

On the Mathematics of Navigational Calculations for Meridian Sailing

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Abstract. Meridian sailing is actually a special case of sailing uniquely determined by relative coordinates $\Delta\varphi, \Delta\lambda$. In this note we discuss explicit finding of certain navigational parameters. These include loxodrome distance, rhumb line and geographic and geocentric latitude of a position on the globe surface. The latter is expressed in terms of the inverse incomplete elliptic integral of the third kind Π ; we discuss and survey certain traditional and recent methods of computation of this function.

Key Words: Meridian (Loxodrome) sailing, Incomplete elliptic integral of the third kind, Numerical quadrature, Binomial series expansion of the integrand, Bulirsch method, Carlson's duplication, Fukushima's J -method.

1. INTRODUCTION

From the nautical point of view, in surface navigation we have two basic ship tracking models when the trajectory is prescribed.

- The first is constant course navigation (or loxodromic navigation), when no course changes occur in a unit of time. Under this assumption we recognize two specific cases:
 - (i) the course line is a meridian (meridian sailing) and
 - (ii) the course line is a parallel or coincides with the equator.
- The second is navigation with a variable rhumb. When the course line cuts meridians pointwise at variable angles, the model is called orthodromic sailing.

We will focus here on loxodromic navigation and its specific case (i). However, because the distance between two parallel lines is the same, regardless of the meridian along which it is measured, the distance L_m along any meridian between two parallels becomes a constant multiple $L_{lox} = \sec \alpha \cdot L_m$ of the secant of the constant angle α between the rhumb line and a meridian on the globe and the meridian distance L_m . Obviously, it is enough to study meridian sailing's length.

Loxodromic navigation is set by a couple of relative coordinates

$$(\Delta\varphi, \Delta\lambda) = (\varphi_2 - \varphi_1, \lambda_2 - \lambda_1),$$

where φ_1, φ_2 , and λ_1, λ_2 stand for the absolute geographic coordinates of latitude and longitude respectively and index 1 denotes the departure, and 2 the arrival positions. The meridian sailing corresponds by definition to $\Delta\lambda = 0$, while the sign of $\Delta\varphi$ determines the general loxodromic course, which is $K_L = 0^\circ$ for positive, and $K_L = 180^\circ$ for negative $\Delta\varphi$.

In this article we study the meridian (loxodromic) arc length on a meridian on the globe spheroid surface starting at latitude φ_1 and finishing at latitude φ_2 . Without loss of generality we shall assume that $\varphi_1 = 0$, that is $\Delta\varphi = \varphi_2 \equiv \varphi$ can be taken, i.e. the navigation begins precisely on the equator. Indeed, we have $\Delta\varphi = (\varphi_2 - \psi) - (\varphi_1 - \psi)$, ψ arbitrary, which reduces the meridian arc length calculation to the case $\varphi_1 \equiv 0$.

2. MERIDIAN SAILING ON THE ELLIPSOID OF REVOLUTION

Figure 1 presents the ellipsoid of revolution (two-axis or rotational ellipsoid) model of the globe with the equatorial radius a and polar radius b . The great ellipse is a planar cross-section of the ellipsoid and the plane containing the symmetry axis. (The associated numerical eccentricity we denote by the usual $e^2 = 1 - b^2/a^2 \leq 1$). On the other hand θ signifies the geocentric latitude of the same position P . Let us mention that $\tan\theta = (1 - e^2) \tan\varphi$.

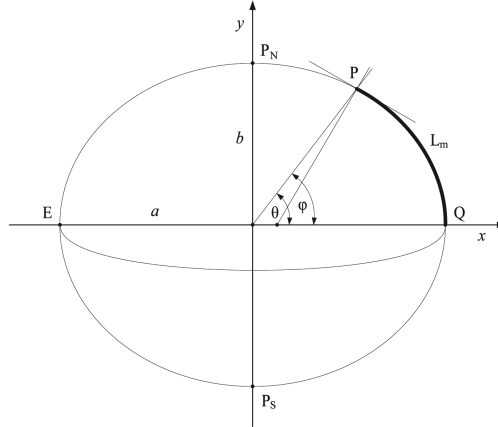


Figure 1. Meridian Ellipse on the Ellipsoid of Revolution.

The meridian ellipse arc length L_m characterizes the path between departure position Q at the equator and the arrival position P and reads as follows

$$L_m = a(1 - e^2) \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}} = \frac{b^2 \Pi(e^2; \varphi | e^2)}{a}, \quad (1)$$

where Π denotes the Legendre incomplete elliptic integral of the third kind defined by

$$\Pi(n; \varphi | m) = \int_0^\varphi \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - m \sin^2 t}} \quad |\varphi| \leq \frac{\pi}{2}, \quad (2)$$

see APPENDIX A. Here φ is the geographic latitude of P ¹, and for the sake of simplicity, here and in what follows, we write $\Pi(\cdot) \equiv \Pi(e^2; \cdot | e^2)$, see [41]. The special function Π is

¹According to the remark upon rhumb line course $\alpha \neq 90^\circ$ we have [45, p. 134, Eq. (6)] that $L_{lox} = L_m \cdot \sec \alpha$. Therefore

$$L_{lox} = \frac{L_m}{\cos \alpha} = \frac{b^2 \Pi(\varphi)}{a \cdot \cos \alpha}.$$

For $\alpha = 90^\circ$ we have equator, or parallel sailing, while for $\alpha = 0^\circ$ it is $L_{lox} \equiv L_m$.

continuous and odd on the range $|\varphi| \leq \pi/2$, and being

$$\frac{d}{d\varphi} \Pi(n; \varphi|m) = \frac{1}{(1 - n \sin^2 \varphi) \sqrt{1 - m \sin^2 \varphi}} > 0 \quad n, m \in (0, 1),$$

it is monotone increasing function in φ . The monotonicity property implies invertibility, that is, $w = \Pi(n; z|m)$ possesses an unique inverse $z = \Pi^{-1}(n; w|m)$. Now, by virtue of (1) we clearly conclude that the meridian distance between positions Q, P corresponds to the latitude

$$\varphi = \Pi^{-1}\left(\frac{a L_m}{b^2}\right). \quad (3)$$

Citations regarding inverse elliptic integral of the third kind are very rare; Allender–Hornreich–Johnson [3, p. 2655] refer to Saupe [43, p. 817, Eq. (7)] concerning the exact solution for the approximated Euler–Lagrange equation which describes the free energy of the uniformly distorted Fréedericksz state, remarking that the exact solution of approximated EL–equation “... it is expressed ... as an inverse elliptic function of the third kind. This is inconvenient for numerical calculations...”. They considered binomial type approximations for $\Pi^{-1}(w) \approx C \cdot (1 - w^\rho)$, ρ real, C an absolute constant (see [3, p. 2657]). However, Saupe followed the usual computational way expressed in section 4.

