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RELATIONSHIPS BETWEEN THE PROFITABILITY AND SAFETY OF SHIPS

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by

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SYMBOLS AND ABBREVIATION

A	Future amount of money
a(t)	Continuous receipts
B	Present worth of future amounts of money
E[g]	Value of expectation of the return or expectation of the return
E[u(g)]	Expectation of the utility
e	Revenue
F(t)	Failure distribution function
f(t)	Failure distribution density
g	Return $g = e - k$
I	Investment
i	Annual interest rate
k	Cost
n	Number of years
R	Probability of safety or reliability
r	$r = \ln(1 + i)$
S	Equivalent of certainty
Sc	Salvage value at end of life T of a ship
T	Life of ship
t	Time
u(g)	Utility function
V	Resultant loss to the shipowner due to collision of his ship
V_n	Resultant loss due to a collision, where the ship remains afloat
V_s	Resultant loss due to a collision, where the ship sinks, $V_s = V_L + V_E$

V_L	Loss of cargo, cost of home transport of crew, etc.
V_E	Cost of replacement ship
W	Probability that a ship will not sink after a collision
$\bar{A}$, $\bar{B}$, $\bar{V}$, etc.	Values of expectation of A, B, V, etc. $\bar{A} = E[A]$, $\bar{B} = E[B]$, $\bar{V} = E[V]$
λ	Failure rate
$\bar{\lambda}_o$	Average number of collisions per year
φ	Density distribution of the loss V
φ_n	Density distribution of the loss V_n, which occurs if the ship does not sink
φ_s	Density distribution of the losses, V_s, which occur if the ship sinks

I

Ships are, generally speaking, a very secure means of transport. Nevertheless, from a modern technological point of view they are not as secure as they might be. Thus, freighters are generally less able to withstand flooding than are passenger vessels of equal size. At the same time, a higher standard of safety could be achieved for many passenger ships. The same may be said about securing a ship against capsizing: If one compares, for example, the pertinent requirements of some navies with current practice in merchant ship design, one comes to the conclusion that merchant ships are less safe in this regard, too.

The reason for this state of affairs is certainly not that technological possibilities are overlooked or that mistakes are actually made; the question is, rather, one of compromise between the requirements of safety and profitability. Unfortunately, it must be said, one rarely mentions this, in my opinion, necessary compromise. This reluctance is understandable. Thus, it would be considered bad advertising to say of a passenger ship that safety had to be sacrificed to keep passenger fares at a low level. Preferred practice in this case would be to talk of maximum safety without being specific.

Yet, in principle, there is no reason not to talk about the relationship between profitability and safety. There is an often voiced but invalid argument that ship safety is concerned mainly with the protection of human life, which cannot be given a monetary value, and consequently, that questions of safety should not be viewed from the standpoint of economics. In actuality it is exactly because human lives are irreplaceable that we have to examine carefully how to attain the greatest feasible safety whenever it is not possible to provide the maximum physically possible safety. This is true in most practical cases. An optimization in this sense, however, is only possible if the relationships between safety and profitability are clearly understood.

One frequently finds the opinion that safety cannot be reconciled with profitability. The idea is that it is indeed possible to calculate the necessary expenditures for a certain standard of safety but that it is impossible to calculate the profit resulting from this higher safety. This opinion could not be contradicted as long as safety was considered an intangible factor. In the recent past, however, safety has been made more amenable to quantitative analysis (see reference 1 and additional reference cited in reference 1). It is possible now to take the next step in trying to find quantitative relationships between safety and profitability. In the following pages methods of finding such relationships are demonstrated. We start with very simple cases and then gradually treat somewhat more complex questions.

Quantitative statements about safety are probability statements (reference 1). If a consideration of profitability follows, this leads to so-called risk situations. These have been treated in detail in numerous papers (e.g., references 2-5). Some familiarity with this field will be presupposed in the following.

II

2.1 Let us consider the simple case of a rocket: If it reaches the target, this means a revenue, e, for the company which produces the rocket. Let k(R) be the cost for the production of the rocket; k is a function of the probability, R, that the rocket will reach its target. Hence, R is a measure of the "safety" that the rocket will work properly and therefore is also called the probability of "safety," or reliability. As a higher reliability also causes higher expenditures, k(R) is a function which increases monotonously with R. If we have a complicated rocket (e.g., in space technology), there will be an upper limit for R, the so-called "safety barrier," which will be less than 1. This limit is set by the present state of our knowledge and cannot be exceeded, no matter how high our efforts and expenses are.* Figure 1 shows an example of the function k(R).

We now have to answer the question of what amount of expenditures, k, is justified for the safety, R, or in other words, which probability of "safety" is, under the given circumstances, the most favorable one for the company that makes the rocket. Let us consider the following: If the rocket works properly, the return is g = e - k; if it does not reach the target, we have a loss of the magnitude k, which also can be denoted as the negative return, g = -k. The probability for the

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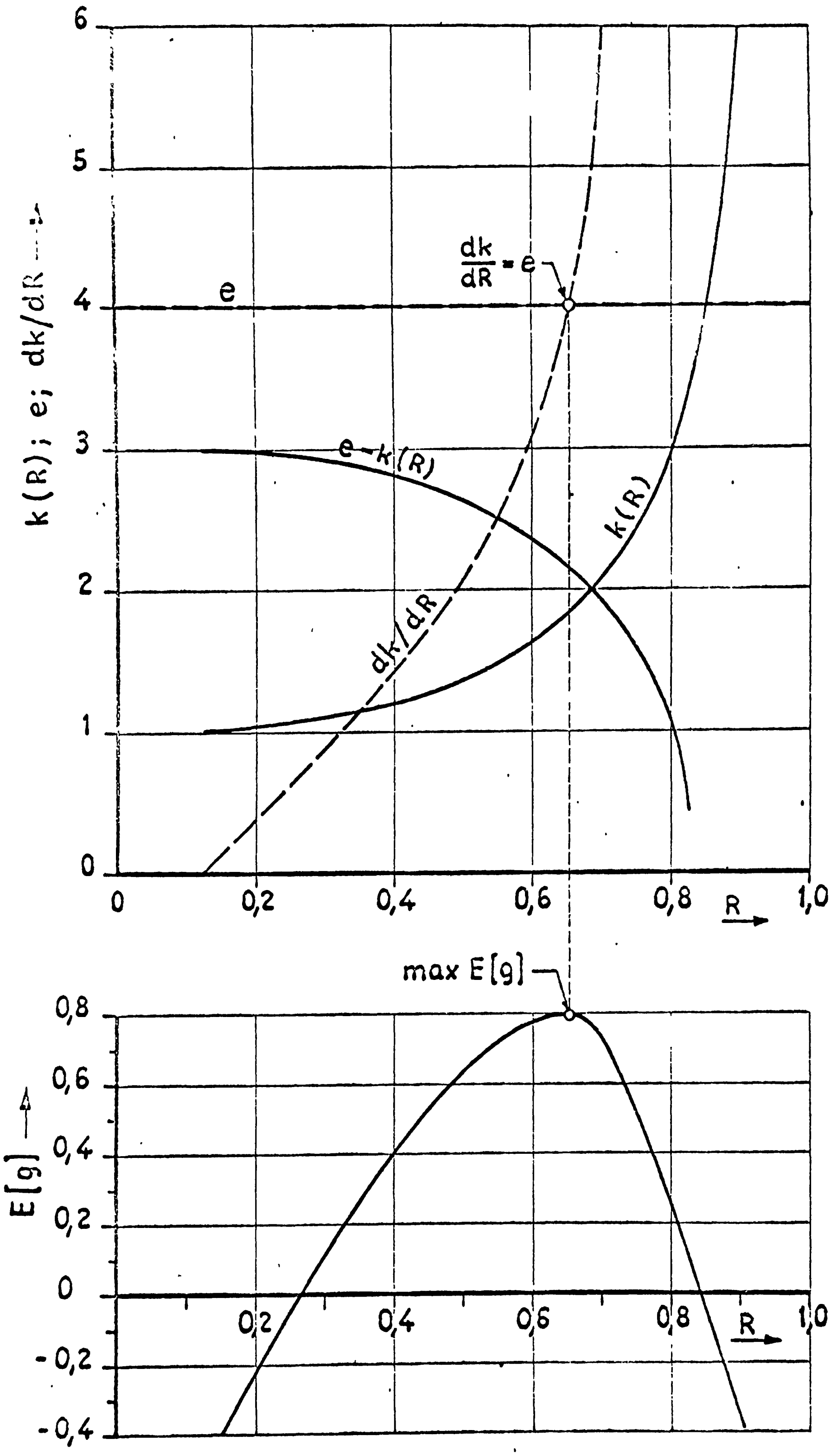
*It will be presumed here that time for the construction of the rocket is limited. If monetary resources and time were unlimited, the "safety barrier" could be shifted more and more toward 1 in the course of time.



6
5
4
3
2
1
0
k(R); e; dk/dR
e
dk/dR = e
e-k(R)
k(R)
dk/dR
0
0,2
0,4
0,6
0,8
1,0
R
max E[g]
0,8
0,6
0,4
0,2
0
-0,2
-0,4
E[g]
0,2
0,4
0,6
0,8
R
1,0

Fig. 1

first event is R; for the second event, 1 - R. A measure of the merit that can be achieved, considering these chances, is the expected value of the return. It is

$$E[g] = (e - k(R)) \cdot R - k(R) \cdot (1 - R) .$$

The function E[g] is drawn in the lower part of Figure 1. The condition that E[g] has a maximum is

$$\frac{dE[g]}{dR} = 0 ,$$

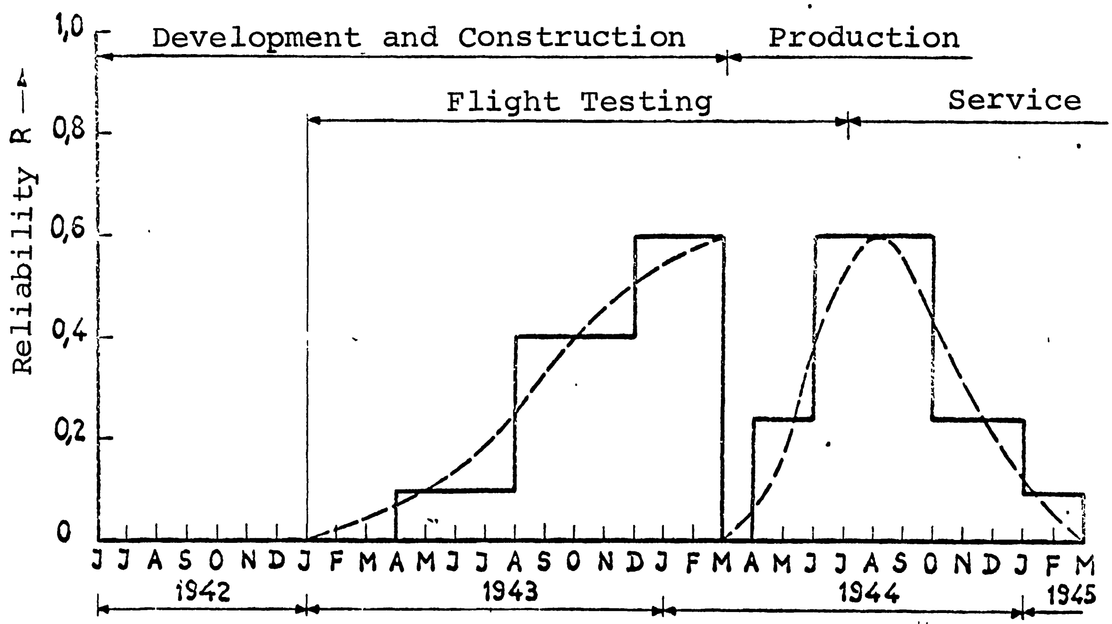
and after substitution of the corresponding magnitudes (see also Figure 1),

$$\frac{dk(R)}{dR} = e .$$

This result shows that the optimal value of the probability of safety depends only on the revenue, e, and on the cost slope and that it is independent of the absolute value of the cost. The latter, however, becomes important if we want to determine the maximum expected value of the return and whether or not it possibly represents a loss.

This example only served to illustrate a way of determining the optimal safety in a simple case, and the questions of if and how we can determine the revenue, e, and the cost, k, and its dependence on the probability of safety was of minor importance in this context. Figure 2 from reference 6, however, shows that it is also practically possible to determine the probability of safety. If R versus cumulative cost had been plotted in Figure 2 instead of R versus time, this would have shown the relationship between k and R.

2.2 In the previous example, the expected value of return had been optimized. This is an approximation



<u>Reliability of V-1 Rockets.</u> The graph shows the increase in reliability achieved by constructive improvements until February 1944. At the start of mass production the reliability dropped to zero. The standard reached by June 1944 through improvements in production methods could not be maintained because of the unfavorable change in the military situation.

Fig. 2

which gives useful results only in cases where the possible loss, k, as well as the possible return, e - k, are small in relation to the assets of the company that produces the rocket. If, for example, the possible loss, k, were so high that it could ruin the company, it would mean much more to the company than its cost in dollars. Many companies would not risk such a loss, which after all has the probability (1 - R), even if the return they could receive with the probability of R would be extraordinarily high. They would not accept the risk of an even somewhat smaller loss and the associated troubles unless they had the chance of a very high return.

This manner of decision-making shows that in situations of risk it is not return and loss that are directly balanced against each other, but the associated utility and damage (which may also be expressed as negative utility). In terms of mathematics this means that it is not the expectation of the return that matters, but the expectations of the utility.

