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ABSTRACT

The method of matched asymptotic expansions is applied to some two- and three-dimensional problems of a ship-to-wall interactions for the case of the small lateral separations from the wall(s).

In the two-dimensional model (i.e., no underkeel clearance), the beam and yaw angle of a ship are assumed comparable to the lateral separation from the wall and much smaller than the ship length. The asymptotic solutions are constructed separately in the confined region between a ship and the wall (channel flow), in the vicinities of the bow and stern (edge flows), and outside of the channel and edge flows (outer flow). These asymptotic solutions are then matched in the overlap regions, and a uniformly valid solution is composed, which is used to obtain formulas for the side-force and yaw-moment coefficients for the ship moving close to the wall and in a canal.

In the three-dimensional case of ship motion in shallow restricted water, the same technique is applied to solve the linearized far field problem on an assumption that the lateral separation from the wall(s) is much smaller than the ship length. Some analytical and numerical results are presented both for steady and unsteady shallow-water motions of the ship close to a bank or in a narrow canal. These results illustrate the influence of the yaw angle, rudder deflection, blockage coefficient and Strouhal number on some hydrodynamic coefficients.

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CONTENTS

I. INTRODUCTION

Due to the increase in size of mammoth ships in recent decades, many waterways that were formerly considered deep and unrestricted have now become relatively shallow and narrow. As a consequence, even slight errors in traffic control of superships may cause costly damage. Because of the complicated character of ship-to-wall hydrodynamic interactions, intuitive decisions in maneuvering are often either misleading or dangerous. Therefore the development and operation of rudder control systems should be based on a thorough examination of the hydrodynamics of the ship in restricted waters.

Due to the problems of scaling, which become very significant in restricted water, experimental investigations are usually difficult and expensive, so that development of the appropriate theoretical approaches becomes rather important.

Many of the theoretical models of the ship-to-wall interactions proposed in recent years have been developed on the basis of singular perturbation techniques, in particular, using the method of matched asymptotic expansions. This method has proven effective in quite a number of applied problems of ship hydrodynamics (see Ogilvie (1977)). It often leads to success in those cases when the convergence of numerical schemes drastically decreases. At the same time, the asymptotic solutions are oftentimes relatively simple and give the possibility to predict the hydrodynamic behavior of the ship for a wide range of parameters and at low computation costs, which is essential for the development of rudder control systems.

Some of the theoretical approaches are based on the idea that in shallow water the flow is asymptotically two-dimensional, even if the clearance is not negligible compared to water depth, provided that we do not look too close to the ship. Tuck (1967) extended his sinkage and trim analysis to include the case of a ship operating in a rectangular canal. Beck, Newman and Tuck (1975) further extended the analysis to include the case of a ship travelling along the centerline of a dredged channel surrounded on both sides by shallow water. Beck (1976) studied the problem of a ship operating at zero yaw angle off the centerline of a canal of rectangular cross-section in which, due to the asymmetry, there exists a cross flow under the bottom of the ship. In this case, the side force and yaw moment are not equal to zero. The cross flow in Beck (1976) is handled in a manner similar to that developed by Newman (1969) for the lateral flow past a slender body between two parallel walls.

A theoretical model for prediction of lateral forces, rudder effectiveness, and course-keeping stability of ships in shallow canals was proposed by Hess (1976). In his model, Hess introduces the effects of yaw angle and rudder deflection, although no numerical results are presented.

The theoretical study of the hydrodynamic interactions of ships in shallow water, including unsteady effects associated with the vortex wake, was carried out by Yung (1978) and King (1977).

All of the previously mentioned investigations have shown the importance of including circulation and the Kutta-Joukowsky condition in the theoretical model.

It should be noted that the theories developed in the studies cited above are only valid if the lateral separation between the ship and the other object (e.g., other ship or bank) is considerably greater than the ship's beam. However, the situations that are of most practical importance are those in which the ship moves close to another ship or close to a bank. Some progress in this direction has been achieved by Yeung and Hwang (1977), who applied the slender-body theory to the problem of ship-to-ship interactions in shallow water. They were able to obtain the solution without solving the outer problem. Their theory is essentially a "near-field" solution. It requires a detailed knowledge of the hull geometry and accounts for it appropriately. However, as pointed out by the authors, "the computation time for such a mathematical model of ship interactions is sufficiently large so that its applications to real time simulation appear to be impractical."

Tuck (1974) formulated the one-dimensional (hydraulic) approach for the case of a ship operating in a very narrow canal, although so far this approach does not seem to be effective in providing information about the forces and moments acting on a ship. Tuck (1975) developed some asymptotic "small gap" solutions of problems for vehicles moving close to a plane surface. His solutions are valid in cases when the gap is very small compared to the lateral dimensions of the vehicle.

Some other publications exist on the small-gap problems. Strand, Royce and Fujita (1962) noted the "hydraulic" or channel-flow character of the tightly-constrained flow between the body and the wall, and Widnall and Barrows (1970) provided a complete asymptotic solution for the steady-flow three-dimensional case, assuming in addition that the body's thickness and camber are small compared to the clearance. Rozhdestvensky (1974, 1976, 1977) extended this analysis to the unsteady three-dimensional case and also to a "non-linear" (camber and thickness comparable to clearance) two-dimensional case. Tuck (1979) considered the lowest order "non-linear" solutions of some steady and unsteady two-dimensional problems for the body moving very close to a plane wall.

In this work we shall consider some problems of ship-to-wall interaction using the method of matched asymptotic expansions, with the aim to obtain relatively simple and readily computable formulae for the side force and yaw moment acting upon the ship.

The report is divided into two main sections.

The first section deals with some two-dimensional ship-to-wall interaction problems, which correspond to the case when the ship is wall sided and there is no bottom gap. In particular, we consider the problem of a ship moving in close proximity to a bank or one of the walls of a canal. The method of matched asymptotic expansions is applied in a way similar to that of Widnall and Barrows (1970) and Rozhdestvensky (1974, 1976). It is assumed that the beam and yaw angle of the ship are comparable to the midstern distance from the bank and at the same time are considerably smaller than the length of a ship ("nonlinear" approach). For the problem of a ship in a canal, we shall consider two possible asymptotic solutions, corresponding to the cases of the canal width (i) comparable to or (ii) much larger than the midstern distance from the walls.

The asymptotic solutions are constructed separately in characteristic regions of the flow and then matched in the overlap regions to account for physical interactions of different parts of the flow and to provide mathematical uniqueness.

The asymptotic formulae for the side force and yaw moment coefficients for a given distribution of beam and prescribed yaw angle are derived with an asymptotic error of $O(\ell^2)$, where ℓ is the midstern distance from the wall, normalized by ship length. Some results of the numerical computations are presented for the theoretical parabolic-beam distribution.

In the second section, a similar, though linearized, approach is applied to the two-dimensional far-field part of the three-dimensional shallow-water flow past a ship in presence of the wall(s). To employ Beck's (1976) far-field formulation, it is assumed that the beam, yaw, and lateral (unsteady) displacements of a ship are comparable to the water depth and at the same time are considerably smaller than the lateral separation from the wall(s). Having thus formulated the far-field problem, and gathering the necessary information from the near field (e.g., the blockage coefficient distribution, etc.), we then solve the far-field problem by matched expansions on the assumption that the separation from the wall(s) is much smaller than the ship's length.

Although the practical validity of the assumptions in the second section must be examined more thoroughly in the future, the method allows us to obtain relatively simple final results for both steady and unsteady cases of ship motion in restricted shallow water.

Some formulae and numerical results are given illustrating the dependence of the hydrodynamic coefficients on such parameters as yaw angle, rudder deflection, blockage coefficient, lateral separation from the wall, and Strouhal number.

II. TWO-DIMENSIONAL SHIP-TO-WALL INTERACTION PROBLEMS*

A. Two-Dimensional Steady Flow Past a Ship Close to a Bank

As shown in Figure 1, consider the steady two-dimensional flow of an inviscid and incompressible fluid past a ship in close proximity to a plane rigid wall. The coordinate system Oxy is moving together with the ship at constant velocity; the x axis is sliding along the wall, and the y axis is directed outward from the wall and passes through midstern.

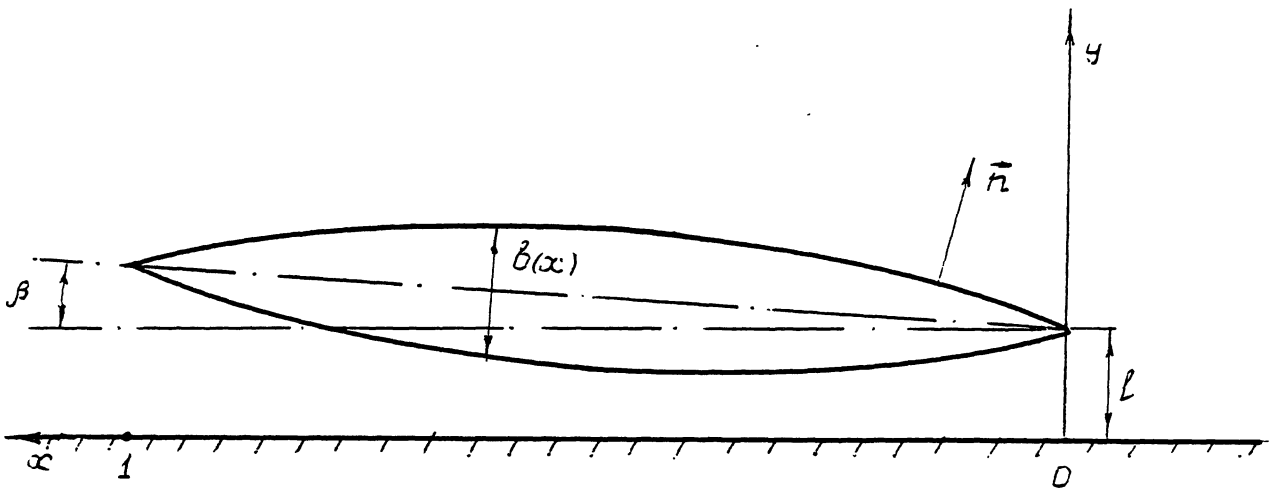

Figure 1. Two-Dimensional Problem for a Ship Moving Close to a Bank

In what follows, all values are nondimensional, the characteristic quantities being the length L of a ship and its velocity U.

Define the distance of the midstern of the ship from the bank as ℓ, local beam as $B(x)$, yaw angle as β (in radians). Introduce $y = y_r(x)$ and $y = y_\ell(x)$, functions describing the contours of the right and left side of the ship. As seen from Figure 1,

$$y_r = \ell + \beta x + \frac{1}{2}b(x) \; , \tag{2.1}$$

$$y_\ell = \ell + \beta x - \frac{1}{2}b(x) \; . \tag{2.2}$$

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*The material of this section was discussed in a presentation to the Panel on Analytical Ship/Wave Relations during the Annual Meeting of the Society of Naval Architects and Marine Engineers in New York on November 15, 1979.

The velocity potential $\Phi(x,y)$ for the perturbation velocities due to the ship has to satisfy the Laplace equation throughout the fluid domain and is subject to a kinematic condition on the wall and ship hull, an appropriate radiation condition, and the Kutta-Joukowsky condition at the stern. Note that the last condition allows us to take into account, though implicitly, some effects of viscosity of the real fluid.

Thus, the problem for the velocity potential can be formulated in the following way:

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad \text{in the fluid domain,} \tag{2.3}$$

$$\frac{\partial\Phi}{\partial n} = \cos(n,x) \quad \text{at} \quad y = y_r(x) \quad \text{and} \quad y = y_\ell(x) \ , \tag{2.4}$$

$$\frac{\partial\Phi}{\partial y} = 0 \quad \text{at} \quad y = 0 + 0 \ , \tag{2.5}$$

$$\nabla\Phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \ , \tag{2.6}$$

plus the Kutta-Joukowsky condition at $x = 0$, $y = \ell$.

Assume that the beam and the yaw angle of a ship are comparable in magnitude to the midstern distance from a wall, all of these dimensions being much smaller than the ship length. This allows us to define the orders of magnitude of the parameters and functions characterizing the problem as follows:

$$\beta \ , \quad b(x) \ , \quad y_r(x) \ , \quad y_\ell(x) = O(\ell) << 1 \ .$$

Introduce the stretched parameters and functions

$$\bar\beta = \beta/\ell \ ; \quad \bar b(x) = b(x)/\ell \ ; \quad \bar y_r(x) = y_r(x)/\ell \ ; \quad \bar y_\ell(x) = y_\ell(x)/\ell \ .$$

Then

$$\bar\beta \ , \quad \bar b(x) \ , \quad \bar y_r(x) \ , \quad \bar y_\ell(x) = O(1) \ .$$

In addition we assume that

$$b'(x) \ , \quad y_r'(x) \ , \quad y_\ell'(x) = O(\ell) \ ,$$

where a prime denotes differentiation with respect to the argument. After the orders of magnitudes have been defined, we can proceed to the solution of the problem, formulated above, by the method of matched asymptotic expansions. Our method is similar to that of Widnall and Barrows (1970), who treated a problem of a wing travelling parallel to a rigid boundary, using ℓ ($\ll 1$) as a small parameter.

At first, the asymptotic solutions will be constructed separately in the confined region between a ship and a bank (channel flow), in the vicinities of the bow and the stern (edge flows), and outside of the channel and edge flows (outer flow). These asymptotic solutions will then be matched in the overlap regions to account for physical interactions of different parts of the flow and to provide the uniqueness of solutions in the above-mentioned regions. Characteristic regions of the flow are shown in Figure 2.

Further on, we shall use the subscripts "ℓ" and "r" to denote quantities characterizing the flow respectively along the left and right sides of the ship.

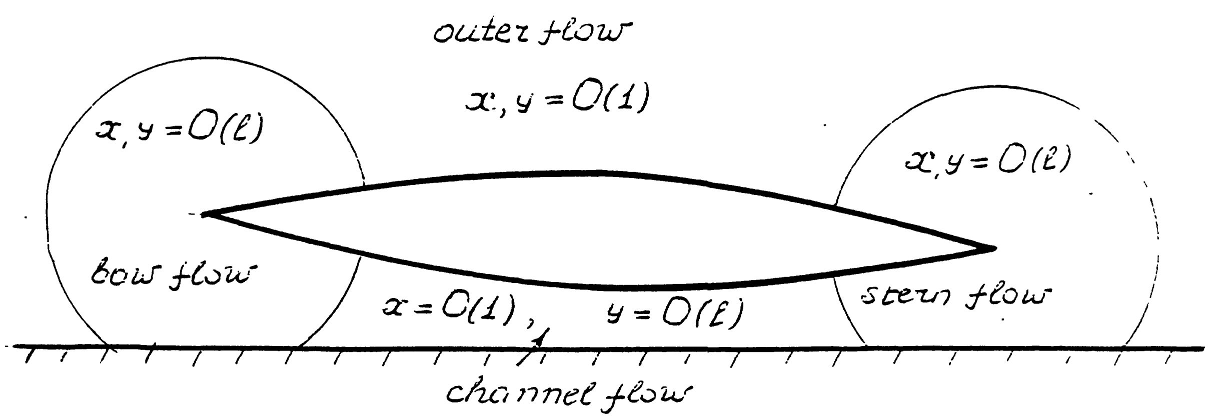

Figure 2. Characteristic Regions of the Flow Past a Ship Near a Bank

The Flow Between the Ship and the Bank (Channel Flow)

In a narrow region between the ship and the bank,

$$x = O(1) \ , \quad y = O(\ell) \ .$$

In order that this region not disappear in the limit $\ell \to 0$, the vertical coordinate y is stretched. Introduce "channel-flow" coordinates:

$$x = O(1) \ , \quad \overline{y} = y/\ell = O(1) \quad \text{as} \quad \ell \to 0 \ .$$

In the channel-flow region, we seek the corresponding potential Φ_ℓ in the form of the following asymptotic expansion:

$$\Phi_\ell = \phi_\ell + \ell^2\phi_\ell^* + \dots \ ; \tag{2.7}$$

$$\phi_\ell \ , \quad \phi_\ell^* = O(1) \ ; \quad \phi_\ell = \phi_{1\ell} + \phi_{2\ell}\ell \ln\frac{1}{\ell} + \ell\phi_{3\ell} \ . \tag{2.8}$$

It will be shown further that the presumed asymptotic structure of ϕ_ℓ given by (2.8) satisfies the condition of matching with the bow and stern flows.

Insert the channel-flow coordinates x and $\overline{y}$ into the formulae of the full problem, (2.3)-(2.6), and consider the problem for the velocity potential $\Phi = \Phi_\ell$ in the channel-flow region. We obtain

$$\frac{\partial^2\Phi_\ell}{\partial x^2} + \frac{1}{\ell^2}\frac{\partial^2\Phi_\ell}{\partial \overline{y}^2} = 0 \qquad \text{in channel-flow region;} \tag{2.9}$$

$$\frac{\partial\Phi_\ell}{\partial \overline{y}^2} = \ell^2\overline{y}_\ell' \left[\frac{\partial\Phi_\ell}{\partial x} - 1\right] \ \text{on} \ \ \overline{y} = \overline{y}_\ell(x) \ , \quad 1 > x > 0 \ ; \tag{2.10}$$

$$\frac{\partial\Phi_\ell}{\partial \overline{y}} = 0 \quad \text{at} \quad \overline{y} = 0 + 0, \quad x < \infty \ . \tag{2.11}$$

The condition at infinity and the Kutta-Joukowsky condition are lost in the channel-flow problem.

After the substitution of the assumed asymptotic expansion (2.7) into (2.9), we obtain

$$\frac{\partial^2\phi_\ell}{\partial x^2} + \ell^2\frac{\partial^2\phi_\ell^*}{\partial x^2} + \frac{1}{\ell^2}\frac{\partial^2\phi_\ell}{\partial \overline{y}^2} + \frac{\partial^2\phi_\ell^*}{\partial \overline{y}^2} = 0 \ , \ \ell \to 0 \ , \quad x \ , \overline{y} \ \ \text{fixed,}$$

wherefrom

$$\frac{\partial^2\phi_\ell}{\partial \overline{y}^2} = 0 \ , \tag{2.12}$$

$$\frac{\partial^2\phi_\ell}{\partial x^2} + \frac{\partial^2\phi_\ell^*}{\partial \overline{y}^2} = 0 \ . \tag{2.13}$$

Upon the substitution of (2.7) into (2.11), we obtain

$$\frac{\partial \phi_\ell}{\partial \bar{y}} = 0 , \qquad \bar{y} = 0 + 0 ; \tag{2.14}$$

$$\frac{\partial \phi_\ell^*}{\partial \bar{y}} = 0, \qquad \bar{y} = 0 + 0 . \tag{2.15}$$

Integrating (2.12) with respect to $\bar{y}$ and taking into account (2.14), we have

$$\frac{\partial \phi_\ell}{\partial \bar{y}} = 0 ,$$

and so

$$\phi_\ell = \phi_\ell(x) ,$$

i.e., with asymptotic error of the order of $O(\ell^2)$, the flow in the channel is one-dimensional. Integrate (2.13) with respect to $\bar{y}$:

$$\frac{\partial \phi_\ell^*}{\partial \bar{y}} = - \bar{y} \frac{\partial^2 \phi_\ell}{\partial x^2} + \tilde{\phi}_\ell^*(x) .$$

where $\tilde{\phi}_\ell^*(x)$ is an unknown function of x . To determine $\tilde{\phi}_\ell^*$, use (2.15), wherefrom

$$\tilde{\phi}_\ell^*(x) = 0 .$$

Therefore

$$\frac{\partial \phi_\ell^*}{\partial \bar{y}} = - \bar{y} \frac{\partial^2 \phi_\ell}{\partial x^2} . \tag{2.16}$$

Substitute the assumed expansion for Φ_ℓ into the boundary condition on the part of the hull surface facing the bank, (2.10):

$$\frac{\partial \phi_\ell}{\partial \bar{y}} + \ell^2 \frac{\partial \phi_\ell^*}{\partial \bar{y}} = \ell^2 \bar{y}_\ell' \left[\frac{\partial \phi_\ell}{\partial x} + \ell^2 \frac{\partial \phi_\ell^*}{\partial x} - 1 \right] .$$

Then

$$\frac{\partial \phi_\ell^*}{\partial \bar{y}} = \bar{y}_\ell' \left[\frac{\partial \phi_\ell}{\partial x} - 1 \right] \qquad \text{on } \bar{y} = \bar{y}_\ell(x) . \tag{2.17}$$

Combining (2.17) with (2.16) and requiring that $\bar{y} = \bar{y}_\ell(x)$ in (2.16), we obtain

$$-\bar{y}_\ell \frac{\partial^2 \phi_\ell}{\partial x^2} = \bar{y}'_\ell \left[\frac{\partial \phi_\ell}{\partial x} - 1\right] .$$

Hence, finally, the equation for ϕ_ℓ takes the form

$$\frac{d}{dx}\left[\bar{y}_\ell \frac{d\phi_\ell}{dx}\right] = \bar{y}'_\ell \quad , \quad 0 < x < 1 \quad , \qquad (2.18)$$

where $\bar{y}_\ell$ is a given function which depends on the form of the hull, distance from the wall, and the yaw angle:

$$\bar{y}_\ell = y_\ell/\ell = 1 + \bar{\beta}x - \frac{1}{2}\bar{b}(x) \quad ,$$

where $\bar{\beta} = \beta/\ell$; $\bar{b}(x) = b(x)/\ell$. Equation (2.18) represents mass conservation in the narrow channel between the port side of the ship and the bank.

The solution of the equation (2.18) is simple:

$$\frac{d\phi_\ell}{dx} = 1 + \frac{R_1}{\bar{y}_\ell(x)} \quad ,$$

$$\phi_\ell = x + R_1 \int \frac{dx}{\bar{y}_\ell(x)} + R_2 \quad .$$

The boundary conditions for ϕ_ℓ at points $x = 0,1$ (which will provide the determination of the unknowns R_1 and R_2) are to be obtained through matching with the bow and stern flows.

The Outer Flow

In the outer flow region, $x,y = O(1)$ as $\ell \to 0$. The corresponding potential must satisfy

* a given normal velocity condition on the right side (starboard) of the ship,
* no-normal-velocity condition on the wall outside of the hull,
* condition at infinity.

