Dynamics on $\mathrm{PSL}(2,\mathbb{C})$-character varieties of certain hyperbolic 3-manifolds

by

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CHAPTER I

Introduction

The study of the dynamics on character varieties and representation varieties originated with Fricke's classical result that the mapping class group acts properly discontinuously on Teichmüller space. Let S be a closed oriented surface of genus at least two. The Teichmüller space $\mathcal{T}(S)$ of S is the set of conjugacy classes of discrete and faithful homomorphisms from $\pi_1(S)$ to the group of orientation-preserving isometries of $\mathbb{H}^2$, such that the hyperbolic surface obtaining from taking the quotient of $\mathbb{H}^2$ by the image of $\pi_1(S)$ has the same orientation as S. In other words,

$$\mathcal{T}(S) = \{\rho : \pi_1(S) \to \mathrm{Isom}^+(\mathbb{H}^2) | \rho \text{ is discrete, faithful and orientation preserving}\}/\sim,$$

where ρ_1 is equivalent to ρ_2 if they differ by conjugation i.e., there exists A in $\mathrm{Isom}^+(\mathbb{H}^2)$ such that for any γ in $\pi_1(S)$, $\rho_1(\gamma) = A\rho_2(\gamma)A^{-1}$. Equivalently, $\mathcal{T}(S)$ can be thought of as the space of marked hyperbolic surfaces homeomorphic to S i.e.,

$$\begin{aligned} \{(X, f)| \quad & X && \text{is a hyperbolic surface,} \\ & f : S \to X && \text{is an orientation-preserving homeomorphism }\}/\sim, \end{aligned}$$

where (X, f) is equivalent to (Y, g) if there exists an isometry $j : X \to Y$ such that $j \circ f$ is homotopic to g. The mapping class group $Mod(S)$ of S is the group of orientation-preserving homeomorphisms of S up to homotopy. It is an index two subgroup of $\mathrm{Out}(\pi_1(S))$ (see [20], Chapter 8).

If G is a linear algebraic Lie group and π is a torsion-free finitely-generated group, then $\mathrm{Hom}(\pi, G)$ has the structure of an algebraic variety. The group G acts on $\mathrm{Hom}(\pi, G)$ algebraically, via conjugation, and the orbit space $\mathcal{R}(\pi, G) := \mathrm{Hom}(\pi, G)/G$ is well-studied (for example, see [25]). When G is complex and reductive, the G-character variety of π, $\mathcal{X}(\pi, G)$ is

$$\mathrm{Hom}(\pi, G)/\!\!/G,$$

the quotient of $\mathrm{Hom}(\pi, G)$ from geometric invariant theory (see [40] for a precise definition). The group of outer automorphisms $\mathrm{Out}(\pi)$ of π, acts on $R(\pi, G)$ and $\mathcal{X}(\pi, G)$ by pre-composition in the following way. If $[f]$ is an element in $\mathrm{Out}(\pi)$ and $[\rho]$ is an element in $R(\pi, G)$ or $\mathcal{X}(\pi, G)$ then $[f][\rho] = [\rho \circ f^{-1}]$.

Goldman ([24]) showed that $\mathcal{R}(\pi_1(S), \mathrm{PSL}(2, \mathbb{R}))$ consists of $4g - 3$ components. The discrete and faithful representations comprise two components $\mathcal{T}(S)$ and $\mathcal{T}(\bar{S})$, where $\bar{S}$ is S with the opposite orientation. Fricke's result shows that $\mathrm{Out}(\pi_1(S))$ acts properly discontinuously on $\mathcal{T}(S) \cup \mathcal{T}(\bar{S})$ and Goldman conjectured that the action on the remaining $4g - 5$ components is ergodic.

There are several generalizations of Teichmüller space on which an analogous result holds. In particular, Labourie ([38]) showed the action of $\mathrm{Mod}(S)$ is properly discontinuous on the so-called Hitchin component of $\mathcal{R}(\pi_1(S), \mathrm{PSL}(n, \mathbb{R}))$, which consists of discrete and faithful representations. Similarly, if G is a Lie group of Hermitian type, Wienhard, Hartnick and Strubel ([58] and [28]) showed that the action of $\mathrm{Mod}(S)$ is properly discontinuous on the set of maximal representations. There are several results supporting a close analogy between these two types of representations and Teichmüller representations; for example, see [37] and [9]. On the other extreme, when G is compact and π is the fundamental group of a not necessarily closed surface of negative Euler characteristic, the action of $\mathrm{Out}(\pi)$ is ergodic ([26], [51] and [22]), except in the case when π is a free group of rank two.

This thesis studies a generalization of Teichmüller space to hyperbolic 3-manifolds, namely when π is the fundamental group of a compact 3-manifold M whose interior admits an infinite volume complete hyperbolic metric and G is $\mathrm{PSL}(2, \mathbb{C})$. In this situation, sitting inside $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ is $AH(M)$, the set of conjugacy classes of discrete and faithful representations, which can be thought of as the space of marked hyperbolic 3-manifolds homotopy equivalent to M. The deformation space $AH(M)$ is well-studied and there is a parametrization of the interior of $AH(M)$ using topological and conformal data (see Section 2.1.8). Using this parametrization and Fricke's result, it is not hard to see that $\mathrm{Out}(\pi)$ acts properly discontinuously on the interior of $AH(M)$. A natural question might be whether $AH(M)$ is a domain of discontinuity. There are several problems with this formulation. First, unlike the situation of Teichmüller space, Hitchin representations and maximal representations, $AH(M)$ is not a collection of connected components of $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$; in fact, it is not open. It is, however, $\mathrm{Out}(\pi)$-invariant and one could still ask if the action is properly discontinuous on $AH(M)$. Canary and Storm ([18]) showed that whenever M contains a primitive essential annulus, $\mathrm{Out}(\pi)$ does not act properly discontinuously on $AH(M)$. If M contains no essential annuli and has incompressible boundary, then $\mathrm{Out}(\pi)$ is finite (see [32] Theorem 27.1), and hence acts properly discontinuously on $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$.

In the case that M is a $S \times [0,1]$, Goldman conjectured that quasi-Fuchsian space, the interior of $AH(S \times [0,1])$ is a maximal domain of discontinuity. In the case that M is a handlebody of genus at least two, H_g, Minsky ([46]) exhibited a domain of discontinuity called the set of primitive-stable representations that is strictly larger than the interior of $AH(H_g)$. Canary, Magid and Storm ([18] and [15]) studied the case when M has incompressible boundary and is not an interval bundle and showed in all these cases that there exists a domain of discontinuity strictly larger than the interior of $AH(M)$. In this thesis, we show the existence of such domains of discontinuity in two of the remaining classes of hyperbolic 3-manifolds: twisted interval bundles and compression bodies.

1.1 Main Results

In this section we describe the main results of this thesis. For ease of notation, in the remainder of this thesis we will use $\mathcal{X}(M)$ to denote $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$.

1.1.1 Twisted I-bundle

A twisted I-bundle M over a non-orientable surface B is

$$\tilde{B} \times I/(x,t) \sim (\theta(x), 1-t),$$

where $\tilde{B}$ is the orientable double cover of B and θ is the orientation reversing involution inducing the covering map $\tilde{B} \to B$. Then, $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ is the same as $\mathcal{X}(\pi_1(B), \mathrm{PSL}(2,\mathbb{C}))$ as $\pi_1(M)$ is isomorphic to $\pi_1(B)$. We will always assume that B is closed.

The first result in this thesis is the following:

Theorem I.1. *If M is a twisted I-bundle over a nonorientable hyperbolic surface, then there exists an open, $\mathrm{Out}(\pi_1(M))$-invariant subset $\mathcal{PS}(M)$, called the set of primitive-stable representations, in $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ containing the interior of $AH(M)$ as well as a subset of $\partial AH(M)$ such that the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{PS}(M)$ is properly discontinuous.*

The set $\mathcal{PS}(M)$ is analogous to the set of primitive-stable representations introduced by Minsky in [46] for the case where M is a handlebody. In our case, a primitive element of $\pi_1(M)$ is one that corresponds to a simple closed curve on the base surface B. A primitive-stable representation is one that sends geodesics corresponding to primitive elements in the Cayley graph $C_S(G)$ to uniform quasi-geodesics in $\mathbb{H}^3$ under an orbit map.

Naturally, we would like to know whether $\mathcal{PS}(M)$ is a maximal domain of discontinuity. Toward this end, we prove Theorem I.2 which characterizes which points in $AH(M)$ lie in $\mathcal{PS}(M)$ and also characterizes which points in $AH(M)$ can lie in a domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$.

Theorem I.2. *Let M be a hyperbolizable twisted I-bundle and let $[\rho]$ be an element of $AH(M)$. Then, $[\rho]$ lies in the complement of $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic. Moreover, if $[\rho]$ lies in $AH(M) - \mathcal{PS}(M)$, then $[\rho]$ does not lie in any domain of discontinuity for the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$.*

Combining Theorem I.2 with the work of Canary, Magid and Storm ([18], [15]) completes the picture when M has incompressible boundary. Namely, the following is true.

Corollary I.3. *Let M be a compact, orientable, hyperbolizable 3-manifold with nonempty incompressible boundary. Then there exists an open,* $\mathrm{Out}(\pi_1(M))$*-invariant set, containing the interior of $AH(M)$ and a subset of the boundary of $AH(M)$, on which* $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously if and only if M is not a trivial I-bundle over an orientable hyperbolic surface. Moreover, in the case that M is not a trivial I-bundle, this set contains all purely hyperbolic points in $AH(M)$.*

Corollary I.3 suggests that when considering the dynamics of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{X}(M)$ the case when M is a trivial I-bundle over an orientable, hyperbolic surface is an anomalous case.

1.1.2 Compression body

A compression body is the boundary connect sum of a 3-ball, a collection of trivial I-bundles over closed surfaces and a handlebody where the other components are attached to the 3-ball along disjoint discs. The fundamental group of a compression body is a free product of surface groups and a free group. The main result in this case is the following.

Theorem I.4. *If M is a nontrivial compression body without toroidal boundary components, then there exists an open,* $\mathrm{Out}(\pi_1(M))$*-invariant subset $\mathcal{SS}(M)$ of* $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$*, called the set of separable-stable representations, containing the interior of $AH(M)$ as well as a subset of $\partial AH(M)$ such that the action of* $\mathrm{Out}(\pi_1(M))$ *on $\mathcal{SS}(M)$ is properly discontinuous.*

This case is a generalization of the case when M is a handlebody, established by Minsky in [46], although, a priori, when M is a handlebody the set of separable-stable representations could be smaller than the set of primitive-stable representations introduced by Minsky. A separable element of $\pi_1(M)$ is one that lies in a proper factor of a free decomposition of $\pi_1(M)$, except when M is a connect sum of two trivial I-bundles. In that case, a separable element is one that misses an essential annulus in one of the two trivial I-bundles. The definition of separable-stability is analogous to the definition of primitive-stability that is, a separable-stable element is one that sends geodesics corresponding to separable elements in the Cayley graph of $\pi_1(M)$ to uniform quasi-geodesics in $\mathbb{H}^3$ under an orbit map.

When M is a compression body, the interior of $AH(M)$ has infinitely many components; they are indexed by marked homeomorphism types. When proving Theorem I.4 we find examples of points on $\partial AH(M)$ that lie in $\mathcal{SS}(M)$ that all correspond to manifolds homeomorphic to M. We also show that there are some components in the interior of $AH(M)$ for which no boundary points lie in $\mathcal{SS}(M)$ (see Proposition V.13).

1.1.3 Outline

Here we briefly outline the rest of the thesis. In chapter II we review some well-known theory of hyperbolic 3-manifolds and prove some elementary but useful results about orbit maps. In chapter III, we define the notion of Q-stable representations, which provides a general framework for defining domains of discontinuity. Then, in chapters IV and V we apply this framework to show that both the set of primitive-stable representations and separable-stable representations are domains of discontinuity. In chapter IV we prove Theorem I.1 and Theorem I.2. In chapter V we prove Theorem I.4.

CHAPTER II

Background

In Section 2.1, we review the background material on hyperbolic 3-manifolds that we will need (for more detailed information see [2], [41], or [43]). In Section 2.2 we prove some basic facts about orbit maps that we will use in Chapters IV and V.

2.1 Hyperbolic Geometry

A hyperbolic 3-manifold N is a Riemannian 3-manifold of constant sectional curvature -1. We will always assume that N is orientable. Then, equivalently, N is a Riemannian manifold isometric to $\mathbb{H}^3/\Gamma$, where Γ is a torsion-free Kleinian group, that is a discrete subgroup of the group of orientation-preserving isometries of $\mathbb{H}^3$. We will also always assume that Γ is finitely-generated and non-elementary (equivalently non-abelian). We can identify $\mathrm{Isom}^+(\mathbb{H}^3)$, the group of orientation-preserving isometries of $\mathbb{H}^3$, with $\mathrm{PSL}(2,\mathbb{C})$, by using the action of an isometry on $\partial\mathbb{H}^3$ (see [41], Chapter 1).

2.1.1 Thick-thin decomposition

The *injectivity radius* of N at a point x is

$$\mathrm{inj}_N(x) = \inf\{\mathrm{length}(c)/2 | c \text{ is a homotopically nontrivial curve passing through } x\}.$$

Given $\epsilon > 0$, the *ϵ-thin part* of N is

$$N_{\mathrm{thin}(\epsilon)} = \{x \in N | \, \mathrm{inj}_N(x) < \epsilon\}.$$

Similarly, the ϵ-*thick part* of N is

$$N_{\text{thick}(\epsilon)} = \{x \in N | \operatorname{inj}_N(x) \geq \epsilon\}.$$

There exists a constant $\mu_3 > 0$, called the *Margulis constant,* such that for any hyperbolic 3-manifold N and any $\epsilon < \mu_3$, each component of $N_{\text{thin}(\epsilon)}$ is one of the following (see [2], Chp. D):

(a) a metric neighborhood of a closed geodesic,

(b) a parabolic cusp homeomorphic $S^1 \times \mathbb{R} \times (0, \infty)$, or

(c) a parabolic cusp homeomorphic to $T \times (0, \infty)$, where T is a torus.

Let N^0_ϵ denote N with all components of type (b) and (c) removed.

2.1.2 Cores

The *convex core* $C(N)$ of N is the smallest convex submanifold of N such that the inclusion of $C(N)$ into N is a homotopy equivalence. The *limit set* $\Lambda(\Gamma) \subset \partial\mathbb{H}^3$ is the smallest, closed Γ-invariant subset of $\partial\mathbb{H}^3$. When Γ is non-elementary, $C(N)$ is $CH(\Lambda(\Gamma))/\Gamma$, where $CH(\Lambda(\Gamma))$ is the convex hull of the limit set $\Lambda(\Gamma)$. N is called *convex cocompact* if $C(N)$ is compact. N is called *geometrically finite* if $C(N) \cap N^0_\epsilon$ is compact and *geometrically infinite* otherwise.

In general, when $\pi_1(N)$ is finitely generated there exists a compact submanifold C, called the *compact core*, whose inclusion induces a homotopy equivalence with N (see [52]). Moreover, C can be chosen such that it intersects each component of the noncompact portion of $N_{\text{thin}(\epsilon)}$ in a single incompressible annulus if the component is homeomorphic to $S^1 \times \mathbb{R} \times (0, \infty)$ or a single incompressible torus if the component is homeomorphic to $T \times (0, \infty)$ (see [44]). A compact core that intersects each component of the noncompact portions of $N_{\text{thin}(\epsilon)}$ in this way is called a *relative compact core.*

2.1.3 Conformal boundary

The *domain of discontinuity* $\Omega(\Gamma)$ is the complement of $\Lambda(\Gamma)$ in $\partial\mathbb{H}^3$; it is the largest open set of $\partial\mathbb{H}^3$ on which Γ acts properly discontinuously. It can be uniquely endowed with a Γ-invariant hyperbolic metric, conformally equivalent to the metric induced by considering $\Omega(\Gamma)$ as a subset of $\partial\mathbb{H}^3 \cong \mathbb{C}P^1$. The *conformal boundary* of N is $\partial_C N = \Omega(\Gamma)/\Gamma$ a collection of hyperbolic surfaces obtained by taking the quotient of $\Omega(\Gamma)$ by Γ. The conformal bordification of N, $(\mathbb{H}^3 \cup \Omega(\Gamma))/\Gamma$ is homeomorphic to $C(N)$, except when Γ is Fuchsian.

2.1.4 Measured laminations

Let T be a closed hyperbolic surface. A *(geodesic) lamination* on T is a closed subset $\lambda \subset T$ which is a union of disjoint simple geodesics. A *leaf* of λ is a simple geodesic in λ. A lamination λ is *minimal* if each half-leaf is dense in λ. A *measured lamination* is a pair (λ, ν) where λ is a geodesic lamination and ν is a Borel measure on arcs transverse to λ such that the support of ν is λ and ν is invariant under isotopies of T preserving λ. Let $ML(T)$ denote the space of measured laminations on T with the weak-$*$ topology on measures and let $PML(T)$ denote $(ML(T) - \{\emptyset\})/\mathbb{R}^+$, the space of projective measured laminations. Weighted simple closed geodesics are dense in $ML(T)$ ([55], Proposition 8.10.7). For γ_1, γ_2 two simple closed geodesics, $i(\gamma_1, \gamma_2)$ the intersection number is the number of points in $\gamma_1 \cap \gamma_2$. This naturally extends to two weighted simple closed geodesics and furthermore to a continuous map $i : ML(T) \times ML(T) \to \mathbb{R}_{\geq 0}$ (see [4] Proposition 4.4). A measured lamination (λ, μ) is *filling* if for any other measured lamination (α, ν) with different support, $i((\lambda, \mu), (\alpha, \nu))$ is nonzero.