This idea of utility goes back to Daniel Bernoulli (1738). Essential foundations for the practical applicability of the concept of utility were laid by von Neumann and Morgenstern in their paper which was published in 1944 (reference 7). It also has been possible to show that optimization of the expectation of the utility can be regarded as a rational basis for decision-making (see, for instance, references 2 and 8). More details on the practical determination of the utility function (i.e., the dependence of utility on financial return or loss) and some examples of the utility functions for various companies are found in reference 4.

Let us now determine the optimal probability of safety for the previously treated example, using the utility function as defined by Bernoulli (or von Neumann and Morgenstern). Let the utility function be shown by the curve $u(g)$ in Figure 3. Using this function we can determine the utility of the return, $u(e - k)$, and of the loss, $u(-k)$, as a function of the probability of safety (see Figure 4). Hence, the expectation of the utility will become

$$E[u(g)] = u(e - k) \cdot R + u(-k) \cdot (1 - R) .$$

The lower part of Figure 4 shows $E[u(g)]$ as a function of the probability of safety, R. The maximum of the utility expectation follows from

$$\frac{dE[u(g)]}{dR} = 0$$

or

$$-\left(\frac{du(e - k)}{dR} R + \frac{du(-k)}{dR} (1 - R)\right) = u(e - k) - u(-k) .$$

A comparison of Figure 1 and Figure 4 shows that the optimal value of the probability of safety has hardly been changed by the introduction of the utility conception.

From the value of expectation of the utility we now can calculate the so-called equivalent of certainty, S:

$$S = u^{-1} (E[u(g)]) .$$

In this equation, u^{-1} denotes the inverse function of u. The equivalent of certainty has the following meaning: It is the certain return which is equivalent to the possibilities of the risk case (here: to make a return $(e - u)$ with probability R or to make a loss k with probability $(1 - R)$).

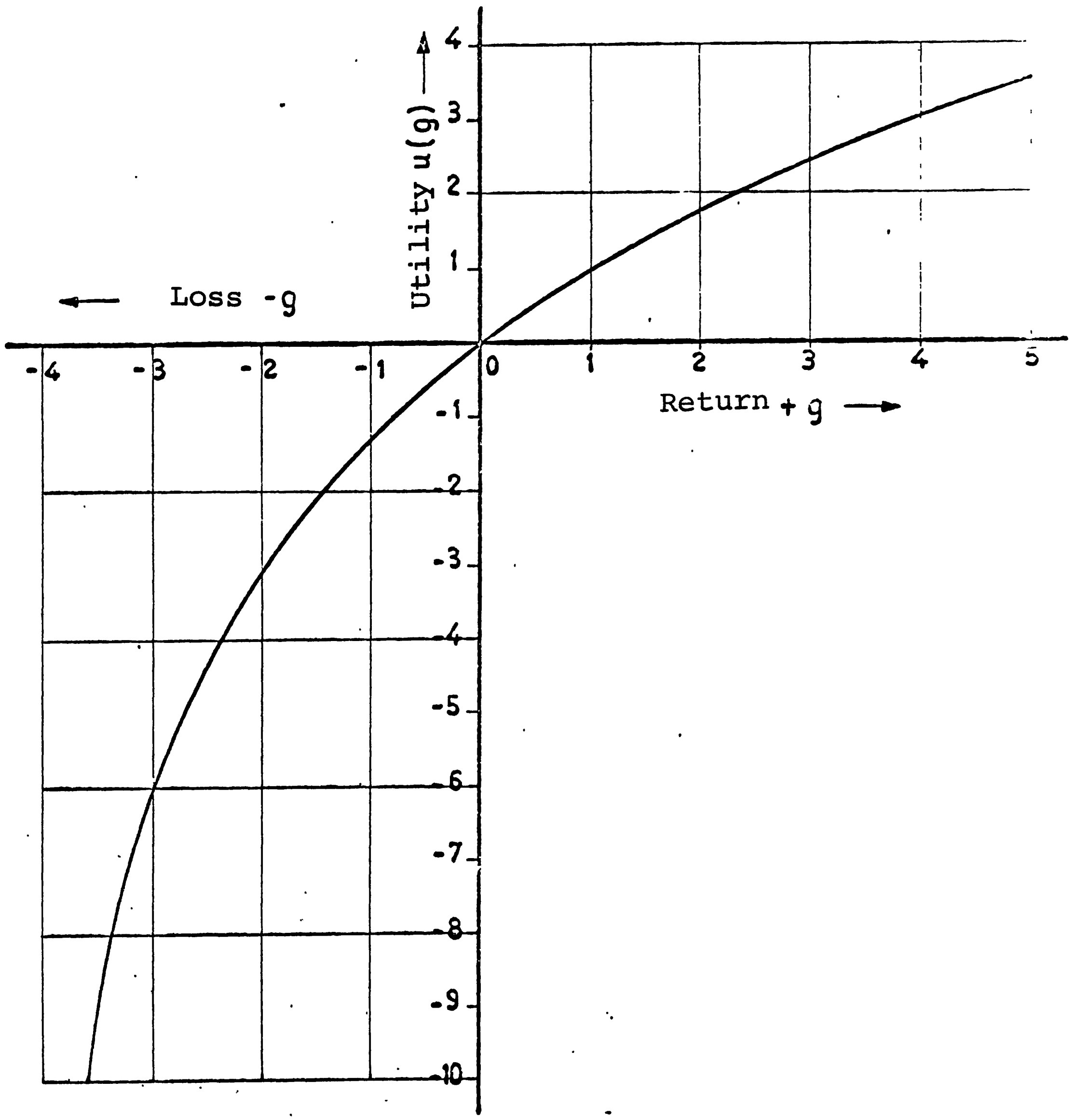
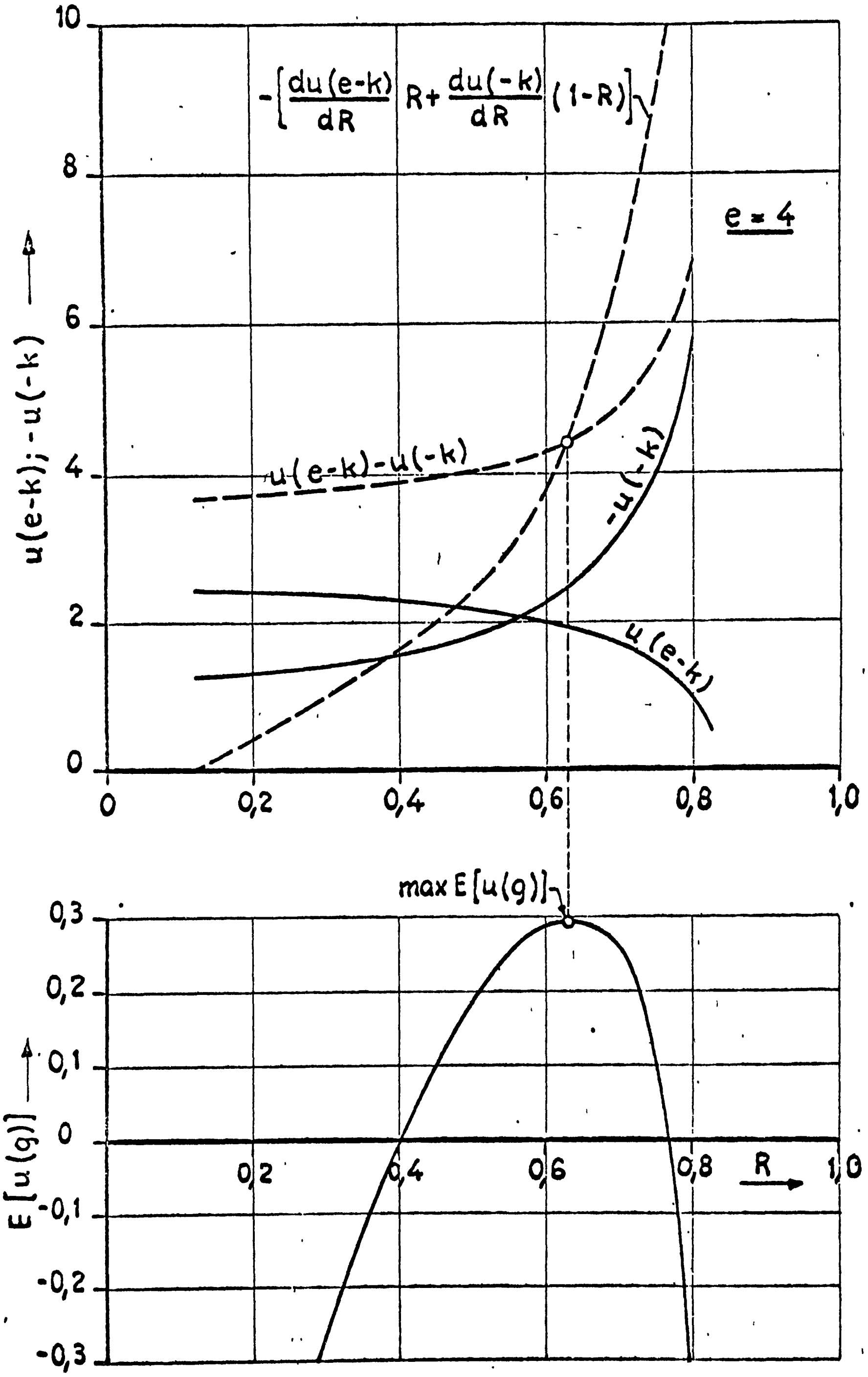


Fig. 3



10
8
6
4
2
0
$-\left[\frac{du(e-k)}{dR}R+\frac{du(-k)}{dR}(1-R)\right]$
e = 4
u(e-k); -u(-k)
u(e-k)-u(-k)
-u(-k)
u(e-k)
0
0,2
0,4
0,6
0,8
1,0
max E[u(g)]
0,3
0,2
0,1
0
-0,1
-0,2
-0,3
E[u(g)]
R

Fig. 4

In our example, the equivalent of certainty, S, is considerably smaller than the expectation of the return, E[g] (see Figure 5). The reason is that we used a utility function that expressed risk aversion.

If the revenue, e, due to the successful performance of the rocket is smaller, the equivalent of safety can become negative even if R is chosen optimally. This means that under the present utility concept of the company, the production of the rocket is equivalent to a certain loss. This statement could not have been made using the expectation of the return (see also Figure 6). So far it has been shown by a simple example that it is possible to measure the economical consequences caused by the uncertainties in the performance and that safety and profitability can be reconciled.

2.3 At the end of this paragraph we want to quote the reasons which allowed us to deal with this example in such a simple manner: It was assumed that return or loss occur at a certain point of time, namely, after the flight of the rocket.* At this point of time there were only two possibilities: either the rocket would work properly and would reach the target or it would not. Dealing with questions of ship safety is a more sophisticated matter. For example, an engine can fail not only once but several times during the life of a ship. The time of failure is always random. The consequences of a failure can be very different too. A ship which is no longer maneuverable may be lost with all its cargo, or during salvage or repair perishable cargo may spoil. But it is also possible that the failure results only

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* In reality, production costs are spread over a longer interval of time; by taking interest into account, however, they could be converted to any appropriate point of time.

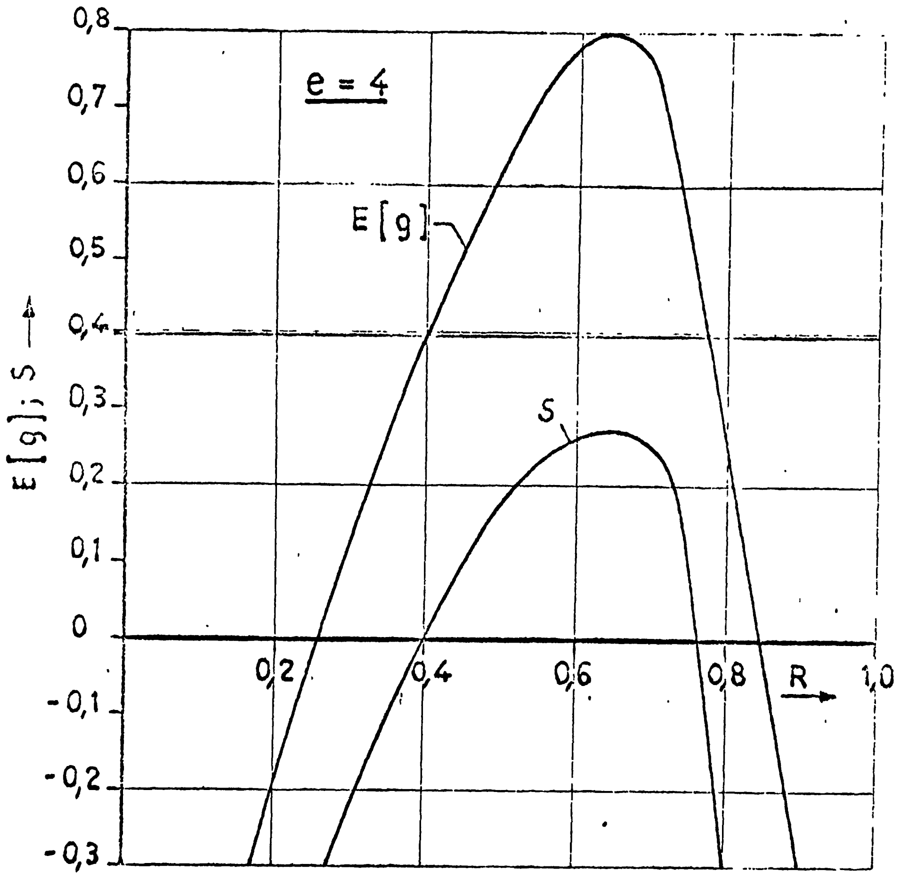

e = 4
E[g]
S
E[g]; S
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
-0,1
-0,2
-0,3
0,2
0,4
0,6
0,8
R
1,0

