to $R^{tr}{}_{tr}$). This suggests a natural connection with an entropy function, as pointed out in [84, 85].

With the definition (III.32), we now see that

$$\begin{aligned} a'(r) &= \frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{-(d-1)A''F + A'F'}{(A')^d} \\ &= \frac{d\pi^{d/2}}{(d/2)!^2}\frac{-(T^t_t - T^r_r) + F''/\kappa^2}{(A')^d}, \end{aligned} \tag{III.33}$$

where the second line follows from the equation of motion (III.27). If it were not for the F'' term, we would then use the null energy condition, $-(T^t_t - T^r_r) \geq 0$, to demonstrate that $a'(r) \geq 0$. This suggests that the higher derivative nature of $f(R)$ gravity directly impacts the fate of the holographic c-theorem. In particular, a non-trivial F'' contribution is a direct sign that the gravitational background incorporates up to four derivatives of A.

Taking a step back, it is perhaps not surprising that in this case monoticity of $a'(r)$ requires not just the weak energy condition on the matter sector, but also a further condition $F'' \geq 0$ on the gravity sector. While we do not see a direct connection with unitarity, it is certainly plausible that this $F'' \geq 0$ condition would be related to the absence of ghost modes in the background of the flow. One way to investigate this would be to map $f(R)$ gravity onto Brans-Dicke theory. In this case, F plays the role of the Brans-Dicke scalar. However, it is not clear to us how F'' may be related to any obvious pathologies of the theory.

In searching for a holographic c-theorem, we may also need to make a distinction between perturbative versus non-perturbative expansions in the higher derivative terms. For example, at the R^2 level, we may take $f = R + d(d-1)/\ell_0^2 + \alpha R^2$, so that $F = 2\alpha R$. The AdS vacuum with radius ℓ is given by the solution to (III.31), and in this case admits two branches

$$\left(\frac{\ell}{\ell_0}\right)^2 = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\alpha}{\ell_0^2}\frac{d(d+1)(d-3)}{d-1}}. \tag{III.34}$$

This may be expanded for small α

$$\begin{aligned} \ell_+ &= \ell_0\left(1 - \frac{\alpha}{2\ell_0^2}\frac{d(d+1)(d-3)}{d-1} + \cdots\right), \\ \ell_- &= \frac{\alpha}{2\ell_0}\frac{d(d+1)(d-3)}{d-1} + \cdots . \end{aligned} \tag{III.35}$$

We thus see that only the positive branch is smoothly connected to the finite radius background in the perturbative limit $\alpha \to 0$. In the context of AdS/CFT, it is natural to view α as an expansion parameter in an effective theory with higher curvature interactions. In this case, we ought to restrict to only the positive branch. Perhaps the sign of F'' can somehow be attributed to this choice. In particular, we have readily found numerical solutions for this example model coupled to a massless scalar where $a'(r)$ changes sign (from positive to negative) along the flow to the IR. However, all such resulting solutions involve a domain wall interpolating between the positive and negative branches of (III.34), and have the form

$$A(r) \to \begin{cases} r/\ell_+, & r \to \infty \\ |r|/\ell_- & r \to -\infty. \end{cases} \tag{III.36}$$

This 'kink up' domain wall solution has the characteristic of a negative tension wall interpolating between two regions that both open up into AdS boundaries. Nevertheless, we have checked that the scalar matter is not responsible for this negative tension. Hence, it must have its origin in the higher derivative gravitational sector. This at least suggests that violation of the c-theorem is closely related to pathologies in the gravity sector of the theory that would not arise when treated in a proper perturbative expansion where the equations of motion can be perturbatively arranged to use no higher than second order derivatives in the expansion.

3.4 Discussion

In both the Lovelock and the $f(R)$ case, we have defined the a-function based on a shifted form of the Lagrangian

$$a_{\text{UV}} = -\frac{\pi^{d/2}}{2\kappa^2}\frac{\ell^{d+1}}{(d/2)!^2}f(\text{AdS}) \qquad \Rightarrow \qquad a(r) = -\frac{\pi^{d/2}}{2\kappa^2(d/2)!^2}\frac{f+2G^r_r}{(A')^{d+1}}, \tag{III.37}$$

where $f = e^{-1}\mathcal{L}$ is the Lagrangian density in the gravity sector (not including matter). Since we take the matter energy density to vanish at AdS fixed points, the addition of $2G^r_r$ does not affect the identification of $a(r)$ with the a anomaly coefficient. However, this improvement allows $a'(r)$ to be related to the difference of the Einstein equations $G^t_t - G^r_r$ along the flow.