Bearing in mind the fact that there is no explicit formula for Π and Π^{-1} expressed *via* certain elementary or higher transcendental functions, we are forced to compute it by numerical methods with prescribed accuracy. So, the meridian arc length $L_m = L_m(\varphi)$ we compute by finding the associated Π function value in (1). Similarly, solving the inverse problem Π^{-1} in (3) we can calculate it only by certain numerical computation procedures that have the computation error under the prescribed accuracy level. Let us remark that e.g. the computational softwares contain built–in subroutines such as `EllipticPi[n, z, m]` in *Mathematica*. In the next chapters we will survey these questions with a list of related links.

3. COMPUTING $\Pi(\varphi)$ BY NUMERICAL QUADRATURES

To compute the meridian arc length L_m , reduce the problem to one of finding values of the elliptic integral of the third kind

$$\Pi(\varphi) = \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}}. \quad (4)$$

Based on a polynomial interpolation of the integrand using uniform grid, we calculate $\Pi(\varphi)$ by certain of the familiar numerical quadratures such as Newton–Cotes formulae, represented by composite rectangle and trapezoidal rules (constant and linear approximation), Simpson’s formula (quadratic polynomial approximation); Gaussian quadrature formula and its numerous variants e.g. Gauss–Legendre, Gauss–Kronrod, Gauss–Lobatto rules etc. For instance,

the composite trapezoidal formula for $\Pi(\varphi)$ is

$$\Pi(\varphi) = \frac{\varphi}{2n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 2 \sum_{j=1}^{n-1} \left[\left(1 - e^2 \sin^2 \left(\frac{\varphi j}{n} \right) \right)^{-3/2} \right] \right\} + R_n^T(\varphi), \quad (5)$$

where the error term is

$$R_n^T(\varphi) = \frac{3e^2 \varphi^3 [4(e^2 - 2) \cos(2\vartheta\varphi) + e^2(2 \cos(4\vartheta\varphi) - 7)]}{32n^2(1 - e^2 \sin^2(\vartheta\varphi))^{7/2}}, \quad \vartheta \in (0, 1).$$

Knowing the computation accuracy $\varepsilon > 0$, we can estimate the grids size n from $R_n(\varphi) < \varepsilon$, for any fixed φ . We mention that the composite Simpson's formula in our case reduces to

$$\begin{aligned} \Pi(\varphi) = \frac{\varphi}{6n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 4 \sum_{j=1}^{n-1} \left[1 - e^2 \sin^2 \left(\frac{\varphi j}{n} \right) \right]^{-3/2} \right. \\ \left. + 2 \sum_{j=1}^n \left[1 - e^2 \sin^2 \left(\frac{\varphi(2j-1)}{2n} \right) \right]^{-3/2} \right\} + R_n^S(\varphi), \end{aligned} \quad (6)$$

and the error term becomes

$$\begin{aligned} R_n^S(\varphi) &= \frac{\varphi^5}{2880 n^4} \left((1 - e^2 \sin^2 t)^{-3/2} \right)' \Big|_{t=\xi} \\ &= \frac{3e^2 \varphi^5}{2880 n^4 (1 - e^2 \sin^2 \xi)^{11/2}} \left[11 e^6 \sin^8 \xi + 7.5 e^6 \sin^4(2\xi) - 18 e^4 \sin^6 \xi \right. \\ &\quad \left. + 2 \cos^2 \xi \cdot (52 e^6 \sin^6 \xi - e^4 \sin^4 \xi - 49 e^2 \sin^2 \xi - 2) \right. \\ &\quad \left. + 3e^2 \sin^4 \xi + 15 e^2 \cos^4 \xi \cdot (12 e^2 \sin^2 \xi + 1) + 4 \sin^2 \xi \right], \quad \xi \in (0, \varphi). \end{aligned}$$

Also by $|R_n^S(\varphi)| < \varepsilon$ we evaluate n and then proceed to (6) with some fixed φ . We note that these two numerical quadratures are built-in [29].

Transforming the integration interval $[0, \varphi]$, $\varphi \leq \pi/2$ onto $[-1, 1]$ putting $\varphi(t+1)/2 \mapsto t$ in the integrand of (4), the Gauss-Legendre quadrature rule gives the approximation

$$\begin{aligned} \Pi(\varphi) &= \frac{\varphi}{2} \int_{-1}^1 \left[1 - e^2 \sin^2 \left(\frac{\varphi}{2}(t+1) \right) \right]^{-3/2} dt \\ &= \frac{\varphi}{2} \sum_{j=1}^n w_j \left[1 - e^2 \sin^2 \left(\frac{\varphi}{2}(t_j+1) \right) \right]^{-3/2} + R_n^{GL}(\varphi), \end{aligned}$$

where the weight-coefficients

$$w_j = \frac{2}{(1 - t_j^2)[P_n'(t_j)]^2}, \quad j = \overline{1, n}$$

are based on the Legendre polynomial $P_n(x)$, see APPENDIX C. The related quadrature error formula turns out to be

$$R_n^{GL}(\varphi) = \frac{(n!)^4 \varphi^{2n+1}}{(2n+1)[(2n)!]^3} \left((1 - e^2 \sin^2 t)^{-3/2} \right)^{(2n)} \Big|_{t=\xi}, \quad \xi \in (0, \varphi).$$

For other kinds of Gaussian quadratures, e.g Gauss–Kronrod quadrature, we refer to the original Kronrod’s monograph [36]; also see the book [31] by Kahaner–Moler–Nash.

The Gauss–Lobatto quadrature is a kind of Gauss–Legendre quadrature procedure, since it is based on the Legendre polynomial $P_n(x)$. For more information please consult the celebrated work by Abramowitz–Stegun [1, pp. 888–900]. The related weights have been given up to $n = 11$ in [15, pp. 63–64].

Gaussian quadrature is used in the QUADPACK library, the GNU Scientific Library, the NAG Numerical Libraries and statistical software R , see [28].

Also, we draw the mathematically better prepared reader’s attention to the excellent monograph by Gil–Segura–Temme [24, §5.3: Gauss quadrature].