2.1.5 Masur domain laminations

For this section, suppose that M is a compression body that is, the boundary connect sum of a 3-ball, a collection of trivial I-bundles over closed surfaces and a handlebody where the other components are attached to the 3-ball along disjoint discs. A *meridian* is a simple closed curve on ∂M that is nontrivial in ∂M but trivial in M. M has one compressible boundary component called the *exterior boundary*, denoted $\partial_{ext}(M)$. Let $\mathcal{M}$ denote the set of meridians on ∂M and let $\mathcal{M}'$ denote the closure of $\mathcal{M}$ in $PML(\partial_{ext}(M))$ (here we fix a convex cocompact hyperbolic structure on M and consequently a hyperbolic structure on ∂M). Let

$$\mathcal{M}'' = \{\lambda \in PML(\partial_{ext}(M))| \text{ there exists a } \nu \in M' \text{ such that } i(\lambda, \nu) = 0\}.$$

A compression body is *small* if it is the boundary connect sum of two trivial I-bundles over closed surfaces or the boundary connect sum of a trivial I-bundle over a closed surface and a solid torus. A compression body is *large* otherwise.

Definition II.1. If M is a large compression body then an element λ in $PML(\partial_{ext}(M))$ lies in the *Masur domain*, denoted $\mathcal{O}(M)$, if it has nonzero intersection number with every element of $\mathcal{M}'$. If M is a small compression body then an element λ in $PML(\partial_{ext}(M))$ lies in $\mathcal{O}(M)$ if it has nonzero intersection number with every lamination in $\mathcal{M}''$.

The Masur domain $\mathcal{O}(M)$ is an open set of full measure in $PML(\partial_{ext}(M))$ ([42] and [34]). Moreover, Mod(M), the group of isotopy classes of homeomorphisms of $\partial_{ext}(M)$ that extend to homeomorphisms of the compression body, acts properly discontinuously on the set of Masur domain laminations ([42] and [50] Proposition 1.4). Since Mod(M) acts properly discontinuously on $\mathcal{O}(M)$, if λ lies in $\mathcal{O}(M)$, then λ intersects every essential annulus in M. Otherwise, Dehn twists about an essential annulus missing λ produce infinitely many elements in the stabilizer of λ in Mod(M), which would contradict the proper discontinuity of the action.

For a general hyperbolizable 3-manifold M with compressible boundary, Lecuire described an extension of the Masur domain, called the set of *doubly incompressible* laminations. A measured lamination (λ, μ) is doubly incompressible if there exists a constant $\eta > 0$ such that $i(\lambda, \partial E) > \eta$ for any essential disc or essential annulus E. When M is a compression body, the pre-image of $\mathcal{O}(M)$ in $ML(S)$ is contained in $\mathcal{D}(M)$ ([39], Lemma 3.1).

2.1.6 Pleated surfaces and simplicial hyperbolic surfaces

Pleated surfaces and simplicial hyperbolic surfaces are two different types of 1-Lipschitz maps from a negatively curved surface into a hyperbolic 3-manifold.

Definition II.2. A *pleated surface* in a hyperbolic 3-manifold N is a surface S with a hyperbolic metric τ of finite area and a map $h : (S, \tau) \to N$ which takes rectifiable arcs in S to rectifiable arcs of the same length in N such that every point x in S lies in the interior of some geodesic arc that is mapped by p to a geodesic arc in N. The *pleating locus* is the set of points in S that lie in the interior of exactly one geodesic arc that is mapped to a geodesic arc in N.

The pleating locus is a geodesic lamination that maps to a union of geodesics in N. Although there can be many pleated surfaces in a fixed homotopy class realizing a geodesic lamination λ, the image of λ in N is unique in that homotopy class ([13], Lemma I.5.3.5). If T is an incompressible boundary component of N and λ is a filling lamination on T, then λ can be realized as the pleating locus of a pleated surface homotopic to the inclusion of T in N. When T is a compressible boundary component of N and $\lambda \subset T$ is the support of a doubly incompressible lamination, then λ can be realized as the pleating locus of a pleated surface homotopic to the inclusion of T in N ([39], Theorem 5.1).

A triangulation $\tau = (V, E)$ of a closed surface S is a collection of points $V = \{v_1, \ldots, v_n\}$ on S, and a maximal collection $E = \{e_1, \ldots, e_m\}$ of pairwise nonisotopic and disjoint arcs with endpoints in V. A *simplicial hyperbolic surface* is a map $h : (S, \tau) \to N$ from S a triangulated hyperbolic surface to N a complete hyperbolic 3-manifold that satisfies the following conditions:

- edges map to geodesics
- faces map to totally geodesic triangles
- the total angle around each vertex is at least 2π.

If h is a simplicial hyperbolic surface, then with the pull-back metric it is locally CAT(-1); in particular, the area of S is at most $2\pi\chi(S)$.

If $f : S \to N$ is an incompressible map and γ is a simple closed curve on S such that $f(\gamma)$ is homotopic to a closed geodesic $f(\gamma)^*$, then there exists a triangulation τ on S and a simplicial hyperbolic surface $h : (S, \tau) \to N$ homotopic to f such that $h(\gamma) = f(\gamma)^*$ ([4] §1.2). Moreover, if $h : (S, \tau) \to N$ is an incompressible simplicial hyperbolic surface, then S, with the pull-back metric, has bounded diameter modulo the thin part ([4], Lemma 1.11). More specifically, the following is true:

Lemma II.3 (Bonahon)**.** *Let $h : S \to N$ be an incompressible simplicial hyperbolic surface. Given $\epsilon > 0$, there exists a constant $B = B(\epsilon, g(S))$, where $g(S)$ is the genus of S, such that for any two points x and y in $h(S) \cap N_{thick(\epsilon)}$, there exists an arc k in $h(S)$ connecting x and y such that the length of $k \cap N_{thick(\epsilon)}$ is less than B.*

2.1.7 Ends of hyperbolic manifolds

The ends of N are in one-to-one correspondence with the components of $\partial C - P$, where C is a relative compact core and P is the intersection of C with the noncompact components of $N_{\mathrm{thin}(\epsilon)}$ (for a precise definition of ends see [33], Section 4.23). Any hyperbolic 3-manifold N with finitely generated fundamental group has finitely many ends. An end is called *geometrically finite* if it has a neighborhood U which does not intersect $C(N)$ and is called *geometrically infinite* otherwise. By the Tameness Theorem ([1], [10]), we can choose a relative compact core C such that $N^0_\epsilon - \mathrm{int}(C)$ is homeomorphic to $\partial C - P \times [0, \infty)$, where $\mathrm{int}(C)$ is the interior of C. Suppose that E is a component of $N^0_\epsilon - \mathrm{int}(C)$ homeomorphic to $T \times [0, \infty)$. If E is geometrically infinite, then there exists a sequence α_i of closed geodesics, homotopic in E to simple closed curves on T that leave every compact set of E. Fix a hyperbolic surface T' and a homeomorphism $T' \to T$. If α'_i is the geodesic on T' corresponding to α_i, then $\alpha'_i / l_{T'}(\alpha'_i)$ converges in $\mathcal{ML}(T')$ to a measured lamination (λ, μ) such that its support λ is independent of the sequence $\{\alpha_i\}$ ([4], [11]). In this situation λ is called the ending lamination for the end E. It is minimal and is the support of a filling doubly incompressible lamination ([11] Corollary 10.2).

2.1.8 PSL$(2,\mathbb{C})$-character varieties and deformation spaces of hyperbolic 3-manifolds

Let M be a hyperbolic 3-manifold. In this section we describe the PSL$(2,\mathbb{C})$-character variety of $\pi_1(M)$ (for a more detailed discussion see [30] or Chapter 4 of [33]). Fix a presentation

$$\pi_1(M) =< x_1, \ldots, x_n | r_1, \ldots, r_m > .$$

Then, Hom$(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ can be given the structure of an affine algebraic subset of $\mathbb{C}^{9n}$ as follows. First, realize PSL$(2,\mathbb{C})$ as an affine subset of $\mathbb{C}^9$, via its adjoint representation. Then, identify Hom$(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ with a subset of $\mathbb{C}^{9n}$ by mapping $\rho \mapsto (\rho(x_1), \ldots \rho(x_n)) \in \mathrm{PSL}(2,\mathbb{C})^n$ and using the relations of $\pi_1(M)$ as the defining equations. The group PSL$(2,\mathbb{C})$ acts algebraically on Hom$(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ via conjugation. The quotient space is often not nice. In particular, it is often non-Hausdorff, and so instead, one can take the quotient from geometric invariant theory $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$, called the character variety. Each representation, $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$, has an associated character map, $tr^2 : \pi_1(M) \to \mathbb{C}$ where $g \mapsto tr(\rho(g))^2$. The character variety $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ is in bijection with the set of characters of representations in Hom$(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$. Clearly, two conjugate representations have the same character map. The converse is true for any irreducible representation ([30] Lemma 3.5).

Given a compact, irreducible 3-manifold M with nonempty boundary, $AH(M)$ is the image of the set of discrete and faithful representations in $\mathcal{X}(M)$. A neighborhood of $AH(M)$ is a smooth complex manifold in $\mathcal{X}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C}))$ ([33], Theorem 8.44). Since $AH(M)$ consists of irreducible representations, it consists of conjugacy classes of discrete and faithful representations. Equivalently, it is the deformation space of marked hyperbolic 3-manifolds homotopy equivalent to M in the following sense:

$$\begin{aligned} AH(M) = \{(N, f)| \quad & N \quad && \text{a hyperbolic 3-manifold,} \\ & f : M \to N && \text{a homotopy equivalence } \}/\sim, \end{aligned}$$

where (N_1, f_1) is equivalent to (N_2, f_2) if there exists $j : N_1 \to N_2$ an isometry such that $j \circ f_1$ is homotopic to f_2.

To see that these two notions of $AH(M)$ are equivalent, suppose that ρ is a discrete and faithful representation of $\pi_1(M)$ into $\mathrm{PSL}(2,\mathbb{C})$. Then, one gets a hyperbolic 3-manifold $N_\rho := \mathbb{H}^3/\rho(\pi_1(M))$. Moreover, as M and N_ρ are both $K(\pi_1(M),1)$-spaces, there exists a homotopy equivalence $h_\rho : M \to N$ that induces ρ on the level of fundamental groups. If ρ' is another discrete and faithful representation such that there exists $A \in \mathrm{PSL}(2,\mathbb{C})$ where $\rho'(g) = A\rho(g)A^{-1}$ for all g in $\pi_1(M)$, then $A : \mathbb{H}^3 \to \mathbb{H}^3$ descends to a map $j_A : N_\rho \to N_{\rho'}$ such that $j_A \circ h_\rho$ is homotopic to $h_{\rho'}$. Conversely, given a pair (N_1, f_1), as described above, then $\pi_1(N_1)$ is a discrete subgroup of $\mathrm{PSL}(2,\mathbb{C})$ and one gets a faithful representation $(f_1)_* : \pi_1(M) \to \pi_1(N_1)$. If (N_1, f_1) and (N_2, f_2) are equivalent, then the isometry $j : N_1 \to N_2$, described above, lifts to an isometry of $\mathbb{H}^3$ conjugating $(f_1)_*$ to $(f_2)_*$.

Classical deformation theory of Kleinian groups describes the interior of $AH(M)$. Assume that M has no toroidal boundary components (the situation is well-understood even when M has toroidal boundary components but this assumption simplifies the statements and suffices for this thesis). In this case, Sullivan ([54]) showed the following.

Theorem II.4 (Sullivan)**.** *The interior of $AH(M)$ is precisely the set of convex cocompact representations.*

Moreover, there is a parametrization of the interior of $AH(M)$ that is described as follows. Let $\mathcal{A}(M)$ be the set of compact, oriented, irreducible marked 3-manifolds homotopy equivalent to M, i.e., the set of pairs (M', f) such that $f : M \to M'$ is a homotopy equivalence, where two pairs (M_1, f_1) and (M_2, f_2) are equivalent if there exists a homeomorphism $j : M_1 \to M_2$ such that $j \circ f_1$ is homotopic to f_2. For each (M', f) in $\mathcal{A}(M)$ let $\mathcal{T}(\partial M')$ be the Teichmüller space of $\partial M'$. If $\partial M'$ has multiple components then $\mathcal{T}(\partial M')$ is the direct product of the Teichmüller spaces of each component. For (M', f) in $\mathcal{A}(M)$, let $\mathrm{Mod}_0(M')$ denote the isotopy classes of homeomorphisms of $\partial M'$ that extend to homeomorphisms of M' homotopic to the identity.

If ρ is a convex cocompact representation such that N_ρ lies in the marked homeomorphism class of M', then there is a homeomorphism $\partial M' \to \partial_C N_\rho$ that corresponds to a point in $\mathcal{T}(\partial M')$ that is only well-defined up to homeomorphisms in $\mathrm{Mod}_0(M')$. By work of Ahlfors, Bers, Kra, Marden, Maskit, Sullivan and Thurston (see [16], Chapter 7), the interior of $AH(M)$ is homeomorphic to

$$\coprod_{[(M',f')]\in\mathcal{A}(M)} \mathcal{T}(M')/\operatorname{Mod}_0(M').$$

Using the parametrization described above, one can show the action of $\operatorname{Out}(\pi_1(M))$ on the interior of $AH(M)$ is properly discontinuous (see [15]). Chapter III offers an alternate proof in the case that M has no tori in its boundary.

In the case that M is homeomorphic to $S \times I$, where S is a closed orientable hyperbolic surface, the interior of $AH(M)$ is parameterized by $\mathcal{T}(S) \times \mathcal{T}(\bar{S})$, where $\bar{S}$ is S with the opposite orientation. Fixing a point Y in the second factor, produces a *Bers slice* B_Y, which is homeomorphic to $\mathcal{T}(S)$. Bers proved that any Bers slice has compact closure ([3]).

2.1.9 Convergence

Let $\rho_i : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ be a sequence of discrete and faithful representations. We say ρ_i converges *algebraically* to $\rho : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ if for any fixed generating set $g_1, \ldots, g_n$ of $\pi_1(M)$, the sequence $\rho(g_i)$ converges to $\rho(g)$ in $\mathrm{PSL}(2,\mathbb{C})$.

We say a sequence of Kleinian groups Γ_i converges *geometrically* to Γ if Γ_i converges to Γ in the Chabauty topology on closed subsets of $\mathrm{PSL}(2,\mathbb{C})$. In other words, Γ_i converges geometrically to Γ if the following two conditions hold:

(a) if x is in Γ, then there exists x_i in Γ_i such that $x_i \to x$ and

(b) if there exists x_i in Γ_i such that $x_i \to x$, then x is in Γ.

If Γ_i converges to Γ geometrically, then the sequence of corresponding manifolds $N_i = \mathbb{H}^3/\Gamma_i$ converges to $N = \mathbb{H}^3/\Gamma$, in the sense of Gromov-Hausdorff convergence.

If a sequence ρ_i converges algebraically, then up to subsequence $\rho_i(\pi_1(M))$ converges geometrically. In general, the algebraic limit is contained in but need not coincide with the geometric limit. In the case that the two limits coincide, we say that the sequence ρ_i converges *strongly.* Combining several results in Kleinian group theory produces a sufficient condition for strong convergence (see [6] Theorem 2.9).

Theorem II.5. *Let M be a compact 3-manifold and let $\{\rho_n\}$ be a sequence in $AH(M)$ converging to ρ in $AH(M)$. If every parabolic element of $\rho(\pi_1(M))$ lies in a rank two free abelian subgroup, then $\{\rho_n\}$ converges strongly to ρ.*

2.2 Cayley graphs and quasi-isometries

If G is a finitely generated group and S a finite symmetric generating set, then the *Cayley graph of G with respect to S*, $C_S(G)$, is a graph where the vertices are in one-to-one correspondence with the elements of G and there is an edge between g and h if there exists an element s in the generating set S such that $gs = h$. A group G acts on $C_S(G)$ by left multiplication on the vertices. If X and Y are two metric spaces, a (K, A)-*quasi-isometry* from X to Y is a map $q : X \to Y$ such that for any two points x and z in X

$$\frac{d(x,z)}{K} - A \leq d(q(x), q(z)) \leq Kd(x,z) + A.$$

A (K, A)-quasi-geodesic is a (K, A)-quasi-isometric image of $\mathbb{R}$. When we refer to quasi-geodesics in $C_S(G)$ we assume it is parametrized by arclength. If G is a convex cocompact torsion-free Kleinian group, then $C_S(G)$ is Gromov hyperbolic. Let $\delta_{(G,S)}$ denote the hyperbolicity constant of $C_S(G)$. As G is torsion-free and hyperbolic, every element of G acts with North-South dynamics on $C_S(G)$. Let $||g||$ denote the minimum of $d(v, gv)$ taken over all vertices v in $C_S(G)$. There does not necessarily exist an axis for the action of g on $C_S(G)$, but the following is true.

