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Articles and Notes
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On Mathematics of Navigational Calculations for Meridian Sailing
Serdjo Kos and Tibor K. Pogany

Part 2. Geosocial Networking: A Case from Ann Arbor, Michigan
David E. Arlinghaus and Sandra L. Arlinghaus

A Short Note
Waldo Tobler

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Update on Varroa Mite Spread
Diana Sammataro

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QR Code Transformations
Sandra L. Arlinghaus

QR Code Alterations
Sandra L. Arlinghaus

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RECENT NEWS, 2012
2. The work above is the first volume in a series of books to be published by CRC Press in its series "Cartography, GIS, and Spatial Science: Theory and Practice." If you have an idea for a book to include, or wish to participate in some other way, please contact the series Editor, Sandra L. Arlinghaus.
3. Virtual Cemetery with William E. Arlinghaus; an ongoing project that continues in development run in the virtual world in parallel with the trust-funded model of a real-world cemetery.

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William C. Arlinghaus shared this link on creating mathematical notation: http://gadgetwise.blogs.nytimes.com/2012/08/03/tip-of-the-week-doing-math-in-windows-7/

David Gitterman shared this link on "Sandy's Island." http://www.tgdaily.com/general-science-brief/67686-island-undiscovered-by-scientists#.ULNV2g6YJJs.email

Karen Hart wrote to share this link to Mark Newman's cartograms of election results: http://www-personal.umich.edu/~mejn/election/2012/

Harold Moellering wrote to share this link on the Hilbert Curve map as possible motivation for other applications of its content: http://www.dcwg.org/new-hilbert-curve-maps-of-the-infections/

From Joseph Kerski: I created 2 StoryMaps (first two links below) in part so I could help others generate these and I thought you might be interested in the first one in particular. They have great potential for education.

Lost Detroit: http://www.josephkerski.com/storymaps/lostdetroit/
20 on 40: 20 Stops along 40 Degrees North Latitude--http://www.josephkerski.com/storymaps/40north/

More generally, link to the main storymaps page: http://storymaps.esri.com

This link is where people can access the templates and create their own storymaps (as well as browse the 100 or so that have already been created on a wide variety of topics).

1. ARCHIVE
2. Editorial Board, Advice to Authors, Mission Statement
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On the Mathematics of Navigational Calculations for Meridian Sailing

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Abstract. Meridian sailing is actually a special case of sailing uniquely determined by relative coordinates $\Delta \varphi, \Delta \lambda$. In this note we discuss explicit finding of certain navigational parameters. These include loxodrome distance, rhumb line and geographic and geocentric latitude of a position on the globe surface. The latter is expressed in terms of the inverse incomplete elliptic integral of the third kind $\Pi$; we discuss and survey certain traditional and recent methods of computation of this function.

Key Words: Meridian (Loxodrome) sailing, Incomplete elliptic integral of the third kind, Numerical quadrature, Binomial series expansion of the integrand, Bulirsch method, Carlson’s duplication, Fukushima’s $J$–method.

1. INTRODUCTION

From the nautical point of view, in surface navigation we have two basic ship tracking models when the trajectory is prescribed.

- The first is constant course navigation (or loxodromic navigation), when no course changes occur in a unit of time. Under this assumption we recognize two specific cases:
  - (i) the course line is a meridian (meridian sailing) and
  - (ii) the course line is a parallel or coincides with the equator.
- The second is navigation with a variable rhumb. When the course line cuts meridians poinwise at variable angles, the model is called orthodromic sailing.

We will focus here on loxodromic navigation and its specific case (i). However, because the distance between two parallel lines is the same, regardless of the meridian along which it is measured, the distance $L_m$ along any meridian between two parallels becomes a constant multiple $L_{lox} = \sec \alpha \cdot L_m$ of the secant of the constant angle $\alpha$ between the rhumb line and a meridian on the globe and the meridian distance $L_m$. Obviously, it is enough to study meridian sailing’s length.

Loxodromic navigation is set by a couple of relative coordinates

$$(\Delta \varphi, \Delta \lambda) = (\varphi_2 - \varphi_1, \lambda_2 - \lambda_1),$$

where $\varphi_1, \varphi_2,$ and $\lambda_1, \lambda_2$ stand for the absolute geographic coordinates of latitude and longitude respectively and index 1 denotes the departure, and 2 the arrival positions. The meridian sailing corresponds by definition to $\Delta \lambda = 0$, while the sign of $\Delta \varphi$ determines the general loxodromic course, which is $K_L = 0^\circ$ for positive, and $K_L = 180^\circ$ for negative $\Delta \varphi$. 

In this article we study the meridian (loxodromic) arc length on a meridian on the globe spheroid surface starting at latitude $\varphi_1$ and finishing at latitude $\varphi_2$. Without loss of generality we shall assume that $\varphi_1 = 0$, that is $\Delta \varphi = \varphi_2 \equiv \varphi$ can be taken, i.e. the navigation begins precisely on the equator. Indeed, we have $\Delta \varphi = (\varphi_2 - \psi) - (\varphi_1 - \psi)$, $\psi$ arbitrary, which reduces the meridian arc length calculation to the case $\varphi_1 \equiv 0$.

2. Meridian Sailing on the Ellipsoid of Revolution

Figure 1 presents the ellipsoid of revolution (two-axis or rotational ellipsoid) model of the globe with the equatorial radius $a$ and polar radius $b$. The great ellipse is a planar cross-section of the ellipsoid and the plane containing the symmetry axis. (The associated numerical eccentricity we denote by the usual $e^2 = 1 - b^2/a^2 \leq 1$). On the other hand $\theta$ signifies the geocentric latitude of the same position $P$. Let us mention that $\tan \theta = (1 - e^2) \tan \varphi$.

Figure 1. Meridian Ellipse on the Ellipsoid of Revolution.

The meridian ellipse arc length $L_m$ characterizes the path between departure position $Q$ at the equator and the arrival position $P$ and reads as follows

$$L_m = a(1 - e^2) \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}} = \frac{b^2 \Pi(e^2; \varphi | e^2)}{a},$$

where $\Pi$ denotes the Legendre incomplete elliptic integral of the third kind defined by

$$\Pi(n; \varphi | m) = \int_0^\varphi \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - m \sin^2 t}} \quad |\varphi| \leq \frac{\pi}{2},$$

see Appendix A. Here $\varphi$ is the geographic latitude of $P$ \(^1\), and for the sake of simplicity, here and in what follows, we write $\Pi(\cdot) \equiv \Pi(e^2; \cdot | e^2)$, see [41]. The special function $\Pi$ is

\(^1\)According to the remark upon rhumb line course $\alpha \neq 90^\circ$ we have [45, p. 134, Eq. (6)] that $L_{\text{lox}} = L_m \cdot \sec \alpha$. Therefore

$$L_{\text{lox}} = \frac{L_m}{\cos \alpha} = \frac{b^2 \Pi(\varphi)}{a \cdot \cos \alpha}.$$  

For $\alpha = 90^\circ$ we have equator, or parallel sailing, while for $\alpha = 0^\circ$ it is $L_{\text{lox}} \equiv L_m$. 

continuous and odd on the range $|\varphi| \leq \pi/2$, and being
\[
\frac{d}{d\varphi} \Pi(n; \varphi|m) = \frac{1}{(1 - n \sin^2 \varphi)\sqrt{1 - m \sin^2 \varphi}} > 0 \quad n, m \in (0, 1),
\]
it is monotone increasing function in $\varphi$. The monotonicity property implies invertibility, that is, $w = \Pi(n; z|m)$ possesses an unique inverse $z = \Pi^{-1}(n; w|m)$. Now, by virtue of (1) we clearly conclude that the meridian distance between positions $Q, P$ corresponds to the latitude
\[
\varphi = \Pi^{-1}\left(\frac{a L_m}{b^2}\right).
\]
Citations regarding inverse elliptic integral of the third kind are very rare; Allender–Hornreich–Johnson [3, p. 2655] refer to Saupe [43, p. 817, Eq. (7)] concerning the exact solution for the approximated Euler–Lagrange equation which describes the free energy of the uniformly distorted Fréedericksz state, remarking that the exact solution of approximated EL-equation "... it is expressed ... as an inverse elliptic function of the third kind. This is inconvenient for numerical calculations...”. They considered binomial type approximations for $\Pi^{-1}(w) \approx C \cdot (1 - w^\rho)$, $\rho$ real, $C$ an absolute constant (see [3, p. 2657]). However, Saupe followed the usual computational way expressed in section 4.