4. BINOMIAL SERIES EXPANSION OF THE INTEGRAND

The most common computational procedure concerns the binomial series expansion of the integrand in (1) (see e.g. Saupe’s paper [43]):

$$\begin{aligned} (1 - e^2 \sin^2 \varphi)^{-3/2} &= \sum_{n \geq 0} \binom{-3/2}{n} (-1)^n e^{2n} \sin^{2n} \varphi \\ &= 1 + \frac{3}{2} e^2 \sin^2 \varphi + \frac{15}{8} e^4 \sin^4 \varphi + \frac{35}{16} e^6 \sin^6 \varphi + \frac{315}{128} e^8 \sin^8 \varphi + \dots, \end{aligned} \quad (7)$$

the termwise integration applied after legitimate exchange the order of summation and integration gives the meridian distance L_m from the equator to a point P of latitude φ (the so-called Delambre expansion)

$$\begin{aligned} L_m = a(1 - e^2) \left\{ A\varphi - \frac{B}{2} \sin(2\varphi) + \frac{C}{4} \sin(4\varphi) \right. \\ \left. - \frac{D}{6} \sin(6\varphi) + \frac{E}{8} \sin(8\varphi) - \frac{F}{10} \sin(10\varphi) + \dots \right\}, \end{aligned} \quad (8)$$

where

$$\begin{aligned} A &= 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11\,025}{16\,384} e^8 + \frac{43\,659}{65\,536} e^{10} + \dots \\ B &= \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2\,205}{2\,048} e^8 + \frac{72\,765}{65\,536} e^{10} + \dots \\ C &= \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2\,205}{4\,096} e^8 + \frac{10\,395}{16\,384} e^{10} + \dots \\ D &= \frac{35}{512} e^6 + \frac{315}{2\,048} e^8 + \frac{31\,185}{131\,072} e^{10} + \dots \\ E &= \frac{315}{16\,384} e^8 + \frac{3\,465}{65\,536} e^{10} + \dots \\ F &= \frac{693}{131\,072} e^{10} + \dots \end{aligned}$$

Higher order partial sums in the expansion (7) gives additional concretized terms in (8), see e.g. [34, p. 1, Eq. (1)]. Deakin [17] reported that for the Geodetic Reference System 1980 (GRS80) ellipsoid ($a = 6\,378.137$ [km]; $b = 6\,356.752\,314\,140$ [km])² the coefficients

$$\begin{aligned} A &= 1.005\,052\,501\,813\,087, & B &= 0.005\,063\,108\,622\,224 \\ C &= 0.000\,010\,627\,590\,263, & D &= 0.000\,000\,020\,820\,379 \\ E &= 0.000\,000\,000\,039\,324, & F &= 0.000\,000\,000\,000\,071. \end{aligned}$$

Therefore the corresponding meridian arc length approximation formula, with terms kept up to degree 10, reads

$$\begin{aligned} L_m^{80} &= 6\,367.449\,139\,507\,109\,\varphi - 16.038\,508\,725\,491\,\sin(2\varphi) + 0.016\,832\,613\,310\,\sin(4\varphi) \\ &\quad - 0.000\,021\,984\,375\,\sin(6\varphi) + 3.1\,142 \cdot 10^{-8}\,\sin(8\varphi) - 4.49 \cdot 10^{-11}\,\sin(10\varphi). \end{aligned} \quad (9)$$

Multiplying coefficients by $1 - e^2$, Lauf [37, p. 36] transformed formula (8) into

$$L_m = a \{ A_0\varphi - A_2\sin(2\varphi) + A_4\sin(4\varphi) - A_6\sin(6\varphi) + A_8\sin(8\varphi) + \dots \}, \quad (10)$$

where

$$\begin{aligned} A_0 &= 1 - \frac{1}{4}e^2 - \frac{3}{64}e^4 - \frac{5}{256}e^6 - \frac{175}{16\,384}e^8 + \dots \approx 0.998\,324\,298\,423\,043 \\ A_2 &= \frac{3}{8} \left(e^2 + \frac{1}{4}e^4 + \frac{15}{128}e^6 + \frac{35}{512}e^8 + \dots \right) \approx 0.002\,514\,607\,124\,555 \\ A_4 &= \frac{15}{256} \left(e^4 + \frac{3}{4}e^6 + \frac{35}{64}e^8 + \dots \right) \approx 0.000\,002\,639\,111\,298 \\ A_6 &= \frac{35}{3072} \left(e^6 + \frac{5}{4}e^8 + \dots \right) \approx 0.000\,000\,003\,446\,837 \\ A_8 &= \frac{315}{131\,072} \left(e^8 + \dots \right) \approx 0.000\,000\,000\,004\,883 \quad \text{etc.} \end{aligned}$$

Powers of e higher than e^6 are ignored in (10). Thus, one obtains the official formula for meridian distance used by Geocentric Datum of Australia Technical Manual (ICSM 2002) [30]

$$\begin{aligned} \tilde{L}_m &= 6\,367.449\,145\,771\,052\,\varphi - 16.038\,508\,741\,588\,\sin(2\varphi) \\ &\quad + 0.016\,832\,613\,417\,\sin(4\varphi) - 0.000\,021\,984\,399\,\sin(6\varphi), \end{aligned} \quad (11)$$

which modestly differs from the L_m^{80} expression; also see [37, p. 36, Eq. (3.55)]³.

In 1837 the celebrated scholar Bessel [7] presented a meridian arc length formula, expressed in terms of the third flattening (or oblateness) of the globe ellipsoid $n = (a - b)/(a + b)$ (not equal to the Legendre modulus characteristic in $\Pi(n; \varphi|m)$). He rewrote (8) as

$$L_m^B = a(1 - n)^2(1 + n) \left\{ \left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 \right) \varphi - \left(\frac{3}{2}n + \frac{45}{16}n^3 + \frac{525}{128}n^5 \right) \sin(2\varphi) \right\}$$

²The currently used official WGS84 ellipsoid has equal a , but $b = 6\,356.752\,314\,2$ [km]; this modest difference does not influence the navigations calculations.

³Deakin pointed out that at $\varphi = 50^\circ$, meridian distances calculated with (9) and by the formula \tilde{L}_m (11) differ for $4.11 \cdot 10^{-4}$ [m], that is, "...GDA Technical Manual will give millimetre accuracy for latitudes covering Australia" [17].

$$+ \left(\frac{15}{16}n^2 + \frac{105}{64}n^4 \right) \sin(4\varphi) - \left(\frac{35}{48}n^3 + \frac{945}{768}n^5 \right) \sin(6\varphi) \Big\}, \quad (12)$$

The main advantage of (12) over the partial sum of order 10 of L_m in (8) is its good convergence, as $n \approx e^2/4$. Bessel's formula employed a reduced number of addends with approximately the same precision, see [34].

