# Essays on Energy Assets Management: Operations, Valuation, and Financing


by

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# ABSTRACT

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Management of energy assets is a critical part of many business processes and has received significant attentions in the operations management area recently. This thesis includes three essays on the operations, valuation and financing of energy assets. How the energy assets are operated determines the value of the assets. Different financing policies impose different constraints on operations and hence affect the valuation of the assets.

The first essay studies the valuation of seasonal energy storage and proposes a new approach to improve a common practice in the industry. According to the industry heuristics, the firm decides its energy injection/withdrawal operations by solving static optimization problems contingent on the forward curve observed in the market, and dynamically adjusts operations as the forward curve changes over time. The new approach improves the industry practice by embedding the option values not captured by the static optimization into adjusted forward curves and applying the static optimization to the adjusted forward curve. Numerical experiments show this price-adjusted approach can significantly close the gap between the industry practice and the optimal valuation. The second essay develops a model to integrate the granular spot market operations into the valuation and risk management of energy storage. The firm takes profit not only from the winter-to-summer futures price differentials but also from the spot-futures price differentials due to higher spot market volatility. I study the structure of the optimal inventory control and trading strategy, and also construct a heuristic policy that is numerically shown to be near-optimal. In the third essay, I develop a multi-period model to explore the interactions between financing method and operations for

non-renewable resource projects. I analyze how different financing approaches (e.g., equity and debt) impose restrictions on project operations and affect the cash-flows in different ways. I describe the conditions under which equity performs better than debt financing and vice versa, and how the financing choice is affected by various market factors.

# CHAPTER 1

# Introduction

This dissertation includes three essays on the operations, valuation and financing of energy assets. The first essay is focused on the management of seasonal energy storage assets, the operations and valuation of which are subject to physical constraints and fluctuations in futures market. The second essay develops a model to integrate the granular spot market into the valuation and risk management of energy storage assets. The third essay examines the appropriate financing approach for projects of exhaustible resource and interactions between financing and operational policy.

The first essay, "Seasonal Energy Storage Operations with Limited Flexibility," studies the management of seasonal energy storage and develops a new method to improve the the conventional management policy. The value of seasonal energy storage depends on how the firm best operates the storage to capture the seasonal price spread. Energy storage operations typically face limited operational flexibility characterized by the speed of storing and releasing energy. A widely used practice-based heuristic, the rolling intrinsic (RI) policy, generally performs well, but can significantly under-perform in some cases. In this paper, I aim to understand the gap between the RI policy and the optimal policy, and design improved heuristic policies to close or reduce this gap. A new heuristic policy, the "price-adjusted rolling intrinsic (PARI) policy," is developed based on theoretical analysis of the value of storage options. This heuristic adjusts prices before applying the RI policy, and the adjusted prices inform the RI policy about the values of various storage options. The numerical experiments show that the PARI policy is especially capable of recovering high value losses of the RI policy. For the instances where the RI policy loses more than 4% of the optimal storage value, the PARI policy on average is able to recover more than 90% of the value loss.

The second essay, "Inventory Control and Risk Management of Energy Storage Assets," builds a model to integrate spot transactions in the valuation of storage assets. Managing a natural gas storage asset involves injection and withdrawal of natural gas and risk management via trading on spot and futures markets. The objective is to shape the probability distribution of end-of-winter profit, so as to balance the down-side risk and up-side profit. The firm takes profit not only from the winter-to-summer futures price differentials but also from the spot-futures price differentials due to higher spot market volatility. Physical constraints are also present: injection and withdrawal of natural gas are subject to the storage capacity constraint, injection/withdrawal rate constraint, and the delivery schedule constraint. In this paper, I analyze a model that captures all the above essential features. I compare the utility maximization objective with the heuristic method used currently in the industry. I study the structure of the optimal inventory control and trading strategy, and also construct a heuristic policy that is numerically shown to be near-optimal.

In the third essay, "Capacity Investment, Production Scheduling and Financing Choice for Non-renewable Resource Projects," I study the interaction among capacity investment, production and financing decisions for projects in a multi-period model. I consider a budget-constrained firm that can finance the capacity investment in the project through either equity (e.g., joint venture) or debt (e.g., loans). The firm operates the project in subsequent periods to earn stochastic cash flows through sales of inventory at fluctuating market prices. I analyze how different forms of financing impose restrictions on project operations and affect the cash-flows in different ways. Debt financing involves bankruptcy risk but also sets a limit on the amount paid back to the creditor, while financing through equity is risk-free but demands a certain fraction of the revenue. I show with the bankruptcy risk the firm's optimal production quantity may be decreasing in its inventory level and increasing in its cash position. I describe the conditions under which equity performs better than debt financing and vice versa. Equity financing is preferred over debt if the project size exceeds some certain threshold. Furthermore, the firm is prone to use debt financing when price volatility increases. I demonstrate that with fixed interest rate on debt, project value first increases and then decreases in debt maturity. Therefore, debt financing performs the best if maturity is at intermediate levels. The firm should choose equity over debt if debt maturity is too long or too short. Moreover, project value is most sensitive with respect to changes in debt maturity when market price is low.

# CHAPTER 2

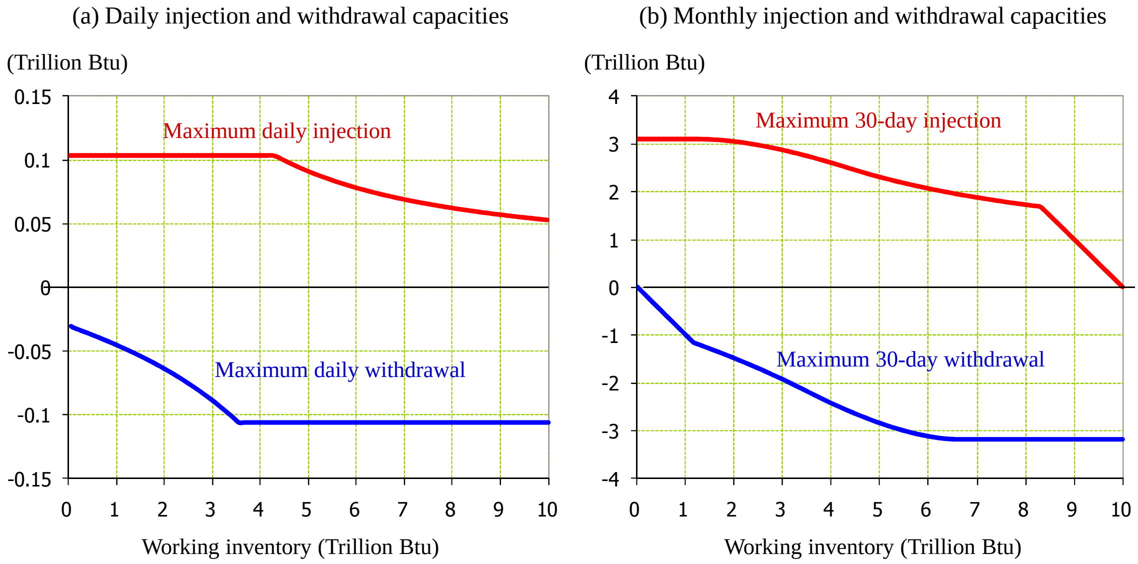
# Seasonal Energy Storage Operations with Limited Flexibility

## 2.1. Introduction

Energy storage plays an essential role in managing the mismatch between energy supply and demand. Because of the seasonality in demand, energy storage operations exhibit seasonal patterns: Natural gas storage (e.g., depleted reservoir, aquifer) operates in annual cycles; electricity storage (e.g., hydroelectric pumped storage, compressed air storage, batteries) typically has daily cycles.

The value of energy storage depends not only on the seasonal price spread, but also on how the firm best operates the storage. Energy storage operations typically face limited operational flexibility: Firms can choose periods with the best energy prices to buy and sell energy, but the quantities are limited by the storing and releasing capacities, which are determined by physical constraints or contractual terms. Figure 2.1 shows an example of physical constraints for a typical natural gas storage facility. Panel (a) shows that the maximum injection rate is initially constant and then declines in response to the higher reservoir pressure as working inventory builds up; a reverse trend is observed for the withdrawal rate. (Gas reservoirs hold working gas and base gas. Working inventory refers to gas that can be withdrawn; base gas is needed as permanent inventory to maintain adequate reservoir pressure.) Panel (b) is derived from panel (a) and shows the monthly injection and withdrawal capacities: An empty storage can receive a maximum of 3.1 trillion Btu (British thermal unit) in the first 30 days and less in the following months. It takes about four months to fill up or deplete the storage, or about eight months to complete a cycle.

Figure 2.1: Injection and withdrawal capacities of a typical natural gas storage facility
Source: Financial Engineering Associates (FEA)




Managing storage with unlimited capacity is known as the warehouse problem, which was first proposed by *Cahn* (1948). With limited flexibility, storage valuation is considerably more challenging because it involves multiple interacting real options, i.e., options to store or withdraw within capacity limits in every period. Analytical solutions for storage valuation typically do not exist; significant development in numerical techniques of valuation has been seen in recent years, e.g., *Manoliu* (2004), *Chen and Forsyth* (2007), *Thompson et al.* (2009), among others.

In general, finding the optimal storage policy is analytically and computationally challenging. Consequently, heuristic methods have been developed in practice and studied in academia. A widely-used heuristic method is the rolling intrinsic (RI) policy, detailed in *Gray and Khandelwal* (2004a,b), and is also referred to as the reoptimized intrinsic policy by *Secomandi* (2010) and *Lai et al.* (2010). Under the RI heuristic, in each period, the storing or releasing quantity is decided by solving a static optimization problem that involves only forward prices or price forecasts; prices are updated every period and the storage is reevaluated. The RI policy has near-optimal performance in many circumstances (*Secomandi* 2010, *Lai et al.* 2010), but can significantly underperform in some cases.

This paper aims to understand the gap between the RI policy and the optimal policy and to design improved heuristic policies to close or reduce the gap. We design a new heuristic policy called the "price-adjusted rolling intrinsic" (PARI) policy, in which prices are adjusted before applying the RI policy. This simple idea turns out to be very effective: In a three-period problem, the PARI policy is proven to be optimal, and in the multiperiod setting, our numerical results show that the PARI policy is especially capable of recovering high value losses caused by the RI policy.

The price adjustment method is derived based on the understanding of four types of option values in storage operations, briefly described below.

(a) *Value of waiting.* Even if the current price is higher than the expected future prices, it may be beneficial to defer sales when the firm has the flexibility to release energy to capture the expected maximum selling prices.

(b) *Value of avoiding adverse price.* Even if the current price is the lowest compared to the expected future prices, selling some inventory right now may be beneficial because it allows the firm to avoid the expected minimum selling prices in the future.

(c) *Value of counter-seasonal operations.* Price fluctuations may create within-season profit opportunities, which can be captured by counter-seasonal operations, e.g., buying in the selling season.

(d) *Value of raising operational capacity.* When the storing (releasing) speed depends on the inventory level, storing (releasing) less energy in the current period allows the firm to have a higher storing (releasing) capacity in the future to profit from better prices.

The value of waiting and the value of raising operational capacity reduce the firm's incentive to sell, whereas the value of counter-seasonal operations and the value of avoiding adverse price increase that incentive. Thus, it is necessary to strike a balance between these values. We formalize these tradeoffs in this paper.

The rest of this paper is organized as follows. The relevant literature is reviewed in §2.2.

The seasonal storage operations are modeled in §4.2. The PARI policy is constructed and analyzed in §2.4 and §2.5. Numerical results are presented in §3.4. We conclude the paper with discussion in §3.5.

## 2.2. Literature Review

Managing a fully flexible storage facility is known as the warehouse problem (*Cahn* 1948). Many researchers have addressed the problem under various settings. The deterministic version of the problem is studied by *Charnes and Cooper* (1955), *Bellman* (1956), *Prager* (1957), and *Dreyfus* (1957). The warehouse problem with stochastic price variations is considered by *Charnes et al.* (1966), who find that the optimal policy is a bang-bang type (if the firm acts, it would either fill up the storage or sell all the inventory). *Kjaer and Ronn* (2008) analyze a model with both spot and futures markets. *Hodges* (2004) solves a continuous-time model of a fully flexible storage facility.

In practice, storage facilities typically have limited flexibility, due to physical constraints or contractual terms. *Secomandi* (2010) shows the optimal policy under injection and withdrawal capacities is characterized by two state-dependent basestock targets: If inventory falls between the two targets, it is optimal to do nothing, otherwise the firm should inject or withdraw to bring the inventory as close to the nearer target as possible. In a continuous-time framework, *Kaminski, Feng, and Pang* (2008) prove the optimal policy has a similar structure.

In essence, energy storage operations are multiple interacting real options, that is, options to store or withdraw within capacity limits in every period. *Dixit and Pindyck* (1994) and *Schwartz and Trigeorgis* (2001) provide the theoretical background of real options. Analytical valuation of storage options typically do not exist due to the injection and withdrawal constraints. Three computational methods have been developed for storage valuation: numerical partial differential equation techniques (*Chen and Forsyth* 2007, *Thompson, Davison, and Rasmussen* 2009), binomial/trinormial trees (*Manoliu* 2004, *Parsons* 2007), and the Monte Carlo simulation (*Boogert and De Jong* 2008, *Carmona and Ludkovski* 2010, *Li* 2009). *Chen and Forsyth* (2007) provide a good survey of these computational methods.

Our work complements the above works by identifying various types of storage options and revealing useful insights to improve heuristic policies.

Practitioners typically employ two heuristic policies to value seasonal energy storage, the rolling intrinsic (RI) approach and the rolling basket of spread options approach (*Gray and Khandelwal* 2004a,b, *Eydeland and Wolyniec* 2003). *Lai et al.* (2010) refer to them as reoptimized intrinsic value policy and reoptimized linear program policy, respectively. *Gray and Khandelwal* (2004b, p. 4) state, "Additionally, we have found empirically that, in general, the rolling intrinsic value is equal to the rolling basket value." *Lai et al.* (2010) employ an approximate dynamic programming approach to value storage with constant capacities and study the effectiveness of the heuristics. They find both heuristics have near-optimal performance. *Lai et al.* (2011) value the real option to store liquefied natural gas at a regasification terminal. Our work complements the above research by identifying the conditions under which the RI heuristic deviates from the optimal policy and by developing methods to bring the RI heuristic closer to optimality.

## 2.3. The Model

Consider an energy storage facility with maximum working inventory level denoted as $K$. The planning horizon lasts $N$ periods, indexed by $t = 1, 2, \ldots, N$. At the beginning of period $t$, let $x_t$ be the inventory level in the storage. In this paper, we interchangeably use 'energy level' and 'inventory level,' which are measured in units of energy. The price-taking firm aims to maximize the profit from storage operations.

### 2.3.1 Operational Constraints and Costs

Let $\overline{\lambda}(x) \geq 0$ and $\underline{\lambda}(x) \leq 0$ be the capacity functions. Their absolute values, $\overline{\lambda}(x)$ and $-\underline{\lambda}(x)$, express the maximum amount of energy that can be stored and released, respectively, in one period when the period-starting energy level is $x$. These capacity functions satisfy the following assumption:

**Assumption 1.** *There exists $H \in (0, K)$ such that $\underline{\lambda}(x) = -x$ when $x \leq H$, and $\underline{\lambda}'(x) \in (-1, 0]$ when $x > H$. There exists $G \in (0, K)$ such that $\overline{\lambda}(x) = K - x$ when $x \geq G$, and*

$\overline{\lambda}'(x) \in (-1, 0]$ *when* $x < G$.

Assumption 1 implies that the storage can be emptied (filled up) within one period if and only if the period-starting inventory level $x \leq H$ ($x \geq G$). The slopes of the capacity functions imply that the period-ending inventory limits, defined as $\underline{y}(x) \stackrel{\text{def}}{=} x + \underline{\lambda}(x)$ and $\overline{y}(x) \stackrel{\text{def}}{=} x + \overline{\lambda}(x)$, are nondecreasing in $x$.

When the injection and withdrawal speeds are constant for all inventory levels, we have $\underline{\lambda}(x) = \max\{\underline{C}, -x\}$ and $\overline{\lambda}(x) = \min\{\overline{C}, K - x\}$ for some $\underline{C} < 0$ and $\overline{C} > 0$. We refer to this case as the constant capacities case, which is examined by *Secomandi* (2010) and *Lai et al.* (2010).

Storing and releasing energy typically involves operational frictions. For example, in natural gas storage operations, the pumps of the storage facility use some of the gas as fuel (*Maragos* 2002). If $q$ units are to be added to the storage, the firm needs to purchase $(1+\alpha)q$ units; if $q$ units are withdrawn from the storage, a fraction $\beta q$ will be lost and $(1-\beta)q$ can be sold, where $\alpha$ and $\beta$ are positive constants. In addition to the volume losses, the firm also incurs a variable cost of $c_\alpha q$ when $q$ units are stored, and a variable cost of $c_\beta q$ when $q$ units are withdrawn, where $c_\alpha$ and $c_\beta$ are positive constants. These costs cover the use of pumps and other equipment (*Maragos* 2002).

Many firms contract gas storage for one year and must remove the gas before the end of the term (usually March 31, the end of the peak season) or pay a penalty (*Buurma* 2010). The penalty is typically proportional to the leftover inventory (*Carmona and Ludkovski* 2010, *Chen and Forsyth* 2007) or in general form (*Boogert and De Jong* 2008). We let $p \geq 0$ denote the penalty per unit of inventory at the end of period $N$; $p$ is realized in period $N$ and may depend on the market prices modeled below.

### 2.3.2 Price Model and Problem Formulation

At the beginning of period $t$, the futures price for delivery in period $t$ is maturing, denoted as $\widetilde{f}_{tt}$. The firm sees this maturing price and other futures prices $\widetilde{f}_{t\tau}$ that mature in period $\tau = t+1, \ldots, N$, and decides the quantity to purchase or sell at price $\widetilde{f}_{tt}$. The settled amount is then stored in or released from the storage over the entire period $t$.

We make the standard no-arbitrage assumption under which the futures prices are martingales under an equivalent martingale measure $Q$ (see, e.g., *Duffie* 2001):

$$\widetilde{f}_{t\tau} = \mathsf{E}_t^Q\big[\widetilde{f}_{s\tau}\big], \qquad t < s \leq \tau, \tag{2.1}$$

where $\mathsf{E}_t^Q$ denotes the expectation under $Q$-measure with information available up to the beginning of period $t$. If the futures market is absent, all results in this paper continue to hold with $\widetilde{f}_{tt}$ interpreted as the spot price in period $t$ and $\widetilde{f}_{t\tau}$ interpreted as the forecast in period $t$ for the price in period $\tau$. We choose to model the futures market because it provides the firm with instruments to hedge the storage value (perfect hedging is achievable in a complete market).

We refer to $(1+\alpha)\widetilde{f}_{t\tau} + c_\alpha$ as the buying price of inventory, the price the firm must pay for having one unit of inventory available in the storage in period $\tau$. This price includes procurement cost, volume losses, and operating costs. Similarly, we refer to $(1-\beta)\widetilde{f}_{t\tau} - c_\beta$ as the selling price of inventory, which is the net profit the firm obtains from releasing one unit of inventory in period $\tau$.

To derive the expected discounted value of the storage, we note that the expected marked-to-market profit/loss from the futures positions held by the firm is zero under $Q$-measure, since futures prices are martingales. Hence, if the firm does not have capital constraints, the no-arbitrage value of the storage is the sum of cash flows at maturity dates evaluated under $Q$-measure and discounted at the risk-free rate (see, e.g., *Duffie* 2001). Operations of large energy storage facilities often require large sums of capital, thereby increasing the possibility of financial distress during the storing season. *Froot and Stein* (1998) show that firms require investments to yield a higher return when all risks cannot be frictionlessly hedged. For the purpose of this paper, we assume that the firm discounts the cash flows at a constant rate $R$. The insights of the paper are intact under any choice of $R$, including the risk-free rate.

Define $f_{t\tau}$ and $f_{t\tau}^b$ respectively as the selling price and buying price of inventory discounted to the first period:

$$f_{t\tau} \stackrel{\text{def}}{=} e^{-R(\tau-1)}\big[(1-\beta)\widetilde{f}_{t\tau} - c_\beta\big], \qquad f^b_{t\tau} \stackrel{\text{def}}{=} e^{-R(\tau-1)}\big[(1+\alpha)\widetilde{f}_{t\tau} + c_\alpha\big]. \tag{2.2}$$

Discounting the prices back to the first period allows not to include the discount factor in the problem formulation in (2.3) below, which simplifies the analytical expressions throughout the paper. Note that for any fixed maturity $\tau$, the discounted selling and buying prices in (2.2) are still martingales.

Let $\mathbf{f}_t = (f_{t\tau} : \tau = t, t+1, \ldots, N)$ be the *discounted forward selling price curve* (or simply *forward curve* when no confusion arises) observed at the beginning of period $t$. Let $V_t(x_t, \mathbf{f}_t)$ be the discounted expected profit-to-go from period $t$ onward. Let $y_t$ be the ending inventory in period $t$, which is decided by the firm at the beginning of period $t$.

The storage valuation problem can be written as:

$$V_t(x_t, \mathbf{f}_t) = \max_{y_t \in [\underline{y}(x_t), \overline{y}(x_t)]} r(y_t - x_t, f_{tt}) + \mathsf{E}^Q_t\big[V_{t+1}(y_t, \mathbf{f}_{t+1})\big], \tag{2.3}$$

where the one-period reward function $r(q, f_{tt}) \stackrel{\text{def}}{=} -f^b_{tt}\, q$, if $q \geq 0$ (purchase), and $r(q, f_{tt}) \stackrel{\text{def}}{=} -f_{tt}\, q$, if $q < 0$ (sell); the period-ending inventory is bounded between $\underline{y}(x_t) = x_t + \underline{\lambda}(x_t)$ and $\overline{y}(x_t) = x_t + \overline{\lambda}(x_t)$. In the last period, the firm sells as much as possible to maximize the profit, and thus,

$$V_N(x_N, f_{NN}) \;=\; -f_{NN}\, \underline{\lambda}(x_N) - \underline{y}(x_N)p. \tag{2.4}$$

In general, solving the problem in (2.3)-(2.4) is complicated. A widely-used heuristic policy is detailed below.

### 2.3.3 Rolling Intrinsic Policy

To define the rolling intrinsic (RI) policy, we first define the *intrinsic policy*, a policy that decides in the first period the actions to be performed in each of the remaining periods. The intrinsic policy is found by solving an optimization problem using *only* the forward prices seen in the first period. The corresponding value is called the *intrinsic value.* The RI policy re-optimizes the action in each period by solving the intrinsic valuation problem using the updated forward prices. We refer to the corresponding value as the *rolling intrinsic value.* The RI policy is commonly used in practice (*Gray and Khandelwal* 2004a,b) and is also referred to as the reoptimized intrinsic policy by *Secomandi* (2010) and *Lai et al.* (2010). Because futures prices are martingales, the RI heuristic essentially replaces uncertain prices by their expected values, which is a type of certainty equivalent control studied by *Bertsekas* (2005). The policy is formally defined below.

Let $V_t^{\mathrm{I}}(x_t, \mathbf{f}_t)$ and $V_t^{\mathrm{RI}}(x_t, \mathbf{f}_t)$ denote the intrinsic value and the rolling intrinsic value of the storage in period $t$, respectively.

In period $t$, given the discounted forward selling prices $\mathbf{f}_t = (f_{t\tau} : \tau \geq t)$, the intrinsic value of the storage $V_t^{\mathrm{I}}(x_t, \mathbf{f}_t)$ is determined by:

$$V_N^{\mathrm{I}}(x_N, \mathbf{f}_t) = -f_{tN}\,\underline{\lambda}(x_N) - \underline{y}(x_N)\mathsf{E}_t^Q[p], \tag{2.5}$$

$$V_s^{\mathrm{I}}(x_s, \mathbf{f}_t) = \max_{y_s \in [\underline{y}(x_s), \overline{y}(x_s)]} r(y_s - x_s, f_{ts}) + V_{s+1}^{\mathrm{I}}(y_s, \mathbf{f}_t), \qquad t \leq s < N. \tag{2.6}$$

When $t = 1$, the recursion in (2.5)-(2.6) yields the intrinsic policy in period 1. If the firm implements the intrinsic policy via futures contracts in period 1 and holds all contracts until maturity, then the policy yields the intrinsic value $V_1^{\mathrm{I}}(x_1, \mathbf{f}_1)$.

In the RI policy, the firm solves (2.5)-(2.6) in every period with updated forward curve $\mathbf{f}_t$, and adjusts the futures positions accordingly. Let $y_t^{\dagger}$ be the futures position on the maturing contract in period $t$, solved from (2.5)-(2.6). Then, the rolling intrinsic value of the storage is defined as:

$$V_N^{\mathrm{RI}}(x_N, \mathbf{f}_N) = V_N^{\mathrm{I}}(x_N, \mathbf{f}_N), \tag{2.7}$$

$$V_t^{\mathrm{RI}}(x_t, \mathbf{f}_t) = r(y_t^{\dagger} - x_t, f_{tt}) + \mathsf{E}_t^Q\left[V_{t+1}^{\mathrm{RI}}(y_t^{\dagger}, \mathbf{f}_{t+1})\right], \qquad 1 \leq t < N. \tag{2.8}$$

## 2.4. Improving the RI Policy: The Three-Period Case

This section introduces the main ideas of improving the RI policy. In §2.4.1, we consider several simple examples that lead to the construction of a new heuristic policy − the price-adjusted rolling intrinsic (PARI) policy. In §2.4.2, we prove the optimality of the PARI policy for the three-period setting.

### 2.4.1 From RI Policy to PARI Policy

The RI policy solves a deterministic optimization problem every period and may miss potential option values rising from the stochastic evolution of the forward curve. The idea of the PARI policy is to adjust the forward curve to inform the RI policy about the value of various options. The following three examples each illustrate a different option value and introduce a price adjustment scheme to capture the option value.

The common settings of all the examples are as follows. The storage size is $K = 4$ units. The storage can release (store) three units per period as long as inventory (space) is available, i.e., $\underline{\lambda}(x) = \max\{-x, -3\}$ and $\overline{\lambda}(x) = \min\{4 - x, 3\}$. The operating cost parameters are: $\alpha = 2\%$, $\beta = 1\%$, $c_\alpha = c_\beta = \$0.02$. Assume the discount rate $R = 0$. Then, the definitions in (2.2) imply that $f^b_{t\tau} = \frac{1+\alpha}{1-\beta}(f_{t\tau} + c_\beta) + c_\alpha = 1.03 f_{t\tau} + 0.04$. We assume the storage is initially full and consider a three-period ($N = 3$) selling season problem.

**Example 1: Value of waiting.** Suppose in period 1 the forward selling price curve is $(f_{11}, f_{12}, f_{13})$ = (\$5.00, \$4.97, \$4.95). The intrinsic policy can be found by a greedy method: sell three units at the highest price \$5.00 and sell one unit at the second highest price \$4.97. Thus, the intrinsic value of the storage is \$19.97. (Operating costs are accounted for in the selling prices.)

Under the RI policy, the firm first sells three units at \$5.00, as prescribed in the intrinsic policy. In the second period, assume the selling prices (martingales) evolve as follows: $(f_{22}, f_{23})$ = (\$5.30, \$5.10) with probability 0.5, and $(f_{22}, f_{23})$ = (\$4.64, \$4.80) with probability 0.5. Upon price increase, the RI policy is to sell the remaining unit at \$5.30. Upon price decrease, the RI policy is to do nothing in the second period (no incentive to buy because

$f^b_{22} = 1.03 \times 4.64 + 0.04 = \$4.82 > f_{23}$) and sell the remaining unit at \$4.80 in the third period. Thus, the remaining unit is sold at an expected price of $(\$5.30 + \$4.80)/2 = \$5.05$. The expected rolling intrinsic value of the storage is \$20.05.

In the above RI policy, the firm effectively sells energy at $\mathsf{E}^Q_1\big[\max\{f_{22}, f_{23}\}\big] = \$5.05$ by exploiting the flexibility of when to sell, but this flexibility is limited: The storage can release at most three units per period. Hence, the optimal policy is to sell one unit at \$5.00 in the first period and sell the remaining three units at \$5.05 in expectation, yielding the optimal expected profit of \$20.15. Thus, although the maturing price $f_{11}$ is the highest on the forward curve, there is a value of delaying sales.

Let us preview one of the key ideas behind the price-adjusted rolling intrinsic (PARI) policy. The original forward curve does not reveal the value of waiting, because $\max\{f_{12}, f_{13}\} < f_{11}$. Suppose we adjust either $f_{12}$ or $f_{13}$ up to \$5.05, and use the adjusted forward curve as the input to the RI policy. Then, because $f_{11} = \$5.00$ is the second highest among the adjusted prices, the RI policy is to sell only one unit at \$5.00. Hence, for this example, adjusting either $f_{12}$ or $f_{13}$ up to $\mathsf{E}^Q_1\big[\max\{f_{22}, f_{23}\}\big]$ informs the RI policy about the value of waiting and brings the RI decision to the optimal. □

**Example 2: Value of potential purchase.** Suppose in period 1 the forward curve is $(f_{11}, f_{12}, f_{13})$ = (\$5.00, \$4.85, \$5.05). The intrinsic policy is to sell one unit at \$5.00 and sell the remaining three units at \$5.05, yielding an intrinsic value of \$20.15. Selling more in the first period and buying in the second period cannot improve the intrinsic value, because the buying price $f^b_{12} = 1.03f_{12} + 0.04 = \$5.04 > f_{11}$.

Under the RI policy, the firm sells one unit in the first period. In the second period, assume the martingale selling prices $(f_{22}, f_{23})$ is (\$5.20, \$5.20) or (\$4.50, \$4.90) with equal probabilities. If $(f_{22}, f_{23})$ = (\$5.20, \$5.20), the firm sells the remaining three units at \$5.20. If $(f_{22}, f_{23})$ = (\$4.50, \$4.90), the firm faces a low buying price $f^b_{22} = 1.03f_{22} + 0.04 = \$4.68$ and can make a profit of $f_{23} - f^b_{22} = \$0.22$ per unit by buying at $f^b_{22}$ and selling at $f_{23}$. However, it can capture this opportunity only if the storage has less than three units at the start of the second period, which is not the case under the RI policy. Hence, the storage value under the RI policy remains \$20.15.

Let us now consider the strategy of selling $1+\varepsilon$ units in the first period, where $\varepsilon \in [0,2]$. Based on Example 1, this strategy gives up some value of waiting: $(\mathsf{E}_1^Q[\max\{f_{22}, f_{23}\}] - f_{11})\varepsilon = \$0.05\varepsilon$, but it brings an extra profit of $\mathsf{E}_1^Q[\max\{f_{23} - f_{22}^b, 0\}]\varepsilon = \$0.11\varepsilon$ from the potential purchase in the second period. The net expected gain is $0.06\varepsilon$. The optimal policy is to sell three units in the first period, i.e., $\varepsilon = 2$, yielding an extra profit of \$0.12 and raising the storage value to \$20.27.

This leads to the second key idea of the PARI policy. The forward buying price $f_{12}^b =$ \$5.04 is too high to reveal the option value of buying inventory in the second period. Let us adjust $f_{12}^b$ down to $\widehat{f}_{12}^b = (\$4.68 + \$5.20)/2 = \$4.94$, implying that $f_{12}$ is lowered to $\widehat{f}_{12} = \$4.76$. Under the adjusted prices $(f_{11}, \widehat{f}_{12}, f_{13}) =$ (\$5.00, \$4.76, \$5.05), the RI policy is to sell three units at the maturing price \$5.00, which coincides with the optimal policy. Note that $f_{13} - \widehat{f}_{12}^b = \$5.05 - \$4.94 = \$0.11$ equals $\mathsf{E}_1^Q[\max\{f_{23} - f_{22}^b, 0\}]$, representing the value of potential purchase. □

**Example 3: Value of avoiding adverse price.** Suppose $(f_{11}, f_{12}, f_{13}) =$ (\$5.00, \$5.05, \$5.02). Note the maturing price $f_{11}$ is the lowest. The intrinsic value is \$20.17, which is the profit of selling three units at \$5.05 and one unit at \$5.02.

The RI policy is to do nothing in the first period. In the second period, assume $(f_{22}, f_{23})$ is (\$5.40, \$5.10) or (\$4.70, \$4.94) with equal probabilities. Upon price increase (or decrease), the RI policy sells three units at \$5.40 (or \$4.94) and one unit at \$5.10 (or \$4.70). The expected value of the storage under the RI policy is \$20.41.

However, if in the first period the firm sells $\varepsilon \in (0,1]$ units at the lowest price $f_{11} = \$5.00$, then upon price increase (or decrease) it sells $1-\varepsilon$ units at \$5.10 (or \$4.70). Thus, by selling $\varepsilon$ units at \$5.00 now, the firm sells $\varepsilon$ units less at an expected price $(\$5.10+\$4.70)/2 = \$4.90$, which equals to the expected minimum price $\mathsf{E}_1^Q[\min\{f_{22}, f_{23}\}]$. The optimal policy is to set $\varepsilon = 1$, and the storage value is improved to \$20.51.

We introduce another idea of the PARI policy that helps the firm avoid selling at the adverse price. The original forward curve does not reveal the adverse price, because

$\min\{f_{12}, f_{13}\} > f_{11}$. Suppose we adjust either $f_{12}$ or $f_{13}$ down to $\mathsf{E}_1^Q[\min\{f_{22}, f_{23}\}] = \$4.90$, and use the adjusted forward curve as the input to the RI policy. Then, because $f_{11} = \$5.00$ is no longer the lowest price among the adjusted prices, the RI policy is to sell one unit at \$5.00, which coincides with the optimal policy. Note that $f_{11} - \mathsf{E}_1^Q[\min\{f_{22}, f_{23}\}] = \$0.10$ is exactly the value difference between the optimal policy and the RI policy. □

The previous examples show three different option values under constant storing and releasing capacities. In Example 3, if the maximum releasing speed increases in the inventory level, there is an incentive not to sell in the first period, because keeping a higher inventory level raises the releasing capacity in the second period, allowing the firm to sell more at $f_{22}$ and less at $f_{23}$ when $f_{22} > f_{23}$. This is the fourth option value – value of raising operational capacity.

We summarize the four option values in Table 2.1. For the value of potential purchase, we use a more general term "value of counter-seasonal operations." The third column shows the impact of the option values on the first-period decision. The fourth and fifth columns show the option values and the related spreads seen on the forward curve in the first period.

Table 2.1: Summary of option values in the selling season

| | | Impact on $y_1^*$ | Option value | Related spread on forward curve | Price adjustment |
|---|---|---|---|---|---|
| $f_{11} > f_{12}$ | Value of waiting | $\uparrow$ | $\mathsf{E}_1^Q[\max\{f_{22}, f_{23}\}] - f_{11}$ | $\max\{f_{12}, f_{13}\} - f_{11}$ | $f_{13}\uparrow$, $f_{12}\downarrow$ |
| | Value of counter-seasonal operations | $\downarrow$ | $\mathsf{E}_1^Q[\max\{f_{23} - f_{22}^b, 0\}]$ | $f_{13} - f_{12}^b$ | |
| $f_{11} < f_{12}$ | Value of avoiding adverse price | $\downarrow$ | $f_{11} - \mathsf{E}_1^Q[\min\{f_{22}, f_{23}\}]$ | $f_{11} - \min\{f_{12}, f_{13}\}$ | $f_{13}\downarrow$, $f_{12}$ stays |
| | Value of raising operational capacity | $\uparrow$ | $\mathsf{E}_1^Q[\max\{f_{22} - f_{23}, 0\}]$ | $f_{12} - f_{13}$ | |

In Table 2.1, the option values (column 4) typically exceed the corresponding spreads on the forward curve (column 5). The idea of the PARI policy is to adjust the forward curve to bring the deterministic spreads closer to the option values. Interestingly, there exists a set of price adjustments under which the deterministic spreads equal the option values. This set of price adjustments is stated in Definition 2.1 below; the last column of Table 2.1 shows the direction of the price adjustments.