Bearing in mind the fact that there is no explicit formula for $\Pi$ and $\Pi^{-1}$ expressed via certain elementary or higher transcendental functions, we are forced to compute it by numerical methods with prescribed accuracy. So, the meridian arc length $L_m = L_m(\varphi)$ we compute by finding the associated $\Pi$ function value in (1). Similarly, solving the inverse problem $\Pi^{-1}$ in (3) we can calculate it only by certain numerical computation procedures that have the computation error under the prescribed accuracy level. Let us remark that e.g. the computational softwares contain built–in subroutines such as EllipticPi[n, z, m] in Mathematica. In the next chapters we will survey these questions with a list of related links.

3. Computing $\Pi(\varphi)$ by Numerical Quadratures

To compute the meridian arc length $L_m$, reduce the problem to one of finding values of the elliptic integral of the third kind
\[
\Pi(\varphi) = \int_0^\varphi \frac{dt}{(1 - e^2 \sin^2 t)^{3/2}}. \tag{4}
\]
Based on a polynomial interpolation of the integrand using uniform grid, we calculate $\Pi(\varphi)$ by certain of the familiar numerical quadratures such as Newton–Cotes formulæ, represented by composite rectangle and trapezoidal rules (constant and linear approximation), Simpson’s formula (quadratic polynomial approximation); Gaussian quadrature formula and its numerous variants e.g. Gauss–Legendre, Gauss–Kronrod, Gauss–Lobatto rules etc. For instance,
the composite trapezoidal formula for \( \Pi(\varphi) \) is

\[
\Pi(\varphi) = \frac{\varphi}{2n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 2 \sum_{j=1}^{n-1} \left[ \left( 1 - e^2 \sin^2 \left( \frac{\varphi j}{n} \right) \right)^{-3/2} \right] \right\} + R_n^T(\varphi),
\]

where the error term is

\[
R_n^T(\varphi) = \frac{3e^2 \varphi^3 [4(e^2 - 2) \cos(2\vartheta \varphi) + e^2(2 \cos(4\vartheta \varphi) - 7)]}{32n^2(1 - e^2 \sin^2(\vartheta \varphi))^7/2}, \quad \vartheta \in (0, 1).
\]

Knowing the computation accuracy \( \varepsilon > 0 \), we can estimate the grids size \( n \) from \( R_n(\varphi) < \varepsilon \), for any fixed \( \varphi \). We mention that the composite Simpson’s formula in our case reduces to

\[
\Pi(\varphi) = \frac{\varphi}{6n} \left\{ 1 + (1 - e^2 \sin^2 \varphi)^{-3/2} + 4 \sum_{j=1}^{n-1} \left[ 1 - e^2 \sin^2 \left( \frac{\varphi j}{n} \right) \right]^{-3/2} \right\} + R_n^S(\varphi),
\]

and the error term becomes

\[
R_n^S(\varphi) = \frac{\varphi^5}{2880 n^4} \left( (1 - e^2 \sin^2 t)^{-3/2} \right)_{t=\xi}^{t=\varphi}
= \frac{3e^2 \varphi^5}{2880 n^4 (1 - e^2 \sin^2 \xi)^{11/2}} \left[ 11 e^6 \sin^8 \xi + 7.5 e^6 \sin^4(2\xi) - 18 e^4 \sin^6 \xi + 2 \cos^2 \xi \cdot (52 e^6 \sin^6 \xi - e^4 \sin^4 \xi - 49 e^2 \sin^2 \xi - 2) + 3e^2 \sin^4 \xi + 15 e^2 \cos^4 \xi \cdot (12 e^2 \sin^2 \xi + 1) + 4 \sin^2 \xi \right], \quad \xi \in (0, \varphi).
\]

Also by \( |R_n^S(\varphi)| < \varepsilon \) we evaluate \( n \) and then proceed to (6) with some fixed \( \varphi \). We note that these two numerical quadratures are built–in [29].

Transforming the integration interval \([0, \varphi], \varphi \leq \pi/2\) onto \([-1, 1]\) putting \( \varphi(t + 1)/2 \mapsto t \) in the integrand of (4), the Gauss–Legendre quadrature rule gives the approximation

\[
\Pi(\varphi) = \frac{\varphi}{2} \int_{-1}^{1} \left[ 1 - e^2 \sin^2 \left( \frac{\varphi}{2} (t + 1) \right) \right]^{-3/2} dt
= \frac{\varphi}{2} \sum_{j=1}^{n} w_j \left[ 1 - e^2 \sin^2 \left( \frac{\varphi}{2} (t_j + 1) \right) \right]^{-3/2} + R_n^{GL}(\varphi),
\]

where the weight–coefficients

\[
w_j = \frac{2}{(1 - t_j^2)(P'_n(t_j))^2}, \quad j = 1, n
\]

are based on the Legendre polynomial \( P_n(x) \), see APPENDIX C. The related quadrature error formula turns out to be

\[
R_n^{GL}(\varphi) = \frac{(n!)^4 \varphi^{2n+1}}{(2n+1)[(2n)!]^3} \left( (1 - e^2 \sin^2 t)^{-3/2} \right)_{t=\xi}^{(2n)} \quad \xi \in (0, \varphi).
\]
For other kinds of Gaussian quadratures, e.g. Gauss–Kronrod quadrature, we refer to the original Kronrod’s monograph [36]; also see the book [31] by Kahaner–Moler–Nash.

The Gauss–Lobatto quadrature is a kind of Gauss–Legendre quadrature procedure, since it is based on the Legendre polynomial $P_n(x)$. For more information please consult the celebrated work by Abramowitz–Stegun [1, pp. 888–900]. The related weights have been given up to $n = 11$ in [15, pp. 63–64].

Gaussian quadrature is used in the QUADPACK library, the GNU Scientific Library, the NAG Numerical Libraries and statistical software R, see [28].

Also, we draw the mathematically better prepared reader’s attention to the excellent monograph by Gil–Segura–Temme [24, §5.3: Gauss quadrature].

4. Binomial Series Expansion of the Integrand

The most common computational procedure concerns the binomial series expansion of the integrand in (1) (see e.g. Saupe’s paper [43]):

\[
(1 - e^2 \sin^2 \varphi)^{-3/2} = \sum_{n=0}^{\infty} \binom{-3/2}{n} (-1)^n e^{2n} \sin^{2n} \varphi
\]

\[
= 1 + \frac{3}{2} e^2 \sin^2 \varphi + \frac{15}{8} e^4 \sin^4 \varphi + \frac{35}{16} e^6 \sin^6 \varphi + \frac{315}{128} e^8 \sin^8 \varphi + \cdots ,
\]

(7)

the termwise integration applied after legitimate exchange the order of summation and integration gives the meridian distance $L_m$ from the equator to a point $P$ of latitude $\varphi$ (the so–called Delambre expansion)

\[
L_m = a(1 - e^2) \left\{ A \varphi - \frac{B}{2} \sin(2 \varphi) + \frac{C}{4} \sin(4 \varphi) \right.
\]

\[
- \frac{D}{6} \sin(6 \varphi) + \frac{E}{8} \sin(8 \varphi) - \frac{F}{10} \sin(10 \varphi) + \cdots \right\},
\]

(8)

where

\[
A = 1 + \frac{3}{4} e^2 + \frac{45}{64} e^4 + \frac{175}{256} e^6 + \frac{11 025}{16 384} e^8 + \frac{43 659}{65 536} e^{10} + \cdots
\]

\[
B = \frac{3}{4} e^2 + \frac{15}{16} e^4 + \frac{525}{512} e^6 + \frac{2 205}{2 048} e^8 + \frac{72 765}{65 536} e^{10} + \cdots
\]

\[
C = \frac{15}{64} e^4 + \frac{105}{256} e^6 + \frac{2 205}{4 096} e^8 + \frac{10 395}{16 384} e^{10} + \cdots
\]

\[
D = \frac{35}{512} e^6 + \frac{315}{2 048} e^8 + \frac{3 1185}{131 072} e^{10} + \cdots
\]

\[
E = \frac{3465}{16 384} e^8 + \frac{131 072}{65 536} e^{10} + \cdots
\]

\[
F = \frac{693}{131 072} e^{10} + \cdots
\]
Higher order partial sums in the expansion (7) gives additional concretized terms in (8), see e.g. [34, p. 1, Eq. (1)]. Deakin [17] reported that for the Geodetic Reference System 1980 (GRS80) ellipsoid \((a = 6378.137 \text{ [km]}; b = 6356.752314140 \text{ [km]}) \) the coefficients
\[
A = 1.00552501813087, \quad B = 0.00506310862224 \\
C = 0.000010627590263, \quad D = 0.000000020820379 \\
E = 0.000000000039324, \quad F = 0.000000000000071.
\]
Therefore the corresponding meridian arc length approximation formula, with terms kept up to degree 10, reads
\[
L^\text{80}_m = 6367.449139507109 \varphi - 16.038508725491 \sin(2\varphi) + 0.016832613310 \sin(4\varphi) \\
- 0.00021984375 \sin(6\varphi) + 3.1142 \times 10^{-8} \sin(8\varphi) - 4.49 \times 10^{-11} \sin(10\varphi).
\] (9)