Lemma II.6. *Let G be a torsion-free hyperbolic group. There exists (K', A') depending on $\delta_{(G,S)}$ and G such that for any element g, there exists a (K', A')-quasi-geodesic l'_g in $C_S(G)$ that satisfies the following:*

(a) $g \cdot l'_g = l'_g$ *and*

(b) $d(z, gz) = ||g||$ *for any z on l'_g.*

Proof. For an element g in G let w_g be a vertex in $C_S(G)$ that realizes the minimum translation distance of g. Let $l'_g = \cup g^n[w_g, gw_g]$, where $[w_g, gw_g]$ is a geodesic connecting w_g

and gw_g. Then, l'_g is a $||g||$-local geodesic, for otherwise there would be a point on l'_g moved less than $||g||$. The translation distance of g along l'_g is $||g||$, by construction. There exists a T such that for $||g|| > T$, l'_g is a (T, T)-quasi-geodesic ([23] Chapter 5 Theorem 13). For an element g with $||g|| < T$, there is (K_g, A_g) such that l'_g is a (K_g, A_g)-quasi-geodesic ([23], Chapter 5). For any conjugate hgh^{-1} of g, the translate $h \cdot l'_g$ is a (K_g, A_g)-quasi-geodesic satsifying (a) and (b). As there are finitely many conjugacy classes of elements $[g]$ with $||g|| < T$, let $K' = \max\{T, K_{g_i}\}$ and $A' = \max\{T, A_{g_i}\}$ where we take one representative g_i from each conjugacy class with $||g_i|| < T$. □

We will call l'_g a (K', A')-*quasi-axis* for g.

When a group G acts on a geodesic metric space X, for a fixed basepoint x in X, let $\tau_x : C_S(G) \to X$ be an orbit map, i.e., a map that sends the identity to a basepoint x in X, is G-equivariant and sends edges to geodesic segments.

Here we will gather some useful facts about such orbit maps in the case that X is $\mathbb{H}^3$. We will drop the subscript x for ease of notation. When we discuss quasi-geodesics in $\tau(C_S(G))$ we assume that the parametrization is the composition of the parametrization by arclength in $C_S(G)$ and τ.

Remark II.7. *We will use the following observation frequently. If* $M = \max\{d(\tau(e), \tau(s))\}$ *where* e *is the identity, and* s *varies over all elements in* S, τ *is* M*-Lipschitz.*

As the following lemma shows, to check whether $\tau(\gamma)$ is a quasi-geodesic in $\mathbb{H}^3$, it suffices to check that the vertices on γ are mapped sufficiently far apart.

Lemma II.8. *Let* G *be a finitely-generated group acting by isometries on* $\mathbb{H}^3$*. Given* (K, A)*, there exists* (K', A') *such that for any* v, w *and* γ*, where* γ *is an infinite path in* $C_S(G)$ *and* v, w *are vertices on* γ *if*

$$d_{\mathbb{H}^3}(\tau(v), \tau(w)) \geq \frac{d_\gamma(v, w)}{K} - A,$$

then $\tau(\gamma)$ *is a* (K', A')*-quasi-geodesic, where* d_γ *is the arclength between* v *and* w *along* γ*.*

Proof. Let M be as in Remark II.7. As τ is M-Lipschitz,

$$d_{\mathbb{H}^3}(\tau(v), \tau(w)) \leq M d_\gamma(v, w).$$

If v', w' are any two points on γ, let v and w be the vertices on γ closest to v and w, respectively. Then

$$\begin{array}{rcl} d_{\mathbb{H}^3}(\tau(v), \tau(w)) & \geq & d_{\mathbb{H}^3}(\tau(v'), \tau(w')) - 2M \\ & \geq & \dfrac{d_\gamma(v', w')}{K} - A - 2M \\ & \geq & \dfrac{d_\gamma(v, w) - 2}{K} - A - 2M. \end{array}$$

Let $K' = \max\{M, K\}$ and $A' = 2/K + A + 2M$. $\square$

Lemma II.9. *Let G be a finitely-generated hyperbolic group acting by isometries on $\mathbb{H}^3$. Let γ be a geodesic in $C_S(G)$ such that $\tau(\gamma)$ is a (K, A)-quasi-geodesic. Let γ' be a (K', A')-quasi-geodesic in $C_S(G)$ with the same endpoints at infinity as γ. Then there exists K'' and A'' such that $\tau(\gamma')$ is (K'', A'')-quasi-geodesic, where K'' and A'' are independent of γ and γ'.*

Proof. Since γ' is a (K', A')-quasi-geodesic, it is contained in the $R' = R'(K', A')$-neighborhood of γ.

Suppose that v' and w' are vertices on γ'. Let v and w be vertices on γ closest to v' and w', respectively. Then $d_{C_S(G)}(v, v')$ and $d_{C_S(G)}(w, w')$ are less than R' and hence $d_{\mathbb{H}^3}(\tau(v), \tau(v'))$ and $d_{\mathbb{H}^3}(\tau(w), \tau(w'))$ are less than $R'M$, where M is as in Remark II.7.

Then,

$$\begin{array}{rcl} d_{\mathbb{H}^3}(\tau(v'), \tau(w')) & \geq & d_{\mathbb{H}^3}(\tau(v), \tau(w)) - 2R'M \\ & \geq & \dfrac{d_{C_S(G)}(v, w)}{K} - A - 2R'M \\ & & \text{since } \tau_x(\gamma) \text{ is } (K, A)\text{-quasi-geodesic} \\ & \geq & \dfrac{d_{C_S(G)}(v', w')}{K} - \dfrac{2R'}{K} - A - 2R'M \\ & \geq & \dfrac{d_{\gamma'}(v', w')}{K'K} - \dfrac{A'}{K} - \dfrac{2R}{K} - A - 2R'M \\ & & \text{since } \gamma' \text{ is } (K', A')\text{-quasi-geodesic} \end{array}$$

By Lemma II.8, we are done. $\square$

The next fact is a useful characterization of quasi-geodesics in $\tau(C_S(G))$, due to Minsky in [46]. Let L be a geodesic in $C_S(G)$, L' the image of L under τ, $\{v_i\}$ the image of the vertex sequence of L, and $P_{j,i}$ the plane that perpendicularly bisects the geodesic segment $[v_{ji}, v_{(j+1)i}]$.

Lemma II.10 (Minsky)**.** *Let G be a finitely-generated hyperbolic group acting by isometries on $\mathbb{H}^3$. Given (K, A) there exists $c > 0$ and $i \in \mathbb{N}$ such that if $L' = \tau(L)$ is a (K, A)-quasi-geodesic, then $P_{j,i}$ separates $P_{(j+1),i}$ and $P_{(j-1),i}$ and $d(P_{j,i}, P_{(j+1),i}) > c$. Conversely, given $c > 0$ and $i \in \mathbb{N}$ there exists (K', A') such that if $L' = \tau(L)$ has the property that $P_{j,i}$ separates $P_{(j+1),i}$ and $P_{(j-1),i}$ and $d(P_{j,i}, P_{(j+1),i}) > c$ then L' is a (K', A')-quasi-geodesic.*

Proof. For the forward direction suppose that L' is a (K, A)-quasi-geodesic. There exists $R' = R'(K, A)$ and a geodesic l such that L' is contained in a R'-neighborhood of l. Let $\pi : \mathbb{H}^3 \to l$ be projection onto l. Then, $\pi \circ \gamma$ is a $(K, A + 2R')$-quasi-geodesic.

The following lemma shows that for i large enough, $d(P_{ji}, P_{(j+1)i})$ is at least $2R'$.

Lemma II.11. *Let l be a geodesic in $\mathbb{H}^3$. Given $R > 0$, there exists $D = D(R)$ such that if x and y are two points on l a distance at least D apart then for any two points x' and y' lying in $B_R(x)$ and $B_R(y)$ respectively, the plane P perpendicularly bisecting $[x', y']$ is contained in the region bounded by the two planes P_x and P_y where P_x is the plane perpendicular to l through the point on $B_R(x)$ closest to y and P_y is the plane perpendicular to l through the point on $B_R(y)$ closest to x (see Figure 2.1).*

Proof. Suppose to the contrary that there exists $R > 0$, D_i approaching infinity and points x_i, y_i, x_i', y_i' such that x_i and y_i are a distance D_i apart, x_i', y_i' lie in $B_R(x_i)$ and $B_R(y_i)$ respectively and P_i, the perpendicular bisector of $[x_i', y_i']$ intersects either P_{x_i} or P_{y_i}. Without loss of generality assume that P_i intersects P_{x_i} for all i. Consider the triangle T formed by the vertices $v_i := [x_i, y_i] \cap P_{x_i}$, $w_i := P_{x_i} \cap P_i$ and $z_i := [x_i, y_i] \cap P_i$. Then, T is a right triangle with a right angle at v_i (see Figure 2.2).

First notice that $d(v_i, z_i)$ approaches infinity as

$$d(v_i, z_i) \geq d(x_i', m_i') - d(x_i', v_i) - d(z_i, m_i'),$$

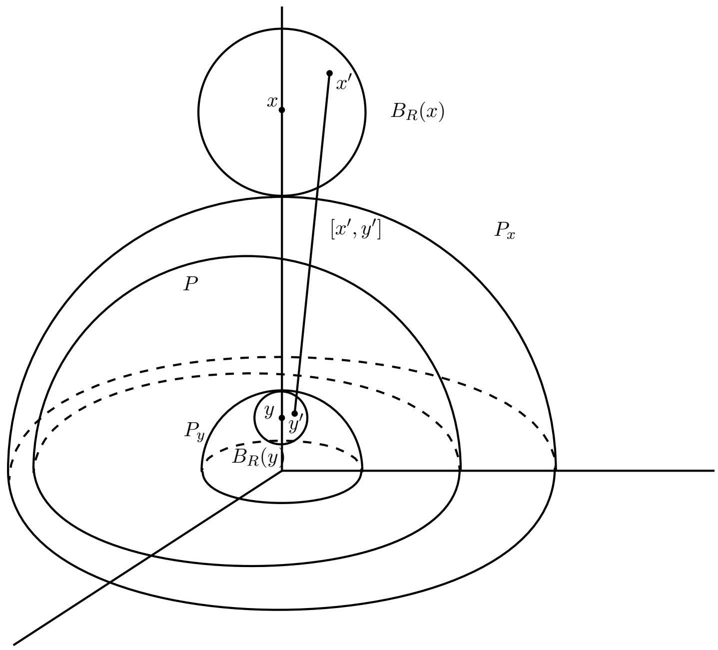

Figure 2.1: If the distance between x and y is large enough P is contained in the region bounded by P_x and P_y.

where m_i' is the midpoint of $[x_i', y_i']$. Since $d(x_i', m_i')$ is at least $\frac{D_i - 2R}{2}$ and $d(x_i', v_i), d(z_i, m_i')$ are bounded above by $2R$, the claim follows.

Next, notice that in order to form a right triangle with the above properties, the angle at z_i must approach zero. This is impossible, as $[x_i', y_i']$ approaches l, the angle between P_i and l approaches $\pi/2$. $\square$

Let $D' = D(R')$ as in Lemma II.11. Then, for i at least $K(D' + A + 2R)$, the distance

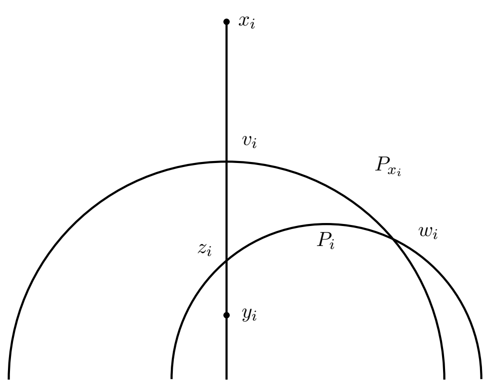

Figure 2.2: If P_i and P_{x_i} intersect, we can form the triangle above with right angle at v_i.

$d(P_{ji}, P_{(j+1)i})$ is at least $2R'$. It remains to show that P_{ji} separates $P_{(j-1)i}$ and $P_{(j+1)i}$. It suffices to show that if we look at the balls $B_{R'}(\pi(v_{ji}))$ they do not backtrack. Suppose to the contrary that $B_{R'}(\pi(v_{(j-1)i}))$ and $B_{R'}(\pi(v_{(j+1)i}))$ are on the same side of $B_{R'}(\pi(v_{ji}))$ along l. If $d(\pi(v_{(j+1)i}), \pi(v_{ji}))$ is less than $d(\pi(v_{ji}), \pi(v_{(j-1)i}))$, then $v_{(j+1)i}$ is within $2R'$ of some vertex w between $v_{(j-1)i}$ and v_{ji} (see Figure 2.3). Since w lies between $v_{(j-1)i}$ and v_{ji}, $d(v_{(j+1)i}, w)$ is at least $\frac{i}{K} - A$, a contradiction for i larger than $K(2R' + A)$. The same argument applies if $d(v_{(j-1)i}, v_{ji})$ is less than $d(v_{ji}, v_{(j+1)i})$.

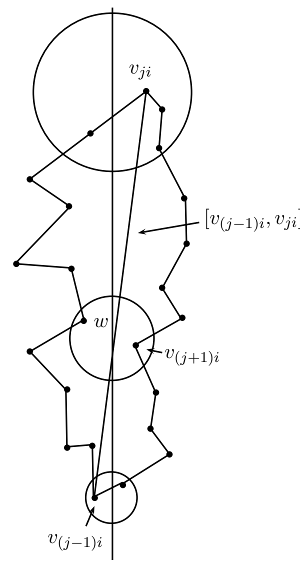

Figure 2.3: The distance between w and $v_{(j+1)i}$ is less than $2R$.

For the backwards direction, suppose that we have $c > 0$ and $i \in \mathbb{N}$ such that $L' = \tau(L)$ has the property that $P_{j,i}$ separates $P_{(j+1),i}$ and $P_{(j-1),i}$ and $d(P_{j,i}, P_{(j+1),i}) > c$. Suppose that v_1 and v_2 are vertices in $C_S(G)$ on L. If $|v_1 v_2^{-1}|$, the word length of $v_1 v_2^{-1}$, is less than $2i$,

$$d_{\mathbb{H}^3}(\tau(v_1), \tau(v_2)) \geq 0 \geq \frac{1}{2i} d_L(v_1, v_2) - 1.$$

If $|v_1 v_2^{-1}|$ is greater than $2i$ by the two properties of $P_{j,i}$,

$$d_{\mathbb{H}^3}(\tau(v_1), \tau(v_2)) \geq (\frac{1}{2i} d_L(v_1, v_2) - 1) * c.$$

By Lemma II.8, this proves the claim. $\square$

2.2.0.1 Cannon-Thurston maps

Let X and Y be Gromov hyperbolic metric spaces and $i : X \to Y$ an embedding. A *Cannon-Thurston* map for i is a continuous extension $\hat{i} : \hat{X} = X \cup \partial X \to \hat{Y} = Y \cup \partial Y$ where ∂X and ∂Y is the geodesic boundary of X and Y, respectively. By continuity, if $\hat{i}$ exists, it is unique. If i is a quasi-isometric embedding, then the existence of $\hat{i}$ is immediate since two geodesics that are within a bounded distance of each other in X map to two quasi-geodesics that are within a bounded distance of each other in Y. Cannon and Thurston ([19]) showed the existence of such maps when Y is the Cayley graph of the fundamental group a closed hyperbolic 3-manifold fibering over the circle, and X is the subgraph associated to the fiber subgroup.

Suppose ρ is a discrete and faithful representation of G into $\mathrm{PSL}(2, \mathbb{C})$ and $\tau_\rho : C_S(G) \to \mathbb{H}^3$ is an orbit map (see Section II for more details on this map). The existence of Cannon-Thurston maps for τ_ρ and the characterization of points that are not mapped injectively by such maps is a well-studied problem. The results we will use in this paper are due to Floyd ([21]) for the case when ρ is geometrically finite, and Mj ([47]) for the case when ρ is geometrically infinite without parabolics; in fact Mj's result is more general.

Theorem II.12 (Floyd)**.** *Let ρ be a geometrically finite representation of G. Then, $\tau_\rho : C_S(G) \to \mathbb{H}^3$ extends continuously to $\bar{\tau}_\rho : \overline{C_S(G)} \to \overline{\mathbb{H}^3}$. Moreover, $\bar{\tau}_\rho$ is $2:1$ onto parabolic points of rank one and injective elsewhere.*

For Mj's characterization of point preimages to make sense, we will need a way to identify endpoints of leaves of ending laminations with points in $\partial C_S(G)$. Suppose that ρ is a purely hyperbolic geometrically infinite representation of G into $\mathrm{PSL}(2, \mathbb{C})$. Let E be a geometrically infinite end of N_ρ. Recall from Section 2.1.7 that we can pick a standard compact core C of N_ρ such that the component of $N_\rho - C$ corresponding to E is homeomorphic to $T \times (0, \infty)$ where T is a boundary component of M. Morever, there is a well-defined ending lamination λ on T. Then, λ is doubly incompressible ([11], Corollary 10.2) if N_ρ has compressible boundary and λ is filling if N_ρ has incompressible boundary. In either case, if ρ' is any convex cocompact representation such that N'_ρ is homeomorphic to N_ρ, then λ is realizable by a pleated surface, $p' : T \to N_{\rho'}$ homotopic to the inclusion of

T in $N_{\rho'}$. If l is a leaf of λ, then $p'(l)$ is a geodesic and its lift in $\mathbb{H}^3$ has two well-defined endpoints in $\Lambda(\rho'(G))$. Since ρ' is convex cocompact, τ'_ρ is a quasi-isometric embedding. Hence it has a continuous extension $\overline{\tau'_\rho}$ that restricts to a homeomorphism from $\partial C_S(G)$ to $\Lambda(\rho'(G))$. Under this homeomorphism, the endpoints of l can be identified with two distinct points in $\partial C_S(G)$.

This identification is independent of the choice of pleated surface, since the image of the pleating locus is independent of the choice of pleated surface. This identification is also independent of the choice of ρ' for if ρ'' is another choice of convex cocompact representation with $N_{\rho''}$ homeomorphic to $N_{\rho'}$, then $\partial\tau_{\rho''} \circ \partial\tau_\rho{}'|^{-1}_{\Lambda(\rho'(G)}$ is a homeomorphism from $\Lambda(\rho'(G))$ to $\Lambda(\rho''(G))$ that sends the attracting fixed point an element $\rho'(g)$ to the attracting fixed point of $\rho''(g)$. If x is an endpoint of a leaf of $p'(l)$, we can find $x_i \to x$ such that x_i is the attracting fixed point of $\rho'(g_i)$. Then, the attracting fixed points of $\rho''(g_i)$ approach an endpoint of $p''(l)$. We are now ready to state Mj's result.