The German geodesist Helmert [25, pp. 44–48] gave an approximation formula for $L_m \approx L_m^H$ expressed *via* the same ellipsoid parameter n . Consider the series expansion ⁴

$$\begin{aligned} L_m = a(1-n)(1-n^2) & \left\{ \left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \dots \right) \varphi - \left(\frac{3}{2}n + \frac{45}{16}n^3 + \dots \right) \sin(2\varphi) \right. \\ & + \frac{1}{2} \left(\frac{15}{8}n^2 + \frac{105}{32}n^4 + \dots \right) \sin(4\varphi) - \frac{1}{3} \left(\frac{35}{16}n^3 + \dots \right) \sin(6\varphi) \\ & \left. + \frac{1}{4} \left(\frac{315}{128}n^4 + \dots \right) \sin(8\varphi) + \dots \right\}, \end{aligned}$$

Helmert multiplied coefficients by $(1-n^2)^2$ and rejected all terms beginning with $\sin(10\varphi)$. He made use of truncated coefficients excluding all terms involving powers of n greater than n^4 [18, p. 70, Eq. (226)], [34, p. 2, Eq. (6)]. Thus,

$$\begin{aligned} L_m^H = \frac{a}{1+n} & \left\{ \left(1 + \frac{1}{4}n^2 + \frac{1}{64}n^4 \right) \varphi - \left(\frac{3}{2}n - \frac{3}{16}n^3 \right) \sin(2\varphi) \right. \\ & \left. + \left(\frac{15}{16}n^2 - \frac{15}{64}n^4 \right) \sin(4\varphi) - \frac{35}{48}n^3 \sin(6\varphi) + \frac{315}{512}n^4 \sin(8\varphi) \right\}. \quad (13) \end{aligned}$$

This alternative Helmert's formula is essentially the same as [5, p. 50, Eq. (5.5)].

A way to compute the latitude φ given a meridian arc length L_m was solved by Lauf [37, p. 38, Eq. (3.72)] by series inversion technique:

$$\begin{aligned} \varphi = \sigma + \left(\frac{3}{2}n - \frac{27}{32}n^3 + \dots \right) \sin(2\sigma) & + \left(\frac{21}{16}n^2 - \frac{55}{32}n^4 + \dots \right) \sin(4\sigma) \\ & + \left(\frac{151}{96}n^3 + \dots \right) \sin(6\sigma) + \left(\frac{1097}{512}n^4 - \dots \right) \sin(8\sigma) + \dots, \end{aligned}$$

where

$$\sigma = \frac{L_m}{G}, \quad G = a(1-n)(1-n^2) \left(1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \dots \right).$$

Here G stands for the mean length of a meridian arc of one radian, see also [18, p. 71, Eq. (229)].

Applying numerical root-finding routines like Newton–Raphson, secant, regula falsi, iteration etc. methods to Helmert's formula (13), we can perform numerical latitude computation with given accuracy, consult Deakin–Hunter's Newton–Raphson iteration MATLAB routines [18, pp. 107–119, §3.2].

Belyakov–Kravtsova–Rappaport [6, p. 5] used two standard power series in n and m to evaluate $\Pi(n; \varphi|m)$. Following their traces, and bearing in mind that we take $n = m = e^2$,

⁴This series expansion in terms of n possesses a faster rate of convergence than the meridian distance formula (8) involving powers of e^2 .

their power series coincide with

$$\Pi(\varphi) = \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} A(j) \left(\frac{e}{2}\right)^{2j} \quad (14)$$

$$= \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} B(j) \left(-\frac{b^2}{4a^2}\right)^j. \quad (15)$$

where

$$A(j) = \int_0^\varphi \frac{\sin^{2j} t}{1 - e^2 \sin^2 t} dt, \quad B(j) = \int_0^\varphi \frac{\sec t \tan^{2j} t}{1 - e^2 \sin^2 t} dt.$$

Because series (14) is not useful for numerical computation for $e|\sin \varphi|$ near to 1 and series (15) slowly converges for $ba^{-1}|\tan \varphi|$ close to 1⁵, Franke [20] reformulated the method by Ai–Harrison [2] to evaluate $\Pi(\varphi)$ for a range of φ critical for the direct use of series (14), (15); he introduced the so-called "double-angle" approach remarking that "... *one application is sufficient for computational purposes*", see [20, p. 495].

We point out that by some longer routine calculation one arrives at

$$A(j) = \frac{\sin^{2j+1} \varphi}{2j+1} F_1\left(j + \frac{1}{2}; j+1, 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi\right) \quad (16)$$

$$B(j) = \frac{\sin^{2j+1} \varphi}{2j+1} F_1\left(j + \frac{1}{2}; \frac{1}{2}, 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi\right), \quad (17)$$

where both Appell's hypergeometric F_1 -functions of two variables converge, being $\Pi(\varphi)$ incomplete elliptic integral, that is, $|\varphi| < \pi/2$, consult APPENDIX B. Let us mention that F_1 is built-in in *Mathematica*, having code `AppellF1[a, b_1, b_2, c, x, y]`, which evaluate F_1 -values with the desired numerical precision inside the domain $|x| < 1, |y| < 1$. Now, approximating $\Pi(\varphi)$ by a suitably long partial sum of series (14), (15) in conjunction with (16), (17), we conclude a new appropriate numerical evaluation tool.

Finally, the convergence rate problem obviously occurs caused by slowly convergent binomial series (7) and the subsequent trigonometric series concluded by termwise integration. So, these methods seem to be inefficient for general input argument values [12, 21]. Moreover, preliminary trials show that the approach using the theta functions [1], is also inferior to the computation procedures applying Gauss or Bartky transformations [21, p. 1963].

