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Plane wave solutions for right-angled interior impedance wedges

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[1] We consider the reflection of a plane electromagnetic wave incident obliquely in a right-angled corner region with impedance walls. The surface impedances are taken in their most general tensor form. We determine the conditions under which the sum of incident, singly and doubly reflected waves provides an exact solution of the problem and report several new explicitly solvable cases.

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1. Introduction

[2] The impedance boundary condition [*Senior and Volakis*, 1995] is a convenient tool for simulating the material properties of a surface. For most materials the surface impedance is a scalar, but there are materials whose properties are anisotropic for which a tensor impedance is required. The simplest cases are those for which the tensors are diagonal but recent work with metamaterials has made possible the creation of very general materials for which the tensor may be non-diagonal. These are the focus of the present study.

[3] The problem considered is a plane electromagnetic wave incident on the interior of a right-angled impedance wedge. This is a geometry that is relevant to the analysis of finitely conducting waveguides and resonators as well as to the propagation of radio waves inside buildings. In general the solution consists of plane waves reflected off the two faces of the wedge and a diffracted field associated with the vertex, but if the surface impedances satisfy certain restrictions the diffracted field disappears. The exact solution is then the sum of four plane waves, one of which is the incident field. We seek the restrictions on the impedances for which this is so, and follow a procedure similar to that by *Senior* [1978].

2. Formulation

[4] The geometry is shown in Figure 1. The tensor impedance boundary condition is

$$\hat{n} \times \bar{E} = \bar{\bar{\eta}} \cdot \hat{n} \times (\hat{n} \times Z\bar{H}) \tag{1}$$

where $\bar{\bar{\eta}}$ is the normalized tensor surface impedance, $\hat{n}$ is the outward unit vector normal to the surface and Z is the intrinsic impedance of the free space. On the horizontal surface ($y = 0$) the tensor impedance is

$$\bar{\bar{\eta}} = \eta_1 \hat{x}\hat{x} + \eta_2 \hat{x}\hat{z} + \eta_3 \hat{z}\hat{x} + \eta_4 \hat{z}\hat{z} \tag{2}$$

and the boundary conditions derived from (1) are

$$\begin{aligned} E_z &= -\eta_1 Z H_x - \eta_2 Z H_z, \\ E_x &= \eta_3 Z H_x + \eta_4 Z H_z. \end{aligned} \tag{3}$$

[5] Similarly, on the vertical face ($x = 0$),

$$\bar{\bar{\eta}} = \bar{\bar{\eta}}' = \eta'_1 \hat{y}\hat{y} + \eta'_2 \hat{y}\hat{z} + \eta'_3 \hat{z}\hat{y} + \eta'_4 \hat{z}\hat{z} \tag{4}$$

and

$$\begin{aligned} E_z &= \eta'_1 Z H_y + \eta'_2 Z H_z, \\ E_y &= -\eta'_3 Z H_y - \eta'_4 Z H_z. \end{aligned}$$

For the boundary conditions to ensure a unique solution it is necessary that $\mathrm{Re}\eta_1 \geq 0$, $\mathrm{Re}\eta_4 \geq 0$ and $4\mathrm{Re}\eta_1 \mathrm{Re}\eta_4 \geq |\eta_2 + \eta_3{*}|^2$ where the asterisk denotes the complex conjugate, with similar restrictions on the primed quantities [see *Senior and Volakis*, 1995, p. 43].

[6] In terms of the single component Hertz vectors

$$\bar{\Pi}_e = \hat{z} U(x, y) e^{-ikz\cos\beta},$$

$$\bar{\Pi}_h = \hat{z} V(x, y) e^{-ikz\cos\beta}$$

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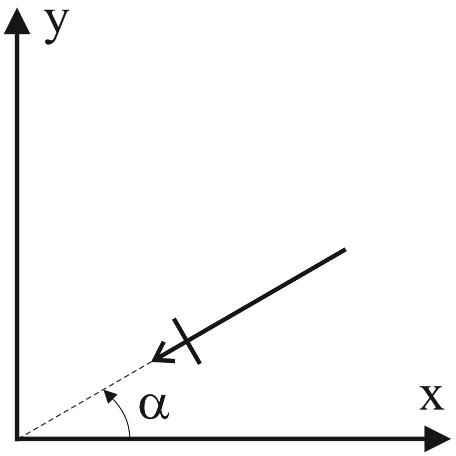

Figure 1. Geometry.

where a time factor $\exp(-i\omega t)$ has been assumed and suppressed, we have

$$
\begin{aligned}
E_x &= -ih\frac{\partial U}{\partial x} + ik\frac{\partial V}{\partial y},\\
ZH_x &= -ih\frac{\partial V}{\partial x} - ik\frac{\partial U}{\partial y},\\
E_y &= -ih\frac{\partial U}{\partial y} - ik\frac{\partial V}{\partial x},\\
ZH_y &= -ih\frac{\partial V}{\partial y} + ik\frac{\partial U}{\partial x},\\
E_z &= \lambda^2 U, \quad ZH_z = \lambda^2 V
\end{aligned}
\tag{5}
$$