**Definition 2.1. Price-adjusted rolling intrinsic (PARI) policy for $N=3$**
Step 1. Price adjustment. Based on the forward curve $\mathbf{f}_1$, define a new forward curve $\widehat{\mathbf{f}}_1$ as follows.

(i) When $f_{11} > f_{12}$, define $\widehat{\mathbf{f}}_1 = (f_{11}, \widehat{f}_{12}, \widehat{f}_{13})$ such that

$$\widehat{f}_{12}^{b} = \mathsf{E}_1^Q\big[\text{median}\{f_{22}, f_{22}^{b}, f_{23}\}\big] \qquad \text{and} \qquad \widehat{f}_{13} = \mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big].$$

(ii) When $f_{11} \le f_{12}$, define $\widehat{\mathbf{f}}_1 = (f_{11}, f_{12}, \widehat{f}_{13})$ where

$$\widehat{f}_{13} = \mathsf{E}_1^Q\big[\min\{f_{22}, f_{23}\}\big].$$

Step 2. In the first period, we solve the intrinsic valuation problem (2.5)-(2.6) with $\mathbf{f}_1$ replaced by $\widehat{\mathbf{f}}_1$, and implement the corresponding first-period decision.
Step 3. Apply the regular RI policy for the remaining two periods.

The three previous examples assume binomial price processes and constant injection and withdrawal speeds. One surprising result is that the above PARI policy is optimal for the three-period model under general price distributions and capacity functions. We now turn to prove this optimality.

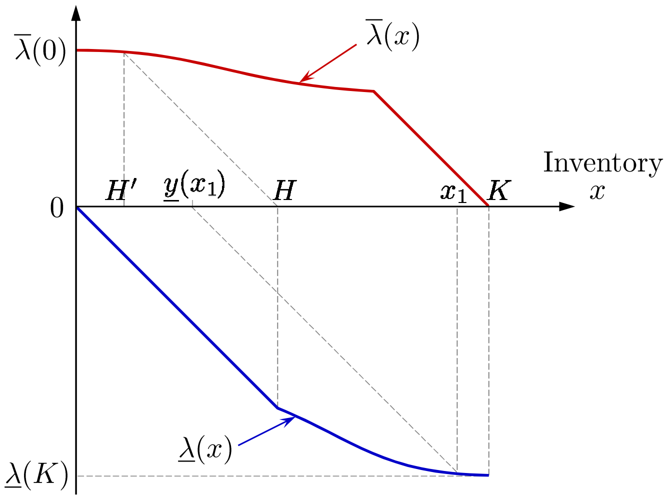
### 2.4.2 Optimality of the PARI Policy

We assume the storage can be emptied in two out of three periods, capturing the limited flexibility of typical seasonal storage. Formally, this assumption is stated as follows:

**Assumption 2.** *(i)* $x_1 > H$. *(ii)* $|\underline{\lambda}(K)| > K - H$.

Part (i) suggests that the initial inventory cannot be sold in a single period. Part (ii) implies that a full storage can release more than $K-H$ in one period. Thus, a full storage can be emptied in two out of three periods. Typical capacity functions satisfying Assumptions 1 and 2 are illustrated in Figure 2.2. In the figure, $H'$ will be defined in Lemma 1.

Figure 2.2: Storing and releasing capacity functions for the three-period model




We first show that Step 3 of the PARI policy is optimal for the last two periods.

**Proposition 1.** *(i) The RI policy is optimal for the last two periods.*
*(ii) If the penalty satisfies* $\mathsf{P}\Big\{p \geq \dfrac{\underline{s} f_{33} - f_{22}}{1-\underline{s}}\Big\} = 1$, *where* $\underline{s} \overset{def}{=} \sup\{-\underline{\lambda}'(x) : x \in (H, K]\}$, *then for any given first-period decision* $y_1$, *the second-period RI (optimal) decision is:*

$$y_2^*(y_1, \mathbf{f}_2) - y_1 = \begin{cases} \underline{\lambda}(y_1), & \text{if } f_{22} \geq f_{23}, \\ \min\{H - y_1,\, 0\}, & \text{if } f_{22} < f_{23} \leq f_{22}^b, \\ \min\{H - y_1,\, \overline{\lambda}(y_1)\}, & \text{if } f_{22}^b < f_{23}. \end{cases} \tag{2.9}$$

*Furthermore,* $y_2^*(y_1, \mathbf{f}_2) \leq H$, *and the storage is emptied in the third period.*

The penalty condition in the above proposition is typically satisfied in practice. Under the constant capacities, we have $\underline{s} = 0$ and the penalty condition clearly holds. When the injection and withdrawal speeds vary with inventory, $\underline{s}$ is typically no more than 0.5 (see Figure 2.1). Thus, the term $\underline{s} f_{33} - f_{22}$ is typically negative, given the fact that the end-of-season selling price $f_{33}$ is typically lower than the mid-season selling price $f_{22}$ (see an example in §2.6.1).

The RI policy in (2.9) reacts to the forward curve as follows: If the forward curve is downward sloping $f_{22} \geq f_{23}$, the firm sells as much as possible at price $f_{22}$. If $f_{22} < f_{23}$, the firm has an incentive to delay sales but needs to sell inventory down to $H$ so that all inventory can be sold in the last period. If the period-starting inventory $x_2$ is already below $H$ and if the forward curve is steeply upward-sloping $f_{22}^b < f_{23}$, then the firm buys inventory up to or as close as possible to $H$.

Using the second-period optimal action in (2.9), we can write the first-period problem as:

$$
\begin{aligned}
V_1(x_1, \mathbf{f}_1) &= \max_{y_1 \in [\underline{y}(x_1), \overline{y}(x_1)]} U_1(x_1, y_1, \mathbf{f}_1), \qquad (2.10)\\
U_1(x_1, y_1, \mathbf{f}_1) &= r(y_1 - x_1, f_{11}) + \mathsf{E}_1^Q\big[r\big(y_2^*(y_1, \mathbf{f}_2) - y_1, f_{22}\big) + f_{23}\, y_2^*(y_1, \mathbf{f}_2)\big]\\
&= r(y_1 - x_1, f_{11}) + f_{13} y_1 + \mathsf{E}_1^Q\big[r\big(y_2^*(y_1, \mathbf{f}_2) - y_1, f_{22}\big) + f_{23}(y_2^*(y_1, \mathbf{f}_2) - y_1)\big]\\
&= r(y_1 - x_1, f_{11}) + f_{13} y_1 + \mathsf{P}\{A_1\}\mathsf{E}_1^Q\big[(-f_{22} + f_{23})\underline{\lambda}(y_1) \mid A_1\big]\\
&\quad + \mathsf{P}\{A_2\}\mathsf{E}_1^Q\big[(-f_{22} + f_{23}) \min\{H - y_1, 0\} \mid A_2\big]\\
&\quad + \mathsf{P}\{A_3\}\mathsf{E}_1^Q\big[r\big(\min\{H - y_1, \overline{\lambda}(y_1)\},\, f_{22}\big) + f_{23}(\min\{H - y_1, \overline{\lambda}(y_1)\}) \mid A_3\big],
\end{aligned}
$$

where $A_1 = \{f_{22} \geq f_{23}\}$ is the downward-sloping forward curve event, $A_2 = \{f_{22} < f_{23} \leq f_{22}^b\}$ is referred to as the slightly upward-sloping forward curve event, and $A_3 = \{f_{22}^b < f_{23}\}$ is the steeply upward-sloping forward curve event.

Next, we prove the optimality of the PARI policy by analyzing the optimal policy and comparing it with the RI policy. We study two cases: $f_{11} > f_{12}$ and $f_{11} < f_{12}$.

#### 2.4.2.1 Case of $f_{11} > f_{12}$

For this case, we show in the appendix that the problem (2.10) can be rewritten as:

$$
\max_{y_1 \in [\underline{y}(x_1), H]} V^w y_1 + V^c \min\{H - y_1, \overline{\lambda}(y_1)\}, \qquad (2.11)
$$

where,

$$V^w \overset{\text{def}}{=} \mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big] - f_{11} = \text{ value of waiting,} \tag{2.12}$$

$$V^c \overset{\text{def}}{=} \mathsf{E}_1^Q\big[\max\{f_{23} - f_{22}^b, 0\}\big] = \text{ value of potential purchase (counter-season operations).} \tag{2.13}$$

By definition, $V^c \geq 0$, and the sign of $V^w$ is unrestricted. The optimal policy for the first period is summarized in the lemma below.

**Lemma 1.** *In the first period, if $f_{11} > f_{12}$, then the optimal decision $y_1^*$ is determined as follows:*

*(a) If $V^w \leq 0$, then $y_1^* = \underline{y}(x_1)$;*

*(b) If $V^w > V^c$, then $y_1^* = H$;*

*(c) If $0 < V^w \leq V^c$, then $y_1^* = \underline{y}(x_1)$ when $\underline{y}(x_1) \geq H'$; when $\underline{y}(x_1) < H'$, $y_1^*$ is determined by $\max_{y_1 \in [\underline{y}(x_1),\, H']} V^w y_1 + V^c \overline{\lambda}(y_1)$, where $H'$ is defined by*

$$H' \overset{\text{def}}{=} \inf\{y \in [0, K] \,:\, y + \overline{\lambda}(y) \geq H\}. \tag{2.14}$$

The value of waiting $V^w$ and the value of potential purchase $V^c$ drive the decision $y_1^*$ in opposite directions, as shown in Table 2.1. Lemma 1(b) and (c) reveal the tradeoff between the two values:

- When $V^w > V^c$, the firm should exercise all options of waiting by keeping $H$ units unsold at the end of the first period, leaving no option of purchase in the second period.
- When $0 < V^w < V^c$, the firm should sell as much energy as possible in the first period, as long as it can buy inventory up to $H$ in the second period (this condition is formally stated as $\underline{y}(x_1) \geq H'$, where $H'$ is the level above which the inventory can be raised to $H$ in one period), thereby giving up the options of waiting while maximizing the opportunity of purchase.

Next, we describe the first-period RI policy in the following lemma.

**Lemma 2.** *In the first period, if $f_{11} > f_{12}$, then under the RI policy, $y_1^{\dagger}$ is determined as follows:*

*(a) If $f_{11} \geq \max\{f_{12}, f_{13}\}$, then $y_1^{\dagger} = \underline{y}(x_1)$;*

*(b) If $f_{11} < \min\{f_{12}^b, f_{13}\}$, then $y_1^{\dagger} = H$;*

*(c) If $f_{13} > f_{11} \geq f_{12}^b$, then $y_1^{\dagger} = \underline{y}(x_1)$ when $\underline{y}(x_1) \geq H'$; when $\underline{y}(x_1) < H'$, $y_1^{\dagger}$ is determined by $\max_{y_1 \in [\underline{y}(x_1),\, H']} (f_{13} - f_{11})y_1 + (f_{13} - f_{12}^b)\overline{\lambda}(y_1)$.*

Comparing the optimal policy and the RI policy, we can prove that if the forward curve in Lemma 2 is adjusted according to Definition 2.1, the resulting PARI policy is the optimal policy in Lemma 1, as stated in the following proposition.

**Proposition 2.** *When $N = 3$ and $f_{11} > f_{12}$, the price-adjusted rolling intrinsic (PARI) policy in Definition 2.1 is optimal. In particular, solving the intrinsic valuation problem* (2.5)-(2.6) *with $\widehat{f}_{12}^b = \mathsf{E}_1^Q\left[\text{median}\{f_{22}, f_{22}^b, f_{23}\}\right]$ and $\widehat{f}_{13} = \mathsf{E}_1^Q\left[\max\{f_{22}, f_{23}\}\right]$ yields the optimal policy for the first period.*

Raising $f_{13}$ allows the RI policy to see the best selling opportunity in the future, thus capturing the value of waiting. Note that $f_{12}$ is adjusted down because $\widehat{f}_{12}^b = \mathsf{E}_1^Q\left[\text{median}\{f_{22}, f_{22}^b, f_{23}\}\right] \leq \mathsf{E}_1^Q\left[\max\{f_{22}, f_{22}^b\}\right] = \mathsf{E}_1^Q\left[f_{22}^b\right] = f_{12}^b$. Lowering $f_{12}$ enlarges the gap between $f_{12}$ and $f_{13}$, which reflects the value of counter-seasonal operations.

#### 2.4.2.2 Case of $f_{11} < f_{12}$

The appendix shows that in this case the problem in (2.10) simplifies to:

$$\max_{y_1 \in [H, \overline{y}(x_1)]} U_1(y_1) = \begin{cases} f_{11}x_1 - V^a y_1 - V^l \underline{\lambda}(y_1), & \text{if } y_1 \in [H, x_1], \\ f_{11}^b x_1 - V^{ab} y_1 - V^l \underline{\lambda}(y_1), & \text{if } y_1 \in (x_1,\, \overline{y}(x_1)], \end{cases} \tag{2.15}$$

where

$$V^a \stackrel{\text{def}}{=} f_{11} - \mathsf{E}_1^Q\big[\min\{f_{22}, f_{23}\}\big] = \text{ value of avoiding adverse price by selling one more unit,} \tag{2.16}$$

$$V^{ab} \stackrel{\text{def}}{=} f_{11}^b - \mathsf{E}_1^Q\big[\min\{f_{22}, f_{23}\}\big] = \text{ value of avoiding adverse price by buying one less unit,} \tag{2.17}$$

$$V^l \stackrel{\text{def}}{=} \mathsf{E}_1^Q\big[\max\{f_{22} - f_{23}, 0\}\big] = \text{ value of raising operational capacity.} \tag{2.18}$$

By definition, $V^l \geq 0$, $V^a < V^{ab}$, and the signs of $V^a$ and $V^{ab}$ are unrestricted. Furthermore, $V^a < V^l$ because $V^a - V^l = f_{11} - \mathsf{E}_1^Q\big[\min\{f_{22}, f_{23}\} + \max\{f_{22}, f_{23}\} - f_{23}\big] = f_{11} - f_{12} < 0$.

The following lemma summarizes the optimal policy in this case.

**Lemma 3.** *In the first period, if $f_{11} < f_{12}$, then the optimal decision $y_1^*$ is determined as follows:*

*(a) If $V^{ab} \leq 0$, then $y_1^* = \overline{y}(x_1)$;*

*(b) If $V^a \leq 0 < V^{ab} \leq V^l$, then $y_1^* \in \underset{y_1 \in [x_1, \overline{y}(x_1)]}{\arg\max} -V^{ab} y_1 - V^l \underline{\lambda}(y_1)$;*

*(c) If $V^a \leq 0 \leq V^l < V^{ab}$, then $y_1^* = x_1$;*

*(d) If $V^a > 0$, then $y_1^* \in \underset{y_1 \in [H, \overline{y}(x_1)]}{\arg\max} U_1(y_1)$, where $U_1(y_1)$ is defined in* (2.15).

Example 3 in §2.4.1 shows that even if $f_{11} < \min\{f_{12}, f_{13}\}$ and all the inventory can be sold in the later periods, selling some inventory in the first period may still be beneficial as it avoids the expected minimum selling price. Similarly, even if $f_{11}^b < \min\{f_{12}, f_{13}\}$, the firm needs to be cautious about buying because the expected minimum price may be below the buying price. We thus refer to $V^{ab}$ in (2.17) as the value of avoiding adverse price by buying one less unit. Only when $V^{ab} \leq 0$, should the firm purchase as much as possible, as confirmed in Lemma 3(a).

The value of avoiding adverse price ($V^a$ or $V^{ab}$) and the value of raising operational capacity ($V^l$) drive the decision $y_1^*$ in opposite directions. When $V^a \leq 0$ (implying that selling inventory brings no benefit), the firm trades off between $V^l$ and $V^{ab}$ to decide the purchase quantity, as prescribed in Lemma 3(b) and (c). When $V^a > 0$, the optimal action may be purchase or sell, determined in part (d).

Next, we summarize the first-period RI policy in the following lemma.

**Lemma 4.** *In the first period, if $f_{11} < f_{12}$, then under the RI policy, $y_1^\dagger$ is determined as follows:*

*(a) If $f_{11}^b \leq \min\{f_{12}, f_{13}\}$, then $y_1^\dagger = \overline{y}(x_1)$;*

*(b) If $f_{11} \leq \min\{f_{12}, f_{13}\} < f_{11}^b \leq f_{12}$, then*
$$y_1^\dagger \in \underset{y_1 \in [x_1, \overline{y}(x_1)]}{\arg\max} \; -\left(f_{11}^b - \min\{f_{12}, f_{13}\}\right) y_1 - \max\{f_{12} - f_{13}, 0\} \underline{\lambda}(y_1);$$

*(c) If $f_{11} \leq \min\{f_{12}, f_{13}\}$ and $f_{12} < f_{11}^b$, then $y_1^\dagger = x_1$;*

*(d) If $f_{11} > f_{13}$, then $y_1^\dagger \in \underset{y_1 \in [H, \overline{y}(x_1)]}{\arg\max} \; U_1^{RI}(y_1)$, where*

$$U_1^{RI}(y_1) = \begin{cases} f_{11}x_1 - (f_{11} - f_{13})y_1 - (f_{12} - f_{13})\underline{\lambda}(y_1), & \text{if } y_1 \in [H, x_1], \\ f_{11}^b x_1 - (f_{11}^b - f_{13})y_1 - (f_{12} - f_{13})\underline{\lambda}(y_1), & \text{if } y_1 \in (x_1, \overline{y}(x_1)]. \end{cases} \tag{2.19}$$

We can prove that if the forward curve in Lemma 4 is adjusted according to Definition 2.1, the resulting PARI policy is the optimal policy in Lemma 3, as stated below.

**Proposition 3.** *When $N = 3$ and $f_{11} < f_{12}$, the price-adjusted rolling intrinsic (PARI) policy in Definition 2.1 is optimal. In particular, solving the intrinsic valuation problem* (2.5)-(2.6) *with $\widehat{f}_{13} = \mathsf{E}_1^Q\left[\min\{f_{22}, f_{23}\}\right]$ yields the optimal policy for the first period.*

Adjusting $f_{13}$ alone captures two values. The adjusted price $\widehat{f}_{13}$ informs the firm about the adverse price in the future. Meanwhile, the difference between $f_{12}$ and $\widehat{f}_{13}$ reflects the value of raising operational capacity.

## 2.5. Improving the RI Policy: The $N$-Period Case

In §2.5.1 and §2.5.2, we consider a multiperiod model ($N \geq 3$) with constant capacities, and show that the value of waiting, counter-seasonal operations, and avoiding adverse price characterize the optimal policy. Because of the constant capacities, the value of raising operational capacity does not appear in the tradeoffs. In §2.5.3, we extend the PARI policy to the $N$-period problem. In §2.5.4, we further extend the PARI policy to multiple seasons, with each season containing multiple periods.

### 2.5.1 Value of Waiting and Value of Avoiding Adverse Price

To focus on the value of waiting and value of avoiding adverse price, we first consider a problem of selling inventory over $N$ periods and delay considering injection (counter-seasonal) operations in §2.5.2. The capacity functions satisfy the following assumption:

**Assumption 3.** *(i)* $\underline{\lambda}(x) = \max\{\underline{C}, -x\}$*, where* $\underline{C} < 0$*; (ii)* $K = T|\underline{C}|$ *for some* $T \in \{2, 3, \ldots, N\}$*; (iii)* $\overline{\lambda}(x) = 0$.

Part (i) suggests that the storage can release $|\underline{C}|$ per period until it is empty, following *Secomandi* (2010) and *Lai et al.* (2010). Part (ii) assumes that a full storage can be emptied in exactly $T$ periods when releasing energy at the maximum rate. Although part (ii) is not crucial, it simplifies the exposition of our analysis. Part (iii) implies injection operations are not considered.

We let $f_t \equiv f_{tt}$ for notational convenience. For period $t$, we introduce a $T$-dimensional vector $\mathbf{u}_t = [u_t^{(1)}, u_t^{(2)}, \ldots, u_t^{(T)}]$, whose $k$-th element $u_t^{(k)}$ represents the *expected $k$-th largest price at which inventory may be sold from period $t$ onward.* Formally,

$$\mathbf{u}_N \stackrel{\text{def}}{=} [f_N, 0, \ldots, 0],$$

$$u_t^{(k)} \stackrel{\text{def}}{=} k\text{-th largest element of } \{f_t, \mathsf{E}_t^Q \mathbf{u}_{t+1}\}, \quad k = 1, \ldots, T, \quad t = 1, \ldots, N-1. \quad (2.20)$$

Let $H_k \stackrel{\text{def}}{=} k|\underline{C}|$, for $k = 0, 1, \ldots, T$. In period $t < N$, when the inventory level is $x_t \in (H_{k-1}, H_k]$, we extend the definitions for the value of waiting and value of avoiding adverse price:

$$V_{tk}^w \stackrel{\text{def}}{=} \mathsf{E}_t^Q u_{t+1}^{(k-1)} - f_t, \qquad k = 2, \ldots, T, \quad (2.21)$$

$$V_{tk}^a \stackrel{\text{def}}{=} f_t - \mathsf{E}_t^Q u_{t+1}^{(k)}, \qquad k = 1, \ldots, T. \quad (2.22)$$

The optimal policy can be characterized using the values in (2.21) and (2.22).

**Proposition 4.** *Under Assumptions 1 and 3, when $x_t \in (H_{k-1}, H_k]$, $k = 2, \ldots, T$, the optimal decision in period $t$ is as follows:*

$$y_t^* = \begin{cases} \underline{y}(x_t), & \text{if } V_{tk}^w \leq 0, \\ H_{k-1}, & \text{if } V_{tk}^w > 0 \text{ and } V_{tk}^a \geq 0, \\ x_t, & \text{if } V_{tk}^a < 0. \end{cases} \tag{2.23}$$

*When $x_t \in (0, H_1]$, $y_t^* = 0$ if $V_{t1}^a \geq 0$, and $y_t^* = x_t$ if $V_{t1}^a < 0$.*

Intuitively, when $x_t \in (H_{k-1}, H_k]$, the storage can be emptied in $k$ periods, and the firm aims to sell inventory at the $k$ largest expected prices. When the maturing price $f_t$ is among the $k-1$ highest expected selling prices ($f_t > \mathsf{E}_t^Q u_{t+1}^{(k-1)}$), there is no value of delaying sales ($V_{tk}^w < 0$) and the firm should sell as much as possible, as in the first case of (2.23).

If the maturing price $f_t$ is lower than the $k$-th largest expected selling price ($f_t < \mathsf{E}_t^Q u_{t+1}^{(k)}$), then $f_t$ itself is an adverse selling price. Thus, there is no value of avoiding adverse price by selling inventory right now ($V_{tk}^a < 0$), and the firm should do nothing, as in the last case of (2.23).

When the maturing price $f_t$ is the $k$-th largest, we have the second case in (2.23). If the firm sells nothing at $f_t$, then to sell all inventory it cannot avoid selling some inventory later at a price lower than $f_t$ in expectation. On the other hand, if the firm sells as much as possible right now, then it does not take full advantage of the larger expected selling prices; waiting has a value. The best strategy is to sell down to $H_{k-1}$, and the remaining $H_{k-1}$ units are expected to be sold at the $k-1$ largest expected selling prices.

The definitions in (2.21) and (2.22) are extensions of the definitions of $V^w$ and $V^a$ in (2.12) and (2.16), respectively. Note when the storage can be emptied in two out of three remaining periods, i.e., when $N = 3$, $t = 1$, and $k = 2$, (2.21) and (2.22) reduce to (2.12) and (2.16), respectively.

### 2.5.2 Value of Counter-Seasonal Operations

We now allow counter-seasonal operations during the selling season. For ease of illustration, we assume the maximum storing and releasing speeds are the same.

**Assumption 4.** *(i)* $\overline{\lambda}(x) = \min\{\overline{C}, K - x\}$ *and* $\underline{\lambda}(x) = \max\{\underline{C}, -x\}$; *(ii)* $K = T\overline{C} = T|\underline{C}|$ *for some* $T \in \{2, 3, \dots, N\}$.

For period $t$, we introduce a vector $\mathbf{v}_t = [v_t^{(1)}, v_t^{(2)}, \dots, v_t^{(T)}]$, whose $k$-th element $v_t^{(k)}$ represents the expected marginal value of inventory in period $t$ when $x_t \in (H_{k-1}, H_k]$. Formally

$$\begin{aligned} \mathbf{v}_N &\stackrel{\text{def}}{=} [\, f_N, 0, \dots, 0\,], \\ v_t^{(k)} &\stackrel{\text{def}}{=} (k+1)\text{-th largest element of } \{f_t,\, f_t^b,\, \mathsf{E}_t^Q \mathbf{v}_{t+1}\}, \quad k = 1, \dots, T, \quad t = 1, \dots, N-1. \end{aligned} \tag{2.24}$$

We inductively prove $\mathbf{u}_t \geq \mathbf{v}_t$. This clearly holds for $t = N$. Suppose $\mathbf{u}_{t+1} \geq \mathbf{v}_{t+1}$. Then, $u_t^{(k)} = k$-th largest element of $\{f_t,\, \mathsf{E}_t^Q \mathbf{u}_{t+1}\} \geq (k+1)$-th largest element of $\{f_t,\, f_t^b,\, \mathsf{E}_t^Q \mathbf{u}_{t+1}\} \geq v_t^{(k)}$. We intuitively explain $\mathbf{u}_t \geq \mathbf{v}_t$: Without injection operations, the value of a marginal unit of inventory is the expected price at which this unit can be sold, captured by $\mathbf{u}_t$. When injection is allowed, the marginal unit of inventory brings extra sales revenue but reduces the value of counter-seasonal operations. Hence, $\mathbf{u}_t - \mathbf{v}_t$ indicates the value of counter-seasonal operations.

In period $t \leq N-2$, for $k = 1, \dots, T$, we define the value of counter-seasonal operations and the value of avoiding adverse price by buying one less unit:

$$V_{tk}^c \stackrel{\text{def}}{=} \mathsf{E}_t^Q \big[u_{t+1}^{(k)} - v_{t+1}^{(k)}\big], \tag{2.25}$$

$$V_{tk}^{ab} \stackrel{\text{def}}{=} f_t^b - \mathsf{E}_t^Q u_{t+1}^{(k)}. \tag{2.26}$$

The optimal policy can be characterized by the values defined in (2.21), (2.22), (2.25), and (2.26).

**Proposition 5.** *Under Assumptions 1 and 4, when* $x_t \in (H_{k-1}, H_k]$, $k = 2, \dots, T-1$, *the* *optimal decision in period t is as follows:*

$$y_t^* = \begin{cases} \underline{y}(x_t), & \textit{if } \; V_{tk}^w \leq V_{t,k-1}^c, \\ H_{k-1}, & \textit{if } \; V_{tk}^w > V_{t,k-1}^c \;\; \textit{and} \;\; V_{tk}^a + V_{tk}^c \geq 0, \\ x_t, & \textit{if } \; V_{tk}^a + V_{tk}^c < 0 \leq V_{tk}^{ab} + V_{tk}^c, \\ H_k, & \textit{if } \; V_{tk}^{ab} + V_{tk}^c < 0 \leq V_{t,k+1}^{ab} + V_{t,k+1}^c, \\ \overline{y}(x_t), & \textit{if } \; V_{t,k+1}^{ab} + V_{t,k+1}^c < 0. \end{cases} \tag{2.27}$$

*When* $x_t \in (0, H_1]$*, the optimal decision is* (2.27) *with the first two cases combined into:* $y_t^* = 0$ *if* $V_{t1}^a + V_{t1}^c \geq 0$*. When* $x_t \in (H_{T-1}, K]$*, the optimal decision is* (2.27) *with the last two cases combined into:* $y_t^* = K$ *if* $V_{tT}^{ab} + V_{tT}^c < 0$*.*

The first three cases in (2.27) parallel (2.23). When counter-seasonal operations are not allowed, the optimal policy in (2.23) considers only the signs of $V_{tk}^w$ and $V_{tk}^a$. Here in (2.27), $V_{tk}^w$ and $V_{tk}^a$ are traded off with the value of counter-seasonal operations.

The last two cases in (2.27) exercise the option of counter-seasonal operations (purchase). The firm should buy as much as possible when buying less provides no combined value of avoiding adverse price and counter-seasonal operations ($V_{t,k+1}^{ab} + V_{t,k+1}^c$). If buying less brings some combined value until inventory hits $H_k$, then the firm should buy only up to $H_k$.

The definition of $V_{tk}^c$ in (2.25) extends that in (2.13). For the three-period model ($N = 3$), we have:

$$\begin{aligned} u_2^{(1)} - v_2^{(1)} &= \max\{f_{22}, f_{23}\} - \text{median}\{f_{22}, f_{22}^b, f_{23}\} \\ &= \begin{cases} f_{23} - f_{22}^b, & \text{if } f_{22}^b < f_{23} \\ 0, & \text{if } f_{22}^b > f_{23} \end{cases} \\ &= \max\{f_{23} - f_{22}^b,\ 0\}. \end{aligned}$$

Thus, $V_{11}^c = \mathsf{E}_1^Q\big[\max\{f_{23} - f_{22}^b, 0\}\big]$, which is exactly $V^c$ defined in (2.13).

### 2.5.3 *N*-Period PARI Policy

Computing the optimal policy for the multiperiod problem faces the curse of dimensionality, manifested in the recursive definition in (2.24). In this section, we design a PARI policy for the $N$-period problem without dramatically increasing the computational burden.

**Definition 2.2. *N*-period price-adjusted rolling intrinsic (PARI) policy**

Step 1. Set $t = 1$.

Step 2. "Min-Max" price adjustment. Let $f_{t\tau_1}$, $f_{t\tau_2}$, $f_{t\tau_3}$, and $f_{t\tau_4}$ be the maximum, the second maximum, the second minimum, and the minimum of the futures prices $\{f_{t\tau} : \tau = t+1, \ldots, N\}$, respectively. Let $t' = \tau_1 \wedge \tau_4$, and $t'' = \tau_1 \vee \tau_4$, where $\wedge$ ($\vee$) refers to the min (max) operator.

(i) When $f_{tt} > f_{tt'}$, we define $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$ such that

$$\widehat{f}^{b}_{tt'} = \mathsf{E}^Q_t\big[\mathrm{median}\{f_{t't'}, f^b_{t't'}, f_{t't''}\}\big], \qquad \widehat{f}_{tt''} = \mathsf{E}^Q_t\big[\max\{f_{\tau_1\wedge\tau_2,\tau_1}, f_{\tau_1\wedge\tau_2,\tau_2}\}\big].$$

(ii) When $f_{tt} \leq f_{tt'}$, we define $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$ such that

$$\widehat{f}_{tt'} = f_{tt'}, \qquad \widehat{f}_{tt''} = \mathsf{E}^Q_t\big[\min\{f_{\tau_3\wedge\tau_4,\tau_3}, f_{\tau_3\wedge\tau_4,\tau_4}\}\big].$$

Step 3. Adjust other prices based on $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$. We adjust $f_{t\tau}$ by multiplying a scalar that is piecewise linear in $\tau$:

(i) For $t < \tau < t'$, define $\widehat{f}_{t\tau} = f_{t\tau}(1 - \delta + \delta\widehat{f}_{tt'}/f_{tt'})$, where $\delta = \frac{\tau-t}{t'-t}$;

(ii) For $t' < \tau < t''$, define $\widehat{f}_{t\tau} = f_{t\tau}\big((1-\delta')\widehat{f}_{tt'}/f_{tt'} + \delta'\widehat{f}_{tt''}/f_{tt''}\big)$, where $\delta' = \frac{\tau-t'}{t''-t'}$;

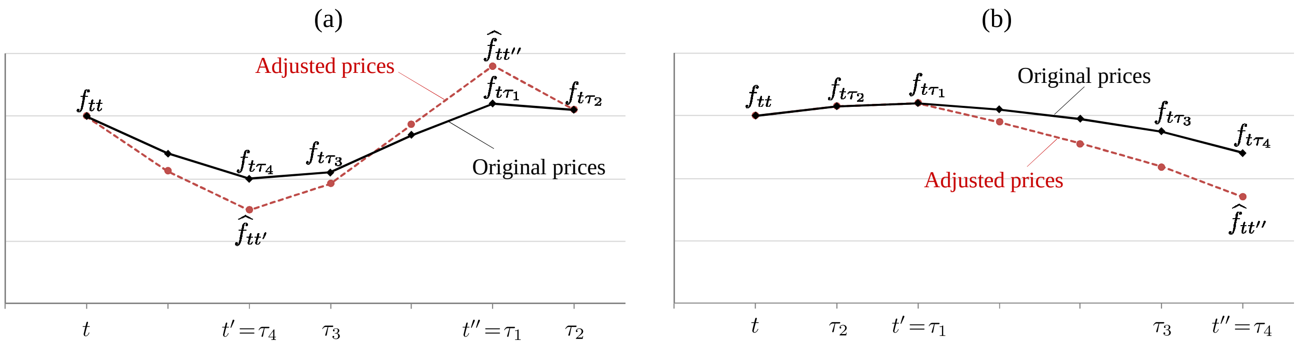
(iii) For $t'' < \tau \leq N$, define $\widehat{f}_{t\tau} = f_{t\tau}\big((1-\delta'')\widehat{f}_{tt''}/f_{tt''} + \delta''\big)$, where $\delta'' = \frac{\tau-t''}{N-t''}$.

Step 4. We solve the intrinsic valuation problem (2.5)-(2.6) with $\mathbf{f}_t$ replaced by $\widehat{\mathbf{f}}_t = (f_{tt}, \widehat{f}_{t,t+1}, \ldots, \widehat{f}_{tN})$, and implement the decision at the maturing price $f_{tt}$.

Step 5. If $t < N-2$, increase $t$ by 1 and go back to Step 2. Otherwise, apply the regular RI policy for the remaining two periods.

Figure 2.3 illustrates two typical instances of price adjustment. Step 2 of the above PARI policy resembles the three-period PARI policy. The three focal prices are $f_{tt}$, $f_{tt'}$, and $f_{tt''}$. The median price formula parallels that in Definition 2.1, whereas the maximum (minimum) expected selling price is estimated based on the two highest (lowest) futures prices. Note that when $N = 3$, the second maximum price $f_{t\tau_2}$ is the minimum price $f_{t\tau_4}$, and the second minimum price $f_{t\tau_3}$ is the maximum price $f_{t\tau_1}$. Then, the price adjustment formulae in Step

Figure 2.3: Price adjustment (steps 2 and 3) in the PARI policy




2 are the same as in Definition 2.1. Indeed, when $N = 3$, the entire policy is identical to that in Definition 2.1.

The focal prices $f_{tt}$, $f_{tt'}$, and $f_{tt''}$ divide the forward curve into three segments. Step 3 specifies how each segment should be adjusted if the segment contains prices other than the three focal prices. In essence, the other prices are "attracted" toward $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$. This adjustment is important for informing the RI policy about the option values. For example, suppose $f_{11}$ is the highest on the forward curve, the inventory can be sold in two periods, but the optimal policy is not to sell right now. Adjusting $f_{1t''}$ upward in Step 2(i) puts $f_{11}$ in the second highest, which does not stop the RI policy from selling at $f_{11}$. Step 3 raises other prices, which may signal enough value of waiting such that the RI policy coincides with the optimal policy. Such a heuristic can significantly close the gap between the RI policy and the optimal policy, as will be examined in §3.4.