Multiplying coefficients by \(1 - e^2\), Lauf [37, p. 36] transformed formula (8) into
\[
L_m = a \{A_0 \varphi - A_2 \sin(2\varphi) + A_4 \sin(4\varphi) - A_6 \sin(6\varphi) + A_8 \sin(8\varphi) + \cdots \},
\] (10)
where
\[
A_0 = 1 - \frac{1}{4} e^2 - \frac{3}{64} e^4 - \frac{5}{256} e^6 - \frac{175}{16384} e^8 + \cdots \approx 0.998324298423043 \\
A_2 = 3 \left(\frac{e^2}{8} + \frac{e^4}{128} + \frac{15}{512} e^6 + \cdots \right) \approx 0.002514607124555 \\
A_4 = \frac{15}{256} \left(\frac{e^4}{4} + \frac{5}{64} e^6 + \cdots \right) \approx 0.000002639111298 \\
A_6 = \frac{35}{3072} \left(\frac{e^6}{4} + \frac{5}{64} e^8 + \cdots \right) \approx 0.0000000046837 \\
A_8 = \frac{315}{131072} \left(e^8 + \cdots \right) \approx 0.0000000004883 \text{ etc.}
\]
Powers of \(e\) higher than \(e^6\) are ignored in (10). Thus, one obtains the official formula for meridian distance used by Geocentric Datum of Australia Technical Manual (ICSM 2002) [30]
\[
\tilde{L}_m = 6367.449145771052 \varphi - 16.038508741588 \sin(2\varphi) \\
+ 0.016832613417 \sin(4\varphi) - 0.00021984399 \sin(6\varphi),
\] (11)
which modestly differs from the \(L^\text{80}_m\) expression; also see [37, p. 36, Eq. (3.55)]

In 1837 the celebrated scholar Bessel [7] presented a meridian arc length formula, expressed in terms of the third flattening (or oblateness) of the globe ellipsoid \(n = (a - b)/(a + b)\) (not equal to the Legendre modulus characteristic in \(\Pi(n; \varphi|m)\)). He rewrote (8) as
\[
L^B_m = a(1 - n)^2(1 + n) \left\{\left(\frac{1}{2} + \frac{9}{4} n^2 + \frac{225}{64} n^4\right) \varphi - \left(\frac{3}{2} n + \frac{45}{16} n^3 + \frac{525}{128} n^5\right) \sin(2\varphi)\right\}
\]

\(^2\)The currently used official WGS84 ellipsoid has equal \(a\), but \(b = 6356.7523142 \text{ [km]}\); this modest difference does not influence the navigations calculations.

\(^3\)Deakin pointed out that at \(\varphi = 50^\circ\), meridian distances calculated with (9) and by the formula \(\tilde{L}_m\) (11) differ for \(4.11 \times 10^{-4} \text{ [m]}\), that is, “...GDA Technical Manual will give millimetre accuracy for latitudes covering Australia” [17].
\[
\left\{ \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \right) \varphi - \left( \frac{3}{2}n + \frac{45}{16}n^3 + \cdots \right) \sin(2\varphi) \right. \\
\left. + \frac{1}{2} \left( \frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots \right) \sin(4\varphi) - \frac{1}{3} \left( \frac{35}{16}n^3 + \cdots \right) \sin(6\varphi) \right. \\
\left. \left. + \frac{1}{4} \left( \frac{315}{128}n^4 + \cdots \right) \sin(8\varphi) \right) \right\},
\]
(12)

The main advantage of (12) over the partial sum of order 10 of \( L_m \) in (8) is its good convergence, as \( n \approx e^2/4 \). Bessel’s formula employed a reduced number of addends with approximately the same precision, see [34].

The German geodesist Helmert [25, pp. 44–48] gave an approximation formula for \( L_m \approx L_m^H \) expressed via the same ellipsoid parameter \( n \). Consider the series expansion

\[
L_m = a(1-n)(1-n^2)\left\{ \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \right) \varphi - \left( \frac{3}{2}n + \frac{45}{16}n^3 + \cdots \right) \sin(2\varphi) \right. \\
\left. + \frac{1}{2} \left( \frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots \right) \sin(4\varphi) - \frac{1}{3} \left( \frac{35}{16}n^3 + \cdots \right) \sin(6\varphi) \right. \\
\left. \left. + \frac{1}{4} \left( \frac{315}{128}n^4 + \cdots \right) \sin(8\varphi) \right) \right\},
\]
(13)

Helmert multiplied coefficients by \((1-n^2)^2\) and rejected all terms beginning with \( \sin(10\varphi) \). He made use of truncated coefficients excluding all terms involving powers of \( n \) greater than \( n^4 \) [18, p. 70, Eq. (226)], [34, p. 2, Eq. (6)]. Thus,

\[
L_m^H = \frac{a}{1+n}\left\{ \left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \right) \varphi - \left( \frac{3}{2}n + \frac{45}{16}n^3 + \cdots \right) \sin(2\varphi) \right. \\
\left. + \frac{1}{2} \left( \frac{15}{8}n^2 + \frac{105}{32}n^4 + \cdots \right) \sin(4\varphi) - \frac{1}{3} \left( \frac{35}{16}n^3 + \cdots \right) \sin(6\varphi) \right. \\
\left. \left. + \frac{1}{4} \left( \frac{315}{128}n^4 + \cdots \right) \sin(8\varphi) \right) \right\},
\]

This alternative Helmert’s formula is essentially the same as [5, p. 50, Eq. (5.5)].

A way to compute the latitude \( \varphi \) given a meridian arc length \( L_m \) was solved by Lauf [37, p. 38, Eq. (3.72)] by series inversion technique:

\[
\varphi = \sigma + \left( \frac{3}{2}n - \frac{27}{32}n^3 + \cdots \right) \sin(2\sigma) + \left( \frac{21}{16}n^2 - \frac{55}{32}n^4 + \cdots \right) \sin(4\sigma) \\
+ \left( \frac{151}{96}n^3 + \cdots \right) \sin(6\sigma) + \left( \frac{1097}{512}n^4 + \cdots \right) \sin(8\sigma) + \cdots,
\]

where

\[
\sigma = \frac{L_m}{G}, \quad G = a(1-n)(1-n^2)\left( 1 + \frac{9}{4}n^2 + \frac{225}{64}n^4 + \cdots \right).
\]

Here \( G \) stands for the mean length of a meridian arc of one radian, see also [18, p. 71, Eq. (229)].

Applying numerical root–finding routines like Newton–Raphson, secant, regula falsi, iteration etc. methods to Helmert’s formula (13), we can perform numerical latitude computation with given accuracy, consult Deakin–Hunter’s Newton–Raphson iteration MATLAB routines [18, pp. 107–119, §3.2].

Belyakov–Kravtsova–Rappaport [6, p. 5] used two standard power series in \( n \) and \( m \) to evaluate \( \Pi(n; \varphi|m) \). Following their traces, and bearing in mind that we take \( n = m = e^2 \),

\footnote{This series expansion in terms of \( n \) posseses a faster rate of convergence than the meridian distance formula (8) involving powers of \( e^2 \).}
their power series coincide with

\[
\Pi(\varphi) = \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} A(j) \left( \frac{e}{2} \right)^{2j}
\]

(14)

\[
= \sum_{j=0}^{\infty} \frac{(2j)!}{(j!)^2} B(j) \left( -\frac{b^2}{4a^2} \right)^{j}.
\]

(15)

where

\[
A(j) = \int_0^{\varphi} \frac{\sin^{2j+1} t}{1 - e^2 \sin^2 t} \, dt,
\]

\[
B(j) = \int_0^{\varphi} \frac{\sec t \tan^{2j+1} t}{1 - e^2 \sin^2 t} \, dt.
\]

Because series (14) is not useful for numerical computation for \(|e| \sin \varphi| \) near to 1 and series (15) slowly converges for \(ba^{-1} |\tan \varphi| \) close to 1 \(^5\), Franke [20] reformulated the method by Ai–Harrison [2] to evaluate \(\Pi(\varphi)\) for a range of \(\varphi\) critical for the direct use of series (14), (15); he introduced the so-called "double–angle" approach remarking that "... one application is sufficient for computational purposes", see [20, p. 495].

