No. 126
January 1972

LOW PAYING VERSUS HIGH PAYING CARGO IN THE LINER TRADE

Stian Erichsen

THE UNIVERSITY OF MICHIGAN 1817

THE DEPARTMENT OF NAVAL ARCHITECTURE AND MARINE ENGINEERING

THE UNIVERSITY OF MICHIGAN
COLLEGE OF ENGINEERING

Publication costs of this report were covered in part by a grant from Mr. Frederic Gibbs of Gibbs and Cox, Inc.

Copies of this report are available from the Department of Naval Architecture and Marine Engineering, West Engineering Building, University of Michigan, Ann Arbor, Michigan 48104, U.S.A. Price: $1.10 on credit, $1.00 with cash payment.

LOW PAYING VERSUS HIGH PAYING CARGO IN THE LINER TRADE

by

Stian Erichsen

INDEX

SYNOPSIS

The liner trade has its own rules and regulations for setting freight rates. High paying and low paying cargoes are carried by the same ship on the same voyage. This study deals with the optimum combination of high and low paying cargoes for different conditions. Some examples are used to form more general rules for deciding whether to accept shipments of low paying cargo.

INTRODUCTION

This note discusses a situation that one may encounter in the liner trade. A liner operator has advertised a sailing and the shippers start to ask for space for their cargo. It may then be that low paying commodities are offered in good time before the ship is scheduled to sail, whereas one expects high paying commodities to be offered shortly before the ship's departure. It is then often a problem to know how far one can go in accepting low paying cargo without running short of space for high paying cargo. The problem is frequently complicated by the fact that the low paying cargo is offered in large lots that have to be accepted in whole or rejected.

In dealing with this situation one more or less deliberately uses past experience as a basis for the decision making. In the following an attempt is made to treat past experience mathematically in order to derive a result that may be of assistance in making the decision. It is assumed that quantitative measurements of past cargo inflow, i. e. statistics, are available.

Based on histograms and statistics, some probability functions may be established, and used as bases for further calculations. Statistics should, however, be used with caution. A histogram based on cargo which has actually been loaded will not always give a correct indication of the true cargo inflow. The ship's capacity and not the cargo inflow may have been the determining factor when accepting cargo.

ASSUMPTIONS

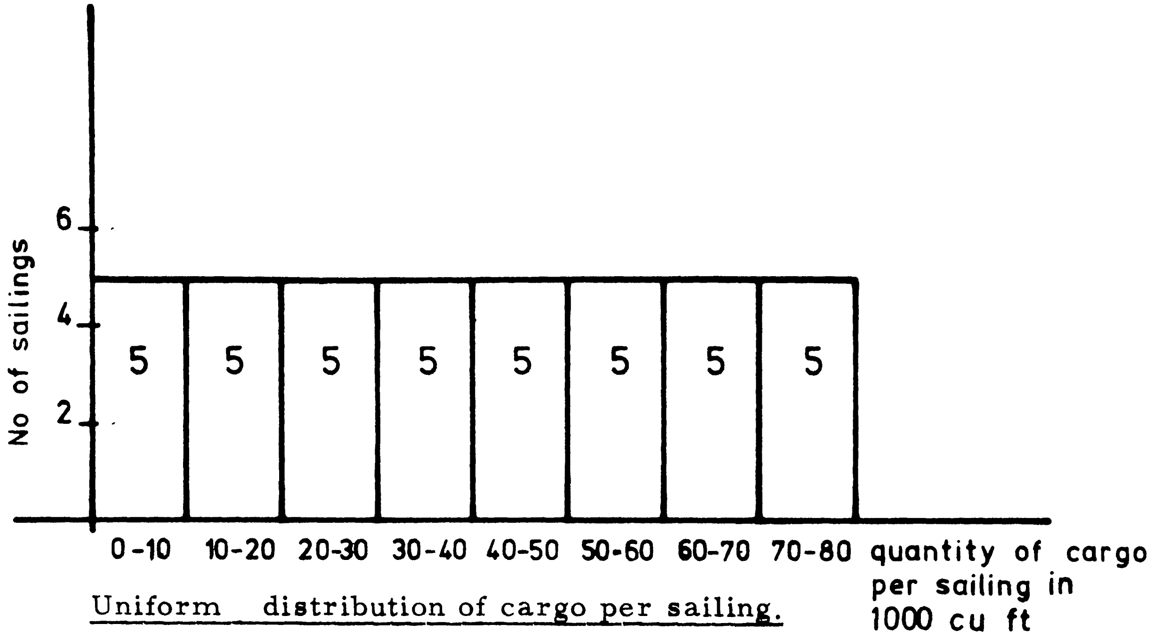
With these reservation, we may attempt to evaluate analytically the problem of low versus high paying cargo. Some simplifying assumptions are made in order to keep the calculation reasonable.

For a cargo liner we may use the cubic capacity as the limiting factor. Further, we may start by assuming that the cargo can be treated as belonging to one of two different categories, one low paying, the other high. Another simplifying assumption that will be applied at least for this and a few more cases, is that the low paying cargo is offered only once. It is also assumed that cargo may be accepted or rejected for any sailing.