Finally, we discuss the computation of the adjusted prices in Step 2. For ease of exposition, assume $\tau_1 < \tau_2$ so that in Step 2(i) we have $\widehat{f}_{tt''} = \mathsf{E}_t^Q\left[\max\{f_{\tau_1\tau_1}, f_{\tau_1\tau_2}\}\right]$. To compute this expectation, we assume $(\log f_{\tau_1\tau_1}, \log f_{\tau_1\tau_2})$ is normally distributed with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)$, where $\mu_i$ and $\sigma_i$ are mean and standard deviation of $\log f_{\tau_1\tau_i}$, $i = 1, 2$, and $\rho$ is the correlation coefficient; these parameters are derived from the forward curve dynamics (see §2.6.1). Let $f_M = \max\{\log f_{\tau_1\tau_1}, \log f_{\tau_1\tau_2}\}$. *Clark* (1961) provides the formulae for the moments of the maximum of two normal random variables:

$$\mathsf{E}_t^Q f_M = \mu_1\Phi(b) + \mu_2\Phi(-b) + a\phi(b),$$
$$\mathsf{E}_t^Q f_M^2 = (\mu_1^2+\sigma_1^2)\Phi(b) + (\mu_2^2+\sigma_2^2)\mu_2\Phi(-b) + (\mu_1+\mu_2)a\phi(b),$$

where $a^2 = \sigma_1^2 + \sigma_2^2 - 2\sigma_1\sigma_2\rho$, $b = (\mu_1 - \mu_2)/a$, and $\phi(\cdot)$ and $\Phi(\cdot)$ are the probability density function and cumulative distribution function of standard normal random variable, respectively. *Clark* (1961) also shows that the maximum of two normal random variables is approximately normally distributed. Thus, the adjusted price $\widehat{f}_{tt''}$ can be calculated as

$$\widehat{f}_{tt''} = \mathsf{E}_t^Q \exp(f_M) \approx \exp\left(\mathsf{E}_t^Q f_M + \tfrac{1}{2}\mathsf{Var}_t^Q f_M\right).$$

The expected minimum of two futures prices in Step 2(ii) can be calculated similarly. To estimate $\widehat{f}_{tt'}^b$ in Step 2(i), note that median$\{f_{t't'}, f_{t't'}^b, f_{t't''}\} = \min\left\{f_{t't'}^b, \max\{f_{t't'}, f_{t't''}\}\right\}$, which can be calculated by repeated use of *Clark* (1961)'s formulae.

### 2.5.4 Multi-Season PARI Policy

Seasonal energy storage operates across seasons. For example, the natural gas industry considers two seasons in storage operation – the withdrawal (peak) season, from November 1 through March 31, and the injection (off-peak) season, from April 1 through October 31 (*Energy Information Administration* 2011). For storage valuation, we divide the valuation horizon into multiple seasons and apply the PARI policy to each season. Thus, the performance of the PARI policy does not deteriorate when the valuation horizon increases. With distinct price seasonality (e.g., Figure 3.1 in §3.4), storage is typically filled during the off-peak season and emptied during the peak season. The off-peak season problem is mathematically equivalent to the peak season problem analyzed in the previous sections, because reducing the inventory level to zero in the peak season is analogous to reducing the space level to zero in the off-peak season. Formally, we define the multi-season PARI policy as follows:

**Definition 2.3. Multi-season PARI policy**

Step 1. Divide the planning horizon into a sequence of alternating peak and off-peak seasons.

Let $N_1$ and $N_2$ be the number of periods in the peak and off-peak seasons, respectively.

Step 2. Solve peak season problems and off-peak season problems alternately. For each peak season, apply the PARI policy in Definition 2.2 with $N = N_1$. For each off-peak season, apply the PARI policy in Definition 2.2 with $N = N_2$ and the following modifications of Step 2:

(i) When $f^b_{tt} < f^b_{tt'}$, we define $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$ such that

$$\widehat{f}_{tt'} = \mathsf{E}^Q_t\big[\text{median}\{f^b_{t't'}, f_{t't'}, f^b_{t't''}\}\big], \qquad \widehat{f}^b_{tt''} = \mathsf{E}^Q_t\big[\min\{f^b_{\tau_3\wedge\tau_4,\tau_3}, f^b_{\tau_3\wedge\tau_4,\tau_4}\}\big].$$

(ii) When $f^b_{tt} \geq f^b_{tt'}$, we define $\widehat{f}_{tt'}$ and $\widehat{f}_{tt''}$ such that

$$\widehat{f}_{tt'} = f_{tt'}, \qquad \widehat{f}^b_{tt''} = \mathsf{E}^Q_t\big[\max\{f^b_{\tau_1\wedge\tau_2,\tau_1}, f^b_{\tau_1\wedge\tau_2,\tau_2}\}\big].$$

In addition, the terminal condition in (2.5) is replaced by $V^{\mathrm{I}}_N(x_N, \mathbf{f}_t) = -f^b_{tN}\,\overline{\lambda}(x_N) + \overline{y}(x_N)p^b$, where $p^b$ is a large constant, which provides incentive to fill up the storage in period $N_2$.

In the modified (i) above, the buying price $f^b_{tt''}$ is adjusted down to $\widehat{f}^b_{tt''}$ to reflect the value of waiting for a lower buying price, and $f_{tt'}$ is adjusted up to reflect the value of potential sales during the buying season. The price adjustment in (ii) captures the value of avoiding adverse buying price.

## 2.6. Application to Natural Gas Storage

### 2.6.1 Data and Setup

The average size (for working gas) of a depleted oil/gas reservoir is about 10 trillion Btu (TBtu). We consider a firm leasing a 10 TBtu storage facility for 12 months.

**Injection and withdrawal capacities.** We consider the case of constant capacities. The capacity pair (injection capacity, withdrawal capacity) takes three values: (2 TBtu/month, 3 TBtu/month), (3 TBtu/month, 4 TBtu/month), and (4 TBtu/month, 5 TBtu/month). Under constant capacities, it is optimal to empty the storage at the end of the horizon regardless of the penalty level (see the proof of Proposition 4). Thus, we set $p = 0$.

**Operating cost parameters.** For depleted reservoirs, the injection loss rate $\alpha$ is typically between 0% and 3%, the withdrawal loss rate $\beta$ is between 0% and 2%. Throughout our analysis, we set $\alpha = 1.5\%$, $\beta = 0.5\%$, and the variable operating costs $c_\alpha = c_\beta = \$0.02$ per million Btu. These parameters are consistent with other studies, e.g., *Maragos* (2002) and *Lai et al.* (2010).

**Discount rate.** The discount rate reflects the firm's cost of capital and is typically benchmarked using the London Interbank Offered Rate (LIBOR, available from http://www.liborated.com). We consider three discount rates: 0%, 1%, and 2% above the six-month LIBOR.

**Storage contract terms.** We consider two different contract terms: (a) the lessee receives an empty storage and returns it empty (such a contract typically starts in April and ends in March); (b) the lessee receives a full storage and returns it full (such a contract typically starts in November and ends in October). These two types of terms are referred to as "seasonal cycling" and "storage carry," respectively, by *Eydeland and Wolyniec* (2003, p. 354).

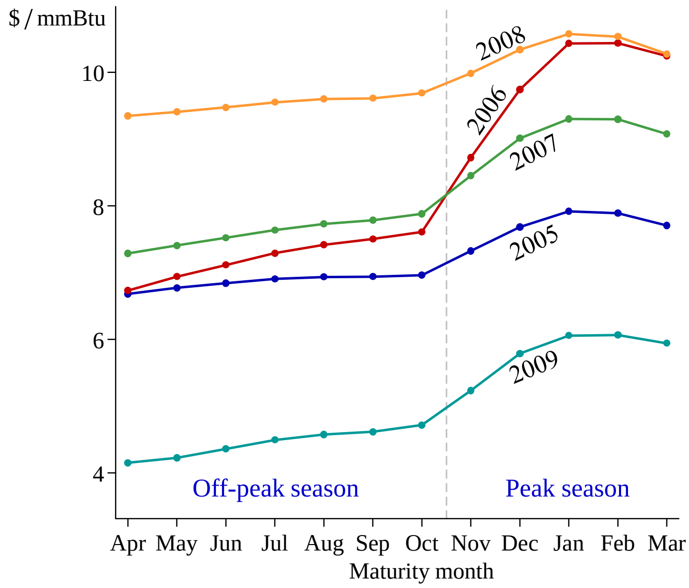
**Storage valuation under various policies.** For the seasonal cycling contracts, the storage value is calculated at the end of March every year for operations from April 1 to March 31. For the storage carry contracts, the value is calculated at the end of October every year. When solving for the optimal policy and the RI policy, we solve the optimization problem without dividing the valuation horizon into peak and off-peak seasons. When implementing the PARI policy, we divide the year into a 7-month off-peak season (April through October) and a 5-month peak season (November through March), and apply the PARI policy in Definition 2.3.

We value the seasonal cycling contracts in each of the 9 years from 2001-2009, and value the storage carry contracts in each of the 8 years from 2002-2009. At each valuation time, we consider 3 capacity pairs and 3 discount rates. This gives us a total of 153 instances.

**Forward curve dynamics.** Figure 3.1 shows the New York Mercantile Exchange (NYMEX) natural gas futures prices observed on the first trading day of March 2005-2009.

Figure 2.4: Natural gas forward curve on the first trading day of each March (2005-2009)

Data source: Bloomberg




We use the NYMEX natural gas futures price data to estimate the following multi-factor martingale model for futures prices (see also *Manoliu and Tompaidis* (2002) and the references therein):

$$\frac{d\widetilde{f}_{t\tau}}{\widetilde{f}_{t\tau}} = \sum_{j=1}^{n} \sigma_j(t,\tau) dW_j(t), \tag{2.28}$$

where $W_j(t), j = 1, \ldots, n$, are independent Brownian motions, and $\sigma_j(t,\tau)$ is the volatility of the futures price $\widetilde{f}_{t\tau}$ contributed by the factor $j$ at time $t$. We employ the principal component analysis (*Basilevsky* 1994) to estimate these volatility functions. See *Clewlow and Strickland* (2000) for examples of principal component analysis for energy prices.

The first two principal components (factors) capture majority of the futures price variations. We build a multi-layer two-factor tree model for the forward curve. Each layer corresponds to a discrete inventory level. This feature is similar to the multi-layer one-factor tree constructed by *Jaillet, Ronn, and Tompaidis* (2004), whereas in our tree each node represents a forward curve. In addition, our tree captures the time-varying volatility feature of the futures prices. The tree construction is described in part 2.8.

### 2.6.2 Performance of the PARI Policy

We measure the performance of a heuristic policy (RI or PARI policy) by the gap between the storage value under the heuristic policy and optimal storage value, expressed as a percentage of the optimal storage value. Figure 2.5 compares the percentage storage value losses under the RI policy and PARI policy when valuation is conducted at the end of March (i.e., seasonal cycling contracts). To save space, the results for storage carry contracts are included in part 2.8.

The value loss of the PARI policy is remarkably lower than the RI policy. For the 153 instances, the PARI policy achieves an average of 99.8% of the optimal value (minimum 99.13% and maximum 99.99%). That is, the value loss under the PARI is no more than 1% of the optimal value, and 0.2% of the optimal value on average.

Among the 153 cases, there are 5 cases where the RI policy leads to more than 4% value loss in a year, and the PARI policy recovers 92% of that value loss on average. In 13 cases, RI policy results in more than 2% value loss, and the PARI policy recovers 85% of the loss on average. In 26 cases, RI policy loses more than 1% of the value, and the PARI policy recovers 75% of the value loss. For all 153 cases, the PARI policy recovers 64% of the value loss.

Figure 2.6 depicts this trend over a wider range of percentage value losses. It also shows the quartiles of the distribution of the loss recovered by the PARI policy (when the RI policy loses more than 5%, there are not enough data points to show the quartiles). Figure 2.6 suggests the higher the value loss under the RI policy, the more capable the PARI policy in recovering the loss.

We remark on the continuity of the storage value in the discount rate. The discount rate bends the forward curve and affects the option values. The optimal policy takes the option values into account (e.g., in (2.11)) and, therefore, the optimal storage value is continuous in the discount rate. However, under the RI policy, a small change in the forward curve can cause the RI policy to miss a lump sum of option values. Thus, the rolling intrinsic value of the storage is, in general, not continuous in the discount rate.

For instance, in Figure 2.5, for year 2001, the value loss of the RI policy under LIBOR+2% is significantly higher than that under LIBOR. Figure 2.7 shows how the storage value in 2001 varies with the discount rate. The value loss of the RI policy clearly does not vary smoothly with the discount rate. Remarkably, the PARI policy consistently performs close to the optimal policy. Figure 2.7 also reinforces the finding in Figure 2.6 that the PARI policy is especially capable of recovering high value losses of the RI policy.

### 2.6.3 Impact of Flexibility

In this section, we study how the operational flexibility of the storage affects the storage value. We vary the flexibility by increasing the injection and withdrawal capacities in tandem, as illustrated in Figure 2.8(a), which shows capacity functions of the form $\overline{\lambda}(x) = \overline{C} \wedge (10-x)$ and $\underline{\lambda}(x) = \underline{C} \vee (-x)$, where $|\underline{C}| = \overline{C}+1$. The storage values are calculated for each capacity function pair indexed by $\overline{C}$.

Figure 2.8(b) shows when the flexibility increases, the gap between the rolling intrinsic value and the optimal value widens, and the PARI policy performs significantly better than the RI policy.

One phenomenon is thought-provoking: More flexibility brings more benefits under the optimal policy, but more flexibility may reduce the storage value under the RI policy. In Figure 2.8(b), the rolling intrinsic value increases and then decreases in flexibility. Intuitively, higher flexibility causes larger deviations of the RI decisions from the optimal decisions, resulting in deteriorating performance. In the online supplement part 2.8, we provide some theoretical support for this finding. We show that if $f_{11} \geq \max\{f_{12}, f_{13}\}$ and $V^w > V^c$, then the expected loss of the RI policy is at least $(V^w - V^c)(H - \underline{y}(x_1))$. If $f_{12} < f_{11} < \min\{f^b_{12}, f_{13}\}$ and $V^w < V^c$, then the expected loss of the RI policy is at least $(V^c - V^w)(H - \max\{\underline{y}(x_1), H'\})$. Note that these lower bounds on the performance gap increase when $\underline{y}(x_1) = x_1 + \underline{\lambda}(x_1)$ decreases or when the releasing capacity $|\underline{\lambda}(x_1)|$ increases. This suggests that more flexibility may cause larger deviation from the optimal policy and lead to higher value loss. Therefore, operational flexibility, if not used with prudence, can be detrimental to the firm. This finding calls for meticulous action to manage relatively flexible storage facilities. The PARI policy does not have the shortcoming of the RI policy: In all of the instances we tested, the storage value under the PARI policy always increases in flexibility.

## 2.7. Conclusion and Extensions

Injection and withdrawal capacities are common operational constraints for energy storage facilities. The presence of these constraints renders the optimization of energy storage operations very difficult. In practice, firms use heuristic policies to capture the seasonal price spread under limited flexibility. This paper identifies when and why the rolling intrinsic (RI) policy leads to significant losses and develops an improved heuristic policy called the price-adjusted rolling intrinsic (PARI) policy. The PARI policy is designed based on the analysis of the option values embedded in the optimal policy. Our numerical analysis shows that the gap between the PARI policy and the optimal policy is consistently small, even when the RI policy leads to significant value losses.

Besides natural gas storage, the ideas in this paper and the resulting heuristic policy can be applied to other types of energy storage, such as hydroelectric pumped storage and compressed air energy storage. An interesting future application is the optimization of the battery recharge process for electrical vehicles. Customers may set a time when the battery needs to be fully charged. The electricity distributor aims to meet customers' needs at the minimum procurement cost for energy. This is essentially the problem of filling up the storage with limited flexibility, i.e., the off-peak season problem, with $f_{t\tau}$ interpreted as the price forecast in period $t$ for the price in period $\tau$. We believe the heuristic policies, such as the PARI policy designed in this paper, have great potential to be used in this application.

There are several limitations of this research. First, we do not analyze the combined spot and futures storing and selling strategy. We refer the reader to *Goel and Gutierrez* (2006), *Kjaer and Ronn* (2008), and *Li* (2009) for analysis of models that involve both spot and futures markets. It would be interesting to study how the insights in this paper extend to the setting where both markets are present, and how one can capture the value of spot trading opportunities. Second, we value storage under the forward curve modeled by a two-factor tree. In recent years, the natural gas futures market has seen more variations that cannot be explained by merely two factors. With higher variations, storage options are expected to be more valuable and, therefore, the PARI policy may be more effective in recovering the value loss of the RI policy. Simulation methods can be used in practice to accommodate more factors in the forward curve model. Finally, the firm considered in this paper is a price-taker. The price is determined by the demand and collective behavior of the production and storage firms (see, e.g., *Wu and Chen* 2010). To consider market equilibrium of storage operations and analyze how energy storage affects energy prices would be another important future direction.

Several extensions to our work are possible. Because explicit analysis of $N$-period ($N > 3$) models with general level-dependent capacities is intractable, extending our work to $N$-period with more restrictive level-dependent capacity functions, such as piecewise constant or linear functions, would be useful.

This paper provides us with a deeper understanding about the tradeoffs involved in storage operations and the managerial insights behind the optimal policies. Using these insights, we have developed a new method to improve the RI policy. This improvement significantly reduces the gap between the rolling intrinsic value and the optimal value.

The method of adjusting the forward curves before applying the RI policy may be implemented in various ways. We find that some other simple forward curve adjustments also lead to noticeable improvement. For example, one can slightly raise the maximum price and lower the minimum price on the forward curve. Such an adjustment can lead to a significant improvement in some instances.

We identify conditions under which the rolling intrinsic (RI) policy is sub-optimal, explain why the RI policy is sub-optimal, and how the optimal policy is able to make the best use of the limited flexibility. We numerically tested a wide range of realistic settings, and examined how the optimal policy differs from the RI policy. We found that the differences between the two policies can be well explained by the theoretical results derived in this paper.

## 2.8. Appendix: Proofs and Derivations

**Derivation of** (2.11) **and** (2.15)

When $f_{11} > f_{12}$, we first show that $y_1^* \leq H$. For any policy with $y_1 > H$, we revise that policy by setting $y_1 = H$, while keeping $y_2$ unchanged (note that $y_2 \leq H$ following Proposition 1). The revised policy sells more in the first period and less in the second period. Because $f_{11} > f_{12} = \mathsf{E}_1^Q[f_{22}]$, the expected profit under the revised policy is higher. Hence, any policy with $y_1 > H$ is sub-optimal, and we must have $y_1^* \leq H$. Thus, to solve (2.10) under $f_{11} > f_{12}$, we need to consider only $y_1 \in [\underline{y}(x_1), H]$. The problem in (2.10) simplifies to:

$$\begin{aligned}\max_{y_1 \in [\underline{y}(x_1),\, H]} \quad & -f_{11}(y_1 - x_1) + f_{13}y_1 + \mathsf{P}\{A_1\}\mathsf{E}_1^Q\big[f_{22} - f_{23} \mid A_1\big]\, y_1 \\ & \qquad - \mathsf{P}\{A_3\}\mathsf{E}_1^Q\big[f_{22}^b - f_{23} \mid A_3\big] \min\{H - y_1, \overline{\lambda}(y_1)\}.\end{aligned} \tag{2.29}$$

Ignoring the constant term $f_{11}x_1$, noting that $-\mathsf{P}\{A_3\}\mathsf{E}_1^Q\big[f_{22}^b - f_{23} \mid A_3\big] \equiv \mathsf{E}_1^Q\big[\max\{f_{23} - f_{22}^b, 0\}\big]$, and employing the following identity:

$$\begin{aligned} f_{13} + \mathsf{P}\{A_1\}\mathsf{E}_1^Q\big[f_{22} - f_{23} \mid A_1\big] &= \mathsf{E}_1^Q\big[f_{23}\big] + \mathsf{P}\{A_1\}\mathsf{E}_1^Q\big[f_{22} - f_{23} \mid A_1\big] \\ &= \mathsf{E}_1^Q\big[f_{23} + \max\{f_{22} - f_{23}, 0\}\big] \;=\; \mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big],\end{aligned} \tag{2.30}$$

we can rewrite the problem in (2.29) as:

$$\begin{aligned}\max_{y_1 \in [\underline{y}(x_1), H]} \quad & \big(\mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big] - f_{11}\big)y_1 + \mathsf{E}_1^Q\big[\max\{f_{23} - f_{22}^b, 0\}\big] \min\{H - y_1, \overline{\lambda}(y_1)\} \\ = \quad & V^w y_1 + V^c \min\{H - y_1, \overline{\lambda}(y_1)\},\end{aligned}$$

which is the problem in (2.11).

When $f_{11} < f_{12}$, we first show $y_1^* \geq H$. For any policy with $y_1 < H$, we can improve the expected profit by raising $y_1$ to $H$, i.e., selling $H - y_1 \equiv \Delta$ less in the first period and selling $\Delta$ more (or buying $\Delta$ less) in the second period. The revised policy is feasible, because Assumptions 1 and 2 imply that, by raising $y_1$ up to $H$, the releasing capacity $|\underline{\lambda}(x)|$ increases by $\Delta$ and the storing capacity $|\overline{\lambda}(x)|$ decreases by at most $\Delta$. Thus, to solve (2.10) we need to consider only $y_1 \geq H$ and the problem in (2.10) simplifies to:

$$\max_{y_1 \in [H, \overline{y}(x_1)]} \quad r(y_1 - x_1, f_{11}) + f_{13} y_1 - \mathsf{P}\{A_1\} \mathsf{E}_1^Q \big[ f_{22} - f_{23} \mid A_1 \big] \underline{\lambda}(y_1) \\ + \mathsf{P}\{A_2 \cup A_3\} \mathsf{E}_1^Q \big[ f_{22} - f_{23} \mid A_2 \cup A_3 \big] (y_1 - H). \tag{2.31}$$

Using $\mathsf{P}\{A_1\}\mathsf{E}_1^Q\big[f_{22} - f_{23} \mid A_1\big] \equiv \mathsf{E}_1^Q\big[\max\{f_{22} - f_{23}, 0\}\big]$ and the following identity,

$$f_{13} + \mathsf{P}\{A_2 \cup A_3\}\mathsf{E}_1^Q\big[f_{22} - f_{23} \mid A_2 \cup A_3\big] = \mathsf{E}_1^Q\big[f_{23} + \min\{f_{22} - f_{23}, 0\}\big] = \mathsf{E}_1^Q\big[\min\{f_{22}, f_{23}\}\big],$$

and ignoring the constant term related to $H$, we can rewrite the problem in (2.31) as in (2.15).

**Two-Factor Tree Model for the Forward Curve Dynamics**

This section describes the estimation of forward curve volatility functions from historical data and a two-factor tree model for the price dynamics.

Our historical estimation of forward curve volatility functions follows the principal component analysis (PCA) described in *Clewlow and Strickland* (2000, §8.6.1). We estimate the volatility functions using the daily futures price data within the three years prior to the date of valuation. For instance, when valuing the storage at the end of March 2005, we use the data from April 2003 to March 2005. The daily futures price data are from Bloomberg.

We construct a two-factor tree model for the evolution of futures prices based on the volatility functions of the first two principal components (factors) that drive the futures price dynamics.

The volatility function for the first factor can be approximated by an exponential function (see, e.g., *Clewlow and Strickland* 2000): $\sigma_1(t, \tau) = \widehat{\sigma} e^{-\widehat{\theta}(\tau - t)}$, where $\tau - t$ is the time to maturity, and $\widehat{\sigma}$ and $\widehat{\theta}$ are positive constants estimated using a least squares regression: $\ln \sigma_1(t, \tau) = \ln \sigma + \theta(t - \tau) + \varepsilon$.

The exponential volatility function suggests that the volatility increases as a futures contract approaches its maturity. This property of increasing volatility over time can be captured by a tree model with decreasing size of time steps, as shown in Figure 2.9. The tree bifurcates at times $t_0 (= 0), t_1, t_2, \ldots, t_M$. The time step $\Delta t_m \equiv t_{m+1} - t_m$ decreases in $m$ in a certain way described shortly. In each time step prior to the maturity date $\tau_i$ of the $i$-th futures, the price $f_{t\tau_i}$ evolves to either $u_i f_{t\tau_i}$ or $d_i f_{t\tau_i}$. For ease of illustration, Figure 2.9 uses only three steps in April. In our actual evaluation, we use many more steps discussed below.

We use the same time steps for all futures contracts, while each futures contract has its own $u_i$ and $d_i$. Because the first factor drives all futures prices toward the same direction (by different amounts), we must ensure that futures prices move up or down with the same probability. Let the probability of moving up at time $t_m$ be $p_m$ for all futures prices. Matching the first and second moments implied by the binomial tree with those implied by the continuous-time price model, we have:

$$p_m u_i + (1 - p_m) d_i = 1 \tag{2.32}$$

$$p_m u_i^2 + (1 - p_m) d_i^2 = \exp(\sigma_1(t_m, \tau_i)^2 \Delta t_m) = \exp(\widehat{\sigma}^2 e^{2\widehat{\theta}(t_m - \tau_i)} \Delta t_m) \tag{2.33}$$

Note that (2.32) suggests that $p_m$ must be time-invariant because $u_i$ and $d_i$ are constants for each futures contract. This, in turn, suggests that the left side of (2.33) is time-invariant, implying that $\widehat{\sigma}^2 e^{2\widehat{\theta}(t_m - \tau_i)} \Delta t_m$ on the right side must be invariant with respect to $m$. This specifies how the size of the time steps should shrink over time:

$$\Delta t_{m+1} = e^{-2\widehat{\theta}\Delta t_m} \Delta t_m. \tag{2.34}$$

In our implementation, we set $\Delta t_0$ to be 0.4% of a year. Because $\widehat{\theta}$ is estimated at each valuation time, the total number of steps over the 11 months (the last future matures at the beginning of the 12th month) depends on the valuation time. The least number of time steps is 495 (when valuing in March 2004); the maximum number of time steps is 760 (when valuing in March 2002).

We set $p_m = 1/2$ for all $m$. Then, we can solve for $u_i$ and $d_i$ from (2.33) as follows:

$$u_i = 1 + \sqrt{\exp(\widehat{\sigma}^2 e^{-2\widehat{\theta}\tau_i} \Delta t_0) - 1}, \qquad d_i = 2 - u_i.$$

The volatility function for the second factor $\sigma_2(t, T)$ estimated using PCA generally cannot be approximated by an exponential function, because this factor typically drives the near-term futures and the long-term futures in opposite directions. Consequently, the tree is no longer recombining. To reduce the burden of computing hundreds of instances studied in the paper, we let the tree take one step per month, which leads to $2^{11} = 2048$ nodes at the beginning of the 12th month.

Storage valuation using the above two-factor tree model can be typically solved within 10 minutes with a 2.4GHz Core 2 processor.

## Storage Carry Contracts

In storage carry contracts, the lessee receives a full storage and returns it full (*Eydeland and Wolyniec* 2003). Storage carry contracts typically start in November and end in October. We conduct storage valuation at the end of each October for storage operations over the 12 months, starting with a 5-month withdrawal season, followed by a 7-month injection season.

Figure 2.11 reports the results, with value in year 2002 referring to the value from November 2001 to October 2002. On average the PARI policy recovers 63% of the value loss. Note that the value loss of the RI policy for the storage carry contracts is lower (less than 2%) compared to the value loss for the seasonal cycling contracts reported in Figure 2.5. This difference is probably because the peak season forward curve observed at the end of October is typically more curved than the off-peak season forward curve observed at the end of March; the RI policy tends to make suboptimal decisions when the forward curve is flatter.

**Proof of Proposition 1.** The RI policy is optimal in the last period, because under both policies the firm sells as much as possible to maximize the last-period profit. Next we show that the RI policy is optimal in the second period. Based on (2.3) and (2.4), the second-period problem can be written as:

$$V_2(x_2, \mathbf{f}_2) = \max_{y_2 \in [\underline{y}(x_2), \overline{y}(x_2)]} U_2(y_2) \stackrel{\text{def}}{=} r(y_2 - x_2, f_{22}) + \mathsf{E}_2^Q\big[ - f_{33}\underline{\lambda}(y_2) - (y_2 + \underline{\lambda}(y_2))p\big], \tag{2.35}$$

where, for ease of exposition, we suppress the dependence of $U_2(y_2)$ on $x_2$ and $\mathbf{f}_2$.

Based on (2.5) and (2.6), the second-period RI policy is determined by:

$$\max_{y_2 \in [\underline{y}(x_2), \overline{y}(x_2)]} r(y_2 - x_2, f_{22}) - f_{23}\underline{\lambda}(y_2) - (y_2 + \underline{\lambda}(y_2))\mathsf{E}_2^Q[p],$$

which is identical to (2.35), noting the martingale property of $f_{t3}$. Thus, the RI policy is optimal in the second period. Next, we prove that the optimal policy has the form in (2.9).

For $y_2 \in (H, K]$ and $y_2 \neq x_2$, the first-order derivative of the objective in (2.35) is

$$\begin{aligned} U_2'(y_2) &= \partial r(y_2 - x_2, f_{22})/\partial y_2 - \underline{\lambda}'(y_2) f_{23} - (1 + \underline{\lambda}'(y_2))\mathsf{E}_2^Q[p] \\ &\leq -f_{22} + \underline{s} f_{23} - (1 - \underline{s})\mathsf{E}_2^Q[p] \ \leq \ 0. \end{aligned}$$

The first inequality follows from two facts: The definition of $r(q, f_{22})$ implies $\partial r(q, f_{22})/\partial q \leq -f_{22}$, and the definition of $\underline{s}$ leads to $-\underline{\lambda}'(y_2) \leq \underline{s}$, for $y_2 \in (H, K]$. The last inequality is because the condition $\mathsf{P}\{p \geq \frac{\underline{s} f_{33} - f_{22}}{1 - \underline{s}}\} = 1$ implies $\mathsf{E}_2^Q[p] \geq \frac{\underline{s} f_{23} - f_{22}}{1 - \underline{s}}$.

Because $U_2'(y_2) \leq 0$ for $y_2 \in (H, K]$, we need to consider only $y_2 \leq H$ in solving (2.35). Assumption 1 implies $\underline{\lambda}(y_2) = -y_2$ when $y_2 \leq H$. Thus, the problem in (2.35) becomes

$$V_2(x_2, \mathbf{f}_2) = \max_{y_2} \big\{ r(y_2 - x_2, f_{22}) + f_{23} y_2 \ : \ \underline{y}(x_2) \leq y_2 \leq \min\{H, \overline{y}(x_2)\} \big\}.$$

The solution to the above problem is:

$$y_2^*(x_2, \mathbf{f}_2) \ = \ \begin{cases} \underline{y}(x_2), & \text{if } f_{22} \geq f_{23}, \\ \min\{H,\, x_2\}, & \text{if } f_{22} < f_{23} \leq f_{22}^b, \\ \min\{H,\, \overline{y}(x_2)\}, & \text{if } f_{22}^b < f_{23}, \end{cases}$$

which leads to the optimal decision expressed in (2.9) in the paper. □

**Proof of Lemma 1.** Consider the objective (2.11) in the paper:

$$\max_{y_1 \in [\underline{y}(x_1), H]} \ V^w y_1 + V^c \min\{H - y_1, \overline{\lambda}(y_1)\}.$$

Under Assumption 1, $\min\{H - y_1, \overline{\lambda}(y_1)\}$ is decreasing in $y_1$ at a rate no faster than the unit rate.

(a) Since $V^c \geq 0$ by definition, the second term in the objective (2.11) is decreasing in $y_1$. When $V^w \leq 0$, the first term is also decreasing in $y_1$ and, therefore, the optimal solution is $y_1^* = \underline{y}(x_1)$.

(b) When $V^w > V^c$, the objective can be written as:

$$(V^w - V^c) y_1 + V^c \big( y_1 + \min\{H - y_1, \overline{\lambda}(y_1)\} \big) \ = \ (V^w - V^c) y_1 + V^c \min\{H,\, \overline{y}(y_1)\},$$

which is increasing in $y_1$, because $\overline{y}(y_1)$ is nondecreasing in $y_1$. Hence, $y_1^* = H$.

(c) When $0 < V^w \le V^c$, the objective can be written as:

$$(V^w - V^c)y_1 + V^c \min\{H, \overline{y}(y_1)\} \;=\; \begin{cases} (V^w - V^c)y_1 + V^c H, & \text{if } y_1 \ge H', \\ V^w y_1 + V^c \overline{\lambda}(y_1), & \text{if } y_1 < H'. \end{cases}$$

Thus, the objective is decreasing in $y_1$ for $y_1 \ge H'$. If $\underline{y}(x_1) \ge H'$, then $y_1^* = \underline{y}(x_1)$. If $\underline{y}(x_1) < H'$, then $y_1^* \in [\underline{y}(x_1),\, H']$ and is determined by maximizing $V^w y_1 + V^c \overline{\lambda}(y_1)$. □

**Proof of Lemma 2.** Had we not known the optimal policy, we would prove Lemma 2 from scratch. With the optimal policy derived in Lemma 1, a short-cut is available. If we set the volatilities of futures prices to be zero, then the optimal policy in Lemma 1 becomes the RI policy. Specifically, under the zero price volatilities assumption, (2.12)-(2.13) in the paper become

$$V^w = \max\{f_{12}, f_{13}\} - f_{11} \qquad \text{and} \qquad V^c = \max\{f_{13} - f_{12}^b, 0\}.$$

We now show that each part of Lemma 1 becomes the corresponding part of Lemma 2:

(a) $V^w \le 0$ is equivalent to $f_{11} \ge \max\{f_{12}, f_{13}\}$.

(b) Because $f_{11} > f_{12}$, $V^w > V^c$ is equivalent to $f_{13} - f_{11} > \max\{f_{13} - f_{12}^b, 0\}$ or $f_{11} < \min\{f_{12}^b, f_{13}\}$.

(c) Based on the equivalence in (a) and (b) above, we can see that $0 < V^w \le V^c$ is equivalent to $f_{13} > f_{11} \ge f_{12}^b$. The maximization problem in Lemma 1(c) is also equivalent to that in Lemma 2(c) because $V^w = f_{13} - f_{11}$ and $V^c = f_{13} - f_{12}^b$. □

**Proof of Proposition 2.** The price is adjusted such that $\widehat{f}_{12}^b = \mathsf{E}_1^Q\big[\text{median}\{f_{22}, f_{22}^b, f_{23}\}\big]$ and $\widehat{f}_{13} = \mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big]$. Using $\widehat{\mathbf{f}}_1 = (f_{11}, \widehat{f}_{12}, \widehat{f}_{13})$ as the input prices of the RI policy, we show that each part of Lemma 2 is equivalent to the corresponding part in Lemma 1:

(a) $f_{11} \ge \max\{\widehat{f}_{12}, \widehat{f}_{13}\} = \widehat{f}_{13} = \mathsf{E}_1^Q\big[\max\{f_{22}, f_{23}\}\big]$ is equivalent to $V^w \le 0$.