Fig. 5

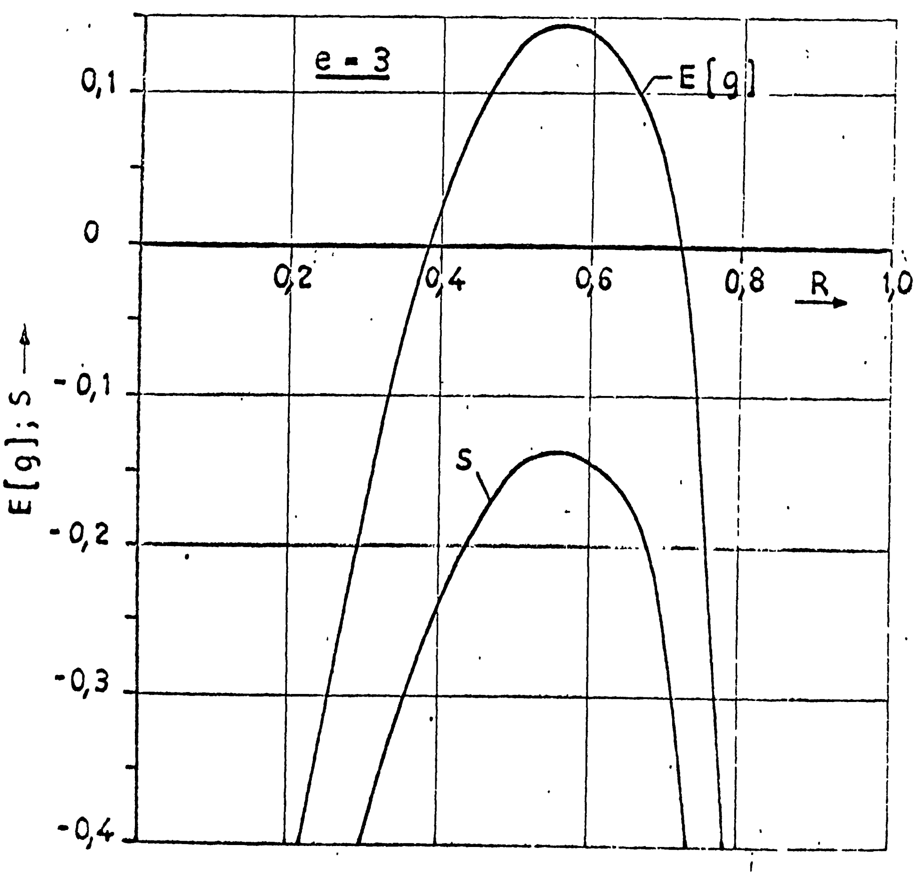

e = 3
E[g]
S
E[g]; S
0,1
0
-0,1
-0,2
-0,3
-0,4
0,2
0,4
0,6
0,8
R
1,0

Fig. 6

in a time loss of hours or a few days. In the following paragraphs we will have a closer look at how to deal with these circumstances.

III

3.1 In this section we will try to find out how the fact that a failure may occur at any random future time influences the profitability of a ship. At first we have to deal with the question of how profitability is to be defined. There are quite a lot of criteria of profitability today. Which one (or ones) we have to choose in each single case depends on the goals of the shipping company, its organization and the subjective preferences of its leading managers (for more details on this subject see, for instance, references 9 and 10). The use of present worth is very common as an economic measure of merit. Its application may be shown in a simple example: disbursements and receipts for a ship are due at different times (see Figure 7). At first an amount, I, has to be invested to get a ship. After this disbursement there will be certain returns, A, in the following time. They are the difference between the revenues for the transport of cargo and the operating cost. In the evaluation of these returns it is important to consider at what time they occur. For example, an amount which is due in ten years has less value than the same amount due immediately. The latter could be invested and would yield a certain interest during these ten years. If we consider an annual interest rate, i, the present amount, B, would be equal in ten years to the amount

$$A = B(1 + i)^n .$$

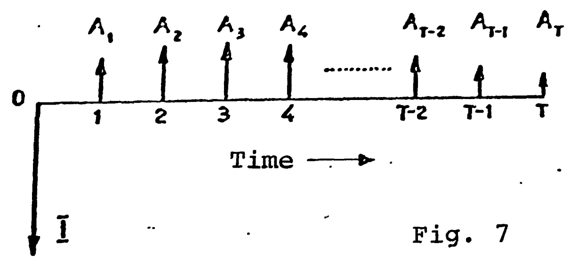


Fig. 7

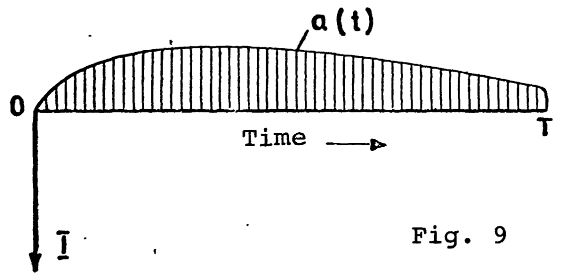


Fig. 9

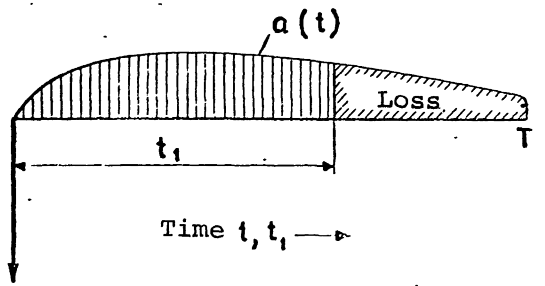


Fig. 10

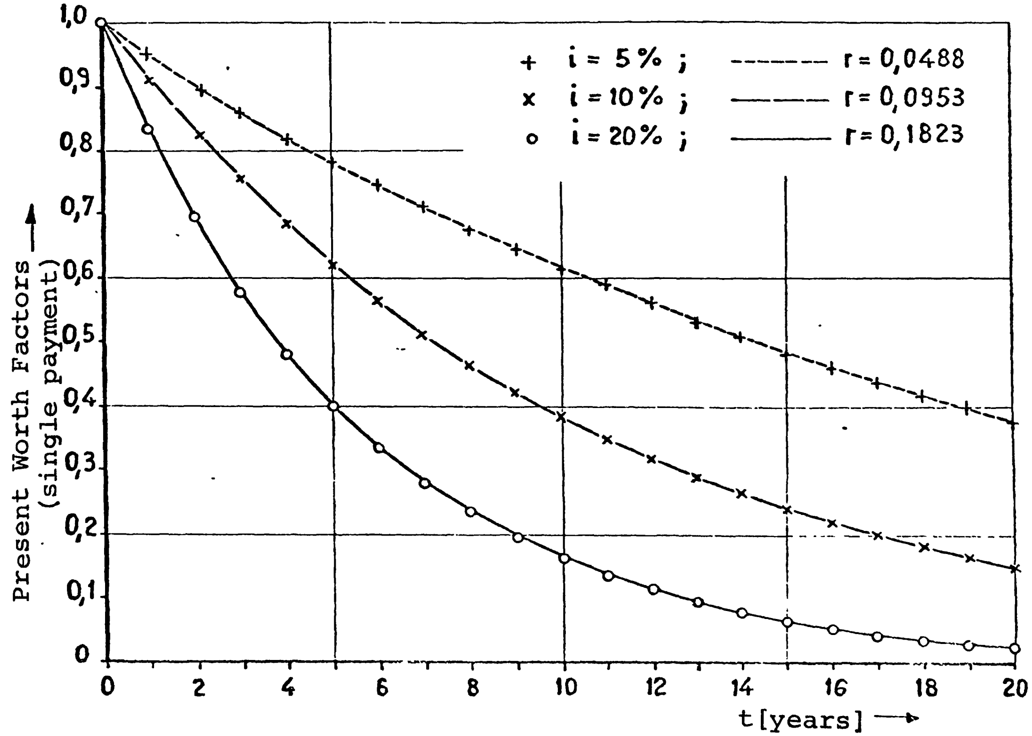


Fig. 8

Conversely, an amount A, n years hence, would have a correspondingly smaller present worth of

$$B = A(1 + i)^{-n} .$$

We denote B as the present worth of the future amount A. The calculation of B from a given amount A is also known as discounting of A, and the term $(1 + i)^{-n}$ is called the present worth factor.

For the following considerations it is useful to use an interest rate based on continuous compounding and equivalent to the annually compounded interest rate, i. In this case

$$B = Ae^{-rt} ,$$

where

$$r = \ln(1 + i)$$

and t is taken in years. Whereas n meant discrete instants of time in units of one year each (i.e., n = 1, 2, 3, ... T years), t denotes continuous time, $0 \leq t \leq T$, where T equals the life of the ship (see Figure 8). If allowance is made for the fact that in annual compounding linear interpolation between two succeeding years is usual, it will be realized that for our purposes both kinds of discounting are completely equivalent. To further simplify calculations, we will replace the discrete receipts, A, by continuous receipts per unit time a(t) (similar to the conversion of many discrete forces to a uniformly distributed load). The investment, I, however, continues to be regarded as a discrete amount although it is spread over a certain space of time too. Compared with the life of a ship, however, this time interval is very small. The error due to these simplifications will certainly be smaller than all the inaccuracies which cannot be avoided in such calculations.

With all these assumptions the cash flow shown in Figure 7 can now be represented as shown in Figure 9. For the present worth, here chosen as a criterion, we get

$$B_o = \int_0^T a(t)e^{-rt}dt - Ie^{-r0} .$$

This equation is valid only in cases where no damage occurs.

3.2 Let us now consider an actual case of damage. The following assumptions are made: at a random time, t_1, the ship is involved in a collision. The ship is assumed to have neither bulkheads nor bilge pumps; so it will sink. From the time of the ship's loss, t_1, until time T (which is the originally chosen life of the ship) the shipowner does not build a new ship. Under these circumstances the loss caused by the damage is equivalent to the loss of the receipts, a(t), during the time interval $t_1 < t \leq T$. The present worth, B_v, of this loss naturally depends on the time at which the damage occurs, t_1. Therefore, it is a function of t_1:

$$B_v(t_1) = \int_{t_1}^T a(t)e^{-rt}dt \qquad \text{for } t_1 < T$$

$$B_v(t_1) = 0 \qquad \text{for } t_1 \geqq T.$$

This leads to a total present worth of

$$B(t_1) = B_o - B_v(t_1) = \int_0^{t_1} a(t)e^{-rt}dt - I \qquad \text{for } t_1 < T$$

$$B(t_1) = B_o = \int_0^T a(t)e^{-rt}dt - I \qquad \text{for } t_1 \geqq T.$$

These relations may be illustrated by the following example. Letting a(t) = constant = a, we have

$$B_v(t_1) = \frac{a}{r}(e^{-rt_1} - e^{-rT}) \qquad \text{for } t_1 < T$$

$$B_v(t_1) = 0 \qquad \text{for } t_1 \geqq T$$

$$B(t_1) = \frac{a}{r}(1 - e^{-rt_1}) - I \qquad \text{for } t_1 < T$$

$$B(t_1) = \frac{a}{r}(1 - e^{-rT}) - I \qquad \text{for } t_1 \geqq T.$$

Figure 11 shows $B(t_1)$ versus t_1. Although it is fairly obvious, it should be pointed out again that this is not the representation of a cash flow, but the representation of all possible present worths which can be received from this ship as a measure of merit for its economic success. Which one of these present worths will actually occur is random. Yet, the following statement can be made: The probability that the actual present worth will lie between $B(t_1)$ and $B(t_1 + dt_1)$ must be equal to the probability that the collision occurs between t_1 and $t_1 + dt_1$.

Reference 1, paragraph 5.1, shows the density distribution of time where a collision occurs:

$$f(t_1) = \bar{\lambda}_o e^{-\bar{\lambda}_o t_1} \qquad \text{for } t_1 \geqq 0$$

$$f(t_1) = 0 \qquad \text{for } t_1 < 0 ,$$

where $\bar{\lambda}_o$ is the average number of collisions per year and per ship exposed to the risk of collision.