5. POWER SERIES EXPANSIONS

The Maclaurin series expansion of the integrand (7) leads to

$$\Pi(\varphi) = \varphi + \frac{3e^2}{2}\varphi^3 + \left(\frac{3e^4}{8} - \frac{e^2}{10}\right)\varphi^5 + \left(\frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105}\right)\varphi^7 + R_8^M(\varphi), \quad (18)$$

⁵Actually, computational problems occur already for $ba^{-1}|\tan \varphi| \geq 0.7$, [20, p. 495].

where the error term in the Lagrange form reads

$$R_8^M(\varphi) = \frac{\Pi^{(8)}(\vartheta\varphi)}{40\,320}\varphi^8, \quad \vartheta \in (0, 1).$$

The related Maclaurin polynomial gives the low order, therefore expectedly not too precise, approximation formula

$$L_m^M = \frac{b^2}{a} \left\{ \varphi + \frac{3e^2}{2}\varphi^3 + \left(\frac{3e^4}{8} - \frac{e^2}{10} \right) \varphi^5 + \left(\frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105} \right) \varphi^7 \right\}, \quad (19)$$

which turns out to be the truncated partial sum of the power series expansion presented in APPENDIX B - up to the constant b^2/a . Obviously, close-to-nil φ guarantees higher precision employing (19).

Completely different kinds of series expansion results are established by Karp–Savenkova–Sitnik [32, 33]. They considered $\Pi(\varphi)$ on the range

$$\varphi \in \mathcal{D} = \left[-\max \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\}, \min \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\} \right],$$

see [32, p. 334, Theorem 1]. Related worked examples can be found in [33].

6. BULIRSCH, AND FUKUSHIMA–ISHIZAKI METHODS

Bulirsch introduced a new way to compute all three elliptic integrals F, E, Π in both complete and incomplete cases. (However, we concentrate here only on his results regarding elliptic integral of the third kind). His approach is based on the Bartky's transformation (actually a by-product of the familiar Landen, or Gauss–Landen transformation) [10]. Bulirsch gave the routine:

$$el3(x, k_c, p) = \int_0^{\arctan x} \frac{d\gamma}{(\cos^2 \gamma + p \sin^2 \gamma) \sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1, \quad (20)$$

where the main branch of the arctan is used, that is, $|\arctan x| \leq \pi/2$, and according to the notations introduced in (4), x, k_c, p correspond to $\tan \varphi, \sqrt{1 - e^2}, 1 - e^2$ respectively. Consequently (compare APPENDIX A):

$$\Pi(\varphi) = el3(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2).$$

The ALGOL procedure for $el3$ is available from [11].

Having cancellation problems with the complementary characteristic $p = 1 - n$ in $el3$, when the parameter $|n|$ of $\Pi(n; \varphi|m)$ is small, Bulirsch unified $el3$ by [11, p. 279, Eq. (4.1.14)]

$$el(x, k_c, p, a, b) = \int_0^{\arctan x} \frac{(a \cos^2 t + b \sin^2 t) d\gamma}{(\cos^2 \gamma + p \sin^2 \gamma) \sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1.$$

He did not give a related computation routine. Fukushima–Ishizaki extended $el(\cdot)$ to the ‘incomplete’ routine function [23, p. 240, Eq. (21)]

$$G(\varphi, n_c, m_c, a, b) = \int_0^\varphi \frac{(a \cos^2 \gamma + b \sin^2 \gamma) d\gamma}{(\cos^2 \gamma + n_c \sin^2 \gamma) \sqrt{\cos^2 \gamma + m_c \sin^2 \gamma}}, \quad (21)$$

where parameters n_c, m_c were introduced to avoid a loss of precision. Hence, they solved the cancellation problem with respect to n positively [23].

Obviouly (consult APPENDIX A)

$$\Pi(\varphi) = G(\varphi, 1 - e^2, 1 - e^2, 1, 1).$$

The complete description of the Fukushima–Ishizaki routine and the calculation algorithm is presented with companion numerical examples in [23].

7. DUPLICATION METHOD BY CARLSON

We have mentioned that Franke initiates a kind of duplication of angles in computing $\Pi(\varphi)$ with a desired accuracy, [20]. In a set of articles, see [13] and his references [3, 4, 5, 6] therein, Carlson developed a new approximation method by introducing a new set of so-called R -integrals in the standardized form. Concentrating on $\Pi(\varphi)$, the adequate standard Carlson integrals of the first and the third kind are [13, p. 2, Eqs. (1.1),(1.3)]

$$\begin{aligned} R_F(x, y, z) &= \frac{1}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} dt \\ R_J(x, y, z, \rho) &= \frac{3}{2} \int_0^\infty [(t+x)(t+y)(t+z)]^{-1/2} (t+\rho)^{-1} dt, \quad \rho \neq 0. \end{aligned}$$

These functions are symmetric in x, y, z and normalized so, that $R_F(x, x, x) = x^{-1/2}$, while $R_J(x, x, x, x) = x^{-3/2}$ and $R_J(0, y, z, \rho)$ is said to be complete. Two cases are signified as

$$R_C(x, y) = R_F(x, y, y), \quad R_D(x, y, z) = R_J(x, y, z, z). \quad (22)$$

The variables x, y, z are nonnegative and at most one of them is zero⁶. The link to $\Pi(\varphi)$ is [13, p. 8, Eq. (4.3)]

$$\Pi(\varphi) = \sin \varphi \cdot R_F(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1) + \frac{e^2}{3} \sin^3 \varphi \cdot R_D(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1). \quad (23)$$

The connection with Bulirsch $el3$ procedure becomes

$$\begin{aligned} \Pi(\varphi) &= el3(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2) = \tan \varphi \cdot R_F\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right) \\ &\quad + \frac{1 - e^2}{3} \tan^3 \varphi \cdot R_D\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right). \end{aligned}$$

⁶We have to point out that Carlson’s set of R -integrals cover a wide family of logarithmic, inverse circular, inverse hyperbolic functions, complete and incomplete elliptic integrals as special cases [13, §4].

Bearing in mind (23), we have to expose Carlson's computation procedure for the duplication formulae [14, Eqs. 19.26.18–19, 22–23]

$$R_F(x, y, z) = 2R_F(x + \lambda, y + \lambda, z + \lambda) = R_F\left(\frac{x + \lambda}{4}, \frac{y + \lambda}{4}, \frac{z + \lambda}{4}\right) \quad (24)$$

$$R_D(x, y, z) = 2R_D(x + \lambda, y + \lambda, z + \lambda) + \frac{3}{\sqrt{z}(z + \lambda)}, \quad (25)$$

where

$$\lambda = \sqrt{xy} + \sqrt{xz} + \sqrt{yz}.$$

After the differences between the variables have been made enough small by successive duplications by (24) and (25), the integrals R_F, R_D are expanded into a triple Maclaurin series approximated by their partial sums of order five. The resulting algorithms consist only of rational operations and square roots. The convergence is fast enough to be superior to methods possessing second order convergence, unless extremely high precision is required, [13, p. 2]. The related algorithms for R_F, R_C, R_J, R_D are given respectively in [13, pp. 2–5, **Algorithm 1-4.**], but they are too massive to be presented here in detail.