We seek the outer-flow potential in the form of the following asymptotic expansion:

$$\Phi_r = \ell\phi_r + O(\ell^2) \quad , \tag{2.19}$$

where ϕ_r is a solution of the problem

$$\frac{\partial^2\phi_r}{\partial x^2} + \frac{\partial^2\phi_r}{\partial y^2} = 0 \quad \text{in the upper half plane,} \quad y > 0 \; ; \tag{2.20}$$

$$\frac{\partial\phi_r}{\partial y} = - \overline{y}_r'(x) \quad , \quad y = 0 + 0, \quad 1 > x > 0 \; ; \tag{2.21}$$

$$\frac{\partial\phi_r}{\partial y} = 0 \; , \quad y = 0 + 0 \; , \quad (x < 0 \; , \quad x > 1) \; ; \tag{2.22}$$

$$\nabla\phi_r \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \; .$$

The solution to the above formulated problem can be readily obtained by the distribution along the segment $1 > x > 0$ of sources (sinks) with density equal to $-2\overline{y}_r'(x)$. In principle, a point source (sink), dipole, or multipole solution can be added at points $x = 1$ and $x = 0$ without violating the normal-velocity conditions (2.21), (2.22) (see Figure 3).

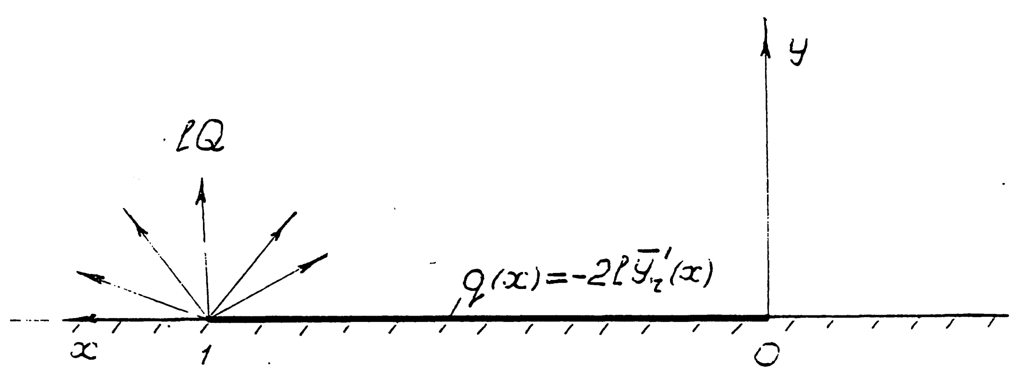

Figure 3. Schematized Outer Problem in Case of a Ship Moving Close to a Bank

By means of matching with the bow and stern potentials, it can be verified that with the same asymptotic accuracy as in (2.19) it is sufficient to keep only concentrated source (sink) solutions, namely, to place a concentrated source (sink) at the point $x = 1$. A source (sink) solution at $x = 0$ (stern) must be excluded because it does not match with the Kutta-Joukowsky stern solution.

Thus, we obtain the following expression for the outer potential on the right side of the hull:

$$\Phi_r = \ell\phi_r = \ell\left[\frac{Q}{2\pi}\ln(1-x) - \frac{1}{\pi}\int_0^1 \bar{y}_r(\xi)\ln|x-\xi|d\xi\right] . \tag{2.23}$$

The outer-flow problem is shown schematically in Figure 3. The strength ℓQ of the concentrated source located at $x = 1$ is to be determined by matching. Note that the expression for Φ_r is not uniformly valid in vicinities of the edges, i.e., we have lost the details of the flow near the edges.

For matching, we shall need the "inner" asymptotic representations of Φ_r near the points $x = 0,1$ (edges). The two-term inner expansion near the bow of the one-term outer potential (2.23) is obtained by expanding in terms of $\bar{\nu} = (x-1)/\ell$:

$$\Phi_r = \Phi_{rb} \sim \frac{\ell Q}{2\pi}\ln(\ell\bar{\nu}) + \frac{\ell^2}{\pi}\bar{y}'_r(1)\bar{\nu}\ln(\ell\bar{\nu}) + \frac{\ell^2}{\pi}A_1\bar{\nu} + \frac{\ell A_2}{\pi} , \tag{2.24}$$

where

$$\bar{\nu} = \nu/\ell ,$$

$$A_1 = -\bar{y}'_r(1) - \int_0^1\left[\bar{y}'_r(\xi) - \bar{y}'(1)\right]\frac{d\xi}{\xi} ,$$

$$A_2 = -\int_0^1 \bar{y}'_r(\xi)\ln(1-\xi)d\xi . \tag{2.24a}$$

The two-term inner expansion near the stern is obtained similarly by expanding (2.23) in terms of $\bar{\nu} = -x/\ell$:

$$\Phi_r = \Phi_{rs} \sim \frac{\ell^2}{\pi}\bar{y}'_r(0)\bar{\nu}\ln(\ell\bar{\nu}) + \frac{\ell^2 B_1}{\pi}\bar{\nu} + \frac{\ell B_2}{\pi} , \tag{2.25}$$

where

$$\bar{\nu} = \nu/\ell ,$$

$$B_1 = \tfrac{1}{2}Q - \bar{y}'_r(0) - \int_0^1\left[\bar{y}'_r(\xi) - \bar{y}'_r(0)\right]\frac{d\xi}{\xi} , \tag{2.25a}$$

$$B_2 = -\int_0^1 \bar{y}'_r(\xi)\ln\xi d\xi . \tag{2.25b}$$

Bow and Stern Flows

Consider first the bow region, where $y = O(\ell)$ and $x - 1 = \nu = O(\ell)$. After stretching of the bow coordinates $(\bar{y} = y/\ell,\ \bar{\nu} = \nu/\ell)$, the magnified bow region looks as in Figure 4.

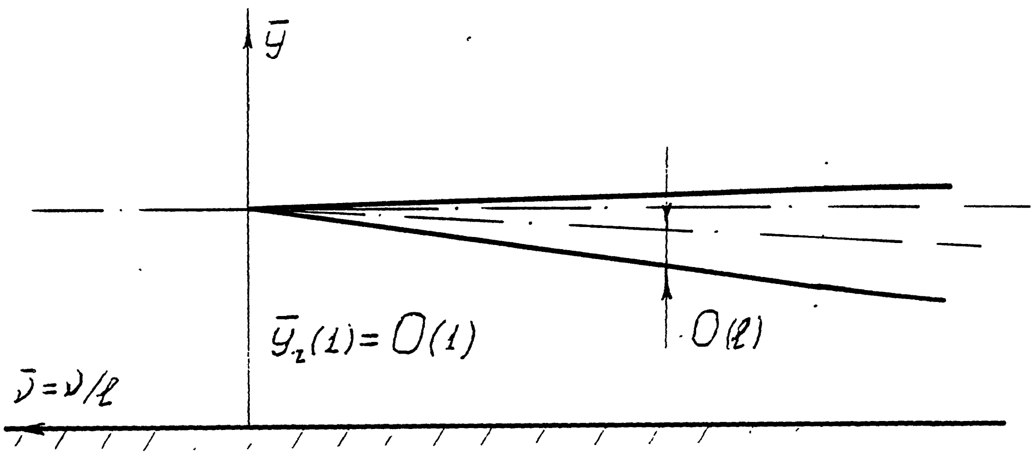

Figure 4. "Magnified" Bow-Flow Region

If, as assumed previously, the beam-distribution x-derivative, $b'(x)$, is of the order of $O(\ell)$, then the distance of a contour point from a horizontal line $\bar{y} = \bar{y}_r(1)$ in a magnified bow region (see Figure 4) is also of the order of $O(\ell)$. Therefore in the edge-flow regions the linear formulation is valid with an asymptotic error that is $O(\ell^2)$. In other words, the normal-velocity condition can be satisfied to such degree of accuracy if it is imposed on the line $\bar{y} = \bar{y}_r(1)$ in the vicinity of the bow and, similarly, on the line $\bar{y} = \bar{y}_r(0)$ near the stern.

The bow-flow potential Φ_b is governed by the 2-D Laplace equation and the following boundary conditions:

$$\frac{\partial \Phi_b}{\partial \bar{y}} = 0 \quad \text{for } |\bar{\nu}| < \infty\,,\quad \bar{y} = 0 + 0\,, \tag{2.26}$$

$$\frac{\partial \Phi_b}{\partial \bar{y}} = -D_r \quad \text{for } \bar{\nu} < 0\,,\quad \bar{y} = \bar{y}_r(1) + 0\,, \tag{2.27}$$

$$\frac{\partial \Phi_b}{\partial \bar{y}} = -D_\ell \quad \text{for } \bar{\nu} < 0\,,\quad \bar{y} = \bar{y}_\ell(1) - 0\,, \tag{2.28}$$

where D_r and D_ℓ are constants to be determined by matching. The solution of this problem can be shown to have the form

$$\Phi_b = a_1 \ell \phi_h + \ell^2 \phi_n + a_3 \ell^2 \bar{\nu} + a_4 \ell \quad , \tag{2.29}$$

where

a_i are constants to be determined by matching;

ϕ_h is the homogeneous solution, which satisfies a zero-normal-velocity condition on the wall and on the hull, representing the circulatory motion of the fluid in the vicinity of the bow (and thus containing a square-root singularity at the edge);

ϕ_n is the nonhomogeneous solution, which satisfies the non-zero normal-velocity conditions on the hull, (2.27), (2.28), and generates no normal velocity on the bank;

$a_3\ell^2\bar{\nu}+a_4\ell$ is a linear combination with respect to $\bar{\nu}$, which automatically satisfies the Laplace equation and creates no normal velocity on the bank or the hull.

The procedure for constructing solutions for ϕ_h and ϕ_n is shown in Appendix A.

On the hull, ϕ_h is given by the expression

$$\phi_h = \frac{\bar{y}_r(1)}{\pi} f_h \quad , \tag{2.30}$$

where

$$\pi\bar{\nu} = \bar{y}_r(1)(1 - e^{f_h} + f_h) \quad .$$

It is demonstrated in the Appendix A that the perturbation velocity corresponding to the homogeneous solution ϕ_h has a square root singularity at the bow, as anticipated.

The nonhomogeneous solution can be obtained in the following form (on the hull):

$$\phi_n = \int_0^{\bar{\nu}} u_n d\bar{\nu} \quad , \tag{2.31}$$

where

$$u_n = \frac{1}{\pi}[(D_r - D_\ell)\ln|1+\xi| + D_\ell \ln|\xi|] \quad ,$$

and ξ is related to $\bar{\nu}$ by the formula

$$\pi\bar{\nu} = \bar{y}_r(1)(1+\xi+\ln\xi) = (1+\bar{\beta})\ (1+\xi+\ln\xi) \quad .$$

The potential in the stern region is constructed similarly to that of the bow region, but the homogeneous solution is excluded, because it does not satisfy the Kutta-Joukowsky condition. With this in mind, we have near the stern:

$$\Phi_s = \ell^2\phi_n + \ell^2 b\bar{\nu} + \ell b_3 \quad , \tag{2.32}$$

where $\bar{\nu} \equiv -x/\ell$ and ϕ_n is given by

$$\phi_n = \int_0^{\bar{\nu}} u_n d\bar{\nu} \quad ,$$

with

$$u_n = \frac{1}{\pi}[(E_r - E_\ell)\ln|1+\xi| + E_\ell \ln|\xi|] \quad .$$

The auxiliary variable ξ is related to $\bar{\nu}$ by the equation

$$\pi\bar{\nu} = \bar{y}_r(0)[1 + \xi + \ln\xi] = 1 + \xi + \ln\xi \quad .$$

The constants E_r and E_ℓ are unknown and must be determined by matching. Note that, in the general case of a flow with circulation, the parameter b_3 in (2.32) takes different values on the right and left sides of the hull, i.e. $b_3^+ \neq b_3^-$.

Matching

The asymptotic solutions constructed so far in different parts of the flow contain the unknown parameters:

channel flow: R_1 and R_2 ,

outer flow: Q ,

bow and stern flows: a_i, D_r, D_ℓ, b_j, E_r, E_ℓ ,

which have to be determined by matching.

First, match the outer-flow potential with the bow-flow potential. Take

$$\Phi_b = a_1 \ell \phi_h + \ell^2 \phi_n + a_3 \ell^2 \bar{\nu} + \ell a_4 \quad . \tag{2.29}$$

Pass over to the outer coordinate $\nu = \bar{\ell}\nu$, fix ν and require that $\ell \to 0$ with $y = y_r(1) + 0$. Obtain the one-term outer expansion of the two-term bow expansion (2.29). The details of this procedure are shown in Appendix A. We write down the final overlap asymptotic expansion for Φ_b :

$$\Phi_b = \Phi_{br} \sim \frac{a_1 \ell}{\pi} \bar{y}_r(1) \ln \left| \frac{\pi\nu}{\ell \bar{y}_r(1)} \right| + \frac{\ell D_r}{\pi} \nu \left[\ln \left| \frac{\pi\nu}{\ell \bar{y}_r(1)} \right| - 1 \right] + a_3 \ell \nu + a_4 \ell \tag{2.33}$$

$$\text{for } \nu \text{ fixed}, \quad \ell \to 0 \ , \quad \bar{y} = \bar{y}_r(1) + 0 \ .$$

Return to the bow variable, $\bar{\nu} = \nu/\ell$, in (2.33):

$$\Phi_{br} \sim \frac{a_1 \ell}{\pi} \bar{y}_r(1) \ln \left| \frac{\pi\bar{\nu}}{\bar{y}_r(1)} \right| + \frac{\ell^2 D_r}{\pi} \bar{\nu} \left[\ln \left| \frac{\pi\bar{\nu}}{\bar{y}_r(1)} \right| - 1 \right] + a_3 \ell^2 \bar{\nu} + a_4 \ell \ ,$$

and compare the resulting expression with the previously obtained two-term bow expansion of the one-term outer potential (2.24),

$$\Phi_{rb} \sim \frac{\ell Q}{2\pi} \ln|\ell\bar{\nu}| + \frac{\ell^2}{\pi} \bar{y}'_r(1) \bar{\nu} \ln(\ell\bar{\nu}) + \frac{\ell^2 A_1}{\pi} \bar{\nu} + \frac{\ell A_2}{\pi} \ ,$$

wherefrom

$$a_1 = \frac{Q}{2\bar{y}_r(1)} \ , \qquad D_r = \bar{y}'_r(1) \ ,$$

$$a_3 = \frac{1}{\pi} \left\{ A_1 - y'_r(1) \left[\ln \left| \frac{\pi}{\ell \bar{y}_r(1)} \right| - 1 \right] \right\} \ ,$$

$$a_4 = \frac{1}{\pi} \left\{ A_2 - a_1 \bar{y}_r(1) \ln \left| \frac{\pi}{\ell \bar{y}_r(1)} \right| \right\} \ ,$$

where, as shown previously,

$$A_1 = -\bar{y}'_r(1) - \int_0^1 [\bar{y}'_r(\xi) - \bar{y}'_r(0)] \frac{d\xi}{1-\xi} \ ,$$

$$A_2 = -\int_0^1 \overline{y}_r'(\xi)\ln|1-\xi|d\xi \quad .$$

Second, match the bow-flow potential with the channel-flow potential. For this purpose write the two-term channel-flow expansion of the two-term bow expansion (2.29) (see Appendix A):

$$\Phi_b = \Phi_{b\ell} \sim \frac{a_1\overline{y}_r(1)}{\pi}\left[\frac{\pi\nu}{\overline{y}_r(1)} - \ell\right] + \frac{D_\ell}{\pi}\left[\frac{\pi\nu^2}{2\overline{y}_r(1)} - \ell\nu\right] + a_3\ell\nu + a_4\ell \quad . \qquad (2.34)$$

This expression must match with the function

$$\Phi_\ell \simeq \phi_\ell = \phi_{1\ell} + \phi_{2\ell}\ell\ \ln\frac{1}{\ell} + \phi_{3\ell} \quad , \qquad (2.35)$$

expanded for very small $\nu = \ell\bar{\nu}$. Comparing (2.34) and (2.35), one can easily deduce the following formulae:

$$\Phi_\ell(1) \simeq \phi_\ell(1) = \phi_{1\ell} + \phi_{2\ell}\ell\ln\frac{1}{\ell} + \phi_{3\ell}\ell = -\frac{a_1\ell}{\pi}\overline{y}_r(1) + \ell a_4 \ , \qquad (2.36)$$

or, taking account of the expression for a_4,

$$\phi_{1\ell}(1) = 0 \quad , \qquad (2.37)$$

$$\phi_{2\ell}(1) = -\frac{a_1}{\pi}\overline{y}_r(1) \quad , \qquad (2.38)$$

$$\phi_{3\ell}(1) = \frac{1}{\pi}\left\{A_2 - a_1\overline{y}_r(1)\left[1+\ln\left|\frac{\pi}{\overline{y}_r(1)}\right|\right]\right\} \quad . \qquad (2.39)$$

In addition we obtain

$$a_1 = \left.\frac{d\phi_{1\ell}}{dx}\right|_{x=1} \quad , \qquad (2.40)$$

$$D_\ell = \overline{y}_r(1)\left.\frac{d^2\phi_{1\ell}}{dx}\right|_{x=1} \quad . \qquad (2.41)$$

The third step consists of the matching of the stern-flow potential with the outer-flow potential. The one-term outer expansion of the two-term stern potential solution has, as shown in Appendix A, the form

$$\Phi_s = \Phi_{sr} \sim \frac{\ell E_r}{\pi}\nu \left[\ln\left|\frac{\pi\nu}{\ell}\right| -1 \right] + b_2\ell\nu + \ell b_3^+ \tag{2.42}$$

for ν fixed, $\ell \to 0$, $\bar{y} = \bar{y}_r(0) + 0 = 1 + 0$. This must match the two-term stern expansion of the one-term outer potential, given by the formula (2.25):

$$\Phi_r = \Phi_{rs} \sim \frac{\ell^2}{\pi}\bar{y}_r'(0)\bar{\nu}\ \ln(\ell\bar{\nu}) + \frac{\ell^2 B_1}{\pi}\bar{\nu} + \frac{\ell B_2}{\pi}$$

for $\bar{\nu}$ fixed, $\ell \to 0$. Rewriting (2.25) in the outer-flow variable, $\nu = \ell\bar{\nu}$, we obtain

$$\Phi_r = \Phi_{rs} \sim \frac{\ell}{\pi}\bar{y}_r'(0)\nu\ln|\nu| + \frac{\ell B_1}{\pi}\nu + \frac{\ell B_2}{\pi} . \tag{2.43}$$

Requiring in the overlap region that $\Phi_{rs} = \Phi_{sr}$, i.e, comparing (2.42) with (2.43), we obtain

$$E_r = \bar{y}_r'(0) \quad , \tag{2.44}$$

$$b_2 = \frac{1}{\pi}\left[B_1 + \bar{y}_r'(0)(1-\ln\frac{\pi}{\ell}) \right] , \tag{2.45}$$

$$b_3^+ = \frac{B_2}{\pi} \quad , \tag{2.46}$$

where, as shown previously,

$$B_1 = -\frac{1}{2}Q - \bar{y}_r'(0) - \int_0^1 [\bar{y}_r'(\xi) - \bar{y}_r(0)]\frac{d\xi}{\xi} ,$$

$$B_2 = -\int_0^1 \bar{y}_r'(\xi)\ln\xi d\xi .$$

Fourth step: As mentioned before, because of the circulatory character of the flow, the potential is discontinuous at the stern. Therefore the matching of the stern flow to the channel flow is fulfilled using velocities rather than potentials. Take the two-term channel expansion of the two-term expansion of the stern velocity (see Appendix A):

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{s\ell}}{dx} \sim -\frac{E_\ell}{\pi}(\pi\nu-\ell) - \ell b_2 \quad . \tag{2.47}$$

This expression must match the function

$$\frac{d\Phi_\ell}{dx} \simeq \frac{d\phi_{1\ell}}{dx} + \frac{d\phi_{2\ell}}{dx}\ell \ln\frac{1}{\ell} + \frac{d\phi_{3\ell}}{dx}\ell \tag{2.48}$$

expanded for very small $\nu = \ell\bar{\nu}$. Comparing (2.47) and (2.48), we obtain

$$\frac{d\Phi_\ell}{dx}(0) \simeq \frac{d\phi_{1\ell}}{dx} + \frac{d\phi_{2\ell}}{dx}\ell \ln\frac{1}{\ell} + \frac{d\phi_{3\ell}}{dx}\ell = \frac{\ell E_\ell}{\pi} - \ell b_2 \ , \tag{2.49}$$

or, recalling the expression for the parameter b_2 ,

$$\frac{d\phi_{1\ell}}{dx}(0) = 0 \quad , \tag{2.50}$$

$$\frac{d\phi_{2\ell}}{dx}(0) = \frac{E_\ell}{\pi} \quad , \tag{2.51}$$

$$\frac{d\phi_{3\ell}}{dx}(0) = -\frac{1}{\pi}[B_1 - \bar{y}'_\ell(0) + \bar{y}'_r(0)(1-\ln\pi)] \quad . \tag{2.52}$$

Besides, we find

$$E_\ell = \left.\frac{d^2\phi_{1\ell}}{dx^2}\right|_{x=0} \quad . \tag{2.53}$$

Now, after the matching has been completed, the uniqueness of the asymptotic solutions obtained in different parts of the flow is provided. A uniformly valid expression for the potential can be constructed with help of the additive composition (see Van Dyke (1975)):

On the right side of the hull (starboard):

$$\Phi_c = \Phi_{cr} = \Phi_r + \Phi_b - \Phi_{br} + \Phi_s - \Phi_{sr} \quad . \tag{2.54}$$

On the left side of the hull (port):

$$\Phi_c = \Phi_{c\ell} = \Phi_\ell + \Phi_b - \Phi_{b\ell} + \Phi_s - \Phi_{s\ell} \quad . \tag{2.55}$$

Φ_c is the composite expansion for the velocity potential.