We point out that by some longer routine calculation one arrives at

\[
A(j) = \frac{\sin^{2j+1} \varphi}{2j + 1} \text{F}_1 \left( j + \frac{1}{2}; j + 1, 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi \right)
\]

(16)

\[
B(j) = \frac{\sin^{2j+1} \varphi}{2j + 1} \text{F}_1 \left( j + \frac{1}{2}; 1; j + \frac{3}{2}; \sin^2 \varphi, e^2 \sin^2 \varphi \right),
\]

(17)

where both Appell’s hypergeometric F1–functions of two variables converge, being \(\Pi(\varphi)\) incomplete elliptic integral, that is, \(|\varphi| < \pi/2\), consult Appendix B. Let us mention that \(\text{F}_1\) is built–in in Mathematica, having code \text{AppellF1}[a, b_1, b_2, c, x, y], which evaluate \(\text{F}_1\)–values with the desired numerical precision inside the domain \(|x| < 1, |y| < 1\). Now, approximating \(\Pi(\varphi)\) by a suitably long partial sum of series (14), (15) in conjunction with (16), (17), we conclude a new appropriate numerical evaluation tool.

Finally, the convergence rate problem obviously occurs caused by slowly convergent binomial series (7) and the subsequent trigonometric series concluded by termwise integration. So, these methods seem to be inefficient for general input argument values [12, 21]. Moreover, preliminary trials show that the approach using the theta functions [1], is also inferior to the computation procedures applying Gauss or Bartky transformations [21, p. 1963].

5. Power Series Expansions

The Maclaurin series expansion of the integrand (7) leads to

\[
\Pi(\varphi) = \varphi + \frac{3e^2}{2} \varphi^3 + \left( \frac{3e^4}{8} - \frac{e^2}{10} \right) \varphi^5 + \left( \frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105} \right) \varphi^7 + R_M^M(\varphi),
\]

(18)

\(^5\)Actually, computational problems occur already for \(ba^{-1} |\tan \varphi| \geq 0.7\), [20, p. 495].
where the error term in the Lagrange form reads
\[ R_8^M(\varphi) = \frac{\Pi^{(8)}(\vartheta \varphi)}{40320} \varphi^8, \quad \vartheta \in (0, 1). \]

The related Maclaurin polynomial gives the low order, therefore expectedly not too precise, approximation formula
\[ L_m = \frac{b^2}{a} \left\{ \varphi + \frac{3e^2}{2} \varphi^3 + \left( \frac{3e^4}{8} - \frac{e^2}{10} \right) \varphi^5 + \left( \frac{35e^6}{112} - \frac{5e^4}{28} + \frac{e^2}{105} \right) \varphi^7 \right\}, \tag{19} \]
which turns out to be the truncated partial sum of the power series expansion presented in Appendix B - up to the constant \( b^2/a \). Obviously, close-to-nil \( \varphi \) guarantees higher precision employing (19).

Completely different kinds of series expansion results are established by Karp–Savenkova–Sitnik [32, 33]. They considered \( \Pi(\varphi) \) on the range \( \varphi \in \mathcal{D} = \left[ -\max \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\}, \min \left\{ \frac{\pi}{2}, \arcsin \frac{a}{\sqrt{a^2 + b^2}} \right\} \right] \), see [32, p. 334, Theorem 1]. Related worked examples can be found in [33].


Bulirsch introduced a new way to compute all three elliptic integrals \( F, E, \Pi \) in both complete and incomplete cases. (However, we concentrate here only on his results regarding elliptic integral of the third kind). His approach is based on the Bartky’s transformation (actually a by-product of the familiar Landen, or Gauss–Landen transformation) [10]. Bulirsch gave the routine:
\[ \text{el3}(x, k_c, p) = \int_0^{\arctan x} \frac{d\gamma}{(\cos^2 \gamma + p \sin^2 \gamma)\sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1, \tag{20} \]
where the main branch of the arctan is used, that is, \( |\arctan x| \leq \pi/2 \), and according to the notations introduced in (4), \( x, k_c, p \) correspond to \( \tan \varphi, \sqrt{1 - e^2}, 1 - e^2 \) respectively. Consequently (compare Appendix A):
\[ \Pi(\varphi) = \text{el3}(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2). \]

The ALGOL procedure for el3 is available from [11].

Having cancellation problems with the complementary characteristic \( p = 1 - n \) in el3, when the parameter \( |n| \) of \( \Pi(n; \varphi|m) \) is small, Bulirsch unified el3 by [11, p. 279, Eq. (4.1.14)]
\[ \text{el}(x, k_c, p, a, b) = \int_0^{\arctan x} \frac{(a \cos^2 t + b \sin^2 t)dt}{(\cos^2 \gamma + p \sin^2 \gamma)\sqrt{\cos^2 \gamma + k_c^2 \sin^2 \gamma}}, \quad |k_c| \leq 1. \]
He did not give a related computation routine. Fukushima–Ishizaki extended \( el(\cdot) \) to the ‘incomplete’ routine function [23, p. 240, Eq. (21)]

\[
G(\varphi, n_c, m_c, a, b) = \int_0^\varphi \frac{(a \cos^2 \gamma + b \sin^2 \gamma)}{(\cos^2 \gamma + n_c \sin^2 \gamma)\sqrt{\cos^2 \gamma + m_c \sin^2 \gamma}} \, d\gamma,
\]

(21)

where parameters \( n_c, m_c \) were introduced to avoid a loss of precision. Hence, they solved the cancellation problem with respect to \( n \) positively [23].

Obviously (consult Appendix A)

\[\Pi(\varphi) = G(\varphi, 1 - e^2, 1 - e^2, 1, 1).\]

The complete description of the Fukushima–Ishizaki routine and the calculation algorithm is presented with companion numerical examples in [23].

7. Duplication Method by Carlson

We have mentioned that Franke initiates a kind of duplication of angles in computing \( \Pi(\varphi) \) with a desired accuracy, [20]. In a set of articles, see [13] and his references [3, 4, 5, 6] therein, Carlson developed a new approximation method by introducing a new set of so-called \( R \)-integrals in the standardized form. Concentrating on \( \Pi(\varphi) \), the adequate standard Carlson integrals of the first and the third kind are [13, p. 2, Eqs. (1.1), (1.3)]

\[
R_F(x, y, z) = \frac{1}{2} \int_0^\infty \left[(t + x)(t + y)(t + z)\right]^{-1/2} \, dt
\]

\[
R_J(x, y, z, \rho) = \frac{3}{2} \int_0^\infty \left[(t + x)(t + y)(t + z)\right]^{-1/2}(t + \rho)^{-1} \, dt, \quad \rho \neq 0.
\]

These functions are symmetric in \( x, y, z \) and normalized so, that \( R_F(x, x, x) = x^{-1/2} \), while \( R_J(x, x, x, x) = x^{-3/2} \) and \( R_J(0, y, z, \rho) \) is said to be complete. Two cases are signified as

\[
R_C(x, y) = R_F(x, y, y), \quad R_D(x, y, z) = R_J(x, y, z, z).
\]

(22)

The variables \( x, y, z \) are nonnegative and at most one of them is zero \(^6\). The link to \( \Pi(\varphi) \) is [13, p. 8, Eq. (4.3)]

\[
\Pi(\varphi) = \sin \varphi \cdot R_F(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1) + \frac{e^2}{3} \sin^3 \varphi \cdot R_D(\cos^2 \varphi, 1 - e^2 \sin^2 \varphi, 1).
\]

(23)

The connection with Bulirsch \( el3 \) procedure becomes

\[
\Pi(\varphi) = el3(\tan \varphi, \sqrt{1 - e^2}, 1 - e^2) = \tan \varphi \cdot R_F\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right) + \frac{1 - e^2}{3} \tan^3 \varphi \cdot R_D\left(1, \frac{1 - e^2 \sin^2 \varphi}{\cos^2 \varphi}, \frac{1}{\cos^2 \varphi}\right).
\]

\(^6\)We have to point out that Carlson’s set of \( R \)-integrals cover a wide family of logarithmic, inverse circular, inverse hyperbolic functions, complete and incomplete elliptic integrals as special cases [13, §4].
Bearing in mind (23), we have to expose Carlson’s computation procedure for the duplication formulae [14, Eqs. 19.26.18–19, 22–23]

\[ R_F(x, y, z) = 2R_F(x + \lambda, y + \lambda, z + \lambda) = R_F\left(\frac{x + \lambda}{4}, \frac{y + \lambda}{4}, \frac{z + \lambda}{4}\right) \]  
\[ R_D(x, y, z) = 2R_D(x + \lambda, y + \lambda, z + \lambda) + \frac{3}{\sqrt{z}(z + \lambda)} \]  

where

\( \lambda = \sqrt{x y} + \sqrt{x z} + \sqrt{y z} \).