8. J -FUNCTION METHOD BY FUKUSHIMA.

Guided by certain practical reasons such as avoiding the loss of significant digits for small $|n|$, Fukushima [21] introduced a linear combination of the Legendre's incomplete elliptic integrals of the first (see APPENDIX A) and third kind instead of $\Pi(n; \varphi|m)$ itself:

$$J(n; \varphi|m) = \frac{1}{n} (\Pi(n; \varphi|m) - F(\varphi|m)) \quad \varphi \in \left(0, \frac{\pi}{2}\right), m, n \in (0, 1),$$

calling it *associated incomplete elliptic integral of the third kind*. Obviously, knowing $J(n; \varphi|m)$ and $F(\varphi|m)$ we have

$$\Pi(n; \varphi|m) = F(\varphi|m) + nJ(n; \varphi|m).$$

Fukushima outlined [21, p. 1963, §1.3] that the standard computation methods employed for $J(n; \varphi|m)$ are Bulirsch's, and Carlson's (applied to R_J). However, from the practical point of view, both methods suffer from either cancellation errors or from large amounts of computation, see the discussions around [21, p. 1964, Eqs. (19), (21)]; the comparison to Bulirsch's and Carlson's methods are given in [21, §4].

Fukushima's J -method is based on selecting one function from the four possible expressions:

$$J(n; \varphi|m) = \begin{cases} J(n; \varphi|m) & \text{(i)} \\ J(n|m) - J\left(n; \arcsin \frac{\cos \varphi}{\sqrt{1 - n \sin^2 \varphi}} \middle| m\right) - R_C(t_s^{-2}, h + t_s^{-2}) & \text{(ii)} \\ J(n|m) - J\left(n; \arccos \sqrt{\frac{1 - m}{1 - m \sin^2 \varphi}} \middle| m\right) - R_C(t_c^{-2}, h + t_c^{-2}) & \text{else} \end{cases}$$

where $J(n|m) = J(n; \pi/2|m)$ is the associate complete elliptic integral of the third kind and

$$\begin{cases} \text{(i)} & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{\sqrt{1+m}+1}} \\ \text{(ii)} & \frac{\cos \varphi}{\sqrt{1-n \sin^2 \varphi}} < y_s \end{cases},$$

while

$$h = n(1-n)(n-m), \quad t_s = \frac{\sin(2\varphi)}{2\sqrt{1-m \sin^2 \varphi}}, \quad t_c = \frac{t_s}{1-n}.$$

Also, y_s, φ_s are constants, where $y_s \in \{0.95, 0.90\}$ in the single and double precision computation modalities respectively, while $\varphi_s = \arcsin \sqrt{y_s} \in \{1.345, 1.249\}$.

In our environment writing $J(\varphi) \equiv J(e^2; \varphi|e^2)$, $J(e^2) \equiv J(\pi/2)$; $F(\varphi) = F(\varphi|e^2)$, we get the associate incomplete elliptic integral of the third kind

$$J(\varphi) = \frac{1}{n} (\Pi(\varphi) - F(\varphi)), \quad (26)$$

and the associated mathematical model

$$J(\varphi) = \begin{cases} J(\varphi) & \text{(i)} \\ J(e^2) - J\left(\arcsin \frac{\cos \varphi}{\sqrt{1-e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2\sqrt{1-e^2 \sin^2 \varphi}} & \text{(ii)} \\ J(e^2) - J\left(\arccos \frac{b}{a\sqrt{1-e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2(1-e^2)\sqrt{1-e^2 \sin^2 \varphi}} & \text{otherwise} \end{cases} \quad (27)$$

for

$$\begin{cases} \text{(i)} & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{\sqrt{1+e^2}+1}} \\ \text{(ii)} & \frac{\cos \varphi}{\sqrt{1-e^2 \sin^2 \varphi}} < y_s \end{cases}.$$

Indeed, $h = 0$ and $R_C(x, x) = R_F(x, x, x) = x^{-1/2}$, according to (22). Now, it remains to apply Fukushima's half and double argument transformations [21, pp. 1966–1970, §3.3–3.7] to the model (26) to earn the desired value of $J(\varphi)$ for φ already given.

9. LUKE'S PADÉ APPROXIMATION METHOD

Using Padé ⁷ approximations for the square root, Luke [38, 39] offered routines for the approximative evaluation of elliptic integrals of all three kinds. His expression looks like (in our setting $n = m = e^2$) [39, p. 193, Eqs. (26)]:

$$\Pi(\varphi) = \frac{H(\varphi, e^2)}{2n+1} \left\{ 1 + \sum_{j=1}^n \frac{2}{\cos^2 \frac{j\pi}{2n+1}} \right\}$$

⁷Henri Eugène Padé (1863–1953) a French mathematician who made important contributions to the theory of continued fractions and introduced what we call today Padé approximants, which are rational approximations to functions given by their power series.

$$\begin{aligned}
& -\frac{2}{2n+1} \sum_{j=1}^n H\left(\varphi, e^2 \sin^2 \frac{j\pi}{2n+1}\right) \tan^2 \frac{\pi j}{2n+1} + R_n^L(\varphi) \\
H(\tau, f) &= \frac{a}{b} \arctan\left(\frac{b}{a} \tan \tau\right) \\
R_n^L(\varphi) &= \frac{2e^{-(2n+1)\zeta} \tan \varphi}{(1-e^2 \sin^2 \varphi)(4n+3)} \left\{ 1 + \frac{\lambda_1}{4n+3} + \mathcal{O}(n^{-2}) \right\} + \mathcal{O}(e^{-4n\zeta}) \\
\lambda_1 &= 1 - \frac{\cosh \zeta - 1}{\sinh \zeta \cdot \cos^2 \varphi} - \frac{2e^2 \sin^2 \varphi \cdot (\cosh \zeta - 1)}{(1-e^2 \sin^2 \varphi) \sinh \zeta}.
\end{aligned}$$

Here $R_n^L(\varphi)$ is the error term which vanishes with growing n [39, p. 194, Eq. (28)]:

$$\lim_{n \rightarrow \infty} R_n^L(\varphi) = 0,$$

and e^ζ is defined by [39, p. 191]

$$e^\zeta = \frac{2 - e^2 \sin^2 \varphi + 2\sqrt{1 - e^2 \sin^2 \varphi}}{e^2 \sin^2 \varphi}.$$

We are drawing the reader's attention to the important difference in notations of the base of natural logarithms e , and the eccentricity e .