The Coefficients of the Side Force and Yaw Moment

The side force and yaw moment (with respect to midship) coefficients are defined as follows:

$$C_y = \frac{2Y}{\rho U^2 Lh} = \int_0^1 (p_{c\ell} - p_{cr})dx \quad , \tag{2.56}$$

$$M_z = \frac{2M_z}{\rho U^2 L^2 h} = \int_0^1 (p_{c\ell} - p_{cr})(x - \frac{1}{2})dx \quad , \tag{2.57}$$

where U is velocity of the ship, L is ship length, h is the undisturbed depth of water (note that ship is assumed wall sided, having zero bottom gap), and $p_{c\ell}$ and p_{cr} are composite pressure coefficients for the left and right sides of the ship, respectively.

In the channel-flow region, where the perturbations are not necessarily small (we did not linearize the channel-flow equations in the vicinity of the horizontal line $\bar{y} = \bar{y}_r(0) = 1$), the pressure coefficient is defined as

$$p = 2\frac{d\Phi}{dx} - \left(\frac{d\Phi}{dx}\right)^2 . \tag{2.58}$$

In the outer and edge regions, where linear formulations were used, we shall employ the linear expression for the pressure coefficient:

$$p = 2\frac{d\Phi}{dx} . \tag{2.59}$$

At first, we calculate the side force and yaw moment to the lowest order, i.e., for very small distances from the bank. In this case

$$\Phi_{c\ell} \approx \phi_{1\ell} = O(1) \; ; \quad \Phi_{cr} = O(\ell) \quad .$$

The function $\phi_{1\ell}$, as seen from (2.18), (2.37), and (2.50), is a solution of the problem

$$\frac{d}{dx}\left[\bar{y}_\ell(x)\frac{d\phi_{1\ell}}{dx}\right] = \bar{y}_\ell'(x) \quad , \qquad 1 \geqslant x \geqslant 0 \ ,$$

$$\phi_{1\ell}(1) = \frac{d\phi_{1\ell}}{dx}(0) = 0 \quad ,$$

where

$$\bar{y}_\ell(x) = 1 + \bar{\beta}x - \frac{1}{2}\bar{b}(x) \ ; \quad \bar{\beta}=\beta/\ell \quad ; \ \bar{b}(x) = b(x)/\ell \quad .$$

It is easy to obtain

$$\frac{d\phi_{1\ell}}{dx}(x) = 1 - \frac{\bar{y}_\ell(0)}{\bar{y}_\ell(x)} = 1 - \frac{1}{\bar{y}_\ell(x)} \quad .$$

Using this result and taking into account the formulae (2.40), (2.41), and (2.53), we can obtain the parameters a_1 , D_ℓ and E_ℓ :

$$a_1 = \frac{d\phi_{1\ell}}{dx}(1) = 1 - \frac{1}{\bar{y}_\ell(1)} \quad ,$$

$$D_\ell = \bar{y}_r(1)\frac{d^2\phi_{1\ell}}{dx^2}(1) = \bar{y}_r(1)\frac{1}{\bar{y}^2(1)}\,\bar{y}_\ell'(1) = \frac{\bar{y}_\ell'(1)}{\bar{y}_\ell(1)} \quad ,$$

$$E_\ell = \frac{d^2\phi_{1\ell}}{dx^2}(0) = \bar{y}_\ell'(0) \quad .$$

Side-force and yaw-moment coefficients to the lowest order can be calculated with help of the following formulae:

$$C_y \simeq C_{y_1} = \int_0^1\left[2\frac{d\phi_{1\ell}}{dx} - \left(\frac{d\phi_{1\ell}}{dx}\right)^2\right]dx$$

$$= 1 - \int_0^1 \frac{dx}{\bar{y}_\ell^2(x)} = 1 - \int_0^1 \frac{dx}{\left[1+\bar{\beta}x-\frac{1}{2}\bar{b}(x)\right]^2} \quad ,$$

$$M_z \simeq M_{z_1} = \int_0^1 (x-\tfrac{1}{2})\left[2\frac{d\phi_{1\ell}}{dx} - \left(\frac{d\phi_{1\ell}}{dx}\right)^2\right] dx$$

$$= \int_0^1 (x-\tfrac{1}{2})\left(1 - \frac{1}{\bar{y}_\ell^2(x)}\right) dx = -\int_0^1 (x-\tfrac{1}{2})\frac{dx}{\bar{y}_\ell^2(x)} \quad .$$

Solving similarly for $\phi_{2\ell}$ and $\phi_{3\ell}$, we obtain

$$\frac{d\phi_{2\ell}}{dx} = \frac{1}{\overline{y}_\ell(x)}\,\frac{d\phi_{2\ell}}{dx}(0) \quad ,$$

$$\frac{d\phi_{3\ell}}{dx} = \frac{1}{\overline{y}_\ell(x)}\,\frac{d\phi_{3\ell}}{dx}(0) \quad ,$$

where $\frac{d\phi_2}{dx}(0)$ and $\frac{d\phi_{3\ell}}{dx}(0)$ are given by the formulae (2.51) and (2.52).

Using the expressions (2.56) and (2.57), the following formulae for C_y and M_z can be obtained with the asymptotic error of the order of $O(\ell^2)$:

<u>Side-force coefficient:</u>

$$C_y = C_{y_1} + C_{y_2}\ell\ln\frac{1}{\ell} + C_{y_3}\ell + O(\ell^2) \ , \tag{2.60}$$

where

$$C_{y_1} = 1 - \int_0^1 \frac{dx}{\overline{y}_\ell^2(x)} \quad ,$$

$$C_{y_2} = \frac{2}{\pi}\left[\overline{y}_r(1) - 1 + \overline{y}_r'(0)(1-C_{y_1})\right] \quad ,$$

$$C_{y_3} = \frac{2}{\pi}\left\{B_2 - A_2 + [\overline{y}_r(1)-1]\left(1 - a_1 + \ln\left|\frac{\pi}{\overline{y}_r(1)}\right|\right) + \left[\overline{y}_r'(0)(\ln\pi - 1) + \overline{y}_\ell'(0) - B_1\right](1-C_{y_1})\right\} \ ;$$

<u>Yaw moment coefficient</u> (calculated with respect to midships):

$$M_z = M_{z_1} + M_{z_2}\ell\ln\frac{1}{\ell} + M_{z_3}\ell \quad , \tag{2.61}$$

where

$$M_{z_1} = \int_0^1 (x-\tfrac{1}{2})\left[1 - \frac{1}{\overline{y}_\ell^2(x)}\right]dx = -\int_0^1 (x-\tfrac{1}{2})\frac{dx}{\overline{y}_\ell^2(x)} \quad ,$$

$$M_{z_2} = -\frac{2}{\pi}\left[\bar{y}_r'(0)M_{z_1} - \frac{1}{2}(\bar{y}_r(1)-1)\right] \quad ,$$

$$M_{z_3} = -\frac{2}{\pi}\left\{\left[\bar{y}_r'(0)(\ln\pi-1) + \bar{y}_\ell'(0)-B_1\right]M_{z_1} - \frac{1}{2}[\bar{y}_r(1)-1]\left[\ln\left|\frac{\pi}{\bar{y}_r(1)}\right| - 1 - a_1\right]\right.$$
$$\left. + \int_0^1 \bar{y}_r'(\xi)\left[\left(\frac{1}{2}-\xi\right)\ln\frac{1-\xi}{\xi} - 1\right] d\xi \right\} .$$

In the above expressions for C_y and M_z, parameters B_1, B_2, A_2 can be determined from the expressions (2.24a), (2.25a) and (2.25b).

Particular case: parabolic beam distribution. Consider a theoretical hull shape with a parabolic beam distribution. In this case,

$$\bar{b}(x) = \frac{b(x)}{\ell} = 4\bar{b}_0 x(1-x) \quad ,$$

where $b_0 = \ell\bar{b}_0$ = beam at midships. The side force coefficient as obtained from the general formula (2.60) is

$$C_y = C_{y_1} + C_{y_2}\ell \ln\frac{1}{\ell} + C_{y_3}\ell \quad ,$$

where

$$C_{y_1} = 1 - \frac{1}{q^2}\left\{\frac{\bar{\beta}+2\bar{b}_0}{1+\bar{\beta}} + 2\bar{b}_0 - \bar{\beta} + \frac{8\bar{b}_0}{q}\left[\arctan\frac{\bar{\beta}+2\bar{b}_0}{q} - \arctan\frac{\bar{\beta}-2\bar{b}_0}{q}\right]\right\},$$

$$q = \sqrt{8\bar{b}_0-(\bar{\beta}-2\bar{b}_0)^2} \quad ,$$

$$C_{y_2} = \frac{2}{\pi}\left\{\bar{\beta} + (\bar{\beta}+2\bar{b}_0)(1-C_{y_1})\right\} \quad ,$$

$$C_{y_3} = \left\{\frac{2}{\pi}\,2\bar{b}_0 + \bar{\beta}\left[1+\ln\left|\frac{\pi}{1+\bar{\beta}}\right|\right] + (1-C_{y1})[(\bar{\beta}+2\bar{b}_0)\ln\pi - 6\bar{b}_0]\right\} .$$

The yaw moment coefficient is

$$M_z = M_{z_1} + M_{z_2}\ell\ln\frac{1}{\ell} + \ell M_{z_3} \quad ,$$

where

$$M_{z_1} = -\frac{\bar{\beta}}{q^2}\left\{\frac{2(1-\bar{b}_0)+\bar{\beta}}{2(1+\bar{\beta})} - \frac{2}{q}\left[\arctan\frac{\bar{\beta}+2\bar{b}_0}{q} - \arctan\frac{\bar{\beta}-2\bar{b}_0}{q}\right]\right\}$$

$$M_{z_2} = -\frac{2}{\pi}\left[(\bar{\beta}+2\bar{b}_0)M_{z_1} - \frac{1}{2}\bar{\beta}\right],$$

$$M_{z_3} = -\frac{2}{\pi}\left\{\left[(\bar{\beta}+2\bar{b}_0)\ln\pi-6\bar{b}_0\right]M_{z_1} - \frac{1}{2}\bar{\beta}\ln\left|\frac{\pi}{1+\bar{\beta}}\right|\right\}.$$

Results of some computations are presented in Figures 5 and 6.

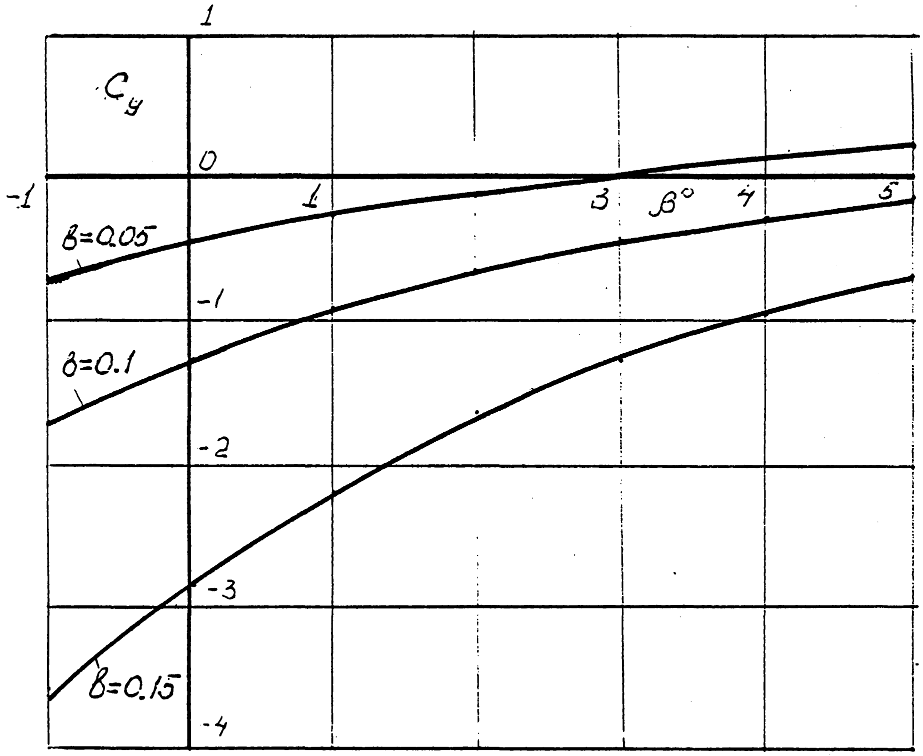

Figure 5. Side Force Coefficient versus Yaw Angle For Different Values of the Beam (Parabolic Ship)

For zero yaw angle, the side force is always negative, representing a suction force toward the bank. Positive yaw angle adds a repulsive bank force, which can be large enough to cancel the suction force. The moment is always in the direction tending to increase yaw angle.

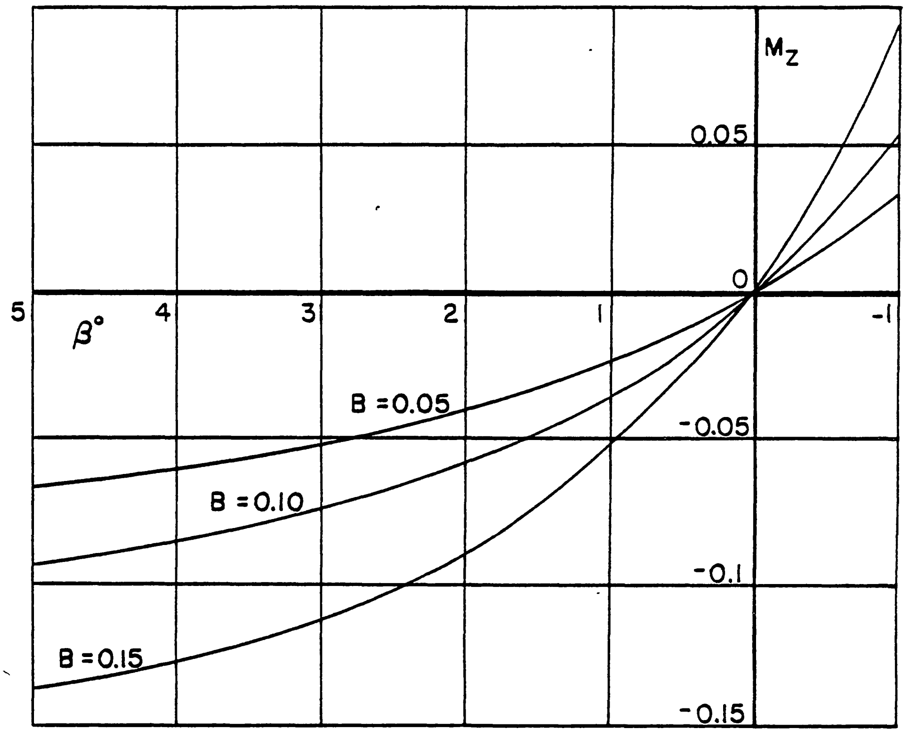

Figure 6. Yaw Moment Coefficient Versus Yaw Angle for Different Values of the Beam (Parabolic Ship)

B. Two-Dimensional Steady Flow Past a Ship in a Canal

Consider the steady potential flow past a ship in a canal bounded by plane, rigid walls. The flow has a constant velocity U far upstream and downstream of the ship. The picture of a ship in a canal, together with the Cartesian coordinate system Oxy is given in Figure 7, where the following notations are introduced: ℓ is the distance of the midstern of the ship from the left bank, $b(x)$ the local beam, β the yaw angle, ε the width of the canal. As in the previous analysis for the case of a ship close to the single wall, all values are nondimensional, characteristic quantities being the length L of the ship and the stream flow velocity U. The coordinate system is attached to the ship and moves with it. The contours of the right and left sides of the ship are described by functions $y_r(x)$ and $y_\ell(x)$,

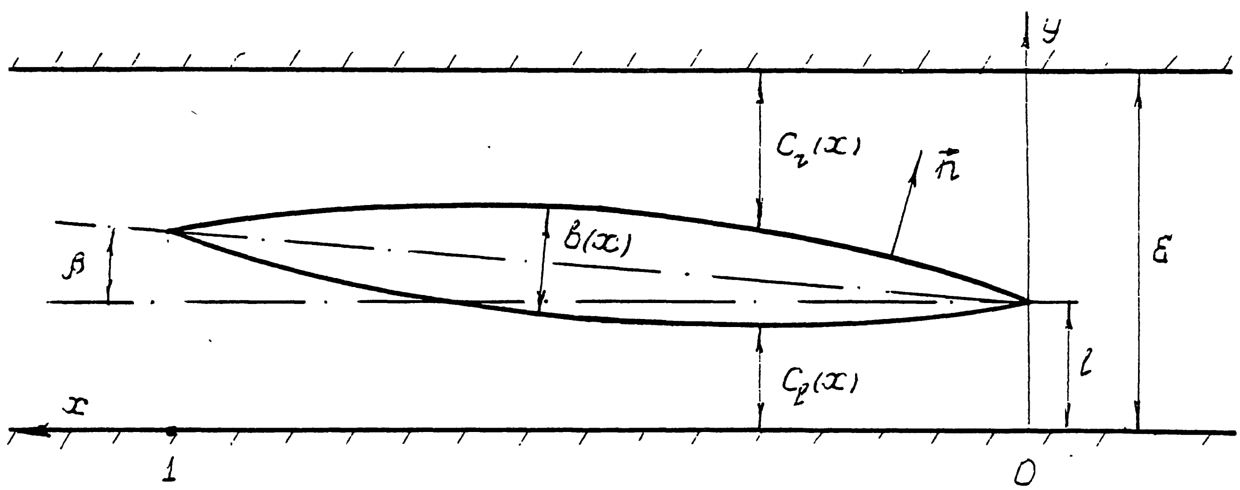

Figure 7. Coordinate System for a Ship in a Canal

defined by the formulae (2.1) and (2.2). The velocity potential $\Phi(x,y)$ is governed by the equations (2.3)-(2.5) of the previous part, with the additional condition of no normal velocity on the right side wall of the canal, that is,

$$\frac{\partial \Phi}{\partial y} = 0 \quad \text{on} \quad y = \varepsilon - 0 \quad . \tag{2.62}$$

We assume that the beam, yaw angle (in radians), and midstern distance from the left wall are all of the same order of magnitude and at the same time are considerably smaller than the length of the ship, that is,

$$\beta \ , \ \ell \ , \ b(x) \ , \ y_r(x) \ , \ y_\ell(x) \ll 1 \ .$$

The asymptotic solution of the problem can take different form, depending on the order relationship between the width of the canal, ε , and the distance of the ship from the left wall, ℓ . We shall consider the two cases: 1) $\varepsilon = O(1)$ and 2) $\varepsilon = O(\ell)$.

1. Ship in a Wide Canal

We consider first the case that $\varepsilon = O(1)$, i.e., the canal width is comparable with the ship length; the ship is near the left bank but distant from the right bank. From the methodological point of view, this problem, when solved by matched expansions with ℓ as a small parameter, is quite similar to that of a ship moving close to a single wall (see Part A of this section). Examining characteristic regions of the flow shown in Figure 8, one can see that the channel-flow problem and the edge problems must be handled exactly as before in Part A. The construction of the outer flow is somewhat different and we shall focus attention on this part of the solution.

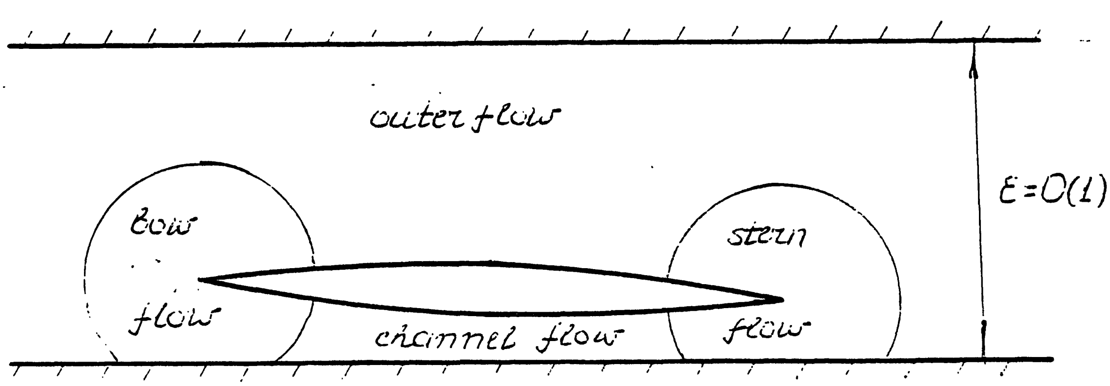

Figure 8. Characteristic Regions of the Flow Past a Ship in a Wide Canal

In the outer region (x,y fixed, $\ell \to 0$), we come to the following problem for the outer flow potential Φ_r (cf. (2.19)-(2.22)):

$$\Phi_r = \ell\phi_r + O(\ell^2) \;;$$

$$\frac{\partial^2\phi_r}{\partial x} + \frac{\partial^2\phi_r}{\partial y} = 0 \quad \text{in} \quad 0 < y < \varepsilon \;;$$

$$\frac{\partial\phi_r}{\partial y} = -\,\overline{y}_r'(x) \;, \quad y = 0 + 0 \;, \quad 1 > x > 0 \;;$$

$$\frac{\partial\phi_r}{\partial y} = 0 \quad \text{for} \quad \begin{cases} y = 0 + 0 \;(x < 0, \;\; x > 1) \;, \\ y = \varepsilon - 0 \;(|x| < \infty) \;; \end{cases} \tag{2.63}$$

$$\nabla\phi_r \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad .$$

The solution of this problem can be constructed by distributing along the segment $1 > x > 0$ sources (sinks) with density equal to $-2\overline{y}_r'$ and in addition placing a concentrated source of strength ℓQ at point $x = 1$. Both the distribution and the concentrated source must satisfy the no-normal-velocity condition on the right side wall of the canal (2.63). The outer problem is schematized in Figure 9. The potential of a unit source located on one wall of a canal of width ε is well known. On the wall $y = 0$, it is given by the function

$$\Phi_0 = \frac{1}{2\pi}\ln\sinh\left[\frac{\pi}{2\varepsilon}(x - x_0)\right] \;, \tag{2.64}$$

where x_0 is the abscissa of the location of the source. Introducing the notation $\delta = \pi/2\varepsilon$ and accounting for (2.64), we arrive at the following expression for the outer flow potential on $y = 0$:

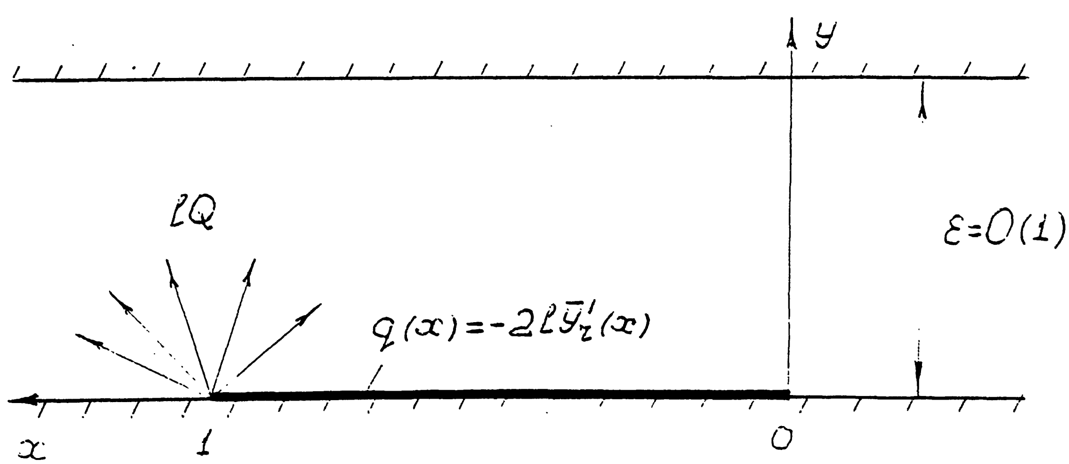

Figure 9. Schematized Outer Problem in a Case of a Ship Moving in a Wide Canal

$$\Phi_r = \ell\phi_r = \frac{\ell}{\pi}\left\{\frac{Q}{2}\ln\sinh[\delta(1-x)] - \int_0^1 \overline{y}_r'(\xi)\ln\sinh[\delta(x-\xi)]d\xi\right\} . \qquad (2.65)$$

The two-term bow and stern expansions of the one-term outer potential (2.65) are given by the formulae (2.24) and (2.25) with A_1 , A_2 , B_1, and B_2 determined from the following expressions:

$$A_1 = -\overline{y}_r'(1)\left[1 + \ln\frac{\sinh\delta}{\delta}\right] - \delta\int_0^1 [\overline{y}_r'(\xi)-\overline{y}_r'(1)]\frac{d\xi}{\tanh[\delta(1-\xi)]} , \qquad (2.66)$$

$$A_2 = \frac{Q}{2}\ln\delta - \int_0^1 \overline{y}_r'(\xi)\ln\sinh[\delta(1-\xi)]d\xi , \qquad (2.67)$$

$$B_1 = \frac{Q\delta}{2\tanh\delta} - \overline{y}_r'(0)\left[1 + \ln\frac{\sinh\delta}{\delta}\right] - \delta\int_0^1 [\overline{y}_r'(\xi)-\overline{y}_r'(0)]\frac{d\xi}{\tanh(\delta\xi)} , \qquad (2.68)$$

$$B_2 = \frac{Q}{2}\ln\sinh\delta - \int_0^1 \overline{y}_r'(\xi)\ln\sinh(\delta\xi)d\xi . \qquad (2.69)$$

It can be shown that, in the limit $\varepsilon \to \infty$ $(\delta \to 0)$, the expressions (2.65)-(2.69) become identical to the corresponding formulae (2.24)-(2.25) of Section A. The side-force and yaw-moment coefficients may be calculated using formulae (2.60) and (2.61) of Section A, with B_1 , B_2 , and A_2 given by equations (2.68), (2.69) and (2.67), respectively.

Considering the asymptotic structure of the solution for the case $\varepsilon = O(1)$, one can conclude that, when the width of a canal is comparable to the ship length and the distance of the ship from one wall goes to zero, the increment in forces acting on a ship caused by the presence of the second wall is $O(\ell)$. In other words, when $\varepsilon = O(1)$ and $\ell \to 0$, the ship-to-wall interaction forces are, with asymptotic error $O(\ell)$, the same as for the ship near the single wall.

2. Ship in a Narrow Canal.

Now let us pass over to the less trivial, more practical case 2), in which the width of the canal is comparable to the distance of the ship from the left side wall, i.e., $\varepsilon = O(\ell)$ as $\ell \to 0$. We shall solve this problem by matched asymptotics with ε (width of the canal) as a small parameter. The division of the flow into characteristic subflows is schematized in Figure 10. In this particular case, we have to construct separate asymptotic solutions in two confined regions between the ship sides and the walls of the canal (channel flows) and near the edges with no-normal-velocity condition on both walls of the canal (edge flows). Then we shall match the edge flows to the channel flows and to the upstream and downstream uniform flows.

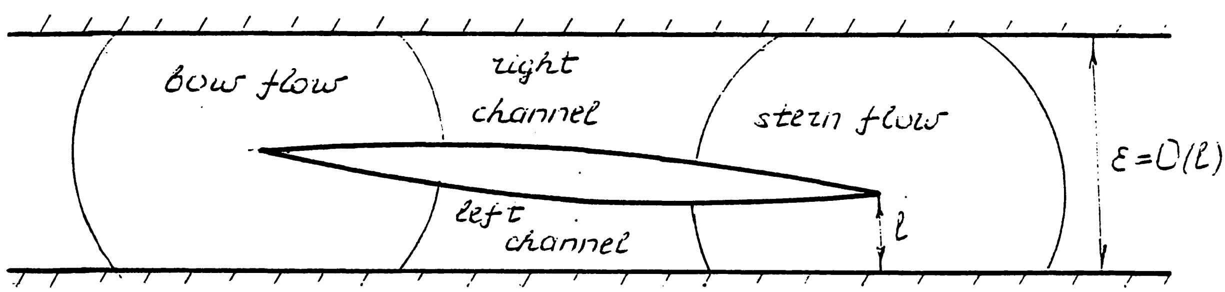

Figure 10. Characteristic Regions of the Flow Past a Ship in a Narrow Canal

The Flows Between the Sides of the Ship and the Canal Walls (Channel Flows)

To consider these flows with $\varepsilon \to 0$ and $x = O(1)$, it is necessary to stretch the vertical coordinate y so that the channels do not disappear in the limit. Introduce:

$$\tilde{y} = y/\varepsilon\ , \quad \tilde{\ell} = \ell/\varepsilon\ , \quad \tilde{b}(x) = b(x)/\varepsilon\ , \quad \tilde{\beta} = \beta/\varepsilon\ , \quad \tilde{y}_r = y_r/\varepsilon\ , \quad \tilde{y}_\ell = y_\ell/\varepsilon\ .$$

In the channel flow regions we assume, as in Section A, the following asymptotic expansions of the velocity potential:

In the right channel,

$$\Phi_r = \phi_r + \varepsilon^2 \phi_r^* + \dots\ ,$$

In the left channel,

$$\Phi_\ell = \phi_\ell + \varepsilon^2 \overset{*}{\phi}_\ell + \dots \quad ,$$

where

$$\phi_r \ , \quad \overset{*}{\phi}_r \ , \quad \phi_\ell \ , \quad \overset{*}{\phi}_\ell = O(1) \quad .$$

Introducing the stretched coordinate $\tilde{y} = y/\varepsilon$ and the assumed asymptotic expansions into the full problem, we obtain in exactly the same way as in Section A the following equations for ϕ_r and ϕ_ℓ :

$$\phi_r = \phi_r(x) \quad , \quad \phi_\ell = \phi_\ell(x) \quad ;$$

$$\frac{d}{dx}[(\tilde{y}_r - 1)\frac{d\phi_r}{dx}] = \tilde{y}_r' \ , \qquad 1 > x > 0 \quad ; \tag{2.70}$$

$$\frac{d}{dx}(\tilde{y}_\ell \frac{d\phi_\ell}{dx}) = \tilde{y}_\ell' \quad , \qquad 1 > x > 0 \quad . \tag{2.71}$$

Equations (2.70) and (2.71) can alternatively be written as follows:

$$\frac{d}{dx}(\tilde{c}_r \frac{d\phi_r}{dx}) = \tilde{c}_r' \quad , \tag{2.72}$$

$$\frac{d}{dx}(\tilde{c}_\ell \frac{d\phi_\ell}{dx}) = \tilde{c}_\ell' \quad , \tag{2.73}$$

where c_r and c_ℓ are the right and left side clearances, respectively,

$$c_r(x) = \varepsilon - y_r(x) \quad , \quad c_\ell(x) = y_\ell(x) \quad , \tag{2.74}$$

and $\tilde{c}_r = c_r/\varepsilon$, $\tilde{c}_\ell = c_\ell/\varepsilon$. The solution of the channel-flow equations is simple:

$$\Phi_r \simeq \phi_r = x + R_{1r} \int \frac{dx}{\tilde{c}_r(x)} + R_{2r} \quad ,$$

$$\Phi_\ell \simeq \phi_\ell = x + R_{1\ell} \int \frac{dx}{\tilde{c}_\ell(x)} + R_{2\ell} \quad .$$

The boundary conditions for ϕ_r and ϕ_ℓ at points $x = 0,1$ (which determine the unknown constants R_{1r} , R_{2r} , $R_{1\ell}$, and $R_{2\ell}$) are to be determined through matching with bow and stern flows.

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Consider the bow region in stretched coordinates:

$$\tilde{y} = y/\varepsilon \ , \quad \tilde{\nu} = \nu/\varepsilon \ , \quad \text{where} \quad \nu \equiv x - 1 \quad .$$

In the magnified region, the bow-flow potential should satisfy the 2-D Laplace equation and the linearized boundary conditions*

$$\frac{\partial\Phi_b}{\partial\tilde{y}} = 0 \quad \text{at} \quad |\tilde{\nu}| < \infty \ , \quad \tilde{y} = 0,1 \quad , \tag{2.75}$$

$$\frac{\partial\Phi_b}{\partial\tilde{y}} = - D_r \quad \text{at} \quad \tilde{\nu} < 0 \ , \quad \tilde{y} = \tilde{y}_r(1) + 0 \quad , \tag{2.76}$$

$$\frac{\partial\Phi_b}{\partial\tilde{y}} = - D_\ell \quad \text{at} \quad \tilde{\nu} < 0 \ , \quad \tilde{y} = \tilde{y}_\ell(1) - 0 \quad , \tag{2.77}$$

where D_r and D_ℓ are unknown parameters.

The solution of this problem has the form given by (2.29), i.e.,

$$\Phi_b = a_1\varepsilon\phi_h + \varepsilon^2\phi_n + a_3\varepsilon^2\tilde{\nu} + a_4\varepsilon \quad , \tag{2.78}$$

where

a_i are constants to be determined by matching;

ϕ_h is the homogeneous solution, which satisfies a zero-normal-velocity condition on the hull and on the canal walls;

ϕ_n is the nonhomogeneous solution, which satisfies a zero-normal-velocity condition on the canal walls and nonzero-normal-velocity conditions, (2.76) and (2.77), on the hull.

The determination of the functions ϕ_h and ϕ_r is given in Appendix B.

For points on the hull, the homogeneous solution ϕ_h is defined by the formula (B.4) (see Appendix B) :

$$\tilde{\nu} = \phi_h - \frac{1}{\pi}\ln\left|1 - q + q\exp\left(\frac{\pi\phi_h}{q}\right)\right| \quad , \tag{B.4}$$

where $q = \tilde{y}_r(1)$. The nonhomogeneous solution is

$$\phi_n = \int_0^{\tilde{\nu}} u_n d\tilde{\nu} \quad \text{on the hull,} \tag{2.79}$$

where

$$u_n = \frac{1}{\pi}[(D_r - D_\ell)\ln|1 + \xi| + D_\ell\ln|\xi|] \ , \quad \tilde{\nu} = \nu/\varepsilon = (x - 1)/\varepsilon \quad ,$$

---

*Linearization of the edge-flow boundary conditions is based upon the same considerations as in Section A.

and the relationship of ξ with $\tilde{\nu}$ is given by

$$\pi\tilde{\nu} = q\ln|\xi| - \ell n|1 - q + q\xi| \ , \quad \xi < 0 \ , \quad q = \tilde{y}(1) \ .$$

The stern flow potential can be shown to be (cf. (2.32))

$$\Phi_s = \varepsilon^2\phi_n + \varepsilon^2 b_2\tilde{\nu} + \varepsilon b_3 \ , \tag{2.80}$$

where $\tilde{\nu} = \nu/\varepsilon = -x/\varepsilon$, $b_3^+ \neq b_3^-$ and ϕ_n is given by the formula

$$\phi_n = \int_0^{\tilde{\nu}} u_n d\tilde{\nu} \ ,$$

where

$$u_n = \frac{1}{\pi}[(E_r - E_\ell)\ln|1 + \xi| + E_\ell \ln|\xi|] \ ,$$

with unknown constants E_r and E_ℓ , and ξ related to $\tilde{\nu}$ by (B.8) with q replaced by $q_1 = \tilde{y}(0) = \tilde{\ell}$.

Matching

Looking through the results obtained so far, we see that they contain several unknown parameters:

channel flows: R_{1r} , R_{2r} , $R_{1\ell}$, $R_{2\ell}$;

bow and stern flows: a_i , D_r , D_ℓ , b_j , E_r , and E_ℓ .

These parameters can be determined if we take into account the physical interaction of different parts of the flow. Mathematically speaking, we have to perform the matching of the corresponding solutions. We shall adopt the same sequence of matching steps as in Section A and, in addition, match the edge solutions with the upstream and downstream uniform flows.

First, match the right-channel-flow potential with the bow-flow potential. Take the two-term right-channel expansion of the two-term bow solution (2.78) (see Appendix B):

$$\Phi_b = \Phi_{br} \sim -\frac{a_1 q}{1-q}\left[\nu + \varepsilon\frac{\ln q}{\pi}\right] + \frac{D_r}{\pi}\left[\frac{\pi\nu^2}{2(q-1)} + \frac{\varepsilon\nu\ln q}{(q-1)}\right] + a_3\varepsilon\nu + a_4\varepsilon \ , \tag{B.9}$$

where $q = \tilde{y}_r(1)$. This expansion must match with the function

$$\Phi_r = \phi_r + O(\varepsilon^2) \tag{2.81}$$

expanded for very small $\nu = \varepsilon\tilde{\nu}$. Comparing (B.9) and (2.81), we find

$$\Phi_r(1) \simeq \phi_r(1) = \phi_{1r}(1) + \varepsilon\phi_{2r}(1) = -\frac{\varepsilon a_1 q \ln q}{\pi(1-q)} + \varepsilon a_4 \quad , \tag{2.82}$$

or

$$\phi_{1r}(1) = 0 \quad , \tag{2.83}$$

$$\phi_{2r}(1) = \frac{a_1 q \ln q}{q-1} + a_4 \quad , \tag{2.84}$$

$$a_1 = \frac{q-1}{q}\frac{d\phi_{1r}}{d\nu} = \frac{q-1}{q}\frac{d\phi_{1r}}{dx}\bigg|_{x=1} \quad ,$$

$$D_r = (q-1)\frac{d^2\phi_{1r}}{d\nu^2} = (q-1)\frac{d^2\phi_{1r}}{dx^2}\bigg|_{x=1} \quad .$$

Second step: Matching the left-channel potential with the bow potential in similar fashion yields:

$$\Phi_\ell(1) \simeq \phi_\ell(1) = \phi_{1\ell}(1) + \varepsilon\phi_{2\ell}(1) = \frac{\varepsilon a_1}{\pi}\ln|1-q| + \varepsilon a_4 \quad , \tag{2.85}$$

or

$$\phi_{1\ell}(1) = 0 \quad , \tag{2.86}$$

$$\phi_{2\ell}(1) = \frac{a_1}{\pi}\ln|1-q| + a_4 \tag{2.87}$$

$$a_1 = \frac{d\phi_{1\ell}}{dx}\bigg|_{x=1} \quad ,$$

$$D_\ell = q\frac{d^2\phi_{1\ell}}{dx^2}\bigg|_{x=1} \quad .$$

Third step: Match the velocities in the right-channel and near the stern. Take the two-term right-channel expansion of the two-term expansion of the stern flow velocity (see Appendix B),

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{sr}}{dx} \sim \frac{E_r}{\pi(1-q_1)}(\pi\nu + \varepsilon \ln q_1) - b_2\varepsilon \quad ; \quad \nu = -x \quad ,$$

and match it with

$$\frac{d\Phi_r}{dx} \simeq \frac{d\phi_r}{dx} = \frac{d\phi_{1r}}{dx} + \varepsilon\frac{d\phi_{2r}}{dx}$$

expanded for very small $\nu = \varepsilon\tilde{\nu}$. We obtain

$$\frac{d\Phi_r}{dx}(0) \simeq \frac{d\phi_r}{dx}(0) = \frac{d\phi_{1r}}{dx}(0) + \varepsilon\frac{d\phi_{2r}}{dx}(0) = \frac{\varepsilon E_r \ln q_1}{\pi(1-q_1)} - \varepsilon b_2 \quad , \tag{2.88}$$

or

$$\frac{d\phi_{1r}}{dx}(0) = 0 \quad , \tag{2.89}$$

$$\frac{d\phi_{2r}}{dx}(0) = \frac{E_r \ln q_1}{\pi(1-q_1)} - b_2 \quad , \tag{2.90}$$

$$E_r = (q_1 - 1)\left.\frac{d^2\phi_{1r}}{dx^2}\right|_{x=0} \quad . \tag{2.91}$$

Fourth step: In analogous manner, match the velocities in the left-channel and near the stern to obtain

$$\frac{d\Phi_\ell}{dx}(0) \simeq \frac{d\phi_\ell}{dx}(0) = \frac{d\phi_{1\ell}}{dx}(0) + \varepsilon\frac{d\phi_{2\ell}}{dx}(0) = -\frac{\varepsilon E_\ell \ln|1-q_1|}{\pi q_1} - \varepsilon b_2 \quad , \tag{2.92}$$

or

$$\frac{d\phi_{1\ell}}{dx}(0) = 0 \quad , \tag{2.93}$$

$$\frac{d\phi_{2\ell}}{dx}(0) = -\frac{E_\ell \ln|1-q_1|}{\pi q_1} - b_2 \quad , \tag{2.94}$$

$$E_\ell = q_1\left.\frac{d^2\phi_{1\ell}}{dx^2}\right|_{x=0} \quad . \tag{2.95}$$

So far, we have the necessary information to compute a_1 , D_r , D_ℓ , E_r , E_ℓ , R_{1r} , R_{2r} , $R_{1\ell}$, $R_{2\ell}$. The parameters a_4 , b_3^+ , b_3^- , a_3 , and b_2 remain undetermined. To go on with our solution at the moment, we need only to define the parameters a_3 and b_2. This can be done through matching of the edge flows to the upstream and downstream uniform flows:

The asymptotic behavior of the edge solutions for upstream and far downstream is analyzed in Appendix B. The one-term upstream expansion of the two-term bow velocity turns out to be (cf. (B.16) and (B.17))

$$\frac{d\Phi_b}{dx} = \frac{d\phi_{bc}}{dx} \sim \frac{1}{\pi}[D_\ell \ln(1-q) - D_r \ln q]\varepsilon + a_3\varepsilon \quad . \tag{2.96}$$

The one-term downstream expansion of the two-term stern velocity has the form

$$\frac{d\Phi_s}{dx} = \frac{d\phi_{sc}}{dx} \sim -\frac{1}{\pi}[E_\ell \ln(1-q_1) - E_r \ln q_1]\varepsilon - b_2\varepsilon \quad . \tag{2.97}$$

The condition at infinity of the full problem requires that the flow be uniform far upstream and far downstream from the ship, i.e., the perturbation velocity at infinity should equal zero:

$$\frac{d\Phi}{dx} = 0 \quad . \tag{2.98}$$

Comparing (2.98) with (2.96) and (2.97), we find the expressions for the parameters:

$$a_3 = \frac{1}{\pi}[D_r \ln q - D_\ell \ln(1-q)] \quad , \quad q = \tilde{y}_r(1) \quad ; \tag{2.99}$$

$$b_2 = \frac{1}{\pi}[E_r \ln q_1 - E_\ell \ln(1-q_1)] \quad , \quad q_1 = \tilde{y}_r(0) = \tilde{\ell} \quad ; \tag{2.100}$$

The uniformly valid expression (additive composition of the velocity potential) is defined as in (2.54), (2.55).