(b) $f_{11} < \min\{\widehat{f}_{12}^b, \widehat{f}_{13}\} = \widehat{f}_{12}^b = \mathsf{E}_1^Q\big[\text{median}\{f_{22}, f_{22}^b, f_{23}\}\big]$ is equivalent to $V^w > V^c$, because

$$\text{median}\{f_{22}, f_{22}^b, f_{23}\} = \max\{f_{22}, f_{23}\} - \max\{f_{23} - f_{22}^b, 0\}. \tag{2.36}$$

One can verify (2.36) by considering three cases: $f_{22} < f_{22}^b < f_{23}$, $f_{22} < f_{23} < f_{22}^b$, and $f_{23} < f_{22} < f_{22}^b$.

(c) Based on the equivalent relations in (a) and (b), $\widehat{f}_{13} > f_{11} \ge \widehat{f}_{12}^b$ is equivalent to $0 <$

$V^w \le V^c$. Furthermore, the maximization problem in Lemma 2(c) is identical to that in Lemma 1(c) because $V^w = \widehat{f}_{13} - f_{11}$ and $V^c = \widehat{f}_{13} - \widehat{f}^b_{12}$, where the latter is due to (2.36). $\square$

**Proof of Lemma 3.** Consider the objective (2.15) in the paper:

$$\max_{y_1 \in [H, \overline{y}(x_1)]} U_1(y_1) = \begin{cases} f_{11}x_1 - V^a y_1 - V^l \underline{\lambda}(y_1), & \text{if } y_1 \in [H, x_1], \\ f^b_{11}x_1 - V^{ab} y_1 - V^l \underline{\lambda}(y_1), & \text{if } y_1 \in (x_1, \overline{y}(x_1)]. \end{cases}$$

(a) Because $V^l \ge 0$ by definition and $\underline{\lambda}(y_1)$ is decreasing in $y_1$ under Assumption 1, the term $-V^l \underline{\lambda}(y_1)$ in the objective is increasing in $y_1$. When $V^{ab} \le 0$, the terms $-V^a y_1$ and $-V^{ab} y_1$ are also increasing in $y_1$ and, therefore, the optimal solution is $y_1^* = \overline{y}(x_1)$.

(b) When $V^a \le 0 < V^{ab}$, $U_1(y_1)$ is increasing for $y_1 \in [H, x_1]$, and the optimal decision is determined by maximizing $-V^{ab} y_1 - V^l \underline{\lambda}(y_1)$ for $y_1 \in [x_1, \overline{y}(x_1)]$.

(c) Continue from part (b). If $V^l < V^{ab}$, then the maximizer of $-V^{ab} y_1 - V^l \underline{\lambda}(y_1)$ is $y_1^* = x_1$.

(d) When $V^a > 0$, the objective is not monotone in general and the optimal solution may lie anywhere between $H$ and $\overline{y}(x_1)$. $\square$

**Proof of Lemma 4.** Parallel to the proof of Lemma 2, when the price volatilities are assumed to be zero, the optimal policy in Lemma 3 becomes the RI policy stated in this lemma. $\square$

**Proof of Proposition 3.** The adjusted price is $\widehat{\mathbf{f}}_1 = (f_{11}, f_{12}, \widehat{f}_{13})$, where $\widehat{f}_{13} = \mathsf{E}^Q_1\big[\min\{f_{22}, f_{23}\}\big]$. Note the following relations:

$$f_{11} - \widehat{f}_{13} = f_{11} - \mathsf{E}^Q_1\big[\min\{f_{22}, f_{23}\}\big] = V^a, \tag{2.37}$$

$$f^b_{11} - \widehat{f}_{13} = f^b_{11} - \mathsf{E}^Q_1\big[\min\{f_{22}, f_{23}\}\big] = V^{ab}, \tag{2.38}$$

$$f_{12} - \widehat{f}_{13} = f_{12} - \mathsf{E}^Q_1\big[\min\{f_{22}, f_{23}\}\big] = \mathsf{E}^Q_1\big[\max\{f_{22} - f_{23}, 0\}\big] = V^l, \tag{2.39}$$

$$f^b_{11} - f_{12} = (f^b_{11} - \widehat{f}_{13}) - (f_{12} - \widehat{f}_{13}) = V^{ab} - V^l. \tag{2.40}$$

Using $\widehat{\mathbf{f}}_1 = (f_{11}, f_{12}, \widehat{f}_{13})$ as the input prices of the RI policy, we show that each part of Lemma 4 is equivalent to the corresponding part in Lemma 3:

(a) $f^b_{11} \le \min\{f_{12}, \widehat{f}_{13}\} = \widehat{f}_{13}$ is equivalent to $V^{ab} \le 0$, due to (2.38).

(b) $f_{11} \le \min\{f_{12}, \widehat{f}_{13}\} = \widehat{f}_{13} < f^b_{11} \le f_{12}$ is equivalent to $V^a \le 0 < V^{ab} \le V^l$ due to (2.37), (2.38), and (2.40). The maximization problem in Lemma 4(b) is identical to that in Lemma 3(b) because $f_{11}^b - \min\{f_{12}, \widehat{f}_{13}\} = f_{11}^b - \widehat{f}_{13} = V^{ab}$ and $\max\{f_{12} - \widehat{f}_{13}, 0\} = f_{12} - \widehat{f}_{13} = V^l$.

(c) $f_{11} \le \min\{f_{12}, \widehat{f}_{13}\} \le f_{12} < f_{11}^b$ is equivalent to $V^a \le 0 \le V^l < V^{ab}$ due to (2.37) and (2.40).

(d) $f_{11} > \widehat{f}_{13}$ is equivalent to $V^a > 0$ due to (2.37). The maximization problem in Lemma 4(d) is identical to that in Lemma 3(d), because (2.37)-(2.39) imply that the objective in (2.19) is identical to the objective in (2.15). □

**Proof of Proposition 4.** The multiperiod problem is formulated in (2.3)-(2.4), and simplified below under Assumption 3.

$$V_t(x_t, \mathbf{f}_t) = \max_{y_t \in [\underline{y}(x_t),\, x_t]} (x_t - y_t) f_t + \mathsf{E}_t^Q \big[V_{t+1}(y_t, \mathbf{f}_{t+1})\big], \tag{2.41}$$

$$V_N(x_N, \mathbf{f}_N) = -f_N\, \underline{\lambda}(x_N). \tag{2.42}$$

Note that under constant capacities, there is no value of raising withdrawal capacity by withholding sales. Thus, it is optimal to empty the storage by the end of period $N$, and the penalty term is not needed in (2.42). Formally, we show that $y_t^* \le (N-t)|\underline{C}|$ and, in particular, $y_N^* = 0$. If $y_t > (N-t)|\underline{C}|$, then for the remaining $N-t$ periods, the best policy is to sell $|\underline{C}|$ every period, leaving $\big(y_t - (N-t)|\underline{C}|\big)$ units unsold in the last period. Thus, $y_t > (N-t)|\underline{C}|$ is a suboptimal decision.

We now inductively prove that for any $\mathbf{f}_t$, $V_t(x_t, \mathbf{f}_t)$ is a concave piece-wise linear function in $x_t$ with slope $u_t^{(k)}$ defined in (2.20) for $x_t \in (H_{k-1}, H_k]$, $k = 1, \ldots, T$.

First, because $\underline{\lambda}(x)$ has slope $-1$ for $x \in (0, H_1]$ and zero slope otherwise, $V_N(x_N, \mathbf{f}_N)$ is concave in $x_N$ and has slope $u_N^{(k)}$ for $x_t \in (H_{k-1}, H_k]$. Suppose $V_{t+1}(y_t, \mathbf{f}_{t+1})$ is concave in $y_t$ with slope $u_{t+1}^{(k)}$ for $y_t \in (H_{k-1}, H_k]$. Then, the objective in (2.41) is concave in $y_t$ with slope $\mathsf{E}_t^Q u_{t+1}^{(k)} - f_t$ for $y_t \in (H_{k-1}, H_k]$.

Let $x_t \in (H_{k-1}, H_k]$, for some $k \in \{2, \ldots, T\}$. Consider three cases:

(i) If the slope $\mathsf{E}_t^Q u_{t+1}^{(k-1)} - f_t \le 0$ (i.e., $V_{tk}^w \le 0$), then the objective in (2.41) is non-increasing for $y_t \ge H_{k-2}$. Thus, it is optimal to sell $|\underline{C}|$. We have $V_t(x_t, \mathbf{f}_t) = |\underline{C}| f_t + \mathsf{E}_t^Q \big[V_{t+1}(x_t - |\underline{C}|, \mathbf{f}_{t+1})\big]$, which is linear in $x_t$ with slope $\mathsf{E}_t^Q u_{t+1}^{(k-1)}$ for $x_t \in (H_{k-1}, H_k]$.

(ii) If the slopes $\mathsf{E}_t^Q u_{t+1}^{(k)} - f_t \le 0$ and $\mathsf{E}_t^Q u_{t+1}^{(k-1)} - f_t > 0$ (i.e., $V_{tk}^a \ge 0$ and $V_{tk}^w > 0$), then the objective in (2.41) is increasing in $y_t$ for $y_t \le H_{k-1}$ and non-increasing for $y_t \ge H_{k-1}$. The optimal decision is $y_t^* = H_{k-1}$; the value function is $V_t(x_t, \mathbf{f}_t) = (x_t - H_{k-1})f_t + \mathsf{E}_t^Q\big[V_{t+1}(H_{k-1}, \mathbf{f}_{t+1})\big]$, which is linear in $x_t$ with slope $f_t$ for $x_t \in (H_{k-1}, H_k]$.

(iii) If the slope $\mathsf{E}_t^Q u_{t+1}^{(k)} - f_t > 0$ (i.e., $V_{tk}^a < 0$), then the objective in (2.41) is increasing in $y_t$ for $y_t \le x_t$, and the optimal decision is $y_t^* = x_t$. Under the optimal decision, $V_t(x_t, \mathbf{f}_t) = \mathsf{E}_t^Q\big[V_{t+1}(x_t, \mathbf{f}_{t+1})\big]$ and has slope $\mathsf{E}_t^Q u_{t+1}^{(k)}$ for $x_t \in (H_{k-1}, H_k]$.

In sum, for $x_t \in (H_{k-1}, H_k]$, $k \ge 2$, $V_t(x_t, \mathbf{f}_t)$ is linear in $x_t$ with slope:

$$\begin{cases} \mathsf{E}_t^Q u_{t+1}^{(k-1)}, & \text{if } f_t \ge \mathsf{E}_t^Q u_{t+1}^{(k-1)}, \\ f_t, & \text{if } \mathsf{E}_t^Q u_{t+1}^{(k)} \le f_t < \mathsf{E}_t^Q u_{t+1}^{(k-1)}, \\ \mathsf{E}_t^Q u_{t+1}^{(k)}, & \text{if } f_t < \mathsf{E}_t^Q u_{t+1}^{(k)}, \end{cases}$$

which is essentially $u_t^{(k)} = k$-th largest element of $\big\{f_t,\ \mathsf{E}_t^Q \mathbf{u}_{t+1}\big\}$.

Finally, when $x_t \in (0, H_1]$, case (iii) above still applies, whereas cases (i) and (ii) are replaced by the following: If $\mathsf{E}_t^Q u_{t+1}^{(1)} \le f_t$ (i.e., $V_{t1}^a \ge 0$), then the optimal decision is $y_t^* = 0$; the value function is $V_t(x_t, \mathbf{f}_t) = x_t f_t + \mathsf{E}_t^Q\big[V_{t+1}(H_{k-1}, \mathbf{f}_{t+1})\big]$, which is linear in $x_t$ with slope $f_t$ for $x_t \in (0, H_1]$. This, together with case (i), implies that $u_t^{(1)} = \max\big\{f_t,\ \mathsf{E}_t^Q \mathbf{u}_{t+1}\big\}$. $\square$

**Proof of Proposition 5.** The $N$-period problem is as follows:

$$V_t(x_t, \mathbf{f}_t) = \max_{y_t \in [\underline{y}(x_t), \overline{y}(x_t)]} \ r(y_t - x_t, f_t) + \mathsf{E}_t^Q\big[V_{t+1}(y_t, \mathbf{f}_{t+1})\big], \tag{2.43}$$

$$V_N(x_N, \mathbf{f}_N) = -f_N\, \underline{\lambda}(x_N). \tag{2.44}$$

We inductively prove that for any $\mathbf{f}_t$, $V_t(x_t, \mathbf{f}_t)$ is a concave piece-wise linear function in $x_t$ with slope $v_t^{(k)}$ defined in (2.24) for $x_t \in (H_{k-1}, H_k]$, $k = 1, \ldots, T$. This is true for $t = N$, as seen in the proof for Proposition 4. Suppose $V_{t+1}(y_t, \mathbf{f}_{t+1})$ is concave in $y_t$ with slope $v_{t+1}^{(k)}$ for $y_t \in (H_{k-1}, H_k]$. Then, the objective in (2.43) is concave in $y_t$ with slope $\mathsf{E}_t^Q v_{t+1}^{(k)} - f_t$ for $y_t \in (H_{k-1}, H_k]$.

Let $x_t \in (H_{k-1}, H_k]$, for some $k \in \{2, \ldots, T-1\}$. Consider five cases below. The first two cases parallel those in the proof of Proposition 4.

(i) If the slope $\mathsf{E}_t^Q v_{t+1}^{(k-1)} - f_t \le 0$ (i.e., $V_{tk}^w \le V_{t,k-1}^c$), it is optimal to sell $|\underline{C}|$. The value function $V_t(x_t, \mathbf{f}_t)$ has slope $\mathsf{E}_t^Q v_{t+1}^{(k-1)}$ for $x_t \in (H_{k-1}, H_k]$.

(ii) If the slopes $\mathsf{E}_t^Q v_{t+1}^{(k)} - f_t \le 0$ and $\mathsf{E}_t^Q v_{t+1}^{(k-1)} - f_t > 0$ (i.e., $V_{tk}^a + V_{tk}^c \ge 0$ and $V_{tk}^w > V_{t,k-1}^c$), the optimal decision is $y_t^* = H_{k-1}$. The value function $V_t(x_t, \mathbf{f}_t)$ has slope $f_t$ for $x_t \in (H_{k-1}, H_k]$.

(iii) If the slopes $\mathsf{E}_t^Q v_{t+1}^{(k)} - f_t > 0$ and $\mathsf{E}_t^Q v_{t+1}^{(k)} - f_t^b \le 0$ (i.e., $V_{tk}^a + V_{tk}^c < 0 \le V_{tk}^{ab} + V_{tk}^c$), then the objective in (2.43) is increasing in $y_t$ for $y_t \le x_t$, and non-increasing for $y_t \ge x_t$. The optimal decision is $y_t^* = x_t$, and $V_t(x_t, \mathbf{f}_t) = \mathsf{E}_t^Q\big[V_{t+1}(x_t, \mathbf{f}_{t+1})\big]$ has slope $\mathsf{E}_t^Q v_{t+1}^{(k)}$ for $x_t \in (H_{k-1}, H_k]$.

(iv) If the slopes $\mathsf{E}_t^Q v_{t+1}^{(k)} - f_t^b > 0$ and $\mathsf{E}_t^Q v_{t+1}^{(k+1)} - f_t^b \le 0$ (i.e., $V_{tk}^{ab} + V_{tk}^c < 0 \le V_{t,k+1}^{ab} + V_{t,k+1}^c$), then the objective in (2.43) is increasing in $y_t$ for $y_t \le H_k$, and non-increasing for $y_t \ge H_k$. The optimal decision is to buy up to $y_t^* = H_k$, and $V_t(x_t, \mathbf{f}_t) = -(H_k - x_t) f_t^b + \mathsf{E}_t^Q\big[V_{t+1}(H_k, \mathbf{f}_{t+1})\big]$ has slope $f_t^b$ for $x_t \in (H_{k-1}, H_k]$.

(v) If the slope $\mathsf{E}_t^Q v_{t+1}^{(k+1)} - f_t^b > 0$ (i.e., $V_{t,k+1}^{ab} + V_{t,k+1}^c < 0$), then the objective in (2.43) is increasing for $y_t \le H_{k+1}$. It is optimal to buy $\overline{C}$, and the resulting value function $V_t(x_t, \mathbf{f}_t) = -\overline{C} f_t^b + \mathsf{E}_t^Q\big[V_{t+1}(x_t + \overline{C}, \mathbf{f}_{t+1})\big]$ has slope $\mathsf{E}_t^Q v_{t+1}^{(k+1)}$ for $x_t \in (H_{k-1}, H_k]$.

In sum, for $x_t \in (H_{k-1}, H_k]$, $k \ge 2$, $V_t(x_t, \mathbf{f}_t)$ is linear in $x_t$ with slope:

$$\begin{cases} \mathsf{E}_t^Q v_{t+1}^{(k-1)}, & \text{if } f_t \ge \mathsf{E}_t^Q v_{t+1}^{(k-1)}, \\ f_t, & \text{if } \mathsf{E}_t^Q v_{t+1}^{(k)} \le f_t < \mathsf{E}_t^Q v_{t+1}^{(k-1)}, \\ \mathsf{E}_t^Q v_{t+1}^{(k)}, & \text{if } f_t < \mathsf{E}_t^Q v_{t+1}^{(k)} \le f_t^b, \\ f_t^b & \text{if } \mathsf{E}_t^Q v_{t+1}^{(k+1)} \le f_t^b < \mathsf{E}_t^Q v_{t+1}^{(k)}, \\ \mathsf{E}_t^Q v_{t+1}^{(k+1)} & \text{if } f_t^b < \mathsf{E}_t^Q v_{t+1}^{(k+1)}, \end{cases}$$

which is essentially $v_t^{(k)} = (k+1)$-th largest element of $\big\{f_t,\, f_t^b,\, \mathsf{E}_t^Q \mathbf{v}_{t+1}\big\}$,

When $x_t \in (0, H_1]$, cases (i) and (ii) are replaced by the following: If the slope $\mathsf{E}_t^Q v_{t+1}^{(1} - f_t \le 0$ (i.e., $V_{t1}^a + V_{t1}^c \ge 0$), we have $y_t^* = 0$, and $V_t(x_t, \mathbf{f}_t)$ has slope $f_t$ for $x_t \in (0, H_1]$.

When $x_t \in (H_{T-1}, K]$, cases (iv) and (v) are replaced by the following: If the slope $\mathsf{E}_t^Q v_{t+1}^{(T)} - f_t^b > 0$ (i.e., $V_{tT}^{ab} + V_{tT}^c < 0$), we have $y_t^* = K$, and $V_t(x_t, \mathbf{f}_t)$ has slope $f_t^b$ for $x_t \in (H_{T-1}, K]$. □

**Lower Bounds on the Value Loss from RI Policy**

In this section, we show that if $f_{11} \ge \max\{f_{12}, f_{13}\}$ and $V^w > V^c$, then the expected loss of the RI policy is at least $(V^w - V^c)(H - \underline{y}(x_1))$. If $f_{12} < f_{11} < \min\{f_{12}^b, f_{13}\}$ and $V^w < V^c$, then the expected loss of the RI policy is at least $(V^c - V^w)(H - \max\{\underline{y}(x_1), H'\})$.

In the appendix of the paper, the derivation of the objective (2.11) suggests that:

$$V(x_1, \mathbf{f}_1) = \max_{y_1 \in [\underline{y}(x_1), H]} U_1(x_1, y_1, \mathbf{f}_1) \equiv V^w y_1 + V^c \min\{H - y_1, \overline{\lambda}(y_1)\} + f_{11} x_1.$$

Proposition 1 shows that the RI policy is optimal for the last two periods. Hence,

$$V_1(x_1, \mathbf{f}_1) - V_1^{\mathrm{RI}}(x_1, \mathbf{f}_1) = U_1(x_1, y_1^*, \mathbf{f}_1) - U_1(x_1, y_1^\dagger, \mathbf{f}_1).$$

We now prove the two statements in sequence.

(i) When $f_{11} \geq \max\{f_{12}, f_{13}\}$ and $V^w > V^c$, Lemma 1(b) and Lemma 2(a) imply that $y_1^\dagger = \underline{y}(x_1) < H = y_1^*$. Then,

$$\begin{aligned} U_1(x_1, y_1^*, \mathbf{f}_1) - U_1(x_1, y_1^\dagger, \mathbf{f}_1) &= V^w H - V^w y_1^\dagger - V^c \min\{H - y_1^\dagger, \overline{\lambda}(y_1^\dagger)\} \\ &\geq V^w (H - y_1^\dagger) - V^c (H - y_1^\dagger) \\ &= (V^w - V^c)(H - \underline{y}(x_1)). \end{aligned}$$

(ii) When $f_{12} < f_{11} < \min\{f_{12}^b, f_{13}\}$, Lemma 2(b) implies that $y_1^\dagger = H$. When $V^w < V^c$, the optimal solution is determined by Lemma 1(a) or (c).

If $\underline{y}(x_1) \geq H'$, then $y_1^* = \underline{y}(x_1)$ and

$$U_1(x_1, y_1^*, \mathbf{f}_1) - U_1(x_1, y_1^\dagger, \mathbf{f}_1) = V^w \underline{y}(x_1) + V^c (H - \underline{y}(x_1)) - V^w H = (V^c - V^w)(H - \underline{y}(x_1)).$$

If $\underline{y}(x_1) < H'$, then $y^* \in [\underline{y}(x_1), H']$ and

$$\begin{aligned} U_1(x_1, y_1^*, \mathbf{f}_1) - U_1(x_1, y_1^\dagger, \mathbf{f}_1) &\geq U_1(x_1, H', \mathbf{f}_1) - U_1(x_1, H, \mathbf{f}_1) \\ &\geq V^w H' + V^c (H - H') - V^w H = (V^c - V^w)(H - H'). \end{aligned}$$

Summarizing the above two cases, we have

$$U_1(x_1, y_1^*, \mathbf{f}_1) - U_1(x_1, y_1^\dagger, \mathbf{f}_1) \geq (V^c - V^w)(H - \max\{\underline{y}(x_1), H'\}).$$

Figure 2.5: Value loss under RI and PARI policies: Valuation at the end of March

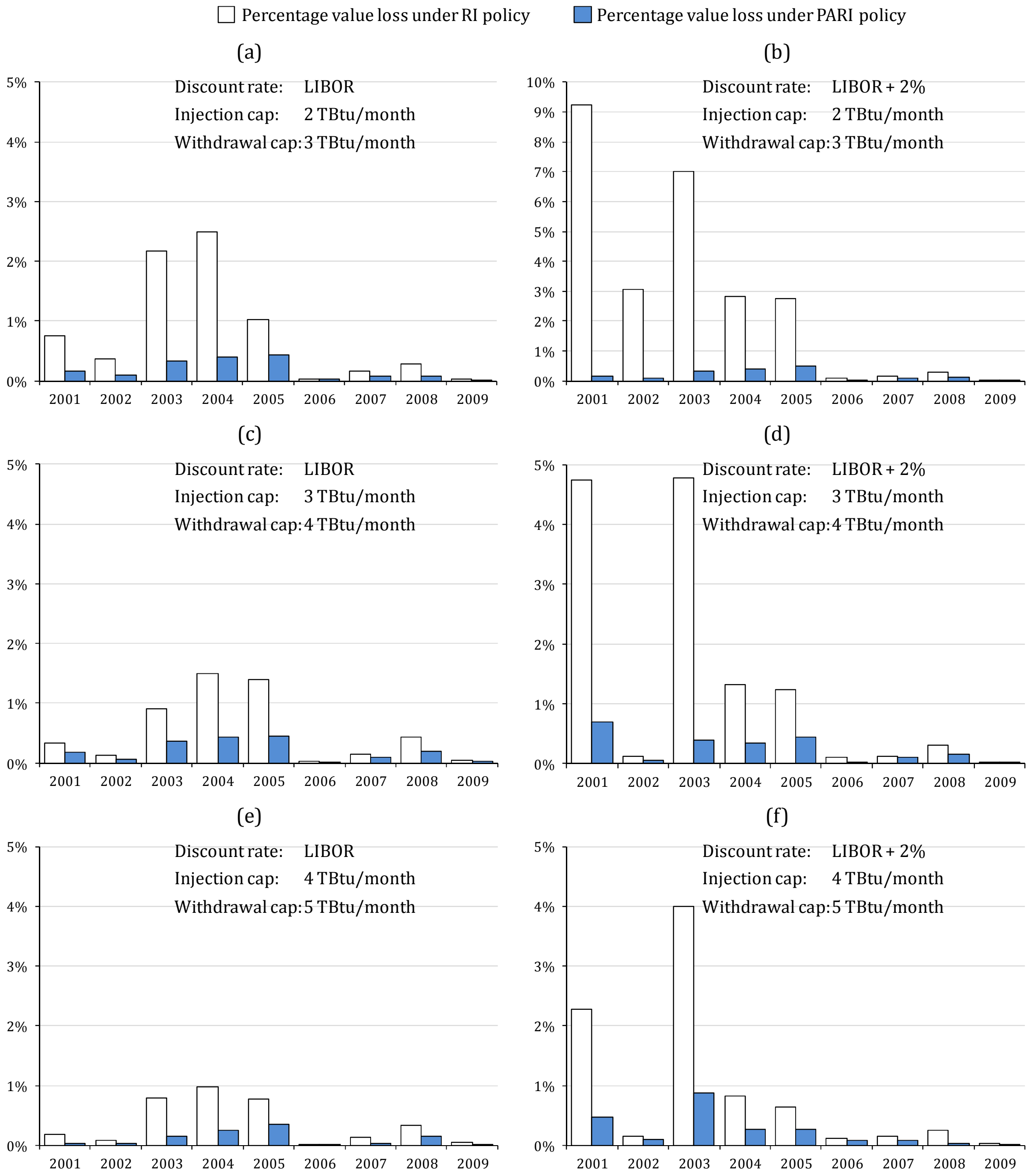


Figure 2.6: Value loss of the RI policy recovered by the PARI policy

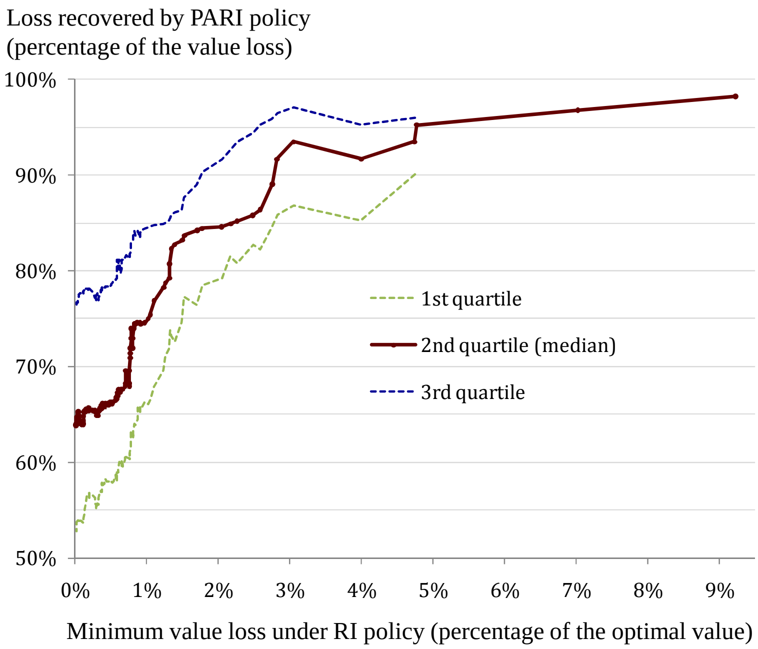



Figure 2.7: Effect of discount rate on storage value

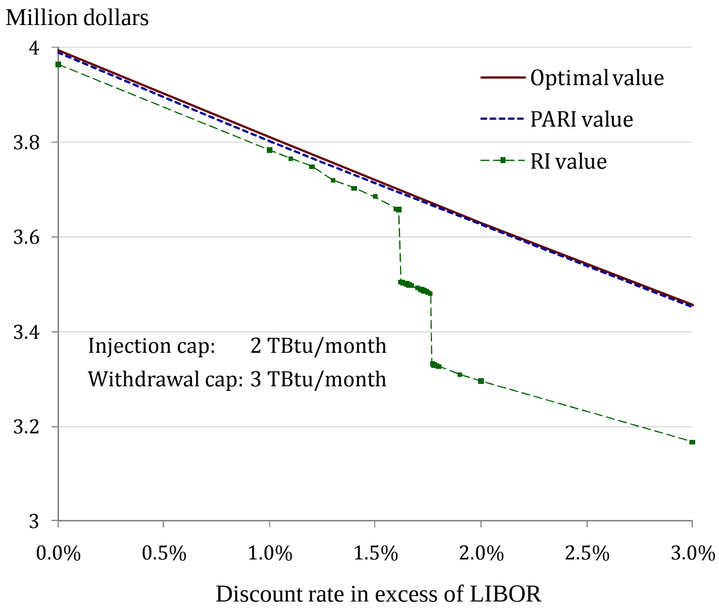


Figure 2.8: Effect of operational flexibility on storage value

The storage values are calculated in March 2007. Discount rate: LIBOR + 1%

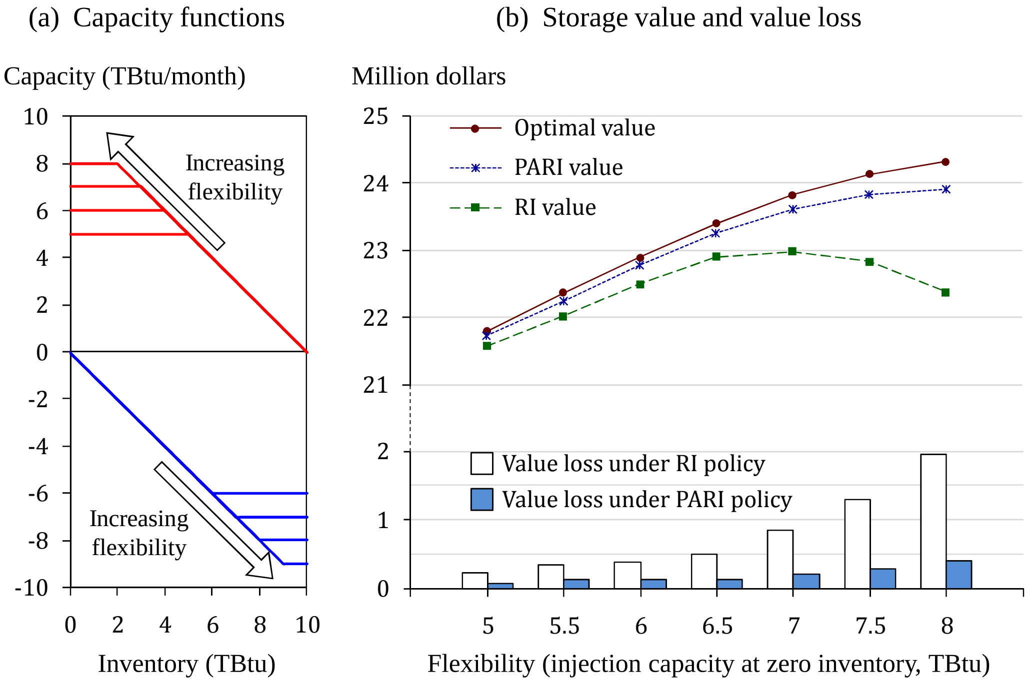



Figure 2.9: Tree model for the first factor

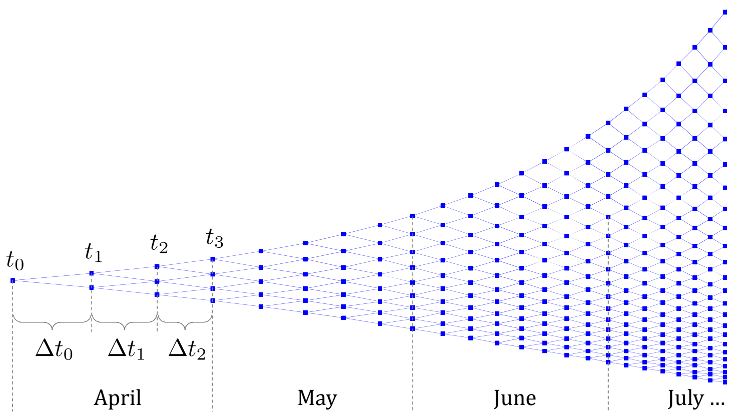
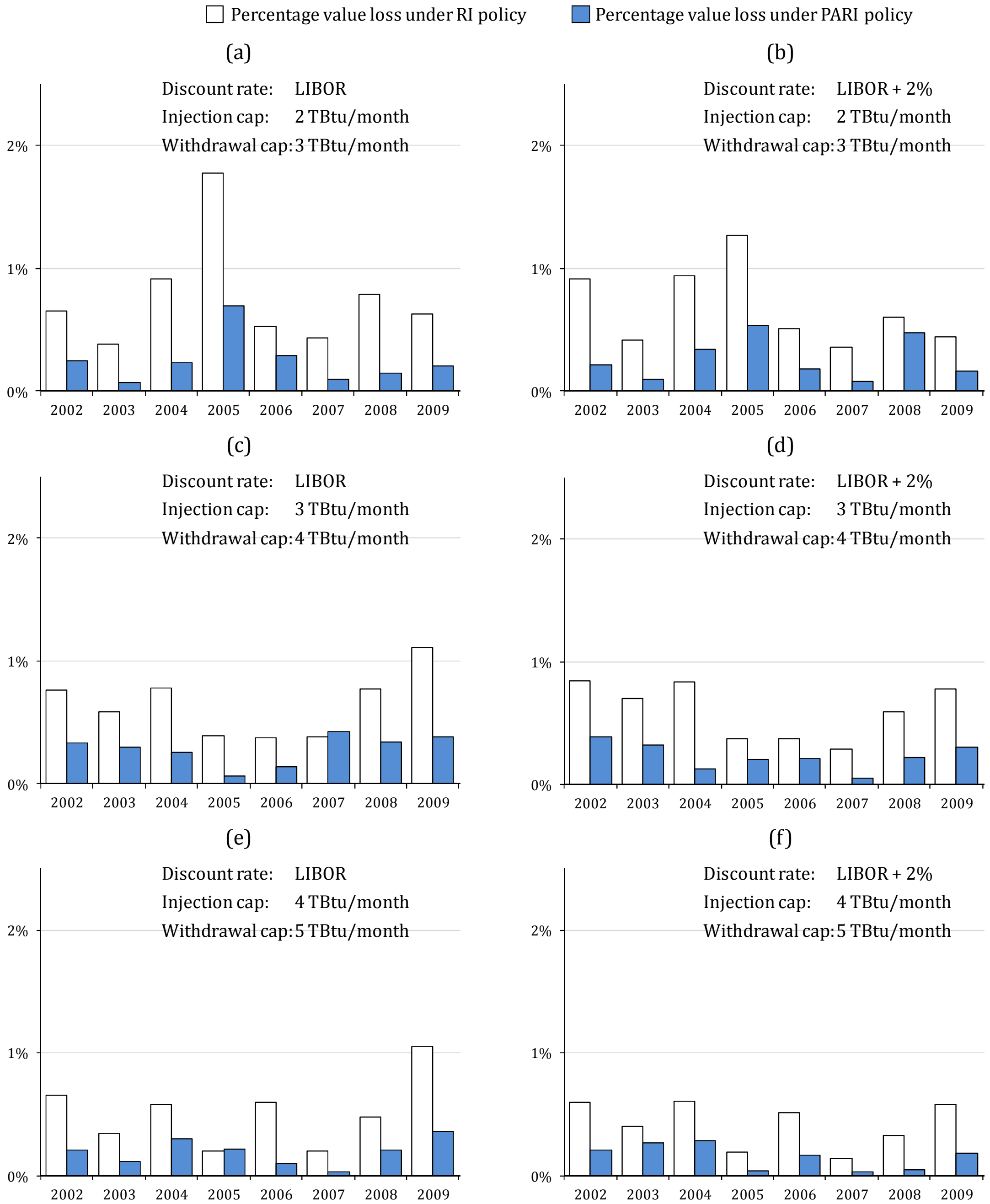



Figure 2.10: Binomial tree for forward curve with time-varying volatility

$T_1$ represents the end of March, when our planning horizon starts and the April contact is about to mature. For illustration purpose, we used only 3 binomial steps from $T_1$ to $T_2$. $T_2$ represents the end of April, when there are 11 remaining prices on the forward curve, the first being the maturing May futures price. At $T_3$ (end of May), 10 prices remain on the forward curve, and so on. The time steps shrink and the number of steps within each month increases as the time goes by, capturing the time-varying volatility. In our actual binomial tree model, we choose a small initial time step $\Delta t_0 = 0.001$ year. When $\sigma = 0.7$ and $\theta = 0.75$, there are 90 steps in the first month, and 480 steps in the last month. The total number of time steps over a year is 2,626.