where $h = k\cos\beta$ and $\lambda = k\sin\beta$ so that

$$h^2 + \lambda^2 = k^2.$$

Substituting (5) into (3), the boundary conditions on $y = 0$ can be written as

$$\left(1 + \eta_1\frac{k}{i\lambda^2}\frac{\partial}{\partial y}\right)U + \left(\eta_2 + \eta_1\frac{h}{i\lambda^2}\frac{\partial}{\partial x}\right)V = 0, \tag{6}$$

$$
\begin{aligned}
&\left(\eta_4 + \frac{k}{i\lambda^2}\frac{\partial}{\partial y} + \eta_3\frac{h}{i\lambda^2}\frac{\partial}{\partial x}\right)V\\
&\quad + \left(-\frac{h}{i\lambda^2}\frac{\partial}{\partial x} + \eta_3\frac{k}{i\lambda^2}\frac{\partial}{\partial y}\right)U = 0,
\end{aligned}
\tag{7}
$$

and similarly the boundary conditions on $x = 0$ are

$$\left(1 + \eta_1'\frac{k}{i\lambda^2}\frac{\partial}{\partial x}\right)U - \left(\eta_2' + \eta_1'\frac{h}{i\lambda^2}\frac{\partial}{\partial y}\right)V = 0, \tag{8}$$

$$
\begin{aligned}
&\left(\eta_4' + \frac{k}{i\lambda^2}\frac{\partial}{\partial x} + \eta_3'\frac{h}{i\lambda^2}\frac{\partial}{\partial y}\right)V\\
&\quad + \left(\frac{h}{i\lambda^2}\frac{\partial}{\partial y} - \eta_3'\frac{k}{i\lambda^2}\frac{\partial}{\partial x}\right)U = 0.
\end{aligned}
\tag{9}
$$

[7] The incident field is a plane wave with z components

$$E_z^{\text{inc}} = \lambda^2 A e^{i\bar{k}\cdot\bar{r}}, \quad ZH_z^{\text{inc}} = \lambda^2 B e^{i\bar{k}\cdot\bar{r}}$$

and direction of propagation

$$\bar{k} = -\lambda(\hat{x}\cos\alpha + \hat{y}\sin\alpha) - h\hat{z}$$

for $0 \le \alpha \le \pi/2$. We seek a plane wave solution

$$
\begin{aligned}
\begin{bmatrix} U \\ V \end{bmatrix} &= \begin{bmatrix} A \\ B \end{bmatrix} e^{-i\lambda(xc+ys)} + \begin{bmatrix} A_1 \\ B_1 \end{bmatrix} e^{-i\lambda(xc-ys)}\\
&\quad + \begin{bmatrix} A_2 \\ B_2 \end{bmatrix} e^{i\lambda(xc-ys)} + \begin{bmatrix} A_3 \\ B_3 \end{bmatrix} e^{i\lambda(xc+ys)}
\end{aligned}
\tag{10}
$$

where, for brevity, $c = \cos\alpha$ and $s = \sin\alpha$. The second and third terms in the right-hand side of (10) represent the single reflections of the incident plane wave (first term) off the two faces of the wedge and the fourth term is the doubly reflected wave. When (10) is substituted into the boundary conditions (6) and (7) on $y = 0$, the coefficients of $\exp(\pm i\lambda xc)$ give

$$
\begin{aligned}
&\left(1 + \eta_1\frac{k}{\lambda}s\right)A_1 + \left(\eta_2 - \eta_1\frac{h}{\lambda}c\right)B_1 =\\
&\quad -\left(1 - \eta_1\frac{k}{\lambda}s\right)A - \left(\eta_2 - \eta_1\frac{h}{\lambda}c\right)B,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&\left(1 + \eta_1\frac{k}{\lambda}s\right)A_3 + \left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)B_3 =\\
&\quad -\left(1 - \eta_1\frac{k}{\lambda}s\right)A_2 - \left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)B_2,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
&\left(\frac{h}{\lambda}c + \eta_3\frac{k}{\lambda}s\right)A_1 + \left(\eta_4 - \eta_3\frac{h}{\lambda}c + \frac{k}{\lambda}s\right)B_1 =\\
&\quad \left(-\frac{h}{\lambda}c + \eta_3\frac{k}{\lambda}s\right)A - \left(\eta_4 - \eta_3\frac{h}{\lambda}c - \frac{k}{\lambda}s\right)B,
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&\left(-\frac{h}{\lambda}c + \eta_3\frac{k}{\lambda}s\right)A_3 + \left(\eta_4 + \eta_3\frac{h}{\lambda}c + \frac{k}{\lambda}s\right)B_3 =\\
&\quad \left(\frac{h}{\lambda}c + \eta_3\frac{k}{\lambda}s\right)A_2 - \left(\eta_4 + \eta_3\frac{h}{\lambda}c - \frac{k}{\lambda}s\right)B_2,
\end{aligned}
\tag{14}
$$

and from the boundary conditions (8) and (9) we obtain

$$
\begin{aligned}
&\left(1 + \eta_1'\frac{k}{\lambda}c\right)A_2 - \left(\eta_2' - \eta_1'\frac{h}{\lambda}s\right)B_2 =\\
&\quad -\left(1 - \eta_1'\frac{k}{\lambda}c\right)A + \left(\eta_2' - \eta_1'\frac{h}{\lambda}s\right)B,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
&\left(1 + \eta_1'\frac{k}{\lambda}c\right)A_3 - \left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)B_3 =\\
&\quad -\left(1 - \eta_1'\frac{k}{\lambda}c\right)A_1 + \left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)B_1,
\end{aligned}
\tag{16}
$$