As noted in [84], the a-function has a second interpretation in terms of entanglement entropy. This can be seen to arise in a natural manner, provided we perform the $2G^r_r$ shift. In particular, consider a general higher curvature action of the form

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}f(R^{ab}{}_{cd}) + S_{\text{matter}}. \tag{III.38}$$

The corresponding Einstein equation may be written as

$$G_{ab} \equiv F_{(a}{}^{cde}R_{b)cde} - \frac{1}{2}f g_{ab} + 2\nabla^c\nabla^d F_{acbd} = \kappa^2 T_{ab}, \tag{III.39}$$

where

$$F_{ab}{}^{cd} = \frac{\delta f(R^{ef}{}_{gh})}{\delta R^{ab}{}_{cd}}. \tag{III.40}$$

This generalizes the $f(R)$ equation of motion given in (III.23). The higher derivative terms are manifestly present in the Einstein tensor. However, they vanish on the AdS background where F_{abcd} is covariantly constant (as it is constructed out of the maximally symmetric Riemann tensor). Further taking T_{ab} to vanish in the asymptotic AdS region, we end up with

$$G^r_r\Big|_{\text{AdS}} = \left[-2A'^2F^{r\mu}{}_{r\mu} - \frac{1}{2}f\right]_{\text{AdS}} = \left[-2dA'^2F^{tr}{}_{tr} - \frac{1}{2}f\right]_{\text{AdS}}. \tag{III.41}$$

The general a-function (III.37) then takes the form

$$a(r) = \frac{2d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{F^{tr}{}_{tr}}{(A')^{d-1}}. \tag{III.42}$$

If we were to consider black hole entropy in the presence of higher curvature corrections, it would be natural to use the Wald entropy formula [99, 66, 67]

$$S = -\frac{2\pi}{2\kappa^2}\int_\Sigma d^{d-1}x\sqrt{-g}\frac{\delta f}{\delta R_{abcd}}\epsilon_{ab}\epsilon_{cd} = \frac{4\pi}{\kappa^2}\int_\Sigma d^{d-1}x\sqrt{h}F^{tr}{}_{tr}, \tag{III.43}$$

where the integral is over the area of the horizon with unit binormal ϵ_{ab} along t and r. This reduces to the familiar one-quarter of the horizon area (in $G_N = \kappa^2/8\pi$ units) in the absence of higher curvatures, where ${F^{ab}}_{cd} = \frac{1}{2}\delta^{ab}_{cd}$. Although this expression is intended to be evaluated at the black hole horizon, it can nevertheless be generalized into an entropy function [46, 21]

$$\tilde{C}(r) = \frac{4\pi}{\kappa^2}{F^{tr}}_{tr}\sqrt{h} = \frac{4\pi}{\kappa^2}e^{(d-1)A}{F^{tr}}_{tr}, \tag{III.44}$$

where we have used the explicit form of the metric (III.7).

Flows of $\tilde{C}(r)$ have been investigated in the context of the second law of black hole thermodynamics in higher curvature gravity, including both Lovelock and $f(R)$ gravity [69, 21, 4]. In the case of $f(R)$ gravity, a c-theorem can be proven which generalizes the Hawking area theorem [57] by use of the Raychaudhuri equation [69, 21]. In particular, we consider an affinely parameterized null congruence given by the tangent vector $k^a\partial_a = d/d\lambda$ and define

$$\tilde{\theta} = \frac{d\log\tilde{C}}{d\lambda} = \theta + k^a\partial_a \log F, \tag{III.45}$$

where θ is the expansion of the null congruence. The Raychaudhuri equation then gives

$$\frac{d\tilde{\theta}}{d\lambda} = -\frac{1}{d-1}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - k^a k^b R_{ab} + k^a k^b \nabla_a \nabla_b \log F, \tag{III.46}$$

and further application of the Einstein equation (III.23) reduces this to

$$\frac{d\tilde{\theta}}{d\lambda} = -\frac{1}{d-1}\theta^2 - \sigma_{ab}\sigma^{ab} + \omega_{ab}\omega^{ab} - \left(\frac{d\log F}{d\lambda}\right)^2 - \frac{\kappa^2 k^a k^b T_{ab}}{F}. \tag{III.47}$$

Provided the congruence is twist-free, and assuming the null energy condition $k^a k^b T_{ab} \geq 0$ along with positivity of F, the terms on the right-hand-side are all non-positive, and as a result we may conclude that $d\tilde{\theta}/d\lambda \leq 0$, which is the statement of the second law in $f(R)$ gravity [69, 21].

For $f(R)$ gravity with the metric written in the explicit form (III.7), we define $k^a\partial_a = -e^{-2A}\partial_t + e^{-A}\partial_r$, in which case

$$\tilde{\theta} = e^{-A}\left[(d-1)A' + \frac{F'}{F}\right], \tag{III.48}$$

so that

$$\tilde{\theta}' = e^{-A}\left[-(d-1)A'^2 - \left(\frac{F'}{F}\right)^2 + \frac{(d-1)A''F - A'F' + F''}{F}\right]$$

$$= -e^{-A}\left[(d-1)A'^2 + \left(\frac{F'}{F}\right)^2 - \frac{\kappa^2(T^t_t - T^r_r)}{F}\right] \leq 0. \tag{III.49}$$

We see here that, even with a higher order equation of motion, the terms involving higher derivatives arrange themselves in just the proper manner to match with the Einstein equation combination (III.27). This is in contrast with the holographic a-function, where $a'(r)$ given in (III.33) picks up an additional contribution proportional to F''. Nevertheless, it would be interesting to see if a connection can be made between the black hole entropy function (III.58) and the holographic a-function (III.42).