Theorem II.13 (Mj)**.** *Let ρ be a purely hyperbolic representation with one geometrically infinite end. Then, $\tau_\rho : C_S(G) \to \mathbb{H}^3$ extends continuously to $\bar{\tau}_\rho : \overline{C_S(G)} \to \overline{\mathbb{H}^3}$. Moreover, $\bar{\tau}_\rho(a) = \bar{\tau}_\rho(b)$ for a and b in $\partial C_S(G)$ if and only if a and b are end-points of a leaf of an ending lamination or boundary points of a complementary ideal polygon.*

CHAPTER III

Q-stable representations

Let $G = \pi_1(M)$ be a torsion-free, Kleinian group that contains no non-cyclic abelian subgroups. We will call a property Q of elements G *prevalent* if it is invariant under automorphisms of G and if there exists a set of generators $X = \{x_1, \ldots, x_n\}$ where x_i and $x_i x_j$, for all $i \neq j$ have property Q. Let H_Q be the subset of elements in G satisfying Q; H_Q will in general not be a subgroup of G.

In this chapter we will show that if Q is prevalent, the set of Q-stable representations is a domain of discontinuity for $\mathrm{Out}(G)$. Roughly speaking, a Q-stable representation will be one that sends geodesics corresponding to elements in H_Q in the Cayley graph to uniform quasi-geodesics in $\mathbb{H}^3$.

When G is the fundamental group of a twisted I-bundle, we will take Q to be primitivity; an element is primitive if it corresponds to a simple closed curve on the base surface. When G is the fundamental group of a compression body, we will take Q to be separability; if G is not $\pi_1(S) * \pi_1(T)$, where S and T are closed surfaces, an element is separable if it lies in a proper factor of a free decomposition of G (see Chapter V for the definition of separability in the remaining case).

The notion of a Q-stable representation is a generalization of the notion of primitive-stable representations of a free group introduced by Minsky in [46]. A primitive element of a free group F_n is an element that is part of a free generating set. A primitive-stable representation is one that takes axes of primitive elements in the Cayley graph of F_n to uniform quasi-geodesics in $\mathbb{H}^3$ under an orbit map.

For each g in G, let g_- and g_+ denote the repelling and attracting fixed points of g

acting on $\partial C_S(G)$. Let $L_S(g)$ denote the set of geodesics connecting g_- and g_+ and $\mathcal{Q}_S$ denote the set of geodesics l in $C_S(G)$ such that l is contained in $L_S(g)$ for some element g in H_Q.

Given a representation $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ and a basepoint x in $\mathbb{H}^3$, let $\tau_{\rho,x}$ be the orbit map for $\rho(G)$ acting on $\mathbb{H}^3$ sending the identity to x (see Section 2.2).

Definition III.1. An irreducible representation $\rho : G \to \mathrm{PSL}_2(\mathbb{C})$ is called (K, A)-Q-*stable* if there exists a basepoint x in $\mathbb{H}^3$ such that $\tau_{\rho,x}$ takes all geodesics in $\mathcal{Q}_S$ to (K, A)-quasi-geodesics.

Recall that we assume that $\tau_{\rho,x}(l)$ is parametrized by the arclength paramerization in $C_S(G)$ composed with $\tau_{\rho,x}$ for any l in $\mathcal{Q}_S$.

We will say a representation ρ is Q-stable, if there exists (K, A) such that ρ is (K, A)-Q-stable. The following lemma shows that Q-stability does not depend on the choice of basepoint x or generating set S, although the constants may change.

Lemma III.2. *The condition Q-stability is independent of the choice of basepoint in $\mathbb{H}^3$ and the choice of generators S. It is also well-defined on $\mathcal{X}(M)$.*

Proof. To see that Q-stability is independent of the choice of basepoint, observe that if $\tau_{\rho,x}$ sends all geodesics in $\mathcal{Q}_S$ to (K, A)-quasi-geodesics, then $\tau_{\rho,y}$ sends all geodesics in $\mathcal{Q}_S$ to $(K, A + 2d(x, y))$-quasi-geodesics.

To see that Q-stability is independent of the choice of generators, suppose that S' is another finite symmetric generating set. The map $\varphi : C_{S'}(G) \to C_S(G)$ that is the identity on vertices and sends an edge in $C_{S'}(G)$ to a geodesic connecting the corresponding vertices is a (K', A')-quasi-isometry for some (K', A'). Suppose γ is a geodesic in $C_{S'}(G)$ in $L_{S'}(g)$ for some element g in H_Q and let $\tau'_{\rho,x}$ be the corresponding orbit map. By Lemma II.9, $\tau_{\rho,x}(\varphi(\gamma))$ is a (K'', A'')-quasi-geodesic. If v and w are vertices on γ, then

$$\begin{aligned} d(\tau'_{\rho,x}(v), \tau'_{\rho,x}(w)) &\geq \frac{d_{\varphi(\gamma)}(v, w)}{K''} - A'' \\ &\geq \frac{d_\gamma(v, w) - A'}{K'K''} - A''. \end{aligned}$$

By Lemma II.8, $\tau'_{\rho,x}(\gamma)$ is a (K''', A''')-quasi-geodesic, where K''' and A''' are independent of γ.

To see that Q-stability is well-defined on $\mathcal{X}(M)$ recall that two irreducible representations lie in the same fiber of $\mathrm{Hom}(\pi_1(M), \mathrm{PSL}(2,\mathbb{C})) \to \mathcal{X}(M)$ if and only if they differ by conjugation (see section 2.1.8). Clearly, (K, A)-Q-stability is preserved under conjugation. $\square$

In light of Lemma III.2, from now on we will drop the subscripts on $L_S(g)$ and $\mathcal{Q}_S$ and we will let $\mathcal{QS}(M)$ denote the set of Q-stable representations in $\mathcal{X}(M)$. The goal of this section is to show the following.

Proposition III.3. *Let $G = \pi_1(M)$ be a torsion-free Kleinian group containing no non-cyclic abelian subgroups. Let Q be a prevalent property of elements in G. Then the set of Q-stable representations in $\mathcal{X}(M)$ is a domain of discontinuity for the action of* $\mathrm{Out}(G)$.

The proof of Proposition III.3 follows an argument given by Minsky in [46] for the case that G is a free group and an element is in H_Q if it lies in a free generating set for G. We want to show that $\mathcal{QS}(M)$ is open and $\mathrm{Out}(\pi_1(M))$-invariant. Openness will follow from the fact that quasi-geodesics remain quasi-geodesics under small perturbations. $\mathrm{Out}(\pi_1(M))$-invariance will follow from the fact that automorphisms of $\pi_1(M)$ preserve the set H_Q.

Lemma III.4. (a) *The set $\mathcal{QS}(M)$ is an open,* $\mathrm{Out}(\pi_1(M))$*-invariant subset of $\mathcal{X}(M)$ containing the interior of $AH(M)$.*

(b) *Moreover, given any $[\rho_0]$ in $\mathcal{QS}(M)$ there exist constants (K_0, A_0) and a neighborhood $U_{[\rho_0]}$ of $[\rho_0]$ such that any element $[\sigma]$ in $U_{[\rho_0]}$ is (K_0, A_0)-Q-stable.*

Proof. We start by showing that $\mathcal{QS}(M)$ is open. By Lemma II.10, to show that $\mathcal{QS}(M)$ is open and to show statement (b), it suffices to show the following:

Lemma III.5. *For any Q-stable representation $[\rho_0]$ in $\mathcal{X}(M)$, there exists a neighborhood $U_{[\rho_0]}$ of $[\rho_0]$ and constants $c' > 0$, $i' \in \mathbb{N}$ such that for any $[\sigma]$ in $U_{[\rho_0]}$ and any geodesic l in $\mathcal{Q}_S$, the hyperplanes, $P_{j,i'}$ corresponding to $\tau_{\sigma,x}(l)$ have the property that $P_{j,i'}$ separates $P_{(j+1),i'}$ and $P_{(j-1),i'}$ and $d(P_{j,i'}, P_{(j+1),i'}) > c'$.*

Proof. Suppose that ρ_0 is (K_0, A_0)-Q-stable. Let i_0 and c_0 be the constants obtained from Lemma II.10. Let $i' = i_0$ and choose $c' < c_0$. Let L be a geodesic in $\mathcal{Q}$.

For any representation $\rho : G \to \mathrm{PSL}(2, \mathbb{C})$, let $v_k(\rho, L)$ denote the image under $\tau_{\rho,x}$ of the k^{th} vertex of L. Since the map $\phi_g : \mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C})) \to \mathbb{H}^3$ that sends $\rho \mapsto \rho(g) \cdot x$ is continuous, for each j, $v_{ji_0}(\rho, L)$ and $v_{(j+1)i_0}(\rho, L)$ vary continuously with respect to ρ. In particular, if ρ_i converges to ρ, then $[v_{ji_0}(\rho_i, L), v_{(j+1)i_0}(\rho_i, L)]$ converges pointwise to $[v_{ji_0}(\rho, L), v_{(j+1)i_0}(\rho, L)]$. If m_i is the midpoint of $[v_{ji_0}(\rho_i, L), v_{(j+1)i_0}(\rho_i, L)]$, then it converges to m the midpoint of $[v_{ji_0}(\rho, L), v_{(j+1)i_0}(\rho, L)]$. It also implies that if w_i is the vector tangent to $[v_{ji_0}(\rho_i, L), v_{(j+1)i_0}(\rho_i, L)]$ at m_i pointing towards $v_{(j+1)i_0}(\rho_i, L)$, then w_i converges to w, the vector tangent to $[v_{ji_0}(\rho, L), v_{(j+1)i_0}(\rho, L)]$ at m in the direction of $v_{(j+1)i_0}(\rho, L)$. Therefore, the point in the unit tangent bundle determining P_{j,i_0} varies continuously over $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$.

As G acts transitively and isometrically on $C_S(G)$, it suffices to check that P_0 separates P_{-i_0} and P_{i_0} and that $d(P_0, P_{i_0}) > c'$ for each triple P_{-i_0}, P_0, and P_{i_0} corresponding to each geodesic L in $\mathcal{Q}$ such that v_0 is the identity. In particular we are only concerned with what happens to a finite number of group elements. Then, there exists an open neighborhood U_{ρ_0} of ρ_0 in $\mathrm{Hom}(G, \mathrm{PSL}(2, \mathbb{C}))$ such that P_{j,i_0} separates $P_{(j+1),i_0}$ and $P_{(j-1),i_0}$ and $d(P_{j,i_0}, P_{(j+1),i_0}) > c'$. The image of U_{ρ_0} in $\mathcal{X}(M)$ is the desired neighborhood of $[\rho_0]$. $\square$

To see that $\mathcal{QS}(M)$ is $\mathrm{Out}(\pi_1(M))$-invariant, recall the assumption that any automorphism f of $\pi_1(M)$ preserves H_Q. Then, the isometry from $C_{f(S)}(G) \to C_S(G)$ that is the identity map on vertices will send the elements of $\mathcal{Q}_{f(S)}$ to $\mathcal{Q}_S$. Since the image of $\tau_{\rho \circ f^{-1},x} : C_{f(S)}(G) \to \mathbb{H}^3$ coincides with the image of $\tau_{\rho,x} : C_S(G) \to \mathbb{H}^3$, if ρ sends elements in $\mathcal{Q}_S$ to (K, A)-quasi-geodesics, then $\tau_{\rho \circ f^{-1},x}$ sends elements in $\mathcal{Q}_{f(S)}$ to (K, A)-quasi-geodesics. Since Q-stability is independent of the choice of generators of G, $\rho \circ f^{-1}$ will also be Q-stable.

By Theorem II.4, for any ρ in the interior, $\tau_{\rho,x}$ is a quasi-isometry into a convex subset of $\mathbb{H}^3$. In particular, $\tau_{\rho,x}$ is a quasi-isometric embedding into $\mathbb{H}^3$ and hence ρ lies in $\mathcal{QS}(M)$. $\square$

We finish this section by showing that the action of $\mathrm{Out}(\pi_1(M))$ on $\mathcal{QS}(M)$ is properly discontinuous. The idea is that Q-stability will imply that translation length of an element in H_Q in the Cayley graph is coarsely the same as translation length of the corresponding isometry in $\mathbb{H}^3$. To show proper discontinuity of the action it will suffice to show that only finitely many automorphisms, up to conjugation, can change the translation length of all elements in H_Q in the Cayley graph by a bounded amount.

Proposition III.6. $\mathrm{Out}(\pi_1(M))$ *acts properly discontinuously on* $\mathcal{QS}(M)$.

Proof. Let $l_\rho(g)$ denote the translation length of $\rho(g)$ in $\mathbb{H}^3$, and let $||g||$ denote the minimum translation length of g in $C_S(G)$. The following lemma shows that for each compact set in $\mathcal{QS}(M)$, the translation length of a Q element in $\mathbb{H}^3$ and in $C_S(G)$ is coarsely the same.

Lemma III.7. *Let C be a compact subset of $\mathcal{QS}(M)$. There exists $r, R > 0$ such that*

$$r \leq \frac{l_\rho(g)}{||g||} \leq R$$

for all g in H_Q and all representations $[\rho]$ in C.

Proof. By Remark II.7, for each $[\rho]$ in C, there exists R_ρ such that $l_\rho(g) \leq R_\rho ||g||$ for any element g. Moreover, R_ρ can be chosen to depend continuously on ρ. Let R be the maximum of R_ρ over C.

To see the lower bound we can first assume by compactness and Lemma III.5 that there exists (K, A) such that every element $[\rho]$ in C is (K, A)-Q-stable. Recall, by Lemma II.6, that there exists (K', A') such that for any element g, there exists a (K', A')-quasi-axis l'_g. Using Lemma II.9, $\tau_{\rho,x}(l'_g)$ is a (K'', A'')-quasi-geodesic in $\mathbb{H}^3$, where K'' and A'' are independent of g. In particular, $\tau_{\rho,x}(l'_g)$ lies in a $R'' = R''(K'', A'')$ neighborhood of the axis for $\rho(g)$. Hence if we take y on $\tau_{\rho,x}(l'_g)$, we have

$$l_\rho(g) \geq d(y, \rho(g) \cdot y) - 2R'' \geq \frac{||g||}{K''} - A'' - 2R''.$$

Therefore,

$$\frac{l_\rho(g)}{||g||} \geq \frac{1}{K''} - \frac{A''}{||g||} - \frac{2R''}{||g||}.$$

For $||g||$ larger than $2(A'' - 2R'')K''$,

$$\frac{1}{K''} - \frac{A''}{||g||} - \frac{2R''}{||g||} > \frac{1}{2K''} > 0.$$

Let $[g_1], \ldots, [g_m]$ be the conjugacy classes of elements of G such that $||g_i||$ is less than $2(A'' - 2R'')K''$. Since $\frac{l_\rho(g_i)}{||g_i||}$ varies continuously over $\mathcal{X}(M)$ for each g_i, there exists a minimum value r_i for $\frac{l_\rho(g_i)}{||g_i||}$ over C. Take r to be the minimum of $\{r_1, \ldots, r_m, \frac{1}{2K''}\}$. $\square$

Now suppose that $[f]$ is an element in $\mathrm{Out}(G)$ such that $f(C) \cap C \neq \emptyset$. Applying the above inequalities we have that for any $[\rho]$ in C and any w in H_Q,

$$||f^{-1}(w)|| \leq \frac{1}{r} l_\rho(f^{-1}(w)) = \frac{1}{r} l_{\rho \circ f^{-1}}(w) \leq \frac{R}{r}||w||.$$

Recall that, by assumption, there exists $X = \{x_1, \ldots, x_m\}$ a set of generators for G such that each x_i is in H_Q and $x_i x_j$ for $i \neq j$ is also in H_Q. Let $\mathcal{W} = \{x_i, x_i x_j | i \neq j\}$. It suffices to show the following lemma.

Lemma III.8. *For any $N > 0$, the set*

$$\mathcal{A} = \{[f] \in \mathrm{Out}(G) |\ ||f(w)|| \leq N||w||| \textit{ for all } w \in \mathcal{W}\}$$

is finite.

Proof. Suppose that $\{[f_k]\}$ is a sequence of infinitely many distinct elements in $\mathcal{A}$. Fix a convex cocompact representation ρ of G. Then $\tau_{\rho,x} : C_S(G) \to CH(\Lambda(\rho(G)))$ is a quasi-isometry. Fix the following notation: Let $\overline{g}$ denote the isometry of $\mathbb{H}^3$ induced by the action of g, $l(\overline{g})$ its translation length and $\mathrm{Ax}(\overline{g})$ its invariant geodesic axis. As G acts cocompactly on $CH(\Lambda(\rho(G)))$, there exists $r > 0$ such that $l(\overline{f_k(x_i)}) \geq r$ on $\mathbb{H}^3$ for all i and k. Since $||f_k(x_i x_j)|| \leq 2N$, there exists R such that $l(\overline{f_k(x_i x_j)}) \leq R$ all k and all pairs i, j such that $i \neq j$.

This implies that there exists an upper bound D on the distance between $\mathrm{Ax}(\overline{f_k(x_i)})$ and $\mathrm{Ax}(\overline{f_k(x_j)})$ for all k and all pairs i, j. Indeed suppose that $\{d(\mathrm{Ax}(\overline{f_k(x_i)}), \mathrm{Ax}(\overline{f_k(x_j)})\}$ was unbounded. Since $l(\overline{f_k(x_i)})$ and $l(\overline{f_k(x_j)})$ are bounded from below by r, $\{l(\overline{f_k(x_i)f_k(x_j)})\}$ would also be unbounded, a contradiction.