$$\left(\frac{h}{\lambda}s + \eta_3'\frac{k}{\lambda}c\right)A_2 - \left(\eta_4' - \eta_3'\frac{h}{\lambda}s + \frac{k}{\lambda}c\right)B_2 = \left(-\frac{h}{\lambda}s + \eta_3'\frac{k}{\lambda}c\right)A + \left(\eta_4' - \eta_3'\frac{h}{\lambda}s - \frac{k}{\lambda}c\right)B, \tag{17}$$

$$\left(-\frac{h}{\lambda}s + \eta_3'\frac{k}{\lambda}c\right)A_3 - \left(\eta_4' + \eta_3'\frac{h}{\lambda}s + \frac{k}{\lambda}c\right)B_3 = \left(\frac{h}{\lambda}s + \eta_3'\frac{k}{\lambda}c\right)A_1 + \left(\eta_4' + \eta_3'\frac{h}{\lambda}s - \frac{k}{\lambda}c\right)B_1. \tag{18}$$

[8] Equations (11) and (13) can be used to express A_1 and B_1 in terms of A and B. The elimination of B_1 gives

$$A_1 = \frac{1}{\Gamma_1}\left\{\left[\Gamma_2 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right]A - 2s\frac{k}{\lambda}\left(\eta_2 - \eta_1\frac{h}{\lambda}c\right)B\right\} \tag{19}$$

where

$$\Gamma_{1,2} = \eta_4 + \left(1 + \det\overline{\overline{\eta}}\right)\frac{k}{\lambda}s \mp (\eta_2 + \eta_3)\frac{h}{\lambda}c + \eta_1\frac{1}{\lambda^2}\left(k^2s^2 + h^2c^2\right)$$

with

$$\det\overline{\overline{\eta}} = \eta_1\eta_4 - \eta_2\eta_3,$$

and similarly, by eliminating A_1, we find

$$B_1 = \frac{1}{\Gamma_1}\left\{2s\frac{k}{\lambda}\left(\eta_3 - \eta_1\frac{h}{\lambda}c\right)A - \left[\Gamma_1 - 2s\frac{k}{\lambda}\left(1 + \eta_1\frac{k}{\lambda}s\right)\right]B\right\}. \tag{20}$$

We can also use (12) and (14) to express A_3 and B_3 in terms of A_2 and B_2, and since the equations differ from (11) and (13) only in the sign of h, we have

$$A_3 = \frac{1}{\Gamma_2}\left\{\left[\Gamma_1 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right]A_2 - 2\,s\frac{k}{\lambda}\left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)B_2\right\}, \tag{21}$$

$$B_3 = \frac{1}{\Gamma_2}\left\{2s\frac{k}{\lambda}\left(\eta_3 + \eta_1\frac{h}{\lambda}c\right)A_2 - \left[\Gamma_2 - 2s\frac{k}{\lambda}\left(1 + \eta_1\frac{k}{\lambda}s\right)\right]B_2\right\}. \tag{22}$$

Taking next the 4 equations derived from the boundary conditions on $x = 0$, (15) and (17) specify A_2 and B_2 in terms of A and B as

$$A_2 = \frac{1}{\Gamma_1'}\left\{\left[\Gamma_2' - 2\left(\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right)\right]A + 2c\frac{k}{\lambda}\left(\eta_2' - \eta_1'\frac{h}{\lambda}s\right)B\right\}, \tag{23}$$

$$B_2 = \frac{1}{\Gamma_1'}\left\{-2c\frac{k}{\lambda}\left(\eta_3' - \eta_1'\frac{h}{\lambda}s\right)A - \left[\Gamma_1' - 2c\frac{k}{\lambda}\left(1 + \eta_1'\frac{k}{\lambda}c\right)\right]B\right\} \tag{24}$$

where

$$\Gamma_{1,2}' = \eta_4' + \left(1 + \det\overline{\overline{\eta}}'\right)\frac{k}{\lambda}c \mp (\eta_2' + \eta_3')\frac{h}{\lambda}s + \eta_1'\frac{1}{\lambda^2}\left(k^2c^2 + h^2s^2\right)$$

with

$$\det\overline{\overline{\eta}}' = \eta_1'\eta_4' - \eta_2'\eta_3',$$

and from (16) and (18)

$$A_3 = \frac{1}{\Gamma_2'}\left\{\left[\Gamma_1' - 2\left(\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right)\right]A_1 + 2c\frac{k}{\lambda}\left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)B_1\right\}, \tag{25}$$

$$B_3 = \frac{1}{\Gamma_2'}\left\{-2c\frac{k}{\lambda}\left(\eta_3' + \eta_1'\frac{h}{\lambda}s\right)A_1 - \left[\Gamma_2' - 2c\frac{k}{\lambda}\left(1 + \eta_1'\frac{k}{\lambda}c\right)\right]B_1\right\}.$$