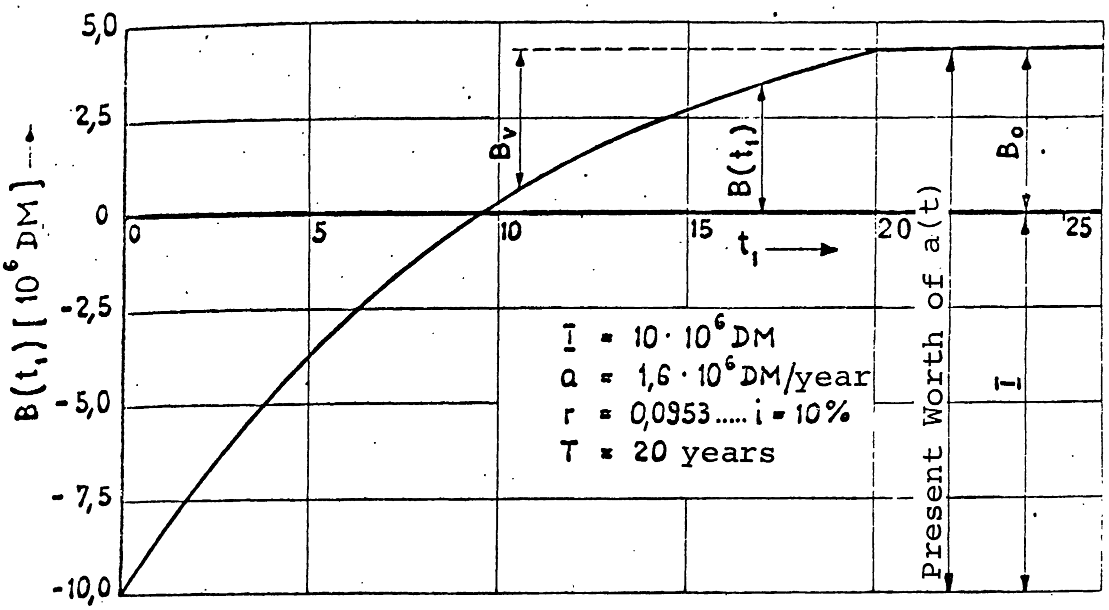

5,0
2,5
0
-2,5
-5,0
-7,5
-10,0
B(t₁) [10⁶ DM]
0
5
10
15
20
25
t₁
Bv
B(t₁)
Bo
Present Worth of a(t)
I
I = 10 · 10⁶ DM
a = 1,6 · 10⁶ DM/year
r = 0,0953..... i = 10%
T = 20 years

Fig. 11

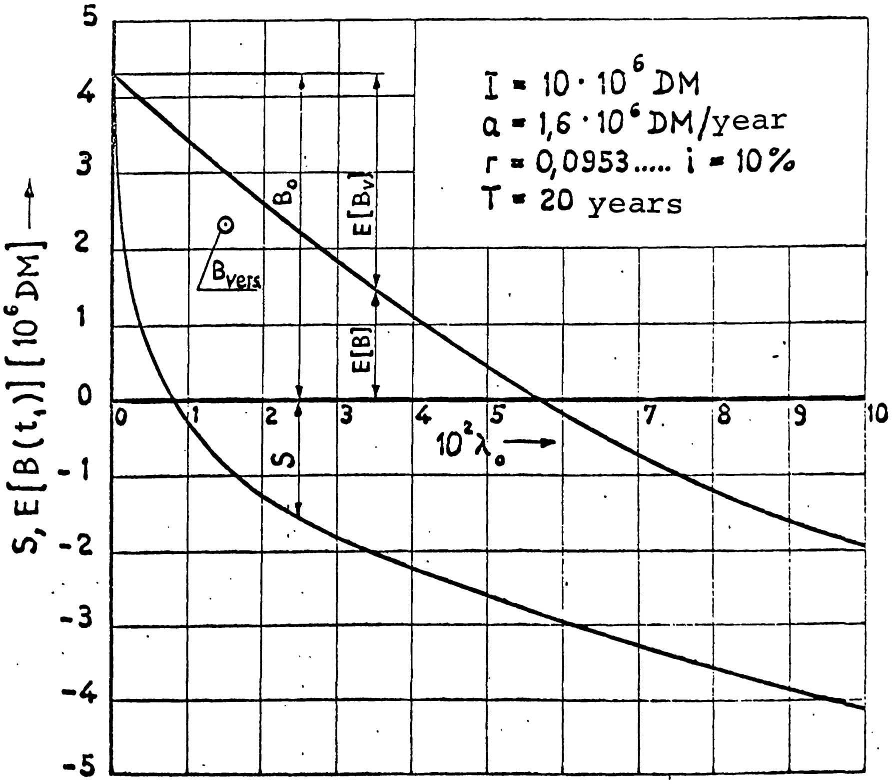

5
4
3
2
1
0
-1
-2
-3
-4
-5
S, E[B(t₁)] [10⁶ DM]
0
1
2
3
4
5
7
8
9
10
10²λ₀
I = 10 · 10⁶ DM
a = 1,6 · 10⁶ DM/year
r = 0,0953..... i = 10%
T = 20 years
Bo
E[Bv]
E[B]
S
Bvers

Fig. 12

We are now able to calculate the expectation of the present worth:

$$E[B(t_1)] = \int_0^\infty B(t_1) \cdot f(t_1)dt_1 \ .$$

For the above example we get

$$E[B(t_1)] = \frac{a}{r}\left(1 - \frac{\bar{\lambda}_0}{r + \bar{\lambda}_0}\right)\left(1 - e^{-(\bar{\lambda}_0 + r)T}\right) - I \ .$$

This result has been plotted versus $\bar{\lambda}_0$ in Figure 12.

3.3 For the shipowner the expectation of the present worth is of minor significance. The amount of money involved is so great that in most cases it will not be possible to assume the utility to be proportional to the return or the loss. The collision of a comparatively new ship would, under the given assumptions (no insurance!), mean the ruin of the shipowner. Instead of calculating the expectation of the present worth, B, it therefore would be better to determine the expectation of the utility, u(B). If the utility function of the shipowner is known (see the corresponding notes in Section 2.2), the value of expectation of the utility is

$$E[u(B(t_1))] = \int_0^\infty u(B(t_1)) \cdot f(t_1)dt_1 \ .$$

This gives us the possibility of calculating the equivalent of certainty. This is the present worth (occurring with certainty), which is equivalent to the possible random present worths:

$$S = u^{-1}\,(E[u(B(t_1))])\ ,$$

where u^{-1} is the inverse function of u. Let us now assume in our example that the utility function of the shipowner may be sufficiently approximated by the following function:

$$u(B) = 0.625 - 0.4(1.25 - B)^2 \qquad \text{for } B \leq 1$$

$$u(B) = 0.6 + 0.2(B - 1) \qquad \text{for } B > 1,$$

where B has to be taken in millions of DM. The equivalent of certainty, S, calculated for this example has been plotted in Figure 12.

3.4 We are now able to solve, for example, the following optimization problem: By better nautical equipment of the ship we can decrease the collision rate $\bar{\lambda}_0$. Yet, at the same time we have to increase the investment, I, for this kind of equipment by the additional cost ΔI. What expenditure ΔI will give us a maximum equivalent of certainty, S?

Thus far, our analysis is based on rather unrealistic suppositions. Let us therefore avoid further treatment of this question now. We will pick up this subject later and discuss it under more realistic suppositions. We will instead draw some conclusions which, in principle, are independent of our previously made assumptions.

Figure 12 shows that the equivalent of certainty, S, versus the collision rate, $\bar{\lambda}_0$, decreases very quickly at the beginning and becomes negative at relatively small values of $\bar{\lambda}_0$. This may be interpreted in the sense that the possible high losses in shipping business may easily discourage a cautious businessman. In reality, however, the shipowner has the possibility of shifting the risk to an insurance company. If an insurance company replaces the loss of the shipowner in exchange for his payment of a premium with the present worth, B_p, the resulting present worth for the shipowner will be

$$B_{Vers} = B_o - B_p \ .$$

This present worth is a determinate amount independent of when or if he loses the ship. Therefore, this present worth, B_{Vers}, equals its equivalent of certainty. The insurance company has to calculate the premium such that B_p equals the expected value of the loss, B_V, plus the insurance company's overhead and profit. This results in a somewhat smaller value of B_{Vers} than the expectation value $E[B(t_1)]$ shown in Figure 12.

If we take an annual premium of 2 percent of the investment, as an example, we get

$$B_p = \int_0^T 0.02I \cdot e^{-rt} dt \ .$$

Taking this result into account and assuming a collision rate $\bar{\lambda}_o = 1.5 \cdot 10^{-2}$ (i.e., out of 100 ships, 15 ships will, on the average, collide within 10 years), and using the results of the previously discussed example, we get the value of B_{Vers}, shown in Figure 12. We realize that despite all the simplifications we made, at least the order of magnitude of B_{Vers} is correct.

IV

4.1 Let us consider a ship's engine plant. Its failure causes a loss, A. It may be that the plant does not fail during the time interval from 0 to T, which represents the life of the ship; or maybe it fails after a time t_1 (where $0 < t_1 < T$); or it fails for the first time after t_1 and after t_2 for the second time (where $0 < t_1 + t_2 < T$), etc. (See Figure 13.)

At first we compute the present worths for the respective losses. They are a function of t_1, t_2, t_3, etc.

$$
\begin{array}{lll}
B_{A1}(t_1) & = Ae^{-rt_1} & \text{for } 0 < t_1 \leqq T \\
B_{A1}(t_1) & = 0 & \text{for } \quad t_1 > T \\
B_{A2}(t_1, t_2) & = Ae^{-r(t_1+t_2)} & \text{for } 0 < t_1+t_2 \leqq T \\
B_{A2}(t_1, t_2) & = 0 & \text{for } \quad t_1+t_2 > T \\
B_{A3}(t_1, t_2, t_3) & = Ae^{-r(t_1+t_2+t_3)} & \text{for } 0 < t_1+t_2+t_3 \leqq T \\
B_{A3}(t_1, t_2, t_3) & = 0 & \text{for } \quad t_1+t_2+t_3 > T,
\end{array}
$$

etc. In order to be able to predict the failure times with greater accuracy, we need the failure density distribution, f(t), of the plant (for additional information see, for

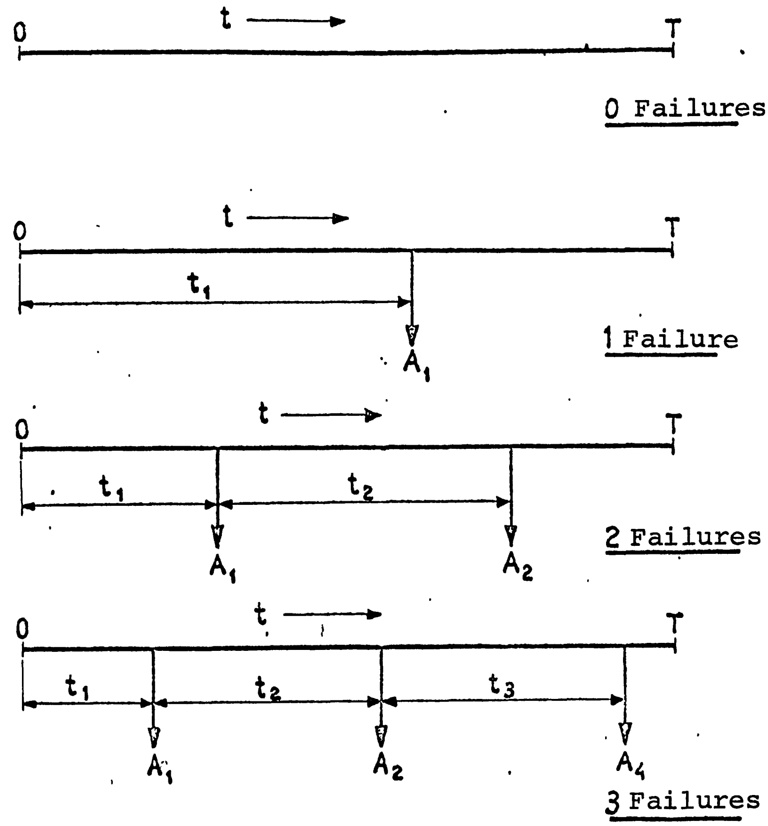

0
t
T
0 Failures
0
t
T
t_1
A_1
1 Failure
0
t
T
t_1
t_2
A_1
A_2
2 Failures
0
t
T
t_1
t_2
t_3
A_1
A_2
A_4
3 Failures

Fig. 13

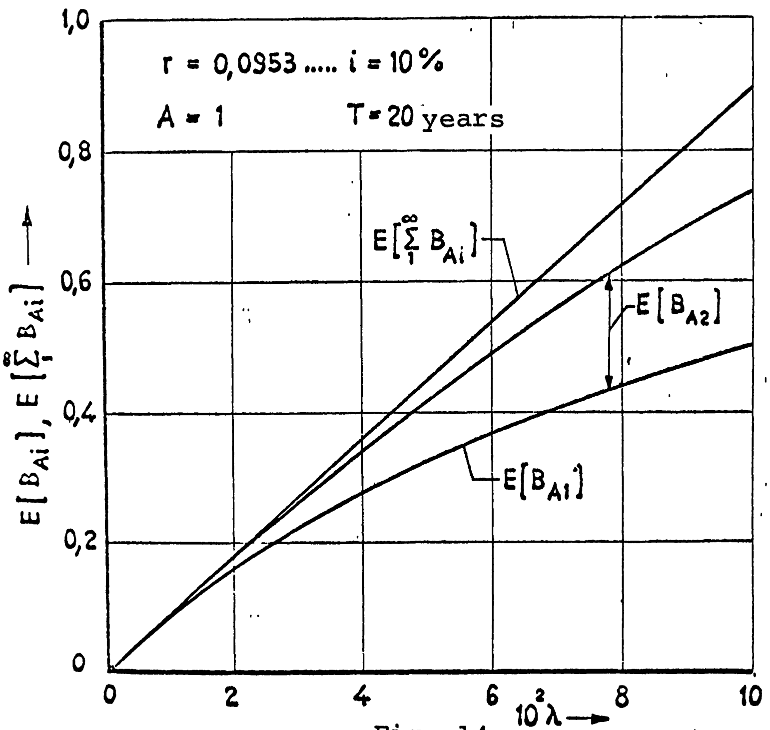

r = 0,0953 ..... i = 10 %
A = 1
T = 20 years
$E[\sum_1^\infty B_{Ai}]$
$E[B_{A2}]$
$E[B_{A1}]$
$E[B_{Ai}]$, $E[\sum_1^\infty B_{Ai}]$
$10^2\lambda$
1,0
0,8
0,6
0,4
0,2
0
2
4
6
8
10