Then, there also exists an upper bound D' on the distance between $\mathrm{Ax}(f_k(x_i))$, a quasi-axis of $f_k(x_i)$ in $C_S(G)$, and $\mathrm{Ax}(f_k(x_j))$, a quasi-axis of $f_k(x_j)$, for all i, j and k. Up to conjugation, we can then assume that $\mathrm{Ax}(f_k(x_i))$ is a uniformly bounded distance D'' from the identity e for all k and i. If y is a point on $\mathrm{Ax}(f_k(x_i))$ closest to e, then we can bound the distance between the identity and $f_k(x_i)$ in the Cayley graph as follows.

$$\begin{aligned} d(e, f_k(x_i)) &\leq D'' + d(y, f_k(x_i)y) + D'' \\ &\leq 2D'' + ||f_k(x_i)|| \\ &\leq 2D'' + N \end{aligned}$$

This implies that up to conjugation, there are only finitely many possibilities for each $f_k(x_i)$. Hence $\mathcal{A}$ must be finite. □

□

Remark III.9. *If H_Q equals G, then $QS(M)$ is the interior of $AH(M)$. This provides an alternate proof that $\mathrm{Out}(\pi_1(M))$ acts properly discontinuously on the interior of $AH(M)$, when M has no toroidal boundary components.*

We end this section with a useful geometric characterization of discrete and faithful Q-stable representations.

Lemma III.10. *Let ρ be a discrete and faithful representation of $\pi_1(M)$ into $\mathrm{PSL}(2, \mathbb{C})$. Then ρ is Q-stable if and only if*

(a) *$\rho(g)$ is hyperbolic for any Q-element g and*

(b) *there exists a compact subset Ω of $N_\rho = \mathbb{H}^3/\rho(\pi_1(M))$ such that the set of geodesics corresponding to Q-elements of $\pi_1(M)$ is contained in Ω.*

Proof. For the forward direction, suppose that ρ is (K, A)-Q-stable. Then, $\rho(g)$ must be hyperbolic for all Q-elements g for if $\rho(g)$ were parabolic, then for any geodesic l connecting the fixed points g_+ and g_- on $\partial C_S(G)$, $\tau_{\rho,x}(l)$ would not be a quasi-geodesic. Moreover, elements of $\mathcal{Q}$ stay within a bounded neighborhood of their corresponding geodesic axes in $\mathbb{H}^3$. In particular, geodesics representing Q elements will stay in a bounded neighborhood of the image of the Cayley graph in N_ρ, which is a compact set.

Conversely, suppose that $\rho(g)$ is hyperbolic for all g in H_Q and that there exists a compact set Ω such that all Q-geodesics of N_ρ are contained in Ω. Without loss of generality, since N_ρ is tame (by [1] or [10]), we can assume that Ω is a compact core C of N_ρ containing the image of $C_S(G)/\rho(G)$ in N_ρ. This implies that $\tilde{\Omega}$, the preimage of Ω in $\mathbb{H}^3$, is connected. For some (K, A), $\tau_{\rho,x} : C_S(G) \to \tilde{\Omega} \subset \mathbb{H}^3$ is a (K, A)-quasi-isometry from $C_S(G)$ to $\tilde{\Omega}$ with the intrinsic metric. In particular, any geodesic l in $\mathcal{P}$ connecting g_- and g_+, the fixed points of g, maps to a (K, A)-quasi-geodesic in $\tilde{\Omega}$, with the intrinsic metric. Then, $\tau_{\rho,x}(l)$ lies in a $R = R_\Omega(K, A)$-neighborhood of $\mathrm{Ax}(g)$, a lift of the geodesic representing g in N_ρ in the intrinsic metric and also with the extrinsic metric. If x, y lie on $\tau_{\rho,x}(l)$ and if π denotes the closest point projection onto $\mathrm{Ax}(g)$ in $\tilde{\Omega}$, then

$$d_{\tilde{\Omega}}(x, y) \leq d_{\tilde{\Omega}}(\pi(x), \pi(y)) + 2R = d_{\mathbb{H}^3}(\pi(x), \pi(y)) + 2R \leq d_{\mathbb{H}^3}(x, y) + 4R$$

This implies that $\tau_{\rho,x}(l)$ is a $(K, A + 4R)$-quasi-geodesic in $\tilde{\Omega}$ with the extrinsic metric. Hence ρ is $(K, A + 4R)$-Q-stable. $\square$

CHAPTER IV

Twisted I-bundle case

In this chapter we study the case when M is a twisted I-bundle. Following the framework in Chapter III, we take the set H_Q to be the set of simple closed curves on the base surface, which we call the set of primitive elements. It easily follows from Proposition III.3 that the set of primitive-stable representations is a domain of discontinuity for the action. Most of the chapter is dedicated to characterizing which points in $AH(M)$ are primitive-stable. We show that a discrete and faithful representation is primitive-stable if and only if no simple closed curve on the base surface is parabolic. This characterization allows us to complete the proof of Theorem I.1. Moreover, combining this characterization with an observation of Minsky, we conclude that the set of primitive-stable representations is maximal with respect to the discrete and faithful representations.

Let B be a nonorientable hyperbolic surface and let $\widetilde{B}$ be its orientable double cover. Let M be a twisted I-bundle over B, namely

$$M = \widetilde{B} \times I/(x,t) \sim (\theta(x), 1-t),$$

where θ is an orientation-reversing, fixed-point free involution of $\widetilde{B}$ such that $\widetilde{B}/\langle\theta\rangle$ is homeomorphic to B.

4.1 Primitive-stable representations

Definition IV.1. We say an element g in $\pi_1(M)$ is *primitive* if it can be represented by a simple closed curve on the base surface B.

Let $\mathcal{PS}(M)$ denote the set of primitive-stable representations in $\mathcal{X}(M)$.

Proposition IV.2. *$\mathcal{PS}(M)$ is a domain of discontinuity for the action of* $\mathrm{Out}(\pi_1(M))$.

Proof. By Proposition III.3, it suffices to show that the set of primitive elements is $\mathrm{Out}(\pi_1(M))$-invariant and that we can find a set of generators for $\pi_1(M)$ such that each generator and each two-fold product of distinct generators is primitive. Since $\pi_1(M)$ is isomorphic to $\pi_1(B)$ and homotopy equivalences of closed surfaces are homotopic to homeomorphisms, the set of primitive elements is $\mathrm{Out}(\pi_1(M))$-invariant.

If the Euler characteristic of B is $2-k<0$, then we can realize B as a $2k$-sided polygon with k pairs of adjacent sides identified. Using the sides of the polygon, $\pi_1(B) \cong \pi_1(M)$ has the following presentation $< a_1, \dots, a_k | \Pi a_i^2 >$. With this presentation, each generator and each two-fold product of distinct generators corresponds to a simple closed curve on B. □

4.2 Primitive-stable points on the boundary of $AH(M)$

In this section we prove Theorem I.2, which combined with Proposition IV.2 completes the proof of Theorem I.1. We break up the proof into Propositions IV.3 and IV.5. We start by characterizing which points in $AH(M)$ lie in $\mathcal{PS}(M)$. We will show that $[\rho]$ in $AH(M)$ lies in $\mathcal{PS}(M)$ if and only if $\rho(g)$ is hyperbolic for all primitive elements g of G. In particular, $\mathcal{PS}(M)$ will contain the interior of $AH(M)$ as well as all purely hyperbolic points on the boundary of $AH(M)$. To complete the proof of Theorem I.2, we will use an observation by Minsky that for a general hyperbolizable 3-manifold M' if $[\sigma]$ in $AH(M')$ maps a core curve of an essential annulus to a parabolic element of $\mathrm{PSL}(2,\mathbb{C})$, then $[\sigma]$ cannot lie in a domain of discontinuity of the action of $\mathrm{Out}(\pi_1(M'))$ on $\mathcal{X}(M')$. Finally, we conclude with the result that no point on the boundary of quasi-Fuchsian space can lie in a domain of discontinuity.

For the reader's convenience we restate Theorem I.2.

Theorem I.2. *Let $[\rho]$ be an element of $AH(M)$. Then $[\rho]$ does not lie in $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic. Moreover,* *if ρ lies in $AH(M) - \mathcal{PS}(M)$, then ρ does not lie in any domain of discontinuity for the action of $Out(\pi_1(M))$ on $\mathcal{X}(M)$.*

We will start by proving the first assertion of Theorem I.2.

Proposition IV.3. *Let $[\rho]$ be an element of $AH(M)$. Then $[\rho]$ does not lie in $\mathcal{PS}(M)$ if and only if there exists a primitive element g of $\pi_1(M)$ such that $\rho(g)$ is parabolic.*

Proof. The backward direction follows from Lemma III.10.

For the forward direction, if $\rho(g)$ is hyperbolic for every primitive g, then, by Lemma III.10, it suffices to check that closed geodesics corresponding to primitive elements remain in a compact set. Let γ_g denote the unique geodesic representative of $\rho(g)$ in $N = N_\rho$. The representation ρ induces a homotopy equivalence $h_\rho : M \to N$. Precompose with the inclusion $B \to M$ to obtain an incompressible map $h'_\rho : B \to N$. Let α_g be a simple closed curve on B such that $h'_\rho(\alpha_g)$ is freely homotopic to γ_g.

Then, the map h'_ρ is homotopic to a simplicial hyperbolic surface h_g such that α_g maps to γ_g (see Section 2.1.6). We claim that there exists an ϵ_0 small enough such that $h_g(B)$ is contained in $N^0_{\epsilon_0}$ for any primitive element g. First, observe that there exists a constant A depending only on the Euler characteristic of B such that for any point x in B there exists a homotopically nontrivial simple curve through x of length less than A. To produce such a curve take a ball centered at x and blow it up until it intersects itself. Since the area of B is bounded, there is a uniform upper bound on the area and radius of such a ball.

For any $L > A$ and any ϵ less than μ_3, if $h(B) \cap (N - \mathcal{N}_L(N^0_\epsilon)) \neq \emptyset$, then there exists a homotopically nontrivial simple curve on $h(B)$ entirely contained within a noncompact component of $N_{\text{thin}(\epsilon)}$, where $\mathcal{N}_L(N^0_\epsilon)$ denotes the L neighborhood of N^0_ϵ. This implies that the curve represents a parabolic element, which is a contradiction. Fix $\epsilon < \mu_3$ and $L > A$ and choose ϵ_0 small enough such that $\mathcal{N}_L(N^0_\epsilon)$ is contained in $N^0_{\epsilon_0}$.

Now suppose that $\{\gamma_i\}$ is a sequence of primitive geodesics not contained in any compact set. Let $h_i : B \to N$ denote a simplicial hyperbolic surface containing γ_i, homotopic to h'_ρ. We can lift h_i to a map $\widetilde{h}_i : S \to \widetilde{N}$ where $S = \partial M$ and $\widetilde{N}$ is the double cover of N associated to the subgroup $\pi_1(S)$. The map $\widetilde{h}_i$ is a simplicial hyperbolic surface containing γ_i and $\widetilde{\theta}(\gamma_i)$, where the associated triangulation is the preimage of the triangulation on B.

Moreover, by construction, $\widetilde{h}_i$ satisfies $\widetilde{\theta} \circ \widetilde{h}_i = \widetilde{h}_i \circ \theta$, where $\widetilde{\theta}$ is the nontrivial covering transformation of $\widetilde{N}$.

Fix C a compact core for N. The preimage $\widetilde{C}$ of C in $\widetilde{N}$ is a compact core for $\widetilde{N}$. As $\widetilde{C}$ is homotopy equivalent to a fiber surface S which separates $\widetilde{N} \cong S \times \mathbb{R}$, $\widetilde{N} - \widetilde{C}$ has two components. Since $\widetilde{C}$ covers C and C has only one boundary component, $\widetilde{\theta}$ must exchange the two boundary components of $\widetilde{C}$, and hence $\widetilde{\theta}$ must exchange the two components of $\widetilde{N} - \widetilde{C}$.

As $\{\gamma_i\}$ is not contained in any compact set, we can assume, up to passing to a subsequence, that there exists a point x_i on γ_i such that x_i lies outside the compact set C_i where C_i is defined as

$$C_i = \{x \in N^0_{\epsilon_0} \mid \text{there exists a path } c \text{ from } x \text{ to } C \text{ such that } l(c \cap N_{thick(\epsilon_0)}) \leq i\}.$$

As $\widetilde{\theta}$ interchanges the two components of $\widetilde{N} - \widetilde{C}$, the two lifts $\widetilde{x}_i, \widetilde{x}'_i$ of x_i lie in different components of $\widetilde{N} - \widetilde{C}$, but by the equivariance property of $\widetilde{h}_i$, they both lie on $\widetilde{h}_i(S)$. Any path c on $\widetilde{h}_i(S)$ connecting $\widetilde{x}_i$ and $\widetilde{x}'_i$ satisfies $l(c \cap \widetilde{N}_{thick(\epsilon_0)}) \geq 2i$, for if not, in N there would be a path c' connecting x_i to C with $l(c' \cap N_{thick(\epsilon_0)}) < i$. For i large enough, this contradicts Lemma II.3. □

This completes the proof of the first assertion of Theorem I.2. This also completes the proof of Theorem I.1; for example, geometrically infinite purely hyperbolic points will lie on $\partial AH(M)$ and in $\mathcal{PS}(M)$. To prove the second assertion we will need the following observation due to Minsky.

Lemma IV.4 (Minsky)**.** *Let M be a compact hyperbolizable manifold with no toroidal boundary components. Let γ be the core curve of an essential annulus in M. Suppose that $\rho : \pi_1(M) \to \mathrm{PSL}_2(\mathbb{C})$ is a discrete and faithful representation such that $\rho(\gamma)$ is parabolic. Then, any neighborhood of $[\rho_0]$ contains points with infinite stabilizers. In particular, $[\rho]$ cannot lie in a domain of discontinuity for the action of $Out(\pi_1(M))$ on $\mathcal{X}(M)$.*

Proof. Consider the map

$$\mathrm{tr}^2_\gamma : \mathcal{X}(M) \to \mathbb{C}$$

where $[\rho] \mapsto \mathrm{tr}([\rho(\gamma)])^2$. As a neighborhood of $AH(M)$ is a smooth complex manifold, on which tr^2_γ is a holomorphic map, tr^2 is either constant or open on each connected component of that neighborhood. Since the interior of $AH(M)$ is dense in $AH(M)$ (see [7] and [48]) and since the interior consists of convex cocompact representations by Theorem II.4, the image of tr^2_γ cannot be constant on any component of $AH(M)$. Therefore, up to passing to a smaller neighborhood, tr^2 must be an open map on that neighborhood of $AH(M)$. Since isometries of $\mathbb{H}^3$ are determined, up to conjugacy, by their trace and there are finite order elliptic isometries with trace arbitrarily close to 2 or -2, there exist representations ρ_i approaching ρ such that $\rho_i(\gamma)$ corresponds to a finite order elliptic isometry. Let n_i denote the order of $\rho_i(\gamma)$. Then $D^{n_i}_\gamma$ the Dehn twist of order n_i about the annulus whose core curve is γ is an element in $\mathrm{Out}(\pi_1(M))$ that fixes $[\rho_i]$. Hence, elements arbitrarily close to $[\rho]$ have infinite stabilizers. $\square$

Proposition IV.5. *If ρ lies in $AH(M) - \mathcal{PS}(M)$, then ρ does not lie in any domain of discontinuity for the action of* $\mathrm{Out}(\pi_1(M))$ *on $\mathcal{X}(M)$.*

Proof. If $[\rho]$ lies in the complement of $\mathcal{PS}(M)$ in $AH(M)$, then there exists a primitive element g such that $\rho(g)$ is parabolic. Then, g is either the core curve of an essential annulus or the core curve of an essential Mobius band. By Lemma IV.4 it suffices to show that the latter is impossible. Suppose that γ is a closed essential curve in M mapping to a parabolic element in $N_\rho = \mathbb{H}^3/\rho(G)$. Then we claim that γ must be homotopic into ∂M. If C is a relative compact core for N_ρ, consider the map $\phi : S = \partial M \to C$ in the homotopy class of $\rho|_{\pi_1(\partial M)}$. If $\widetilde{C}$ is the cover of C associated to the subgroup $\phi_*(\pi_1(S))$ it is a compact manifold with $\pi_1(\widetilde{C}) \cong \pi_1(S)$. This implies that $\widetilde{C}$ must be a trivial I-bundle over S ([29] Theorem 10.6). The lift $S \to \widetilde{C}$ can be homotoped into $\partial\widetilde{C}$. Hence the map $S \to C$ is also homotopic into the boundary of C. Then, we can homotope the map $M \to C$ to a map that sends ∂M into ∂C. This map is either homotopic to a homeomorphism or is homotopic to a map $M \to \partial C$ ([29] Theorem 13.6). The latter cannot happen as this would imply that ∂C is a nonorientable closed surface. Therefore, any curve in M mapping to a parabolic element in N_ρ is homotopic into ∂M. As the core curve of an essential Mobius band cannot be homotoped into ∂M, it cannot be mapped to a parabolic element in N_ρ. $\square$

This completes the proof of Theorem I.2. We end this section with an application of Minsky's observation in the quasi-Fuchsian case.