[9] We now have 8 equations linking the 4 pairs of coefficients A, B, A_1, B_1, A_2, B_2 and A_3, B_3. By substituting (23) and (24) into (21) we can express A_3 in terms of A and B as

$$\begin{aligned} A_3 = {} & \left\{\left[\Gamma_1 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right]\right. \\ & \cdot\left[\Gamma_2' - 2\left(\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right)\right] \\ & \left. + 4\frac{k^2}{\lambda^2}sc\left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)\left(\eta_3' - \eta_1'\frac{h}{\lambda}s\right)\right\}\frac{A}{\Gamma_2\Gamma_1'} \\ & + \left\{c\left(\eta_2' - \eta_1'\frac{h}{\lambda}s\right)\left[\Gamma_1 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right]\right. \\ & + \left[\Gamma_1' - 2\frac{k}{\lambda}c\left(1 + \eta_1'\frac{k}{\lambda}c\right)\right] \\ & \left. \cdot\, s\left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)\right\}\frac{2kB}{\lambda\Gamma_2\Gamma_1'}, \end{aligned} \tag{26}$$

and the substitution of (19) and (20) into (25) gives

$$\begin{aligned}A_3 = {} & \left\{ \left[\Gamma_2 - 2\left(\eta_4 + \frac{k}{\lambda} s + \eta_1 \frac{h^2}{\lambda^2} c^2 \right) \right] \right. \\ & \cdot \left[\Gamma_1' - 2\left(\eta_4' + \frac{k}{\lambda} c + \eta_1' \frac{h^2}{\lambda^2} s^2 \right) \right] \\ & \left. + 4\frac{k^2}{\lambda^2} sc \left(\eta_3 - \eta_1 \frac{h}{\lambda} c \right) \left(\eta_2' + \eta_1' \frac{h}{\lambda} s \right) \right\} \frac{A}{\Gamma_1 \Gamma_2'} \\ & - \left\{ c\left(\eta_2' + \eta_1' \frac{h}{\lambda} s \right) \left[\Gamma_1 - 2\frac{k}{\lambda} s \left(1 + \eta_1 \frac{k}{\lambda} s \right) \right] \right. \\ & + \left[\Gamma_1' - 2\left(\eta_4' + \frac{k}{\lambda} c + \eta_1' \frac{h^2}{\lambda^2} s^2 \right) \right] \\ & \left. \cdot s \left(\eta_2 - \eta_1 \frac{h}{\lambda} c \right) \right\} \frac{2kB}{\lambda \Gamma_1 \Gamma_2'}. \end{aligned} \tag{27}$$

[10] The analogous expressions for B_3 are

$$\begin{aligned}B_3 = {} & \left\{ s\left(\eta_3 + \eta_1 \frac{h}{\lambda} c \right) \right. \\ & \cdot \left[\Gamma_2' - 2\left(\eta_4' + \frac{k}{\lambda} c + \eta_1' \frac{h^2}{\lambda^2} s^2 \right) \right] \\ & + \left[\Gamma_2 - 2\frac{k}{\lambda} s \left(1 + \eta_1 \frac{k}{\lambda} s \right) \right] \\ & \left. \cdot c \left(\eta_3' - \eta_1' \frac{h}{\lambda} s \right) \right\} \frac{2kA}{\lambda \Gamma_2 \Gamma_1'} \\ & + \left\{ \left[\Gamma_2 - 2\frac{k}{\lambda} s \left(1 + \eta_1 \frac{k}{\lambda} s \right) \right] \right. \\ & \cdot \left[\Gamma_1' - 2\frac{k}{\lambda} c \left(1 + \eta_1' \frac{k}{\lambda} c \right) \right] \\ & \left. + 4\frac{k^2}{\lambda^2} sc \left(\eta_3 + \eta_1 \frac{h}{\lambda} c \right) \left(\eta_2' - \eta_1' \frac{h}{\lambda} s \right) \right\} \frac{B}{\Gamma_2 \Gamma_1'}, \end{aligned} \tag{28}$$

$$\begin{aligned}B_3 = {} & -\left\{ s\left(\eta_3 - \eta_1 \frac{h}{\lambda} c \right) \right. \\ & \cdot \left[\Gamma_2' - 2\frac{k}{\lambda} c \left(1 + \eta_1' \frac{k}{\lambda} c \right) \right] \\ & + \left[\Gamma_2 - 2\left(\eta_4 + \frac{k}{\lambda} s + \eta_1 \frac{h^2}{\lambda^2} c^2 \right) \right] \\ & \left. \cdot c \left(\eta_3' + \eta_1' \frac{h}{\lambda} s \right) \right\} \frac{2kA}{\lambda \Gamma_1 \Gamma_2'} \\ & + \left\{ \left[\Gamma_1 - 2\frac{k}{\lambda} s \left(1 + \eta_1 \frac{k}{\lambda} s \right) \right] \right. \\ & \cdot \left[\Gamma_2' - 2\frac{k}{\lambda} c \left(1 + \eta_1' \frac{k}{\lambda} c \right) \right] \\ & \left. + 4\frac{k^2}{\lambda^2} sc \left(\eta_2 - \eta_1 \frac{h}{\lambda} c \right) \left(\eta_3' + \eta_1' \frac{h}{\lambda} s \right) \right\} \frac{B}{\Gamma_1 \Gamma_2'}. \end{aligned} \tag{29}$$

[11] If the field inside the wedge is to consist of the 4 plane waves shown in (10), the above expressions for A_3 must be identical, as must those for B_3. We now seek the restrictions on $\overline{\overline{\eta}}$ and $\overline{\overline{\eta}}'$ to make this so, and start with the simple case of normal incidence.