A NUMERICAL EXAMPLE

Before proceeding, let us consider a numerical example to see how we can use information about the cargo to calculate expected revenue. Say the low net revenue (after stevedoring and other expenses have been deducted) is \$0. 15 per cu ft, the high revenue \$0. 30. If we have 80, 000 cu ft of ship capacity at our disposal, how much could we then accept of the low paying cargo without spoiling our chances of making a reasonable revenue on the high paying one ? This depends on how much high paying cargo we expect.

We need a probability distribution for the high paying cargo, i. e., a graph that shows how frequently we may expect different quantities.



Uniform distribution of cargo per sailing.

FIGURE 1

This frequency may be based on past experience as demonstrated by a histogram. Let us for the sake of simplicity assume that our histogram is as shown in Figure 1, i. e., we have had an equal number of sailings with any quantity of high paying cargo in the range 0 to 80, 000 cu ft . The case is treated mathematically in Appendix I ; the result may be presented as shown in Table I. A look at this table leads to the following conclusions:

a) Any quantity of low paying cargo is better than none.

b) Maximum expected profit occurs when volume of low paying cargo is equal to half of the total available space.

c) When the accepted quantity of low paying cargo varies from 30, 000 to 50, 000 cu ft , i. e. with $\pm$ 25% of the optimum quantity, the expected revenue varies only between \$ 14, 810 and \$ 15, 000 (with less than $\pm$ 1. 5% of the optimum revenue).

TABLE I

EXPECTED REVENUE FOR DIFFERENT QUANTITIES OF LOW PAYING CARGO

Net revenue on low paying cargo = \$0. 15 per cu ft
Net revenue on high paying cargo = \$0. 30 per cu ft
Available space = 80, 000 cu ft
The probability distribution of the high paying cargo is uniform and has the range 0 - 80, 000 cu ft

Accepted quantity of low paying cargo, cu ft	Expected profit, U. S. dollars
0	12, 000
10, 000	13, 312
20, 000	14, 250
30, 000	14, 810
40, 000	15, 000
50, 000	14, 810
60, 000	14, 250
70, 000	13, 312
80, 000	12, 000

d) One may restrict acceptance of low paying cargo to well below half of the allocated space without risking loss of substantial revenue. This is important in case the inflow of high paying cargo is pessimistically estimated.

GENERAL CASES ASSUMING ONE OFFER OF LOW PAYING CARGO

Some of the above conclusions are not strictly limited to the numerical example, but before we try to expand their range of validity, let us develop more general expressions.

Uniform distribution of high paying cargo.

We use the following symbols:

Maximum available cargo space, cu ft ,	S
Maximum expected quantity of high paying cargo cu ft ,	A
High paying cargo, cu ft ,	y
Accepted quantity of low paying cargo, cu ft ,	x
Net revenue on low paying cargo, dollars per cu ft ,	p
Net revenue on high paying cargo, dollars per cu ft,	q
Expected revenue, dollars,	E (R)

The distribution of y has the form,

$$P(y) = \frac{1}{A} \quad , \quad y = 1 - A \ , \quad P(y) = 0, \text{ elsewhere}$$

This leads to the following expression for the expected revenue:

$$E(R) = p\,x + \int_{0}^{(A-x)} q\,\frac{y}{A}\,dy + \int_{A-x}^{A} q\,\frac{(A-x)}{A}\,dy$$

The integrations are performed in Appendix II. The results are as indicated by Table II and Figure 2. Although the acceptable range of the low paying cargo varies with the ratio of low to high revenues, a study of Table II leads to conclusions similar to those of the numerical example.

TABLE II

EXPECTED REVENUE AS FUNCTION OF ACCEPTED QUANTITY OF LOW PAYING CARGO.

The probability distribution of the high paying cargo is uniform and has the range 0 - A cu ft
Net revenue on low paying cargo = p $/ cu ft
Net revenue on high paying cargo = q $/ cu ft
Maximum expected quantity of high paying cargo = A cu ft

Accepted quantity of low paying cargo as fraction of maximum expected quantity of high paying cargo	Expected profit for different relations low/high net revenue		
	p/q=1/4	p/q=1/2	p/q = 1
	Expected profit as multiples of q A		
x = 0	0.500	0.500	0.50
1/8	0.524	0.555	0.63
1/4	0.530	0.595	0.73
3/8	0.524	0.617	0.81
1/2	0.500	0.625	0.875
5/8	0.460	0.617	0.930
3/4	0.406	0.595	0.969
7/8	0.336	0.555	0.992
1/1 A	0.250	0.500	1.00