Proposition IV.6. *Let F be an orientable hyperbolic surface. Then no point on the boundary of $AH(F \times I)$ can lie in a domain of discontinuity for the action of $Out(\pi_1(F \times I))$ on $\mathcal{X}(F \times I)$.*

Proof. Since geometrically finite points are dense on the boundary of $AH(F \times I)$ (see [8] and [14]) and since any simple closed curve on F is the core curve of an essential annulus, the result follows from Lemma IV.4. $\square$

CHAPTER V

Compression body case

The main goal of this chapter is to prove Theorem I.4. After discussing some background on compression bodies in Section 5.1 we will show that the set of separable-stable representation is a domain of discontinuity by using Proposition III.3. The majority of the chapter is devoted to showing that the set of separable-stable representations contains points on the boundary of $AH(M)$. In doing this, we find two types of representations on $\partial AH(M)$ that are separable-stable; these representations correspond to manifolds homeomorphic to compression bodies. We end the chapter with a brief discussion of representations on $\partial AH(M)$ whose corresponding manifolds are not homeomorphic to compression bodies.

5.1 Compression bodies

A *compression body* is a compact, orientable, irreducible 3-manifold M with a boundary component $\partial_{ext}M$, called the *exterior boundary*, whose inclusion induces a surjection $\pi_1(\partial_{ext}M) \to \pi_1(M)$. The other boundary components are called *interior boundary* components. Equivalently, a compression body is a boundary connect sum of a 3-ball, a collection of solid tori and a collection of trivial I-bundles over closed surfaces such that the other summands are attached to the 3-ball along disjoint discs.

A compression body is *trivial* if it is a trivial I-bundle over a closed surface. A *meridian* is a simple, closed curve on $\partial_{ext}(M)$ that is nontrivial in $\partial_{ext}(M)$ but trivial in M. By Dehn's lemma (see [29], Chapter 4) every meridian bounds an embedded disc in M. A compression body is *small* if there exists an essential, properly embedded disc D such that

$M - D$ is either two trivial I-bundles over closed surfaces or one trivial I-bundle over a closed surface; otherwise M is a *large* compression body.

The fundamental group of a compression body can be expressed as $G_1 * G_2 * \cdots * G_n$ where G_i is isomorphic to a closed surface group for $1 \leq i \leq k$ and G_j is infinite cyclic $k < j \leq n$. By Grushko's theorem ([27]) and Kurosh's subgroup theorem ([36]) any other decomposition of the fundamental group into a free product, $H_1 * H_2 * \cdots H_m$, where each factor is freely indecomposable, satisfies $n = m$ and $H_i \cong G_i$, up to re-ordering. An essential separating disc D in M realizes a splitting of $\pi_1(M)$ into a free product in the following sense. If $M - \mathcal{N}(D) = M_1 \sqcup M_2$, where $\mathcal{N}(D)$ is a regular neighborhood of D, then $\pi_1(M) \cong i_*(\pi_1(M_1 \cup \mathcal{N}(D))) * i_*(\pi_1(M_2 \cup \mathcal{N}(D)))$, where i is inclusion, and the basepoint is chosen to lie in $\mathcal{N}(D)$; as the basepoint may change, this splitting is only well-defined up to conjugation. As the following lemma shows, the converse is also true.

Lemma V.1. *Let M be a compression body and $\pi_1(M) = H * K$ a nontrivial splitting of $\pi_1(M)$ into a free product. Then, there exists a properly embedded disc D realizing the splitting in the sense described above.*

Proof. As M is a boundary connect sum of a collection of solid tori, a collection of trivial I-bundles and a 3-ball such that the other summands are attached to the 3-ball along disjoint discs, the splitting corresponding to this connect sum, $G = G_1 * G_2 * \cdots * G_n$, where G_i is isomorphic to the fundamental group of a closed surface for $1 \leq i \leq k$ and G_j is infinite cyclic for $k < j \leq n$ has the following property. If σ is any permutation of $\{1, \ldots, n\}$ then the splitting $G' * G''$ where $G' = G_{\sigma(1)} * \cdots * G_{\sigma(l)}$ and $G'' = G_{\sigma(l+1)} * \cdots * G_{\sigma(n)}$ is realizable by an essential disc Δ_σ. A result of McCullough-Miller ([45] Corollary 5.3.3) shows that the image of $\mathrm{Homeo}^+(M)$, the group of orientation preserving homeomorphisms, in $\mathrm{Out}(\pi_1(M))$ has index 2^k where the cosets can be described in the following way. For each surface group factor G_i, $i \leq k$, let $f_i : G \to G$ be an automorphism such that $f_i|_{G_j} = id|_{G_j}$ for $i \neq j$ and $f_i|_{G_i}$ is an orientation-reversing automorphism; notice that f_i and f_j commute for all $i, j \leq k$. The cosets of $\mathrm{Homeo}^+(M)$ in $\mathrm{Out}(\pi_1(M))$ are $\{f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_l} \cdot \mathrm{Homeo}^+(M)\}$ for $1 \leq i_1 < i_2 < \cdots < i_l \leq k$ and $l \leq k$.

Let σ be a permutation of $\{1, \ldots, n\}$ such that if $G_H = G_{\sigma(1)} * \cdots * G_{\sigma(l)}$ and $G_K =$

$G_{\sigma(l+1)} * \cdots * G_{\sigma(n)}$, then G_H is isomorphic to H and G_K is isomorphic to K; by the uniqueness of a maximal decomposition of G into a free product, such a permutation must exist. By the discussion above, there exists a disc D' realizing the splitting $G = G_H * G_K$. We can find an automorphism $\phi : G \to G$ such that $\phi(G_H) = H$ and $\phi(G_K) = K$. Notice that the automorphisms f_i do not affect the splitting of G. By pre-composing with f_i, if necessary, we can assume that $\phi|_{G_i}$ for $i \leq k$ is orientation-preserving. Hence, ϕ is realizable by a homeomorphism f_ϕ and we can take D to be the image of $f_\phi(D')$. □

5.2 Separable-stable representations

Definition V.2. If M is a compression body that is not the connect sum of two trivial I-bundles over closed surfaces, an element g in $\pi_1(M)$ is *separable* if it corresponds to a loop in M that can be freely homotoped to miss an essential disc. If M is the connect sum of two trivial I-bundles over closed surfaces, an element g in $\pi_1(M)$ is *separable* if it corresponds to a loop in M that can be freely homotoped to miss an essential annulus contained in one of the two trivial I-bundles.

Proposition V.3. *$\mathcal{SS}(M)$ is a domain of discontinuity for the action of* $\mathrm{Out}(\pi_1(M))$.

Proof. It suffices to show that the set of separable elements is $\mathrm{Out}(\pi_1(M))$-invariant and that we can find a set of generators for $\pi_1(M)$ such that each generator and each two fold product of distinct generators is separable.

If M is a compression body that is not the connect sum of two trivial I-bundles, then an element g is separable if and only if g lies in a proper factor of a decomposition of M into a free product. Indeed, if g corresponds to a curve that is freely homotopic to a curve missing an essential separating disc corresponding to the splitting $H*K$, then g lies in either H or K, up conjugation. If g corresponds to a curve that is freely homotopic to a curve missing an essential nonseparating disc, then there is a splitting $H*_1$ such that g lies in H, up to conjugation. Since $H*_1 \cong H * \mathbb{Z}$, g lies in a proper factor of a free decomposition. The converse follows from Lemma V.1. This implies that separability is preserved under automorphisms.

For the remaining case, suppose that M is the connect sum of $S_1 \times I$ and $S_2 \times I$ where S_i

is a closed surface of genus at least two. First, we want to see that g is separable if and only if there is a decomposition of $\pi_1(M)$ as $\pi_1(M) \cong (H *_{<c>} K) * L$ or $\pi_1(M) \cong (K*_{<c>}) * L$ such that g lies in $K * L$ satisfying the following

(a) $L \cong \pi_1(S_i)$

(b) $H *_{<c>} K \cong \pi_1(S_j)$ in the first type of decomposition or $(K*_{<c>}) \cong \pi_1(S_j)$ in the second type of decomposition where $i \neq j$

(c) c is freely homotopic to a simple closed curve on S_j

If g is separable, then it misses an essential annulus A in $S_j \times I$. If c is the core curve of A, $\pi_1(S_j)$ decomposes as $H *_{<c>} K$ or $K*_{<c>}$. Then $\pi_1(M)$ decomposes as $\pi_1(M) = H *_{<c>} K * \pi_1(S_i)$ or $\pi_1(M) = \pi_1(S_i) * K*_{<c>}$ such that g lies in $K * \pi_1(S_i)$. Conversely, if g is an element in $\pi_1(M)$ that lies in $K * L$ for a decomposition of one of the above types, then g misses the essential annulus $c \times I$.

Now, it suffices to show that such a decomposition is preserved under automorphisms. If ϕ is an automorphism of $\pi_1(M)$, then $\phi((H *_{<c>} K) * L) = (\phi(H) *_{<\phi(c)>} \phi(K)) * \phi(L)$. By the Kurosh subgroup theorem ([36]), $\phi(H) *_{<\phi(c)>} \phi(K)$ is conjugate to $\pi_1(S_j)$ and $\phi(L)$ is conjugate to $\pi_1(S_i)$. Up to composition with an inner automorphism and potentially switching the factors, $\phi(\pi_1(S_j)) = \pi_1(S_j)$ and $\phi(L) = \pi_1(S_i)$ ([16] Lemma 9.1.2). Since homotopy equivalence of closed surfaces are homotopic to homeomorphisms, $\phi(c)$ is a simple closed curve on S_j.

If M is a large compression body, take a maximal decomposition of G into a free product, $G = G_1 * \cdots * G_n$. Let X be the union of finite generating sets for each factor. Since n is at least three, any two fold product of distinct generators is separable. If M is a small compression body, let X be the union of the standard generators of each closed surface group factor and a generator for the infinite cyclic factor if there is a handle. In the case that M is a connect sum of two trivial I-bundles over closed surfaces, it is clear that any two fold product of such generators is separable. In the case that M is a trivial I-bundle over a closed surface with a one handle, we only need to concern ourselves with products of the form $x_{i_1}x_{i_2}$ where x_{i_1} is part of the generating set for the closed surface group and x_{i_2}

is the generator for the infinite cyclic factor. Then, x_{i_2} misses D an essential disc. Let f be an automorphism of G that is the identity on the surface group factor and maps x_{i_2} to $x_{i_1}x_{i_2}$. By the discussion in the proof of Lemma V.1, f is realizable by a homeomorphism f'. Hence $x_{i_1}x_{i_2}$ misses the essential disc $f'(D)$. $\square$

5.3 Separable-stable points on $\partial AH(M)$

The goal of this section is to prove the existence of separable-stable points on $\partial AH(M)$ (Proposition V.11). These points will correspond to pinching either a Masur domain curve or a Masur domain lamination on the exterior boundary of M. Using Lemma III.10, it suffices to show that all separable geodesics lie in a compact set. Roughly speaking, if this were not the case, then we could find a sequence of fixed points of separable elements in $\partial C_S(G)$ approaching an endpoint of an end invariant. We use the Whitehead graph to form a dichotomy between separable elements and Masur domain laminations to show that such a situation is impossible.

5.3.1 The Whitehead graph for a compression body

In this section we define the Whitehead graph for a closed geodesic or Masur domain lamination with respect to a fixed system of meridians α on M. The generalization of Whitehead graphs to compression bodies was developed in Otal's Thèse d'Etat ([50]).

5.3.1.1 The handlebody case

We will start by describing Whitehead's original construction in the case when M is a handlebody ([57], [56], see [53]). As this is discussed in detail in [46], we will sketch this case and discuss the case of compression bodies that are not handlebodies in detail. For a fixed free symmetric generating set $X = \{x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}\}$ of $\pi_1(M)$ and a word $w = w_1 \cdots w_k$ in $\pi_1(M)$, the Whitehead graph of w with respect to X is the graph with $2n$ vertices $x_1, x^{-1}, \ldots, x_n, x_n^{-1}$ and an edge from x to y^{-1} for each string xy in w or any cyclic permutation of w. In this situation Whitehead ([57]) proved the following.

Lemma V.4. *(Whitehead) Let g be a cyclically reduced word. If the Whitehead graph of g*

with respect to X is connected and has no cutpoint, then g is not primitive.

A word $w_1 \cdots w_k$ is cyclically reduced if it is reduced and satisfies $w_1 \neq w_k^{-1}$. A cutpoint is a vertex whose complement is a disconnected graph. A primitive element in a free group is an element that lies in a free generating set for the group. In particular, a primitive element is separable.

Otal extended Whitehead's condition to laminations on the boundary of the handlebody as follows. If X is a free generating set for $\pi_1(M)$, then it is dual to a system of properly embedded disks on M whose complement is a 3-ball. If $D = \{D_1, \ldots, D_n\}$ is such a system of disks, then Otal calls a lamination λ in *tight position* with respect to D if there are no *waves* on D disjoint from λ. A wave is an arc k properly embedded in $\partial M - \partial D$ such that k is homotopic in M but not in ∂M into ∂D. Cutting ∂M along D, produces a planar surface with $2n$ boundary components $D_1^+, D_1^-, D_2^+, D_2^-, \ldots, D_n^+, D_n^-$. The vertices of the Whitehead graph are in one-to-one correspondence with these boundary components. There is an edge between two vertices if there is an arc of $\lambda - \partial D$ connecting the corresponding boundary components. Otal proved that for laminations in the Masur domain of M that are in tight position with respect to D, the Whitehead graph is connected and has no cutpoints ([50] Proposition 3.10, see [46] Lemma 4.5). Moreover, if λ is a Masur domain lamination, then there always exists a system of meridians α such that λ is in tight position with respect to α.

5.3.1.2 Compression bodies that are not handlebodies

Here we will discuss the Whitehead graph of compression bodies that are not handlebodies. Fix $\sigma : G \to PSL(2, \mathbb{C})$ a convex cocompact representation of G such that $N_\sigma = \mathbb{H}^3/\sigma(G)$ is homeomorphic to the interior of M.

Definition V.5. A *system of meridians* is a collection $\alpha = \{\alpha_1, \ldots, \alpha_n\}$ of disjoint, pairwise nonisotopic, simple closed curves on S, the exterior boundary of M, that bound discs $D = \{D_1, \ldots, D_n\}$ in M such that $M - \mathcal{N}(D)$ consists of a collection of trivial I-bundles over closed surfaces, where $\mathcal{N}(D)$ is a regular neighborhood of D.

Since σ is convex cocompact, $\overline{N_\sigma} = N_\sigma \cup \partial_C N_\sigma$ is homeomorphic to M. We will often identify ∂M with $\partial_C N_\sigma$. Let α be a system of meridians bounding the discs $D = D_1 \cup \cdots \cup D_n$. Let $\Sigma_1 \times I, \ldots, \Sigma_k \times I$ be the components of $\overline{N_\sigma} - \mathcal{N}(D)$. Let μ be a subset of $\Lambda(\sigma(G)) \times \Lambda(\sigma(G))$ that is $\sigma(G)$-invariant and also invariant under switching the two factors. Most of the time μ will be one of the two following sets:

- If γ is a closed geodesic in N_σ, let μ_γ be all pairs of endpoints of the lifts of γ.
- If λ is a Masur domain lamination, then it is realizable by a pleated surface $h : S \to N_\sigma$ homotopic to the inclusion map. Let μ_λ be all pairs of endpoints of lifts of leaves in $h(\lambda)$.

Recall from Section 2.1.6 that $h(\lambda)$ is independent of the choice of pleated surface. $\Gamma_\alpha(\mu)$, the Whitehead graph of μ with respect to α, is a collection of not necessarily connected graphs, $\Gamma_\alpha(\mu)^{\Sigma_1}, \ldots, \Gamma_\alpha(\mu)^{\Sigma_k}$, where the elements in the collection are in one-to-one correspondence with the components of $\overline{N_\sigma} - D$. In $\overline{N_\sigma} - \mathcal{N}(D)$, there are two copies D_i^+ and D_i^- of each D_i in D. Given a component $\Sigma \times I$ of $\overline{N_\sigma} - \mathcal{N}(D)$, the vertices in the corresponding graph $\Gamma_\alpha(\mu)^\Sigma$ are in one-to-one correspondence with the components of D_i^+ and/or D_i^- of D in the frontier of $\Sigma \times I$. Fix a Jordan curve C in $\Lambda(\sigma(G))$ that is invariant under a conjugate of $\pi_1(\Sigma)$, which we will continue to denote $\pi_1(\Sigma)$. To avoid superscripts, we will abuse notation and let $D_1, \ldots, D_m$ denote the vertices of this component. Let F denote the boundary component of $\Sigma \times I$ coming from S. Fix a lift $\widetilde{\partial D_i}$ of each ∂D_i in $\partial \mathbb{H}^3$ such that $\widetilde{\partial D_i}$ lies in the component of the preimage of F in ∂H^3 containing C on its boundary. Let U_i be the open set in $\partial \mathbb{H}^3$, bounded by $\widetilde{\partial D_i}$ not containing C. The edges from D_i to D_j will be in one-to-one correspondence with elements g in $\pi_1(\Sigma)$ such that $\mu \cap (U_i \times gU_j)$ is nonempty. We will denote such an edge (U_i, gU_j). Notice that although these edges are directed, for each edge from U_i to U_j labeled g, there is an edge from U_j to U_i labeled g^{-1}.

Definition V.6 (Otal)**.** A connected component of $\Gamma_\alpha(\mu)^\Sigma$ is *strongly connected* if there exists a cycle that represents a nontrivial element of $\pi_1(\Sigma)$. A connected component of $\Gamma_\alpha(\mu)^\Sigma$ has a *strong cutpoint* if we can express the graph as the union of two graphs G_1 and G_2 that intersect in a single vertex such that either G_1 or G_2 is not strongly connected.