3. Normal Incidence

[12] If the plane wave is incident in a plane perpendicular to the edge, $\beta = \pi/2$ implying $h = 0$ and therefore $\lambda = k$. Then

$$\Gamma_2 = \Gamma_1 = \eta_4 + \left(1 + \det \overline{\overline{\eta}}\right) s + \eta_1 s^2,$$

$$\Gamma_2' = \Gamma_1' = \eta_4' + \left(1 + \det \overline{\overline{\eta}}'\right) c + \eta_1' c^2$$

and (26) and (27) become

$$\begin{aligned}A_3 = {} & \{ [\Gamma_1 - 2(\eta_4 + s)] [\Gamma_1' - 2(\eta_4' + c)] \\ & + 4\eta_2 \eta_3' sc \} \frac{A}{\Gamma_1 \Gamma_1'} + \{ \eta_2' c [\Gamma_1 - 2(\eta_4 + s)] \\ & + \eta_2 s [\Gamma_1' - 2c(1 + \eta_1' c)] \} \frac{2B}{\Gamma_1 \Gamma_1'}, \end{aligned} \tag{30}$$

$$\begin{aligned}A_3 = {} & \{ [\Gamma_1 - 2(\eta_4 + s)] [\Gamma_1' - 2(\eta_4' + c)] \\ & + 4\eta_3 \eta_2' sc \} \frac{A}{\Gamma_1 \Gamma_1'} - \{ \eta_2' c [\Gamma_1 - 2s(1 + \eta_1 s)] \\ & + \eta_2 s [\Gamma_1' - 2(\eta_4' + c)] \} \frac{2B}{\Gamma_1 \Gamma_1'}. \end{aligned} \tag{31}$$

[13] Similarly, from (28) and (29),

$$\begin{aligned}B_3 = {} & \{ \eta_3 s [\Gamma_1' - 2(\eta_4' + c)] + \eta_3' c \\ & \cdot [\Gamma_1 - 2s(1 + \eta_1 s)] \} \frac{2A}{\Gamma_1 \Gamma_1'} + \{ [\Gamma_1 - 2s(1 + \eta_1 s)] \\ & \cdot [\Gamma_1' - 2c(1 + \eta_1' c)] + 4\eta_3 \eta_2' sc \} \frac{B}{\Gamma_1 \Gamma_1'}, \end{aligned} \tag{32}$$

$$\begin{aligned}B_3 = {} & -\{ \eta_3 s [\Gamma_1' - 2c(1 + \eta_1' c)] + \eta_3' c \\ & \cdot [\Gamma_1 - 2(\eta_4 + s)] \} \frac{2A}{\Gamma_1 \Gamma_1'} + \{ [\Gamma_1 - 2s(1 + \eta_1 s)] \\ & \cdot [\Gamma_1' - 2c(1 + \eta_1' c)] + 4\eta_2 \eta_3' sc \} \frac{B}{\Gamma_1 \Gamma_1'}. \end{aligned} \tag{33}$$

[14] Comparing (30) and (31), the coefficients of A are the same if

$$sc\left(\eta_2 \eta_3' - \eta_3 \eta_2'\right) = 0 \tag{34}$$

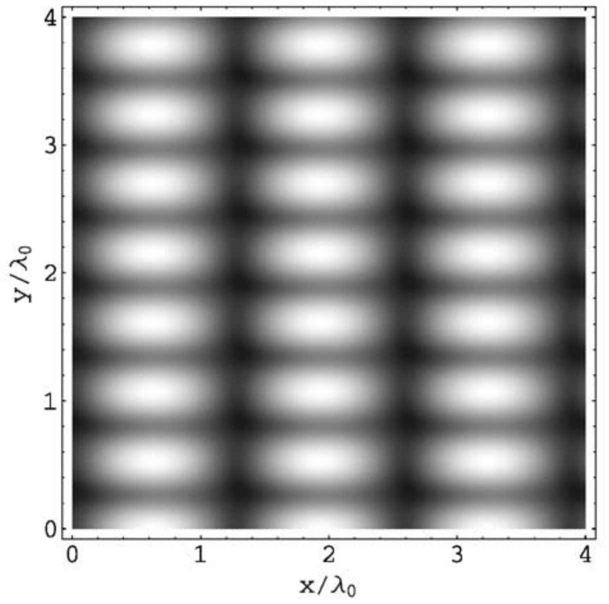

Figure 2. Total field distribution $|E_z(x, y)|$ for an E-polarized plane wave of unit amplitude ($|E_z^{\mathrm{inc}}| = 1$, $|H_z^{\mathrm{inc}}| = 0$) incident from direction $\alpha = 3\pi/8$, $\beta = \pi/2$ in a sector $0 \le x, y \le 4\lambda_0$ with $\eta_1 = 2$, $\eta_2 = 0.5$, $\eta_3 = -0.5$, $\eta_4 = 0.5$, $\eta'_1 = 1$, $\eta'_2 = i$, $\eta'_3 = -i$, $\eta'_4 = 2 - 0.5i$. The field level varies between the following minimum (black) and maximum (white) values: min $|E_z| = 0.222$, max $|E_z| = 2.169$.