After the differences between the variables have been made enough small by successive duplications by (24) and (25), the integrals \( R_F, R_D \) are expanded into a triple Maclaurin series approximated by their partial sums of order five. The resulting algorithms consist only of rational operations and square roots. The convergence is fast enough to be superior to methods possessing second order convergence, unless extremely high precision is required, [13, p. 2]. The related algorithms for \( R_F, R_C, R_J, R_D \) are given respectively in [13, pp. 2–5, Algorithm 1-4.], but they are to massive to be presented here in detail.


Guided by certain practical reasons such as avoiding the loss of significant digits for small \( |n| \), Fukushima [21] introduced a linear combination of the Legendre’s incomplete elliptic integrals of the first (see APPENDIX A) and third kind instead of \( \Pi(n; \varphi|m) \) itself:

\[ J(n; \varphi|m) = \frac{1}{n} (\Pi(n; \varphi|m) - F(\varphi|m)) \quad \varphi \in \left(0, \frac{\pi}{2}\right), \quad m, n \in (0, 1), \]

calling it associated incomplete elliptic integral of the third kind. Obviously, knowing \( J(n; \varphi|m) \) and \( F(\varphi|m) \) we have

\[ \Pi(n; \varphi|m) = F(\varphi|m) + nJ(n; \varphi|m). \]

Fukushima outlined [21, p. 1963, §1.3] that the standard computation methods employed for \( J(n; \varphi|m) \) are Bulirsch’s, and Carlson’s (applied to \( R_J \)). However, from the practical point of view, both methods suffer from either cancellation errors or from large amounts of computation, see the discussions around [21, p. 1964, Eqs. (19), (21)]; the comparison to Bulirsch’s and Carlson’s methods are given in [21, §4].

Fukushima’s \( J \)–method is based on selecting one function from the four possible expressions:

\[ J(n; \varphi|m) = \begin{cases} 
J(n; \varphi|m) \\
J(n|m) - J\left(n; \arcsin \frac{\cos \varphi}{\sqrt{1 - n \sin^2 \varphi}}\right|m) - R_C(t_s^{-2}, h + t_s^{-2}) \\
J(n|m) - J\left(n; \arccos \frac{1 - m}{1 - m \sin^2 \varphi}\right|m) - R_C(t_c^{-2}, h + t_c^{-2}) \\
\end{cases} \]

(i) (ii)
where \( J(n|m) = J(n; \pi/2|m) \) is the associate complete elliptic integral of the third kind and

\[
\begin{cases}
(i) & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{1 + n + 1}} \\
(ii) & \frac{\cos \varphi}{\sqrt{1 - n \sin^2 \varphi}} < y_s
\end{cases}
\]

while

\[ h = n(1 - n)(n - m), \quad t_s = \frac{\sin(2\varphi)}{2\sqrt{1 - m \sin^2 \varphi}}, \quad t_c = \frac{t_s}{1 - n}. \]

Also, \( y_s, \varphi_s \) are constants, where \( y_s \in \{0.95, 0.90\} \) in the single and double precision computation modalities respectively, while \( \varphi_s = \arcsin \sqrt{y_s} \in \{1.345, 1.249\} \).

In our environment writing \( J(\varphi) \equiv J(e^2; \varphi|e^2), J(e^2) \equiv J(\pi/2); F(\varphi) = F(\varphi|e^2) \), we get the associate incomplete elliptic integral of the third kind

\[ J(\varphi) = \frac{1}{n}(\Pi(\varphi) - F(\varphi)), \quad \text{(26)} \]

and the associated mathematical model

\[
J(\varphi) = \begin{cases}
J(\varphi) & \text{(i)} \\
J(e^2) - J\left(\arcsin \frac{\cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2\sqrt{1 - e^2 \sin^2 \varphi}} & \text{(ii)} \\
J(e^2) - J\left(\arccos \frac{b}{a \sqrt{1 - e^2 \sin^2 \varphi}}\right) - \frac{\sin(2\varphi)}{2(1 - e^2)\sqrt{1 - e^2 \sin^2 \varphi}} & \text{otherwise}
\end{cases}
\]

for

\[
\begin{cases}
(i) & \varphi < \varphi_s \quad \text{or} \quad \varphi > \arcsin \frac{1}{\sqrt{1 + e^2 + 1}} \\
(ii) & \frac{\cos \varphi}{\sqrt{1 - e^2 \sin^2 \varphi}} < y_s
\end{cases}
\]

Indeed, \( h = 0 \) and \( R_C(x, x) = R_F(x, x, x) = x^{-1/2} \), according to (22). Now, it remains to apply Fukushima’s half and double argument transformations [21, pp. 1966–1970, §3.3–3.7] to the model (26) to earn the desired value of \( J(\varphi) \) for \( \varphi \) already given.

9. LUKE’S PĂDĂ APPROXIMATION METHOD

Using Pădă \(^7\) approximations for the square root, Luke [38, 39] offered routines for the approximative evaluation of elliptic integrals of all three kinds. His expression looks like (in our setting \( n = m = e^2 \)) [39, p. 193, Eqs. (26)]:

\[
\Pi(\varphi) = \frac{H(\varphi, e^2)}{2n + 1} \left\{ 1 + \sum_{j=1}^{n} \frac{2}{\cos^2 \frac{j\pi}{2n+1}} \right\}
\]

\(^7\)Henri Eugène Pădă (1863–1953) a French mathematician who made important contributions to the theory of continued fractions and introduced what we call today Pădă approximants, which are rational approximations to functions given by their power series.
\[- \frac{2}{2n+1} \sum_{j=1}^{n} H(\varphi, e^2 \sin^2 \frac{j\pi}{2n+1}) \tan^2 \frac{\pi j}{2n+1} + R_n^L(\varphi) \]

\[ H(\tau, f) = \frac{a}{b} \arctan \left( \frac{b}{a} \tan \tau \right) \]

\[ R_n^L(\varphi) = \frac{2e^{-2(n+1)\zeta} \tan \varphi}{(1 - e^2 \sin^2 \varphi)(4n+3)} \left( 1 + \frac{\lambda_1}{4n+3} + \mathcal{O}(n^{-2}) \right) + \mathcal{O}(e^{-4n\zeta}) \]

\[ \lambda_1 = 1 - \frac{\cosh \zeta - 1}{\sinh \zeta \cdot \cos^2 \varphi} - \frac{2e^2 \sin^2 \varphi \cdot (\cosh \zeta - 1)}{(1 - e^2 \sin^2 \varphi) \sinh \zeta} . \]

Here \( R_n^L(\varphi) \) is the error term which vanishes with growing \( n \) [39, p. 194, Eq. (28)]:

\[ \lim_{n \to \infty} R_n^L(\varphi) = 0 , \]

and \( e^\zeta \) is defined by [39, p. 191]

\[ e^\zeta = \frac{2 - e^2 \sin^2 \varphi + 2\sqrt{1 - e^2 \sin^2 \varphi}}{e^2 \sin^2 \varphi} . \]

We are drawing the reader’s attention to the important difference in notations of the base of natural logarithms \( e \), and the eccentricity \( e \).

A further variant of this method, useful for \( \varphi \) near to \( \pi/2 \), was proposed in [39, pp. 194–196, Eqs. (29)–(40)].

### 10. Comparison of the Presented Methods

Illustration of the presented methods will be the comparison of meridian distance calculation’s efficiency for composite trapezoidal (5) and Simpson’s (6) rules for few subdivisions \( n \). These employ Geodetic Reference System 1980 (GRS80) ellipsoid formula \( L_{m}^{80} \) (9), Geocentric Datum of Australia Technical Manual (ICSM 2002) official formula \( \tilde{L}_{m} \) (11), Bessel’s \( L_{m}^{B} \) (12), Helmert’s \( L_{m}^{H} \) (13) formulae, and the power series approximation formula \( L_{m}^{M} \) (18), using referent values of WGS84 ellipsoid characteristics \( a = 6378.137 \, [km] \), \( e = 0.0818192 \) and the latitude of Senj \( ^8 \varphi(S) = 45^\circ \) N. The corresponding value

\[ \Pi(\pi/4) = \text{EllipticPi}[0.006694381,\pi/4,0.006694381] = 0.786 834 838 411 \]

was evaluated by the WolframAlpha computational engine. By (1) we get \(^9\)

\[ L_m = a(1 - e^2)\Pi(\pi/4) = 4.984.944.374.286 \quad [km] = 2.691.654.630 \quad [M] . \]

\(^8\)Senj is a small town situated on the Adriatic coast near Rijeka, Croatia.