Figure 2.11: Value loss under RI and PARI policies: Valuation at the end of October



# CHAPTER 3

# Inventory Control and Risk Management of Energy Storage Assets

## 3.1. Introduction

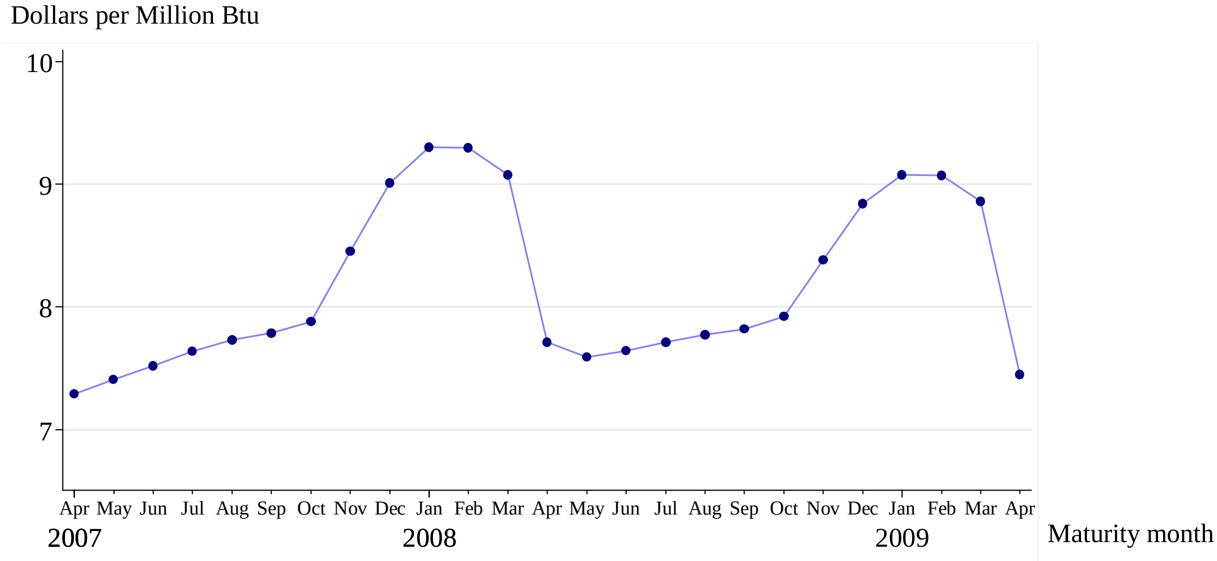
In the United States, daily production of natural gas is relatively constant at 52 billion cubic feet per day (BCF/d), while consumption exhibits significant seasonal variations: 55 BCF/d during the non-heating season (April through October), and 70 BCF/d during the heating season (November through March)[1]. The seasonal supply-demand imbalance makes it necessary to build underground natural gas storage facilities throughout the U.S., in particular in the Gulf production area and the North East and Midwest consumption areas. Reflecting this seasonal supply-demand imbalance, the natural gas futures market on the New York Mercantile Exchange (NYMEX) prices summer contracts at a significant discount relative to the winter contracts, providing incentives for physical players to acquire and use the storage assets to capture the summer-to-winter price differentials.

Figure 3.1 shows the natural gas futures prices observed in March 2007. In March, the firm (owner of a natural gas storage asset) can decide an injection and withdrawal schedule from April through to the next March by taking long or short positions on futures contracts. Once these futures positions are determined (e.g., by solving a static profit maximization problem subject to certain physical constraints discussed in detail in the next section), the firm essentially locks in a risk-free profit. This is sometimes referred to as the day-1 intrinsic value of the storage asset. On top of that value, the firm can re-balance its portfolio of

[1]Source: http://www.eia.doe.gov/pub/oil_gas/natural_gas/analysis_publications/ngprod/ngprod.pdf

Figure 3.1: Natural gas futures price observed on March 1, 2007

Data source: New York Mercantile Exchange

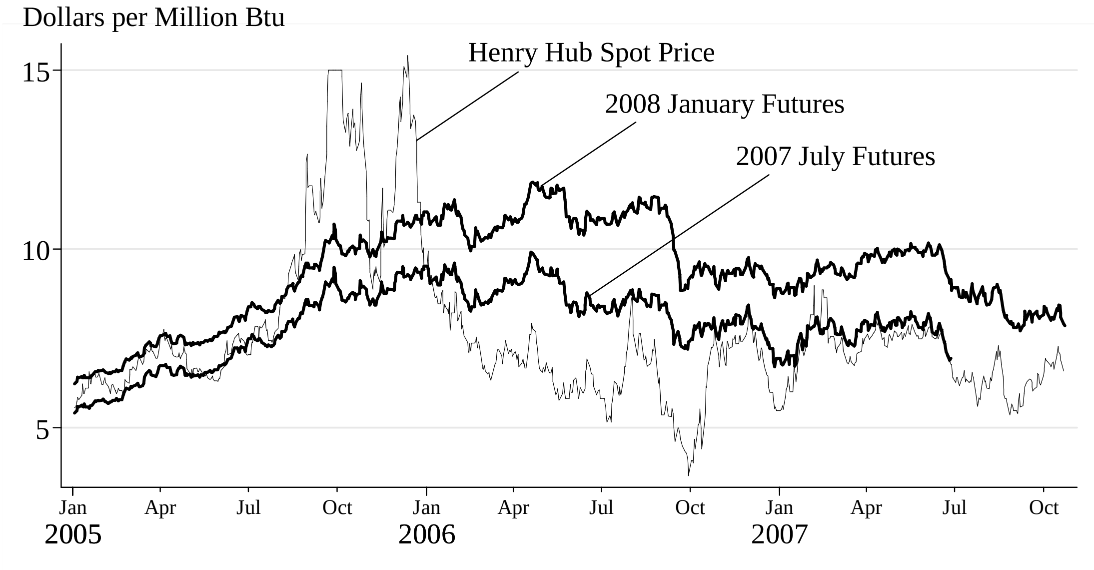



futures contracts over time to lock in more profit as futures prices move. However, a potential problem of this approach is that the storage asset value associated with spot price volatility is ignored.

Figure 3.2 shows evolution of the Henry Hub spot price and the prices two futures that mature in 2007 summer and 2008 winter. It is clear that spot price is much more volatile than the futures prices. It can also been seen that the correlation between spot price and futures prices are less than the correlation among futures with different maturity dates. Firms backed by storage assets can take advantage of the volatile spot price and its low correlation with the futures to make profit from the price spread between the two. This feature need to be considered in the storage asset valuation.

There are several physical constraints that need to be considered in order to make the futures and spot trading operationally feasible. First, a natural gas storage facility has its maximum capacity. If the firms buy more futures than the storage space allows, the firm will be forced to sell the excess volume on the spot market. Second, the injection and withdrawal rates depend on the inventory level in the storage facility. The more natural gas is in the storage, the higher the pressure in the reservoir, hence the slower the maximum injection rate and the faster the maximum withdrawal rate. Third, New York Mercantile Exchange

Figure 3.2: Natural gas futures and spot prices




regulates the timing of delivery of natural gas futures contract[2] : (1) delivery shall take place no earlier than the first calendar day of the delivery month and shall be completed no later than the last calendar day of the delivery month; (2) all deliveries shall be at as uniform an hourly and daily rate of flow over the course of the delivery month as is possible under the operating procedures and conditions of the transporting pipelines. These regulations mean that the decision made at the last trading day of each futures contract will immediately affect the decision throughout the delivery month.

In this paper, we incorporate all of the above profit opportunities and physical constraints into a stochastic dynamic programming problem. The underlying uncertainties are those in the spot and futures markets, which are modeled as a multi-dimensional stochastic process. The objective of the firm is to maximize the expected utility of the winter-end profit. The energy company we contacted with has its fiscal year end in March, and therefore it is most relevant to concentrate our attention on the shape of the probability distribution of the winter-end profit.

Our paper is closely related to literature on commodity procurement problem. *Seifert et al.* (2004) consider a risk-neutral decision maker who can either buy the commodity in advance with forward contracts or buy on spot market with a negligible lead time. Under

[2]http://www.nymex.com/rule_main.aspx?pg=33

a single period setting with stochastic spot price and demand, they derive a closed-form expression for the optimal order quantity. *Golovachkina and Bradley* (2002) consider coordinating supply chain of a single supplier and a single manufacturer with the presence of spot market. By modeling the negotiation procedure as a Stachelberg game, they obtain a closed-form expression for the optimal quantity of contracts the manufacturer should buy. All the above papers deal with one- or two-period setting. *Secomandi* (2010) studies the procurement and sales policy for commodity storage assets under a multi-period model. Moreover, he introduces inventory dependent injection and withdrawal capacities into the problem, and commodity spot price is modeled as an exogenously given Markov process. The optimal trading policy is shown to be characterized by two base-stock levels. *Haksoz and Seshadri* (2007) provide an extensive review on recent literature about supply chain management with the presence of spot market.

None of the above papers consider procurement on futures markets. *Goel and Gutierrez* (2006) considered procurement decision making and stochastic inventory control problem in a periodic review model with the presence of futures market. Different from most other papers in the field, they include convenience yield and transaction costs into the model. Their results suggest a significant reduction on storage costs by incorporating spot and futures market information and cast light on the way how convenience yield affects procurement strategy. However, their paper doesn't feature physical constraint common in commodity operations. Their model admits only one future contract, and does not involve seasonality issues.

The rest of this paper is organized as follows. We start with a formal description of our model in Section 4.2. The analysis of the inventory control and trading policy is conducted in Section 3.3. The numerical results are presented in Section 3.4. Concluding remarks are summarized in Section 3.5.

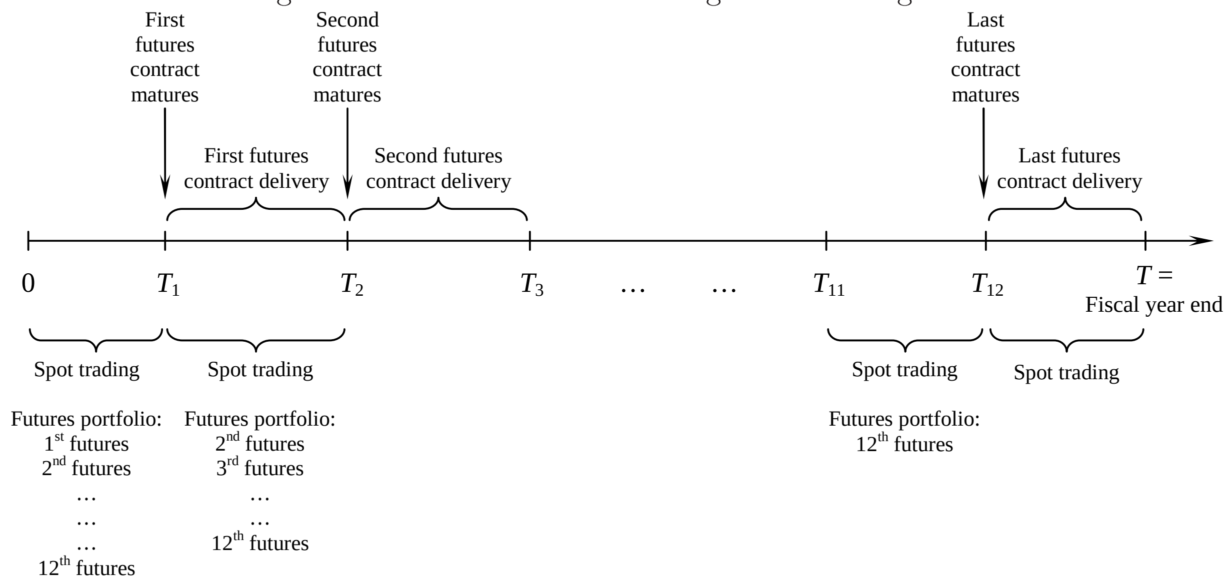
## 3.2. Model Description

We consider a firm managing a natural gas storage asset for an entire fiscal year. The fiscal year ends on March 31st[3], so that the management problem starts with injection (from April to October) and then withdrawal (from November to next March). The firm's

[3]The energy company we contacted with has fiscal-year end on March 31.

objective is to maximize the profit realized at the end of the fiscal year. As discussed in the introduction, the firm can lock in a risk-free profit by contracting for future deliveries. But the firm aims at achieving a better profit profile by balancing between the down-side risk and upside-side potentials. This requires certain control of the risk in the management objective. In this paper, we control the risk by maximizing the expected utility of fiscal-year ending profit generated from the natural gas storage asset.

Figure 3.3: Time line of the storage asset management

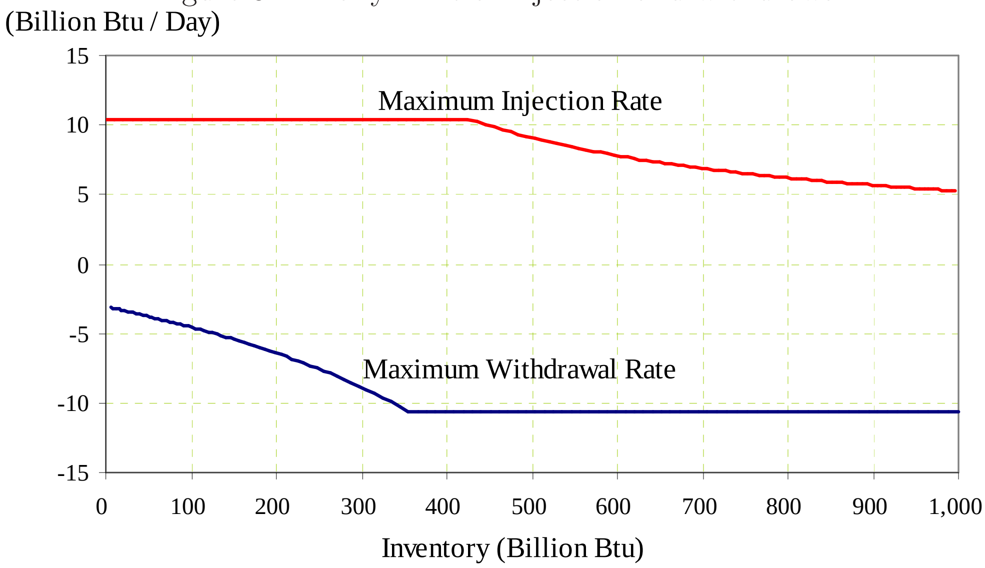



The time line of the firm's decision process is depicted in Figure 3.3. A total of $N = 12$ futures contracts (corresponding to 12 months) are available for injection or withdrawal during the fiscal year, with maturity date denoted as $T_i$ $(i = 1, 2, ..., N)$. When a futures contract matures at the end of a month, in addition to the monetary transactions, the firm must fulfill the contract obligation by delivering natural gas to or receiving natural gas from the commodity exchange. The exchange typically requires a uniform delivery schedule over the entire delivery month.

The firm also engages in spot market trading every day (in practice, spot delivery takes place within 24 hours after the transaction occurs). Hence, the daily injection or withdrawal amount is the sum of the futures contract delivery and the spot delivery. That daily amount has a physical constraint described below.

The total injection or withdraw rate is constrained by the physics of the natural gas storage facility. Figure 3.4 shows the daily injection and withdrawal limits for a 1000 BBtu (billion British thermal unit) storage facility. When the storage has less than 430 BBtu of natural gas, the maximum injection rate is only constrained by the flow rate at the valve (10.35 BBtu/day). After that, the more natural gas in the storage, the higher pressure makes it more difficult to inject. In the case of withdrawing natural gas, a similar logic can be understood. We note that due to the injection/withdrawal limits, spot trading volume depends on the futures contract delivery. For example, if the contract schedule is to withdrawal 5 BBtu of natural gas every day, then the upper and lower limits for daily spot trading volume have the same shape as in Figure 3.4, but shifted up by 5 BBtu. In particular, if the inventory is below 120 BBtu, then the firm will be forced to purchase on the spot market in order to fulfil its obligation.

Figure 3.4: Daily limit of injection and withdrawal




Data Source: Courtesy of DTE Energy

During injection and withdrawal procedure, a fraction of natural gas is lost due to various reasons (fuel burning, leakage, etc.). The fuel loss is relatively small (about 1% for injection and 0.5% for withdrawal), but including it in the model would introduce unnecessarily long terms that are not essential to the model. For the reader's ease, we do not include them in the model description below. For the same reason, we ignore the transaction cost related to the financial trading.

The model can be formally written as follows:

**Decision epochs**:

$t = 0, 1, ..., T$.

**State variables**:

$p_t$: spot price in period $t$;

$f_t = \{f(t, T_i) : t \leq T_i,\ i = 1, 2, ..., N\}$: vector of futures prices, where $T_i < T$ is the maturity date of the $i$-th futures. For notational convenience, let us define $T_{N+1} \equiv T$. The delivery of the $i$-th futures occurs between $T_i$ and $T_{i+1}$;

$I_t$: inventory (i.e., amount of natural gas in the storage) at the beginning of period $t$;

$w_t$: wealth (i.e., initial wealth plus cumulative cash flows associated with natural gas tradings) at the beginning of period $t$;

$d_t$: futures contract delivery in period $t$, $d_t > 0$ corresponds to injection, $d_t < 0$ corresponds to withdrawal; this quantity is determined by the futures position on the previous maturity date (see below for details).

**Decisions**:

$x_t = \{x(t, T_i) : t \leq T_i,\ i = 1, 2, ..., N\}$: vector of futures positions in period $t$; $x(t, T_i) > 0$ corresponds to long positions, $x(t, T_i) < 0$ corresponds to short positions;

$y_t \in \mathcal{A}(I_t) \equiv \left[\, I_t + \underline{\lambda}(I_t),\ I_t + \overline{\lambda}(I_t) \,\right]$: inventory level at the end of period $t$. Note that the change of inventory within a period is $y_t - I_t$, which must fall within $\left[\, \underline{\lambda}(I_t),\ \overline{\lambda}(I_t) \,\right]$, where $\overline{\lambda}(I_t)$ (resp. $-\underline{\lambda}(I_t)$) is the maximum amount can be injected (resp. withdrawn) in a single period when the starting inventory level is $I_t$. Figure 3.4 depicts a real example of $\overline{\lambda}(I_t)$ and $-\underline{\lambda}(I_t)$. Note that deciding the ending inventory level $y_t$ is equivalent to deciding the spot market trading volume, which is $y_t - I_t - d_t$.

**State Transitions:**

The state $(I_t, d_t, w_t)$ has an initial value $(I_0, d_0, w_0) = (0, 0, w)$ and transits as follows:

$$
\begin{aligned}
I_{t+1} &= y_t && (3.1)\\
d_{t+1} &= \begin{cases} d_t, & \text{for } t \neq T_i \\ x(T_i, T_i)/(T_{i+1} - T_i), & \text{for } t = T_i \end{cases} && (3.2)\\
w_{t+1} &= (1+r)w_t + \sum_{i:T_i>t} x(t, T_i)\big(f(t+1, T_i) - f(t, T_i)\big) - p_t(y_t - I_t - d_t) - d_t f(T_j, T_j) && (3.3)\\
& && (3.4)
\end{aligned}
$$

where $r$ is the one-period interest rate, and $T_j$ is the last maturity date.

Transition equation (3.2) indicates that the futures delivery is even: $d_{t+1} = d_t$ as long as $t$ is not a maturity date, and when $i$-th futures contracts mature, the firm has a position of $x(T_i, T_i)$, which will be evenly spread out between $T_i$ and $T_{i+1}$.

In (3.4), the first term is the wealth cumulated at the risk-free interest rate $r$, the second term is the cash flows associated with the futures positions (futures are marked to market), the third term is the cash flows related to the spot market purchase/sales, and the last term is the cash flow associated with the futures delivery, where we assume pay-on-delivery.[4]

We use a price model that is commonly seen in practice (see, e.g., *Clewlow and Strickland* (2000)). In continuous time, the futures prices are modeled as an $n$-factor process:

$$df(t, T) = f(t, T) \sum_{j=1}^{n} \sigma_j(t, T) dz_j(t) \tag{3.5}$$

where $z_j(t), j = 1, \ldots, n$, are independent Brownian motions and $\sigma_j(t, T)$ is the volatility of $T$-maturing futures price due to the factor $j$ at time $t$. The volatility function can be estimated from historical forward price data. In practice, it has been found that using a small number of factors (e.g., $n = 2$ or 3) can usually capture most of the futures price dynamics. As empirical observation shows, volatility factors can shape the forward curves with different maturities in three fundamental ways. One way is to shift up all curves under a positive shock. A second way is to tilt the curve by moving short maturity and long maturity curves in opposite directions. The third way is to bend the curve so that two ends move in the same direction while the middle section moves in opposite direction.

[4] In reality, the payment is typically delayed until the 20th of the next month. Pay-on-delivery is an innocuous assumption, which amounts to a slight difference due to the interests earned on early payment.

The spot price in our context is the one of the regional spot market. It is not the spot price implied from futures prices. (When implied from the futures prices, spot price itself is served as an unobservable factor and backed out based on theoretical spot-futures relations. See, e.g., *Schwartz* (1997).) In practice, the spot price is typically much more volatile than the futures price, and the correlation between spot and futures is also less than the correlations among futures contracts. Firms backed by storage assets often try to make profit from the price spread between the two.[5] In our model, the spot price is assumed to follow an mean-reverting process with a time-varying mean:

$$dp_t = -\kappa(p_t - m_t)dt + \sum_{j=1}^{n+1} \alpha_j(t) dz_j(t), \tag{3.6}$$

where $\kappa > 0$ is the mean-reverting coefficient. Note that an additional factor $z_{n+1}$ presents in the spot price dynamics, but not in the futures price. This additional factor captures the above-mentioned fact that spot is more volatile. The mean process $m_t$ reflects the spot price trend, which should be estimated from the data.

**Objective**: The firm aims to maximize the expected utility on its fiscal year end wealth $w_T$:

$$\max \quad \mathsf{E}_0 U(w_T) \tag{3.7}$$

where $U$ is a concave and increasing utility function, and $\mathsf{E}_t$ denotes expectation given the information available at time $t$.

**Dynamic Program formulation:**

Let $V_t(w, I, p, f, d)$ be the maximum expected utility at time $t$ when the state of the system is $(w, I, p, f, d)$. Then

$$\begin{aligned} V_t(w_t, I_t, p_t, f_t, d_t) &= \max_{x_t, y_t} \mathsf{E}_t[V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})] \qquad \text{for } t < T & (3.8) \\ V_T(w_T, \cdot, \cdot, \cdot, \cdot) &= U(w_T) & (3.9) \end{aligned}$$

where the state variables evolve according to (3.1)-(3.6). Note that inventory left at the end of the fiscal year $I_T$ has no contribution to the fiscal year end wealth.

[5]Based on our conversation with the energy company, this fact that spot is more volatile than and less correlated with futures price is an essential feature that must be captured in valuing a storage asset.

## 3.3. Inventory Control and Trading Policies

### 3.3.1 Optimal Policy

In this section, we examine some basic properties of the optimal inventory control and trading policy. We show that the value function is concave under mild assumptions, and then derive the optimality conditions.

**Assumption 5.** *$\underline{\lambda}(I)$ is convex and decreasing in $I$, and $\overline{\lambda}(I)$ is concave and decreasing in $I$.*

**Assumption 6.** *$\overline{\lambda}$ and $\underline{\lambda}$ are differentiable, and $-1 \leq \frac{\partial \overline{\lambda}}{\partial I} \leq 0$, $-1 \leq \frac{\partial \underline{\lambda}}{\partial I} \leq 0$.*

The above assumption essentially imposes analytical properties on the injection and withdrawal constraints in Figure 3.4. The shape of the injection/withdrawal limits in Figure 3.4 can be approximately assumed to satisfy Assumption 5.

**Proposition 6.** *Under Assumption 5, the value function $V_t(w, I, p, f, d)$ defined in (3.8) is concave in $(w, I, d)$ for any $(p, f)$.*

*Proof.* From (3.9), $V_T(w_T, I_T, p_T, f_T, d_T) = U(w_T)$. The result clearly holds for $t = T$. Suppose that the result holds for $V_{t+1}, t < T$. We now show that it holds for $V_t$. Let the maximand in (3.8) be

$$J_t(w_t, I_t, p_t, f_t, d_t, x_t, y_t) \quad := \quad \mathsf{E}_t[V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})]$$

We first show $J_t$ is concave in $(w_t, I_t, d_t, x_t, y_t)$ for any price vector $(p_t, f_t)$. Consider any two distinct points: $(w_t^i, I_t^i, d_t^i, x_t^i, y_t^i), i = 1, 2$, and for any given $\alpha \in (0, 1)$, let

$$(w_t^0, I_t^0, d_t^0, x_t^0, y_t^0) \quad = \quad \alpha(w_t^1, I_t^1, d_t^1, x_t^1, y_t^1) + (1 - \alpha)(w_t^2, I_t^2, d_t^2, x_t^2, y_t^2)$$

Let $(w_{t+1}^i, I_{t+1}^i, d_{t+1}^i), i = 0, 1, 2$ be the corresponding values of the state variables in the next period. Then by (3.1)-(3.4), we have

$$\begin{aligned} I_{t+1}^0 &= \alpha I_{t+1}^1 + (1 - \alpha) I_{t+1}^2 \\ d_{t+1}^0 &= \alpha d_{t+1}^1 + (1 - \alpha) d_{t+1}^2 \\ w_{t+1}^0 &= \alpha w_{t+1}^1 + (1 - \alpha) w_{t+1}^2, \qquad \text{for any realization of } (p_{t+1}, f_{t+1}) \end{aligned}$$

Since $V_{t+1}$ is concave in $(w_{t+1}, I_{t+1}, d_{t+1})$, we have for any realization of $(p_{t+1}, f_{t+1})$:

$$\begin{aligned}&V_{t+1}(w_{t+1}^0, I_{t+1}^0, p_{t+1}, f_{t+1}, d_{t+1}^0) \geq \\ &\qquad \alpha V_{t+1}(w_{t+1}^1, I_{t+1}^1, p_{t+1}, f_{t+1}, d_{t+1}^1) + (1-\alpha)V_{t+1}(w_{t+1}^2, I_{t+1}^2, p_{t+1}, f_{t+1}, d_{t+1}^2)\end{aligned}$$

Taking expectation on both sides, we have the concavity of $J_t$ in $(w_t, I_t, d_t, x_t, y_t)$:

$$J_t(w_t^0, I_t^0, p_t, f_t, d_t^0, x_t^0, y_t^0) \;\; \geq \;\; \alpha J_t(w_t^1, I_t^1, p_t, f_t, d_t^1, x_t^1, y_t^1) + (1-\alpha)J_t(w_t^2, I_t^2, p_t, f_t, d_t^2, x_t^2, y_t^2).$$

Next, we show $V_t$ is concave in $(w_t, I_t, d_t)$. To simplify notation, we omit subscript $t$ in the arguments. According to the dynamic program in (3.8), for any $r > 0$, there exist $(x^{ir}, y^{1r})$ and $(x^{2r}, y^{2r})$, such that

$$V_t(w^i, I^i, p, f, d^i) \;<\; J_t(w^i, I^i, p, f, d^i, x_t^{ir}, y^{ir}) + r, \qquad i = 1, 2. \tag{3.10}$$

Let $(w^0, I^0, d^0, x^{0r}, y^{0r}) = \alpha(w^1, I^1, d^1, x^{1r}, y^{1r}) + (1-\alpha)(w^2, I^2, d^2, x^{2r}, y^{2r})$. By Assumption 1, we have:

$$\begin{aligned} I^0 + \underline{\lambda}(I^0) \;&\leq\; \alpha(I^1 + \underline{\lambda}(I^1)) + (1-\alpha)(I^2 + \underline{\lambda}(I^2)) \\ &\leq\; \alpha y^{1r} + (1-\alpha)y^{2r} \;=\; y^{0r} \\ &\leq\; \alpha(I^1 + \overline{\lambda}(I^1)) + (1-\alpha)(I^2 + \overline{\lambda}(I^2)) \\ &\leq\; I^0 + \overline{\lambda}(I^0) \end{aligned}$$

Hence, $y^{0r} \in \mathcal{A}(I^0)$, i.e., $y^{0r}$ is a feasible policy. Now

$$\begin{aligned} V_t(w^0, I^0, p, f, d^0) \;&\geq\; J_t(w^0, I^0, p, f, d^0, x^{0r}, y^{0r}) \\ &\geq\; \alpha J_t(w^1, I^1, p, f, d^1, x^{1r}, y^{1r}) + (1-\alpha)J_t(w^2, I^2, p, f, d^2, x^{2r}, y^{2r}) \\ &>\; \alpha V_t(w^1, I^1, p, f, d^1) + (1-\alpha)V_t(w^2, I^2, p, f, d^2) - r \end{aligned}$$

where the first inequality is due to the feasibility of the policy $(x^{0r}, y^{0r})$, the second inequality is due to the concavity of $J_t$ proved earlier, and the last inequality follows from (3.10). Letting $r \to 0$ in the above inequalities yields the concavity of $V_t$ in $(w, I, d)$. □

**Proposition 7.** *Under Assumption 6, the value function $V_t(w, I, p, f, d)$ defined in (3.8) is increasing in $w$ and $I$ for any $(p, f)$.*

*Proof.* For $t = T$ the claim holds since $V_T = U(w_T)$ is an increasing function of $w_T$. Suppose $V_{t+1}$ is increasing in $w_{t+1}$ and $I_{t+1}$. If $w_t$ increases, then $w_{t+1}$ increases. Therefore $V_t$ is increasing in $w_t$. If $I_t$ increases from $I_t$ to $I_t + \Delta I$, by Assumption 6,

$$I_t + \Delta I + \overline{\lambda}(I_t + \Delta I) > I_t + \overline{\lambda}(I_t) \tag{3.11}$$

where we have used property that $\overline{\lambda}(I)$ decreases under-proportionally compared to $I$. Let $y_t^*$ be the original optimal inventory level when starting inventory is $I_t$: (1) if $y_t^*$ is still feasible for $I_t + \Delta I$, then by setting inventory to $y_t^*$, we have a scenario where $w_{t+1}$ increases and the other state variables for $t+1$ remain unchanged, (2) if $y_t^*$ is no longer feasible for $I_t + \Delta I$, then it must be that $y_t^* < I_t + \Delta I + \underline{\lambda}(I_t + \Delta I)$. Since $y_t^* + \Delta I \geq I_t + \underline{\lambda}(I_t) + \Delta I \geq I_t + \Delta I + \underline{\lambda}(I_t + \Delta I)$, $y_t^* + \Delta I$ is feasible for $I_t + \Delta I$. Under the inventory level $y_t^* + \Delta I$, $I_{t+1}$ increases and the other state variables remain unchanged. In both scenarios, the value function $V_t$ will increase. $\square$

The concavity result gives the following characterization of the optimal policy. Based on the dynamic program in (3.8) and the state evolution in (3.1) and (3.4), we have the first order condition for $y_t$:

$$\mathsf{E}_t\Big[ - p_t \frac{\partial V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})}{\partial w} + \frac{\partial V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})}{\partial I} \Big] = 0$$

Let $y_{t0}$ be the point that satisfies the above first-order condition, then the optimal $y_t^*$ can then be written as

$$y_t^* = \begin{cases} y_{t0} & \text{if } y_{t0} \in \mathcal{A}(I_t) \\ I_t + \underline{\lambda}(I_t) & \text{if } y_{t0} < I_t + \underline{\lambda}(I_t) \\ I_t + \overline{\lambda}(I_t) & \text{if } y_{t0} > I_t + \overline{\lambda}(I_t) \end{cases}$$

The optimal $x(t, T_j)$ satisfies:

$$\mathsf{E}_t\Big[\frac{\partial V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})}{\partial w}\big(f(t+1, T_j) - f(t, T_j)\big)\Big] = 0, \qquad \text{if } t \neq T_j, \tag{3.12}$$

$$\mathsf{E}_t\Big[\frac{\partial V_{t+1}(w_{t+1}, I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})}{\partial d}\Big] = 0, \qquad \text{if } t = T_j. \tag{3.13}$$

The first-order condition in (3.12) means that at non-maturity date the firm tries to profit from the futures price dynamics, whereas the first order condition in (3.13) implies that at the futures maturity date the firm focuses on setting up the optimal delivery rate for the next month.

### 3.3.2 A Heuristic Policy

The firm's futures position decided at the maturity date is the firm's final decision on how much natural gas to buy or sell during the next month. This position affects the allowable daily spot trading volume over the next month, since total daily injection/withdrawal amount is physically constrained. This coupling effect between spot and futures trading exacerbates the "curse of dimensionality."

In this section, we develop a heuristic policy under which spot and futures decisions are decoupled, and the required computational effort is 60-70% less than a full optimization. More importantly, in Section 3.4, we numerically show that this heuristic policy is actually near-optimal in the sense that the resulting fiscal-year-end profit distribution is very close to that under the optimal policy.

In searching for the optimal policy, we notice the following computational burden: at maturity date $T_i$, to evaluate each trial futures position $x(T_i, T_i)$, the firm has to resolve the entire dynamic program from $T_i$ onwards, which is computationally cumbersome. The idea of the heuristic policy is essentially to obviate the need for resolving the entire dynamic program for each futures position $x(T_i, T_i)$. In fact, the heuristic algorithm first solves for the optimal policy for a certain fixed $x(T_i, T_i)$. Then, for other values of $x(T_i, T_i)$, we simply apply that policy rather than solve for the optimal policy.

Formally, at maturity date $T_i$, the system state is $(w_{T_i}, I_{T_i}, p_{T_i}, f_{T_i}, d_{T_i})$, and the firm is to decide $(x_{T_i}, y_{T_i})$. We first fix $x(T_i, T_i) = (T_{i+1} - T_i)d^0$ for certain fixed $d^0$ (implying $d_t = d^0$ for $t = T_i + 1, \ldots, T_{i+1}$), and we use the usual backward induction to solve for the optimal decisions at $T_i$ and for the next month, denoted as $\{\big(x_t^*(w_t, I_t, p_t, f_t, d^0), y_t^*(w_t, I_t, p_t, f_t, d^0)\big) : t = T_i, \ldots, T_{i+1}\}$.