A further variant of this method, useful for φ near to $\pi/2$, was proposed in [39, pp. 194–196, Eqs. (29)–(40)].

10. COMPARISON OF THE PRESENTED METHODS

Illustration of the presented methods will be the comparison of meridian distance calculation's efficiency for composite trapezoidal (5) and Simpson's (6) rules for few subdivisions n . These employ Geodetic Reference System 1980 (GRS80) ellipsoid formula L_m^{80} (9), Geocentric Datum of Australia Technical Manual (ICSM 2002) official formula \tilde{L}_m (11), Bessel's L_m^B (12), Helmert's L_m^H (13) formulae, and the power series approximation formula L_m^M (18), using referent values of WGS84 ellipsoid characteristics $a = 6378.137$ [km], $e = 0.0818192$ and the latitude of Senj ⁸ $\varphi(S) = 45^\circ$ N. The corresponding value

$$\Pi(\pi/4) = \text{EllipticPi}[0.006694381, \pi/4, 0.006694381] = 0.786834838411$$

was evaluated by the **WolframAlpha** computational engine. By (1) we get ⁹

$$L_m = a(1 - e^2)\Pi(\pi/4) = 4984.944374286 \quad [km] = 2691.654630 \quad [M].$$

⁸Senj is a small town situated on the Adriatic coast near Rijeka, Croatia.

⁹Here and in what follows, all computed values of meridian distances are expressed in [km] units; by L_m [km] = $L_m/1.852$ [M] we clearly transform it into nautical miles.

First, by obvious rough estimates $1 - e^2 \sin^2(\vartheta\varphi) \geq \cos^2(\vartheta\varphi) \geq \cos^2 \varphi$, which hold true for all $\varphi \in (0, \pi/2)$, $\vartheta \in (0, 1)$, we conclude that

$$\begin{aligned} |R_n^T(\pi/4)| &\leq \left. \frac{3e^2(13e^2 + 8)\varphi^3}{32n^2 \cos^7 \varphi} \right|_{\varphi=\frac{\pi}{4}} = \frac{0.003430}{n^2} \\ |R_n^S(\pi/4)| &\leq \left. \frac{e^2\varphi^5(122.5e^6 + 200e^4 + 116e^2 + 4)}{960n^4 \cos^{11} \varphi} \right|_{\varphi=\frac{\pi}{4}} = \frac{0.000828557}{n^4}. \end{aligned}$$

Denote L_m^T, L_m^S the right-hand-side sums in the trapezoidal (5) and Simpson's (6) rules, respectively. We get the following table

n	L_m^T	$ R_n^T <$	L_m^S	$ R_m^s <$
2	4985.777412	$8.6 * 10^{-4}$	4984.907385	$5.2 * 10^{-5}$
3	4985.312473	$3.8 * 10^{-4}$	-	-
4	4985.151015	$2.1 * 10^{-4}$	4984.942218	$3.2 * 10^{-6}$
5	4985.076504	$1.3 * 10^{-4}$	-	-
6	4985.036084	$9.5 * 10^{-5}$	4984.943954	$6.4 * 10^{-7}$
10	4984.977367	$3.4 * 10^{-5}$	4984.944321	$8.3 * 10^{-8}$
20	4984.952620	$8.6 * 10^{-6}$	4984.944372	$5.2 * 10^{-9}$
50	4984.945696	$1.4 * 10^{-6}$	4984.944372	$1.3 * 10^{-10}$

Table 1. TRAPEZOIDAL AND SIMPSON'S RULES CALCULATIONS OF MERIDIAN DISTANCES

Next, we approximate L_m by $L_m^{80}, \tilde{L}_m, L_m^B$ and L_m^H by quoted formulae (9), (11), (12) and (13) respectively. First, the third flattening parameter of the globe WGS84 ellipsoid is equal to

$$n = \frac{a-b}{a+b} = 0.001679220.$$

This value, together with the associated eccentricity e transforms Bessel's, Helmert's and the Maclaurin expansion formulae into

$$\begin{aligned} L_m^B &= 6367.449148277502\varphi - 16.038504978751 \sin(2\varphi) \\ &\quad + 0.016832605426 \sin(4\varphi) - 0.000021984392 \sin(6\varphi) \\ L_m^H &= 6367.449148277505\varphi - 16.038504978753 \sin(2\varphi) \\ &\quad + 0.016832605523 \sin(4\varphi) - 0.000021984408 \sin(6\varphi) \\ &\quad + 0.00000031148 \sin(8\varphi) \\ L_m^M &= 6335.432364595861\varphi + 63.617697072503\varphi^3 \\ &\quad - 2.999026589075\varphi^5 + 0.353815715411\varphi^7. \end{aligned}$$

Latitude $\varphi = 45^\circ$ of Senj gives

$$L_m^{80} = 4984.944373 \quad [km], \quad \tilde{L}_m = 4984.944378 \quad [km]$$

$$L_m^B = L_m^H = 4\,984.944\,384 \quad [km], \quad L_m^M = 5\,005.826\,977 \quad [km].$$

Now, it is not hard to see all estimated values of L_m differ in $12 [mm]$; exceptions are the composite trapezoidal formula with approximately $1.3 [m]$ deviation, and the very slowly convergent Maclaurin polynomial approximant L_m^M with near to $21 [km]$ overlength. Therefore we conclude that all exposed and successfully examined formulae have the same navigational and geodesical significance and applicability.

11. CONCLUSION

In this article we presented, without using further computation, methods by Burlisch, Carlson, Fukushima–Ishizaki and Fukushima, since they are superior to the classical series expansion computation methods for the incomplete elliptic integral of the third kind $\Pi(\varphi)$.

The use of all presented methods is highly appreciated in numerous applications, because $\Pi(n; \varphi|m)$ appears in various mathematical models. These include the approximated Euler–Lagrange equation [3, p. 2655], [43, p. 817, Eq. (7)], the model of the gravitational or electromagnetic field associated with scalar or vector potential of a simple distribution such as annular disks with finite thickness [21]; the model of a magnetic field caused by thick coil [19]; the torque-free rotation of triaxial rigid body [22, Appendix C] and the periodic solutions of the Schrödinger equation [16].