We take the convention that a cycle

$$(U_{i_1}, g_1 U_{i_2}), (U_{i_2}, g_2 U_{i_3}), \ldots, (U_{i_k}, g_k U_{i_1})$$

corresponds to the group element $g_1 \cdots g_k$. We made two choices when defining the Whitehead graph, the lifts U_i of D_i and the Jordan curve C. Suppose that we pick a different set of lifts $U'_1, \ldots, U'_m$ of $D_1, \ldots, D_m$. Then $U'_i = h_i U_i$ for some h_i in $\pi_1(\Sigma)$. There is an edge (U_{i_1}, gU_{i_2}) in the original graph if and only if there is an edge $(U'_{i_1}, h_{i_1} g h_{i_2}^{-1} U'_{i_2})$ in the new graph. In particular, there is a cycle $(U_{i_1}, g_1 U_{i_2}), \ldots, (U_{i_k}, g_k U_{i_1})$ in the original graph if and only if there is a cycle

$$(U'_{i_1}, h_{i_1} g_1 h_{i_2}^{-1} U'_{i_2}), \ldots, (U'_{i_k}, h_{i_k} g_k h_{i_1}^{-1} U'_{i_1})$$

in the new graph. Since $g_1 \cdots g_k$ is nontrivial if and only if $h_{i_1} g_1 \cdots g_k h_{i_1}^{-1}$ is nontrivial, the above definitions do not depend on the choice of lifts U_i.

Suppose we choose a different Jordan curve C'. Then there exists an element a in G such that $C' = aC$. The lifts aU_i of D_i will lie in the appropriate component of the preimage of F in ∂H^3, namely the one containing C' on its boundary. Since μ is $\sigma(G)$-invariant there is an edge between U_i and gU_j if and only if there is an edge between aU_i and $aga^{-1}aU_j$; in particular, any edge labeled g in the original graph is now an edge in the new graph labeled aga^{-1}. So the above definitions do not depend on the choice of Jordan curve C.

5.3.1.3 Topological Interpretation

In the case when $\mu = \mu_\lambda$ for a Masur domain lamination λ, Otal describes a topological interpretation of the Whitehead graph in terms of the exterior boundary. As in the handlebody case, there is a notion of *tight position.*

Definition V.7. A measured lamination λ is in *tight position relative to* α a system of meridians if there does not exist a *wave* k disjoint from λ properly embedded in $S - \alpha$, where a wave is an arc satisfying the following.

- the interior of k is disjoint from α.

- k can be homotoped in N_σ but not in S relative to its endpoints to an arc contained in some α_i.

Observe that if λ is in tight position, then there are no waves in λ. If λ is a Masur domain lamination then there exists a system of meridians α such that λ is in tight position with respect to α ([42], Section 3 for handlebodies, [50] Theorem 1.3 for general compression bodies). Such a system is obtained by minimizing the intersection number with λ. Assume that λ is in tight position with respect to α. We will start by defining a related collection of graphs denoted $\Gamma'_\alpha(\lambda)$. Consider the collection of surfaces with boundary obtained by cutting S along α. For each α_i in α, there are two boundary components α_i^+ and α_i^- in the new collection of surfaces with boundary. For each component F of $S - \alpha$, we will define a graph of $\Gamma'_\alpha(\lambda)^F$ as follows. The vertices will be in one to one correspondence with the copies of α_i^+ and/or α_i^- in the frontier of F. We will abuse notation and relabel the boundary components α_i to avoid superscripts. The edges from the vertex α_i to the vertex α_j are in one-to-one correspondence with the isotopy classes of arcs on S in λ connecting α_i and α_j.

There is a natural surjective map from $\Gamma'_\alpha(\lambda) \to \Gamma_\alpha(\mu_\lambda)$ defined as follows. Take the obvious map on the vertices. Suppose that $[k]$ is an edge connecting α_i and α_j. Let D_i and D_j denote the corresponding vertices in $\Gamma_\alpha(\mu_\lambda)$ and let U_i and U_j be the fixed lifts of D_i and D_j, respectively. Take the lift $\widetilde{k}$ of k intersecting U_i. Since λ is in tight position with respect to α, $\widetilde{k}$ will have one endpoint in U_i and the other in gU_j for some g in $\pi_1(\Sigma)$. Map the edge $[k]$ in $\Gamma'_\alpha(\lambda)$ to the edge (U_i, gU_j) in $\Gamma_\alpha(\mu_\lambda)$. To see that (U_i, gU_j) is an edge in $\Gamma_\alpha(\lambda)$ we will need the following two facts.

- Any lift of a leaf l of λ has two well-defined endpoints x_1 and x_2 in $\Lambda(\sigma(G))$.
- (x_1, x_2) are the endpoints of $h(l)$, where $h : S \to N_\sigma$ is a pleated surface realizing λ.

To see a proof of the first fact see Lemma 1 in [35]. The second fact is clear since it is true for simple closed curves in the Masur domain and we can approximate λ by such curves.

Now, if we consider the leaf $\widetilde{l}$ of $\widetilde{\lambda}$ containing the arc $\widetilde{k}$ its endpoints must be contained in U_i and gU_j by tightness.

To see that the map is surjective, given an edge (U_i, gU_j) in $\Gamma_\alpha(\mu_\lambda)$, there exists a leaf $\widetilde{l}$ with endpoints in U_i and gU_j. This will give an arc between α_i and α_j. Two edges $[k]$ and $[k']$ in $\Gamma'_\alpha(\lambda)$ are identified in $\Gamma_\alpha(\mu_\lambda)$ if and only if they are homotopic in N_σ (see Figure 5.1). Hence edges in the Whitehead graph between D_i and D_j correspond to homotopy classes of arcs of λ joining ∂D_i and ∂D_j.

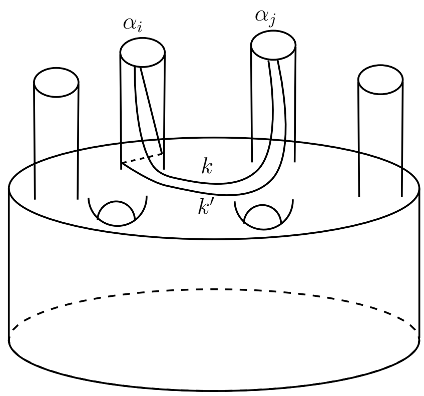

Figure 5.1: The edges $[k]$ and $[k']$ in $\Gamma'_\alpha(\mu)$ get identified in $\Gamma_\alpha(\mu)$.

5.3.1.4 Whitehead graphs of separable curves and Masur domain laminations

In this section we give Otal's generalization of Whitehead's lemma, namely that for a separable element g there exists a connected component of $\Gamma_\alpha(\mu_g)$ that is either not strongly connected or has a strong cutpoint. On the other hand, Otal also showed that the Whitehead graph of a Masur domain curve in tight position with respect to α is strongly connected and without a strong cutpoint. We will use this dichotomy in Section 5.3.2.

Proposition V.8 (Otal, Proposition A.3)**.** *Let M be a nontrivial compression body that is neither the connect sum of two trivial I-bundles over closed surfaces nor a handlebody. If g is a separable element of $G = \pi_1(M)$, then some connected component of $\Gamma_\alpha(\mu_g)$ is not strongly connected or has a strong cutpoint.*

Proof. Let γ be the geodesic representative of $\sigma(g)$ in N_σ. Since M is not the connect sum of two trivial I-bundles, by Lemma V.1, there exists an essential disc Δ disjoint from γ. As Δ is only well-defined up to isotopy we will often abuse notation and use Δ to refer to different representatives of the isotopy class of Δ.

First consider the case where Δ does not intersect D, up to isotopy. Then Δ is isotopic (rel boundary) into the boundary of some component $\Sigma \times I$ of $\overline{N_\sigma} - \mathcal{N}(D)$. Consider the sets $\mathcal{A} = \{D_i | D_i \subset \Delta\}$ and $\mathcal{A}^c = \{D_i | D_i \notin \mathcal{A}\}$. If $\mathcal{A}$ is empty, then Δ is isotopic to some D_i. This implies that D_i is an isolated vertex in the Whitehead graph. Moreover, any edge connecting D_i to itself, is labeled with the trivial element as γ does not intersect D_i. In particular, the connected component containing D_i is not strongly connected. If $\mathcal{A}$ is nonempty, first observe that no vertex in $\mathcal{A}$ can be connected to a vertex in $\mathcal{A}^c$. Let C be any connected component of $\Gamma_\alpha(\mu_\gamma)^\Sigma$ containing a vertex in $\mathcal{A}$. C is not strongly connected for if $(U_{i_1}, g_1 U_{i_2}), \ldots, (U_{i_k}, g_k U_{i_1})$ is a cycle in C and if $\widetilde{\Delta}$ is the lift of Δ containing U_{i_1}, then $\widetilde{\Delta}$ also contains $g_1 \cdots g_k U_{i_1}$ since γ does not intersect Δ. In particular, the cycle is trivial as the stabilizer of $\widetilde{\Delta}$ is trivial.

If Δ and D intersect nontrivially, up to isotopy, $\Delta \cap D$ is a finite collection of disjoint properly embedded arcs. Take k_0 an innermost arc in this collection, meaning that one of the discs Δ_0 formed by $\partial\Delta$ and k_0 has interior disjoint from D. Δ_0 lies in some component $\Sigma \times I$ of $\overline{N_\sigma} - \mathcal{N}(D)$. $\partial\Delta_0$ intersects one of the D_i in the frontier of $\Sigma \times I$ nontrivially. Let D_0 denote that disc. Moreover, Δ_0 is isotopic relative to its boundary into the boundary of $\Sigma \times I$. Let C denote the connected component of $\Gamma_\alpha(\mu_\gamma)^\Sigma$ containing D_0. Consider $\mathcal{B}$ the set of vertices in C such that the corresponding discs D_i in $\Sigma \times I$ are contained in Δ_0. Notice that if $\mathcal{B}$ is empty, then we can isotope Δ to remove k_0 from the intersection and repeat the procedure above. Let $\mathcal{C}$ denote the set of vertices in C such that the corresponding discs D_i in $\Sigma \times I$ are disjoint from Δ_0. The only vertex not lying in either set is D_0. We claim that D_0 is a strong cutpoint of C where the graph associated to the vertices in $\mathcal{B}$ is not strongly connected. In particular, if $\mathcal{C}$ is empty, then C is not strongly connected.

First, we want to show that no vertex in $\mathcal{B}$ is connected by an edge to a vertex in $\mathcal{C}$. Suppose that there is an edge (U_b, gU_c) where U_b is the fixed lift of a vertex in $\mathcal{B}$ and U_c is the fixed lift of a vertex in $\mathcal{C}$, i.e., there is a lift $\widetilde{\gamma}$ of γ such that one endpoint lies in U_b and the other endpoint lies in gU_c. This implies that γ must intersect Δ_0 nontrivially (see Figure 5.2), which contradicts how we chose Δ_0.

Secondly we want to show that the subgraph associated to $\mathcal{B}$ is not strongly connected. Suppose there is a cycle $(U_0, g_0 U_1), \ldots, (U_k, g_k U_0)$ such that $g_0 \cdots g_k$ is nontrivial. If $\widetilde{\Delta}_0$ is

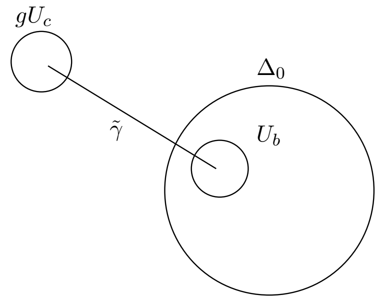

Figure 5.2: If there were an edge connecting a vertex in $\mathcal{B}$ to one in $\mathcal{C}$ then γ would intersect Δ_0 nontrivially.

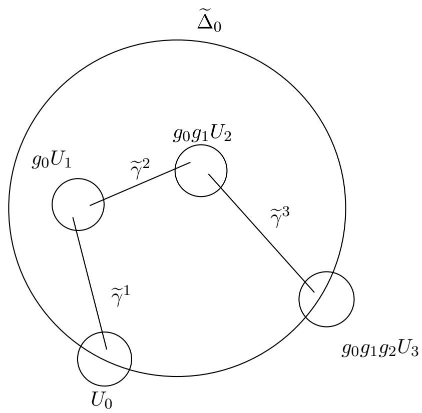

Figure 5.3: If there is a non-trivial cycle $(U_0, g_0U_1), \ldots, (U_k, g_kU_0)$, then U_0 and $g_0g_1g_2\cdots g_kU_0$ will both intersect $\widetilde{\Delta}_0$.

the lift of Δ_0 containing g_0U_1, then U_0 and $g_1\cdots g_kU_0$ intersect Δ_0 (see Figure 5.3). This is impossible as it implies that either there is a nontrivial curve in Δ_0 or $\Delta_0 \cap D_0$ consists of two connected components, contradicting how we chose Δ_0. $\square$

Proposition V.9 (Otal, Propostion A.5)**.** *Let λ be a measured lamination in the Masur domain that is in tight position with respect to a system of meridians α. Then, each connected component of $\Gamma_\alpha(\mu_\lambda)$ is strongly connected and without a strong cutpoint.*

Proof. We will use the topological interpretation of the Whitehead graph discussed in Section 5.3.1.3. Suppose that a component C of $\Gamma_\alpha(\mu_\lambda)^\Sigma$ is not strongly connected. Let $D_1, \ldots, D_k$ be the components of D that correspond to the vertices in C. Let $\mathcal{A}$ be the union of the D_i and $\lambda \cap \Sigma$. Take $\mathcal{N}(\mathcal{A})$ a regular neighborhood of $\mathcal{A}$. The boundary of $\mathcal{N}(\mathcal{A})$ consists of simple closed curves that each bound a disc in Σ, as C is not strongly connected. One of these boundary components b must bound a disc containing some D_i. Then, b is nontrivial on S and so we have found a meridian that misses λ, a contradiction.

Suppose that C has a strong cutpoint. Let F denote $\Sigma - \cup\, \text{int}(D_i)$. Let D_0 correspond to the strong cutpoint and let G_1 and G_2 be the two graphs whose intersection is D_0 such that G_1 is not strongly connected. Let $\beta_1, \ldots, \beta_t$ be the merdians in F corresponding to vertices of G_1. Let $\lambda' \subset \lambda \cap F$ consisting of arcs intersecting at least one β_i. Let $\mathcal{N}$ denote a regular neighborhood of $\lambda' \cup (\cup \beta_i)$. The boundary of $\mathcal{N}$ consists of closed curves $c_1, \ldots, c_l$ and arcs $a_1, \ldots, a_s$ with endpoints lying on α_0. Since G_1 is not strongly connected, each c_i bounds a disc. We claim that at least one of the arcs a_i is a wave, i.e., an arc disjoint from λ, homotopic relative to its endpoints, in M but not in S into α_0. Indeed, any a_i is disjoint from λ, by construction and homotopic in M into α_0, since G_1 is not strongly connected. For each arc a_i choose an arc b_i in α_0 sharing the same endpoints as b_i such that $a_i \cup b_i$ bounds a disc not containing α_0. At least one of the loops $c_1, \ldots, c_l, a_1 \cup b_1, \ldots, a_s \cup b_s$ contains some β_i, since they form the boundary components of $\mathcal{N}$. If c_i contained some β_i, then β_i would not be connected to α_0, which contradicts how we chose β_i. Therefore, some $a_k \cup b_k$ bounds a disc containing at least one β_i. In particular, a_k will not be homotopic in S into α_0. So a_k is a wave disjoint from λ, a contradiction to the assumption that λ is in tight position (see Figure 5.4).

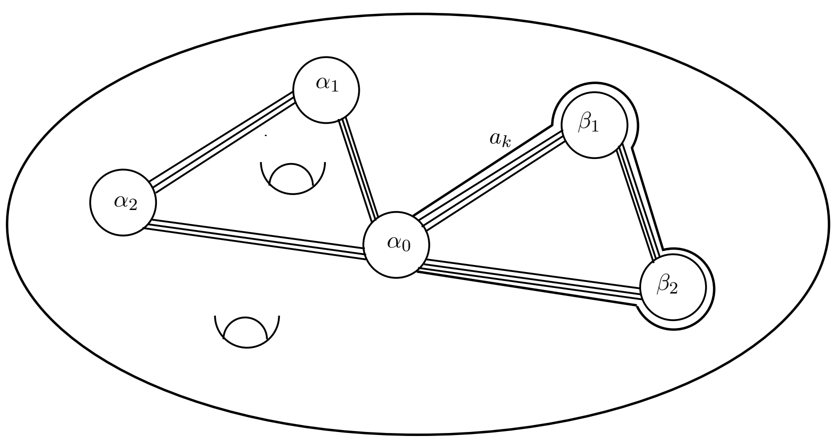

Figure 5.4: The arc a_k is a wave disjoint from λ.

□

5.3.2 Examples of separable-stable points on $\partial AH(M)$

In this section we prove Proposition V.11, which shows that two types of points on $\partial AH(M)$ are separable-stable; namely that a geometrically finite point with one cusp associated to a Masur domain curve is separable-stable and a purely hyperbolic geometrically infinite point with one geometrically infinite end corresponding to the exterior boundary component is separable-stable. The case that a geometrically finite point with one cusp associated to a Masur domain curve for handlebodies is separable stable was proven by Minsky [46]. The case that a purely hyperbolic geometrically infinite point is separable-stable for handlebodies was proven by Jeon-Kim in [31].