Ideally, it would be desirable to construct a general proof of a holographic c-theorem for the a-function of (III.42) using techniques such as the generalized Raychaudhuri equation (III.46). This would allow the c-theorem to be separated from the particular bulk AdS flow parameterization of (III.7). However, unlike the Raychaudhuri equation itself, which is independent of dynamics, the incorporation of $F^{tr}{}_{tr}$ into the generalized expansion $\tilde{\theta}$ necessarily brings higher derivative dynamics into the picture. Thus it appears unlikely that the proof of a holographic c-theorem for higher curvature gravity can be fully decoupled from the exact form of the higher order interactions. (In fact, we do not expect any c-theorem to hold unless additional unitarity constraints are imposed.) Nevertheless, we anticipate that it would be possible to make a general statement in the restricted case where the linearized equations of motion remain second order in the AdS background.

3.5 A holographic c-theorem for arbitrary $f(R^{ab}{}_{cd})$ gravity

In previous sections, we investigated the possibility of constructing a holographic c-theorem for $f(R)$ gravity [75], and demonstrated that monoticity of the a central charge will be ensured by a combination of the null energy condition on the bulk matter and a condition related to the higher derivative nature of the gravitational sector. This second condition appears to be a weaker version of the unitarity requirement of [85]. In the remaining sections of the chapter, we extend the investigation of [75] by constructing a general a-function which is applicable to a large class of higher derivative gravity theories in the bulk coupled to matter. As in the $f(R)$ case, monoticity is ensured by a combination of conditions on the bulk matter and gravitational sectors of the theory.

3.6 Defining the a-function

Consider a general bulk action of the form

$$S = \frac{1}{2\kappa^2}\int d^{d+1}x\sqrt{-g}f(R^{ab}{}_{cd}) + S_{\mathrm{matter}}. \tag{III.50}$$

Here, $a, b, \ldots = 0, 1, \ldots, d$ correspond to bulk indices, and below we will use $\mu, \nu, \ldots = 0, 1, \ldots, d-1$ to denote boundary indices. For simplicity, we take $f(R^{ab}{}_{cd})$ to be built out of the Riemann tensor with raised and lowered indices as indicated. In particular, it does not contain any implicit metric factors nor derivatives of the Riemann tensor. In this case, the Einstein equation reads

$$G_{ab} \equiv F_{(a}{}^{cde}R_{b)cde} - \frac{1}{2}fg_{ab} + 2\nabla^c\nabla^d F_{acbd} = \kappa^2 T_{ab}, \tag{III.51}$$

where

$$F_{ab}{}^{cd} = \frac{\delta f(R^{ef}{}_{gh})}{\delta R^{ab}{}_{cd}}. \tag{III.52}$$

Following [85, 75], we consider the metric

$$ds^2 = e^{2A(r)}(-dt^2 + d\vec{x}^2_{d-1}) + dr^2, \tag{III.53}$$

and define the a-function

$$a(r) = \frac{2d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{F^{tr}{}_{tr}}{(A')^{d-1}}. \tag{III.54}$$

As shown in [85, 75], this function reproduces the holographic a-anomaly [58, 64] when evaluated at an AdS fixed point of the holographic flow. To see this, consider, for example, the case of Einstein gravity where $F^{tr}{}_{tr} = 1/2$. In this case, the a-function takes the form

$$a(r) = \frac{d\pi^{d/2}}{\kappa^2 (d/2)!^2} \ell_{\text{eff}}^{d-1}, \tag{III.55}$$

where $\ell_{\text{eff}} = 1/A'$ is the 'effective' AdS radius. Setting $d = 4$ and taking ℓ_{eff} to be a constant AdS radius L then gives the familiar holographic expression [58]

$$a = \frac{\pi^2}{\kappa_5^2} L^3 = \frac{N^2}{4}, \tag{III.56}$$

where we have used the AdS$_5$/CFT$_4$ dictionary $L^4 = 4\pi\alpha'^2 g_s N$ and $2\kappa_5^2 = 2\kappa_{10}^2/\text{Vol}(S^5) = (2\pi)^7 g_s^2 \alpha'^4/\pi^3 L^5$.

In the more general higher curvature case, the expression (III.54) is somewhat reminiscent of the Wald entropy formula [99, 66, 67]

$$S = \frac{4\pi}{\kappa^2} \int_\Sigma d^{d-1}x \sqrt{h} F^{tr}{}_{tr}, \tag{III.57}$$

where the integral is taken over the horizon of the black hole. Although the bulk spacetimes we are interested in are not necessarily that of black holes, we may nevertheless define an entropy function [46, 21]

$$\tilde{C}(r) = \frac{4\pi}{\kappa^2} e^{(d-1)A} F^{tr}{}_{tr}, \tag{III.58}$$

corresponding to the metric (III.53). While we are primarily focused on the holographic a anomaly, we will also comment on the behavior of $\tilde{C}(r)$ below.