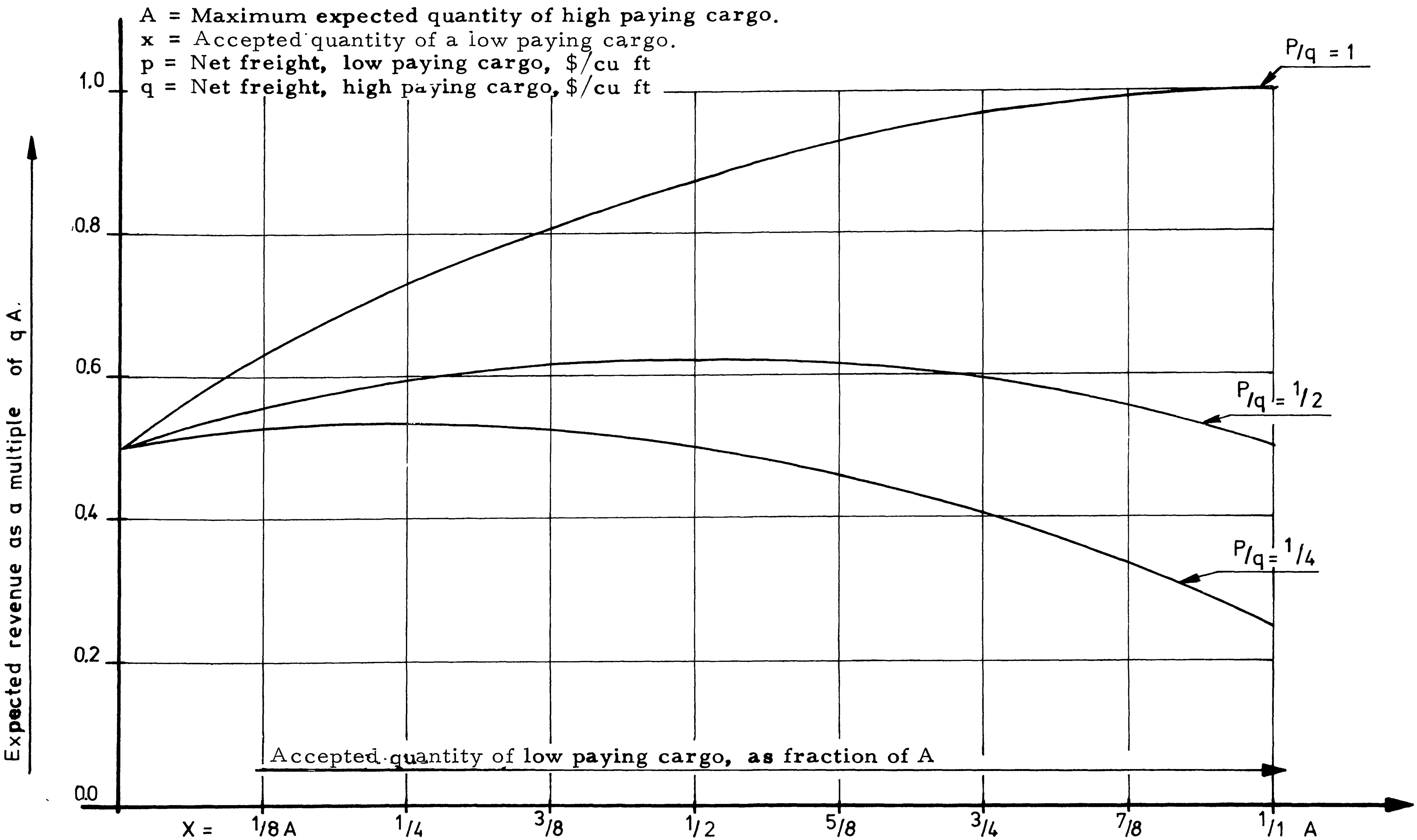


FIGURE 2

Expected profit for different relations low/high net freight for uniform probability distribution of high paying cargo in the range 0 - A.

The general form of the numerical example may be seen from column three of the table ($\frac{p}{q} = \frac{1}{2}$). A lower net revenue on the low paying cargo will reduce its optimum quantity; a higher revenue will increase it. In the numerical example the accepted quantity of low paying cargo might vary between 75 and 125 percent of the optimum quantity without affecting the expected revenue by more than $\pm$ 1. 5% (shown by point c page 4). When the relative net revenue on the low paying cargo is lower than in the example, the accepted quantity of low paying cargo may vary between relatively wider limits without having any greater effect on the expected net revenue. In addition to this conclusion, the table may be used in concrete cases like the following two examples.

First example:

The total allocation for a port is 100, 000 cu ft
The expected maximum quantity of the high paying cargo is 80, 000 cu ft.
There is an equal chance of getting any quantity between 0 and 80, 000 cu ft of the high paying cargo, i. e. a uniform distribution in the range 0 - 80, 000 cu ft.
The net revenue of the high paying cargo is \$ 0. 40 per cu ft. Then a lot of 30, 000 cu ft of low paying cargo is offered. The net revenue on this lot will be \$0. 10 per cu ft.

The calculation will be as follows:

Of a total allocation of 100, 000 cu ft , a maximum of 80, 000 cu ft will be needed for the high paying cargo.
Then 20, 000 cu ft of the low paying cargo may be accepted without hesitation, but what with the remaining 10, 000 cu ft ? Here is a risk of giving away space that may be needed for high paying cargo later on.
A study of Table II , however, indicates that the whole lot should be accepted. The approach is as follows:

10, 000 cu ft = 1/8 of 80, 000 cu ft (The maximum expected quantity of high paying cargo).