\(^9\)Here and in what follows, all computed values of meridian distances are expressed in \([km]\) units; by \( L_m \) \([km]\) = \( L_m / 1.852 \) \([M]\) we clearly transform it into nautical miles.
First, by obvious rough estimates \(1 - e^2 \sin^2(\vartheta \varphi) \geq \cos^2(\vartheta \varphi) \geq \cos^2 \varphi\), which hold true for all \(\varphi \in (0, \pi/2), \vartheta \in (0, 1)\), we conclude that

\[
|R^T_n(\pi/4)| \leq \frac{3e^2(13e^2 + 8)}{32n^2 \cos^2 \varphi}_{\varphi = \pi/2} = 0.003430
\]

\[
|R^S_n(\pi/4)| \leq \frac{e^2 \varphi^5(122.5e^6 + 200e^4 + 116e^2 + 4)}{960n^4 \cos^{11} \varphi}_{\varphi = \pi/2} = 0.000828557
\]

Denote \(L^T_m, L^S_m\) the right–hand–side sums in the trapezoidal (5) and Simpson’s (6) rules, respectively. We get the following table

| \(n\) | \(L^T_m\) | \(|R^T_n| <\) | \(L^S_m\) | \(|R^S_n| <\) |
|---|---|---|---|---|
| 2 | 4985.777412 | 8.6 \times 10^{-8} | 4984.907385 | 5.2 \times 10^{-5} |
| 3 | 4985.312473 | 3.8 \times 10^{-4} | - | - |
| 4 | 4985.151015 | 2.1 \times 10^{-6} | 4984.942218 | 3.2 \times 10^{-6} |
| 5 | 4985.076504 | 1.3 \times 10^{-4} | - | - |
| 6 | 4985.036084 | 9.5 \times 10^{-8} | 4984.943954 | 6.4 \times 10^{-7} |
| 10 | 4984.977367 | 3.4 \times 10^{-5} | 4984.944321 | 8.3 \times 10^{-8} |
| 20 | 4984.952620 | 8.6 \times 10^{-6} | 4984.944372 | 5.2 \times 10^{-9} |
| 50 | 4984.945696 | 1.4 \times 10^{-6} | 4984.944372 | 1.3 \times 10^{-10} |

Table 1. Trapezoidal and Simpson’s Rules Calculations of Meridian Distances

Next, we approximate \(L_m\) by \(L^80_m, \tilde{L}_m, L^B_m\) and \(L^H_m\) by quoted formulae (9), (11), (12) and (13) respectively. First, the third flattening parameter of the globe WGS84 ellipsoid is equal to

\[
n = \frac{a - b}{a + b} = 0.001679220.
\]

This value, together with the associated eccentricity \(e\) transforms Bessel’s, Helmert’s and the Maclaurin expansion formulae into

\[
L^B_m = 6367.449148277502 \varphi - 16.038504978751 \sin(2\varphi)
+ 0.016832605462 \sin(4\varphi) - 0.000021984392 \sin(6\varphi)
\]

\[
L^H_m = 6367.449148277502 \varphi - 16.038504978753 \sin(2\varphi)
+ 0.016832605523 \sin(4\varphi) - 0.000021984408 \sin(6\varphi)
+ 0.00000033148 \sin(8\varphi)
\]

\[
L^M_m = 6335.432364595861 \varphi + 63.617697072503 \varphi^3
- 2.99902589075 \varphi^5 + 0.353815715411 \varphi^7.
\]

Latitude \(\varphi = 45^\circ\) of Senj gives

\[
L^80_m = 4984.944373 \text{ [km]}, \quad \tilde{L}_m = 4984.944378 \text{ [km]}
\]
\[ L^B_m = L^H_m = 4984.944384 \ [km], \quad L^M_m = 5005.826977 \ [km]. \]

Now, it is not hard to see all estimated values of \( L_m \) differ in 12 [mm]; exceptions are the composite trapezoidal formula with approximately 1.3 [m] deviation, and the very slowly convergent Maclaurin polynomial approximant \( L^M_m \) with near to 21 [km] overlength. Therefore we conclude that all exposed and successfully examined formulae have the same navigational and geodesical significance and applicability.

11. Conclusion

In this article we presented, without using further computation, methods by Burlisch, Carlson, Fukushima–Ishizaki and Fukushima, since they are superior to the classical series expansion computation methods for the incomplete elliptic integral of the third kind \( \Pi(\varphi) \).

The use of all presented methods is highly appreciated in numerous applications, because \( \Pi(n; \varphi|m) \) appears in various mathematical models. These include the approximated Euler–Lagrange equation [3, p. 2655], [43, p. 817, Eq. (7)], the model of the gravitational or electromagnetic field associated with scalar or vector potential of a simple distribution such as annular disks with finite thickness [21]; the model of a magnetic field caused by thick coil [19]; the torque–free rotation of triaxial rigid body [22, Appendix C] and the periodic solutions of the Schrödinger equation [16].

Our main purpose here was to present enough precise computational tools and procedures for navigational calculations in both nautical and geodesical topics. Regarding nautical themata, has to be mentioned the very recent article by Weintrit–Kopacz [44] in which the authors gave an exhaustive up-to-date presentation, together with the numerous articles therein e.g. [46, 47]. We point out that all these articles work with formulae and routines in terms of the geographic latitude \( \varphi \), the exception is Petrović’s note [42], where the meridian arc length has been expressed via the incomplete elliptic integral of the second kind as \( L_m = a \cdot E(\theta|e^2) \), see [42, p. 88, Eq. (21)], also see the paper by Kos–Filjar–Hess [35, p. 959].

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26) http://functions.wolfram.com/EllipticIntegrals/EllipticPi3/03/01/01/0006

27) http://functions.wolfram.com/EllipticIntegrals/EllipticPi3/06/06/07/07/0002


APPENDIX A. To express $L_m$ explicitly, we employ the Legendre’s incomplete elliptic integrals of the first, second and third kind defined respectively by

$$F(z|m) = \int_0^z \frac{dt}{\sqrt{1 - m \sin^2 t}}$$

$$E(z|m) = \int_0^z \sqrt{1 - m \sin^2 t} \, dt$$

$$\Pi(n; z|m) = \int_0^z \frac{dt}{(1 - n \sin^2 t) \sqrt{1 - m \sin^2 t}} \quad |z| \leq \frac{\pi}{2};$$
related functions
\[ F\left(\frac{\pi}{2} | m\right) = F(m), \quad E\left(\frac{\pi}{2} | m\right) = E(m), \quad \Pi\left(n; \frac{\pi}{2} | m\right) = \Pi(n|m) \]
signify the complete elliptic integrals of all three kinds. Because the function \( \Pi \) is odd and quasi-periodic, the following properties hold [26]
\[ \Pi(n; z|m) = -\Pi(n; -z|n) \]
\[ \Pi(n; z + k\pi|m) = \Pi(n; z|m) + k\Pi(n|m), \quad k \in \mathbb{Z}, |n| \leq 1 \]
\[ \Pi(0; z|m) = F(z|m) \]
\[ \Pi(n; z|n) = \frac{1}{1-n} \left\{ E(z|n) - \frac{n \sin(2z)}{2\sqrt{1-n \sin^2 z}} \right\} \]
The transcendental equation \( \Pi(n; z|n) = w \) has no explicit solution in \( z \).

Rewriting (2) as
\[ \Pi(n; \varphi|m) = \int_0^\varphi \frac{(\sin^2 t + \cos^2 t)dt}{(\cos^2 t + (1-n) \sin^2 t) \sqrt{\cos^2 t + (1-m) \sin^2 t}}, \]
then introducing general parameters \( p = 1-n, k_x^2 = 1-m, x = \tan \varphi \), Burilsch in [8, 9, 10, 11], subsequently \( a,b,n_e = 1-n, m_e = 1-n \), Fukushima–Ishizaki in [23] have proposed more general and redirected computation routines \( e \ell 3 \) (20) and \( G \) (21) respectively, for the incomplete elliptic function of the third kind \( \Pi \).