and those of B are the same if

$$sc\left[\eta_2\left(1-\det\overline{\overline{\eta}}'\right)+\eta'_2\left(1-\det\overline{\overline{\eta}}\right)\right]=0.$$

[15] For (32) and (33) the coefficients of A are the same if

$$sc\left[\eta_3\left(1-\det\overline{\overline{\eta}}'\right)+\eta'_3\left(1-\det\overline{\overline{\eta}}\right)\right]=0$$

and (34) is sufficient to make the coefficients of B identical. Hence for a plane wave having any angle of incidence α, i.e., arbitrary c and s, the requirements for the existence of a plane wave solution are

$$\eta_2\eta'_3-\eta_3\eta'_2=0, \tag{35}$$

$$\eta_2\left(1-\det\overline{\overline{\eta}}'\right)+\eta'_2\left(1-\det\overline{\overline{\eta}}\right)=0, \tag{36}$$

$$\eta_3\left(1-\det\overline{\overline{\eta}}'\right)+\eta'_3\left(1-\det\overline{\overline{\eta}}\right)=0. \tag{37}$$

[16] In view of (35) the condition (36) implies (37) and vice versa, so that (35) and (36) are sufficient. We now have two relations connecting the 8 quantities $\eta_{1,2,3,4}$ and $\eta'_{1,2,3,4}$. This allows us to freely choose 6 of them, e.g., $\eta_{1,2,3,4}$ and $\eta'_{1,2}$, with the other two specified as

$$\eta'_3=\eta_3\frac{\eta'_2}{\eta_2},$$

$$\eta'_4=\frac{1}{\eta'_1}\left[1+\frac{\eta'_2}{\eta_2}\left(\eta_2\eta'_3+1-\det\overline{\overline{\eta}}\right)\right].$$

[17] A few special cases are worthy of note. If the impedance tensors are diagonal ($\eta_2 = \eta_3 = \eta'_2 = \eta'_3 = 0$), the problem is easily solved using Maliuzhinets' technique [*Maliuzhinets*, 1958], and since the angular spectra are 2π-periodic functions, the diffracted field is clearly zero. The conditions are also satisfied by polarization-independent surfaces whose impedances are such that $\eta_2 = \eta_3$ and $\det\overline{\overline{\eta}} = 1$, and if $\eta_2 = \eta_3 = 0$ with $\det\overline{\overline{\eta}} = 1$, a plane wave solutions exists regardless of the impedance of the other face.

[18] The plots in Figures 2 and 3 are sample distributions of total electric and magnetic fields in a right-angled interior wedge with tensor impedances compliant with (35) and (36). The patterns in the field distributions are explained by the analytical structure of expression (10), which is a continuous function of x and y whose magnitude is periodic in the x and y directions with

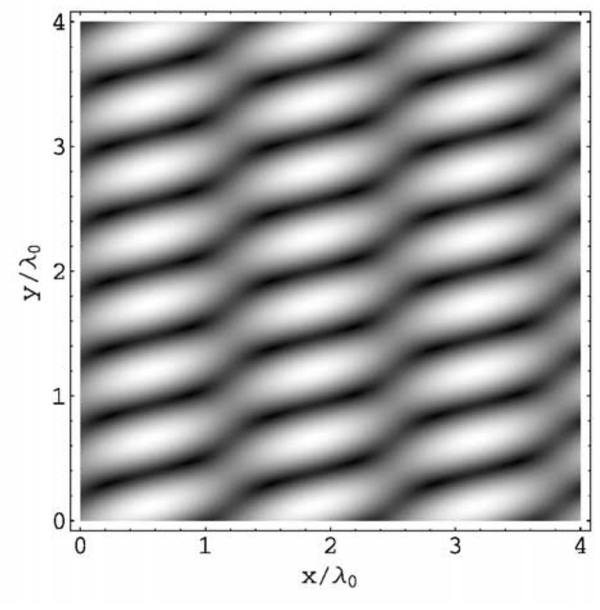

Figure 3. Same as in Figure 2 but for $|ZH_z(x, y)|$, min $|ZH_z| = 0.003$, max $|ZH_z| = 0.610$.

periods $\delta x = \lambda_0/(2\cos\alpha \sin\beta)$ and $\delta y = \lambda_0/(2\sin\alpha \sin\beta)$, respectively ($\lambda_0$ is the wavelength).

4. Arbitrary Incidence

[19] This is the most general case and the analysis is more laborious, primarily because Γ_1 and Γ_2 differ in the sign of $\eta_2 + \eta_3 = \Delta$ (say), as do Γ_1' and Γ_2' in the sign of $\eta_2' + \eta_3' = \Delta'$.