Next, we search for the optimal position $x(T_i, T_i)$. Rather than search for optimal policy again for each value of $x(T_i, T_i)$, we apply $\{\big(x_t^*(w_t, I_t, p_t, f_t, d^0), y_t^*(w_t, I_t, p_t, f_t, d^0)\big) : t = T_i, \ldots, T_{i+1}\}$ in the following ways. For each $t = T_i, \ldots, T_{i+1} - 1$, we compute the next-period wealth as if the delivery schedule is $d^0$ every period:

$$w_{t+1} = (1+r)w_t + \sum_{j=i+1}^{N} x^*(t, T_j)\big(f(t+1, T_j) - f(t, T_j)\big) - p_t(y_t^* - I_t - d^0) - d^0 f(T_i, T_i)$$

and the next-period decisions $(x^*_{t+1}, y^*_{t+1})$ are based on the above $w_{t+1}$. This policy can be seen as feasible, because the physical constraint $y^*_t \in \big[\, I_t + \underline{\lambda}(I_t),\ I_t + \overline{\lambda}(I_t) \,\big]$ is clearly still satisfied. In fact, the firm would sell the difference $x(T_i, T_i)/(T_{i+1} - T_i) - d^0$ on the spot so as to maintain its period-end inventory to be $y^*_t$.

The above heuristic policy significantly reduces the computational effort, yet achieves near-optimal performance (see Section 3.4).

Another important implication of the above heuristic policy is that it allows us to further derive insights associated with the firm's management decision. For clarity of exposition, for the rest of this section, we assume that the daily interest rate $r = 0$. (When considering the tradeoff in short time horizon, e.g., a month, $r = 0$ is an innocuous assumption.)

Using the above heuristic policy, for $t = T_i + 1, \ldots, T_{i+1}$, wealth evolves according to,

$$w_{t+1} \;=\; w_t + \sum_{j=i+1}^{N} x^*(t, T_j)\big(f(t+1, T_j) - f(t, T_j)\big) - p_t(y^*_t - I_t - d_t) - d_t f(T_i, T_i).$$

Taking sum of the above equation from $t = T_i$ to $T_{i+1}-1$ and noting that $d_t = x(T_i, T_i)/(T_{i+1} - T_i)$, we obtain

$$\begin{aligned} w_{T_{i+1}+1} \;=\; & w_{T_i+1} + \sum_{t=T_i+1}^{T_{i+1}} \sum_{j=i+1}^{N} x^*(t, T_j)\big(f(t+1, T_j) - f(t, T_j)\big) \\ & - \sum_{t=T_i+1}^{T_{i+1}} p_t\Big(y^*_t - I_t - \frac{x(T_i, T_i)}{T_{i+1} - T_i}\Big) \\ & - x(T_i, T_i) f(T_i, T_i) \end{aligned}$$

Rearranging terms, we have

$$\begin{aligned} w_{T_{i+1}+1} \;=\; & w_{T_i+1} + x(T_i, T_i)\Big[\frac{1}{T_{i+1} - T_i} \sum_{t=T_i+1}^{T_{i+1}} p_t - f(T_i, T_i)\Big] \\ & + \sum_{t=T_i+1}^{T_{i+1}} \sum_{j=i+1}^{N} x^*(t, T_j)\big(f(t+1, T_j) - f(t, T_j)\big) \\ & - \sum_{t=T_i+1}^{T_{i+1}} p_t(y^*_t - I_t) \end{aligned} \tag{3.14}$$

In (3.14), we note that by the definition of the heuristic policy, $(x^*_t, y^*_t)$ is independent of

$x(T_i, T_i)$. Hence, the decision $x(T_i, T_i)$ affects wealth only through the term

$$x(T_i, T_i)\left[\overline{p}_{[T_i,T_{i+1}]} - f(T_i, T_i)\right],$$

where $\overline{p}_{[T_i,T_{i+1}]} \equiv \frac{1}{T_{i+1}-T_i} \sum_{t=T_i+1}^{T_{i+1}} p_t$ is the average spot price over the delivery month.

This term captures the essential tradeoff the firm faces when deciding futures delivery volume $x(T_i, T_i)$. It shows that by signing $x(T_i, T_i)$ number of futures contracts, the firm exchanges a deterministic cash flow of the amount $x(T_i, T_i)f(T_i, T_i)$ for a stochastic cash flow of the amount $x(T_i, T_i)\overline{p}_{[T_i,T_{i+1}]}$.

The above result not only reveals the tradeoff of the firm's decision, but also have significant implication to our understanding of the futures markets. In finance, it is well-accepted that the futures price converges to the spot market price when it matures, otherwise arbitrage opportunity exists. In the commodity markets, however, the delivery lag casts doubt on the validity of this convergence. As the delivery of matured futures contracts need to be evenly spread over the month, the simple arbitrage strategy of getting delivery at low price and selling high on spot does not work. Instead, the firm is concerned about the average spot level over the entire delivery month. And we conjecture that, under certain conditions, in the equilibrium the futures price would converge to the expected average spot price.[6]

### 3.3.3 Simplified Model

In this section we simplify the general model. We assume that instead of maximizing the utility on the end-of-period wealth, the firm is risk-neutral and chooses to maximize the expected wealth itself. From the futures price process (3.5), $f(t, T_i)$ is martingale with $\mathsf{E}_t f(t+1, T_i) = f(t, T_i)$. Thus the expected marking-to-market return from futures is always zero, and we only need to decide the spot trading on non-maturity days. Let $V_t(I_t, p_t, f_t, d_t)$ be the maximal expected revenue-to-go function from period $t$. The DP formula can then

[6]Based on our conversation with the energy company, the relationship between the matured futures price and the average spot price may have strong relations.

be written as,

$$
\begin{aligned}
V_t(I_t, p_t, f_t, d_t) &= \max_{x_t, y_t} \; -p_t(y_t - I_t - d_t) - d_t f(T_j, T_j) \\
&\quad + \mathsf{E}_t[V_{t+1}(I_{t+1}, p_{t+1}, f_{t+1}, d_{t+1})] \qquad \text{for } t \le T \qquad (3.15) \\
V_{T+1}(\cdot, \cdot, \cdot, \cdot) &= 0 \qquad (3.16)
\end{aligned}
$$

where the four state variables evolve according to (3.1), (3.2), (3.5) and (3.6).

**Proposition 8.** *Under Assumptions 5 and 6, $V_t$ is concave in $(I, d)$ and increasing in $I$.*

*Proof.* The simplified model is a special case of the original model with a linear utility $U(w) = w$. So all the properties derived in the previous section also hold here. □

The optimal spot trading a non-maturity day is characterized by a base stock level $y_t^*$ independent of the starting inventory $I_t$. If the inventory level is such that $y_t^*$ is attainable within the delivery limit, then the inventory is increased or decreased to the base stock level. If $y_t^*$ exceeds the upper limit, then the firm should inject up to capacity. If $y_t^*$ drops below the lower limit, then the firm should withdraw down to capacity.

At maturity $T_i$, the firm needs to decide $y_{T_i}$ and $x(T_i, T_i)$. Let $(y_{T_i}^*, x^*(T_i, T_i))$ denote the maximizer of the RHS of (3.15). If $y_{T_i}^*$ is attainable, the optimal policy is to set inventory position at $y_{T_i}^*$, and long $x^*(T_i, T_i)$ futures. If $y_{T_i}^* < I_{T_i} + \underline{\lambda}(I_{T_i})$, choose the limit $I_{T_i} + \underline{\lambda}(I_{T_i})$, and long $\hat{x}(T_i, T_i)$ futures, where $\hat{x}(T_i, T_i)$ is the maximizer of the maximand when $y_{T_i} = I_{T_i} + \underline{\lambda}(I_{T_i})$. If $y_{T_i}^* > I_{T_i} + \overline{\lambda}(I_{T_i})$, choose the limit $I_{T_i} + \overline{\lambda}(I_{T_i})$, and long $\hat{x}(T_i, T_i)$ futures, where $\hat{x}(T_i, T_i)$ maximizes the RHS of (3.15) when $y_{T_i} = I_{T_i} + \overline{\lambda}(I_{T_i})$.

**Proposition 9.** *The optimal spot trading volume decreases in $I_t$.*

*Proof.* Let $s_t = y_t - I_t - d_t$ be the spot trading volume. The the maximand in (3.15) can be written as

$$
\begin{aligned}
J_t(I_t, p_t, f_t, d_t, x_t, s_t) &:= -p_t s_t - d_t f(T_j, T_j) \\
&\quad + \mathsf{E}_t[V_{t+1}(I_t + s_t + d_t, p_{t+1}, f_{t+1}, d_{t+1})]
\end{aligned}
$$

$J_t$ is submodular in $(I_t, s_t)$ since

$$
\frac{\partial^2 J_t}{\partial I_t \partial s_t} = \mathsf{E}_t \frac{\partial^2 V_{t+1}^2}{\partial I^2} \le 0
$$

Note that $s_t \in [\underline{\lambda}(I_t), \overline{\lambda}(I_t)]$, and $\underline{\lambda}$ and $\overline{\lambda}$ are both decreasing in $I_t$. So $s_t^*$ decreases in $I_t$. $\square$

**Proposition 10.** *Under Assumption 6, the marginal value of inventory $\frac{\partial V_t}{\partial I_t}$ is constrained by the inequality*

$$\frac{\partial V_t}{\partial I_t} \leq \max\{E_t p_{t+i} | i = 0, 1, ..., T-t\}$$

*Proof.* To be completed. For $t = T$, $\frac{\partial V_T}{\partial I_T} = \frac{\partial}{\partial I_T}(-p_T \underline{\lambda}(I_T)) = -p_T \underline{\lambda}'$. By Assumption 6 $|\underline{\lambda}'| \leq 1$, so $\frac{\partial V_T}{\partial I_T}$ satisfies the inequality. Now suppose the claim holds for $t+1$, we will show that it holds for $t$ as well. There are three cases to consider.

(1) If $y_t^* = I_t + \overline{\lambda}(I_t)$. $V_t(I_t, p_t, f_t, d_t) = -p_t \overline{\lambda}(I_t) - d_t f(T_j, T_j) + \mathsf{E}_t[V_{t+1}(I_t + \overline{\lambda}(I_t), p_{t+1}, f_{t+1}, d_{t+1})]$. $\frac{\partial V_t}{\partial I_t} = -p_t \overline{\lambda}' + \mathsf{E}_t[\frac{\partial V_{t+1}}{\partial I_{t+1}}](1 + \overline{\lambda}')$. By Assumption 6, $1 \geq 1 + \overline{\lambda}' \geq 0$. So $\frac{\partial V_t}{\partial I_t} \leq (-\overline{\lambda}' + 1 + \overline{\lambda}') \max\{E_t p_{t+i} | i = 0, 1, ..., T-t\} = \max\{E_t p_{t+i} | i = 0, 1, ..., T-t\}$.

(2) If $I_t + \underline{\lambda}(I_t) < y_t^* < I_t + \overline{\lambda}(I_t)$. $V_t(I_t, p_t, f_t, d_t) = -p_t(y_t^* - I_t - d_t) - d_t f(T_j, T_j) + \mathsf{E}_t[V_{t+1}(y_t^*, p_{t+1}, f_{t+1}, d_{t+1})]$. So $\frac{\partial V_t}{\partial I_t} = p_t \leq \max\{E_t p_{t+i} | i = 0, 1, ..., T-t\}$.

(3) If $y_t^* = I_t + \underline{\lambda}(I_t)$. Proof is the same as (1). $\square$

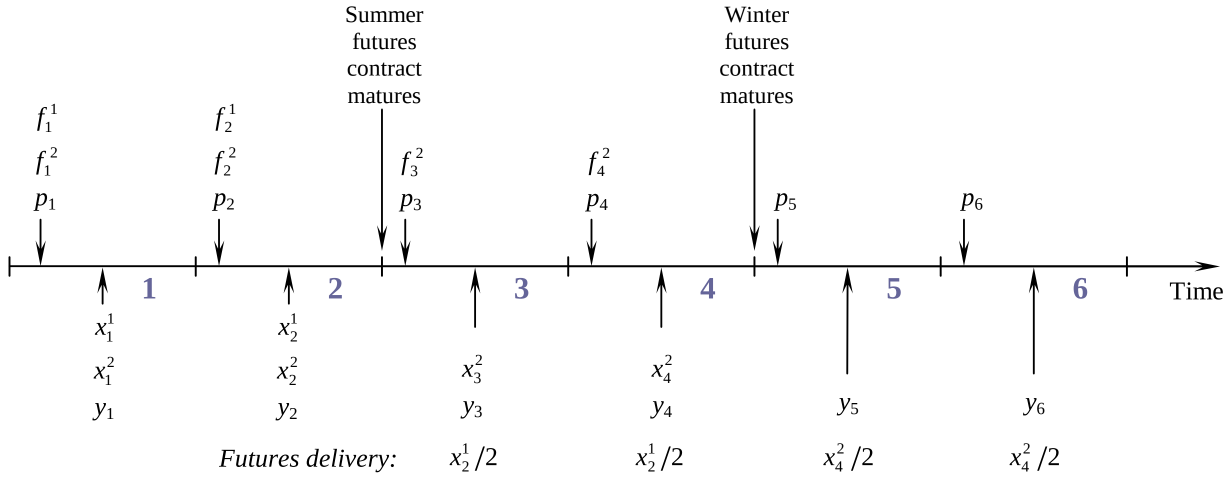
## 3.4. Numerical Results

In this section, we numerically analyze a six-period model. The periods are indexed as 1, 2, ..., 6. We consider only two futures, which we call "summer futures" and "winter futures." The summer futures are traded in period 1 and 2, maturing at the end of period 2, and delivered from period 3 to 4. The winter contracts are traded from period 1 to 4, maturing at the end of period 4, and the delivery takes place in period 5 to 6. The summer futures prices are generally lower than the winter futures prices, and spot commodity is traded in all periods with a mean-reverting price process (see the price model below).

The detailed time line is shown in Figure 3.5. For notational convenience, we denote futures prices as $f_t^i = f(t, T_i)$, and futures positions as $x_t^i = x(t, T_i)$, where $i = 1$ for summer futures and $i = 2$ for winter futures. In the first two periods, the firm decides the spot trading and the futures positions based on the market information. At the end of period 2, $x_2^1$ number of summer contracts mature, and will be delivered in equal amounts over period 3 and 4. The firm continues to adjust its position on the winter futures that mature at the end of period 4, and fulfill the contract delivery over period 5 and 6. The firm's objective, as before, is to maximize the expected utility of the wealth at the end of period 6.

The utility function we impose on end wealth is an exponential utility $U(w) = -\alpha e^{-rw}$ $(\alpha > 0, r > 0)$. In practice, considering the range of ending wealth and precision of results, we choose $\alpha = 20000$ and $r = 1/80$.

Figure 3.5: The six-period model




To model the futures and spot prices, we apply the technique described in Section 4.2 with two factors (i.e., $n = 2$ in (3.5)). Futures prices can then be written as

$$df_t^1 = a_1 f_t^1 dz_1 + b_1 f_t^1 dz_2 \tag{3.17}$$

$$df_t^2 = a_2 f_t^2 dz_1 + b_2 f_t^2 dz_2 \tag{3.18}$$

In order to capture the "shift" and "tilt" dynamics, we choose parameters such that $a_1 a_2 > 0$ and $b_1 b_2 < 0$. So the two prices moves in the same (opposite) direction under shocks on $z_1$ ($z_2$). Moreover, as indicated in Section 4.2, summer contracts are priced at a significant discount relative to winter contracts. Hence we choose volatility factors and initial prices to ensure that second forward curve is above the first one. These parameters are shown in Table 3.1.

Table 3.1: Parameters of futures price

| | | |
|---|---|---|
| Future 1 | $a_1 = 0.01$ | $b_1 = -0.01$ |
| Future 2 | $a_2 = 0.01$ | $b_2 = 0.005$ |

*Notes.* The initial prices $f_1^0 = 6.2$ and $f_2^0 = 6.8$

The spot price evolution (3.6) now becomes:

$$dp_t = \kappa(p_t - m_t)dt + a_0 p_t dz_1 + b_0 p_t dz_2 + c_0 p_t dz_3 \tag{3.19}$$

As discussed in Section 1, correlation between spot price and futures prices is less than correlation among futures prices with different maturities, so we assign smaller coefficients $a_0$ and $b_0$ for spot price relative to parameters of futures. Also for the heuristic policy, we have derived the tradeoff between futures price and expected average spot price in equation (3.14). In order to avoid arbitrage in the tradeoff term and considering the risky nature of spot price, we choose $\kappa$ and $m_t$ such that average spot price over period 3 and 4 (5 and 6) is slightly higher than $f_2^1$ ($f_4^2$). Parameters of spot price are shown in Table 3.2.

Table 3.2: Parameters of spot price

| t | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $m_t$ | $f_1^1+0.2$ | $f_2^1-0.3$ | $f_3^2-0.1$ | $f_4^2+0.1$ | $f_5^2+0.2$ | $f_6^2$ |
| $\kappa$ | $-0.5$ | $-0.4$ | $-0.6$ | $-0.5$ | $-0.5$ | $-0.5$ |
| $a_0$ | 0.005 | 0.001 | 0.005 | 0.003 | 0.002 | 0.005 |
| $b_0$ | 0.005 | 0.001 | 0.001 | 0.002 | 0.004 | 0.005 |
| $c_0$ | 0.02 | 0.02 | 0.15 | 0.05 | 0.05 | 0.02 |

*Notes.* The initial spot price is set to 6.4

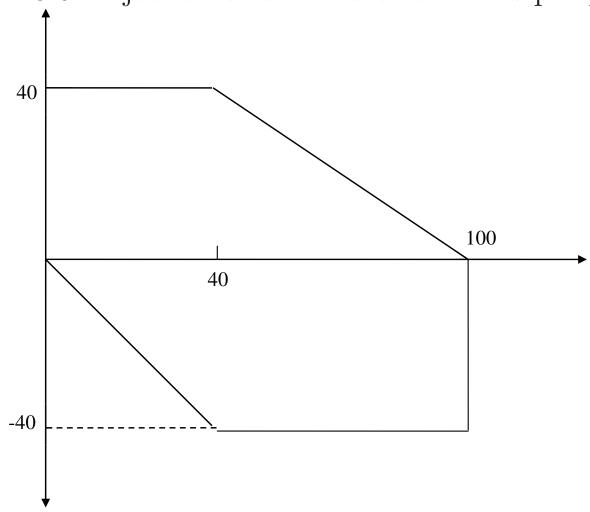
The firm initially has zero inventory and wealth. The total storage capacity is assumed to be 100 contracts. We assume that injection/withdrawal limits are the following functions of inventory,

$$\overline{\lambda}(I_t) = \begin{cases} 40 & \text{if } I_t \leq 40 \\ \frac{2}{3}(100 - I_t) & \text{if } I_t > 40 \end{cases} \tag{3.20}$$

$$\underline{\lambda}(I_t) = \begin{cases} -I_t & \text{if } I_t \leq 40 \\ -40 & \text{if } I_t > 40 \end{cases} \tag{3.21}$$

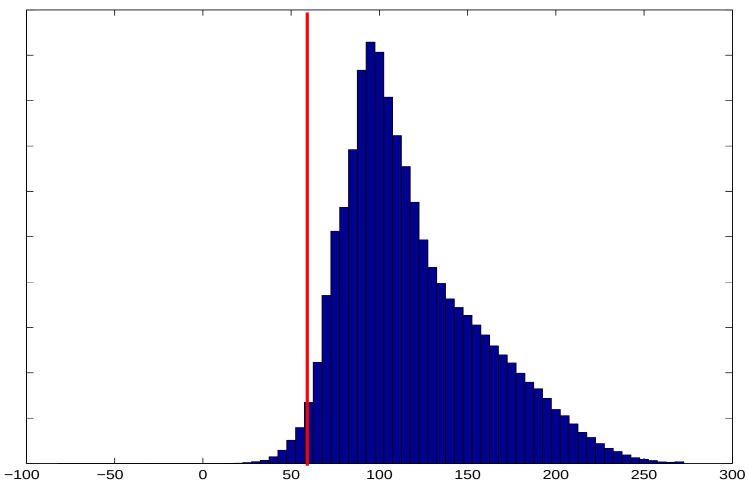
The curve shown in Figure 3.6 has a similar shape as the empirical plots shown in Figure 3.4, but the delivery limits magnitude relative to total capacity has been significantly scaled up. The reason is that we are using this six-period model to simulate for the entire fiscal year, and accordingly each period represents not a single day but a longer period of time covering tens of days. As a result, period-wise constraint should be scaled to mimic the total injection/withdrawal capacity over months.

Figure 3.6: Injection and withdrawal limits per period



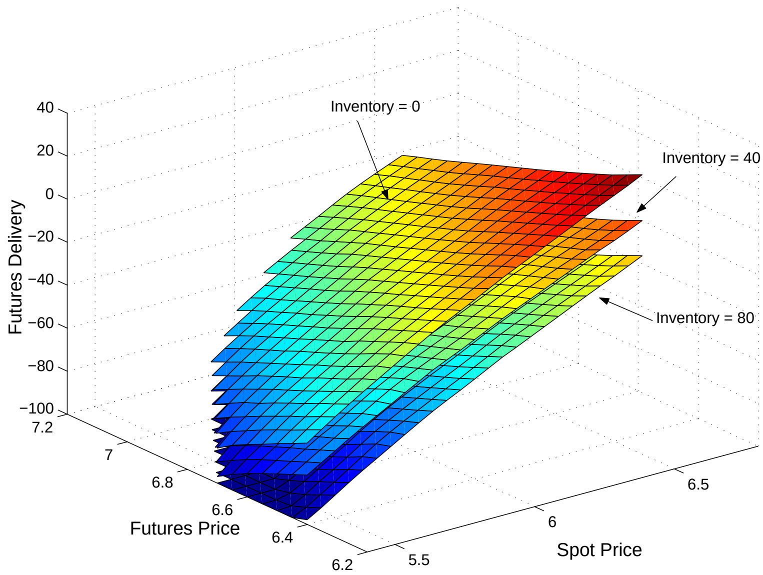

This six-period model is simple, yet captures all the essential features of concern: (a) the summer-winter price spread and mean-reverting spot price are essential features of the natural gas prices; (b) the delivery of the futures contract does not happen immediately, but evenly spreads over the next a few periods; (c) injection and withdrawal are subject to certain physical constraints.

Figure 3.7: Histogram of the winter-end wealth under the optimal policy




We first evaluate the performance of optimal policy. Figure 3.7 shows the histogram of winter-end profit under optimal policy. The vertical line in the figure corresponds to the

Figure 3.8: Futures delivery per period in period 5 and 6



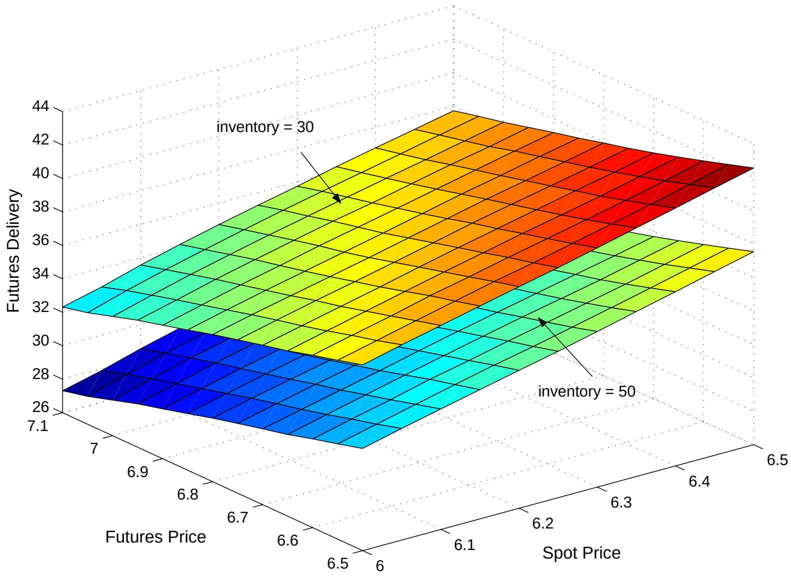

"day-1 intrinsic value" that firm can lock on futures market at the beginning of horizon. This risk-free profit equals the futures price spread (0.6) in the first period multiplied by total capacity (100).

Figure 3.7 shows that the risk-free profit is at about the 3rd percentile of the histogram. The upside gain under optimal policy can be as high as 450% of the risk-free profit while the maximum downside loss is about 60%. Hence the optimal policy offers significant benefits.

Figure 3.8 depicts delivery volume of the second future per period(i.e., $x_4^2/2$ in Figure 3.5) as a function of spot price and futures price for different starting inventory levels in period 4. We can see that the surface is negative, which means that the firm sells natural gas through the second future during the last two periods. When inventory increases, futures position becomes more negative, which implies that the firm sells more natural gas on futures market. Also when spot price increases, absolute value of futures delivery decreases, and so does the withdrawal quantity on futures market.

Figure 3.9 depicts delivery volume of the first future per period(i.e., $x_2^1/2$ in Figure 3.5) for different prices when the starting inventory in period 2 is 30 or 50. Different from the results of the second future, futures delivery is positive now, which means that the firm buys natural gas through the first future. The shape of the surface indicates that the firm buys

Figure 3.9: Futures delivery per period in period 3 and 4




more on futures market when spot price is high and futures price is low.

All these results are rather intuitive. Because second future price is generally higher than the first future price, the firm would like to buy inventory assets using the first future and sell these assets through the second future later. During the withdrawal season, the more starting inventory the firm has, the faster she withdraws natural gas. Furthermore, a higher spot price induces the firm to reduce futures withdrawal and sell more inventory on spot market. And if futures price increases, selling through futures market becomes more preferable. A similar explanation extends to the first future in Figure 3.9.

Figure 3.10 shows the histogram under the heuristic policy. The first period intrinsic value is at the 5th percentile. We can see that the profit distribution under heuristic policy is very close to the optimal. So heuristic policy is near-optimal for our model. Table 3.3 shows the percentage of the heuristic histogram relative to the optimal histogram in mean, 5th, 25th, 50th, 75th, 95th percentiles. It shows that difference between two histograms decreases from low percentile to high percentile. At the 95th percentile the difference is very small. Thus the heuristic policy can capture most of the upside benefits. On the other hand, the greater difference at low percentile suggests a higher downside risk under heuristic policy.

Figure 3.10: Histogram of the winter-end wealth under heuristic policy

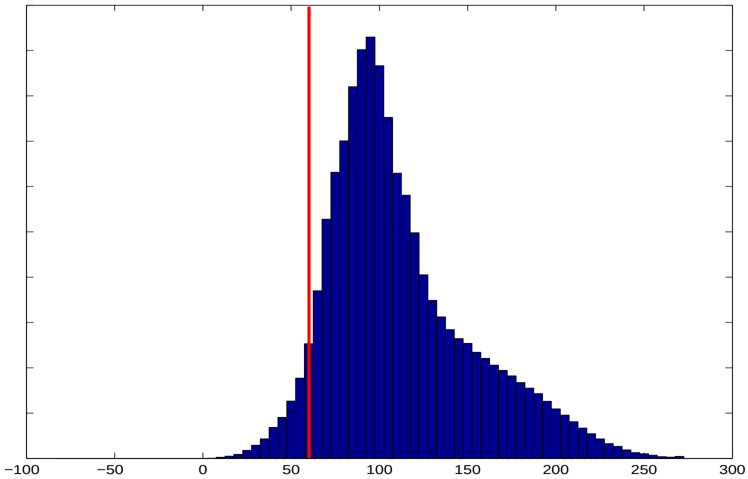



Table 3.3: Performance of the heuristic policy

| mean | 5th | 25th | 50th | 75th | 95th |
|---|---|---|---|---|---|
| 93.4% | 85.8% | 92.3% | 93.2% | 93.5% | 99.3% |

## 3.5. Conclusions

This paper studies an inventory control and risk management problem that is of immediate concern for firms managing natural gas storage assets. Our model takes into account the delivery mechanism of natural gas futures and the physical constraints in the operation of storage facilities. We formulate the problem as a stochastic dynamic programming problem, and characterize the optimal solutions. We numerically determine the optimal policy for a six-period model. The results demonstrate that optimal policy offers substantial benefits, and futures delivery at maturities has some certain monotone properties. However, the curse of dimensionality and the coupling effect between trading in futures market and spot market make it extremely hard to solve for the optimal policy in practice.

To partially overcome the "curse of dimensionality", we develop a more time-efficient heuristic policy. For the same six-period model, we show that the heuristic policy yields a profit distribution very close to that of an optimal policy. Furthermore, under this heuristic policy, we identify a crucial tradeoff the firm faces when deciding futures delivery at maturities. According to the tradeoff, the firm exchanges a deterministic cash flow evaluated at known futures price for a stochastic cash flow evaluated at unknown average spot price. This tradeoff also implies that under a financial market equilibrium, natural gas futures price might converge to its expected average spot price. This is different from the well-accepted belief in finance that futures price converges to spot price at maturity, and suggests directions for future research in the interface between operations and finance.

Our model can be easily extended to include injection/withdrawal losses. Another immediate focus is to solve for the problem under a more practical setting, with a much longer horizon and more futures.

# CHAPTER 4

# Capacity Investment, Production Scheduling and Financing Choice for Nonrenewable Resource Projects

## 4.1. Introduction and Literature Review

Many nonrenewable projects require significant investment in their early setup stage, especially in energy and mining sections. The acquisition of mineral rights, exploration and construction of infrastructure constitute the bulk of setup investment before any revenue can be realized. The size of the investment made during the setup stage determines the total amount of resource that the firm has access to during the following production stage. Usually the firm does not have enough capital to pursue the project on its own and need to seek external capital to finance the investment. After the project is set up, the project enters the production stage that is much longer than the setup stage. During production stage, the firm extracts the resource and sells it at fluctuating market prices. Part of the revenue is paid back to external investors according to predetermined arrangements in financing contract and the firm earns the rest.

Equity financing (e.g., joint venture) and debt financing (e.g., loans) are the two primary forms of financing used in the industry. The essential distinction between equity financing and debt financing lies in the way how the external investors should be paid back. In equity financing, the firm, as the original owner of the project, can obtain necessary capital from outside investor through equity financing partnership, such as joint venture. The external equity investor and the firm share the cost and profit of the project. In doing so, the external investor obtains rights to a fraction of the sale revenue by providing part of the initial investment, and receives a pre-specified fraction of the revenue as a return. For instance, in a 60-40 joint venture, the firm pays 60% of the setup cost with its own capital and its joint venture partner pays the remaining 40%. Reflecting on each party's share of the cost, the firm and its joint venture partner split the revenue in production periods according to some agreed upon ratio, say 60% and 40% in a proportional split, or 70% and 30% in an unproportional split. As a special case of equity financing, whenever the firm has enough capital to finance the project on its own, it will do so and act as the single stakeholder in the project. In terms of the effect of equity financing on operation, no matter how much stake the firm has in the project, the operations of the project are free from any financial obligations. Therefore, equity financing allows the firm to choose the operation policy to maximize its revenue. However, the absolute amount paid to external investor as a fixed share of the revenue can be huge when revenue soars.

Debt financing also plays an important role in energy/mining industry. Compared to equity financing, debt financing imposes direct restrictions on the firm's operation strategy because of the debt repayment obligation. For example, oil and natural gas producers heavily depend on external loans to finance the project and such loans can have maturities ranging from shorter terms of less than 12 months to longer terms of more than 5 years. The producers repay their debts through the sales of crude oil and natural gas.[1] In a booming market, the producer can easily pay off the debt and retain all the remaining revenue. If the market is bad, the firm may need to adjust up its production in order to generate enough revenue to meet the debt repayment obligation. If the firm fails to pay off the debt at maturity, it declares bankruptcy and loses all the accumulative revenue. While debt financing may result in bankruptcy, in a booming market, the revenue the firm has to pay back to loan creditor is capped by a fixed amount, compared to the uncapped amount in equity financing. In this case, the firm can earn a higher profit under debt financing than equity financing.

Take the shale natural gas industry as an example. A typical shale gas play consists of the following steps: land acquisition, drilling, hydraulic fracturing, completion and production.

[1]Producers can hedge a certain fraction of their production. But oil and natural gas producers usually hedge production only for the next year or so. Given that the project's life can be longer than twenty years, we do not consider hedging in our model.

In the land acquisition stage, the producer leases land from land owners at a cost that can be as high as $15000 per acre, and it needs 40-80 acres to drill a well. Exploration, drilling, hydraulic fracturing and completion can cost the producer $3-7 million per well dependent on the depth of the well and the exact technology used, and can be done in 4-6 months. Production of the gas can last for at least 10-20 years, and it takes 3-4 years to recover the cost assuming a $4 per mmBtu natural gas price. Shale gas play demands huge upfront investment in the phases of land grabbing, exploration, drilling and hydraulic fracturing. Shale gas producers depend heavily on external sources for financing, such as joint venture, debt, intercompany advances, etc. Producers embarked on a shale gas land rush in the past several years. An interesting ensuing phenomenon in the industry is that even though the natural gas price collapsed in 2008 and never returned to the pre-peak $6-8 level, instead of cutting back their production and waiting for the price to recover, producers are actually extracting more and more natural gas. Among the reasons for producing more in a dire market is that some producers is forced to produce under financial pressure to pay off their debts. [2] What happened in the shale gas industry presents provoking research questions to us. How is the firm's operation affected by different financing schemes? What are the factors driving the firm's financing choice? Under what conditions will the firm choose equity financing over debt financing and vice versa?

The broad problem of supply chain financing has received substantial attention recently. We refer readers to *Xu and Birge* (2004), *Buzacott and Zhang* (2004), *Lai et al.* (2009), *Kouvelis and Zhao* (2009) for the most recent progress in this area. *Xu and Birge* (2004) studies joint production and financing decision making in the presence of market uncertainty and imperfection, and demonstrates the significant value of integrating production and financing decisions. *Buzacott and Zhang* (2004) studies the interactions between a firm's financing and operation decisions in a multi-period model where the firm's borrowing capacity is determined by its assets. *Kouvelis and Zhao* (2009) compares bank financing against trade credit, and concludes that trade credit is superior to debt financing. Most paper in the operations management area discuss the financing problem using the newsvendor framework, which is

[2]A Business Week article states that: Shale-gas producers may very well be forced to produce no matter what the price of natural gas because so much money has already been staked on these various projects. - Gas Output in Lower 48 Declined in October, EIA Says, *Business Week*, Dec 29, 2010.

not suitable for the shale natural gas industry. Also, none of the papers discuss about joint venture.

The interaction of firm's capital structure and operational policy is an active research area in corporate finance literatures. *Leland and Toft* (1996) examines the optimal amount and maturity of the debt that the firm should borrow to maximize the firm value in a continuous-time model. *Hennessy and Whited* (2005) and *Gamba and Triantis* (2008) study the optimal investment and financial policies of a firm in a discrete-time infinite horizon model. In our basic model we use debt of arbitrary maturity rather than the single-period debt in *Hennessy and Whited* (2005) or the perpetual debt in *Gamba and Triantis* (2008). Moreover, in our model inventory is exhaustible and the production is capacity constrained.

We consider a stylized model in which a firm needs to finance the project to initiate. Once the project kicks off, the firm accrues the revenue that is a function of price (which is exogenous) and production (endogenous). In deciding the optimal level of output in each period, the firm trades off the value of resource sold in current market and the expected value of resource to be extracted in the future. If the project is financed by risky debt, then the obligation to pay back the debt also plays a role in determining the optimal output. The setting of a financially-constrained firm operating in multiple periods enables us to study the inter-temporal production pattern and distinguishes our paper from most existing literature.