**APPENDIX B.** The Maclaurin series expansion of the elliptic integral of the third kind coincides with a triple series [27]
\[ \Pi(n; \varphi|m) = \frac{\sqrt{\pi} (m/2)^{3/2}}{n(n-2)(1+\sqrt{1-m})^{3/2}} \sum_{q=0}^{\infty} \frac{(-4)^q}{(2q+1)!} \sum_{k=0}^{2q} S_2^{(k)} \sum_{j=0}^{k} (-n)^{-j} j! \frac{k^{-j}((1+\sqrt{1-n})^{j+1} - (1+\sqrt{1-m})^{j+1})}{(1-\sqrt{1-m})^{k-j}} \times F_1\left(\frac{1}{2}; \frac{1}{2}, -\frac{3}{2}; \frac{1}{2}, \frac{1}{2} + \frac{1}{2} \right) - \frac{1}{2\sqrt{1-m}} \frac{1+\sqrt{1-m}}{m} \right) \varphi^{2q+1}, \]
where the Stirling numbers of the second kind
\[ S_b^{(a)} = \frac{(-1)^a}{a!} \sum_{m=0}^{a} (-1)^m \left(\begin{array}{c} a \\ m \end{array}\right) m^b, \quad a, b - 1 \in \mathbb{N} \]
and the Appell function [40, p. 21, Eq. (2.1.1)]
\[ F_1(a; b; b'; c; x, y) = \sum_{r,s=0}^{\infty} \frac{(a)_{r+s}(b)_{r'}(b')_{s} (c)_{r+s} r! s!}{r! s!} x^r y^s, \quad |x| < 1, |y| < 1, \]
play significant roles. Here the familiar Pochhammer symbol (or shifted factorial)
\[(a)_n = a(a + 1) \cdots (a + n - 1) \quad a \in \mathbb{C}, n \in \mathbb{N}_0 = \{0, 1, 2, \ldots \},\]
and 0_0 = 1 is used by convention. Specifying \( n = m = e^2 \), the previous formulae give (5).

**APPENDIX C.** The Legendre polynomial \( y = P_n(x) \) is a solution of the Legendre differential equation, reads as follows
\[((1 - x^2)y')' + n(n + 1)y = 0 ;\]
the solution may be expressed by the Rodrigues formula [4, p. 252]
\[P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} (1 - x^2)^n.\]
The associated Gauss–Legendre low order \((n = 1, 2, 3, 4)\) quadrature weight–coefficients
\[w_j = \frac{2}{(1 - t_j^2)[P_n(t_j)]^2}, \quad j = 1, n, \quad n = 1, 2, 3, 4\]
are presented on the following table

<table>
<thead>
<tr>
<th>( n )</th>
<th>nodes ( t_j )</th>
<th>weights ( w_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>± ( \frac{1}{\sqrt{3}} )</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>± ( \frac{1}{5} ), 0</td>
<td>( \frac{5}{3}, \frac{8}{5} )</td>
</tr>
<tr>
<td>4</td>
<td>± ( \frac{1}{7} \left( 3 - 2 \sqrt{\frac{6}{5}} \right); \pm \sqrt{\frac{1}{7} \left( 3 + 2 \sqrt{\frac{6}{5}} \right)} )</td>
<td>( \frac{1}{2} + \frac{\sqrt{30}}{36}; \frac{1}{2} - \frac{\sqrt{30}}{36} )</td>
</tr>
</tbody>
</table>

Table 2. **Low–order Gauss–Legendre Quadrature weight–coefficients**
As the authors noted in the previous volume of Solstice, when the County of Washtenaw embarked on a major stream bank erosion control project, environmentally-sensitive residents quite naturally became concerned for the trees and wildlife that will be destroyed or disturbed. The project continued, and the geosocial network described in this article was the result of the project.

The project was critical—a simple red/blue confusion could cost a tree its life! All vegetation within the easement, except trees or shrubs carrying red tags, were to be removed. Color tags were used to differentiate between vegetation that was to be left alone. All vegetation within the easement, except trees or shrubs carrying red tags, were to be removed. Color tags were used to differentiate between vegetation that was to be left alone.

Plantings were an attempt to continue to track the progress of the project. Figures 2 through 8 show a sequence of screen captures from the field photographs. Follow up in July of 2012, following the removal of trees and stream bank stabilization enabled the neighborhood to continue to track the progress of the project. Figures 2 through 6 show the successes and failures (related to a drought) of larger, staked plantings. Figures 7 and 8 shows two of the many stream bank photos designed to illustrate the broader restoration.

When the camera GPS coordinates were obtained, a photo of the tagged item was also taken. The figures below show the photo pointing to a location. These pointing associations are all accurate. The accuracy of the geotagging of the photos was limited by several factors. First, the software in the smartphone has limits. Second, the geotagging of the tree is actually the geotagging of the camera where the smartphone was located, rather than the tree itself. David stood to take the picture of the tree, rather than of the tree position, itself. He attempted to stand at a consistent distance from trees to ensure precision (but that is difficult in this terrain). The level of precision, consistent with the need for a social network, was considered sufficiently precise (although not accurate) to offer local residents a clear picture of what did happen in their local wooded areas.

One neighborhood used Google Earth, together with a GPS-enabled smartphone, to make an inventory of trees present, along a half-mile stretch of the creek, before the project began. David E. Arlinghaus did all the photography with a smartphone that geotagged the images. He then transmitted the images to Sandra L. Arlinghaus who did the mapping using a combination of GeoSetter and Google Earth (Figure 1).

When the camera GPS coordinates were obtained, a photo of the tagged item was also taken. The figures below show the photo pointing to a location. These pointing associations are all accurate. The accuracy of the geotagging of the photos was limited by several factors. First, the software in the smartphone has limits. Second, the geotagging of the tree is actually the geotagging of the camera where the smartphone was located, rather than the tree itself. David stood to take the picture of the tree, rather than of the tree position, itself. He attempted to stand at a consistent distance from trees to ensure precision (but that is difficult in this terrain). The level of precision, consistent with the need for a social network, was considered sufficiently precise (although not accurate) to offer local residents a clear picture of what did happen in their local wooded areas.

Captions reveal some of what can be noted; however, to get a full view, download the linked .kmz file at the top of the article and open it in Google Earth. In that way, the reader of this article can follow along with what is happening in this territory without having to walk through the somewhat difficult terrain! Figure 1. The original survey, prior to stream bank restoration. Pink arrows mark flags showing County drain easements. Red balloons mark trees to be saved within the easement. Blue balloons mark trees to be cut. Figure 2. A large number of new trees were planted; these plants were larger than shrubs but not huge trees. They needed to be staked but could easily be planted with a shovel.

http://www.mylovedone.com/image/solstice/win12/Geosocial2.html
Institute of Mathematical Geography

The neighborhood association established a tree monitoring committee. The committee was given a Google Earth file showing tree location and associated tag color. The easement was also geocoded. Prior to using the file, the neighborhood association president and the creator of the Google Earth display met with the lead County official and the lead engineer on the project to ensure a cooperative approach to file usage. Subsequently, the tree monitoring committee used the information in conjunction with field-checking vegetation. Geosocial networking was, and is (through remaining tree restoration scheduled in spring 2013), critical in developing a constructive relationship among the various parties adjacent to this well-meant and successful environment stream-bank restoration project.

References


Acknowledgements
Numerous people were involved directly and indirectly in this project that brought together County and City officials, Engineers from a local engineering firm, and members of the public from a variety of neighborhoods including differing residential types and zoning. We thank: Harry Sheehan, Greg Marker, Jane Lumm, Roger Rayle, Janice Bobrin, Matt Naud, and all the members of the Huron Valley Neighborhood Alliance and the individuals from those neighborhoods who participated in so many helpful ways.
A way to look at geographical movement:

In this note movement is considered between geographical places identified by geographical coordinates of the beginning and ending locations. This movement may consist of a single person (or other entity: an idea, artifact, commodity, etc.) or multiple persons. The path taken is not considered and the movement takes place during some specified time interval. The assumption is made that there may be many moves, from dozens to millions.

The pair of locations, as point locations can be rendered as one directed line (arrow), as between points on a map, and the entire group of movements as many arrows. An example is shown in the first illustration: Journey to Work in the Detroit vicinity. All this is well known. It is also conventional, when many movements are involved, to group beginning and ending locations into aggregated locations: separate little pieces of territory. These are often counties, census tracts, states or countries, etc., irregularly shaped polygons on the earth's surface. The movement between these areas can be shown in a similar manner to that of point location data, generally using area centroids. This is shown in the next figure, having made using the Flow Mapper program from CSISS.org.

What I would like to suggest now is that one may imagine shrinking these areal units into small size, really infinitesimally small, and then to consider the movement pattern in a spatially continuous fashion. This can be done by introducing a grid, or raster, over the domain and reassigning all of the movements to the nodes in this grid. This is accomplished by smoothly spreading all of the movement from (or to) an area over the nodes bounded by each of the movement reporting areas. Such a raster covering the contiguous United States is illustrated here.