[20] If the expressions for A_3 in (26) and (27) are labeled $A_3^{(1)}$ and $A_3^{(2)}$ respectively, we have

$$\Gamma_1\Gamma_1'\Gamma_2\Gamma_2'\left(A_3^{(1)} - A_3^{(2)}\right) = \delta_1 A + \delta_2 B \qquad (38)$$

where

$$\begin{aligned}
\delta_1 = {} & \Gamma_1\left[\Gamma_1 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right] \\
& \cdot \Gamma_2'\left[\Gamma_2' - 2\left(\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right)\right] \\
& - \Gamma_2\left[\Gamma_2 - 2\left(\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right)\right] \\
& \cdot \Gamma_1'\left[\Gamma_1' - 2\left(\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right)\right] \\
& + 4\frac{k^2}{\lambda^2}sc\left[\left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)\left(\eta_3' - \eta_1'\frac{h}{\lambda}s\right)\Gamma_1\Gamma_2'\right. \\
& \left. - \left(\eta_3 - \eta_1\frac{h}{\lambda}c\right)\left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)\Gamma_2\Gamma_1'\right] \\
= {} & \left\{\left[\frac{k}{\lambda}s\left(\det\overline{\overline{\eta}} + \eta_1\frac{k}{\lambda}s\right) - \Delta\frac{h}{\lambda}c\right]^2\right. \\
& \left. - \left[\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right]^2\right\} \\
& \cdot \left\{\left[\frac{k}{\lambda}c\left(\det\overline{\overline{\eta}}' + \eta_1'\frac{k}{\lambda}c\right) + \Delta'\frac{h}{\lambda}s\right]^2\right. \\
& \left. - \left[\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right]^2\right\} \\
& - \left\{\left[\frac{k}{\lambda}s\left(\det\overline{\overline{\eta}} + \eta_1\frac{k}{\lambda}s\right) + \Delta\frac{h}{\lambda}c\right]^2\right. \\
& \left. - \left[\eta_4 + \frac{k}{\lambda}s + \eta_1\frac{h^2}{\lambda^2}c^2\right]^2\right\} \\
& \cdot \left\{\left[\frac{k}{\lambda}c\left(\det\overline{\overline{\eta}}' + \eta_1'\frac{k}{\lambda}c\right) - \Delta'\frac{h}{\lambda}s\right]^2\right. \\
& \left. - \left[\eta_4' + \frac{k}{\lambda}c + \eta_1'\frac{h^2}{\lambda^2}s^2\right]^2\right\} + 4\frac{k^2}{\lambda^2}sc \\
& \cdot \left[\left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)\left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)\left(\Gamma_2\Gamma_1' - \Gamma_1\Gamma_2'\right)\right. \\
& \left. + \left(\eta_2 + \eta_1\frac{h}{\lambda}c\right)\Delta'\Gamma_1\Gamma_2' - \left(\eta_2' + \eta_1'\frac{h}{\lambda}s\right)\Delta\Gamma_2\Gamma_1'\right].
\end{aligned}$$

[21] When multiplied out this is found to be the sum of sixth power polynomials in s multiplied by s and c, and while the coefficients of the highest power turn out to be zero, the coefficients of the next (and all subsequent) powers vanish only if $\Delta = \Delta' = 0$.

[22] This greatly simplifies the analysis. Since $\Gamma_2 = \Gamma_1$ and $\Gamma_2' = \Gamma_1'$ the coefficients of A in the expressions for A_3 are clearly equal, and from (26), (27) and (38),

$$\begin{aligned}
\delta_2 = {} & 4\Gamma_1\Gamma_1'\frac{k}{\lambda} \\
& \cdot \left\{\eta_2' c\left[\Gamma_1 - \eta_4 - 2\frac{k}{\lambda}s - \eta_1\frac{1}{\lambda^2}\left(k^2s^2 + h^2c^2\right)\right]\right. \\
& + \frac{h}{\lambda}\eta_1' cs\left[\eta_4 + \eta_1\frac{1}{\lambda^2}\left(h^2c^2 - k^2s^2\right)\right] \\
& + \eta_2 s\left[\Gamma_1' - \eta_4' - 2\frac{k}{\lambda}c - \eta_1'\frac{1}{\lambda^2}\left(k^2c^2 + h^2s^2\right)\right] \\
& \left. + \frac{h}{\lambda}\eta_1 cs\left[\eta_4' + \eta_1'\frac{1}{\lambda^2}\left(h^2s^2 - k^2c^2\right)\right]\right\} \\
= {} & 4\Gamma_1\Gamma_1'\frac{k}{\lambda}sc\left[h\left(\eta_1\eta_4' + \eta_4\eta_1' - \eta_1\eta_1'\right)\right. \\
& \left. - k\eta_2\left(1 - \det\overline{\overline{\eta}}'\right) - k\eta_2'\left(1 - \det\overline{\overline{\eta}}\right)\right]
\end{aligned}$$

which vanishes for all angles of incidence if (36) is satisfied and

$$\frac{\eta_4}{\eta_1} + \frac{\eta_4'}{\eta_1'} = 1. \qquad (39)$$

[23] This is the standard impedance compatibility condition [*Dybdal et al.*, 1971]. In the case of B_3 the coefficients of B are equal and those of A agree if (37) and (39) are satisfied.