Classical finance papers treat financing and operation separately. In other words, the form of financing should not affect how these firms operate. But, our results show that operation policy depends on the firm's financing method. We show that with financial obligations, the firm's actions will deviate from the unconstrained case. For instance, the firm that finances the project with equity always follows the optimal policy that a centralized firm will follow, but the firm under debt financing may modify production quantity with the obligation to pay back the debt in mind. With bankruptcy risk, the firm chooses its output level to balance its exposure to bankruptcy risk and its potential to take profits in the future. There does not exist a monotonic relationship between the firm's output and its inventory/wealth. We find that in contrast to the results from prior asset selling models, the optimal output level may decrease for a higher inventory position if the firm has debt obligation. Besides, when the firm's debt position decreases, instead of reducing production as it is under less pressure to pay back the debt, sometimes the firm should increase its output.

We are interested in determining the optimal financing method for a project of given size. We find that under secured debt financing with no default risk, there exists a threshold project size below (above) which debt financing (equity financing) should be used. If borrowing debt can cause bankruptcy, there may not exist a single threshold that defines two regions, but rather two or even more critical project sizes to define more than two regions. The firm may alternate between equity financing and debt financing when project size goes from one region to another.

Within the context of this multi-period model, we also investigate the effect of debt term structure and price dynamics on the firm's financing policy. When debt interest rate is fixed, we show the maturity leading to the highest project value may be at intermediate maturity levels. Therefore, equity financing can beat debt financing when the debt maturity is in the short-term end or long-term end. We show the effect of changing the debt maturity is most significant when the price is low. We find that as the drift and volatility of price increase, the firm is more prone to using debt financing. The assumption of finite maturity distinguishes our study from the majority of papers, which assume either perpetual debt (e.g., *Hennessy and Whited* (2005)) or single-period debt (e.g., *Buzacott and Zhang* (2004) and *Boyabath and Toktay* (2011)). We also find that if the price follows a geometric Brownian motion of a mean-reversion process as in *Schwartz* (1997), higher volatility leads to higher project value and the firm is more prone to use debt financing.

The rest of the paper is organized as follows. Section 4.2 describes the multi-period model used in the paper. We analyze the optimal production policy during the operating periods in Section 4.3. Section 4.4 discusses the relationship between the term structure of debt and the firm's operations and investment policy. Section 4.5 examines the impact of price dynamics on the firm's financing choice. Throughout the paper, we consolidate and complement analytical results with numerical experiments.

## 4.2. Model Description

We consider a discrete-time finite-horizon model, with periods labeled as $t$ =0, 1, ..., $T$. At the beginning of the horizon ($t$ = 0), the firm owns the right to invest in a exhaustible resource project. The project size is measured by its capacity $K$, which is the total amount of non-renewable resource in the entire field that the project can produce. Consequently, the term "capacity" is used to describe the "project size", and the term "inventory" is used to denote the amount of resource left in the reserve. Investment in the project is irreversible. And there is no option to expand or shrink the project size later.

The investment is made at the beginning of period 0. Pre-production preparations, such as installment of equipments and construction of infrastructures, will take place in period 0. Period 0 can be much longer than each operating period, reflecting the time needed to set up the project. The firm starts extracting and selling the resource at the stochastic market price from time 1 to $T$. The project has a finite life-time of $T$ periods. The salvage value of inventory at the end of the last period is zero. No matter what financing vehicle is used, all parties in our model have the same information. The firm's objective is to maximize its own discounted expected profit, by choosing the optimal project size and the appropriate financing method in period 0, and optimally producing from 1 to $T$. The profit from the project consists of the revenue from the project less the capacity investment cost, loan paid back to creditors and cash flow paid to outside equityholders depending on the financing method used.

The firm acts as a price-taker. It extracts the resource and sells the output in the stochastic spot market.[3] The spot price of the resource, denoted as $p_t$, follows a stochastic Markovian process. At the beginning of period $t$, after observing the current inventory in reserve $x_t$ and the spot price $p_t$, the firm decides the production quantity $q_t$. The per-period operational profit (revenue flow minus variable cost) of the firm is denoted as $r(q_t, p_t)$. We assume $r(q_t, p_t)$ is continuous, strictly increasing, and differentiable almost everywhere in $q_t$ and $p_t$. Also, we assume $r(q_t, p_t)$ is concave in $q_t$, and $r(0, p_t) = 0$. We comment that some widely-used profit functions in existing literature can be regarded as special cases of $r(q_t, p_t)$. We refer the reader to the Appendix for detailed discussion of possible forms and relevance of the various profit functions.

Let $w_t$ denote the firm's wealth (total cash on hand) at the beginning of period $t$. At the

[3]In reality, natural resource companies may dynamically hedge their production by using various financial derivatives over time. However, they only hedge for a period of time that is short relative to the horizon of the project. For instance, in 2010 Chesapeake hedged about 50% of its year 2011 production according to its SEC filing. Therefore, we do not consider hedging of price in our model.

beginning of period 0, the firm is endowed with initial capital $w_0$ and needs to decide project size $K$. The required capital cost for this size is $cK$, where $c$ is the unit cost of capacity. If the firm does not have sufficient initial wealth to cover the project cost, i.e., $w_0 < cK$, then the firm needs to seek external sources for financing. It finances through either equity-type (joint venture) or liability-type (debt-financing) method. We assume the firm has no further financing opportunities from time 1 to end of project. [4] In our basic model, we consider a frictionless capital market where taxes, transaction costs and bankruptcy costs are zero. The firm can invest its revenue in a risk-free security.

According to whether and how the project is financed, we describe the problems for equity-financed firm and debt-financed firm as follows. Notations used in the paper are summarized in Table 4.1.

Table 4.1: Notations

| | | |
|---|---|---|
| $K$ | : | Project size, |
| $\widehat{K}$ | : | Optimal project size when the firm has enough capital to finance on its own, |
| $c$ | : | Capacity investment cost per unit, |
| $p_t$ | : | The price of the resource in period $t$, |
| $r_f$ | : | Risk-free interest rate, |
| $x_t$ | : | The inventory at the beginning of period $t$, |
| $q_t^d$ | : | The production in period $t$ under debt financing, |
| $q_t^e$ | : | The optimal production when the project is financed by equity, |
| $\alpha$ | : | The fraction of revenue reserved for the original firm in equity financing, |
| $R$ | : | The interest rate charged by the creditor on debt, |
| $m$ | : | The maturity of debt, |
| $D$ | : | The amount of cash that has to be paid to creditor at maturity, |
| $w_t$ | : | The cash holding of the firm at the beginning of period $t$. |

*A. The Equity-financed Firm*

If the project is financed by raising equity, the firm shares the expenses and revenues of the project with a counter-party equity-holder. If the firm has enough cash holdings to kick off the project, it will finance the project on its own and consequently become the sole claimant to the revenue. We also assume if the firm does not have enough capital and decides

[4] In practice, the firm may repay its existing short-term debt by borrowing or debt rollover. The ability of the firm to so depends the credit market situation and its own financial strength. Rollover of debt can be risky because the creditor may refuse to roll over the debt contracts.

to finance through equity financing, it will dump all its capital in the project. We assume the fraction of revenue, $\beta$, received by the firm in the equity financing partnership has an affine structure,

$$\beta(w_0, K) = \alpha + (1-\alpha)\frac{\min(w_0, cK)}{cK}, \tag{4.1}$$

where $\alpha$ is the fraction reserved for the firm and the remaining fraction of $1-\alpha$ is split proportionally. The term $\frac{\min(w_0,cK)}{cK}$ is the share of capacity investment cost contributed by the firm. The share $\beta(w_0, K)$, once determined in period 0, will remain fixed throughout the ensuing operating periods.

If the firm has a sufficiently deep pocket $w_0 \geq cK$, it finances and operates the project on its own. In this case, the firm has sole ownership of the project. The capacity investment and production decisions are unconstrained. The firm maximizes the project's discounted expected market value, $V_0^e(K, p_0)$, determined as follows:

$$V_t^e(x_t, p_t) = \max_{0\leq q_t \leq x_t} \left\{ r(q_t, p_t) + \frac{1}{1+r_f}\mathsf{E}_t V_{t+1}^e(x_t - q_t, p_{t+1}), \right\} \quad t = 1, ..., T \tag{4.2}$$

$$V_{T+1}^e(.,.) = 0. \tag{4.3}$$

where $r_f$ is the risk-free interest rate per operating period, $\mathsf{E}_t$ is the conditional expectation under risk neutral measure $\mathbb{Q}$ at period $t$. For the purpose of simplicity, we suppress the notation of $\mathbb{Q}$ throughout the paper. The terminal condition means that the residual value of left-over inventory is zero. We let $q_t^e$ denote the optimal production in period $t$, which depends on inventory and price. At time 0, the project's expected revenue is $V_0^e(K, p_0) = \frac{1}{1+r_f}\mathsf{E}_0 V_1^e(K, p_1)$. Here we have normalized the discount factor in period 0 to $r_f$. At the beginning of period 0, the firm decides the optimal project size $\widehat{K}$ by solving the following optimization,

$$\widehat{K} = \arg\max_{K\geq 0} \left\{ V_0^e(K, p_0) - cK \right\}. \tag{4.4}$$

If the firm needs to finance the project with the help of external equity, we assume the firm initiating the project has control right and is responsible for the operations. [5] Unlike

[5]For instance, in a recent joint venture, "Devon Energy Corp. agreed to sell 30% of its interest in about 650,000 net acres in the oil-rich Cline and Midland-Wolfcamp shales in West Texas to Japan's Sumitomo

debt financing, equity financing partnership does not involve bankruptcy risk and operations are not financially constrained. Since the firm earns a fixed share of the revenue, maximizing the firm's revenue is equivalent to maximizing the entire project's revenue. Therefore, the firm should follow identical production policy as the unconstrained case given in (4.2), and the revenue of the entire project will be $V_0^e(K, p_0)$.

In equity financing, if the project is too big, the firm has to get more funding from its equity financing partner and consequently forfeits a larger fraction of revenue. If the project is too small, it may not generate enough revenue. At time 0, the firm decides the optimal project size by solving the following problem,

$$\max_{K \geq 0} \left\{ \beta(w_0, K) V_0^e(K, p_0) - cK \right\}. \tag{4.5}$$

*B. The Debt Financed Model*

If the firm starts a project of size $K$ with $cK > w_0$, the firm borrows $cK - w_0$, the amount exactly needed to cover capacity investment. The debt needs to be paid off at the end of period $m$, $1 \leq m \leq T$ ($m$ stands for maturity). Let $R$ be the per-period interest rate paid to the creditor. Let $D$ denote the total amount that the firm needs to pay the creditor at the end of period $m$. $R$ and $D$ satisfy the condition,

$$(1 + R)^m (cK - w_0) = D. \tag{4.6}$$

For simplicity, we assume the debt does not demand any intermediate coupon payment. There is no transaction cost of raising/liquidating debt. Depending on how the project is financially constrained, the operation of the project can be divided into three stages, after, before or at debt maturity.

After maturity $m$, if the firm survives and has already paid off all debt at $m$, it becomes the sole owner of the project and earns the entire revenue. Its operation policy is identical to the unconstrained policy under equity financing and the project value is given by (4.2).

We now focus on the decision that the firm makes in period $m$. The firm needs to pay back $D$ at the beginning of period $m$ from sales of inventory and cash holdings. In

Corp. ...... Devon will serve as the operator of the project and be responsible for marketing." - *Wall Street Journal*, August 1st 2012.

the worst case, the firm is unable to pay $D$ even if it extracts and sells all inventory, i.e., $r(x_m, p_m) + w_m < D$. Then, the firm goes bankrupt and the value of the project diminishes to zero.[6]

If the firm can manage to pay off all debt $r(x_m, p_m) + w_m \geq D$, the project value is determined by the following optimization problem,

$$V_m^d(x_m, w_m, p_m) = \max r(q_m, p_m) + w_m - D + \frac{1}{1+r_f}\mathsf{E}_m V_{m+1}^e(x_m - q_m, p_{m+1}) \tag{4.7}$$

$$s.t. \qquad 0 \leq q_m \leq x_m \tag{4.8}$$

$$r(q_m, p_m) + w_m \geq D \tag{4.9}$$

From period 1 to $m-1$, the firm extracts the inventory, sells it in spot market and invests the proceedings in a risk-free security. The firm decides production quantity conditional on current inventory, capital and price,

$$V_t^d(x_t, w_t, p_t) = \max_{0 \leq q_t \leq x_t} \frac{1}{1+r_f}\mathsf{E}_t V_{t+1}^d(x_t - q_t, (1+r_f)(w_t + r(q_t, p_t)), p_{t+1}), \quad 1 \leq t < m. \tag{4.10}$$

At time 0, the firm borrows loan of the amount $(cK - w_0)^+$ with maturity $m$. The time 0 value function can be written as

$$V_0^d(K, w_0, p_0) = \frac{1}{1+r_f}\mathsf{E}_0 V_1^d(K, (1+r_f)(w_0 - cK)^+, p_1). \tag{4.11}$$

The firm chooses the optimal project size

$$\max_{K \geq 0} V_0^d(K, w_0, p_0) - \min(w_0, cK). \tag{4.12}$$

Next we turn to describe how to determine $R$, the interest on debt, and $D$, the amount paid back to creditor. We consider two situations, i.e., interest $R$ is either exogenously given or endogenously determined. In the case of exogenously given interest, $R$ is a fixed constant regardless of the project size and maturity.

[6] We define bankruptcy in the sense of *Wruck* (1990), i.e., "as a situation where cash flow is insufficient to cover current obligations." This flow-type definition is distinct from the stock-type insolvency, where "the present value of its cash flows is less than its total obligations." Since the project is financed with zero coupon debt, bankruptcy is triggered only at maturity if the firm does not earn enough revenue from the project to pay debt.

In the case of endogenously determined interest, we impose the assumption that the credit market is competitive, debt is fairly priced and the creditor earns an expected return equal to risk-free rate (see, e.g., *Dotan and Ravid* (1985), *Xu and Birge* (2006) and *Boyabath and Toktay* (2011)). We turn to the cash flow received by the creditor at maturity $m$. The creditor receives $D$ if the firm has enough cash and sales revenue to pay $D$. Otherwise the firm defaults and the creditor grabs all the accumulative revenue. Therefore, the cash flow to the creditor is

$$Y_d(x_m, w_m, D) = \min\{r(x_m, p_m) + w_m, D\}. \tag{4.13}$$

In a perfectly competitive market, debt is fairly priced in that the expected return earned by the creditor under rate $R$ should be equal to the risk-free rate $r_f$. Therefore, we have

$$\mathsf{E}_0 Y_d(x_m, w_m, D) = (1 + r_f)^m (cK - w_0). \tag{4.14}$$

The condition (4.14) guarantees that the borrower can not transfer wealth from the creditor. From (4.13), (4.6) and (4.14), the interest of the fairly priced debt can be endogenously determined. Due to the possibility of bankruptcy, $R$ should be no less than risk-free rate $r_f$. If the firm can always manage to avoid bankruptcy, $R$ would be equal to $r_f$. If the project size is too big, fairly priced debt may not exist. From a creditor's perspective, the maximum profit it can earn from the project is $V_0^e(K, p_0)$ through an initial investment of $cK - w_0$.[7] If $cK - w_0 > V_0^e(K, p_0)$, there is no way debt could be fairly priced. The firm's borrowing capacity is constrained by the maximum value that can be generated by the project. Throughout the paper, we focus our attention on the case where the firm stays within its borrowing capacity.

We comment that in the single-period financing models (e.g.,*Dotan and Ravid* (1985), *Xu and Birge* (2004), *Kouvelis and Zhao* (2009) and *Boyabath and Toktay* (2011)), production decision is independent of debt obligation. Because of independence, the fair interest rate can be analytically derived given the distribution of post-production revenue. In our model, production decisions before maturity depend on outstanding debt. Therefore, we generally do not have an analytical representation of $R$ except for some special cases.

[7]Here we have assumed that the creditor also uses risk-free rate to discount the cash-flow.

## 4.3. The Operational and Investment Policy

In this section we examine the optimal operational policy for the project under different financing methods. We illustrate how the risk of bankruptcy may alter the production policy and its dependence on inventory/wealth. We describe the condition under which the equity financing results in higher return than debt financing. In this section, the project size is exogenously specified and does not come into the firm's decision.

The operations under equity financing are not financially constrained, no matter how much stake the firm invests in the project. Because of bankruptcy, the value function under debt financing is generally not concave. However, if we restrict our attention to the region where the firm can always manage to avoid bankruptcy, the value function is still concave.

**Proposition 11.** *(i) For an unconstrained firm, the value function $V_t^e(x_t, p_t)$ $(t = 1, ..., T)$ is increasing and concave in inventory level $x_t$. The optimal production $q_t^e$ is increasing in $x_t$.*
*(ii) At debt maturity the value function $V_m$, as described by (4.7), is jointly concave in $(x_m, w_m, D)$ in the region $r(x_m, p_m) + w_m \geq D$.*
*(iii) Before maturity, in the set of states $(x_t, w_t, p_t)$ $(1 \leq t < m)$ where there is no default risk, (i.e., the firm can always operate the project to avoid bankruptcy no matter what price scenario happens),the project value is jointly concave in $(x_t, w_t)$.*

Due to the concavity of value function for unconstrained firm, the optimal production $q_t^e$ is characterized by the first-order condition,

$$\frac{\partial r}{\partial q} - \frac{1}{1 + r_f} \mathsf{E}_t \frac{\partial V_{t+1}^e}{\partial x} = 0. \tag{4.15}$$

The above equation means that at optimality the marginal value of current extraction should equal the expected marginal value of inventory in the future.

Now we examine the behavior of the optimal production policy under debt financing. We assume the project size and debt term structure are such that the firm can always manage to avoid bankruptcy. Therefore the value function $V_t^d$ is concave and the first-order condition is valid. If the firm increases production, in that period it will earn a higher cash flow. But at the same time, the firm will have less inventory left. The optimal production $q_t^*$ is characterized by the first-order condition,

$$\mathsf{E}_t\left\{-\frac{\partial V^d_{t+1}}{\partial x}+(1+r_f)\frac{\partial r}{\partial q}\frac{\partial V^d_{t+1}}{\partial w}\right\}=0. \tag{4.16}$$

The above equation states that the firm should choose a production quantity such that the expected marginal value of inventory is equal to the expected marginal value of cash holding.

How the inventory level $x_t$ affects production $q^*_t$ is characterized by the second-order cross derivative,

$$\mathsf{E}_t\left\{-\frac{\partial^2 V^d_{t+1}}{\partial x^2}+(1+r_f)\frac{\partial r}{\partial q}\frac{\partial^2 V^d_{t+1}}{\partial w \partial x}\right\}. \tag{4.17}$$

The sign of (4.17) is undetermined. Therefore, it is hard to determine if the firm should produce more with higher inventory/wealth. If the cross-derivative is negative, then the firm should reduce production for higher inventory level.

**Lemma 5.** *The value function under debt financing satisfies* $V^d_t(x_t,w_t,p_t,D)=V^d_t(x_t,0,p_t,D-(1+r_f)^{m-t}w_t)$ *for* $t=1,..,m$.

The above lemma shows that we can use a single variable to record the net cash position (debt-wealth) of the firm.

Based on the constraint imposed on operations, Figure 4.1 divides states at maturity into three regions. For a given price $p_m$, the value function is

$$V^d_m(x_m,0,p_m,D) \;=\; \begin{cases} 0, & \text{if } r(x_m,p_m)<D,\\ \frac{1}{1+r_f}\mathsf{E}_m V^e_{m+1}(x_m-r^{-1}(D,p_m),p_{m+1}), & \text{if } r(x_m,p_m)\ge D \text{ and } r(q^e_m,p_m)<D,\\ V^e_m(x_m,p_m)-D, & \text{if } r(q^e_m,p_m)\ge D, \end{cases} \tag{4.18}$$

where $r^{-1}$ is the inverse function of $r$ and $q^e_m$ is the optimal production of the unconstrained case.

In deciding the output level, the firm always tries to balance its exposure to market risk and its potential to profit from a volatile market. Increasing production to attain a higher cash position provides better protection for the firm against financial distress. Decreasing production to reserve more inventory implies the project has a higher value in the future. If

Figure 4.1: The constrained operations at maturity

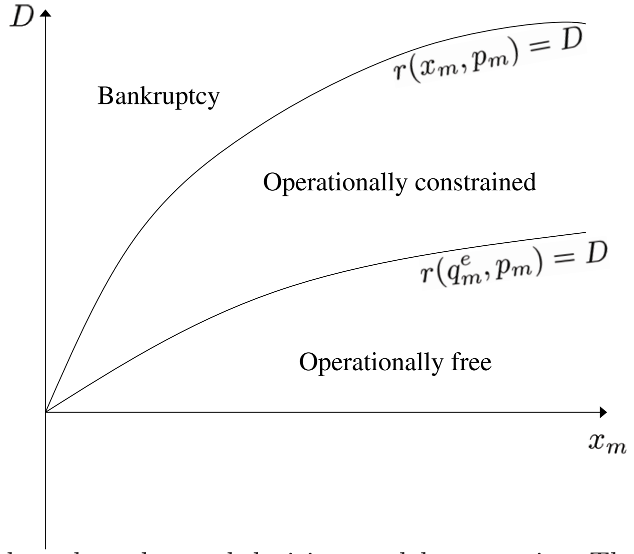


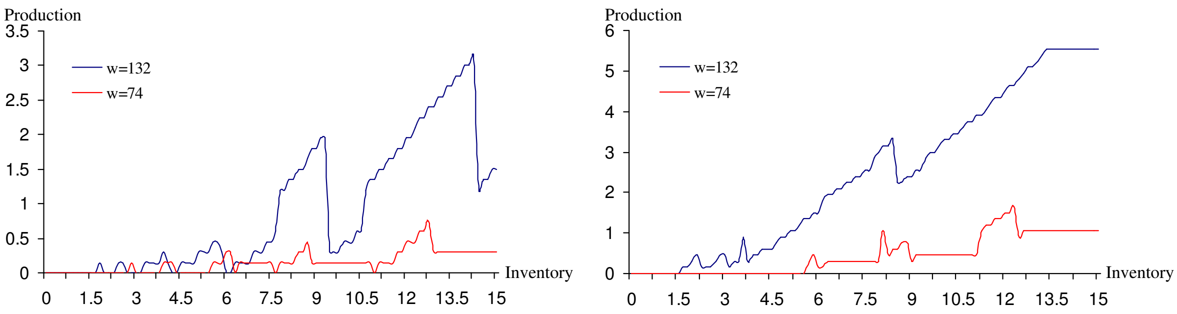

Figure 4.1 plots the value and decision at debt maturity. There are three distinctive regions. If the debt is high enough or inventory is low enough, the project goes bankrupt. The project is default-free but the production is operationally constrained if $r(x_m, p_m) \geq D$ and $r(q_m^e, p_m) < D$. Here $q_m^e$ is the optimal production under equity financing when inventory is $x_m$. When the firm is operationally constrained, it must produce the minimum amount to pay off the debt. If $r(q_m^e, p_m) < D$, the firm produces $q_m^e$ which is enough to pay the debt. Note that the boundaries between the regions will change if $p_m$ changes.

the firm has low inventory, it tends to take the safe way and produce more to have a higher cash position. This is because with more cash it can degrade the effect of negative market situation rather than be forced to produce more in financial distress. If the the firm has high inventory, it will be less concerned about market risk, because it can always have enough inventory left to capture profit opportunities in the future. Hence, the firm may cut back production when it has more inventory. It is worthwhile to note that if inventory continues to grow higher, eventually the firm should produce more.

In a similar way, we argue that the optimal production may increase if the cash position decreases or debt increases. If cash position is too low and debt is high enough, increasing the output in the current period may have little effect to relieve the firm from financial distress later. Instead, the firm, realizing there is a high chance of bankruptcy/firesale, would rather cut back production so it can take advantage of higher inventory to capture profit opportunities in the following periods. The bankruptcy loss resulted from decreasing production can be compensated by the gain from higher inventory.

We further examine the optimal production policy through numerical experiments. The numerical study helps us derive additional managerial insights and characterize the investment and production policies under different financing approaches. We use the one-factor model in *Schwartz* (1997) and the parameters estimated therein to model the commodity price. Details of the design and implementation of numerical experiments are described in Appendix B.

Figure 4.2: Optimal production as a function of inventory




(a) Sampled at $p_2$=5.606 (b) Sampled at $p_2$=16.446

Figure 4.2 plots the optimal production as a function of inventory at different prices and wealth levels. It is assumed that $T = 20$, debt maturity $m = 3$, project size $K = 15$, initial capital $w_0 = 100$, revenue function $r(q, p) = pq^{0.6}$, variable capacity cost $c = 18$. The interest rate on debt is exogenously given, $R = 3.5\%$. The two panels all illustrate the optimal production quantities one period before maturity at $t = 2$, and are sampled at two different prices respectively.

Figure 4.2 depicts how the optimal production under debt financing $q_t^d$ changes in $x_t$ and $w_t$ at different prices. When the wealth position is high enough, the debt obligation becomes less stringent and the firm simply follows the unconstrained optimal policy. When the wealth position is too low, the risk of bankruptcy becomes bigger and raising production can only marginally reduce bankruptcy risk. The non-monotonic relationship between production and inventory/wealth is most obvious when the market price is at low or intermediate levels.

### 4.3.1 Financing Policy

Whether equity financing performs better than debt financing and vice versa depends on various factors. In this section we analyze the effect of project size, cost/revenue split in equity financing and risk-free rate on financing choice. We postpone the analysis of the impact of debt term structure and price dynamics to the next two section. First we look at the performance of equity financing.

**Lemma 6.** *equity financing profit $\beta(w_0, K)V_0^e(K, p_0) - \min(w_0, cK)$ is an increasing function of reserved share $\alpha$. If the reserved share for the firm is $\alpha = 0$, the profit is a decreasing function of project size $K$.*

According to the definition of $\beta(w_0, K)$, the expected profit earned by the counter-party in equity financing is

$$(1-\alpha)\frac{cK - w_0}{cK}V_0^e(K, w_0, p_0). \tag{4.19}$$

If the counter-party firm makes an expected zero return from the equity financing, we must have that the above expression is equal to $cK - w_0$, which implies

$$(1-\alpha)V_0^e(K, w_0, p_0) = cK. \tag{4.20}$$

We define $K^o$ as the maximum project size that can satisfy condition (4.20). If $K^o \leq \frac{w_0}{c}$, the firm finances the project on its own. Only when the project is smaller than $K^o$, does the counter-party firm make a positive return through the equity financing.

Next, we compare the two financing methods with respect to the changes in the key parameters, including project size, interest rate, fixed equity financing share and debt maturity. We start with discussion of the simpler case where the firm has no default risk, then the analysis is extended to the case with default risk.

**Proposition 12.** *Assume no default risk, and project size is given at $K$.*
*(i) There exists a threshold project size $K^o$. If $K > \max(K^o, \frac{w_0}{c})$, equity financing is preferred over debt finance; If $\frac{w_0}{c} \leq K \leq \max(K^o, \frac{w_0}{c})$, debt financing is preferred over equity financing; If $K < \frac{w_0}{c}$, the firm finances the project on its own. The threshold $K^o$ is determined by*[8]

$$\frac{cK^o}{V_0^e(K^o, p_0)} = 1 - \alpha. \tag{4.21}$$

[8]The result holds because the cost $cK$ is a straight line and has two intersection points with a concave value function (One intersection is 0). If capacity investment cost is a convex function, the threshold result still holds. If the capacity cost function is concave, then there may be multiple intersection points.

*(ii) Furthermore, there exists a level $\alpha^o$, such that for a given project size, equity financing (debt financing) should be used if $\alpha$ is above (below) $\alpha^o$.*
*(iii) If debt is fairly priced, then there exists a date, with the maturity beyond (below) which the firm should choose debt financing (equity financing).*
*(iv) $K^o$ decreases as the risk-free rate $r_f$ increases. Therefore, the firm is more prone to adopting equity financing when interest rate is higher.*

The threshold capacity level in (4.21) is given as the point where the fraction of revenue proportionally shared in equity financing is equal to the total capacity investment cost. If $\alpha = 0$, at the threshold the expected revenue equals the capacity investment cost. However, this will never happen because It implies that if $\alpha = 0$, equity financing should not be used. As $\alpha$ increases, $\widehat{K}$ will increase.

Based on the proof, for part $(i)$ to hold it only requires that the value function is increasing and concave in capacity. Therefore part $(i)$ is a quite general result. The threshold is independent of the firm's capital $w_0$. For a fixed capacity, equity financing and debt financing will contribute the same amount of capital $cK - w_0$ to the project. The firm pays $cK - w_0$ back to creditor in debt financing. The threshold $K^o$ is the size of project at which the counter-party in equity financing makes zero expected profit. This is not surprising because under debt financing without default risk, the expected return for the creditor is zero. Therefore, equity financing can only beat debt financing when the counter-party's expected return is negative.

In a equity financing with $\alpha = 0$, project value is decreasing in $K$. If $\alpha = 1$, project value is increasing and always higher than debt financing. For intermediate $\alpha$, project value is between the two extremes. The value under debt financing with no default risk, as shown by the dotted line, attains its maximum at $\widehat{K}$, and intersects with equity financing at $K^o$. As $\alpha$ increases, equity financing value goes up and the intersection point $K^o$ becomes smaller. If $\alpha$ is big enough, equity financing value is consistently higher than debt financing and the intersection point does not exist.

Part $(iii)$ is straightforward because the value of debt-financed project is increasing in debt maturity. So far, change in $\alpha$ and $m$ only affects the performance of debt financing. But in part $(iv)$, change in risk-free rate will affect performance of both debt financing and equity financing. While a higher interest rate will result in lower value for both equity financing and debt financing, the drop in debt financing value is more than equity financing. Under debt financing, the expected present value of the cash-flow paid back to creditor remains the same no matter what the interest rate is. However, under equity financing, the firm paid less to the counter-party since the project value has decreased due to higher interest rate.

We now turn to the comparison of financing methods if the firm may go bankrupt in debt financing. It can be shown that the the value function under debt financing and the value function under equity financing can have more than one intersection point.

**Proposition 13.** *If debt is fairly priced and the firm faces bankruptcy risk, there exist double thresholds $K^o$ and $K^l$ ($K^o \geq K^l$), such that if $K > K^o$, equity financing should be used; if $K < K^l$, debt financing should be used. The upper threshold $K_h$ is the same as the capacity given in* (4.21). *(But what is the financing decision for K in the middle is not clear yet) $K^o$ and $K^l$ coincide if there is no default risk.*

## 4.4. Impact of Debt Term Structure

In this section we analyze how the debt term structure (e.g., maturity and interest rate) affect the firm's project value and capacity investment decision. We consider both exogenously given fixed rate and endogenously determined fair pricing rate. Throughout the paper we use $V_0^d(K, w_0, p_0 \mid m, R)$ to denote the value of the project if it is financed through debt of maturity $m$ and interest rate $R$.

**Proposition 14.** *(i) Consider two types of debts with different maturities $m_l < m_h$ and the corresponding interest rates $R_l$ and $R_h$. If $(1 + R_h)^{m_h} < (1 + R_l)^{m_l}(1 + r_f)^{m_h - m_l}$, then the long-term debt should be used,*

$$V_0^d(K, w_0, p_0 \mid m_l, R_l) \leq V_0^d(K, w_0, p_0 \mid m_h, R_h). \tag{4.22}$$

*(ii) If a project of given size with $cK > w_0$ is financed through fairly priced debt (e.g., the debt satisfies* (4.14)*), longer debt maturity will result in higher project value. Therefore, there exists a date, with the maturity beyond (below) which the firm should choose debt financing (equity financing).*

Proposition 14*(i)* specifies the the condition under which one term structure performs better than the other one. Note that the condition is a sufficient condition. Also note that we derive the above result without imposing any restrictions on the price process.

If interest rate is exogenously given, debt maturity can impact the project value in two ways. First, a longer maturity implies a higher degree of operational flexibility for the firm before the firm can produce for longer time before debt is due. Second, a longer maturity may change the amount of debt the firm has to repay. A longer maturity will results in a higher face value. If the interest rate is fixed, depending on the specific price process under consideration, the firm's ability to pay back the face value may increase or decrease. Therefore the relationship between maturity and project value is generally undermined unless additional information is known about the price process. However, if the expected return on debt is the risk-free rate, shorter maturity does lower the project value.

We qualitatively analyze how the maturity affects the value of the project. Let $V_t^d(x_t, w_t, p_t \mid m, R)$ denote the value function at $t$ when the debt maturity is $m \geq t$ and interest is $R$. For notational convenience, we denote the case $r(x_m, p_m) + w_m < D$, $r(q_m^e, p_m) + w_m \geq D$ as $\Omega_1$ and $\Omega_2$ respectively, and define $\Omega_3 = (\Omega_1 \cup \Omega_2)^c$. With outstanding debt of maturity $m$, the optimal policy is denoted as $\pi_m$. We denote the state attained in period $m$ under policy $\pi_m$ as $(x_m, w_m, p_m)$. In order to analyze the variation of value due to change of maturity, we construct a feasible policy with debt of maturity $m+1$. Specifically, we assume under debt of maturity $m+1$, the firm still operates according to $\pi_m$ until $m$, but operates optimally from $m$ to $T$. The difference between the value functions of different maturities can be decomposed as

$$\begin{aligned}
&V_0^d(K, 0, p_0 \mid m+1, R) - V_0^d(K, 0, p_0 \mid m, R) \\
&\geq \frac{1}{(1+r_f)^m} \mathsf{E}_0 \left( V_m^d(x_m, w_m, p_m \mid m+1, R) - V_m^d(x_m, w_m, p_m \mid m, R) \right) \\
&= \frac{1}{(1+r_f)^m} \Bigg[ \mathsf{E}_0 \Big( V_m^d(x_m, w_m, p_m \mid m+1, R) - 0 \Big| \Omega_1 \Big) \mathrm{Prob}(\Omega_1) \\
&+ \mathsf{E}_0 \Big( V_m^d(x_m, w_m, p_m \mid m+1, R) - V_m^d(x_m, w_m, p_m \mid m, R) \Big| \Omega_2 \cup \Omega_3 \Big) \mathrm{Prob}(\Omega_2 \cup \Omega_3) \Bigg].
\end{aligned} \tag{4.23}$$

The term $\mathsf{E}_0\big(V_m^d(x_m, w_m, p_m \mid m+1, R) - 0 \big| \Omega_1\big)$ calibrates the benefit of longer debt maturity where shorter maturity leads to bankruptcy. This term is positive since the firm cannot perform worse with a longer maturity. The second term calibrates the change of value where the shorter maturity does not result in bankruptcy. The sign of the second term is undetermined. If the probability of bankruptcy is high enough, the sum of the two terms is positive and the value function increases in longer maturity $V_0^d(K, 0, p_0 \mid m+1, R) \geq V_0^d(K, 0, p_0 \mid m, R)$. Based on the above analysis, we conjecture that under fixed interest, the project value will first increase and then decrease in maturity.

If we assume the price $p_t$ is capped by an upper bound $p^u$, then the effects of debt maturity and interest rate on project value can be summarized in the following proposition.