Net unit revenue on the low paying cargo is 1/4 of the net revenue of the high paying cargo.

If the entire lot of low paying cargo is accepted, the table gives an expected profit of:

$0.524\, q\ A = 0.524 \cdot \$0.40 \cdot 80{,}000$	= \$ 16, 780
In addition, we have the revenue on the 20,000 cu ft that will not occupy space needed for the high paying cargo	= \$ 2, 000
Total expected revenue	= \$ 18, 780
The expected revenue when the low paying cargo is rejected is $0.5\, q\, A$	= \$ 16, 000

Second example:

Assume the booking has passed the initial stage, and the situation is as follows:

10, 000 cu ft of ship capacity is still available shortly before sailing. The probability of filling the space with high paying cargo is uniformly distributed in the range 0 - 10, 000 cu ft.

Then 5, 000 cu ft of a commodity, whose net revenue is one quarter of the net revenue of the high paying cargo, is offered.

The table indicates the following:

Expected revenue rejecting the low paying cargo: $0.5\, q\, A = 0.5 \cdot \$0.40 \cdot 10{,}000$	= \$ 2, 000
Expected revenue accepting the low paying cargo: also, $0.5\, q\, A$	= \$ 2, 000

Then one could answer the shipper as follows:

1) The freight is too low, it should be increased to make it profitable to accept such a large quantity. (A 100% increase of the net revenue will make $p/q = 1/2$ and the expected revenue will have its maximum for $x = 1/2\, A = 1/2 \cdot 10{,}000 = 5{,}000$ cu ft.)

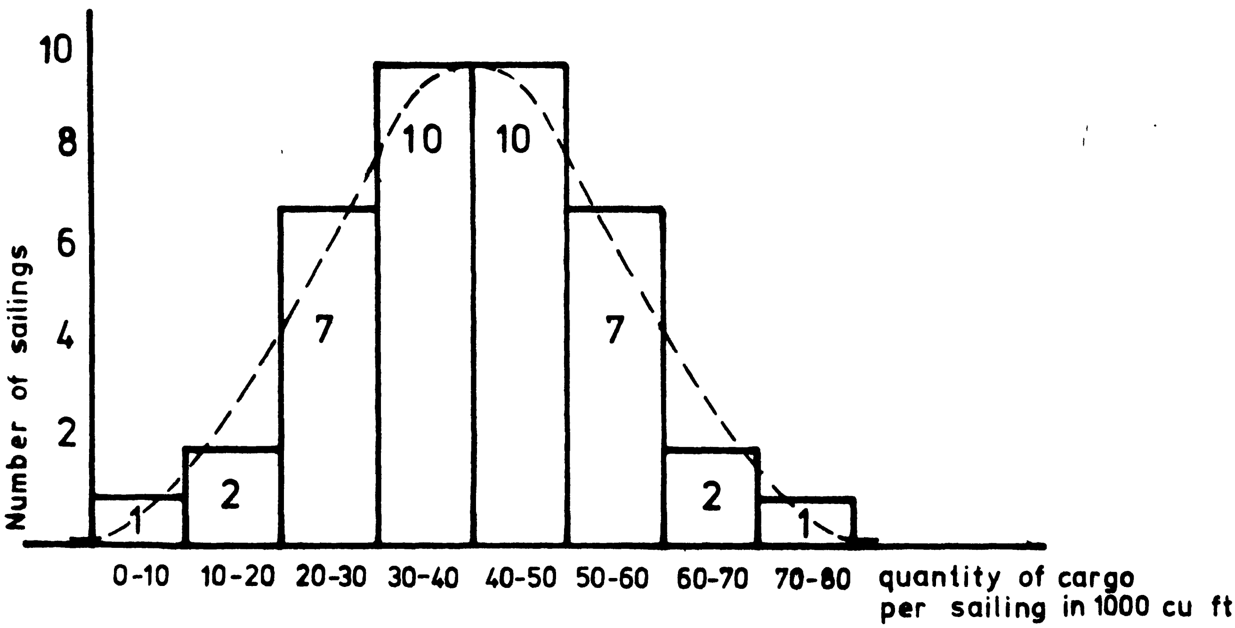
2) Or the answer could be:

The lot is too large, we are just as well off by taking nothing, or maybe you can reduce the quantity to 2, 500 cu ft and let the rest go by another sailing. (2, 500 cu ft makes $x = 1/4 \cdot 10\ 000 = 1/4\ A$; the corresponding expected revenue is $0.53\ q\ A = \$\ 2,120$, which is higher than for any other quantity at the rate of \$0. 10 per cu ft.)

Normal distribution of high paying cargo.

The above examples may be limited in their general use owing to the assumptions that have been made. One of these assumptions is that the high paying cargo has a uniform distribution. Sometimes this will not be the case, but only a study of cargo statistics can, as mentioned, give proper basis for a cargo distribution. In many cases it will be possible to approximate a histogram by some known mathematical distribution function, in other cases numerical solution techniques may be applied. The examples above indicate, however, - and this is also confirmed by the following case - that the accepted quantity of the low paying cargo may differ rather widely from the optimum quantity without affecting the expected revenue too much. From this one may deduct that it is not too important to find a distribution function that fits the histogram exactly. In many cases the normal distribution will fit reasonable well, as in the example shown in Figure 3. Table III and Figure 4 based on this distribution are, therefore, included in the following. The mathematics may be found in Appendix III. The symbols are as for the previous case.