Fig. 14

example, reference 1, especially equations 2 and 5a). Thus, the probability of a failure in the time interval $(t_1; t_1 + dt_1)$ is

$$W\{\text{failure in } (t_1, t_1 + dt_1)\} = f(t_1)dt_1 \ .$$

If one assumes that repair of the plant does not influence the failure density distribution, the following probabilities for two or more failures can be given (for the definition of t_1, t_2, t_3, etc., see Figure 13):

$$W\{\text{a failure in } (t_1, t_1 + dt_1) \ \underline{\text{and}} \\ \text{a failure in } (t_2, t_2 + dt_2)\} = f(t_1)f(t_2)dt_1dt_2$$

$$W\{\text{a failure in } (t_1, t_1 + dt_1) \ \underline{\text{and}} \\ \text{a failure in } (t_2, t_2 + dt_2) \ \underline{\text{and}} \\ \text{a failure in } (t_3, t_3 + dt_3)\} = f(t_1)f(t_2)f(t_3)dt_1dt_2dt_3 \ ,$$

etc. We assume further that the present worth of the losses A_i is relatively small, so that Bernoulli's utility of the present worths may be treated as linear. The safety equivalent of the present worths of the losses is then equal to the expectation of these present worths:

$$E\left[\sum_{i=0}^{\infty} B_{Ai}\right] = \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \ldots (B_{A1}(t_1) + B_{A2}(t_1 + t_2)$$

$$+ \ldots) f(t_1) f(t_2) \ldots (dt_1 dt_2) \ldots .$$

(With respect to the limits of this integral, compare the definition of the B_{Ai} as functions of the respective time intervals.)

4.2 Let us illustrate the above formula by an example. For many well-maintained plants, the assumption of a time independent mean failure rate, λ, is a very useful approximation that serves our purpose. With this assumption the failure density distribution is (see reference 1)

$$f(t) = \lambda e^{-\lambda t} .$$

For the calculation of the preceding integral we use the relationship that the expectation of a sum equals the sum of the expectations of the terms of this sum:

$$E\left[\sum_{i=0}^{\infty} B_{Ai}\right] = E[B_{A1}] + E[B_{A2}] + \ldots ,$$

where

$$E[B_{A1}] = \int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty} \cdots B_{A1}(t_1)f(t_2)\cdots dt_1 dt_2 \cdots$$

$$= \int_{t_1=0}^{\infty} B_{A1}(t_1)f(t_1)dt_1$$

$$E[B_{A2}] = \int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty} \cdots B_{A2}(t_1, t_2)f(t_1)f(t_2)\cdots dt_1 dt_2 \cdots$$

$$= \int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty} B_{A2}(t_1, t_2)f(t_1)f(t_2)dt_1 dt_2 ,$$

etc. If we now replace B_{Ai} with the expressions derived in 4.1 for B_{Ai} and also insert the function of the failure density distribution, we get

$$E[B_{A1}] = \int_0^T Ae^{-rt_1}\ \lambda e^{-\lambda t_1}\ dt_1 = A\frac{\lambda}{\lambda + r}\left(1 - e^{-(\lambda+r)T}\right)$$

$$E[B_{A2}] = \int_0^T\int_0^{T-t_1} Ae^{-r(t_1+t_2)}\ \lambda^2 e^{-\lambda(t_1+t_2)}\ dt_1 dt_2$$

$$= A\left\{\left(\frac{\lambda}{\lambda + r}\right)^2\left(1 - e^{-(\lambda+r)T}\right) - \frac{\lambda}{\lambda + r}\lambda Te^{-(\lambda+r)T}\right\}$$

$$E[B_{A3}] = \int_0^T \int_0^{T-t_1} \int_0^{T-t_1-t_2} Ae^{-r(t_1+t_2+t_3)} \lambda^3 e^{-\lambda(t_1+t_2+t_3)}$$

$$dt_1 dt_2 dt_3$$

$$= A\left\{\left(\frac{\lambda}{\lambda + r}\right)^3\left(1 - e^{-(\lambda+r)T}\right) - \left(\frac{\lambda}{\lambda + r}\right)^2 \lambda T e^{-(\lambda+r)T}\right.$$

$$\left. - \frac{1}{2}\left(\frac{\lambda}{\lambda + r}\right) \lambda^2 T^2 e^{-(\lambda+r)T}\right\},$$

etc. After some simple transformations we get the following simple relationship for the sum of the expectation values:

$$E\left[\sum_1^\infty B_{Ai}\right] = \sum_1^\infty E[B_{Ai}] = A \frac{\lambda}{r}\left(1 - e^{-rT}\right) .$$

Figure 14 shows a graph of this sum and of its first two terms as a function of the failure rate, λ (for $0 < \lambda < 0.1$). The figure shows that the sum converges quickly, especially for small failure rates. This means that the expectation of the present worth of the losses is caused primarily by the first failure and to a lesser extent by the second failure. Further failures during the time T are so unlikely that they contribute little to the expectation of the total present worth of the losses.

4.3 The supposition that the present value of the losses is relatively small, or, in other words, the assumption of a linear utility function in the relevant range makes it possible, in a simple way, to adapt our subject still better to reality. In many cases the damage caused by the failure of a plant will not be of a precise magnitude, but will be a random variable. This means that the magnitude of the loss depends on the circumstances at the time of the failure. To take this into account, let us formulate again the sum of the expectation values of the present worths of the first, second, third, etc. losses:

$$\sum_{1}^{\infty} E[B_{Ai}] = A \cdot b_1(\lambda, r, T) + A \cdot b_2(\lambda, r, T) + A \cdot b_3(\lambda, r, T) + \ldots.$$

The factors b_i, functions of λ, r, T, follow from the solutions of the respective integrals in Section 4.2. A is a random variable; therefore, the sum on the left side of the above equation is a random variable too. According to a theorem in probability theory, the expectation of a sum of linear functions of random variables is equal to the sum of the same functions of the expectation of the random variables. This yields

$$E\left[\sum_{1}^{\infty} E[B_{Ai}]\right] = E[A](b_1 + b_2 + b_3 + \ldots.)$$
$$= E[A] \frac{\lambda}{r} \left(1 - e^{-rt}\right).$$

To account for the random distribution of the losses, A, it is sufficient to substitute, instead of a determinate value of the loss, the expectation of A.

4.4 It would not be difficult to proceed similarly to Section 2 from a relationship between investment cost and the failure rate and thus determine the plant with the optimal safety. To avoid a repetition, however, let us consider now a different example. We want to discuss the question of whether or not it is worthwhile to install a second plant in order to increase the safety of the first plant.

Where we have a choice between two possibilities:

1. We can keep the second plant permanently in operation along with the first one (parallel operation)

2. We can install it as a reserve, to be used only if the first plant fails.

The previously derived formulas are valid in cases of parallel operation, as well as of a reserve plant, if we use the appropriate failure density distributions for the respective cases.

Denoting the distribution function for one plant as $F(t)$, the failure distribution function for two plants operating parallel, F_{2P}, is (see reference 1, section 4.3)

$$F_{2P}(t) = F(t) \cdot F(t)$$

and the failure density distribution for two plants operating parallel, f_{2P}, is

$$f_{2P}(t) = \frac{F_{2P}(t)}{dt} = 2f(t)F(t) \ .$$

If we use the second plant as a reserve, we get the failure density distribution for both plants together by convolution of the densities of each single plant separately (see reference 1, section 4.4):

$$f_{2Re}(t) = f(t) * f(t) \ .$$

Figure 15 shows as an example the failure density distributions for the case of a single plant as well as for the case of parallel operation or reserve when two plants are used.

As an example, let us regard a small cargo refrigeration plant. If it fails, the losses will vary according to the value of the cargo which has been affected. Let the loss expectation be $E[A] = \bar{A}$. Furthermore, let the failure density distribution for one refrigeration plant be

$$f(t) = \lambda e^{-\lambda t} \ .$$

For the calculation of the expectation values of the present worths of the losses in case only one refrigeration plant is used, we may apply the results of Section 4.2.

As an alternative, let us regard a plant which has a second refrigeration machine serving as a reserve. At first we calculate the expectation of the losses for this case. The failure density distribution is

$$f_{2Re}(t) = f(t) * f(t) = \lambda^2 t e^{-\lambda t} \ .$$

If we insert this result in the integrals on page 19 (we will denote the expectation value with E^* now, to indicate that it has been calculated by using f_{2Re}),

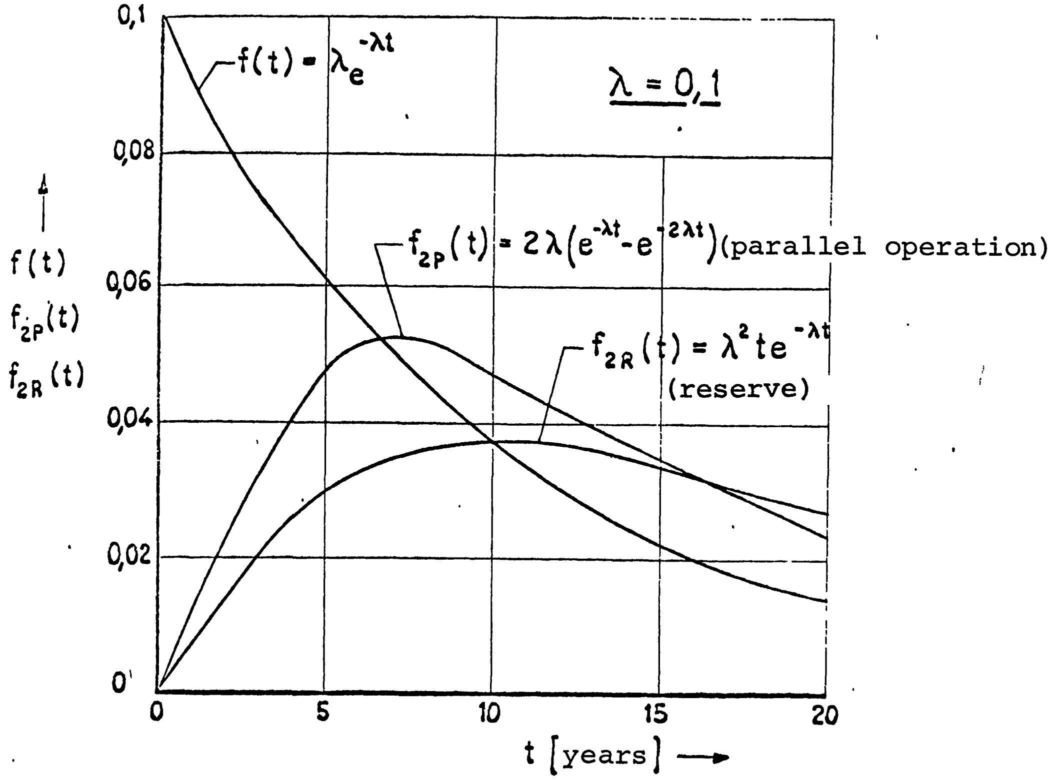

0,1
0,08
0,06
0,04
0,02
0
f(t)
$f_{2P}(t)$
$f_{2R}(t)$
$f(t) = \lambda e^{-\lambda t}$
$\lambda = 0,1$
$f_{2P}(t) = 2\lambda(e^{-\lambda t} - e^{-2\lambda t})$ (parallel operation)
$f_{2R}(t) = \lambda^2 t e^{-\lambda t}$
(reserve)
0
5
10
15
20
t [years]

Fig. 15

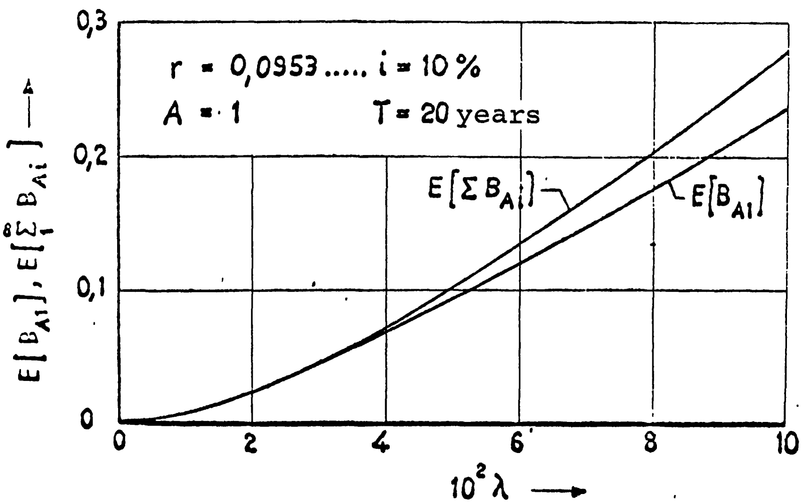

0,3
0,2
0,1
0
$E[B_{A1}], E[\sum_1^8 B_{Ai}]$
r = 0,0953 ..... i = 10 %
A = 1
T = 20 years
$E[\Sigma B_{Ai}]$
$E[B_{A1}]$
0
2
4
6
8
10
$10^2 \lambda$