[24] Thus, for arbitrary angles of incidence, the requirements for a plane wave solution are

$$\eta_2 + \eta_3 = 0, \qquad \eta_2' + \eta_3' = 0 \qquad (40)$$

plus (36), (37) and (39). We note that (40) implies (35) but not the reverse and that (36) in conjunction with (40) implies (37). A sufficient set of conditions is therefore

(36), (39) and (40) and this allows the free choice of 4 elements of the impedance tensors. If, for example, we choose η_1, η_2, η_4 and η'_2, the remaining elements are

$$\eta_3 = -\eta_2, \qquad \eta'_3 = -\eta'_2,$$

$$\eta'_4 = \eta'_1\left(1 - \frac{\eta_4}{\eta_1}\right),$$

$$\eta'_1 = \pm\left[\frac{1 + \eta'_2\left(1 - \det\bar{\bar{\eta}}\right)/\eta_2 - \left(\eta'_2\right)^2}{1 - \eta_4/\eta_1}\right]^{\frac{1}{2}}.$$

[25] Either sign of the square root is permissible subject, of course, to the requirements of physical realizability [*Senior and Volakis*, 1995].

[26] Figures 4 and 5 present sample field distributions for a configuration with tensor impedances which are in agreement with conditions (36), (39), and (40).

[27] A case of special interest is a wedge having one side, e.g., the vertical, perfectly conducting ($\eta'_1 = \eta'_2 = \eta'_3 = \eta'_4 = 0$) with the other having a diagonal impedance tensor ($\eta_2 = \eta_3 = 0$). A plane wave solution then exists for arbitrary η_1 and η_4.

5. Conclusions

[28] For a plane wave incident on the interior of a right-angled wedge with impedance walls we have established the conditions for the existence of a solution consisting only of four plane waves without any diffracted field. In general, a diffracted field is necessary to compensate for the discontinuity in the plane waves doubly reflected off the vertical and horizontal and horizontal and vertical faces of the wedge. The discontinuity exists across the plane ϕ (= arctan y/x) = α, but when the conditions that we have derived are satisfied, there is no discontinuity and, hence, no diffracted field. The results are an extension of those previously found [*Senior*, 1978] for diagonal impedance tensors.

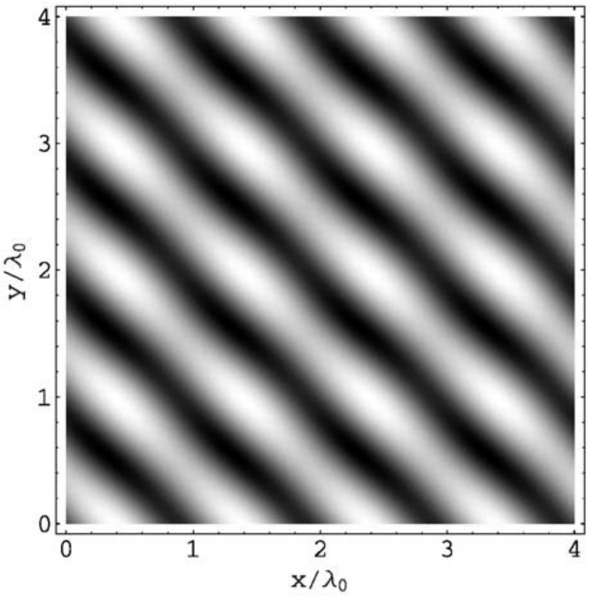

Figure 4. Same as in Figure 2 but for $\alpha = \pi/4$, $\beta = \pi/4$, $\eta_1 = 2$, $\eta_2 = 0.5i$, $\eta_3 = -0.5i$, $\eta_4 = 0.5$, $\eta'_1 = \sqrt{2}$, $\eta'_2 = 0.5i$, $\eta'_3 = -0.5i$, $\eta'_4 = 1.5/\sqrt{2}$. The field level is such that $0.726 \leq |E_z| \leq 1.299$.

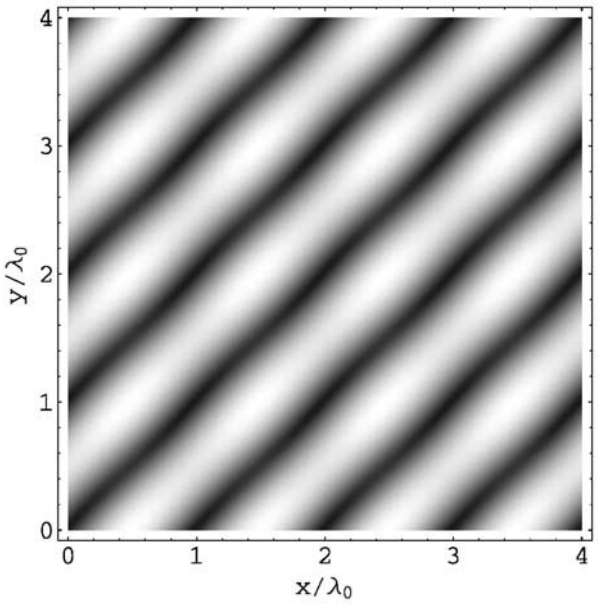

Figure 5. Same as in Figure 4 but for $|ZH_z(x, y)|$ with $0.107 \leq |ZH_z| \leq 1.035$.

[29] An immediate consequence is that in a rectangular waveguide whose neighboring walls have the surface impedances (2) and (4), the conditions (36), (39) and (40) are necessary to ensure the existence of a separable modal solution. In other words, the conditions guarantee that the modal spectrum is discrete with no continuous component.

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