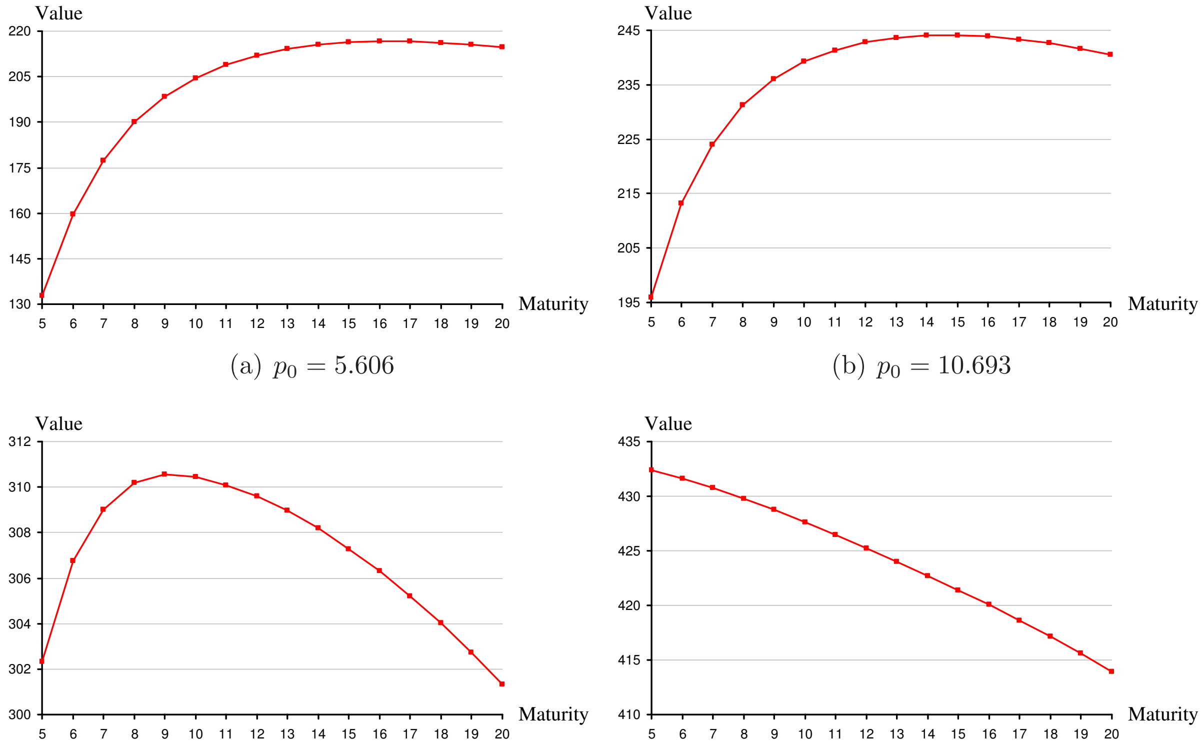
**Proposition 15.** *(i) The project value $V_0^d$ decreases upon the extension of debt maturity from $m$ to $m+1$, if the project size, interest rate and $m$ satisfy the inequality*

$$\frac{(1+r_f)r(K, p^u)}{cK - w_0} \leq R(1+R)^m. \tag{4.24}$$

*(ii) For any fixed $R$, there exists a threshold maturity such that $V_0^d$ decreases in $m$ if $m$ is greater than this threshold maturity.*
*(iii) There exists a threshold interest rate, such that $V_0^d$ decreases as debt maturity increases from $m$ to $T$ for any $R$ greater than this threshold interest.*
*(iv) If $\lim_{K\to\infty} r_K(K, p^u) = 0$, for any given $R$ and $m$, there exists a threshold project size, such that $V_0^d$ decreases as debt maturity increases from $m$ to $T$ for any project larger than this threshold project size.*

Figure 4.3 and Figure 4.4 provide numerical illustration relevant to above proposition. Figure 4.3 illustrates variation of project value as a function of debt maturity at different initial prices. Panels (a)(b)(c) indicate that the project value first increases and then decrease in debt maturity. In panel (d), the project value always decreases. In panel (a), as the initial price $p_0$ is low, the project value monotonically grows from 132.6 for $m = 5$ to 216.7 for $m = 16$. Comparing the curves in (a)(b)(c), we find that as debt maturity increases, the project value grows faster under a lower initial price. The reasoning is as follows. If the initial price is low, the firm is under huge pressure to pay off the debt, and more inventory will be sold and hence less inventory will be left. Extension of maturity will then provide with the opportunity to save more inventory. If the initial price is higher, the firm may still save some inventory given a longer maturity. But the effect is not as significant as the case when price is low. If the initial price is the highest, the firm is not that concerned about debt repayment obligation and having a longer maturity does little to save the inventory.

Figure 4.3: Impact of debt maturity on project value at different prices




(a) $p_0 = 5.606$ (b) $p_0 = 10.693$

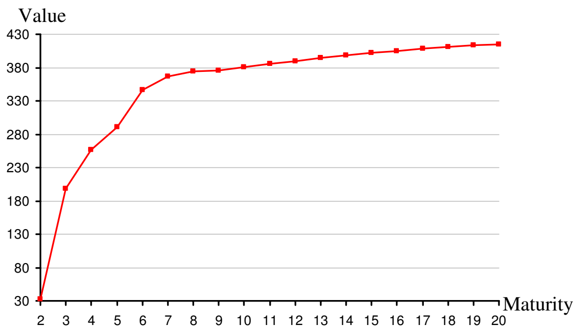
(c) $p_0 = 25.294$ (d) $p_0 = 59.829$

Figure 4.3 plots the project value as a function of debt maturity at different initial prices. It is assumed that $T = 20$, debt maturity varies from 5 to 20, project size $K = 30$, initial capital $w_0 = 100$, revenue function $r(q,p) = pq^{0.6}$, variable capacity cost $c = 8$. The firm borrows $cK - w_0 = 140$ to initiate the project. The risk-free rate $r_f = 2.5\%$; the interest rate on debt is exogenously given, $R = 2.8\%$. The four panels are obtained at four different starting prices respectively.

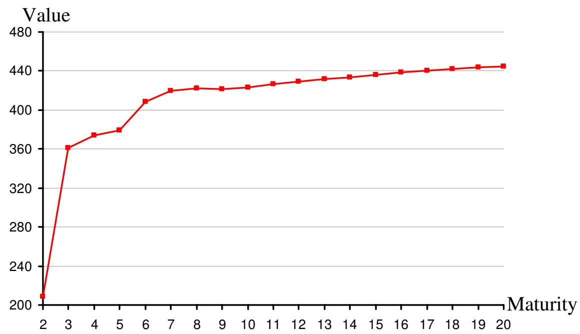
A longer maturity impacts the project value in two ways. It implies higher operational flexibility for the firm and a higher amount to pay to creditor. Figure 4.3 shows that the optimal maturity $m$ at which the project attains the highest value decreases as the initial price becomes bigger. Longer maturity is generally better when the price is low. Shorter maturity is favored when the price is high enough. Medium price implies that optimal maturity is somewhere in the middle. The project value can decrease in maturity because the effect of a higher amount paid to creditor will dominate. The benefit of better operational flexibility is more significant for lower prices and gradually degrades as maturity increases. The higher the price, the earlier the loss from a higher amount paid to creditor outweighs the benefit of flexibility. How longer maturity affects the project value depends on the price. The firm should be cautious against longer debt maturity if the initial price is high.

Figure 4.3 also implies that there may exist two maturities $m_l < m_h$, such that debt financing performs better than equity financing only when the debt maturity $m$ falls between $m_l$ and $m_h$. This is different from the case of fairly priced debt where the debt financing is better if $m$ is larger than some threshold maturity. Figure 4.4 is obtained with the output capacitated by an upper limit. It indicates that the relationship between debt maturity and project value can be even more complex.

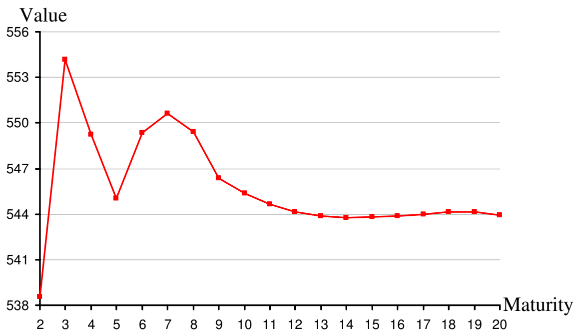
Figure 4.4: Impact of debt maturity on project value at different prices



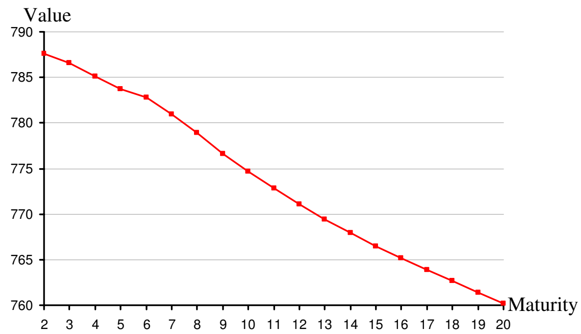

(a) $p_0 = 5.606$




(b) $p_0 = 10.693$




(c) $p_0 = 25.294$




(d) $p_0 = 59.829$

Figure 4.4 plots the project value as a function of debt maturity at different initial prices. It is assumed that $T = 20$, debt maturity varies from 2 to 20, project size $K = 30$, initial capital $w_0 = 100$, revenue function $r(q,p) = p\min(q,\overline{C})$ with $\overline{C} = 5$, variable capacity cost $c = 8$. The firm borrows $cK - w_0 = 140$ to initiate the project. The risk-free rate $r_f = 2.5\%$; the interest rate on debt is exogenously given, $R = 2.8\%$. The four panels are obtained at four different starting prices respectively.

## 4.5. Impact of Price Dynamics

In this section, we investigate how the price dynamics impact the firm's investment decision and financing strategy. To derive analytical results, we assume the cash flow from the project is $r(q,p) = pq^r$ with $r \in (0,1)$. We will draw some conclusions with respect to very general price processes first, and then focus the discussion on some specific price processes that have been widely used to model commodity price.

**Lemma 7.** *If the revenue function $r(q,p) = pq^r$ for some $r \in (0,1)$,*
*(i) the value of the project for an unconstrained firm is $V_t^e(x_t, p_t) = \delta_t(p_t)x_t^r$. The coefficient $\delta_t$ can be defined recursively,*

$$\delta_t(p_t) = \frac{\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1(p_{t+1})}}{\left(1 + \left(\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{(1+r_f)p_t}\right)^{\frac{1}{r-1}}\right)^{r-1}}, \qquad t = 1, ..., T-1. \tag{4.25}$$

*with the boundary condition $\delta_T(p_T) = p_T$.*
*(ii) The project value for an unconstrained firm at time 0 is*

$$V_0^e(K, p_0) = \frac{1}{1+r_f}K^r\mathsf{E}_0\delta_1(p_1). \tag{4.26}$$

*The corresponding optimal project size is*

$$\widehat{K} = \left(\frac{(1+r_f)c}{r\mathsf{E}_0\delta_1(p_1)}\right)^{\frac{1}{r-1}} \tag{4.27}$$

*(iii) The optimal project size $\widehat{K}$ decreases in risk-free rate $r_f$.*

We comment that Lemma 7$(i)(ii)$ does not require any additional assumptions on the price dynamics. The only parameter depending on price dynamics in the expression of $V_0^e$ and $\widehat{K}$ is $\delta_1(p_1)$. Now we analyze how $\delta_1(p_1)$ changes with respect to price dynamics. Specifically, we consider two types of price processes,

$$p_{t+1} = p_t e^{\mu + \sigma\epsilon}, \tag{4.28}$$

$$\log(p_{t+1}) = \rho\log(p_t) + \sigma\epsilon, \quad (\rho > 0) \tag{4.29}$$

where $\epsilon \sim \mathcal{N}(0,1)$ i.i.d. is the random shock, $\sigma$ models the volatility, $\mu$ models the drift, and $\rho$ is the decay rate. The first process can be viewed as a discrete-time version of continuous-time geometric Brownian motion. The second process specifies that the log price follows an AR(1) process. The AR(1) process can be obtained by sampling a continuous-time mean-reversion process. Both processes have been widely used to model evolution of commodity price (see e.g. *Deaton and Laroque* (1996)). In the Appendix, we show the AR(1) process can be regarded as a discrete-time version of the one-factor commodity price model in *Schwartz* (1997).

When the price evolves according to (4.28), we analyze the relationship between drift/volatility and $\delta_t(p_t)$. First we note $E_{T-1}p_T$ and $\frac{\mathsf{E}_{T-1}p_T}{(1+r_f)(p_{T-1})} = \frac{1}{1+r_f}e^{\mu+\frac{1}{2}\sigma^2}$ are increasing in $\mu$ and $\sigma$. Hence, $\delta_{T-1}(p_{T-1})$ is an increasing function of $\mu$ and $\sigma$. By induction it can be proved that $\delta_t(p_t)$ is increasing in $\mu$ and $\sigma$. Therefore, the optimal project size $\widehat{K}$ increases in $\mu$ and $\sigma$.

When the price evolves according to (4.29), we can prove that $\delta_t(p_t)$ is monotonically increasing in volatility $\sigma$. Therefore, the firm with a sufficient capital should invest more aggressively if the market is more volatile. However, the relationship between $\delta_t(p_t)$ and $\rho$ is not monotonic. We will examine the impact of $\rho$ by numerical method later.

**Proposition 16.** *If the price follows geometric Brownian motion, the threshold inventory level $K^o$ increases in $\mu$ and $\sigma$. In other words, if the price drift $\mu$ or volatility $\sigma$ increases, the firm is more likely to use debt financing.*

The reason is that the project value increases in $\mu$ and $\sigma$. As a result, the partner of the firm in equity financing will make more profit if $\mu$ and $\sigma$ increase. But the creditor who lends money to finance the project always earns the same amount no matter what the project value is.

## 4.6. Conclusions

In this paper we develop a multi-period model to study how the firm makes real investment, financing and production decisions, and how these decisions are affected by various market factors. The firm has an irreversible investment opportunity in a project in the very beginning. Since the firm has a limited budget, external capital is needed if the setting up the project demands more than the firm's own capital. The firm can raise additional capitals to cover the project setup cost through either equity financing or debt financing. There are no further opportunities to raise additional capitals after the operations start. Operations of the project generate stochastic cash-flows, out of which the firm repays its partner in equity financing or creditor in debt financing. The two financing approaches differ in the way the cash-flows are garnered from the project and the restrictions imposed on the operations. If a firm with outstanding debt fails to meet the debt repayment obligation, it declares bankruptcy and is divested of any wealth generated by the project. The firm is immune from bankruptcy if it finances through equity financing, but it may give up a significant share of the revenue to its equity financing partner.

We show and explain that debt obligation can significantly alter the firm's operations policy. With higher inventory or lower wealth, the firm with outstanding debt may decrease its output to maximize the expected profit. We show how price dynamics impact the firm's investment decision. Increased drift and volatility make the firm more prone to use debt financing. We demonstrate how the firm's financing choice and operation policy are affected by the term structure of debt. Extension of debt maturity provides the firm more flexibility in scheduling its production but also demands more amount to be paid back to creditor. With the interest rate fixed, we show that a longer maturity generally results in higher value if the project is financed by short-term debt but can be detrimental if the project is financed by long-term debt. The effect that longer maturity leads to higher project value is most significant when the market price is low. The non-monotonic relationship between project value and debt maturity indicates that debt financing may be used only when the maturity falls into certain intermediate range. The non-monotonic relationship provides managerial insights for managers to choose appropriate maturity under debt financing. We discuss the conditions under which equity financing is better than debt financing and vice versa. Without bankruptcy risk, equity financing is preferred over debt financing if the project size exceeds some certain threshold.

There are several directions that the current work can be extended in the future. For instance, we may consider a model where debt demands periodic coupon payment and the firm can default before maturity if it misses coupon payment. While the production decision is made dynamically in our model, the firm is given a single opportunity to make irreversible investment and financing decisions. It may be possible to develop a model where the firm can dynamically adjust its capital structure by issuing new debt and equity in later periods.

## 4.7. Appendix

### 4.7.1 Examples of Revenue Function

The production $q_t$ may be limited by either a physical limit, e.g., pipeline capacity, or the economics of the production is such that $q_t$ has an upper bound. The general form of $r(q,p)$ covers various types of revenue function, such as

$$r(q,p) = \begin{cases} qp & \text{if } q \leq \bar{C} \\ \bar{C}p & \text{if } q > \bar{C} \end{cases} \tag{4.30}$$

$$r(q,p) = \begin{cases} qp - aq - \frac{1}{2}bq^2 & \text{if } q \leq \frac{p-a}{b} \\ \frac{(p-a)^2}{2b} & \text{if } q > \frac{p-a}{b} \end{cases} \tag{4.31}$$

$$r(q,p) = pq^r \quad \text{for } r \in (0,1) \tag{4.32}$$

The formula in (4.30) corresponds to the case when the output per period is capacitated by a constant limit $\bar{C}$. For instance, $\bar{C}$ may represent the maximum flow rate of the pipeline for an oil field. We can also include a quadratic production cost $ax + \frac{1}{2}bx^2$ in the function as in (4.31). With a quadratic cost function, production is naturally capacitated as the firm should stop producing when marginal cost is equal to the current price. The function in (4.32) represents an iso-elastic revenue. All three types of function have been used in previous literature.

### 4.7.2 Setup of Numerical Study

*Price.* In our numerical study, we derive the discrete-time price $p_t$ from the the one-factor commodity price model in *Schwartz* (1997), which specifies that the commodity price $S$ in continuous-time follows,

$$dS = \kappa(\mu - \log S)Sdt + \overline{\sigma}SdZ^P,$$

where $\kappa$ is the mean-reverting rate, $\mu$ is long-term mean, $\overline{\sigma}$ is volatility and $Z^P$ is the Brownian motion under physical measure. The logarithmic price $\log S$ follows the Ornstein-Uhlenbeck process, with its representation under risk-neutral measure as

$$d\log S = \kappa(\mu - \frac{\overline{\sigma}^2}{2\kappa} - \lambda - \log S)dt + \overline{\sigma}dZ.$$

Here $\lambda$ is the market price of risk. The discrete-time price $p_t$ is sampled from $S_t$ with time step size $\delta t = 1$. Then we have $\log p_t$ follows an AR(1) process

$$\log(p_{t+1}) = \eta + \rho \log(p_t) + \sigma\epsilon_t, \tag{4.33}$$

where $\epsilon_t \sim N(0, \sigma)$ is the random shock, and

$$\rho = e^{-\kappa}, \qquad \eta = (1 - e^{-\kappa})(\mu - \frac{\overline{\sigma}^2}{2\kappa} - \lambda), \qquad \sigma = \overline{\sigma}\sqrt{\frac{1 - e^{-2\kappa}}{2\kappa}}.$$

Following the method in *Tauchen* (1986) and *Hennessy and Whited* (2005), we approximate the AR(1) process (4.33) with a Markov-chain of $N_p$ states $(p^1, ..., p^{N_p})$, which are defined as follows,

$$p^i = \exp\left\{\left(i - \frac{N_p + 1}{2}\right)\frac{6\sigma}{N_p\sqrt{1-\rho^2}}\right\}, \qquad i = 1, ..., N_p$$

We then divide the price space into $N_p$ cells $\{[\theta_i, \theta_{i+1}]\}$, $i = 1, ..., N_p$, where the boundary points $\theta_i$s are

$$\theta_1 = -\infty, \quad \theta_i = \frac{\log p^{i-1} + \log p^i}{2}, \quad \theta_{N_p+1} = \infty.$$

If the price falls into $[\theta_i, \theta_{i+1}]$, we regard it as in state $i$. The state transition matrix is

$$\pi(i,j) = N\left(\frac{\theta_{j+1} - \rho \log p^i}{\rho}\right) - N\left(\frac{\theta_j - \rho \log p^i}{\rho}\right).$$

We also discretize state variable inventory $x$ and cash holding $w$ into $N_x$ and $N_w$ sections. The decision variable $q$ takes values in $[0, x]$ and we discretize it into $N_q$ sections. All parameters used in the basic model are summarized in Table 4.2. The values of price parameters $(\kappa, \mu, \overline{\sigma}, \lambda)$ are from *Schwartz* (1997) Table IV.

### 4.7.3 Proofs

**Proof of Proposition 11:** *(i)* It is straightforward to establish that $V_t^e$ is increasing in $x_t$. Now we prove concavity. In the last period, given the zero salvage value of inventory, the firm simply sells all inventory on hand and its profit $V_T^e(x_T, p_T) = r(x_T, p_T)$ is concave in $x_T$. Supposing $V_{t+1}^e(x_{t+1}, p_{t+1})$ is concave in $x_{t+1}$, we now prove $V_t^e(x_t, p_t)$ is concave in

Table 4.2: Basic model parameters

| | | |
|---|---|---|
| $T$ | Production horizon | 20 |
| $\kappa$ | Mean-reverting rate | 0.301 |
| $\mu$ | Long-term mean | 3.093 |
| $\overline{\sigma}$ | Volatility of price | 0.334 |
| $\lambda$ | Market price of risk | -0.242 |
| $r$ | Interest rate | 2.0% |
| $\rho$ | AR(1) process decay rate | 0.6 |
| $\sigma$ | AR(1) process random shock volatility | 0.15 |
| $N_p$ | Number of states for price | 12 |
| $N_w$ | Number of cash holding discretization steps | 50 |
| $N_x$ | Number of x discretization steps | 200 |
| $N_q$ | Number of production $q$ discretization steps | 100 |

$x_t$. The maximand in (4.2)

$$r(q_t, p_t) + \frac{1}{1+r_f}\mathsf{E}_t V^e_{t+1}(x_t - q_t, p_{t+1})$$

is jointly concave in $(x_t, q_t)$. After optimization over $q_t$, we have that $V^e_t(x_t, p_t)$ is concave in $x_t$. Checking the cross derivative of the maximand with respect to $x_t$ and $q_t$, we find the derivative is non-negative. Therefore optimal production $q^e_t$ increases in inventory position $x_t$.

*(ii)* We note that $V_m$ in the region $r(x_m, p_m) + w_m \geq D$ can be written as

$$V^d_m(x_m, w_m, p_m; D) = \max_{q_m} \left\{ r(q_m, p_m) + w_m - D + \frac{1}{1+r_f} V^e_{m+1}(x_m - q_m, p_{m+1}) \right\}, \quad (4.34)$$

$$\begin{aligned} s.t. \quad & 0 \leq q_m \leq x_m, \\ & r(q_m, p_m) + w_m \geq D \end{aligned}$$

where we have made the dependence of $V_m$ on $D$ explicit. The maximand in (4.34) is jointly concave in $(x_m, w_m, D, q_m)$. The feasible set defined by the two inequalities in the above optimization problem is a convex set. Therefore we have $V_m$ is jointly concave in $(x_m, w_m, D)$.

*(iii)* The value function before maturity is given by (4.10). The maximand $\frac{1}{1+r_f}\mathsf{E}_t V^d_{t+1}(x_t - q_t, (1+r_f)(w_t+r(q_t, p_t)), p_{t+1})$ in jointly concave in $(x_t, w_t, q_t)$. In fact, let us consider the max-

imand defined at $(x_t^a, w_t^a, q_t^a)$, $(x_t^b, w_t^b, q_t^b)$ and $(x_t^c, w_t^c, q_t^c) = \theta(x_t^a, w_t^a, q_t^a) + (1-\theta)(x_t^b, w_t^b, q_t^b)$ for $\theta \in (0,1)$. Then

$$\begin{aligned}
&\theta V_{t+1}^d(x_t^a - q_t^a, (1+r_f)(w_t^a + r(q_t^a, p_t)), p_{t+1}) + (1-\theta)V_{t+1}^d(x_t^b - q_t^b, (1+r_f)(w_t^b + r(q_t^b, p_t)), p_{t+1}) \\
\leq & V_{t+1}^d(x_t^c - q_t^c, (1+r_f)(w_t^c + \theta r(q_t^a, p_t) + (1-\theta)r(q_t^b, p_t)), p_{t+1}) \qquad \text{(because } V_{t+1}^d \text{ is concave)} \\
\leq & V_{t+1}^d(x_t^c - q_t^c, (1+r_f)(w_t^c + r(q_t^c, p_t)), p_{t+1}). \qquad \text{(because } r(q_t, p_t) \text{ is concave)}
\end{aligned}$$

The optimization of a concave function in a convex set will yield $V_t$, which is jointly concave in $(x_t, w_t)$. ■

**Proof of Lemma 6:** Because $\beta(w_0, K)$ is increasing in $\alpha$, the profit under equity financing simply increases as $\alpha$ increases. If $\alpha = 0$, the derivative of profit with respect to $K$ is

$$\frac{w_0}{c}\frac{\frac{\partial V_0^e}{\partial K}K - V_0^e}{K^2}. \tag{4.35}$$

Since $V_0^e$ is concave in $K$, the nominator in the above expression is not positive $\frac{\partial V_0^e}{\partial K}K - V_0^e \leq 0$. Hence, the derivative (4.35) is not positive and expanding the project through equity financing can only decrease the project value earned by the firm. ■

**Proof of Proposition 12:** *(i)* For project size $K \leq \frac{w_0}{c}$, the firm simply uses its own capital. For bigger project $K > \frac{w_0}{c}$, the profit under equity financing and debt financing are

$$\left(\alpha + (1-\alpha)\frac{w_0}{cK}\right) V_0^e(K, w_0, p_0) - w_0 \quad \text{and} \quad V_0^e(K, w_0, p_0) - cK$$

respectively. The difference between the above two values is,

$$\begin{aligned}
&\left(\left(\alpha + (1-\alpha)\frac{w_0}{cK}\right) V_0^e(K, w_0, p_0) - w_0\right) - (V_0^e(K, w_0, p_0) - cK) \\
= \quad &(cK - w_0)\left(1 - \frac{1-\alpha}{cK}V_0^e(K, w_0, p_0)\right).
\end{aligned}$$

Which financing method is better depends on the sign of $1 - \frac{1-\alpha}{cK}V_0^e(K, w_0, p_0)$. If $\frac{1-\alpha}{cK}V_0^e(K, w_0, p_0)$ is consistently greater than or less than 1 for any $K > 0$, either debt financing or equity financing prevails. Otherwise, there exists $K^o > 0$ at which $\frac{1-\alpha}{cK^o}V_0^e(K^o, w_0, p_0) = 1$. Moreover, $K^o$ is unique because $V_0^e$ is concave.

*(ii)* The profit under equity financing is a strictly increasing function of $\alpha$. Further, if $\alpha = 1$, equity financing prevails over debt financing. If $\alpha = 0$, either debt financing yields higher

profit than equity financing or equity financing is still better. The threshold $\alpha^o$ is either 0 or the number at which equity financing profit is equal to debt financing.

*(iii)* The value under debt financing is an increasing function of $m$.

*(iv)* $V_0^e$ is a decreasing function of $r_f$. ■

**Proof of Lemma 7:** *(i)* We prove by induction. In the last period $T$, the firm extracts and sells all remaining inventory, $V_T^e(x_T, p_T) = p_T x_T^r$ and the result holds. Assuming $V_{t+1}^e(x_{t+1}, p_{t+1}) = \delta_{t+1}(p_{t+1}) x_{t+1}^r$, the optimal production in period $t$ is determined by

$$V_t^e(x_t, p_t) = \max_{0 \le q_t \le x_t} q_t^r p_t + \frac{1}{1+r_f}(x_t - q_t)^r \mathsf{E}_t \delta_{t+1}(p_{t+1}) \tag{4.36}$$

The optimal production

$$q_t^e = \frac{(\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}}{p_t^{\frac{1}{r-1}} + (\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}} x_t. \tag{4.37}$$

Therefore, $q_t^e$ is a certain fraction of $x_t$. Plugging $q_t^e$ into (4.36), we have

$$\begin{aligned}
& V_t^e(x_t, p_t) \\
&= \left( \frac{(\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}}{p_t^{\frac{1}{r-1}} + (\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}} \right)^r x_t^r p_t + \frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f} \left( \frac{p_t^{\frac{1}{r-1}}}{p_t^{\frac{1}{r-1}} + (\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}} \right)^r x_t^r \\
&= \frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f} p_t \frac{(\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}} + p_t^{\frac{1}{r-1}}}{\left(p_t^{\frac{1}{r-1}} + (\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}\right)^r} x_t^r \\
&= \frac{\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f} p_t}{\left(p_t^{\frac{1}{r-1}} + (\frac{1}{1+r_f}\mathsf{E}_t\delta_{t+1}(p_{t+1}))^{\frac{1}{r-1}}\right)^{r-1}} x_t^r \\
&= \frac{\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f}}{\left(1 + (\frac{1}{1+r_f}\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{p_t})^{\frac{1}{r-1}}\right)^{r-1}} x_t^r \\
&= \delta_t(p_t) x_t^r
\end{aligned}$$

*(ii)* The project value net of capacity investment cost at time 0 is $\frac{1}{1+r_f} K^r \mathsf{E}_0 \delta_1(p_1) - cK$, which is concave in $K$. Using the first-order condition, we find the optimal project size as $\widehat{K} = \left(\frac{(1+r_f)c}{r\mathsf{E}_0\delta_1(p_1)}\right)^{\frac{1}{r-1}}$.

*(iii)* We prove $\delta_t(p_t)$ is a decreasing function of $r_f$ by induction. Suppose it is true for $\delta_{t+1}(p_{t+1})$ is decreasing in $r_f$. Then $\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f}$ is a decreasing function of $r_f$. By checking the first-order derivative, we can prove that $\delta_t(p_t)$ is an increasing function of $\frac{\mathsf{E}_t\delta_{t+1}(p_{t+1})}{1+r_f}$. Therefore, higher interest rate will lead to lower optimal project size. ■

**Proof of Proposition 13:** We consider three different value functions, $V_0^e$, $V_0^d$ and $V_0^\dagger$. We derive upper and lower bounds to benchmark the profit when the project can go bankrupt. In order to derive the lower bound, we construct a value function $V_0^\dagger$. The only difference between $V_0^\dagger$ and $V_0$ is the treatment of bankruptcy. Specifically, we define

$$V_m^\dagger(x_m, w_m, p_m) = \begin{cases} V_m^e(x_m, p_m) + w_m - D, & \text{if } r(x_m, p_m) + w_m < D \\ V_m^d(x_m, w_m, p_m) & \text{if } r(x_m, p_m) + w_m \geq D \end{cases} \tag{4.38}$$

Due to the way it is constructed, the inequality $V_m^\dagger \leq V_m^d$ holds for all states. Therefore, we have $V_0^\dagger \leq V_0^d$.

The value under bankruptcy risk is lower than the value without bankruptcy risk, $V_0^d(K, w_0, p_0) - w_0 \leq V_0^e(K, w_0, p_0) - cK$. Therefore, $V_0^e(K, w_0, p_0) - cK$ provides an upper bound on the value.

If there is no default risk, $V_0^\dagger$ and $V_0^e - cK$ coincide, $V_0^\dagger = V_0^e - cK$. ■

**Proof of Proposition 14:** *(i)* We let $\pi_t$ denote the optimal operation policy when debt maturity is $t$. If debt maturity is $m_h$, we construct a policy based on $\pi_{m_l}$ and show this policy can yield a value no less than $V_0(K, w_0, p_0 \mid m_l, R_l)$. Specifically, we let the firm operates in the way exactly the same as $\pi_{m_l}$. We denote the distribution of profit in period $t$ under policy $\pi$ by $Q^{pi}(\omega_t)$, which is contingent on the price realization $\omega_t$.

*(ii)* We consider two projects of the the same size but different debt maturities. Let $R_t$ denote the interest for debt of maturity $t$. To study the effect of maturity on project value, we need to compare $V_0^d(K, 0, p_0 \mid m, R_m)$ and $V_0^d(K, 0, p_0 \mid m+1, R_{m+1})$. Let $\pi_t$ denote the optimal operation policy when debt maturity is $t$. The realization of profit in period $m$ under policy $\pi$ and price evolution $\omega_m$ is denoted by $Q^\pi(\omega)$. Therefore we have $\mathsf{E}_0 \min(Q^{\pi_m}(\omega_m), D_m) = (1+r_f)^m(cK - w_0)$.

If the maturity is extended to $m+1$, we construct a hypothetical policy and interest rate such that the creditor earns the same risk-free return and the firm earns a value not less than $V_0(K, 0, p_0 \mid m, R_m)$. Specifically, we consider an interest rate

$$R_{m+1} = ((1 + r_f)(1 + R_m)^m)^{\frac{1}{m+1}}. \tag{4.39}$$

Under this rate, $D_{m+1} = (1 + r_f)D_m$. We assume the firm still follows policy $\pi_m$. If for some state the firm goes bankrupt under maturity $m$ debt, it will go bankrupt in the same state with maturity $m + 1$ debt. If for some state the firm is solvent under maturity $m$ debt, it will still be solvent under maturity $m+1$ debt. Then the creditor earns $\mathsf{E}_0 \min((1 + r_f)Q^{\pi_m}(\omega_m), (1 + r_f)D_m) = (1 + r_f)\mathsf{E}_0 \min(Q^{\pi_m}(\omega_m), D_m) = (1 + r_f)^{m+1}(cK - w_0)$, which means the return on debt is the risk-free rate. At the same time, the value earned by the firm is still $V_0(K, 0, p_0; m, R_m)$. Therefore, given that the firm has alternative policies other than the above hypothetical policy, it will earn a higher value than $V_0(K, 0, p_0 \mid m, R_m)$. $\blacksquare$

# CHAPTER 5

# Conclusions

This dissertation consists of three essays, with the first two dealing with operations and valuations of energy storage assets and the third essay dealing with the interactions of financing and operations in the development of non-renewable resource projects.

In the valuation problem based on futures market, the firm operates the storage on a monthly schedule with the injection and withdrawal quantity constrained by inventory-dependent limits. In practice, practitioners use heuristic policies to capture the seasonal price spread under limited flexibility. The first essay identifies when and why the industry heuristics lead to significant losses. A new heuristic policy called the price-adjusted rolling intrinsic (PARI) policy is developed to capture the optimal values embedded in the optimal policy. The second essay develops a model to integrate the granular spot market. The firm can take profits not only from seasonal price spreads, but also futures/spot price differentials. The problem is considerably more complex due to the coupling effect of trading in futures market and the spot market. We develop a more time-efficient heuristic policy to overcome "curse of dimensionality".

In the third essay, I develop a multi-period model to study how the firm makes real investment, financing and production decisions jointly, and how these decisions are affected by various market factors. Operations and financing decisions are intertwined even if the project does not bear any default risk. I show that with higher inventory or lower wealth, the firm with outstanding debt may decrease its output to maximize the expected profit. Higher drift and volatility of price make the firm more prone to use debt financing. Extension of debt maturity provides the firm more flexibility in scheduling its production but also demands more amount to be paid back to creditor. There does not exist a monotonic relationship between debt maturity and project value. I show that increasing debt maturity generally results in higher project value in the short term. But increasing maturity can be detrimental if the debt maturity is longer than some certain date. Project value is most sensitive to debt maturity when the market price is low. The non-monotonic relationship between project value and debt maturity also implies that the firm may choose equity over debt if debt maturity is too long or too short.

Several extensions to the above essays are possible. For the energy storage assets valuation problem, it is worthwhile to derive a hedging strategy since in practice the manager is concerned about the distribution of profit. For the financing problem, the model may be extended to include periodic coupon payment, with which the firm can default before maturity if it misses coupon payment. In the current model the firm is given a single opportunity to make irreversible investment and financing decisions. It may be possible to develop a model where the firm can dynamically adjust its capital structure by issuing new debt, paying out dividends and raising equity in any periods.

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