The Coefficients of the Side Force and Yaw Moment

The side-force and yaw-moment (amidships) coefficients are determined as in Section A, the pressure coefficients in the channels being defined by (2.58). To the lowest order, i.e., for a very narrow canal, the composite velocity potential can be approximated as

$$\Phi_{cr} \simeq \phi_{1r} + O(\varepsilon) \; , \quad \Phi_{c\ell} \simeq \phi_{1\ell} + O(\varepsilon) \quad ,$$

the functions ϕ_{1r} and $\phi_{1\ell}$ satisfying the equations

$$\frac{d}{dx}(\tilde{c}_r \frac{d\phi_{1r}}{dx}) = \tilde{c}'_r \quad ,$$

$$\frac{d}{dx}(\tilde{c}_\ell \frac{d\phi_{1\ell}}{dx}) = \tilde{c}'_\ell \quad ,$$

with boundary conditions

$$\phi_{1r}(1) = \phi_{1\ell}(1) = 0 \quad , \quad \frac{d\phi_{1r}}{dx}(0) = \frac{d\phi_{1\ell}}{dx}(0) = 0 \quad ,$$

where

$$\tilde{c}_r = 1 - \tilde{\ell} - \tilde{\beta}x - \frac{1}{2}\tilde{b}(x) \quad , \qquad (2.101)$$

$$\tilde{c}_\ell = \tilde{y}_\ell = \tilde{\ell} + \tilde{\beta}x - \frac{1}{2}\tilde{b}(x) \quad , \qquad (2.102)$$

$$\tilde{\ell} = \ell/\varepsilon \ , \quad \tilde{\beta} = \beta/\varepsilon \ , \quad \tilde{b}(x) = b(x)/\varepsilon \quad .$$

It is easy to see that

$$\frac{d\phi_{1r}}{dx}(x) = 1 - \frac{\tilde{c}_r(0)}{\tilde{c}_r(x)} \quad ; \quad \frac{d\phi_{1\ell}}{dx}(x) = 1 - \frac{\tilde{c}_\ell(0)}{\tilde{c}_\ell(x)} \quad , \qquad (2.103)$$

where

$$\tilde{c}_r(0) = 1 - \tilde{\ell} \ , \quad \tilde{c}_\ell(0) = \tilde{\ell} \quad .$$

Therefore, from the formulae given earlier, it follows that

$$a_1 = \frac{q-1}{q}\frac{d\phi_{1r}}{dx}(1) = \frac{d\phi_{1\ell}}{dx}(1) = \frac{\tilde{c}_\ell(1)-\tilde{c}_\ell(0)}{\tilde{c}_\ell(1)} = \frac{\tilde{\beta}}{\tilde{\ell}+\tilde{\beta}} = \frac{q-\tilde{\ell}}{q} \quad ,$$

$$D_r = (q - 1)\frac{d^2\phi_{1r}}{dx^2}(1) = - \frac{\tilde{c}_r(0)}{\tilde{c}_r(1)}\tilde{c}'_r(1) = \frac{1-\tilde{\ell}}{1-\tilde{\ell}-\tilde{\beta}}[\tilde{\beta} + \frac{1}{2}\tilde{b}'(1)] \quad ,$$

$$D_\ell = q\frac{d^2\phi_{1\ell}}{dx^2}(1) = [1 - \tilde{c}_r(1)]\frac{\tilde{c}_\ell(0)}{\tilde{c}_\ell^2(1)}\tilde{c}'_\ell(1)$$

$$= \frac{\tilde{\ell}(1-\tilde{\ell}-\tilde{\beta})}{(\tilde{\ell}+\tilde{\beta})^2}[\tilde{\beta} - \frac{1}{2}\tilde{b}'(1)] \quad ,$$

$$E_r = (q_1 - 1)\frac{d^2\phi_{1r}}{dx^2}(0) = (\tilde{\ell} - 1)\frac{\tilde{c}'_r(0)}{\tilde{c}_r(0)} = \tilde{\beta} + \frac{1}{2}\tilde{b}'(0) = \tilde{y}'_r(0) \quad ,$$

$$E_\ell = q_1\frac{d^2\phi_{1\ell}}{dx^2}(0) = \tilde{\ell}\,\frac{\tilde{c}'_\ell(0)}{\tilde{c}_\ell(0)} = \tilde{\beta} - \frac{1}{2}\tilde{b}'(0) = \tilde{y}'_\ell(0) \quad .$$

Side-force and yaw-moment coefficients are, to the lowest order (very narrow canal),

$$C_y \simeq C_{y_1} = \int_0^1 \left[\frac{\tilde{C}_r^2(0)}{\tilde{C}_r^2(x)} - \frac{\tilde{C}_\ell^2(0)}{\tilde{C}_\ell^2(x)}\right] dx \ ,$$

$$M_z \simeq M_{z_1} = \int_0^1 (x - \tfrac{1}{2}) \left[\frac{\tilde{C}_r^2(0)}{\tilde{C}_r^2(x)} - \frac{\tilde{C}_\ell^2(0)}{\tilde{C}_\ell^2(x)}\right] dx \ .$$

Solving similarly for ϕ_{2r} and $\phi_{2\ell}$, we obtain

$$\frac{d\phi_{2r}}{dx} = \frac{\tilde{C}_r(0)}{\tilde{C}_r(x)} \frac{d\phi_{2r}}{dx}(0) \ ,$$

$$\frac{d\phi_{2\ell}}{dx} = \frac{\tilde{C}_\ell(0)}{\tilde{C}_\ell(x)} \frac{d\phi_{2\ell}}{dx}(0) \ ,$$

where $\dfrac{d\phi_{2r}}{dx}(0)$ and $\dfrac{d\phi_{2\ell}}{dx}(0)$ can be computed from formulae (2.90) and (2.94).

Using the expressions (2.56), (2.57), we arrive at the following formulae for C_y and M_z (with respect to midships) in the case of a narrow canal:

$$C_y = C_{y_1} + \varepsilon C_{y_2} + O(\varepsilon^2) \ , \tag{2.104}$$

where

$$C_{y_1} = K_r - K_\ell \ ,$$

$$C_{y_2} = -\frac{2}{\pi}\left\{[\tilde{y}_r'(0)\tilde{\ell}\ln\tilde{\ell} + \tilde{y}_\ell'(0)(1-\tilde{\ell})\ln|1-\tilde{\ell}|]\left[\frac{K_r}{1-\tilde{\ell}} + \frac{K_\ell}{\tilde{\ell}}\right] + (q-\tilde{\ell})\,\frac{\tilde{\ell}}{q}\left[\frac{\ln|1-q|}{q} + \frac{\ln q}{1-q}\right]\right\} ,$$

$$K_r = \int_0^1 \frac{\tilde{c}_r^2(0)}{\tilde{c}_r^2(x)}\, dx \ , \quad K_\ell = \int_0^1 \frac{\tilde{c}_\ell^2(0)}{\tilde{c}_\ell^2(x)}\, dx \ ,$$

$$q = \tilde{y}_r(1) \ ;$$

$$M_z = M_{z_1} + \varepsilon M_{z_2} + O(\varepsilon^2) \ , \tag{2.105}$$

where

$$M_{z_1} = S_r - S_\ell \ ,$$

$$M_{z_2} = -\frac{2}{\pi}\left\{[\tilde{y}_r'(0)\tilde{\ell}\ln\tilde{\ell} + \tilde{y}_\ell'(0)(1-\tilde{\ell})\ln(1-\tilde{\ell})]\left[\frac{S_r}{1-\tilde{\ell}} + \frac{S_\ell}{\tilde{\ell}}\right] - \frac{(q-\tilde{\ell})}{2}\ \frac{\tilde{\ell}}{q}\left[\frac{\ln|1-q|}{q} + \frac{\ln q}{1-q}\right]\right\},$$

$$S_r = \int_0^1 (x-\frac{1}{2})\,\frac{\tilde{c}_r^2(0)}{\tilde{c}_r^2(x)}dx \quad , \quad S_\ell = \int_0^1 (x-\frac{1}{2})\,\frac{\tilde{c}_\ell^2(0)}{\tilde{c}_\ell^2(x)}dx \quad .$$

For the case when the yaw angle is much smaller than the canal width, i.e., $\tilde{\beta} << 1$ and $b(x) = 0$, we obtain the linearized formulas for a flat plate between the two walls:

$$C_y = \frac{\tilde{\beta}}{\tilde{\ell}(1-\tilde{\ell})}\left\{1 - \frac{4\varepsilon}{\pi}[\tilde{\ell}\ln\tilde{\ell} + (1-\tilde{\ell})\ln(1-\tilde{\ell})]\right\} + O(\varepsilon^2) \quad , \qquad (2.106)$$

$$M_z = \frac{\tilde{\beta}}{6\tilde{\ell}(1-\tilde{\ell})}\left\{1 - \frac{6\varepsilon}{\pi}[\tilde{\ell}\ln\tilde{\ell} + (1-\tilde{\ell})\ln(1-\tilde{\ell})]\right\} + O(\varepsilon^2) \quad . \qquad (2.107)$$

Passing to $\bar{\varepsilon} = \varepsilon/\ell = 1/\tilde{\ell}$, we derive from (2.106):

$$C_y = \frac{\beta}{\ell}\,\frac{\bar{\varepsilon}}{\bar{\varepsilon}-1}\left\{1 + \frac{4\ell}{\pi}[(1-\bar{\varepsilon})\ln(\bar{\varepsilon}-1)+\bar{\varepsilon}\ln\bar{\varepsilon}]\right\} \quad . \qquad (2.108)$$

The expression (2.108) is identical to the formula for the lift coefficient of a flat plate moving between two parallel walls (Minami et al. (1974)).

It should be noted that the only case to which the previous two-dimensional analysis can be applied directly is that of a wall-sided ship moving in water so shallow that the clearance between the ship bottom and the bottom of the water can be neglected. In this case, provided free-surface effects are negligible, i.e., for sufficiently small Froude numbers based on water depth, the flow becomes truly two-dimensional everywhere.

At the same time, the crossflow under the keel is reported to be of great importance in shallow-water problems. Numerical calculations performed by Beck (1976) and Yung (1978) indicate that even a slight gap can alter the flow considerably. Crossflow effects are dealt with in the following chapter, both for steady and unsteady linearized flows, where an attempt is made to combine Beck's (1976) shallow-water formulation with the method outlined above, aiming at simplifying the final results.

III. THREE-DIMENSIONAL LINEARIZED PROBLEMS OF SHIP-TO-WALL INTERACTIONS IN SHALLOW WATER

Consider the motion, steady or unsteady, of a slender ship near a bank in shallow water or in a canal of rectangular cross section. The Oxy plane is coincident with the calm-water level. The corresponding coordinate systems are shown in Figures 11 and 12. The velocity potential, $\Phi(x,y,z,t)$, representing the perturbations due to the ship motion must satisfy Laplace's equation everywhere in the fluid domain (except possibly in a trailing-vortex-wake region). This velocity potential is also subject to a free-surface boundary condition, kinematic conditions on the walls and on the ship hull, kinematic and dynamic conditions in the wake, the Kutta-Joukowsky condition at the stern, and an appropriate radiation condition.

Throughout this chapter, non-dimensional parameters and functions will be used, characteristic quantities being L, the ship length, and U, the steady forward velocity of the ship parallel to the wall(s). We shall introduce the following notations:

β = yaw angle,

$S(x)$ = cross-sectional area of ship at x,

ℓ = midstern distance at the waterline from the bank or, in case of a canal, from the left wall,

h = water depth,

ε = canal width,

$b(x)$ = local waterline beam of ship.

It is assumed that the beam and yaw angle of the ship are comparable to the water depth and at the same time are considerably smaller than the midstern distance from the wall, the latter being much smaller than the length of a ship, that is

$$b(x),\ \beta = O(h) << \ell << 1 \quad . \qquad (3.1)$$

In effect, we have two small parameters, (i) ℓ, as defined previously, and (ii) a parameter, say α, indicating the [small] order of magnitude of beam and yaw angle with respect to the ship/bank clearance. We keep only terms that are linear in α, which yields an airfoil-type problem with boundary conditions imposed on the undisturbed location of the center plane. With respect to ℓ, we develop an asymptotic solution in a manner quite similar to that of the previous section.

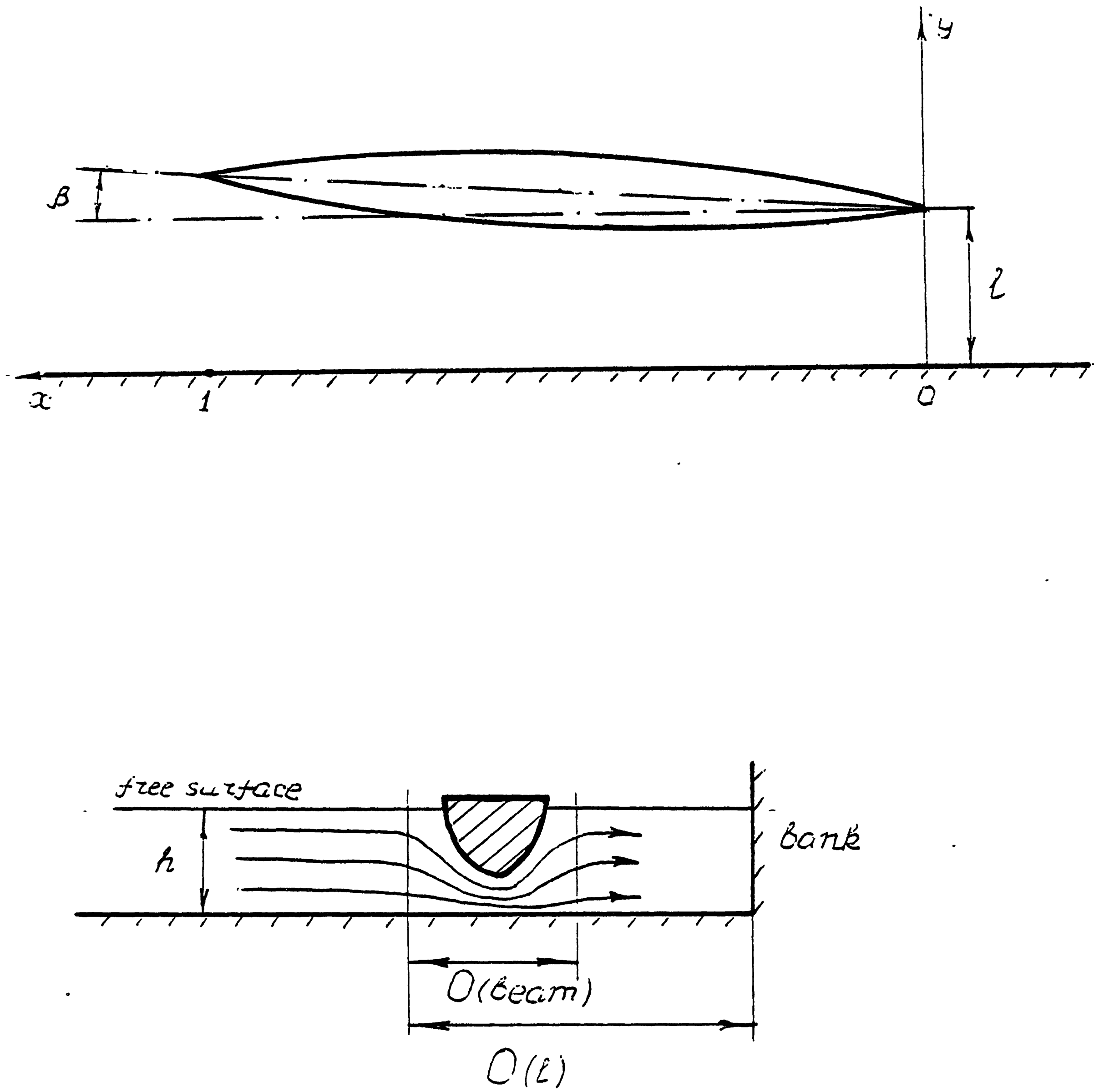

Figure 11. Three Dimensional Linearized Flow Past a Ship Close to a Bank in Shallow Water

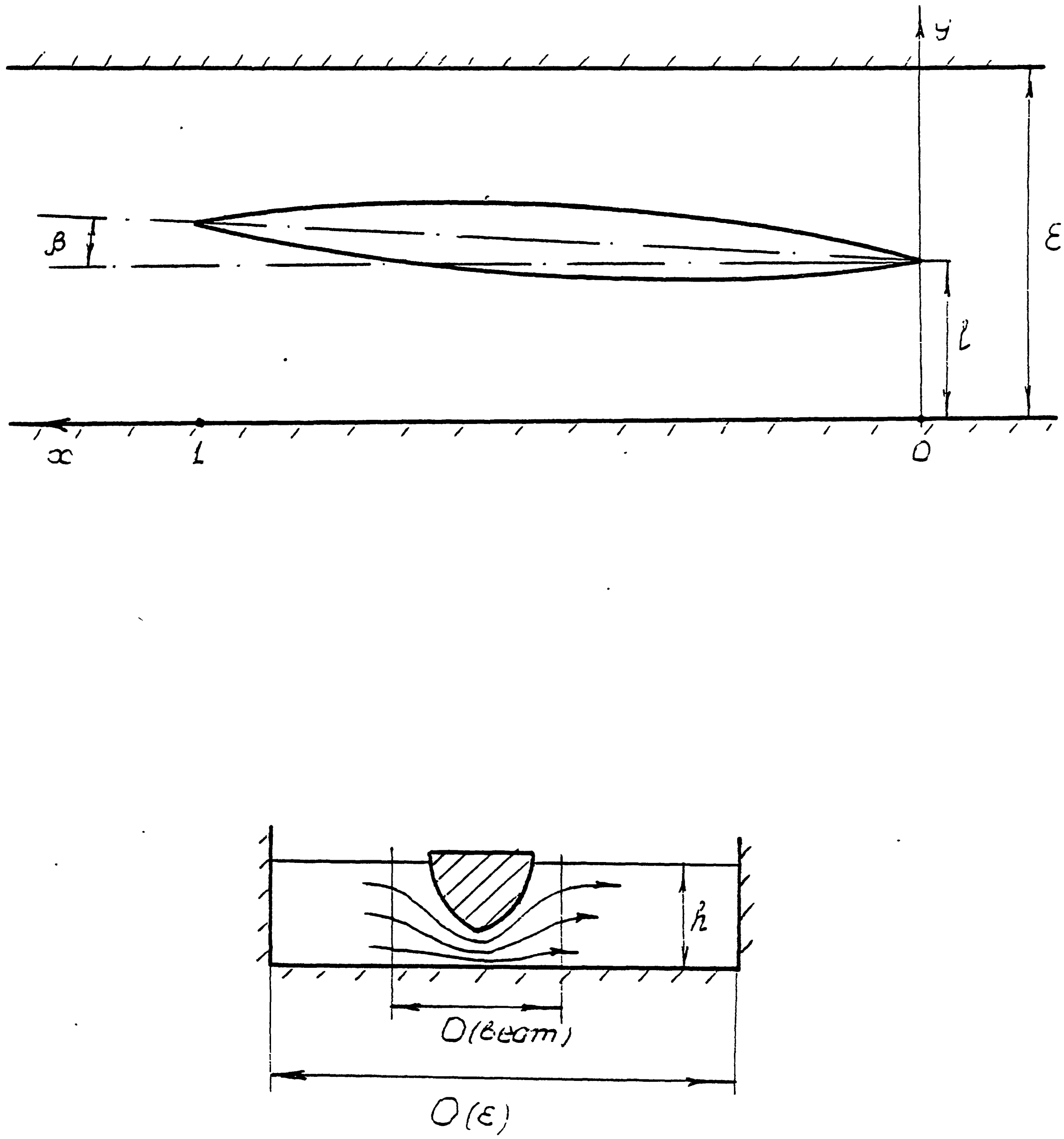

Figure 12. Three Dimensional Linearized Flow Past a Ship in a Canal in Shallow Water

The practical usefulness of this approach must be examined more thoroughly in the future. From a mathematical point of view, it allows us to extend Beck's (1976) approach to the case that the distance of the ship from a wall is small compared with ship length, i.e., $\ell << 1$. See Appendix D.

As $\alpha \to 0$, we can consider a near-field problem and a far-field problem in the sense of Newman (1969), Beck (1976), and Hess (1977), without regard to whether or not ℓ is small. The near-field problem is formulated in the region very close to the ship, where $y - \ell = O(\text{beam})$ and $x = O(1)$. It involves solving for a two-dimensional flow past the ship in planes transverse to the ship. In the far field, i.e., at lateral distances much greater than $b(x)$, the flow is also two-dimensional but parallel to the (x,y)-plane, and the wetted hull (plus its reflections with respect to the water surface and the water floor) behaves as a "porous" airfoil (Newman (1969)). The term "porosity" implies that there exists leakage through the airfoil, which corresponds to the crossflow under the ship bottom.

The matching of the far-field with the near-field shows that the leakage velocity, say v_0 , of the crossflow is proportional to the jump of potential, $\Delta\Phi$, across the airfoil:

$$v_0 = \frac{\Delta\Phi}{2C(x)} \quad ,$$

where $C(x)$ is the so-called "blockage" coefficient (c.f. Tuck and Taylor (1968)). The jump in potential is just the circulation around the part of the ship forward of the station at x , i.e.,

$$\Gamma(x) = \int_x^1 \gamma(\xi)d\xi \quad ,$$

where $\gamma(x)$ is the vorticity in the far-field representation of the flow. Then the leakage velocity is given by

$$v_0 = \frac{\Gamma(x)}{2C(x)} \quad . \tag{3.2}$$

The jump in potential comes from the solution of the two-dimensional Neumann problem of a unit-stream flow past a body in a channel. If $C(x) = 0$, there is no blockage and hence no body present. The case $C(x) = \infty$ corresponds to complete blockage (the body touches the bottom). The magnitude of $C(x)$ can be uniquely determined from the form of the ship cross-section and draft and the depth of the water. It should be noted that the blockage coefficient $C(x)$ is related to the section lateral added mass λ (Newman, (1969)):

$$\lambda = -2\rho SL^2 + 4\rho hLC \quad .$$

In the case of rectangular sections, the formulae for added mass were obtained by Gurevich (1940) and Sedov (1965). Numerical values for λ were computed and published by Flagg and Newman (1971). The asymptotic matching technique for obtaining blockage coefficients was developed by Taylor (1973).

Matching with the near-field also shows that "the push-aside flow," as Yung (1978) calls it, due to thickness effects in the near-field produces in the far-field a jump, Δv_1, of the transverse velocity component across the ship. The jump is proportional to the slope of the sectional area curve $S'(x)$, namely,

$$\Delta v_1 = \frac{1}{h} S'(x) = O(\text{beam})^* \quad .$$

Now, having gathered all necessary information supplied by matching with the near-field, we shall focus our attention on the far-field two-dimensional problem for the "porous" airfoil moving near one wall or between the two walls, either steadily and unsteadily. In what follows the far-field problem will itself be treated as a singular perturbation problem and its solution will be constructed by matched asymptotics with ℓ (midstern distance from the wall) or ε (the width of canal) as a small parameter. This problem (and its method of solution) is similar to that of Section II. There are only two significant differences: (i) The boundary conditions on the ship are linearized with respect to α, and (ii) there is a transverse flow, the leakage flow, through the body. Because of the first of these, we find some apparent differences in orders of magnitude of the solution that follows.

A. Solution of the Far-Field Problem for the Case of Steady Motion of a Ship Close to a Bank

Considering the steady motion of a ship close to a bank, we can formulate the far-field two-dimensional problem in terms of a velocity potential as

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*For the general unsteady case this formula should be modified and takes the form given by (3.54a).

follows:

$$\Phi(x,y,z) \simeq \Phi(x,y) \quad \text{in the far-field} \quad ; \tag{3.3}$$

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad \text{in the far-field flow region} \quad ; \tag{3.4}$$

$$\frac{\partial\Phi}{\partial y} = -\beta - \frac{1}{2}\sigma(x) + \kappa(x)\Gamma(x) \ , \quad 1 > x > 0 \ , \quad y = \ell + 0 \quad ; \tag{3.5}$$

$$\frac{\partial\Phi}{\partial y} = -\beta + \frac{1}{2}\sigma(x) + \kappa(x)\Gamma(x) \ , \quad 1 > x > 0 \ , \quad y = \ell - 0 \quad ; \tag{3.6}$$

$$\frac{\partial\Phi}{\partial y} = 0 \quad , \quad |x| < \infty \ , \quad y = 0 + 0 \quad ; \tag{3.7}$$

$$\nabla\Phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad ; \tag{3.8}$$

Kutta-Joukowsky condition at the stern $(x = 0,\ y = \ell)$.