3.7 A holographic c-theorem and the null energy condition

Given the a-function (III.54), we now wish to demonstrate that $a'(r) \geq 0$, or at least understand the conditions for this to hold. Motivated by the holographic c-theorem in ordinary Einstein gravity [3, 45, 35, 91], which makes crucial use of the null energy condition, we focus on the combination $G^t_t - G^r_r$, where the generalized Einstein tensor G_{ab} is defined in (III.51). For the metric (III.53), this takes the form

$$G^t_t - G^r_r = 2(d-1)(F^{tx}{}_{tx}R^{tx}{}_{tx} - F^{tr}{}_{tr}R^{tr}{}_{tr}) + 2\nabla_a\nabla^b(F^{ta}{}_{tb} - F^{ra}{}_{rb}). \tag{III.59}$$

Note that, because of the isometries of the metric, we have taken x to be an arbitrary spatial direction (which we can take to be x^1).

Computing the Riemann tensor corresponding to the metric (III.53) is straightforward, and gives

$$R^{\mu\nu}{}_{\rho\sigma} = -A'^2(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho), \qquad R^{\mu r}{}_{\nu r} = -(A'' + A'^2)\delta^\mu_\nu. \tag{III.60}$$

The covariant derivatives acting on $F^{ab}{}_{cd}$ are somewhat more cumbersome to evaluate. We find

$$\begin{aligned}\nabla_a\nabla^b F^{ta}{}_{tb} &= F^{tr}{}_{tr}{}'' + (d-1)[A'(2F^{tr}{}_{tr}{}' - F^{tx}{}_{tx}{}') + (A'' + (d-1)A'^2)(F^{tr}{}_{tr} - F^{tx}{}_{tx})],\\ \nabla_a\nabla^b F^{ra}{}_{rb} &= dA'F^{tr}{}_{tr}{}' + d(d-1)A'^2(F^{tr}{}_{tr} - F^{tx}{}_{tx}).\end{aligned} \tag{III.61}$$

Combining the above, we obtain

$$G^t_t - G^r_r = 2(d-1)A''F^{tr}{}_{tr} - 2A'F^{tr}{}_{tr}{}' + 2F^{tr}{}_{tr}{}'' + 2(d-1)[A'(F^{tr}{}_{tr} - F^{tx}{}_{tx})]'. \tag{III.62}$$

We are now in a position to examine flows of the a-function. Taking a radial derivative of (III.54) and substituting in (III.62) yields

$$a'(r) = \frac{d\pi^{d/2}}{\kappa^2(d/2)!^2}\frac{-(G^t_t - G^r_r) + 2[(d-1)A'(F^{tr}{}_{tr} - F^{tx}{}_{tx}) + F^{tr}{}_{tr}{}']'}{(A')^d}. \tag{III.63}$$

Assuming d is even, the sign of $a'(r)$ is then given by the sign of the numerator above. By imposing the null energy condition on bulk matter, $-(T^t_t - T^r_r) \geq 0$, and using the Einstein equation, we see that the first term in the numerator is indeed non-negative.

However, there does not appear to be any obvious constraint on the sign of the second term. What this demonstrates is that the null energy condition by itself is no longer sufficient to guarantee monoticity of the a-function in a higher derivative bulk theory of gravity. Instead, what is required is that the entire numerator is non-negative. At the same time, however, there appears to be a clear physical separation between the two terms in the numerator; the first term is related to the bulk matter, while the second is related to the higher derivative gravitational interactions. As suggested in [75], the sign of the latter term may be connected to unitarity and ghost issues in the gravitational sector.

One way to ensure that the a-function defined in (III.54) is monotonic increasing in flows to the UV is to impose separate conditions on the matter and gravitational sectors: (i) the bulk matter must satisfy the null energy condition $-(T_t^t - T_r^r) \geq 0$, and (ii) the gravitational sector must satisfy $\Delta' \geq 0$ where

$$\Delta \equiv (d-1)A'(F^{tr}{}_{tr} - F^{tx}{}_{tx}) + F^{tr}{}_{tr}{}'. \tag{III.64}$$

Of course $\Delta = 0$ for Einstein gravity. Furthermore, for $f(R)$ gravity, the function $F_{ab}{}^{cd}$ is simply

$$F^{ab}{}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c)F(R). \tag{III.65}$$

Hence $\Delta = F'(R)/2$, and the condition $\Delta' \geq 0$ is identical to the condition $F'' \geq 0$ obtained in [75]. Note that this condition is entirely expressed in terms of the scalar curvature R, and in particular does not involve explicit factors of the metric scale factor A.