Fig. 16

$$E^*[B_{A1}] = \int_0^T \bar{A}e^{-rt_1}\,\lambda^2 t_1 e^{-\lambda t_1}\,dt_1$$

$$= \bar{A}\left[\left(\frac{\lambda}{\lambda + r}\right)^2 - e^{-(\lambda+r)T}\left(\left(\frac{\lambda}{\lambda + r}\right)^2 + \left(\frac{\lambda}{\lambda + r}\right)\lambda T\right)\right]$$

$$E^*[B_{A2}] = \int_0^T \int_0^{T-t_1} \bar{A}e^{-r(t_1+t_2)}\,\lambda^4 t_1 t_2 e^{-\lambda(t_1+t_2)}dt_1 dt_2$$

$$= \bar{A}\left[\left(\frac{\lambda}{\lambda + r}\right)^4 - e^{-(\lambda+r)T}\left(\left(\frac{\lambda}{\lambda + r}\right)^4 + \left(\frac{\lambda}{\lambda + r}\right)^3 \lambda T\right.\right.$$

$$\left.\left. + \left(\frac{\lambda}{\lambda + r}\right)^2 \frac{\lambda^2 T^2}{2!} + \left(\frac{\lambda}{\lambda + r}\right)\frac{\lambda^3 T^3}{3!}\right)\right]$$

$$E^*[B_{A3}] = \int_0^T \int_0^{T-t_1} \int_0^{T-t_1-t_2} \bar{A}e^{-r(t_1+t_2+t_3)}\,\lambda^6 t_1 t_2 t_3$$

$$\cdot\ e^{-\lambda(t_1+t_2+t_3)}dt_1 dt_2 dt_3$$

$$= \bar{A}\left[\left(\frac{\lambda}{\lambda + r}\right)^6 - e^{-(\lambda+r)T}\left(\left(\frac{\lambda}{\lambda + r}\right)^6 + \left(\frac{\lambda}{\lambda + r}\right)^5 \lambda T\right.\right.$$

$$+ \left(\frac{\lambda}{\lambda + r}\right)^4 \frac{\lambda^2 T^2}{2!} + \left(\frac{\lambda}{\lambda + r}\right)^3 \frac{\lambda^3 T^3}{3!} + \left(\frac{\lambda}{\lambda + r}\right)^2 \frac{\lambda^4 T^4}{4!}$$

$$\left.\left. + \left(\frac{\lambda}{\lambda + r}\right)\frac{\lambda^5 T^5}{5!}\right)\right],$$

etc. After some transformations, the sum of the expected values is

$$E^*[\sum_{1}^{\infty} B_{Ai}] = \sum_{1}^{\infty} E^*[B_{Ai}]$$

$$= \bar{A}\left[\frac{\lambda^2}{r(2\lambda + r)} - e^{-(\lambda+r)T}\left(\frac{\lambda^2}{r(2\lambda + r)}\cosh \lambda T + \frac{\lambda(\lambda + r)}{r(2 + r)}\sinh \lambda T\right)\right].$$

Figure 16 shows the sum of the expectation values and the first term of this sum as a function of the failure rate, λ. The convergence in the regarded range of λ is even better than in our previous example (see Figure 14). The losses to be expected are now also considerably lower.

For a valid comparison of the two alternatives, it is not sufficient to consider only the losses caused by the failure of one or of both refrigeration plants. The receipts are the same in either case. The alternative with two refrigeration plants, however, requires additional disbursements for investment and maintenance. Let us denote the present worth of these additional disbursements with Z. The economic difference between these two alternatives may now be determined for the respective resultant present worths of all those quantities which are not equal in the two alternatives:

Without reserve plant:

$$B_1 = E\left[\sum B_{Ai}\right] = \bar{A}\,\frac{\lambda}{r}\,(1 - e^{-rT})$$

With reserve plant:

$$B_2 = E^{*}\left[\sum B_{Ai}\right] + Z$$

$$= \bar{A}\left[\frac{\lambda^2}{r(2\lambda + r)} - e^{-(\lambda+r)T}\left(\frac{\lambda^2}{r(2\lambda + r)}\cosh \lambda T\right.\right.$$

$$\left.\left. + \frac{\lambda(\lambda + r)}{r(2\lambda + r)}\sinh \lambda T\right)\right] + Z\ .$$

As these present worths are losses, the plant with the smaller present worth is the better one.

Figure 17 shows the results of calculations for different values of λ, $\bar{A}$, and Z. We realize that a high failure rate, λ; a high cargo value, $\bar{A}$; and comparatively low cost, Z, favor the installation of a reserve plant. This result agrees with our intuition. The advantage of calculation, however, is a quantitative evaluation of these otherwise unmeasurable influences and tendencies. This now allows us to make decisions on a rational rather than intuitive basis.

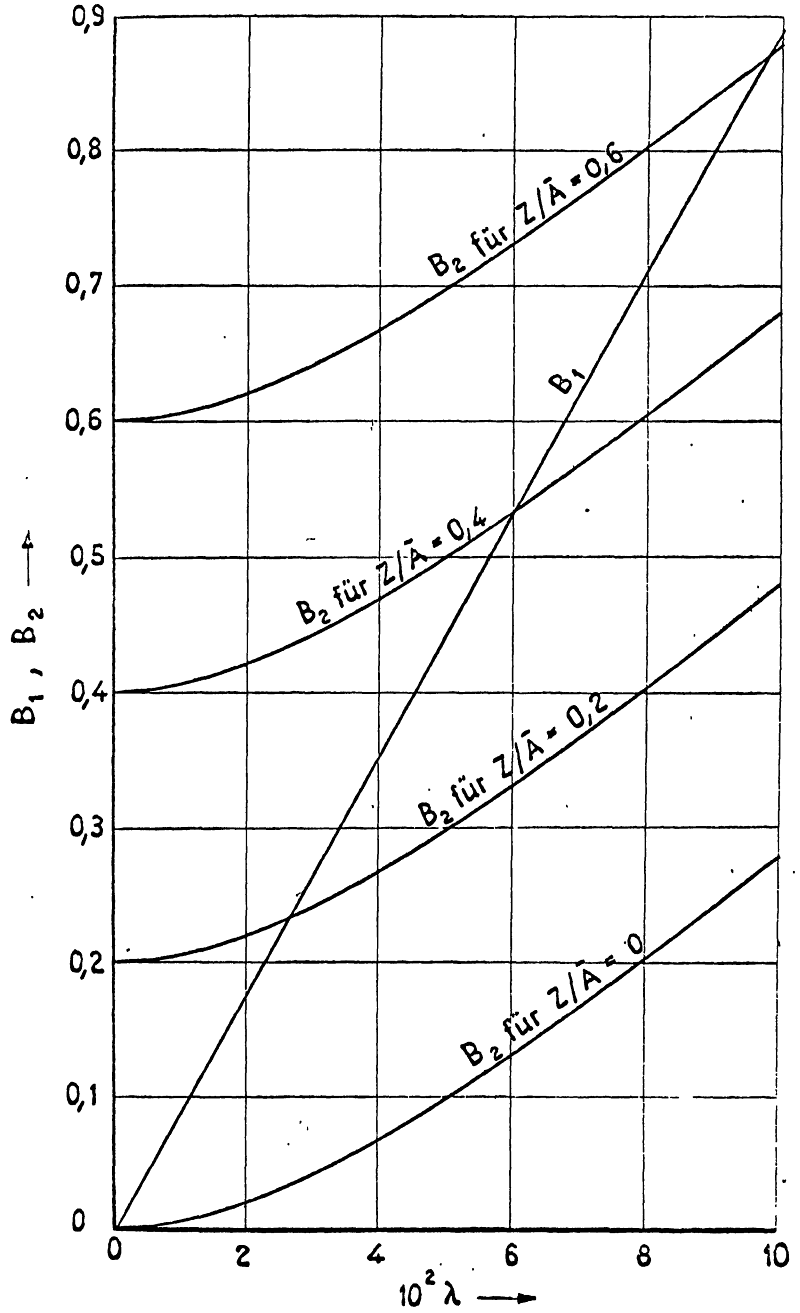

0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
0
2
4
6
8
10
B_1 , B_2
$10^2 \lambda$
B_2 für $Z/\bar{A} = 0,6$
B_2 für $Z/\bar{A} = 0,4$
B_2 für $Z/\bar{A} = 0,2$
B_2 für $Z/\bar{A} = 0$
B_1

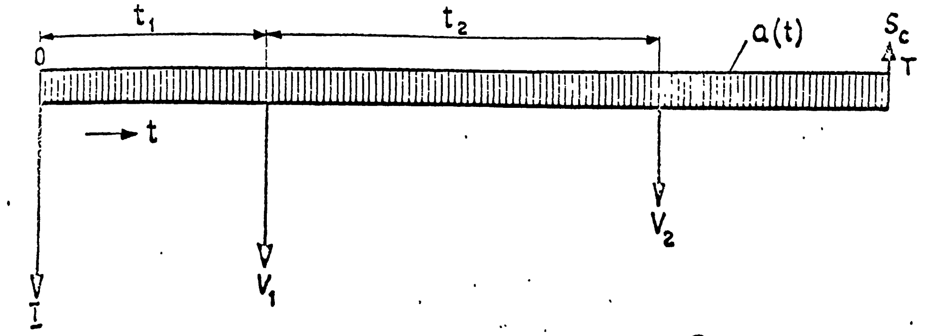
Fig. 17

V

5.1 In this section we will resume the discussion of some problems concerned with the safety of ships in case of collision (see Section 3.2). Let us consider the following model (see also Figure 18): Let I be the initial investment necessary to acquire the ship. The difference between operating revenues and operating costs is assumed to be continuous and is denoted $a(t)$. At the end of its life, T, the ship is assumed to have a scrap value Sc.

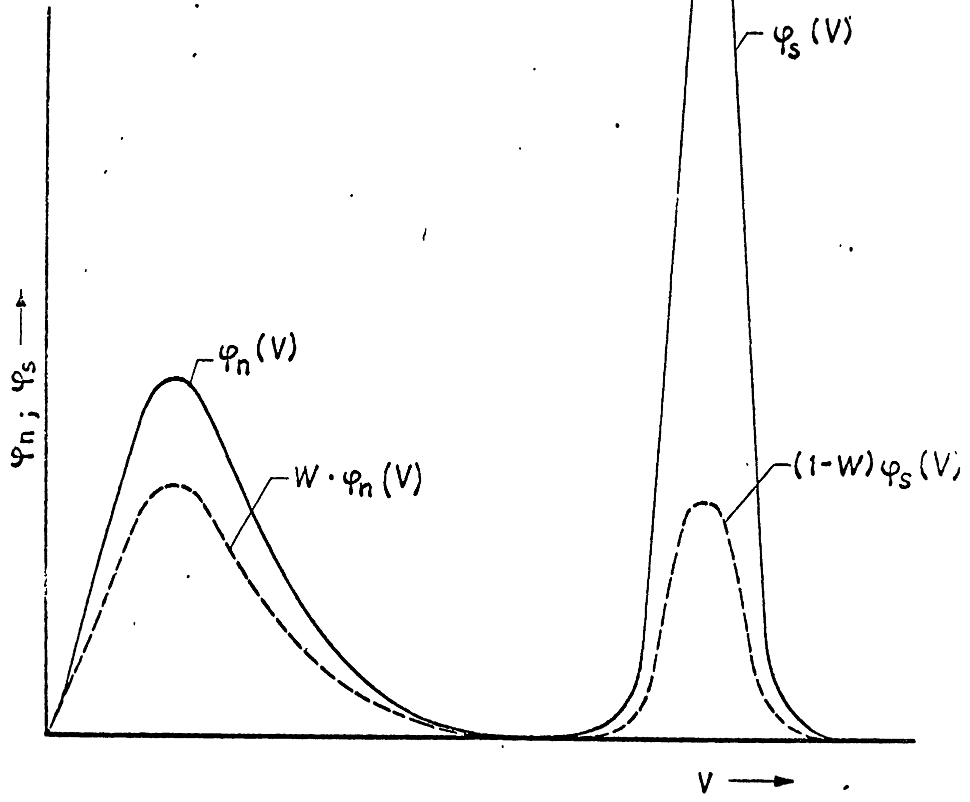
During the life, T, of the ship there may be no collision, or a collision may occur after the time t_1, or after the time t_2, following the first collision, there may be a second collision, etc. In case the ship should sink after a collision we assume that it will be replaced by an equivalent ship. Equivalent here means that the life of the replacement ship shall be equal to the time which the first ship could still have been operated if it had not sunk.

Every collision means a loss, V. The cost of an eventual insurance premium for ship and cargo may be taken into account with $a(t)$. In case of damage, the payment of the insurance company will be subtracted from the actual damage. Therefore, V represents the resultant loss to the shipowner. Statements about V in the following sections shall only serve as examples for further clarification and shall not mean a restriction with regard to other interpretations.