Our main purpose here was to present enough precise computational tools and procedures for navigational calculations in both nautical and geodesical topics. Regarding nautical themata, has to be mentioned the very recent article by Weintrit–Kopacz [44] in which the authors gave an exhaustive up-to-date presentation, together with the numerous articles therein e.g. [46, 47]. We point out that all these articles work with formulae and routines in terms of the geographic latitude φ , the exception is Petrović’s note [42], where the meridian arc length has been expressed *via* the incomplete elliptic integral of the second kind as $L_m = a \cdot E(\theta|e^2)$, see [42, p. 88, Eq. (21)], also see the paper by Kos–Filjar–Hess [35, p. 959].

ACKNOWLEDGEMENTS

Both authors were partially supported by the *Ministry of Science, Education and Sports of the Republic of Croatia* under Research Projects No. 112-1121722-3066 and No. 112-1121722-3314, respectively.

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APPENDIX A. To express L_m explicitly, we employ the Legendre’s incomplete elliptic integrals of the first, second and third kind defined respectively by

$$\begin{aligned}
 F(z|m) &= \int_0^z \frac{dt}{\sqrt{1-m\sin^2 t}} \\
 E(z|m) &= \int_0^z \sqrt{1-m\sin^2 t} dt \\
 \Pi(n; z|m) &= \int_0^z \frac{dt}{(1-n\sin^2 t)\sqrt{1-m\sin^2 t}} \quad |z| \leq \frac{\pi}{2};
 \end{aligned}$$

related functions

$$F\left(\frac{\pi}{2} \middle| m\right) = F(m), \quad E\left(\frac{\pi}{2} \middle| m\right) = E(m), \quad \Pi\left(n; \frac{\pi}{2} \middle| m\right) = \Pi(n|m)$$

signify the complete elliptic integrals of all three kinds. Because the function Π is odd and quasi-periodic, the following properties hold [26]

$$\begin{aligned} \Pi(n; z|m) &= -\Pi(n; -z|n) \\ \Pi(n; z + k\pi|m) &= \Pi(n; z|m) + k \Pi(n|m), \quad k \in \mathbb{Z}, |n| \leq 1 \\ \Pi(0; z|m) &= F(z|m) \\ \Pi(n; z|n) &= \frac{1}{1-n} \left\{ E(z|n) - \frac{n \sin(2z)}{2\sqrt{1-n \sin^2 z}} \right\}. \end{aligned}$$

The transcendental equation $\Pi(n; z|n) = w$ has no explicit solution in z .

Rewriting (2) as

$$\Pi(n; \varphi|m) = \int_0^\varphi \frac{(\sin^2 t + \cos^2 t) dt}{(\cos^2 t + (1-n) \sin^2 t) \sqrt{\cos^2 t + (1-m) \sin^2 t}},$$

then introducing general parameters $p = 1 - n, k_c^2 = 1 - m, x = \tan \varphi$, Burilich in [8, 9, 10, 11], subsequently $a, b, n_c = 1 - n, m_c = 1 - m$, Fukushima-Ishizaki in [23] have proposed more general and redirected computation routines $el3$ (20) and G (21) respectively, for the incomplete elliptic function of the third kind Π .

APPENDIX B. The Maclaurin series expansion of the elliptic integral of the third kind coincides with a triple series [27]

$$\begin{aligned} \Pi(n; \varphi|m) &= \frac{\sqrt{\pi} (m/2)^{3/2}}{n(m - 2(1 + \sqrt{1-m}))^{3/2} \sqrt{1-n}} \sqrt{1 + \frac{1}{\sqrt{1-m}}} \sum_{q=0}^{\infty} \frac{(-4)^q}{(2q+1)!} \sum_{k=0}^{2q} \frac{S_{2q}^{(k)}}{2^k} \\ &\times \sum_{j=0}^k \frac{(-n)^{-j} j! \binom{k}{k-j}}{\Gamma(j - k + \frac{1}{2})} \frac{m^{k-j} ((1 + \sqrt{1-n})^{j+1} - (1 + \sqrt{1-n})^{j+1})}{(1 - \sqrt{1-m})^{k-j}} \\ &\times F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{3}{2}; j - k + \frac{1}{2}; \frac{1}{2} - \frac{1}{2\sqrt{1-m}}, 2 \frac{1 + \sqrt{1-m}}{m}\right) \varphi^{2q+1}, \end{aligned}$$

where the Stirling numbers of the second kind

$$S_b^{(a)} = \frac{(-1)^a}{a!} \sum_{m=0}^a (-1)^m \binom{a}{m} m^b, \quad a, b - 1 \in \mathbb{N}$$

and the Appell function [40, p. 21, Eq. (2.1.1)]

$$F_1(a; b, b'; c; x, y) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s} (b)_r (b')_s}{(c)_{r+s} r! s!} x^r y^s, \quad |x| < 1, |y| < 1,$$

play significant roles. Here the familiar Pochhammer symbol (or shifted factorial)

$$(a)_n = a(a+1)\cdots(a+n-1) \quad a \in \mathbb{C}, n \in \mathbb{N}_0 = \{0, 1, 2, \dots\},$$

and $0_0 = 1$ is used by convention. Specifying $n = m = e^2$, the previous formulae give (5).

APPENDIX C. The Legendre polynomial $y = P_n(x)$ is a solution of the Legendre differential equation, reads as follows

$$((1-x^2)y')' + n(n+1)y = 0;$$

the solution may be expressed by the Rodrigues formula [4, p. 252]

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{d^n x} (1-x^2)^n.$$

The associated Gauss–Legendre low order ($n = 1, 2, 3, 4$) quadrature weight–coefficients

$$w_j = \frac{2}{(1-t_j^2)[P'_n(t_j)]^2}, \quad j = \overline{1, n}, \quad n = 1, 2, 3, 4$$

are presented on the following table

n	nodes t_j	weights w_j
1	0	2
2	$\pm \frac{1}{\sqrt{3}}$	1
3	$\pm \sqrt{\frac{3}{5}}; 0$	$\frac{5}{9}; \frac{8}{9}$
4	$\pm \sqrt{\frac{1}{7}} \left(3 - 2\sqrt{\frac{6}{5}}\right); \pm \sqrt{\frac{1}{7}} \left(3 + 2\sqrt{\frac{6}{5}}\right)$	$\frac{1}{2} + \frac{\sqrt{30}}{36}; \frac{1}{2} - \frac{\sqrt{30}}{36}$

Table 2. LOW-ORDER GAUSS-LEGENDRE QUADRATURE WEIGHT-COEFFICIENTS