To further develop our understanding of the $\Delta' \geq 0$ condition, we may examine a general R^2 action of the form

$$f(R^{ab}{}_{cd}) = R + \Lambda + \alpha_1 R^2 + \alpha_2 R_{ab}R^{ab} + \alpha_3 R_{abcd}R^{abcd}. \tag{III.66}$$

In this case, $F^{ab}{}_{cd}$ takes the form

$$F^{ab}{}_{cd} = (1+2\alpha_1 R)\frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) + 2\alpha_2\frac{1}{4}(\delta^a_c R^b_d - \delta^a_d R^b_c - \delta^b_c R^a_d + \delta^b_d R^a_c) + 2\alpha_3 R^{ab}{}_{cd}. \tag{III.67}$$

Decomposing this into μ and r components, and using (III.60), we obtain

$$\Delta = \frac{4d\alpha_1 + (d+1)\alpha_2 + 4\alpha_3}{4d} R', \tag{III.68}$$

where $R = -d(2A'' + (d+1)A'^2)$. What is curious is that this is again written only in terms of the curvature scalar R, even though $f(R^{ab}{}_{cd})$ involves the full Ricci tensor.

Note that Δ vanishes identically when $4d\alpha_1 + (d+1)\alpha_2 + 4\alpha_3 = 0$, as noted in [85]. This encompasses the Gauss-Bonnet combination $\{\alpha_1, \alpha_2, \alpha_3\} = \{1, -4, 1\}$, but also allows for a two-parameter family of R^2 theories that satisfy the c-theorem with only the null energy condition.

3.8 More discussion

As we have seen, monoticity of flows of $a(r)$ for a general higher derivative bulk theory follows from both the null energy condition and a gravitational sector condition $\Delta' \geq 0$. The latter condition is explicitly higher than second order in derivatives, and encodes the content of the higher curvature terms in the bulk gravitational action. As argued in [85], we would like to impose the physical requirement that the boundary theory is unitary, both at fixed points and along the RG flow. This requirement corresponds to the constraint that the bulk metric fluctuations are second order in derivatives, and is thus equivalent to demanding that $\Delta = 0$.

While the restriction to bulk theories with only second order fluctuations makes physical sense, we nevertheless believe additional information can be obtained even in the presence of higher order fluctuations. After all, we ought to view the bulk theory as an effective gravity theory, in which case any breakdown in unitarity could potentially be relegated to energy scales above those of interest. For example, in the case of R^2 gravity, we have found a non-trivial Δ given in (III.68). However, at linearized order, a field redefinition

$g_{ab} \to g_{ab} + \lambda_1 R_{ab} + \lambda_2 g_{ab} R$ can be used to put $f(R^{ab}{}_{cd})$ into the Gauss-Bonnet form

$$f(R^{ab}{}_{cd}) = R + \Lambda + \alpha_3(R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2) + \cdots . \tag{III.69}$$

In this frame, the linearized action is second order in derivatives, and Δ vanishes identically. Of course, what we have done is pushed the higher derivative interactions to higher orders. Nevertheless, this demonstrates that in some cases a theory with non-trivial Δ (and hence explicit higher derivative terms) may be equivalent up to any finite order in the higher curvature expansion to a theory with $\Delta = 0$.

In fact, the issue of field redefinitions is somewhat more involved for the matter coupled gravity system in the bulk. For R^2 gravity at linearized order, one expects that only α_3 is physical, and yet both α_1 and α_2 show up in the expression (III.68) for Δ. While Δ itself is not a direct physical observable, its behavior along radial flows does have an impact on the monoticity of $a(r)$. Thus its dependence on α_1 and α_2 is perhaps somewhat unexpected. We believe that the resolution to this apparent paradox is that the above field redefinition of the metric necessarily mixes the gravitational and matter sectors of the original action (III.50), so that in particular S_{matter} will now depend on both curvature and actual matter fields. This suggests that there is in fact no sharp distinction between the gravity and matter sectors of the bulk theory, and that unitarity of the gravity theory cannot be completely disentangled from unitarity of the bulk matter.

With the above in mind, we note that the Einstein equation (III.51) may be used to rewrite (III.63) as

$$a'(r) = \frac{d\pi^{d/2}}{\kappa^2 (d/2)!^2} \frac{-\kappa^2(T^t_t - T^r_r) + 2\Delta'}{(A')^d}. \tag{III.70}$$

In this case, a weaker form of the c-theorem may be obtained by demanding only that the numerator is non-negative

$$-\kappa^2(T^t_t - T^r_r) + 2\Delta' \geq 0. \tag{III.71}$$

We believe this may be viewed as a generalized null energy condition that takes both matter and gravity into account. It would be interesting to see if this form of a generalized null energy condition can be made more precise. In particular, we believe this ought to be directly related to the unitarity of dilaton scattering in the field theoretic proof of the c-theorem presented in [72].

In addition to the holographic Weyl anomaly and the a-function, it is also possible to define an entropy function, (III.58), based on the Wald entropy formula. Although both $a(r)$ and $\tilde{C}(r)$ depend linearly on ${F^{tr}}_{tr}$, they differ in their dependence on metric factors: $a(r)$ depends on the effective radius $\ell_{\text{eff}}^{d-1} = 1/(A')^{d-1}$, while $\tilde{C}(r)$ depends on the horizon volume $e^{(d-1)A}$. As a result, the proof of the second law of black hole thermodynamics involving $\tilde{C}(r)$ differs from that of the holographic c-theorem given above.