A suggested next step is to consider the difference between the incoming and outgoing moves within each area, in the raster assignment. Thus some of the areas (i.e., raster nodes) will get positive numbers and some will have negative numbers. Imagine that the negative numbers are high hills and the positive numbers represent low valleys - i.e., you have a kind of topographic surface. Now let the movement quantities move downhill, like topography eroding. The movement arrows are now the gradients to the "movement surface", and can be shown as a vector field. Connecting these gradient vectors can render this as a streamline movement pattern.

The example shows these ideas using a U.S. Census Bureau 48 by 48 contiguous state-to-state migration table converted in this fashion. The hills are the out-migration quantities and the low places, e.g., Florida, are the in-migration places. The resulting movement pattern clearly delineates source and sink domains describing the net spatial pattern of migration. In this particular depiction sixteen million people are migrating: the trajectories are ensemble averages, not individual moves. But migration to the Northwest, Southwest, Southeast, and Northeast are clearly distinguished. That this migration map resembles a map of wind or ocean currents is not surprising given that we speak of migration "flows" and "backwaters" and use many such hydrodynamic terms when discussing migration and movement phenomena. The appearance is that of laminar flow, due in part to the coarse resolution of the data.

The suggestion is that this continuous type of movement pattern analysis can be used in other situations, such as movement within a city, perhaps based on data assembled from cellular phone observations when connected to GPS locations. Many other such situations come to mind. For example, taxes collected and entitlements disbursed, both by congressional district.

Waldo Tobler
http://www.geog.ucsb.edu
1935-1940
Net Migration
**Update on Varroa Mite Spread**

Diana Sammataro

Nigeria has been added to the list of impacted countries.


Hawaii, part of the Big Island and all of Oahu are now impacted.


Current work with the Swiss Bee Research Center is ongoing to map and update Varroa spread.

Stay tuned for more information and for updates of the animated Varroa mite mapping project.
Map transformation is an idea that is familiar to most geographers (Tobler, 1961; Thompson, 1917; Coxeter, 1961). A QR code might be viewed as a "map"—it employs a mathematical code to direct understanding of content. While the geographical map uses mathematics to guide understanding of content of the Earth, the QR code map uses mathematics to guide understanding within electronic worlds (link to general pattern of meaning within a QR code).

One might ask questions about QR codes in the same way one does about geographical maps. Not all regions of a map carry equal weight (land masses on one map might be more important than water masses). The same observation is true about QR codes—different regions of the image carry different weights (as indicated in material in the link above). The thoughtful reader will think of numerous parallels. Is there some sort of parallel world of QR-cartography?

Geometric Transformations

Rotation (turn), reflection (flip), and translation (slide) form a basic set of movements of figures in the plane (Coxeter, 1961). In the case of a QR code pattern in a box, it is clear that translation does not affect the code—simply slide the smartphone to catch up with the sliding motion of the QR code box. Do rotation and reflection of the surface pattern in the box alter the linkage to a website? Does the embedded link to a website get destroyed by such motion? If not, does the link remain constant under such transformation. Again, it seems clear that if the link is not destroyed that then the rotated or reflected QR code would point to the same url as the original QR code box....it is simply the surface pattern and not the embedded code that is being transformed. Figure 1 shows a single QR code that has had transformations applied to it. Figure 1a is the "original" pattern. Figure 1b shows the pattern with a horizontal flip. Figure 1c shows the pattern with a vertical flip. Figure 1d shows it rotated 45 degrees clockwise.

Figure 1a. Base QR code. Links to IMaGe home site.
Figure 1b. Horizontal flip of Figure 1a.
Figure 1c. Vertical flip of Figure 1a.
Figure 1d. Rotation of Figure 1a through 45 degrees clockwise.

This simple experiment suggests that rotation does not destroy the linkage of the pattern to a website but that reflection does do so. At first, this result might seem surprising. If, however, one returns to the analogy of QR codes and geographic maps, it might not. The difference is that QR code patterns are not sensible to our eye whereas many map patterns are. Thus, in the sequence of maps in Figures 2a through 2d, it makes sense that the original and the rotated map (Figures 2a and 2d) are equivalent in terms of content but that the flipped maps in Figures 2b and 2c are not sensible in that regard. Nonetheless, QR codes such as the flipped ones that are not sensible from an overhead view might become useful in situations where one wishes to look at the QR code from the back or from underneath. The animation in Figure 2c shows the distinction between rotating the map through 180 degrees (so that north and south are interchanged) and reflecting, or flipping, the map.

Figure 2b. Horizontal reflection of the map in Figure 2a.
Figure 2c. Animation showing difference between a vertical reflection of the map in Figure 2a and a rotation through 180 degrees of the map in Figure 2a.
Figure 2d. Rotation through 45 degrees of the map in Figure 2a.

If one looks further, at the interleaving of data and error correcting blocks that lie behind the QR code pattern, then one sees immediately—due to lack of symmetry, that the Escher-like tiles of data are preserved under rotation but not under reflection (Figures 3a through 3d).
Permutation Group of the Symmetries of a Square

Legal motions of a QR code form a subgroup of the group of motions of symmetries of a square. It is the subgroup of rotations. The QR code is invariant under the geometric transformations of rotation and translation, but not under reflection.

References


link

scans but it does not scan when set to 65%. In Figure 6c, the two overlays are set at 36% each. The image scans
hexagonal overlay is 100% opaque. The image does not scan. In Figure 6b, the overlay is 60% opaque and the image

Changing color may slightly improve design appearance. As with a map, four colors suffice. Too many colors create

were to consider tattoos on growing humans, etching QR codes into the bark of growing trees, and no doubt a variety of

scanner could not decode it while another could do so. Clearly there are limits to this sort of distortion but determining

Figure 5 prompts questioning the extent to which one image may be slid (translated) across a QR code with opacity

Meantime, the interested reader might work with the attached file

linked

As translation is considered, what logos or other images in fully opaque form may be inserted within the QR code? The

It is important to understand what sorts of alterations of QR code pattern will still permit decoding of the image. From

distortion will be tied to error correction capability). In Figures 4a and 4b the decoding is swift. In Figure 4c, one

animation of QR codes. Do the intermediate "tweening" frames resolve correctly?

geometric styles of pattern disruption that are permitted? Figure 1 shows simple pattern destruction that

question lies in the extent of embedded error correction within the underlying code. The greater the correction

What alterations of a QR image do not destroy the embedded link to a website (or other)? Some of the answer to that

sort of parallel world of QR-cartography?

In Figure 3b: A 50% increase in height dimension while keeping width the same.

Figure 7 shows one existing example; there are many others.

Figure 3c.

Embedding

Map alteration, through distortion, interruption, overlays, and so forth, is an idea that is familiar to most geographers. A

QR code might be viewed as a "map"—it employs a mathematical code to direct understanding of content. While the

QR code uses mathematics to guide understanding of content of the Earth, the QR code map uses

Geometric Alterations beyond the Plane

While from an abstract standpoint this strategy overcomes a variety of difficulties, it is really only useful on a computer

It is tempting to want to tip the image so that the 3D buildings stand out more clearly. Doing so, however, interferes with

layers of roads superimposed on the QR code

It is important to be able to make QR codes clearly linked to a business. And, from a municipal viewpoint, it

It is easy to get QR codes to show up on flat paper. That is the source of their greatest utility, currently. They also work well

QR codes are often presented on flat paper. That is the source of their greatest utility, currently. They also work well

or TV screen. Thus, it might not see the same set of practical utility (except perhaps in a store window on a wide screen

One way to create interesting visual effects is to layer images. The extent to which it is possible to layer images of

layers of roads superimposed on the QR code

animation of QR codes. Do the intermediate "tweening" frames resolve correctly?

Interruption and Opacity

in height dimension while keeping width the same.

in general pattern of meaning within a QR code)

http://www.almalach.com/

Looking original pieces of art, designed to have an embedded QR code are present in a variety of locales: one that

seemed particularly attractive to this author were the examples present in the extensive PATH network linking

underground downtown Toronto, Ontario locations.

Animation

Example from Seurat, 1884. Pointillism

A Sunday on La Grande Jatte
Figure 11a. QR code covers Grant Park, Chicago Illinois. The code should scan properly.

Figure 11b. Code is adjusted so that it serves now as a cloud over the tall buildings which would otherwise interfere with proper scanning.

Figure 11c. Wave of the future...combine elements to create codes that scan from below...to scan the QR Cloud from the Earth!

References


Pereira, I. 09/24/2012. New York is going to start putting QR codes on city permits. AM New York.