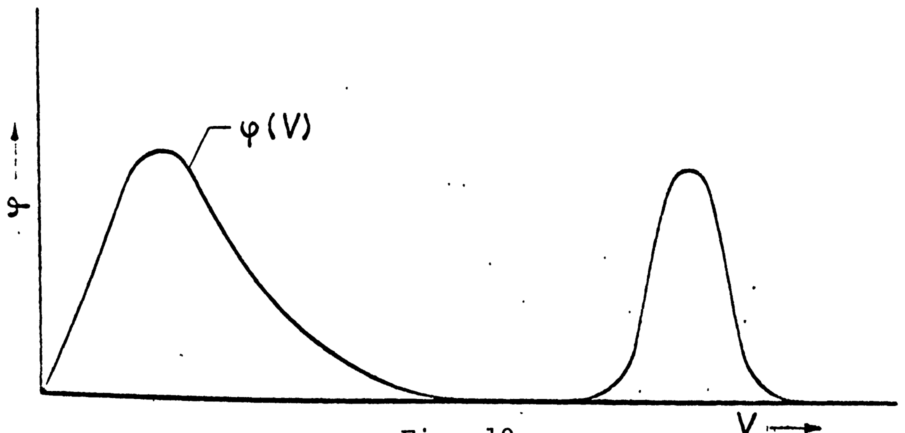


t_1
t_2
0
$a(t)$
S_c
T
t
I
V_1
V_2

Fig. 18



$\varphi_n ; \varphi_s$
$\varphi_n(V)$
$W \cdot \varphi_n(V)$
$\varphi_s(V)$
$(1-W)\varphi_s(V)$
V



φ
$\varphi(V)$
V

Fig. 19

5.2 Losses resulting from collision may be very different in magnitude. The ship may remain afloat or may sink. In the first case, costs for its salvage arise. These may be low if the collision occurs near a harbor and the ship is still maneuverable. They may be very high if the ship has to be towed over a long distance under unfavorable weather conditions. Furthermore, there are costs for repair. These also may be very different depending on the extent and the nature of the damage. Costs, for instance, are substantially higher if the damage leads to a flooding of the engine room than if only the plating has been damaged in the area of a tank. Additional costs which may be very different are loss of returns, damage of the cargo, etc. In a manner similar to the determination of the distribution of the extent of damage to be expected after a collision on the basis of collision damage statistics, it is possible in our problem to determine a distribution of the costs, V_n, which are caused by a collision in which the ship does not sink.

In the second case, where the ship sinks after a collision, the loss can be very different, too. We will denote these costs as V_s and we will split them up into two separate amounts. The first part, V_L, shall comprise the loss of the cargo, the home transport of the crew and similar items. It is quite clear that V_L is a random variable. The second part will be denoted V_E. It comprises the cost of the replacement ship which has to be acquired in case of a total loss, according to the above assumptions. V_E depends on the collision time: If the collision takes place shortly after the ship started its service, then an almost new ship has to be replaced; if it is after a time which is not much shorter than T, the cost for the "equivalent" replacement ship will be much smaller than I. Therefore, V_s is a random variable and its distribution contains the collision time as parameter:

$$V_s(t) = V_L + V_E(t) .$$

From the distribution φ_n of the losses V_n which occur if the ship does not sink, and from the distribution φ_s of the losses V_s in case the ship sinks, we may now determine the distribution of the loss V which applies to both cases. To accomplish that, we still need the probability, W, that a ship will not sink after a collision and the complementary probability, (1 - W), that it will sink. (For the determination of this probability, see, for instance, references 13-15.) We can now establish the following equation (see also Figure 19):

$$\varphi(V;t) = W \cdot \varphi_n(V) + (1 - W) \cdot \varphi_s(V;t) ,$$

where t represents the collision time. The relationship between the expectation $\bar{V}$ of V and the expectation $\bar{V}_n$ and $\bar{V}_s(t)$ of V_n and V_s, respectively, can be easily given now:

$$\bar{V}(t) = W\bar{V}_n + (1 - W)\bar{V}_s(t) .$$

Let us now consider a more concrete example and introduce the following relation. The replacement cost of the ship may decrease in direct proportion to t from the initial value of I at time $t = 0$ to the final value Sc at time $t = T$:

$$V_E(t) = I - \frac{I - Sc}{T} t .$$

Correspondingly, we may say that

$$V_s(t) = V_L + I - \frac{I - Sc}{T} t$$

and

$$\bar{V}_s(t) = \bar{V}_L + I - \frac{I - Sc}{T} t .$$

5.3 To determine the expectation of the resultant present worth we need the present worths of all amounts occurring during the time of consideration. For the calculation of the present worths of the losses we may use what has been said in Section 4.3: Instead of using the random losses, V, we can take their expectation values, $\bar{V}$. For the first, second, and third, etc. collision we get the following present worths for $\bar{V}$:

$$\bar{B}_{V1}(t_1) = \bar{V}(t_1)e^{-rt_1} \qquad \text{for } 0 < t_1 \leq T\text{; or else } 0$$

$$\bar{B}_{V2}(t_1, t_2) = \bar{V}(t_1 + t_2)e^{-r(t_1+t_2)}$$

$$\text{for } 0 < t_1 + t_2 \leq T\text{; or else } 0$$

$$\bar{B}_{V3}(t_1, t_2, t_3) = \bar{V}(t_1 + t_2 + t_3)e^{-r(t_1+t_2+t_3)}$$

$$\text{for } 0 < t_1 + t_2 + t_3 \le T; \text{ or else } 0,$$

etc. If we now insert the relations we have chosen before as an example, we get

$$\bar{B}_{V1}(t_1) = \left[W(\bar{V}_n) + (1 - W)\left(\bar{V}_L + I - \left(\frac{I - Sc}{T}\right)t_1\right)\right] e^{-rt_1}$$

$$\bar{B}_{V2}(t_1, t_2) = \left[W(\bar{V}_n) + (1 - W)\left(\bar{V}_L + I - \left(\frac{I - Sc}{T}\right)(t_1 + t_2)\right)\right] \left(e^{-r(t_1+t_2)}\right)$$

$$\bar{B}_{V3}(t_1, t_2, t_3) = \left[W(\bar{V}_n) + (1 - W)\left(\bar{V}_L + I - \left(\frac{I - Sc}{T}\right)(t_1 + t_2 + t_3)\right)\right] e^{-r(t_1+t_2+t_3)},$$

etc.; $\bar{B}_{Vi} = 0$ in the same intervals, as previously stated! The present worth of the returns a(t) is

$$B_a = \int_0^T a(t)e^{-rt}dt .$$

Now we need the distributions of the times when a collision occurs. If we take the density distribution of the time when a collision occurs, introduced on page 12, and use analogously the statements on the probability of one, two, etc. failures made, on page 17, we get

$$W\{\text{collision at } t_1\} = f(t_1)dt_1 = \bar{\lambda}_o e^{-\bar{\lambda}_o t_1}\, dt_1 \;,$$

$$W\{\text{collision at } t_1 \text{ \underline{and} } t_2\} = f(t_1)f(t_2)dt_1dt_2$$

$$= \bar{\lambda}_o^2 e^{-\bar{\lambda}_o(t_1+t_2)}\, dt_1dt_2 \;,$$

$$W\{\text{collision at } t_1 \text{ \underline{and} } t_2 \text{ \underline{and} } t_3\} = f(t_1)f(t_2)f(t_3)dt_1dt_2dt_3$$

$$= \bar{\lambda}_o^3 e^{-\bar{\lambda}_o(t_1+t_2+t_3)} dt_1dt_2dt_3 \;,$$

etc. This results in the following equation for the expectation of the resulting present worth:

$$E[B_{res}] = \int_{t_1=0}^{\infty} \int_{t_2=0}^{\infty} \cdots \Big(B_a - I - \bar{B}_{V1}(t_1) - \bar{B}_{V2}(t_1,\, t_2)$$

$$- \ldots\Big)\; f(t_1)f(t_2)\ldots dt_1dt_2\ldots .$$

$$= B_a - I - \int_{t_1=0}^{\infty} \bar{B}_{V1}(t_1)f(t_1)dt_1 - \int_{t_1=0}^{\infty}\int_{t_2=0}^{\infty} \bar{B}_{V2}$$

$$(t_1, t_2)f(t_1)f(t_2)dt_1dt_2 - \ldots.$$

If we insert the relations of our example, we get

$$E[B_{res}] = B_a - I - W(\bar{V}_n)\left\{\int_0^T e^{-rt_1}\,\bar{\lambda}_o e^{-\bar{\lambda}_o t_1}\,dt_1\right.$$

$$+\int_0^T\int_0^{T-t_1} e^{-r(t_1+t_2)}\,\bar{\lambda}_o^2 e^{-\bar{\lambda}_o(t_1+t_2)}dt_1dt_2$$

$$\left.+\ \ldots\right\} - (1 - W)\left[\left(\bar{V}_L + I\right)\left\{\int_0^T e^{-rt_1}\,\bar{\lambda}_o e^{-\bar{\lambda}_o t_1}\,dt_1\right.\right.$$

$$+\int_0^T\int_0^{T-t_1} e^{-r(t_1+t_2)}\,\bar{\lambda}_o^2 e^{-\bar{\lambda}_o(t_1+t_2)}\,dt_1dt_2$$

$$\left.+\ \ldots\right\} + \left(\frac{I - Sc}{T}\right)\left(\int_0^T t_1e^{-rt_1}\,\bar{\lambda}_o e^{-\bar{\lambda}_o t_1}dt_1\right.$$

$$\left.\left.+\int_0^T\int_0^{T-t_1}(t_1 + t_2)\bar{\lambda}_o^2 e^{-\bar{\lambda}_o(t_1+t_2)}\,dt_1dt_2 + \ldots\right)\right].$$

If we now analogously apply the result derived on page 20, we can replace the above sums written in {...}:

$$\{./.\} = \frac{\bar{\lambda}_o}{r}(1 - e^{-rT}) \; .$$

The integrals of the last sum in the equation for $E[B_{res}]$ yield

$$J_1 = \int_0^T t_1 e^{-rt_1}\,\bar{\lambda}_o e^{-\bar{\lambda}_o t_1}\,dt_1$$

$$= \frac{\bar{\lambda}_o}{(\bar{\lambda}_o + r)^2} - e^{-(\bar{\lambda}_o + r)T}\left(\frac{\bar{\lambda}_o}{(\bar{\lambda}_o + r)} + \frac{1}{(\bar{\lambda}_o + r)}\,\bar{\lambda}_o T\right)$$

$$J_2 = \int_0^T \int_0^{T-t_1} (t_1 + t_2) e^{-r(t_1+t_2)}\,\bar{\lambda}_o^2 e^{-\bar{\lambda}_o(t_1+t_2)}\,dt_1 dt_2$$

$$= 2\left[\frac{\bar{\lambda}_o^2}{(\bar{\lambda}_o + r)^3} - e^{-(\bar{\lambda}_o + r)T}\left(\frac{\bar{\lambda}_o^2}{(\bar{\lambda}_o + r)^3} + \frac{\bar{\lambda}_o}{(\bar{\lambda}_o + r)^2}\,\bar{\lambda}_o T\right.\right.$$

$$\left.\left. + \frac{1}{(\bar{\lambda}_o + r)}\left(\frac{\bar{\lambda}_o^2 T^2}{2!}\right)\right)\right]$$

$$J_3 = \int_0^T \int_0^{T-t_1} \int_0^{T-t_1-t_2} (t_1 + t_2 + t_3) e^{-r(t_1+t_2+t_3)}$$

$$\bar{\lambda}_o^3 e^{-\bar{\lambda}_o(t_1+t_2+t_3)} \quad dt_1 dt_2 dt_3$$

$$= 3\left[\frac{\bar{\lambda}_o^3}{(\bar{\lambda}_o + r)^4} - e^{-(\bar{\lambda}_o+r)T}\left(\frac{\bar{\lambda}_o^3}{(\bar{\lambda}_o + r)^4} + \frac{\bar{\lambda}_o^2}{(\bar{\lambda}_o + r)^3} \bar{\lambda}_o T\right.\right.$$

$$\left.\left. + \frac{\bar{\lambda}_o}{(\bar{\lambda}_o + r)^2} \frac{\bar{\lambda}_o^2 T^2}{2!} + \frac{1}{(\bar{\lambda}_o + r)} \frac{\bar{\lambda}_o^3 T^3}{3!}\right)\right] ,$$

etc. After some transformations we get for the sum

$$\sum J_i = \frac{\bar{\lambda}_o}{r^2} [1 - e^{-rT} (1 + rT)] .$$

If we let $a(t) = a$, we get for the expectation of the resulting present worth

$$E[B_{res}] = \frac{a}{r} (1 - e^{-rT}) - I - W(\bar{V}_n) \frac{\bar{\lambda}_o}{r} (1 - e^{rT}) - (1 - W)$$

$$\left(\left\{(V_L + I) \frac{\bar{\lambda}_o}{r} (1 - e^{-rT}) - \frac{I - Sc}{T} [1 - e^{-rT}\right.\right.$$

$$\left.\left.(1 - rT)]\right\}\right) .$$

In Figures 20 and 21 the results of a numerical example are plotted. Figure 20 shows how the parts of $E[B_{res}]$, owing to $\bar{V}_n$ and $\bar{V}_s$, change versus the probability, W. Figure 21 shows the dependency of $E[B_{res}]$ on the collision rate, $\bar{\lambda}_0$, and on the probability of surviving damage, W.

The increase of $E[B_{res}]$, shown in Figures 20 and 21 (equivalent to an improvement of profitability with increasing probability, W), is only valid when I is constant. It could be used to find out what is economically feasible if it should be possible to increase the probability, W, without additional expenses (for example, by better placing of bulkheads).