It is in fact instructive to contrast flows of $a(r)$ with flows of $\tilde{C}(r)$. For the latter, we consider an affinely parameterized null congruence given by the tangent vector $k^\mu\partial_\mu = d/d\lambda$ and define

$$\tilde{\theta} = \frac{d\log\tilde{C}}{d\lambda} = \theta + k^\mu\partial_\mu \log {F^{tr}}_{tr}, \tag{III.72}$$

where θ is the expansion of the null congruence. In particular, for the metric given in (III.53), we take $d/d\lambda = -e^{-2A}\partial_t + e^{-A}\partial_r$, in which case

$$\tilde{\theta} = e^{-A}\left[(d-1)A' + \frac{{{F^{tr}}_{tr}}'}{{F^{tr}}_{tr}}\right]. \tag{III.73}$$

Taking a further radial derivative gives

$$\tilde{\theta}' = -e^{-A}\left[(d-1)A'^2 + \left(\frac{{{F^{tr}}_{tr}}'}{{F^{tr}}_{tr}}\right)^2 - \frac{(d-1)A''{F^{tr}}_{tr} - A'{{F^{tr}}_{tr}}' + {{F^{tr}}_{tr}}''}{{F^{tr}}_{tr}}\right]. \tag{III.74}$$

We now substitute in the combination $G_t^t - G_r^r$ given in (III.62) to obtain

$$\tilde{\theta}' = -e^{-A}\left[(d-1)A'^2 + \left(\frac{{{F^{tr}}_{tr}}'}{{F^{tr}}_{tr}}\right)^2 + \frac{-\kappa^2(T_t^t - T_r^r) + 2(d-1)[A'({F^{tr}}_{tr} - {F^{tx}}_{tx})]'}{2{F^{tr}}_{tr}}\right]. \tag{III.75}$$

Since the first two terms in the square brackets are non-negative, monoticity of $\tilde{\theta}'$ will be ensured so long as the third term is also non-negative. This again involves something like a generalized null energy condition

$$-\kappa^2(T_t^t - T_r^r) + 2\tilde{\Delta}' \geq 0, \tag{III.76}$$

however with a different $\tilde{\Delta} = (d-1)A'(F^{tr}{}_{tr} - F^{tx}{}_{tx})$ from that given in (III.64). Note that $\tilde{\Delta}$ vanishes for $f(R)$ gravity [21, 69], as in this case $F^{tr}{}_{tr} = F^{tx}{}_{tx} = F(R)/2$. However, for R^2 gravity, we have

$$\tilde{\Delta} = -\frac{(d-1)[(d-1)\alpha_2 + 4\alpha_3]}{4}(A'^2)'. \tag{III.77}$$

In particular, this does not vanish for the Gauss-Bonnet combination. This leads to the curious observation that the null energy condition is sufficient to prove monoticity of the a-function, yet does not appear to be sufficient for ensuing monoticity of the entropy function.

Finally, recent developments suggest that the a-anomaly is closely related to the holographic entanglement entropy of the boundary CFT [84, 85, 28, 25, 63, 16]. This connection suggests that the holographic c-theorem is a universal means of capturing the effective number of degrees of freedom along renormalization group flows of the boundary theory.

3.9 Final remarks

In this thesis we have discussed several aspects of the AdS/CFT correspondence. First we presented consistent supersymmetric truncations of IIB supergravity on squashed Sasaki-Einstein manifolds up to the second Kaluza-Klein level. We presented the equations of motion of both bosonic and fermionic sectors of the truncated theory. The full Lagrangian based on these equations of motion was then constructed. We also determined the supersymmetry variation of fermions. We organized the bosons and fermions into $D = 5$,

$\mathcal{N} = 2$ supermultiplets and proved the consistency of the truncation. We demonstrated some further truncations, one of which is particularly interesting because it describes a holographic superconductor system.

In the second part of the thesis, we proved a holographic c-theorem for higher curvature Lovelock gravity, where the bulk equations of motion remain second order. Based on the null energy condition, we also discussed additional sufficient conditions for monoticity of the a-anomaly coefficient flow where the bulk is described by a gravitational action built out of an arbitrary function $f(R)$ of the Ricci scalar, or $f(R^{ab}{}_{cd})$ of the Riemann tensor coupled to bulk matter.

APPENDICES

APPENDIX A

Fermion Conventions and Reduced Action: Dirac Matrix Conventions

A.1 Dirac Matrix Conventions

We work with a mostly plus metric signature, and take the conventional Clifford algebra $\{\Gamma^A, \Gamma^B\} = 2\eta^{AB}$. Note, in particular, that Γ^0 is anti-hermitian, so that $(\Gamma^0)^\dagger = -\Gamma^0$ and $(\Gamma^i)^\dagger = \Gamma^i$. The ten-dimensional Chirality matrix is given by

$$\Gamma^{11} \equiv \frac{1}{10!}\epsilon_{A_1 \cdots A_{10}}\Gamma^{A_1} \cdots \Gamma^{A_{10}} = \Gamma^0 \cdots \Gamma^9, \tag{A.1}$$

and squares to the identity.