The table and the figure are self-explanatory. In relation to the examples based on the uniform distribution of high paying cargo, there is one marked difference, and that is that the optimum quantity of low paying cargo is higher for extremely low net revenues.



Distribution of cargo per sailing approximately according to a "normal" distribution.

FIGURE 3

TABLE III

EXPECTED REVENUE AS FUNCTION OF ACCEPTED QUANTITY OF LOW PAYING CARGO.

The probability distribution of the high paying cargo is "normal" with mean = A/2, standard deviation A/5.

Net revenue on low paying cargo	= p \$/cu ft
Net revenue on high paying cargo	= q \$/cu ft
Maximum expected quantity of high paying cargo	= A cu ft

Accepted quantity of low paying cargo, x, as fraction of maximum expected quantity of high paying cargo.	Expected profit for different relations low/high net revenue.		
	p/q= 1/4	p/q = 1/2	p/q = 1
	Expected profit as multiples of qA		
x = 0	0. 4938	0. 4938	0. 4938
1/8	0. 5251	0. 5563	0. 6188
1/4	0. 5487	0. 6107	0. 7357
3/8	0. 5575	0. 6512	0. 8387
1/2	0. 5417	0. 6667	0. 9167
5/8	0. 4970	0. 6532	0. 9657
3/4	0. 4263	0. 6138	0. 9888
7/8	0. 3410	0. 5597	0. 9972
1/1 A	0. 25	0. 50	1. 00