Here,

$$\sigma(x) = S'(x)/h, \quad \kappa(x) = 1/2C(x) \quad . \tag{3.8a}$$

The formulated problem will be solved by matched expansions with ℓ as a small parameter. As in Section II, we subdivide the flow region into outer, channel, and edge regions, solve for the velocity potential in each of the regions, and then match the asymptotic solutions in the overlap zones.

In the channel-flow region, we stretch the vertical coordinate, $\bar{y} = y/\ell$, and assume the following aymptotic expansions for the potential:

$$\Phi_\ell = \frac{1}{\ell}\,\phi_\ell + \ell\phi_\ell^* \quad , \tag{3.9}$$

with

$$\phi_\ell = \phi_{1\ell} + \phi_{2\ell}\ell\,\ln\frac{1}{\ell} + \phi_{3\ell}\,\ell \quad . \tag{3.10}$$

Substitution of the stretched coordinate $\bar{y}$ and the assumed expansion (3.9) into the far-field equations (3.3)-(3.8) yields the following relationships for ϕ_ℓ :

$$\phi_\ell = \phi_\ell(x) \quad ,$$

$$\frac{d^2\phi_\ell}{dx^2} = \beta - \frac{1}{2}\sigma(x) - \kappa(x)\Gamma(x) \quad , \quad 1 > x > 0 \quad . \tag{3.11}$$

Recalling that the circulation $\Gamma(x)$ outside of the points $x = 0,1$ can be written as

$$\Gamma(x) = \Phi_r - \Phi_\ell = \Phi_r - \frac{1}{\ell}\phi_{1\ell} - \phi_{2\ell}\ \ln\frac{1}{\ell} - \phi_{3\ell} \quad ,$$

where Φ_r is the outer flow potential, we arrive at the following sequence of equations for $\phi_{1\ell}$, $\phi_{2\ell}$ and $\phi_{3\ell}$:

$$\frac{d^2\phi_{1\ell}}{dx^2} = \beta - \frac{1}{2}\sigma(x) + \bar{\kappa}(x)\phi_{1\ell}(x) \quad , \tag{3.12}$$

$$\frac{d^2\phi_{2\ell}}{dx^2} = \bar{\kappa}(x)\phi_{2\ell} \quad , \tag{3.13}$$

$$\frac{d^2\phi_{3\ell}}{dx^2} = \bar{\kappa}(x)(\phi_{3\ell}-\Phi_r) \quad , \tag{3.14}$$

where

$$\bar{\kappa} = \kappa/\ell = 1/2\ell C(x) = O(1) \quad .$$

We need two conditions for each of these equations, one at $x = 0$ and one at $x = 1$. These must be determined by matching with the outer-flow potential at the extremities of the body.

The problem for the outer-flow potential Φ_r has the form

$$\frac{\partial^2\Phi_r}{\partial x^2} + \frac{\partial^2\Phi_r}{\partial y^2} = 0 \quad ,$$

$$\begin{aligned}
\frac{\partial\Phi_r}{\partial y} &= -\beta - \frac{1}{2}\sigma(x) + \kappa(x)\Gamma(x) \\
&= -\beta\ \frac{1}{2}\ -\sigma(x) - \bar{\kappa}(x)\phi_{1\ell}(x) + O(\ell) \\
&= -f_r\ ,\ 1 > x > 0\ ,\quad y = 0 + 0\ ,
\end{aligned} \tag{3.15}$$

$$\frac{\partial\Phi_r}{\partial y} = 0\ ,\quad y = 0 + 0\ ,\quad x < 0\ ,\quad x > 1\ ,$$

$$\nabla\Phi_r \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad .$$

The solution is constructed as in Section II-A. On $y = 0 + 0$, it is as follows:

$$\Phi_r = \frac{Q}{2\pi}\ln|1-x| - \frac{1}{\pi}\int_0^1 f_r(\xi)\ln|x-\xi|d\xi = O(1) \quad . \tag{3.16}$$

Near the bow ($x \to 1 - 0$; $\nu = x - 1 \to 0 - 0$) , the expression (3.16) takes the following asymptotic form:

$$\Phi_r = \Phi_{rb} \simeq \frac{Q}{2\pi}\ln|\nu| + \frac{1}{\pi} f_r(1)\nu\ln|\nu| + \nu\frac{A_1}{\pi} + \frac{A_2}{\pi} \quad ,$$

where

$$A_1 = - f_r(1) - \int_0^1 [f_r(\xi) - f_r(1)]\frac{d\xi}{1-\xi} \quad ,$$

$$A_2 = - \int_0^1 f_r(\xi)\ln|1-\xi|d\xi \quad ,$$

and, as in (3.15),

$$f_r(x) = \beta + \frac{1}{2}\sigma(x) + \overline{K}(x)\phi_{1\ell}(x) \quad .$$

Near the stern ($x = -\nu \to 0 + 0$) ,

$$\Phi_r = \Phi_{rs} \simeq \frac{1}{\pi}f_r(0)\nu\ln|\nu| + \nu\frac{B_1}{\pi} + \frac{B_2}{\pi} \quad ,$$

where

$$B_1 = \frac{1}{2}Q - f_r(0) - \int_0^1 [f_r(\xi) - f_r(0)]\frac{d\xi}{\xi} \quad ,$$

$$B_2 = - \int_0^1 f_r(\xi)\ln\xi d\xi \quad .$$

Edge flows. In the bow region, we have a problem governed by the Laplace equation and the following boundary conditions:

$$\frac{\partial \Phi_b}{\partial \bar{y}} = 0 \; , \quad \bar{y} = 0 + 0 \quad ,$$

$$\frac{\partial \Phi_b}{\partial \bar{y}} = - \ell[\beta + \frac{1}{2}\sigma(1) - \kappa(1)\Gamma(1)] = - \ell[\beta - \frac{1}{2}\sigma(1)] = - \ell f_r(1) \; , \quad \bar{y} = 1 + 0 \quad ,$$

$$\frac{\partial \Phi_b}{\partial \bar{y}} = - \ell[\beta - \frac{1}{2}\sigma(1) - \kappa(1)\Gamma(1)] = - \ell[\beta + \frac{1}{2}\sigma(1)] = - \ell f_\ell(1) \; , \quad \bar{y} = 1 - 0 \quad .$$

The fact has been used that the circulation (potential jump) at the bow is zero. The solution has the general form

$$\Phi_b = a_1\phi_h + \ell\phi_n + a_3\ell\bar{\nu} + a_4 \quad ,$$

where $\bar{\nu} = (x-1)/\ell$, ϕ_h and ϕ_n are the homogeneous and nonhomogeneous solutions obtained as in Appendix A (although we have to put $\bar{y}_r(1) = \bar{y}_r(0) = 1$) ,

$$\pi\bar{\nu} = 1 - e^{\pi\phi_h} + \pi\phi_h \quad ,$$

$$\phi_n = \int_0^{\bar{\nu}} u_n d\bar{\nu} \quad , \quad u_n = \frac{1}{\pi}[-\sigma(1)\ln|1+\xi|+f_\ell(1)\ln\xi] \quad ,$$

$$\pi\bar{\nu} = 1 + \xi + \ln\xi \quad , \quad \xi < 0 \quad \text{on the hull} \quad ,$$

and a_i are constants to be determined by matching.

Similarly, near the stern, omitting the homogeneous term, we have

$$\Phi_s = \ell\phi_n + b_2\ell\bar{\nu} + b_3 \quad ,$$

where $\bar{\nu} = -x/\ell$,

$$\phi_n = \int_0^{\bar{\nu}} u_n d\bar{\nu} \; , \quad u_n = \frac{1}{\pi}[-\sigma(0)\ln|1+\xi| + f_\ell(0)\ln\xi] \quad ,$$

$$f_\ell(0) = \beta + \frac{1}{2}\sigma(0) + \bar{\kappa}(0)\phi_{1\ell}(0) \quad .$$

Matching of the flows as in Section II yields the following information:

$$\phi_\ell(1) = - \frac{a_1\ell}{\pi}(1 + \ln\frac{\pi}{\ell}) + \frac{\ell}{\pi}A_2 \quad ,$$

or

$$\phi_{1\ell}\ (1) = 0 \quad ,$$

$$\phi_{2\ell}\ (1) = -\frac{a_1}{\pi} \quad , \tag{3.17}$$

$$\phi_{3\ell}\ (1) = -\frac{a_1}{\pi}(1 + \ln\pi) + \frac{1}{\pi}A_2 \quad , \tag{3.18}$$

with
$$a_1 = \frac{1}{2}Q = \frac{d\phi_{1\ell}}{dx}(1) \quad ; \tag{3.18a}$$

$$\frac{d\phi_\ell}{dx}(0) = \frac{\ell}{\pi}[f_\ell(0) - B_1 + f_r(0)(\ln\frac{\pi}{\ell} - 1)] \quad ,$$

or

$$\frac{d\phi_{1\ell}}{dx}\ (0) = 0 \quad ,$$

$$\frac{d\phi_2}{dx}(0) = \frac{f_r(0)}{\pi} \quad , \tag{3.19}$$

$$\frac{d\phi_3}{dx}(0) = -\frac{1}{\pi}[B_1 - f_\ell(0) + f_r(0)(1 - \ln\pi)] \quad . \tag{3.20}$$

The uniformly valid expressions for the velocity potentials on the right, Φ_{cr} , and left, $\Phi_{c\ell}$, sides of the hull are given by (2.54) and (2.55).

The side-force and yaw-moment (with respect to midships) coefficients are defined as in (2.56) and (2.57). Because of the linearization of the problem they take the form*

$$C_y = 2 \int_0^1 \left[\frac{d\Phi_{c\ell}}{dx} - \frac{d\Phi_{cr}}{dx}\right] dx = 2[\Phi_{cr}(0) - \Phi_{c\ell}(0)] \quad ,$$

$$M_z = 2 \int_0^1 (x - \frac{1}{2}) \left[\frac{d\Phi_{c\ell}}{dx} - \frac{d\Phi_{cr}}{dx}\right] dx$$

---

*Recall that $\Phi_{c\ell}$ and Φ_{cr} represent the composite solutions on left- and right-hand sides of the body.

$$= -\Phi_{cr}(0) + \Phi_{c\ell}(0) - 2\int_0^1 [\Phi_{c\ell}(x) - \Phi_{cr}(x)]dx \quad .$$

To the lowest order, we have

$$C_Y \simeq C_{Y_1} = -\frac{2}{\ell}\phi_{1\ell}(0) \quad ,$$

$$M_Z \simeq M_{Z_1} = +\frac{1}{\ell}\phi_{1\ell}(0) - \frac{2}{\ell}\int_0^1 \phi_{1\ell}(x)dx \quad ,$$

where $\phi_{1\ell}(x)$ satisfies the equation

$$\frac{d^2\phi_{1\ell}}{dx^2} - \bar{\kappa}(x)\phi_{1\ell} = \beta - \frac{1}{2}\sigma(x) = r(x) \quad , \qquad (3.21)$$

$$\phi_{1\ell}(1) = 0 \quad ,$$

$$\frac{d\phi_{1\ell}}{dx}(0) = 0 \quad .$$

For a given yaw angle β, cross-sectional area curve $S(x)$, and distribution of the blockage coefficient $C(x)$ along the hull, the equation (3.21) can be readily integrated numerically, for example, by the method of Runge-Kutta.

It is worthwhile to mention Beck's (1976) indication that the difference in the side force and yaw moment between the case of constant blockage coefficient and the actual blockage coefficient (which varies along the ship length) is small. Now, assuming that the blockage coefficient $C(x)$ is constant (for example, equal to the value of the blockage coefficient at midship), we can obtain an analytic solution of Equation (3.21) for any right-hand-side function $r(x)$, using the method of variation of parameters, in the form

$$\phi_{1\ell} = \frac{1}{a}\int_1^x r(\xi)\sinh a(x-\xi)\, d\xi$$

$$+ \frac{\sinh a(x-1)}{a\cosh a}\int_0^1 r(\xi)\cosh a\xi\, d\xi \quad , \qquad (3.22)$$

where

$$a = \sqrt{\kappa} \quad .$$

Side force and yaw moment coefficients to the lowest order are

$$C_Y \simeq C_{Y_1} = \frac{2}{\ell a \cosh a} \int_0^1 r(\xi) \sinh a(1 - \xi)\, d\xi \quad ; \tag{3.23}$$

$$z \simeq z_1 = -\frac{1}{2} C_{Y_1} - \frac{2}{\ell a} \left\{ \int_0^1 \int_1^x r(\xi) \sinh a(x - \xi)\, d\xi dx \right.$$

$$\left. - \frac{\tanh a \tanh \frac{a}{2}}{a} \int_0^1 r(\xi) \cosh a\xi\, d\xi \right\} \quad . \tag{3.24}$$

As the formulation of the problem is linear, the effects of yaw angle, rudder deflection, and beam can be considered separately.

To determine the coefficients C_y and z due to the yaw angle, put $r(x) = \beta$. In this case the formulae (3.22)-(3.24) yield

$$\phi_{1\ell} = \frac{\beta}{a^2} \left[\frac{\cosh ax}{\cosh a} - 1 \right] \quad , \quad a = \sqrt{\kappa} \quad ,$$

$$a_1 = \frac{1}{2} Q = \frac{d\phi_{1\ell}}{dx}(1) = \frac{\beta}{a} \tanh a \quad , \tag{3.25}$$

$$C_Y \simeq C_{Y1} = \frac{2\beta}{\ell} \frac{\tanh a \tanh \frac{a}{2}}{a^2} \quad , \tag{3.26}$$

$$z \simeq z_1 = -\frac{1}{2} C_{Y1} + \frac{2\beta}{\ell a^2} \left[1 - \frac{\tanh a}{a} \right] \quad . \tag{3.27}$$

To compute the coefficients C_y and z in case of no yaw angle and zero rudder deflection, we require that $r(x) = -\frac{1}{2}\sigma(x) = -S'(x)/2h$. For example, if the ship has constant draft d , rectangular cross sections and parabolic beam distribution, we have

$$S(x) = 4db_0x(1 - x) \quad ,$$

$$r(x) = -\frac{2db_0}{h}(1 - 2x) \quad ,$$

$$C_Y \simeq C_{Y_1} = \frac{4db_0}{\ell h a^2 \cosh a}\left[\frac{2}{a}\sinh a - 1 - \cosh a\right] , \qquad (3.28)$$

$$z \simeq z_1 = -\frac{1}{2}C_Y + \frac{8b_0 d}{\ell h a^2}\,\frac{\cosh a - 1 - a\sinh\frac{a}{2}}{\cosh a} \quad . \qquad (3.29)$$

When $a = \sqrt{\kappa} \to 0$ (purely two-dimensional case)

$$C_{Y_1} \to -\frac{2b_0 d}{3\ell h} \quad , \qquad z_1 \to 0 \quad .$$

Note that in the three-dimensional case the yaw moment is not equal to zero even if the ship is symmetrical fore and aft.

For the analysis of rudder effectiveness in maneuvering control one may need to know the side-force and yaw-moment coefficients due to the rudder deflection (e.g., see Hess (1979)). To obtain these coefficients to the lowest order, we require that

$$r(x) = \begin{cases} 0 & \text{when} \quad 1 > x > s \ , \\ \delta & \qquad\quad s > x > 0 \ , \end{cases}$$

where δ is rudder deflection angle (positive for rudder to port), s is rudder length. Using the formulae (3.19) and (3.20), we obtain

$$C_Y \simeq C_{Y_1} = \frac{2\delta}{\ell a^2}\left[1 - \frac{\cosh a(1-s)}{\cosh a}\right] \quad , \qquad (3.30)$$

$$z \simeq z_1 = -\frac{1}{2}C_{Y_1} - \frac{2\delta}{\ell a^3}\left[\frac{\sinh as}{\cosh a} + \sinh a(1 - s) - a\right] \quad , \qquad (3.31)$$

with $a = \sqrt{\kappa}$.

Some computational results are presented in Figure 13 to show the effect of underkeel clearance on the side-force coefficient. For $\kappa = 0$, there is no clearance and thus no transverse flow (no "porosity" or "leakage"). As κ increases from zero, the side force resulting from either a ship yaw angle or a rudder deflection decreases rapidly.

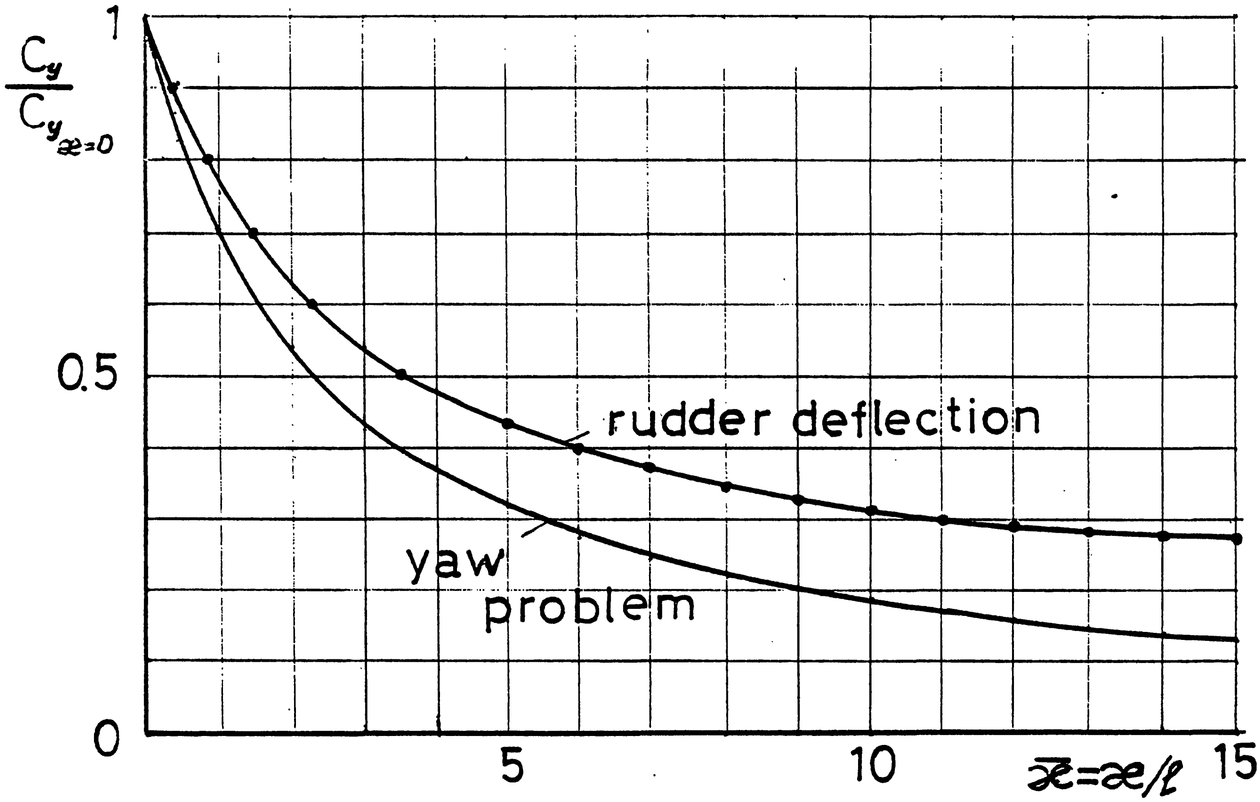

Figure 13. Side Force Coefficient Related to Its Value at No "Porosity"

Solving for the next term, which is of the order $O(\ln\frac{1}{\ell})$ in (3.10), we arrive at the following expression for $\phi_{2\ell}$:

$$\phi_{2\ell}(x) = \phi_{2\ell}(1)\frac{\cosh ax}{\cosh a} + \frac{d\phi_{2\ell}}{dx}(0)\ \frac{\sinh a(x-1)}{a \cosh a}\ , \qquad (3.32)$$

where $\phi_{2\ell}(1)$ and $\frac{d\phi_{2\ell}}{dx}(0)$ are given by formulae (3.17) and (3.19). The corresponding increments in C_y and ${}_z$ are equal to

$$\text{increment } C_y = \frac{2\ln\frac{1}{\ell}}{\cosh a}\left[\frac{d\phi_{2\ell}}{dx}(0)\ \frac{\sinh a}{a} - \phi_{2\ell}(1)\right] ,$$

$$\text{increment } {}_z = -\frac{1}{2}[\text{incr. } C_y] - 2\ln\frac{1}{\ell}\left[\phi_{2\ell}(1)\frac{\sinh a}{a\cosh a} + \frac{d\phi_{2\ell}}{dx}(0)\ \frac{(1-\cosh a)}{a^2\cosh a}\right] .$$

For example, to this order, the side-force and moment coefficients for the yaw problem are

$$C_Y \approx \frac{2\beta}{\ell}\left\{\frac{\tanh a \tanh \frac{a}{2}}{a^2} + \frac{\ell \ln\frac{1}{\ell}}{\pi a}\left[\frac{\sinh a + \tanh a}{\cosh a}\right]\right\}, \tag{3.33a}$$

$$M_z \approx -\frac{1}{2}C_Y + \frac{2\beta}{\ell a^2}\left\{1 - \frac{\tanh a}{a} + \frac{\ell \ln\frac{1}{\ell}}{\pi}(\tanh\frac{a}{2} - \tanh a)\tanh a\right\}. \tag{3.33b}$$

In order to determine the next term, which is of the order $O(1)$, in expansion (3.10), we have to solve for $\phi_{3\ell}$, which was found in the form

$$\phi_{3\ell}(x) = [C_1 + U_1(x)]e^{ax} + [C_2 + U_2(x)]e^{-ax} \quad ,$$

where

$$U_1(x) = -\frac{a}{2}\left\{\frac{Q}{2\pi}R(x,1) - \frac{1}{\pi}\int_0^1 f_r(\xi_1)R(x,\xi_1)d\xi_1\right\} \quad ,$$

$$U_2(x) = \frac{a}{2}\left\{\frac{Q}{2\pi}R(x,-1) - \frac{1}{\pi}\int_0^1 f_r(\xi_1)R(x,-\xi_1)d\xi_1\right\}$$

$$R(x,\xi_1) = \frac{e^{-a\xi_1}}{a}\Big\{Ei[a(\xi_1 - x)] - Ei[a(\xi_1 - 1)]$$

$$+ e^{a(\xi_1-1)}\ln|\xi_1 - 1| - e^{a(\xi_1 - x)}\ln|\xi_1 - x|\Big\} \quad .$$

The parameters C_1 and C_2 are determined from the system of equations

$$C_1 - C_2 = \frac{1}{a}\frac{d\phi_{3\ell}}{dx}(0) - U_1(0) + U_2(0) \quad ,$$

$$C_1e^a + C_2e^{-a} = \phi_{3\ell}(1) \quad ,$$

where $\phi_{3\ell}(1)$ and $\dfrac{d\phi_{3\ell}}{dx}(0)$ are given by (3.18) and (3.20) and the function $f_r(x)$ and the parameter Q are defined by the formulae (3.15) and (3.18a), respectively.