Lemma V.10. *Let ρ be a discrete faithful representation such that $\rho(g)$ is hyperbolic for all separable elements g. If ρ is not separable-stable, then there exists a sequence of separable elements g_i such that the endpoints of $\mathrm{Ax}(\rho(g_i))$ the axis of $\rho(g_i)$ in $\mathbb{H}^3$ converge to a single point z in $\partial\mathbb{H}^3$ but the points g_i^+ and g_i^- in $\partial C_S(G)$ converge to distinct points z^+ and z^- in $\partial C_S(G)$.*

Proof. By Lemma III.10, since ρ is not separable-stable, the set of geodesics homotopic to separable curves is not contained in any compact set of N_ρ. Let $\{\gamma_i\}$ be a sequence of separable geodesics such that $\{\gamma_i\}$ is not contained in any compact set. Recall that the image of the Cayley graph under $\tau_{\rho,x}$ in N_ρ has only one vertex, v. Choose D_i approaching infinity such that γ_i does not lie in a ball of radius D_i around v.

Fix a set of lifts $\widetilde{\gamma}_i$ of γ_i. Then $\widetilde{\gamma}_i$ is an axis for $\rho(g_i)$ for some separable element g_i. Let l_i be a geodesic in the Cayley graph connecting g_i^+ and g_i^-. There exists a vertex v_i on $\tau_{\rho,x}(l_i)$ such that the distance to $\widetilde{\gamma}_i$ is at least D_i. Shift l_i to $l_i' := v_i^{-1} \cdot l_i$. Then, l_i' passes through e the identity element and connects the fixed points of $v_i^{-1} g_i v_i$, which is still a separable element. If $\widetilde{\gamma}_i'$ is $\rho(v_i^{-1}) \cdot \widetilde{\gamma}_i$, the distance from x to $\widetilde{\gamma}_i'$ is D_i. This implies that, up to subsequence, the endpoints of $\widetilde{\gamma}_i$ approach a single point z on $\partial\mathbb{H}^3$. There exists z^+ and z^- on $\partial C_S(G)$ such that up to subsequence, $v_i^{-1} g_i v_i^+ \to z^+$ and $v_i^{-1} g_i v_i^- \to z^-$. Since each l_i' passes through the e, z^+ and z^- are distinct. □

Proposition V.11. *Let ρ be a discrete and faithful representation such that $N_\rho = \mathbb{H}^3/\rho(G)$ is homeomorphic to the interior of M satisfying one of the following two conditions.*

(a) *ρ is a geometrically finite representation with one cusp associated to a Masur domain curve or*

(b) *ρ is a purely hyperbolic representation where the end corresponding to the exterior boundary is geometrically infinite and all other ends are convex cocompact.*

Then ρ is separable-stable.

Proof. Let $\lambda \subset S$ be the cusp curve if ρ is of type (a) or the ending lamination if ρ is of type (b). If ρ is not separable-stable, by Lemma V.10 there exists a sequence of separable elements g_i such that the endpoints of $\mathrm{Ax}(\rho(g_i))$ converge to a single point z in $\partial\mathbb{H}^3$ but the points g_i^+ and g_i^- in $\partial C_S(G)$ converge to distinct points z^+ and z^- in $\partial C_S(G)$. In particular, z^+ and z^- are identified under the Cannon–Thurston map for $\tau_{\rho,x}$.

Consider our fixed convex cocompact representation $\sigma : G \to \mathrm{PSL}(2, \mathbb{C})$ used to define the Whitehead graph. If $\overline{\tau_{\sigma,x}}$ is the Cannon–Thurston map for $\tau_{\sigma,x}$, using the results of Floyd and Mj (see Section 2.2.0.1), $\overline{\tau_{\sigma,x}}(z^+)$ and $\overline{\tau_{\sigma,x}}(z^-)$ are either

(a) endpoints of $\mathrm{Ax}(\sigma(g))$ where $\rho(g)$ is parabolic or

(b) endpoints of a leaf of the ending lamination or ideal endpoints of a complementary polygon

where the first case occurs if ρ is of type (a) and the second case occurs if ρ is of type (b) (as in the statement of the proposition).

Let $\mu_\infty \subset \Lambda(\sigma(G)) \times \Lambda(\sigma(G))$ be the set of limit points of $\{\mu_{g_i}\}$. Then μ_∞ is $\sigma(G)$-invariant and invariant under switching the two factors. Moreover, $(\overline{\tau_{\sigma,x}}(z^+), \overline{\tau_{\sigma,x}}(z^-))$ lies in μ_∞.

Since ending laminations lie in the Masur domain (see Section 2.1.7), we can choose α a system of meridians such that λ is in tight position with respect to α (see Section 5.3.1.3). We first claim that $\Gamma_\alpha(\mu_\lambda)$ is contained in $\Gamma_\alpha(\mu_\infty)$. In case (a) this is obvious as μ_λ is exactly the $\sigma(G)$ translates of $(\overline{\tau_{\sigma,x}}(z^+), \overline{\tau_{\sigma,x}}(z^-))$. In case (b), let L be the geodesic connecting $\overline{\tau_{\sigma,x}}(z^+)$ and $\overline{\tau_{\sigma,x}}(z^-)$ and l be the geodesic in $\Omega(\sigma(G))$ that is a leaf of the preimage of the ending lamination with one endpoint $\overline{\tau_{\sigma,x}}(z^+)$. Let w be the other endpoint of l. Recall that to define the Whitehead graph we fixed a system of meridians α on $\partial_C N_\sigma$

that bound discs D. Let $\widetilde{\alpha}_i$ be a lift of one of the meridians α_i with the following property. $\partial\mathbb{H}^3 - \widetilde{\alpha}_i$ has two components W_1 and W_2 such that $\overline{\tau_{\sigma,x}}(z^+)$ lies in W_1 and $\overline{\tau_{\sigma,x}}(z^-)$ and w lie in W_2. Let r be the ray of l that starts at $\widetilde{\alpha}_i$ and ends at $\overline{\tau_{\sigma,x}}(z^+)$ (see Figure 5.5).

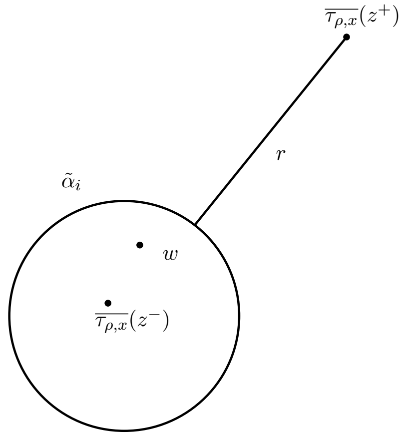

Figure 5.5: r is a ray in $\partial\mathbb{H}^3$ that starts at $\widetilde{\alpha}_i$ and ends at $\overline{\tau_{\rho,x}}(z^+)$.

Edges in $\Gamma_\alpha(\mu_\lambda)$ correspond to homotopy classes of arcs of λ connecting the components of α. Since λ is minimal, r' the image of r in S is dense in λ. Then, for any edge in $\Gamma_\alpha(\mu_\lambda)$ there is an arc of r' corresponding to that edge. Let (U, gV) be an edge of $\Gamma_\alpha(\mu_\lambda)$ and r_0 the arc of r' corresponding to that edge, i.e., there is a lift $\widetilde{r}_0$ of r_0 with one endpoint on ∂U and the other on ∂gV. This means that there is a translate $h \cdot r$ of r such that $h \cdot r$ intersects ∂U and ∂gV. This implies that $h \cdot \overline{\tau_{\sigma,x}}(z^+)$ lies in U or gV. Without loss of generality assume that $h \cdot \overline{\tau_{\sigma,x}}(z^+)$ lies in gV. Then it suffices to show that $h \cdot \overline{\tau_{\sigma,x}}(z^-)$ lies in U. Since r intersects U and is in tight position with respect to α, $h \cdot \widetilde{\alpha}_i$ must lie inside U. This implies that $h \cdot \overline{\tau_{\sigma,x}}(z^-)$ lies in U (see Figure 5.6). This completes the proof of the claim.

Secondly, we observe that $\Gamma_\alpha(\mu_\lambda)$ is a finite graph. In case (a) this is obvious as λ is a closed curve. For case (b) recall the topological interpretation of the Whitehead graph (see Section 5.3.1.3), where edges in the Whitehead graph correspond to homotopy classes of arcs of λ with endpoints on ∂D, relative to those endpoints. Since there can only be finitely many homotopy classes of arcs with endpoints on ∂D that can be realized disjointly on $\partial_{ext}(M)$, there can only be finitely many edges in $\Gamma_\alpha(\mu_\lambda)$.

Since $\Gamma_\alpha(\mu_\lambda)$ is a finite graph contained in $\Gamma_\alpha(\mu_\infty)$ for i large enough, $\Gamma_\alpha(\mu_{g_i})$ contains

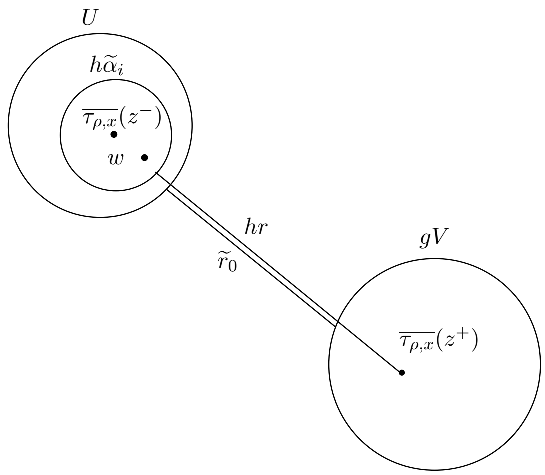

Figure 5.6: Since λ is in tight position with respect to α there is a leaf with one endpoint in U and the other in gV.

$\Gamma_\alpha(\mu_\lambda)$ as a subgraph. Notice that any vertex in $\Gamma_\alpha(\mu_{g_i})$ is also a vertex in $\Gamma_\alpha(\mu_\lambda)$. We claim that this implies that $\Gamma_\alpha(\mu_{g_i})$ must be strongly connected and without a strong cutpoint by Proposition V.9. Indeed, since any vertex in $\Gamma_\alpha(\mu_{g_i})$ is part of a nontrivial cycle in $\Gamma_\alpha(\mu_\lambda)$, it is part of the same nontrivial cycle in $\Gamma_\alpha(\mu_{g_i})$. If $\Gamma_\alpha(\mu_{g_i})$ had a strong cutpoint v, then the component of $\Gamma_\alpha(\mu_\lambda)$ containing v would either be not strongly connected or also have a strong cutpoint. When M is not the connect sum of two trivial I-bundles over closed surfaces, this contradicts Proposition V.8.

For the case when M is the connect sum of two trivial I-bundles over closed surfaces, we first claim that each edge of $\Gamma_\alpha(\mu_\lambda)$ "intersects" every essential annulus, A, contained in each trivial I-bundle in the following sense. Let $M = S \times I \# T \times I$. Suppose that A is an annulus in $S \times I$ or $T \times I$. Let $\partial A = c_1 \sqcup c_2$. In defining the Whitehead graph, we fixed lifts $\widetilde{S}$ and $\widetilde{T}$ of S and T. If we take a lift $\widetilde{c}_1$ of c_1 and the lift $\widetilde{c}_2$ of c_2 with the same endpoints as $\widetilde{c}_1$, then $\widetilde{c}_1 \cup \widetilde{c}_2$ forms a loop in $\partial\mathbb{H}^3$. We will say that an edge $e = (U_i, gU_j)$ intersects A if there exists lifts $\widetilde{c}_1$ and $\widetilde{c}_2$ in either $\widetilde{S}$ or $\widetilde{T}$ as above such that U_i and gU_j lie in different components of $\partial\mathbb{H}^3 - (\widetilde{c}_1 \cup \widetilde{c}_2)$.

If λ is a Masur domain lamination, then it intersects every essential annulus. Using the topological interpretation of the Whitehead graph (see Section 5.3.1.3), $\Gamma_\alpha(\mu_\lambda)$ "intersects" every essential annulus in the sense above.

Since $\Gamma_\alpha(\mu_\lambda)$ is a finite graph contained in $\Gamma_\alpha(\mu_\infty)$ for i large enough, $\Gamma_\alpha(\mu_{g_i})$ "intersects" any essential annulus contained in one of the two trivial I-bundles. This implies that the geodesic representative γ_i of g_i intersects any essential annulus contained in one of the two factors, a contradiction. □

The assumption in Proposition V.11 that each end corresponding to a component of the interior boundary is incompressible is necessary as the following proposition shows. Recall that $\pi_1(M) = \pi_1(S_1) * \ldots * \pi_1(S_k) * F_j$, where S_i is a closed surface and F_j is the free group on j elements.

Proposition V.12. *If $[\rho]$ lies in $\partial AH(M)$ such that $\rho|_{\pi_1(S_i)}$ is not convex cocompact, then $[\rho]$ does not lie in $\mathcal{SS}(M)$. Moreover, if $\rho|_{\pi_1(S_i)}$ is not convex cocompact, then $[\rho]$ cannot lie in a domain of discontinuity*

Proof. If $\rho|_{\pi_1(S_i)}$ has a cusp, then there is a separable element that maps to a parabolic element. So ρ cannot be separable-stable. If $\rho|_{\pi_1(S_i)}$ is geometrically infinite, then in N_ρ there is a sequence of separable geodesics exiting every compact set, so ρ cannot be separable-stable by Lemma III.10.

To see the second statement, suppose that $\rho(g)$ is parabolic for a separable element g. Then, there exists a sequence of representations ρ_i in $\mathcal{X}(M)$ such that $\rho_i(c)$ is elliptic of finite order, n_i (see the proof of Lemma IV.4). Since g is separable, there exists a nontrivial splitting, $G = G_1 * G_2$ such that g lies in G_1. The automorphism f_{n_i} that restricts to conjugation by g^{n_i} on G_1 and restricts to the identity on G_2 fixes ρ_i. In particular, each ρ_i has an infinite stabilizer, so, the limit $[\rho]$ cannot lie in any domain of discontinuity.

Now suppose that $\rho(g)$ is not parabolic for any separable element g but $\rho|_{\pi_1(S_i)}$ is geometrically infinite for some i. Then, we will describe a sequence of representations ρ_k approaching ρ such that there exists a separable curve g_k with $\rho_k(g_k)$ parabolic. Since each ρ_k cannot lie in a domain of discontinuity, neither can ρ. To find such a sequence first observe that $\rho|_{\pi_1(S_i)}$ is purely hyperbolic. By the covering theorem ([12]), it can have only one geometrically infinite end and so it lies in the closure of a Bers slice, $\overline{B}$. Let λ be its ending lamination, and let γ_j be a sequence of simple closed curves on S_i approaching λ. Define a sequence of representations ρ_k satisfying the following:

- $\rho_k|_{\pi_1(S_i)}$ lies in $\overline{B}$,
- $\rho_k(\gamma_j)$ is parabolic and
- $\rho_k|_{\pi_1(S_1)*\ldots*\widehat{\pi_1(S_i)}*\ldots*\pi_1(S_k)*F_j} = \rho|_{\pi_1(S_1)*\ldots*\widehat{\pi_1(S_i)}*\ldots*\pi_1(S_k)*F_j}$.

Then, as $\overline{B}$ is compact, up to subsequence $\rho_k|_{\pi_1(S_i)}$ converges to some ρ' in $\overline{B}$. As the length function is continuous on $AH(S_i \times I)$ ([5]), the length of λ in ρ' must be zero. In particular, λ must be an ending lamination for ρ'. By the ending lamination theorem ([8]), possibly after conjugating, $\rho_k|_{\pi_1(S_i)}$ must converge to $\rho|_{\pi_1(S_i)}$. On the other factors of $\pi_1(M)$, ρ_k obviously converges to ρ. Hence, ρ_k converges to ρ where ρ_k has a separable curve pinched. Since $[\rho_k]$ cannot lie in a domain of discontinuity, neither can $[\rho]$. □

5.3.3 Other homeomorphism types in $AH(M)$

So far we have found separable-stable points on $\partial AH(M)$ that have the same homeomorphism type as M. In this section, we show that if M is a large compression body, then there exists M' homotopy equivalent but not homeomorphic to M such that the for each component C of the interior of $AH(M)$ corresponding to M', no point on ∂C is separable-stable, even though every point in C is separable-stable.

Proposition V.13. *Suppose that M' is homotopy equivalent to M such that*

(a) *M' is not homeomorphic to M*

(b) *for each compressible component B of $\partial M'$, $i_*(\pi_1(B))$ is a free factor of $\pi_1(M')$.*

If C is a component of the interior of $AH(M)$ corresponding to M', then $\overline{C} - C$ has no separable stable points.

Proof. Let $[\rho]$ be a purely hyperbolic point in $\overline{C} - C$. Then ρ is the algebraic limit of ρ_i in C such that N_{ρ_i} is homeomorphic to the interior of M'. Since ρ has no parabolics, ρ_i converges to ρ geometrically (Theorem II.5). Then, N_ρ is homeomorphic to N_{ρ_i} ([17]). Since each boundary component of M' maps to a proper factor of a free decomposition of $\pi_1(M)$, there is a boundary component B of M' such that the end corresponding to B is geometrically infinite. Then, there exists a sequence of simple closed curves on B whose geodesic representatives leave every compact set of N_ρ. By Lemma III.10, ρ cannot be separable-stable, as any simple closed curve on B is separable. Since purely hyperbolic points are dense in $\overline{C} - C$ ([14], Lemma 4.2, [48], [49]), this completes the proof. $\square$

If M is a large compression body, then there always exists such an M'. Any M' homotopy equivalent to M with more than one compressible boundary component will suffice. Suppose that B and B' are two compressible boundary components of M'. Let m be a meridian in B' bounding a disc D. If D separates M' into M_1' and M_2', then $\pi_1(M) \cong \pi_1(M_1') * \pi_1(M_2')$ and $\pi_1(B)$ lies in one of the two factors. If D is non-separating and $M'' = M' - D$, then $\pi_1(M) \cong \pi_1(M'')*_1 \cong \pi_1(M'') * \mathbb{Z}$ and $\pi_1(B)$ lies in the first factor.

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