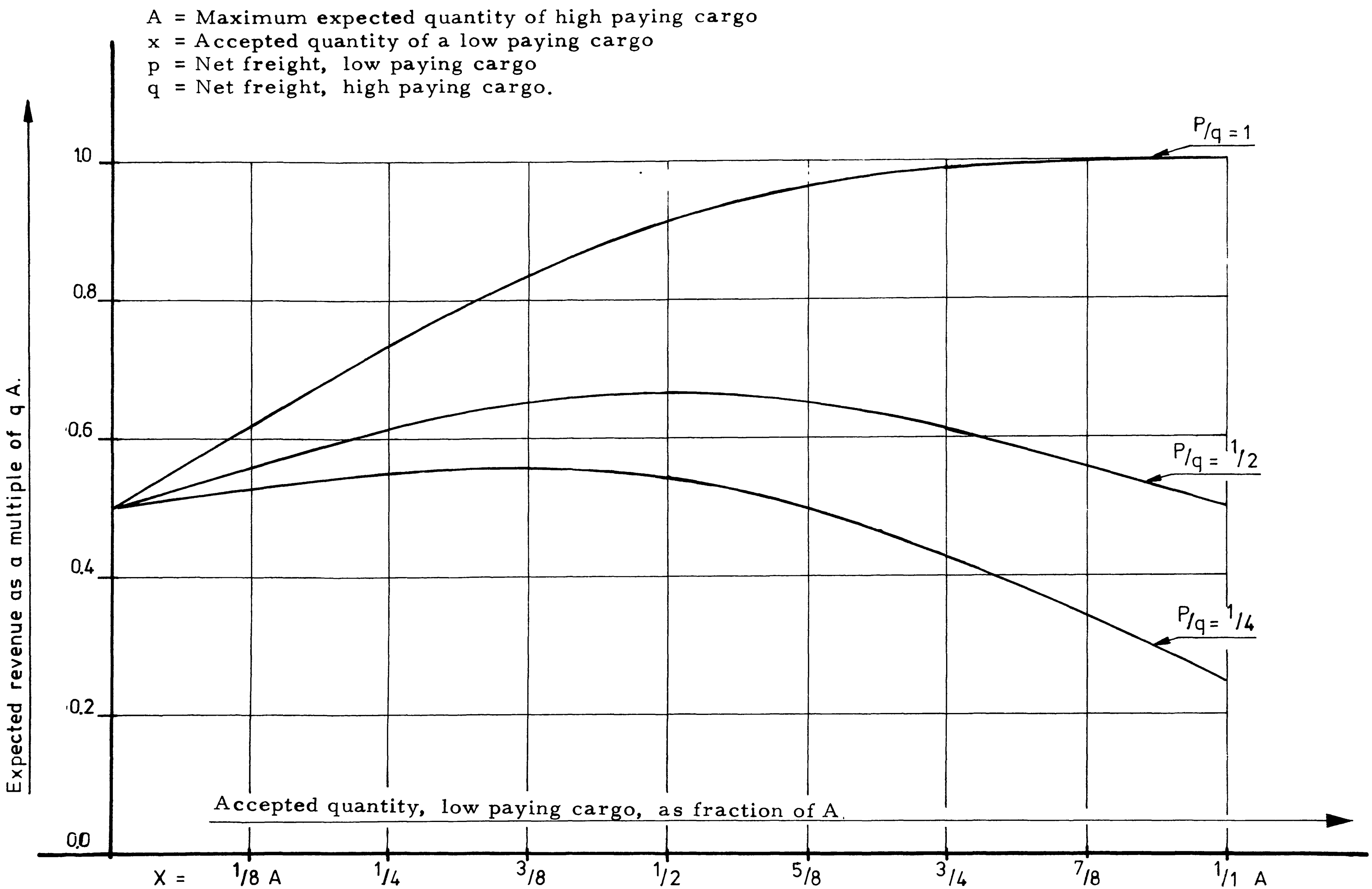


FIGURE 4

Expected profit for different relations low/high net freight for normal distribution of high paying cargo; mean = A/2, standard deviation = A/5

GENERAL CASES ASSUMING TWO OR MORE EXPECTED OFFERS OF LOW PAYING CARGO

A more general case than the examples discussed above, is when one may expect more than one offer of the low paying cargo. There may be, for example, more than one shipper having low paying cargo, and if one lot is rejected, there may still be a certain probability of another being offered. It may also be the case that the low paying cargo is offered in many different shipments and more or less simultaneously with the shipments of high paying cargo.

Low and high paying cargoes offered during the same period.

The situation with low and high paying cargoes being offered simultaneously is simple to deal with. When the low paying cargo is not offered during distinct periods, well ahead of ships' departures, but is offered during the same period as other cargo, we should deal with each individual offer of low paying eargo as before. However, instead of using the distribution of high paying cargo to calculate the expected revenue, we should use a distribution of the mixture of high and low paying cargo and the average net revenue of this mixture. We can use the same tables and graphs as before. The histogram, i. e. the distribution of the high paying cargo, must be replaced by the distribution of the mixture of cargoes, and the net revenue of the high paying cargo be replaced by the average net revenue of the mixture. Whenever we get an offer whose net revenue is higher than the average, we should accept it. Whenever we get an offer where the net revenue is lower than the average, we should deal with it in the same way as we dealt with the low paying cargo in the preceding section. In principle there is no change.

It is likely that the majority of real life cases will fall in one of the categories that have now been discussed. Other cases should, however, not be totally neglected and a few of them are therefore discussed in the following.

Multiple shipments of low paying cargo offered before high paying cargo.

This section presents some examples based on the assumption that one expects to be offered two or more shipments of low paying cargo before one gets any offer of high paying cargo. It is not intended to cover all possible combinations of lot size, distribution of lot size and so on, but it is hoped that the discussion of the examples will also give some guidance in dealing with cases that are not directly mentioned in the following.

As the nature of the lot size distribution in some cases may influence the decision making, two different distributions of the lot size of the low paying cargo have been used in the examples.

Uniform distribution of low paying cargo over the entire range.

If the low paying cargo lot size is uniformly spread over the available space, i. e. the entire interval, one should accept any offer of low paying cargo as long as its optimum quantity is not exceeded. The probability of getting a *new* offer which is closer to the optimum than the one under consideration is namely not greater than the probability of getting an *additional* offer which brings the accumulated quantity of low paying cargo closer to the optimum. An inspection of the graphs in Figure 2 and 4 will reveal this.

If the offer of low paying cargo would bring the total quantity *beyond* the optimum, the problem is more complicated. Let us assume that the conditions are such that we may use column 3 of Table III (or the middle contour of Figure 4) as our basis for decision making, and that we are offered a lot of low paying cargo equal to 3/4 of the maximum expected quantity of high paying cargo. If we accept it, it would increase the expected revenue by 23%, but we would still be about 10% below its optimum. Should we take it or not? A crucial question here is how certain we are to get more offers.

It appears from the table that any lot measuring between 1/4 and 3/4 of the available space will produce an equally good or higher expected revenue; any lot measuring between 3/8 and 5/8 of the available space will increase the expected revenue by at least 0.037 q A. As the lot size of the low paying cargo is uniformly distributed over the available space, the probability that any other offer will be at least equally good or better than the one under consideration (i.e. fall in the 1/4 to 3/4 range) is 0.5, and the probability that it will increase the expected revenue by at least 0.037 q A (fall in the 3/8 to 5/8 range) is 0.25.

If we are certain to get n more offers, the probability of <u>not</u> getting a better or equally good offer is $(1 - 0.5)^n$. The probability of <u>not</u> getting an offer that will increase the expected revenue by at least 0.037 q A is $(1 - 0.25)^n$. The importance of knowing the number of additional offers, n, may here be demonstrated. If n equals 1, $(1 - 0.50)^n = 0.50$; that is, in one out of two cases, the additional offer will <u>fail</u> to make an improvement. If n = 5, however, $(1 - 0.50)^n = 1/32$ and one will get an equally good or better offer in 31 out of 32 cases. Similarly if n = 1, three out of four cases will <u>fail</u> to produce an offer that improves the expected revenue by 0.037 q A; but if n = 5, three out of four cases <u>will</u> produce an offer that improves the expected revenue by at least 0.037 q A.