B. Steady Motion of a Ship in a Shallow Canal

It was mentioned earlier (see Section II-A) that the asymptotic solution of the problem for a ship in a canal depends on the order relationship between the width of a canal, ε , and the ship distance from the left side wall, ℓ . Returning to our two-dimensional far-field description of the three-dimensional flow past a ship in a canal, we can consider two cases: (i) $\varepsilon = O(1)$ (wide canal); (ii) $\varepsilon = O(\ell)$ (narrow canal). As shown in Section II in case of a wide canal, for which $\varepsilon = O(1)$, the additional forces due to the presence of the right side wall are of the order of $O(\ell)$, i.e., they are rather small. Therefore we shall confine ourselves to the case of a narrow canal, which is more interesting from the practical point of view. It appears that for this very case the solution of the problem for the "porous" airfoil can be carried out easily to the order of $O(\varepsilon)$.

As the procedure of the solution is very similar to what was demonstrated previously, we shall simply give the outline and some final results.

The far-field potential $\Phi = \Phi(x,y)$ is described by the same set of relationships as for the ship close to a bank (see (3.3)-(3.8)), although the no-normal-velocity condition must be satisfied on both walls of the canal, i.e., (3.7) applies on $y = \varepsilon - 0$ as well as on $y = 0 + 0$. For $\varepsilon = O(\ell) << 1$ we shall seek the asymptotic solution of this problem by matched asymptotic expansions with ε as a small parameter. The characteristic zones of the flow in the far field are the same as shown in Figure 10. We have two edge flows (bow and stern) and two confined channel flows.

In the channel flows, using the same technique as before, we obtain the following relationships for the potentials of the flow in the right channel Φ_r and the left channel Φ_ℓ :

$$\Phi_r = \frac{1}{\varepsilon}\phi_r + O(\varepsilon) \ ; \quad \Phi_\ell = \frac{1}{\varepsilon}\phi_\ell + O(\varepsilon) \quad ; \tag{3.34}$$

$$\frac{d^2\phi_r}{dx^2} = \frac{1}{1-\tilde{\ell}}[-\beta - \frac{1}{2}\sigma(x) + \tilde{\kappa}(\phi_r - \phi_\ell)] \quad , \quad 1 > x > 0 \ ; \tag{3.35}$$

$$\frac{d^2\phi_\ell}{dx^2} = \frac{1}{\tilde{\ell}}[\beta - \frac{1}{2}\sigma(x) - \tilde{\kappa}(\phi_r - \phi_\ell)] \quad , \quad 1 > x > 0 \ , \tag{3.36}$$

where the tilda (˜) is used to denote quantities divided by ε , e.g., $\tilde{\ell} = \ell/\varepsilon$, $\tilde{\kappa} = \kappa/\varepsilon$, etc. One can readily see that we can conveniently write the equation for the difference $\Phi_r - \Phi_\ell = \Gamma$, which represents the circulation. Substracting (3.36) form (3.35), we obtain

$$\varepsilon(\Gamma'' - a_0^2\Gamma) = r_0(x) \quad , \tag{3.37}$$

where

$$a_0^2 = \frac{\tilde{\kappa}}{\tilde{\ell}(1-\tilde{\ell})} \quad ; \tag{3.37a}$$

$$r_0(x) = \frac{1}{\tilde{\ell}(\tilde{\ell}-1)}[\beta + \frac{1}{2}\sigma(x)(2\tilde{\ell} - 1)] \quad . \tag{3.37b}$$

Passing over to the edge solutions and using techniques similar to those of Appendix B, we obtain the following expressions for the bow and stern potentials:

The <u>bow-flow</u> potential is

$$\Phi_b = a_1\phi_h + \varepsilon\phi_n + a_3\varepsilon\tilde{\nu} + a_4 \quad ,$$

where

$$\tilde{\nu} = \nu/\varepsilon \quad , \quad \nu = x - 1 \quad .$$

The homogeneous solution is determined from the equation

$$\tilde{\nu} = \phi_h - \frac{1}{\pi}\ln\left|1 - \tilde{\ell} + \tilde{\ell}\exp\left(\frac{\pi\phi_h}{\tilde{\ell}}\right)\right| \quad .$$

The nonhomogeneous term is given by the expression

$$u_n = -\frac{\varepsilon}{\pi}\sigma(1)\ln|1 + \xi| + \frac{\varepsilon}{\pi}[\beta + \frac{1}{2}\sigma(1)]\ln|\xi| \quad , \tag{3.38}$$

$$\phi_n = \int_0^{\tilde{\nu}} u_n d\tilde{\nu} \quad , \tag{3.39}$$

where, as in the previous analysis for the ship in a canal,

$$\pi\tilde{\nu} = \frac{1}{\pi}[\tilde{\ell}\ln|\xi| - \ln|\tilde{\ell}\xi + \tilde{\ell} - 1|] \quad .$$

The <u>stern-flow</u> potential is defined as follows:

$$\Phi_s = \varepsilon\phi_n + b_2\varepsilon\tilde{\nu} + b_3 \quad ,$$

where

$$\nu = -x \quad ,$$

$$b_3^+ \neq b_3^- \quad ,$$

ϕ_n is given by the formulae (3.38) and (3.39).

Matching

To solve the equation (3.37) for Γ we have to obtain the boundary conditions on Γ at $x = 0$ and $x = 1$. This can be done through matching. Expanding the bow-flow potential in the channel-flow variable ν far from the bow in the right and left channel, we find the following asymptotic expansion for the circulation Γ in the bow overlap region:

$$\Gamma_b \sim \Phi_{br} - \Phi_{b\ell} \approx a_1\left[\frac{\nu}{\varepsilon(\tilde{\ell}-1)} - \frac{\ln|1-\tilde{\ell}|}{\pi} - \frac{\tilde{\ell}\ln\tilde{\ell}}{\pi(1-\tilde{\ell})}\right] \quad , \tag{3.40}$$

wherefrom we determine the asymptotic structure of Γ far from the edges:

$$\Gamma = \frac{1}{\varepsilon}\Gamma_1 + \Gamma_2 + O(\varepsilon) \quad . \tag{3.41}$$

Comparing (3.40) with (3.41) expanded for small $\nu = \tilde{\nu}\varepsilon$, we obtain

$$\Gamma(1) = \frac{1}{\varepsilon}\Gamma_1(1) + \Gamma_2(1) = -\frac{a_1}{\pi}\left[\ln|1-\tilde{\ell}| + \frac{\tilde{\ell}\ln\tilde{\ell}}{1-\tilde{\ell}}\right] , \tag{3.42}$$

or

$$\Gamma_1(1) = 0 \quad , \tag{3.43}$$

$$\Gamma_2(1) = -\frac{a_1}{\pi}\left[\ln|1-\tilde{\ell}| + \frac{\tilde{\ell}\ln\tilde{\ell}}{1-\tilde{\ell}}\right] \quad . \tag{3.44}$$

At the same time the parameter a_1 is found to be

$$a_1 = (\tilde{\ell} - 1)\Gamma_1'(1) \quad . \tag{3.45}$$

Matching in the stern overlap region is performed similarly and gives the following results:

$$\Gamma'(0) = \frac{1}{\varepsilon}\Gamma_1'(0) + \Gamma_2'(0) = \frac{1}{\pi}\left[\frac{f_r(0)\ln\tilde{\ell}}{1-\tilde{\ell}} + \frac{f_\ell(0)\ln|1-\tilde{\ell}|}{\tilde{\ell}}\right] \quad , \tag{3.46}$$

or

$$\Gamma_1'(0) = 0 \quad , \tag{3.47}$$

$$\Gamma_2'(0) = \frac{1}{\pi}\left[\frac{f_r(0)\ln\tilde{\ell}}{1-\tilde{\ell}} + \frac{f_\ell(0)\ln|1-\tilde{\ell}|}{\tilde{\ell}}\right] \quad , \tag{3.48}$$

where

$$f_r(0) = \beta + \frac{1}{2}\sigma(0) - \tilde{\kappa}(0)\Gamma_1(0) \quad , \tag{3.49}$$

$$f_\ell(0) = \beta - \frac{1}{2}\sigma(0) - \tilde{\kappa}(0)\Gamma_1(0) \quad . \tag{3.50}$$

Now it is not difficult to solve the equations for Γ_1 and Γ_2 using the boundary conditions determined by the matching. The solution for Γ_1 is given by the expression (3.22) with $\phi_{1\ell}$ replaced by Γ_1 , $r(x)$ by $r_0(x)$ (see (3.37b)), a by a_0 (see (3.37a)). The side-force and yaw-moment coefficients to the lowest order can be computed employing the formulae (3.23) and (3.24) where again $r(x)$ should be replaced by $r_0(x)$ and a should be replaced by a_0 .

For the particular case when only the yaw angle effects are considered, we obtain

$$C_Y \approx C_{Y_1} = \frac{2\beta}{\varepsilon\tilde{\ell}(1-\tilde{\ell})}\,\frac{\tanh a_0 \tanh \frac{a_0}{2}}{a_0^2} \quad , \tag{3.51}$$

$$M_z \approx M_{z_1} = -\frac{1}{2}C_{Y_1} + \frac{2\beta}{\varepsilon\tilde{\ell}(1-\tilde{\ell})a_0^2}\left[1 - \frac{\tanh a_0}{a_0}\right] \quad . \tag{3.52}$$

It follows from the inspection of the formulae (3.26), (3.27) and (3.51), (3.52), that to determine the yaw angle coefficients for the case of the ship in a narrow canal to the lowest order we may simply take the coefficients obtained for the case of a ship close to a bank for $a = a_0$ and multiply them by the factor

$$\frac{1}{1-\tilde{\ell}} = \frac{\varepsilon}{\varepsilon-\tilde{\ell}} \quad .$$

Turning to the next order of approximation, the following expression for $\Gamma_2(x)$ can be obtained:

$$\Gamma_2(x) = \Gamma_2(1)\frac{\cosh a_0 x}{\cosh a_0} + \Gamma_2'(0)\frac{\sinh a_0(x-1)}{a_0\cosh h a_0} \quad ,$$

where $\Gamma_2(1)$ and $\Gamma_2'(0)$ are given by the formulae (3.44), (3.48). The increment in C_y of the order $O(1)$ is found to be

$$\frac{2}{\cosh a_0}\left[\Gamma_2(1) - \Gamma_2'(0)\frac{\sinh a_0}{a_0}\right] .$$

The increment in M_z of the order $O(1)$ is equal to

$$+\frac{2}{a_0}\tanh a_0\left[\Gamma_2(1) - \Gamma_2'(0)\frac{\tanh\frac{a_0}{2}}{a_0}\right] .$$

For the yaw-angle problem we arrive at the following formulae for C_y and M_z :

$$C_y \approx \frac{2\beta}{\varepsilon\tilde{\ell}(1-\tilde{\ell})}\left[\frac{\tanh a_0 \tanh\frac{a_0}{2}}{a_0^2} - \frac{2\varepsilon\tanh a_0}{\pi a_0\cosh a_0}\,[\tilde{\ell}\ln\tilde{\ell} + (1-\tilde{\ell})\ln|1-\tilde{\ell}|]\right] ,$$

$$M_z \approx -\frac{1}{2}C_y - \frac{2\beta\tanh a_0}{\pi\tilde{\ell}(1-\tilde{\ell})a_0\cosh a_0}\left[1 + \frac{\sinh a_0}{a_0}\right][\tilde{\ell}\ln\tilde{\ell} + (1-\tilde{\ell})\ln|1-\tilde{\ell}|] .$$

It can be verified that these formulae, valid with the asymptotic error of the order of $O(\varepsilon^2)$, coincide in the limit $a_0 \to 0$ (no "porosity", purely two-dimensional case) with the expressions (2.106) and (2.107) of Section I.

Some results of the computations of the function $\ell C_y/\beta$, with the help of formulae derived above, are presented in Figure 14.

C. Unsteady Motion of a Ship in a Shallow Canal

It is known that unsteady problems are much more difficult to analyze in the case of zero or small bottom gap, where the circulation becomes particularly important and, as in the corresponding aerodynamic problem, vorticity is shed continually from the trailing edge of the moving body. Progress in solving such problems for ship-ship interactions has been made recently by King (1977), Yung (1978) and Kijima (1979). Within the limitations of the assumptions adopted throughout this chapter, it becomes possible to obtain relatively simple analytic results (at least to the lowest order) for some cases of unsteady motions of a ship in a canal or close to a bank in shallow water. In the following paragraphs we shall outline the solutions of the unsteady problems, omitting the parts of the solution techniques that are identical to those displayed in detail in preceding sections. At the same time, the pecu-

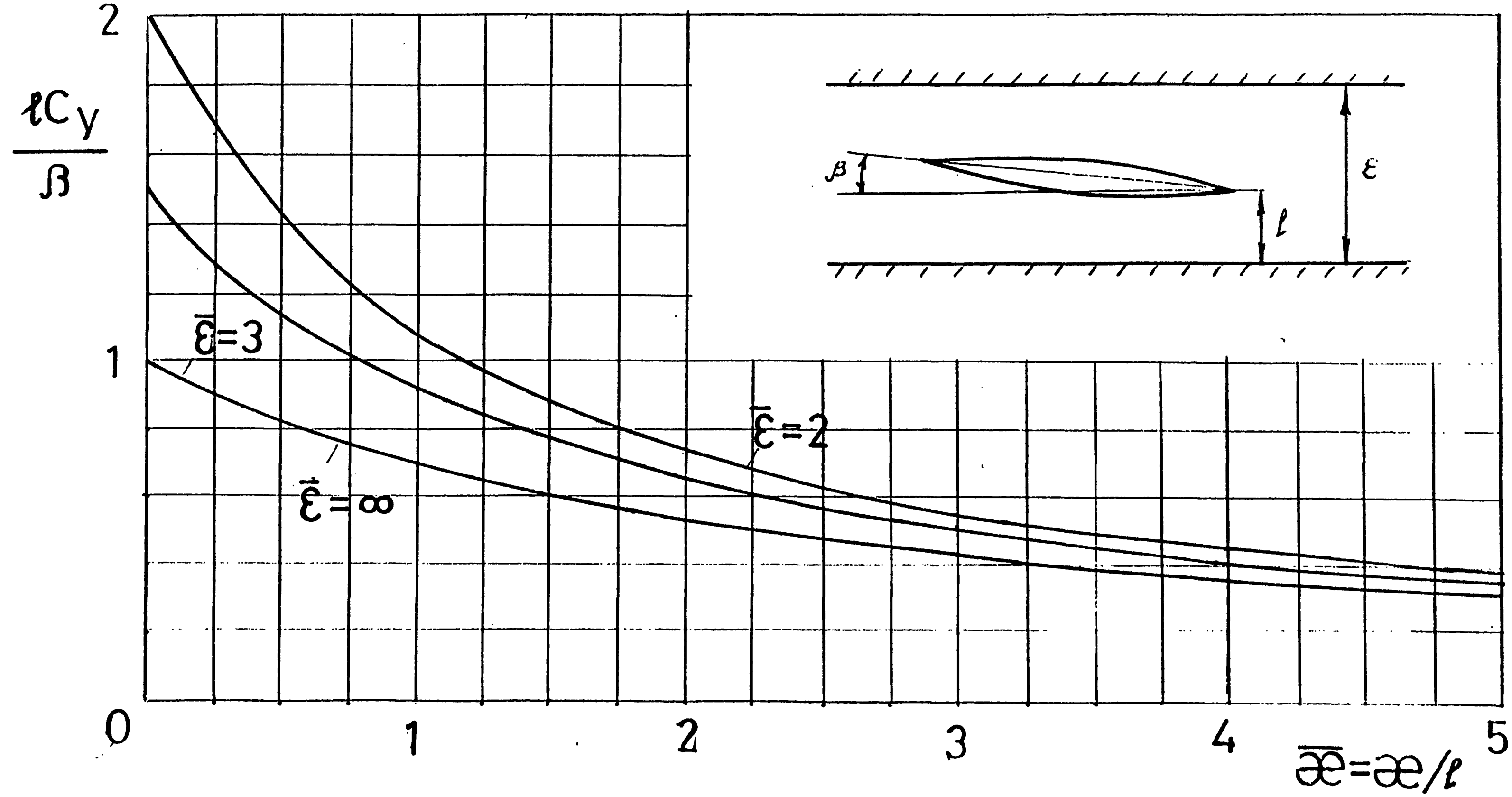

Figure 14. The Function $\ell C_y/\beta$ versus κ/ℓ for Different Widths of Canal

liarities of the solution due to the unsteady character of the problem will be discussed in more detail.

Suppose the ship is moving with a constant velocity parallel to the canal walls and at the same time performs oscillatory motions of small amplitude. In practice, these oscillatory motions may be the result of the superposition of sway and yaw oscillations and also of heaving and pitching oscillations of the ship.

The far-field two-dimensional problem for the corresponding "porous" airfoil in unsteady motion between two rigid walls in terms of velocity potential has the form

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} = 0 \quad ,$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial y_c}{\partial x} + \frac{\partial y_c}{\partial t} - \frac{1}{2}\sigma(x,t) + \kappa(x)\Gamma(x,t) \quad , \quad 1 > x > 0 \; , \quad y = \ell + 0 \quad ,$$

$$\frac{\partial\Phi}{\partial y} = -\frac{\partial y_c}{\partial x} + \frac{\partial y_c}{\partial t} + \frac{1}{2}\sigma(x,t) + \kappa(x)\Gamma(x,t) \quad , \quad 1 > x > 0 \; , \quad y = \ell - 0 \quad ,$$

$$\frac{\partial\Phi}{\partial y} = 0 \quad , \quad |x| < \infty \quad , \quad y = 0 + 0 \quad , \quad y = \varepsilon - 0 \quad .$$

In the wake we should take account of the Thompson theorem:

$$\Phi_+(0,\ell,t^*) - \Phi_-(0,\ell,t^*) = \Phi_+(x,\ell,t) - \Phi_-(x,\ell,t) \quad , \tag{3.53}$$

for

$$y = \ell \quad , \quad x < 0 \quad , \quad t^* = t + x \quad .$$

Condition (3.53) is equivalent to the condition of the continuity of pressure across the wake.

The Kutta-Joukowsky condition at the stern is formulated as

$$p_+ = p_- \quad \text{at} \quad x = 0 \; , \quad y = \ell \quad . \tag{3.54}$$

Also, we require that

$$\nabla\Phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad .$$

In the above formulation we have used the following notations: $y_c(x,t)$ is the function describing the instantaneous position of the ship centerline (note that rudder deflections can be included by considering the rudder surface as a continuation of the centerline), $\kappa(x) = 1/2C(x)$, $C(x)$ is the blockage coefficient, $p = 2\left(\frac{\partial\Phi}{\partial x} - \frac{\partial\Phi}{\partial t}\right)$ the pressure coefficient, and $\Gamma(x,t)$ the circulation.

It can be shown by matching with the near field, where $y - \ell =$ O(amplitude of the oscillations), that the function $\sigma(x,t)$ is related to the change of the cross sectional area $S(x,t)$ both along the ship and in time and is equal to

$$\sigma(x,t) = \frac{1}{h}\left(\frac{\partial S}{\partial x} - \frac{\partial S}{\partial t}\right) \quad . \tag{3.54a}$$

The solution to the problem is obtained as in the case of a steady motion of the ship in a canal by matched expansions with ε as a small parameter. The asymptotic velocity potentials in the bow and channel regions are determined as in the steady case, although the parameters in unsteady case depend on time (e.g., the parameters a_1, a_3 and a_4 of the bow-flow potential are functions of time). The analysis of the channel flows allows us to write the equation for the circulation in the far field in the form

$$\varepsilon\left(\frac{\partial^2\Gamma}{\partial x^2} - a_0^2\Gamma\right) = r_0(x,t) \quad , \tag{3.55}$$

where

$$\varepsilon\Gamma = \Gamma_1 + \varepsilon\Gamma_2 + O(\varepsilon^2) \quad ,$$

$$a_0^2 = \frac{\tilde{\kappa}}{\tilde{\ell}(1-\tilde{\ell})} \quad ,$$

$$r_0(x,t) = \frac{1}{\tilde{\ell}(\tilde{\ell}-1)}[\alpha_s(x,t) + \frac{1}{2}\sigma(x,t)(2\tilde{\ell} - 1)] \quad ,$$

$$\alpha_s(x,t) = \frac{\partial y_c}{\partial x} - \frac{\partial y_c}{\partial t} \quad .$$

Matching with the bow-flow potential gives, as in the steady case, the boundary condition for Γ at $x = 1$ and the value of the parameter a_1 in the bow solution:

$$\varepsilon\Gamma(1,t) = \Gamma_1(1,t) + \varepsilon\Gamma_2(1,t) = -\frac{\varepsilon a_1(t)}{\pi}\left[\ln|1-\tilde{\ell}| + \frac{\tilde{\ell}\ln\tilde{\ell}}{1-\tilde{\ell}}\right] ,$$

$$a_1(t) = (\tilde{\ell}-1)\frac{\partial\Gamma_1}{\partial x}(1,t) .$$

Near the stern, the unsteady-flow velocity potential must satisfy not only the normal velocity conditions on the hull and canal walls but also the condition in the wake (3.53).