More important in practice, however, is the case that in increasing the probability, W, the construction cost, and, therefore, I, will increase. In this connection it would be interesting to know how much W has to be increased to justify the expenditure ΔI necessary to increase W, that is, how much W may decrease when expenditures are cut down by $-\Delta I$ without decreasing profitability at the same time.

To answer this question we have to proceed as follows: We calculate for a ship I, a(t), W, and $E[B_{res}]$. With these data we may now compute the least amount that W has to rise lest an expenditure to raise W (i.e., additional cost, $+\Delta I$; possibly reduction of returns, a(t)) cause a decrease of $E[B_{res}]$ (i.e., the profitability). In a similar manner, we can find out the most that W may decrease through a reduction of the expenditures by $-\Delta I$ before profitability drops. Figure 21 shows an example of such a limit of profitability. If in this example one changes the ship under consideration with respect to its

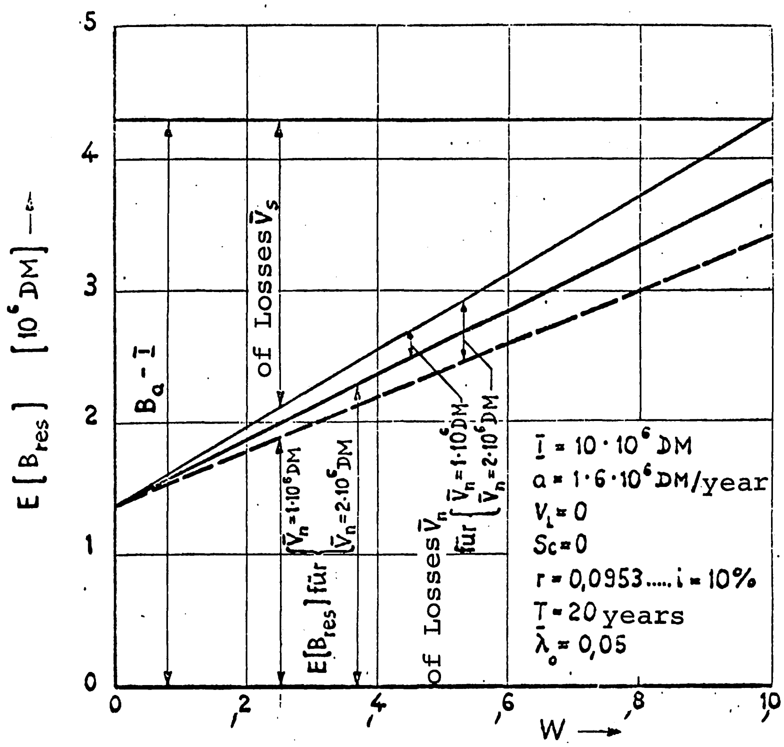

E [B_res] [10^6 DM] →
W →
B_a - Ī
of Losses V̄_s
E[B_res] für {V̄_n=1·10^6 DM / V̄_n=2·10^6 DM
of Losses V̄_n für {V̄_n=1·10^6 DM / V̄_n=2·10^6 DM
Ī = 10·10^6 DM
a = 1·6·10^6 DM/year
V_L = 0
Sc = 0
r = 0,0953 ..... i = 10%
T = 20 years
λ̄_o = 0,05

Fig. 20

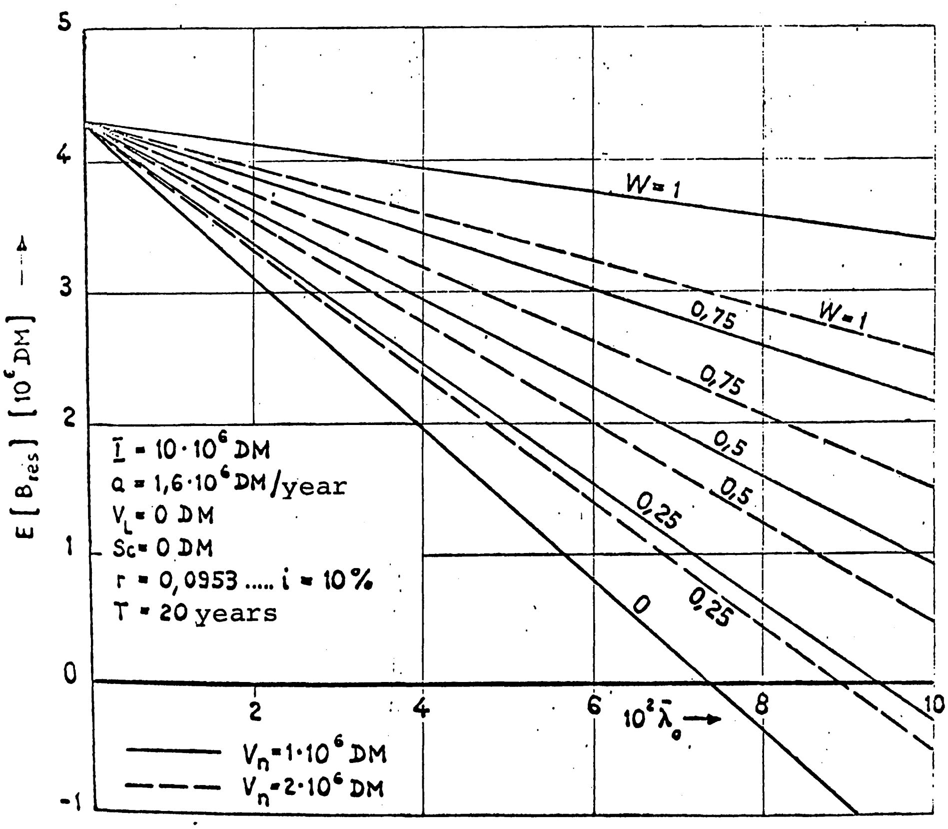

E [B_res] [10^6 DM] →
10² λ̄_o →
W = 1
0,75
W = 1
0,75
0,5
0,5
0,25
0
0,25
Ī = 10·10^6 DM
a = 1,6·10^6 DM/year
V_L = 0 DM
Sc = 0 DM
r = 0,0953 ..... i = 10%
T = 20 years
V_n = 1·10^6 DM
V_n = 2·10^6 DM

Fig. 21

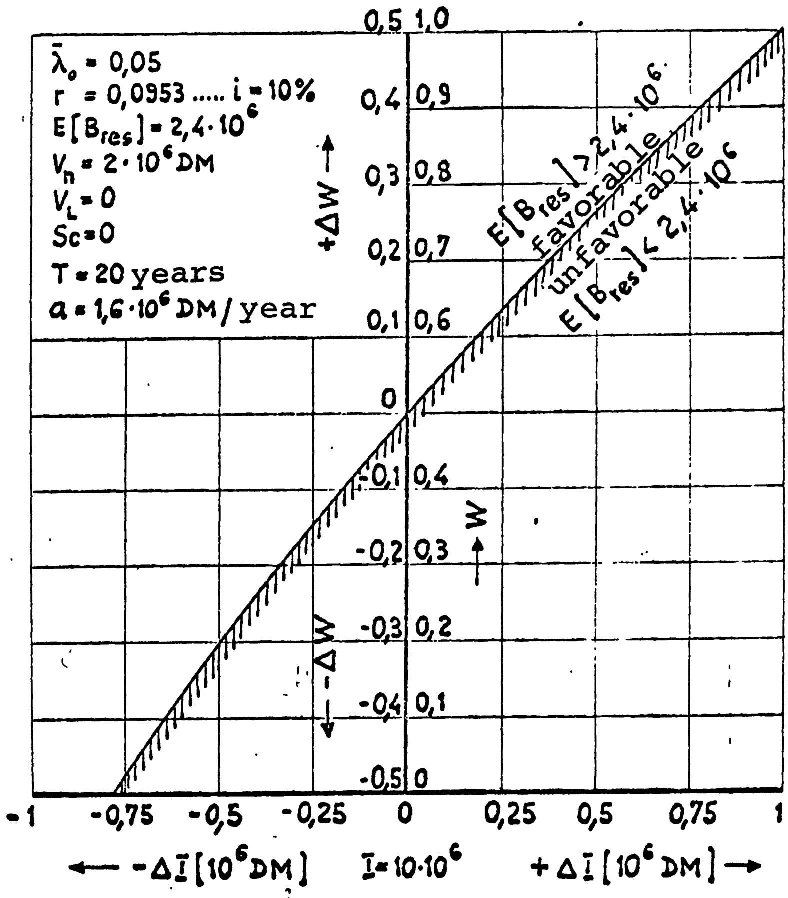

$\bar{\lambda}_o$ = 0,05
r = 0,0953 ..... i = 10%
E[B_{res}] = $2,4 \cdot 10^6$
V_n = $2 \cdot 10^6$ DM
V_L = 0
Sc = 0
T = 20 years
a = $1,6 \cdot 10^6$ DM/year
E[B_{res}] > $2,4 \cdot 10^6$
favorable
unfavorable
E[B_{res}] < $2,4 \cdot 10^6$
+ΔW
-ΔW
W
0,5 1,0
0,4 0,9
0,3 0,8
0,2 0,7
0,1 0,6
0
-0,1 0,4
-0,2 0,3
-0,3 0,2
-0,4 0,1
-0,5 0
-1
-0,75
-0,5
-0,25
0
0,25
0,5
0,75
1
-ΔĪ [10^6 DM]
Ī = $10 \cdot 10^6$
+ΔĪ [10^6 DM]

Fig. 22

ability to survive flooding, the cost, $+\Delta I$, or savings, $-\Delta I$, resulting from this change are only justified if the probability due to this change exceeds the limit curve.

5.4 Let us now calculate the expectation of the utility of the resulting present worth. First, we have to look at the resulting present worth. Its definition depends on the times when a collision occurs, t_i:

$$B_{res} = B_a - I \qquad \text{for } t_1 > T$$

$$B_{res} = B_a - I - B_{v1}(t_1) \qquad \text{for } t_1 < T,\ t_1 + t_2 > T$$

$$B_{res} = B_a - I - B_{v1}(t_1) - B_{v2}(t_1, t_2) \qquad \text{for } t_1 + t_2 < T,\ t_1 + t_2 + t_3 > T.$$

In the third and in all following equations of those just given for B_{res}, the sums contain two or more random variables, B_{vi}. The density distributions of B_{vi}, f_{Bi}, can easily be evaluated from the density distribution of V, $\varphi(V; t)$, which was introduced on page 28. The density distributions f_{Bi} contain t_1, t_2, ...t_i as parameters. With the values of f_{Bi}, the density distribution of the sum

$$\sum_n B_{vi} = B_{v1} + B_{v2} + \ldots B_{vn} ,$$

$f_{\Sigma n}$, may be easily determined by convolution of the density distributions of the terms of the sum

$$f_{\Sigma n}\left(\sum_n B_{vi};\ t_1,\ t_2,\ \ldots t_n\right) = f_{B1}(f_{B2})(\ldots f_{Bn}) \ .$$

If also the utility function $u = u(B_{res})$ is given, we will get the following equation for the value of expectation of the utility:

$$E[u(B_{res})] = \int_T^\infty u(B_a - I) f_1(t_1) dt_1 + \int_{t_1=0}^\infty \int_{t_2=T-t_1}^\infty$$

$$\int_{B_{v1}=0}^\infty u(B_a - I - B_{v1}(t_1)) f_{B1}(B_{v1};\ t_1) f_1$$

$$(t_1) f_2(t_2) dB_{v1} dt_1 dt_2 + \int_{t_1=0}^{T} \int_{t_2=0}^{T-t_1} \int_{t_3=T-t_1-t_2}^\infty$$

$$\int_{\sum_2 B_{vi}=0}^\infty u(B_a - I - \sum_2 B_{vi}) f_{\Sigma 2}(\sum_2 B_{vi};\ t_1,\ t_2)$$

$$f_1(t_1) f_2(t_2) f_3(t_3) d(\sum_2 B_{vi}) dt_1 dt_2 dt_3, \quad + \text{ etc.}$$

We will not evaluate this equation any further here. Its solution for a special case is more elaborate than would be suitable for an example within the limits of this paper. This will be the subject of a separate paper, to be written later.

VI

Finally, to round off my quantitative considerations in this paper, I want to add some general conclusions about the relationship between safety and profitability. In reference 16* the statement is made that from the viewpoint of the best possible protection of human lives, the strength of structures in today's ships is too great. This is established as follows: Because the dimensions of the ship's hull girders are more than ample, a breakdown of ships and, thus, danger to, or loss of, the crew is made almost impossible today. It would, however, be one-sided to draw the conclusion from this fact that we have thus achieved the maximum possible protection of human lives. The production of ship construction steel also involves dangers. (We only have to remember the inevitable accidents in coal and ore mines as well as industrial accidents in steelworks.) Taking this into account, the optimal protection of human lives, according to reference 16, will be achieved by building ships of less weight (hence, which are less safe against breaking).

Without agreeing completely on the conclusions drawn in reference 16, I can say, nevertheless, that these considerations are very useful or even necessary. Cheap possibilities of transport are an important means of reaching material welfare, which, if wisely used, is essential to secure human life (for example, by securing sufficient food at the lower end of the class scale or by utilization of expensive medical facilities, etc. at the upper end).

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*The hint at this paper I owe to Professor Harry Benford.

Taken all together, the general demand for especially high safety, therefore, may be very well disadvantageous. This fact applies even more to naval construction than to commercial shipping. In either case, it seems necessary to me to consider explicitly the relationships between safety and profitability.

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