In addition to the probability of getting one single offer that improves the expected revenue, there is a certain probability that the sum of some offers will produce a higher expected revenue; but this probability may be neglected without causing great errors in our comparisons.

If n = 5 (if we are certain to get five more offers of low paying cargo) the case may be summarized as follows:

If we reject the offer, the expected revenue will at least be 0.5 qA + (0.65 - 0.5) q A · 0.75	<u>0.625 qA</u>
If we accept it, it is	<u>0.615 qA</u>

These results differ only by 0.01 q A. For A = 100,000 cu ft and q = $ 0.4 per cu ft, the expected gain by rejecting the offer becomes $ 400. If we are certain to get five more offers and equally certain that our distribution assumptions are correct, we should reject the offer. But in real life some of our assumptions may be based on estimates made under uncertain conditions; and in some cases one may feel that the expected gain of $ 400 is too small to justify a rejection of the present offer.

The numerical outcome of the calculation may, however, dictate another action under other conditions.

Uniform distribution of low paying cargo over a part of the range.

As an example of a distribution that is concentrated around a certain value, we may mention a uniform distribution that covers only a limited portion of the range under consideration.

Say that our maximum quantity of high paying cargo, i.e. our total range, is A, that the lot size of low paying cargo is uniformly distributed between 3/8 to 5/8 A, and that it so happens that we are offered a shipment of 3/8 A. Let us again assume that the conditions are such that we can use column three in Table III as guidance in the decision making. It appears that any other offer would be as good as or better than the one under consideration, and besides, if any additional offer is added to the one under consideration, we get an expected revenue well below the one we can get by any single offer. Under these circumstances, and if we are certain to receive more offers, we should reject the offer.

Normal distribution of low paying cargo.

Another typical example of a distribution concentrated around a certain value is a normal distribution with a mean equal to 1/2 and a variance equal to 1/4 of A. For the sake of simplicity let us again assume that we may use column three in Table III. If we then are offered a shipment of low paying cargo measuring 1/4 of the available space, should we accept it ?

An inspection of the table reveals that, if the total quantity of low paying cargo is between 3/8 and 5/8 of the available space, the expected revenue will be near its maximum.

This range may again be reached either by accepting the offer and hoping for additional offers that will bring the total quantity up in the desired range, or by rejecting the offer, hoping for a new offer that will be in the desired range.

In the first case we need an additional offer in the range of 1/8 to 3/8, in the second case an offer in the range 3/8 to 5/8 A. By using the described normal distribution, we find the probability that one offer will fall in the desired range is respectively 0.229 and 0.497.

If we are certain to get at least one more offer, the expected revenue becomes not less than:

If we accept the offer, $E(R) = (0.615 + 0.035 \cdot 0.229)\, q A \quad = \underline{0.623\, qA}$

If we reject the offer, $E(R) = (0.5 + 0.15 \cdot 0.497)\, q A \quad = \underline{0.575\, qA}$

If we are certain to get five more offers, the probability of <u>not</u> getting one offer in the 1/8 to 3/8 range is $(1 - 0.229)^5 = 0.272$, and the probability of <u>not</u> getting an offer in the 3/8 to 5/8 range is $(1 - 0.497)^5 = 0.032$. The corresponding expected revenues become not less than:

If we accept the offer:

$$E(R) = (0.615 + 0.035\,(1 - 0.272))\, q A \qquad = \underline{0.640\, q A}$$

If we reject the offer:

$$E(R) = (0.5 + 0.15\,(1 - 0.032))\, q A \qquad = \underline{0.645\, q A}$$

Again the difference is so small that we may as well accept the offer. This seems to lead to the conclusion that one should accept an offer as long as the optimum quantity is not exceeded. If one does not have sufficient information to perform a calculation, one may use this as a rule of thumb provided that the case is not extreme in any direction.

Let us complete the study by investigating what happens if we get an offer which is greater than the optimum quantity. Say that we are offered a low paying lot equal to three quarters of A, and that we again may use column three of Table III.

If we accept, the expected revenue will be 0.615 qA. If we refuse, there is a probability of 0.497 that each additional offer will fall in the 3/8 A to 5/8 A range and increase the expected revenue to at least 0.65 qA. The calculation becomes:

1) If we are certain to get one more offer,

   If we accept the offer, $E(R) =$ $\underline{0.615\ qA}$

   If we reject : $E(R) = (0.5 + 0.15 \cdot 0.497)\ qA$ $= \underline{0.575\ qA}$

   As before, the offer should be accepted.

2) If we are certain to receive five more offers:

   If we accept the offer $E(R)$ $= \underline{0.615\ qA}$

   If we reject, $E(R) = (0.5 + 0.15\ (1-0.32))\ q\ A$ $= \underline{0.645\ qA}$

Here, there is moderate indication that we should reject the offer.

CONCLUSION

It is possible to estimate the expected revenue for any cargo combination provided one has sufficient statistics for past cargo offers and freight rates.

The expected revenues (hence profits) will vary only modestly over a wide range above or below the optimum quantity of low paying cargo. Therefore it is not important to take only exactly optimum amounts of low paying cargo.