Corresponding to the metric reduction (II.64), we decompose the ten-dimensional Dirac matrices according to

$$\begin{aligned}
\Gamma^\alpha &\equiv \gamma^\alpha \otimes 1_4 \otimes \sigma_1, \\
\Gamma^a &\equiv 1_4 \otimes \tau^a \otimes \sigma_2, \\
\Gamma^9 &\equiv 1_4 \otimes \tau^9 \otimes \sigma_2,
\end{aligned} \tag{A.2}$$

where γ^α are Dirac matrices in the $5d$ spacetime with $\gamma^4 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$ and τ^a are Dirac matrices in the $5d$ internal space with $\tau^9 \equiv \tau^5\tau^6\tau^7\tau^8$. The Chirality matrix Γ^{11} is then given by

$$\Gamma^{11} = \Gamma^0 \cdots \Gamma^9 = 1_4 \otimes 1_4 \otimes \sigma_3. \tag{A.3}$$

We furthermore take the following conventions for the A, C and D intertwiners which map between different representations of the Dirac matrices

$$A_{10}\Gamma_M A_{10}^{-1} = \Gamma_M^\dagger, \qquad C_{10}^{-1}\Gamma_M C_{10} = -\Gamma_M^T, \qquad D_{10}^{-1}\Gamma_M D_{10} = -\Gamma_M^*. \tag{A.4}$$

Here C_{10} denotes the charge conjugation matrix. These may be decomposed as

$$A_{10} = A_{4,1} \otimes A_5 \otimes \sigma_1, \qquad C_{10} = C_{4,1} \otimes C_5 \otimes \sigma_2, \qquad D_{10} = iD_{4,1} \otimes D_5 \otimes \sigma_3, \tag{A.5}$$

where the five-dimensional intertwiners are defined as

$$\begin{aligned} A_{4,1}\gamma_\mu A_{4,1}^{-1} &= -\gamma_\mu^\dagger, \qquad & C_{4,1}^{-1}\gamma_\mu C_{4,1} &= \gamma_\mu^T, \qquad & D_{4,1}^{-1}\gamma_\mu D_{4,1} &= -\gamma_\mu^* \\ A_5\tau_a A_5^{-1} &= \tau_a^\dagger, & C_5^{-1}\tau_a C_5 &= \tau_a^T, & D_5^{-1}\tau_a D_5 &= \tau_a^*. \end{aligned} \tag{A.6}$$

It turns out the following is a consistent decomposition:

$$A_{10} = \Gamma_0 = \gamma_0 \otimes 1 \otimes \sigma_1, \qquad C_{10} = C_{4,1} \otimes C_5 \otimes \sigma_2, \qquad D_{10} = i\gamma_0 C_{4,1} \otimes C_5 \otimes \sigma_3. \tag{A.7}$$

The five dimensional charge conjugation matrices on both spacetime and the internal manifold satisfy

$$C_5 = -C_5^T = C_5^* = -C_5^{-1}. \tag{A.8}$$

Finally, we define the charge conjugate of a spinor in any dimension to be $\psi^c = CA^T\psi^*$, which is equivalent to $\psi^c = -\Gamma_0 C_{10}\psi^*$. Therefore, letting χ and η be spinors on M and SE_5, respectively, the charge conjugates are given by $\chi^c = -\gamma_0 C_{4,1}\chi^*$ and $\eta^c = C_5\eta^*$.

APPENDIX B

Fermion Conventions and Reduced Action: The Reduced Lagrangian

B.1 The Reduced Lagrangian

The bosonic Lagrangian with the massive gravitino multiplet removed was presented in [77], and takes the form

$$\begin{aligned}\mathcal{L}_b \quad = \quad & R*1+(24e^{2A-2B}-4e^{5A+3C}-\frac{1}{2}e^{8A}(4+\phi_0)^2)*1-\frac{28}{3}dB\wedge *dB-\frac{8}{3}dB\wedge *dC \\ & -\frac{4}{3}dC\wedge *dC-\frac{1}{2\tau_2^2}d\tau\wedge *d\bar{\tau}-\frac{1}{2}e^{2C-2A}F_2\wedge *F_2-e^{A-C}(F_2+\frac{1}{4}\mathbb{F}_2')\wedge *(F_2+\frac{1}{4}\mathbb{F}_2') \\ & -\frac{1}{2}e^{-8B}[\mathbb{A}_1'-\frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)]\wedge *[\mathbb{A}_1'-\frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)] \\ & -2\mathcal{M}_{ij}[e^{5A-C}(f_0^i\bar{f}_0^j+\bar{f}_0^i f_0^j)*1+e^{-4B}(f_1^i\wedge *\bar{f}_1^j+\bar{f}_1^i\wedge *f_1^j)] \\ \text{(B.1)} \quad & -A_1\wedge(F_2+\frac{1}{4}\mathbb{F}_2')\wedge(F_2+\frac{1}{4}\mathbb{F}_2'),\end{aligned}$$

where $\mathbb{A}_1'=\mathbb{A}_1+\frac{2i}{3}\epsilon_{ij}(f_0^i\bar{f}_1^j-\bar{f}_0^i f_1^j)$, and where $\mathbb{F}_2'=d\mathbb{A}_1'$.