If one has no exact information on the distribution of the lot size of low paying cargo, one may, as a rule of thumb, accept offers of low paying cargo as long as the optimum quantity is not exceeded.

ACKNOWLEDGEMENT

I want to express my gratitude to Mr. Tor Segaard and Professor Harry Benford, who encouraged me to make this study. The original idea was conceived in correspondence with Mr. Segaard. Professor Benford took the trouble to go through my manuscripts and has made much effort in improving the language and in inspiring me to complete the work.

Ann Arbor/Oslo 1971

Stian Erichsen

APPENDIX I

CALCULATION OF EXPECTED REVENUE WITH A UNIFORM DISTRIBUTION OF HIGH PAYING CARGO (A SPECIFIC CASE)

Expected revenue	$= E(R)$
Quantity of low paying cargo	$= x$
Quantity of high paying cargo	$= y$
Net revenue on x	= \$0. 15 per cu ft
Net revenue on y	= \$0. 30 per cu ft
x and y are booked for a maximum allocated space of 80, 000 cu ft	

Distribution of y: uniform between 0 and 80,000 cu ft :

$P(y) = \frac{1}{80,000}$, $y = 1 - 80,000$; $P(y) = 0$, elsewhere

$$E(R) = 0.15\,x + \int_{0}^{(80,000-x)} \frac{0.30\,y}{80,000}\,dy + \int_{(80,000-x)}^{80,000} \frac{0.30}{80,000}\,(80,000 - x)\,dy$$

$$E(R) = 0.15\,x + \Big[\frac{0.30}{80,000}\;\frac{y^2}{2}\Big]_{0}^{(80,000-x)} + \Big[\frac{0,30}{80,000}\,(80,000 - x)\,y\Big]_{(80,000-x)}^{80,000}$$

$$= 0.15\,x + \frac{0.30}{80,000}\left[\frac{(80,000-x)^2}{2} + (80,000 - x)\,80,000 - (80,000 - x)^2\right]$$

$$= 0.15x + \frac{0.30}{8 \cdot 10^4}\left[32 \cdot 10^8 - \frac{x^2}{2}\right] = 12,000 + 0.15\,x - \frac{0.30}{16 \cdot 10^4}\,x^2$$

This formula has been solved for different values of x shown in Table I.

APPENDIX II

CALCULATION OF EXPECTED REVENUE WITH A UNIFORM DISTRIBUTION OF HIGH PAYING CARGO (THE GENERAL CASE)

Expected revenue	$= E(R)$
Quantity of low paying cargo	$= x$
Quantity of high paying cargo	$= y$
Net revenue on x, \$/cu ft	$= q$
Net revenue on x, \$/cu ft	$= p$
Maximum expected quantity of y	$= A$
exp[x]	$= e^{x}$

$$E(R) = p \cdot x + \int_{0}^{(A-x)} q \cdot \frac{y}{A}\, dy + \int_{(A-x)}^{A} \frac{q}{A} (A-x)\, dy$$

$$= p\,x + \left[\frac{q}{A} \frac{y^2}{2}\right]_{0}^{(A-x)} + \left[\frac{q}{A} (A-x)\, y\right]_{(A-x)}^{A}$$

$$= p\,x + \frac{q}{A}\left[\frac{(A-x)^2}{2} + A^2 - Ax - (A-x)^2\right]$$

$$= px + \frac{q}{A}\left[A^2 - Ax - \frac{A^2}{2} + Ax - \frac{x^2}{2}\right]$$

$$= px + \frac{q\,A}{2}\left[1 - \frac{x^2}{A^2}\right]$$

APPENDIX III

CALCULATION OF EXPECTED REVENUE WITH A NORMAL DISTRIBUTION OF HIGH PAYING CARGO (THE GENERAL CASE)

The symbols are the same as in Appendix II. It is assumed that the high paying cargo is normally distributed about a mean of A/2, with a standard deviation of A/5. This is a fairly good approximation of the histogram that appears in Figure 3. The E(R) integral becomes the following:

$$E(R) = p\,x + q\int_0^{A-x} \frac{1}{\sqrt{2\pi}\,A/5} \cdot \exp\left[-1/2\,\frac{(y - A/2)^2}{(A/5)^2}\right] y\,dy$$

$$+ q\int_{A-x}^{A} \frac{1}{\sqrt{2\pi}\,A/5} \cdot \exp\left[\frac{-1/2\,(y - A/2)^2}{(A/5)^2}\right] (A - x)\,dy$$

Setting $t = \dfrac{y - A/2}{A/5}$

$$E(R) = p\,x + q\Bigg[\int_0^{A-x} \frac{A/2}{\sqrt{2\pi}} \cdot \exp\left[\frac{-t^2}{2}\right] dt$$

$$- \int_0^{A-x} \frac{A/5}{\sqrt{2\pi}} \cdot \exp\left[\frac{-t^2}{2}\right] + \int_{(A-x)}^{A} \frac{(A - x)}{\sqrt{2\pi}} \cdot \exp\left[\frac{-t^2}{2}\right] dt\Bigg]$$

The values of this expression may be found from tables for the normal distribution. The solutions for different values of x may be found in Table III.