The corresponding fermionic Lagrangian may be obtained from the equations of motion presented in Section 2.8.2. At quadratic order in the fermions, we have

$$
\begin{aligned}
e^{-1}\mathcal{L}_f \;=\;& \bar{\hat{\psi}}_\mu \gamma^{\mu\nu\rho}\mathcal{D}_\nu\hat{\psi}_\rho \\
&+\Big[-\frac{8}{15}\bar{\psi}^{m=11/2}\gamma^\mu K(\psi^{m=11/2})\hat{\psi}_\mu - \frac{4}{5}\bar{\psi}^{m=-9/2}\gamma^\mu K(\psi^{m=-9/2})\hat{\psi}_\mu \\
&\qquad -\frac{1}{2}\bar{\lambda}'\gamma^\mu K(\lambda')\hat{\psi}_\mu + h.c.\Big] \\
&+\frac{8}{15}\bar{\psi}^{m=11/2}\Big[\gamma^\mu D_\mu + \frac{3i}{5}e^{-4B}\gamma^\mu\mathbb{A}_\mu - \frac{i}{120}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{11i}{60}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad +e^A\left(-\frac{17}{12}(4+\phi_0)e^{-4B-C} + \frac{1}{15}e^{-2B+C} - \frac{1}{10}e^{-C}\right)\Big]\psi^{m=11/2} \\
&+\frac{4}{5}\psi^{m=-9/2}\Big[\gamma^\mu D_\mu + \frac{2i}{5}e^{-4B}\gamma^\mu\mathbb{A}_\mu - \frac{3i}{40}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{3i}{20}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad +e^A\left(\frac{1}{4}(4+\phi_0)e^{-4B-C} + \frac{13}{5}e^{-2B+C} + \frac{9}{20}e^{-C}\right)\Big]\psi^{m=-9/2} \\
&+\frac{1}{2}\bar{\lambda}'\Big[\gamma^\mu D_\mu + \frac{i}{8}\gamma^{\mu\nu}\left(e^{C-A}F_{\mu\nu} - 2e^{-A-2B-C}p_{\mu\nu}\right) \\
&\qquad -\frac{1}{4}(4+\phi_0)e^{A-4B-C} + e^{A-2B+C} + \frac{3}{2}e^{A-C}\Big]\lambda' \\
&+\frac{8}{15}\Big[\bar{\psi}^{m=11/2}\Big(\frac{3i}{5}e^{-4B}\gamma^\mu\mathbb{A}_\mu + \frac{i}{5}e^{C-A}\gamma^{\mu\nu}F_{\mu\nu} - \frac{i}{10}e^{-A-2B-C}\gamma^{\mu\nu}p_{\mu\nu} \\
&\qquad +e^A\left(\frac{12}{5}e^{-2B+C} - \frac{12}{5}e^{-C}\right)\Big)\psi^{m=-9/2} + h.c.\Big] \\
&+\frac{8}{15}\Big[v_i e^{-2B}\bar{\psi}^{m=11/2}\left(-\frac{2}{5}\gamma^\mu f^i_\mu + \frac{34i}{15}e^{A-C}f^i_0\right)\psi^{c\,m=11/2} + h.c.\Big] \\
&+\frac{8}{15}\Big[v_i e^{-2B}\bar{\psi}^{m=11/2}\left(\frac{3}{5}\gamma^\mu f^i_\mu - \frac{7i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2} + h.c.\Big] \\
&+\frac{4}{5}\Big[v_i e^{-2B}\bar{\psi}^{m=-9/2}\left(-\frac{3}{5}\gamma^\mu f^i_\mu - \frac{3i}{5}e^{A-C}f^i_0\right)\psi^{c\,m=-9/2} + h.c.\Big] \\
&+\frac{8}{15}\Big[\bar{v}_i e^{-2B}\bar{\psi}^{m=11/2}\left(\frac{3}{4}\gamma^\mu f^i_\mu + \frac{7i}{4}e^{A-C}f^i_0\right)\lambda' + h.c.\Big] \\
&+\frac{4}{5}\Big[\bar{v}_i e^{-2B}\bar{\psi}^{m=-9/2}\left(\frac{1}{2}\gamma^\mu f^i_\mu - \frac{i}{2}e^{A-C}f^i_0\right)\lambda' + h.c.\Big],
\end{aligned}
\tag{B.2}
$$

and the full Lagrangian up to quadratic order in the fermions is given by

$$
\mathcal{L} = \mathcal{L}_b + \mathcal{L}_f. \tag{B.3}
$$

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BIBLIOGRAPHY

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