Passing over to the variable $\tilde{\nu} = \nu/\varepsilon$ in the equation (3.53), we obtain the following asymptotic condition in the wake in the vicinity of the stern:

$$(\Phi_{s+} - \Phi_{s-})_{\tilde{\nu},t} = (\Phi_{s+} - \Phi_{s-})_{0,t} + \varepsilon\tilde{\nu}\frac{\partial}{\partial t}(\Phi_{s+} - \Phi_{s-})_{0,t} + O(\varepsilon^2) \quad . (3.56)$$

It can be seen from (3.55) that to the order of $O(\varepsilon)$ the values of the stern-flow potential on the right and left sides of the wake differ by a linear function of $\tilde{\nu}$. The expression for the stern-flow potential can be written as follows:

$$\Phi_s = \varepsilon\phi_n + b_2\varepsilon\tilde{\nu} + b_3 ,$$

where ϕ_n is defined by formula (3.39), $b_2^+ \neq b_2^-$, $b_3^+ \neq b_3^-$. Satisfying the Kutta-Joukowsky condition at the stern ($\tilde{\nu} = -x/\varepsilon = 0$, $\phi_n = 0$) , we arrive at the following relationship between b_2 and b_3 on the right and left sides:

$$b_2^- + \frac{\partial b_3^-}{\partial t} = b_2^+ + \frac{\partial b_3^+}{\partial t} .$$

Matching of the pressure jumps in the stern overlap region gives the necessary boundary condition for the equation (3.55) at $x = 0$:

$$\varepsilon\left(\frac{\partial\Gamma}{\partial x} - \frac{\partial\Gamma}{\partial t}\right) = \frac{\varepsilon}{\pi}\left(\frac{f_r(0,t)\ln\tilde{\ell}}{1-\tilde{\ell}} + \frac{f_\ell(0,t)\ln|1-\tilde{\ell}|}{\tilde{\ell}}\right) , \qquad (3.56)$$

where

$$f_r(0,t) = \alpha_s(0,t) + \frac{1}{2}\sigma(0,t) - \tilde{\kappa}(0)\Gamma_1(0,t) , \qquad (3.57)$$

$$f_\ell(0,t) = \alpha_s(0,t) - \frac{1}{2}\sigma(0,t) - \tilde{\kappa}(0)\Gamma_1(0,t) . \qquad (3.58)$$

Accounting for the asymptotic structure of Γ , we obtain from (3.56)

$$\frac{\partial\Gamma_1}{\partial x} - \frac{\partial\Gamma_1}{\partial t} = 0 \quad \text{at} \quad x = 0 , \qquad (3.59)$$

$$\frac{\partial\Gamma_2}{\partial x} - \frac{\partial\Gamma_2}{\partial t} = \frac{1}{\pi}\left[\frac{f_r(0,t)\ln\tilde{\ell}}{1-\tilde{\ell}} + \frac{f_\ell(0,t)\ln\left(1-\tilde{\ell}\right)}{\tilde{\ell}}\right] \quad . \tag{3.60}$$

It is not difficult to obtain the analytic solution of the equation (3.55) with boundary condition, assuming that the blockage coefficient is constant and that the unsteady motions are harmonic, i.e.,

$$\tilde{\kappa} = \text{const.} \quad ,$$

$$r_0(x,t) = \hat{r}_0(x)e^{ikt} \quad ,$$

where

$$i = \sqrt{-1} \quad ,$$

$$k = \frac{\omega L}{U} = \text{Strouhal number } (\omega \text{ is the circular frequency}) \quad .$$

Here we shall present some final results for one particular case of sway oscillations of the hull (i.e., we assume that the ship translates at a constant speed U parallel to the canal walls and at the same time performs very small lateral harmonic oscillations with an amplitude ℓ_0). For this case, the instantaneous position of the centerline can be written as

$$y_c(x,t) = \ell + \ell_0 \sin kt = \ell - i\ell_0 e^{ikt} \quad .$$

Then

$$\alpha_s(x,t) = \frac{\partial y_c}{\partial x} - \frac{\partial y_c}{\partial t} = -\ell_0 k e^{ikt} \quad .$$

The sway-force coefficient to the lowest order is

$$\varepsilon C_{y_1} = \varepsilon \hat{C}_{y_1} e^{ikt} = 2e^{ikt}\int_0^1 (\Gamma_1' - ik\Gamma_1)dx$$

$$= -\frac{2e^{ikt}k\ell_0}{\tilde{\ell}(1-\tilde{\ell})a_0}\left\{\frac{1+ik}{a_0} - \hat{A}_1[a_0 + ik(e^{a_0}-1)] - \hat{B}_1[a_0 + ik(1-e^{-a_0})]\right\} \quad , \tag{3.61}$$

where

$$\hat{A}_1 = \frac{a_0+ik(1-e^{-a_0})}{2a_0^2(a_0\cosh a_0+ik\sinh a_0)} \quad ,$$

$$\hat{B}_1 = \frac{a_0 - ik(1-e^{-a_0})}{2a_0^2(a_0\cosh a_0 + ik\sinh a_0)} \quad .$$

Turning to the approximation of the order $O(1)$, we finally find the following expression for the sway force coefficient

$$\varepsilon C_y = \varepsilon C_{y_1} + \varepsilon C_{y_2} + O(\varepsilon^2)$$

$$= e^{ikt}(\varepsilon\hat{C}_{y_1} + \varepsilon\hat{C}_{y_2}) + O(\varepsilon^2) \quad , \tag{3.62}$$

where

$$C_{y_2} = -\frac{2e^{ikt}}{a_0}\{\hat{A}_2[a_0 + ik(e^{a_0}-1)] + \hat{B}_2[a_0 - ik(e^{-a_0}-1)]\} \quad ,$$

and

$$\hat{A}_2 = \frac{\hat{\Gamma}_2(1)(a_0+ik)+\hat{P}_2(0)e^{-a_0}}{2(a_0\cosh a_0+ik\sinh a_0)} \quad ,$$

$$\hat{B}_2 = \frac{\hat{\Gamma}_2(1)(a_0-ik)-\hat{P}_2(0)e^{a_0}}{2(a_0\cosh a_0+ik\sinh a_0)} \quad ,$$

$$\hat{\Gamma}_2(1) = \Gamma_2(1,t)e^{-ikt} \quad ,$$

$$\hat{P}_2(0) = \left[\frac{\partial\Gamma_2}{\partial x}(0) - \frac{\partial\Gamma_2}{\partial t}(0)\right] e^{-ikt} \quad .$$

For purposes of ship traffic control, it might be convenient to represent the unsteady sway-force coefficient in the form

$$C_y = \dot{y}_c C_y^{\dot{y}_c} + \ddot{y}_c C_y^{\ddot{y}_c} \quad , \tag{3.63}$$

where

$$\dot{y}_c = \ell_0 k\cos kt \quad ,$$

$$\ddot{y}_c = -\ell_0 k^2 \sin kt \quad ,$$

$C_y^{\dot{y}_c}$, $C_y^{\ddot{y}_c}$ are derivatives of the sway force coefficients with respect to the velocity $\dot{y}_c$ and acceleration $\ddot{y}_c$ of the sway oscillation.

It can be shown that

$$\varepsilon C_y^{\dot{y}_c} = \frac{1}{\ell_0 k} Re(\varepsilon\hat{C}_y) \quad , \tag{3.64}$$

$$\varepsilon C_y{}^{\ddot{y}_C} = \frac{1}{\ell_0 k^2} Im(\varepsilon \hat{C}_y) \quad . \tag{3.65}$$

Some results of the computations of the coefficients $C_y{}^{\dot{y}_C}$ and $C_y{}^{\ddot{y}_C}$ versus the parameter $\tilde{\kappa} = 1/2\varepsilon C$ for different values of Strouhal number are presented in Figures 15 and 16 (only the lowest-order solution being taken into account).

The effects of yaw oscillations or of rudder oscillatory deflection can be considered similarly. It is interesting to mention that, at least to the considered order of the solution, the unsteady vortex wake located at distances of the order $O(1)$ behind the stern does not influence the values of force coefficients. This is due to the fact that the perturbations induced by wake vortices located in a narrow canal decay exponentially at distances of the order of $O(1)$. Therefore, the unsteady effects in case of ship motion in a narrow canal arise because of the free unsteady vortices within the ship hull.

D. Unsteady Motion of a Ship Close to a Bank

Some intermediate results for this problem are presented in Appendix C. The asymptotic solution is constructed with respect to the small parameter ℓ . As shown in Rozhdestvensky (1977, 1979), the influence of the unsteady wake manifests itself only at the approximation level of the order $O(1)$, that is, it is very small compared to the lowest order, $O(\frac{1}{\ell})$. Physically it means that as $\ell \to 0$ the velocities induced on the hull by the unsteady vortices in the wake at distances of the order $O(1)$ behind the stern are neutralized by the induced velocities of the image vortices. Therefore when the ship moves very close to a bank ($\ell \to 0$) , to the lowest order, the unsteady effects are caused only by the free unsteady vortices generated on the hull within the length of the ship.

Below we shall present some results for one particular case of the sway oscillations of the amplitude ℓ_0 ($y_C = \ell + \ell_0 \sin kt = \ell - i\ell_0 \exp(ikt)$). The sway-force coefficient to the lowest order, as in steady case, can be computed using the corresponding result for the ship in narrow canal (3.61) and replacing a_0 by $a = \sqrt{\bar{\kappa}}$ ($\bar{\kappa} = \kappa/\ell$) and $\ell_0/(1 - \tilde{\ell})$ by ℓ_0 .

For the case when $\bar{\kappa} \to 0$, the problem becomes identical to the two-dimensional problem of the thin airfoil oscillating near a rigid wall while moving at constant speed parallel to the wall, for which, in Rozhdestvensky (1976,

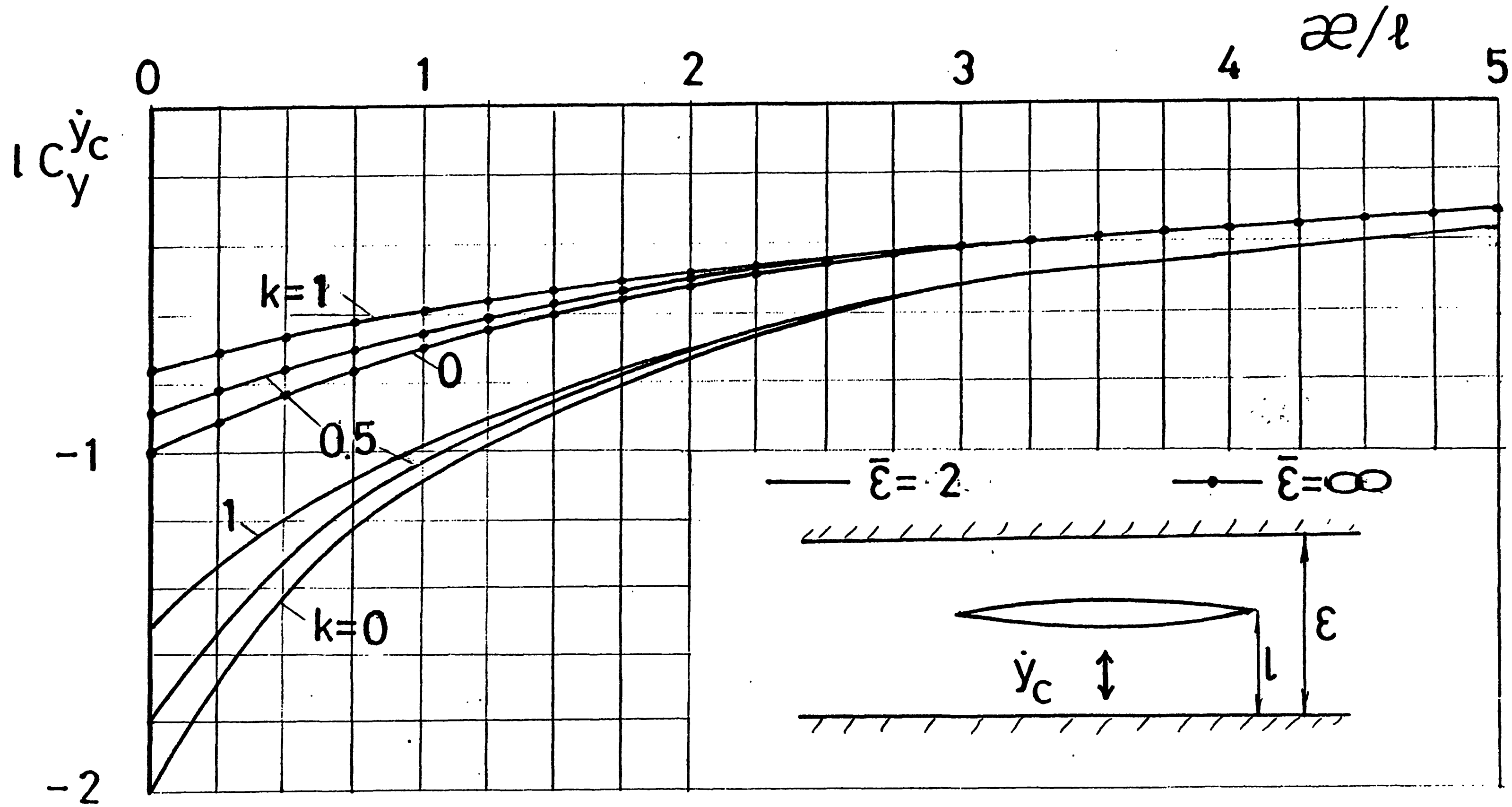

Figure 15. Unsteady Sway Derivative $\ell C_y^{y_c}$ Versus κ/ℓ at Different Values of Strouhal Number k (Lowest Order)

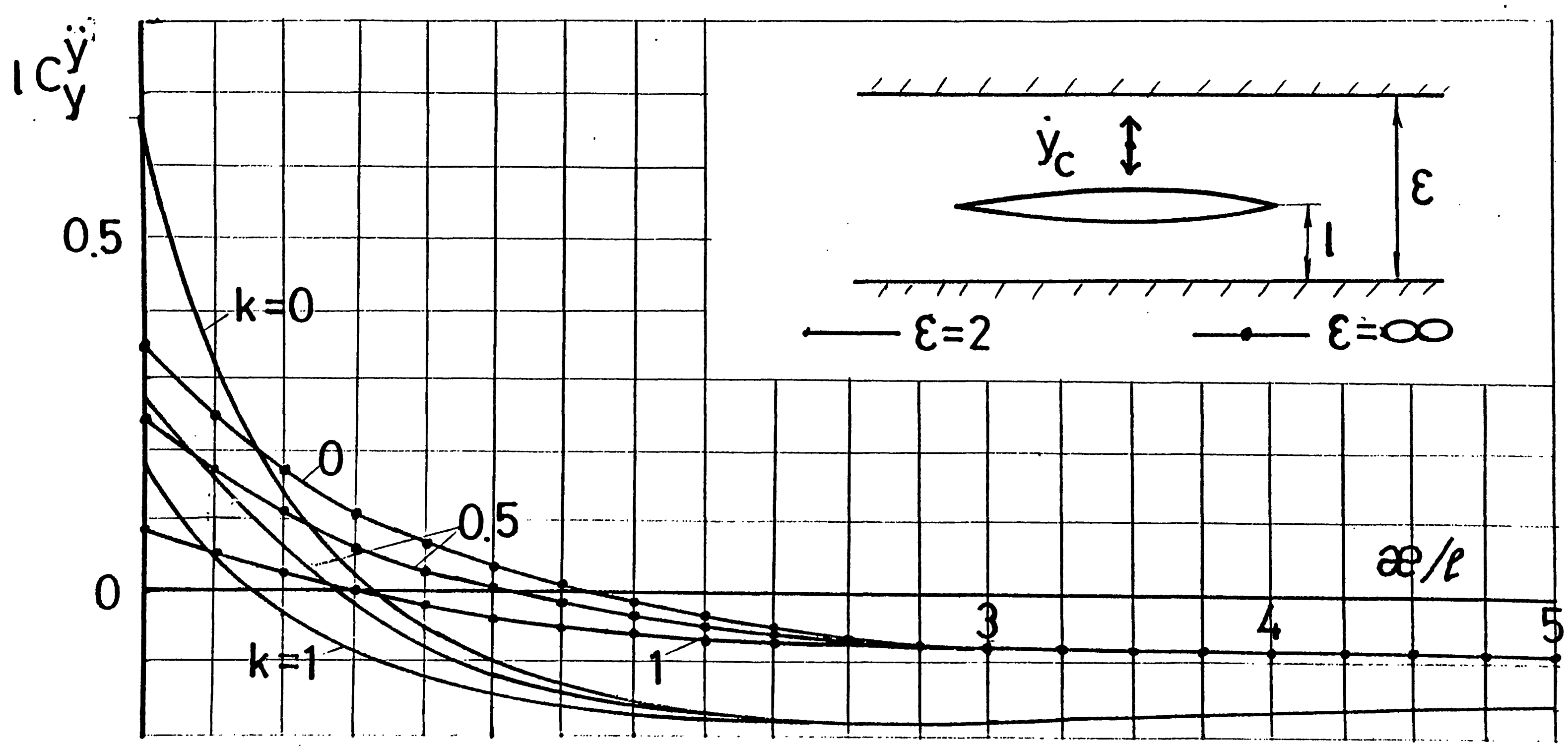

Figure 16. Unsteady Sway Derivative $\ell C_y^{\ddot{y}_c}$ Versus κ/ℓ at Different Values of Strouhal Number k (Lowest Order)

1979), the following formulae for the sway force coefficient was obtained

$$C_y = \dot{y}_c C_y^{\dot{y}_c} + \ddot{y}_c C_y^{\ddot{y}_c} \quad ,$$

$$\ell C_y^{\dot{y}_c} = -\frac{2+k^2}{2(1+k^2)} - \frac{\ell \ln\frac{1}{\ell}}{\pi} \frac{2(1+k^2)^2+2-k^2}{(1+k^2)^2}$$
$$- \frac{\ell}{\pi}\left\{\frac{3k^2(1+k^2)+2(1+\ln\pi)[k^2+(2+k^2)(1+k^2)] - k^3[(1-k^2)J_{11}+2kJ_{21}]}{2(1+k^2)^2}\right\} , \quad (3.66)$$

$$\ell C_y^{\ddot{y}_c} = \frac{2-k^2}{6(1+k^2)} + \frac{\ell \ln\frac{1}{\ell}}{\pi} \frac{(1-k^2)(2+k^2)}{(1+k^2)^2}$$
$$+ \frac{\ell}{\pi}\left\{\frac{2(1+\ln\pi)-2k^2\ln\pi(2+k^2)-k^2[2kJ_{11}+(k^2-1)J_{21}]}{2(1+k^2)^2}\right\} , \quad (3.67)$$

where

$$J_{11} = si(k)\cos k + (2\ \ln k - ci(k))\sin k \quad ,$$

$$J_{21} = si(k)\sin k - (2\ \ln k - ci(k))\cos k \quad ,$$

$k = \frac{\omega b}{U}$ as previously (Strouhal number),

si and ci are sine and cosine integrals.

Results of some computations are represented in Figures 17 and 18 by continuous lines. For comparison, in the same figures we plot some results obtained by I.I. Efremoff (1975) by a straightforward numerical solution of the integral equation of the airfoil near the rigid boundary for $\ell = 0,1$ (dotted line).

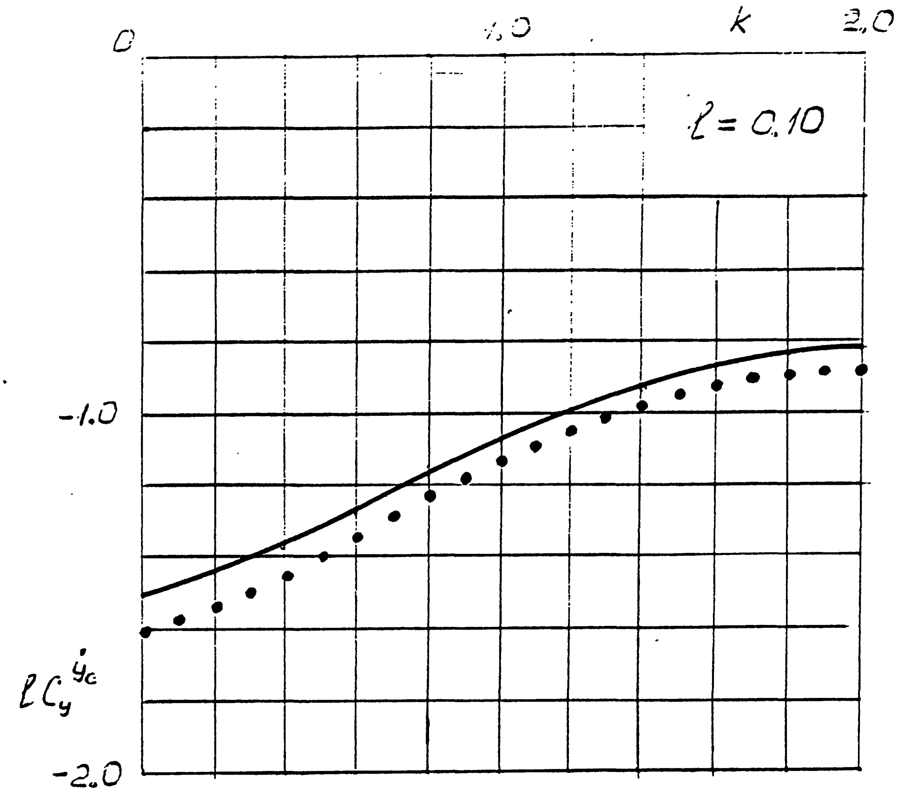

Figure 17. Unsteady Sway Derivative $\ell C_y^{\dot{y}_c}$ (to the Order of $O(\ell)$) Versus Strouhal Number for the Case of Complete Blockage. (Comparison with the Results of the Numerical Solution of the Integral Equation)

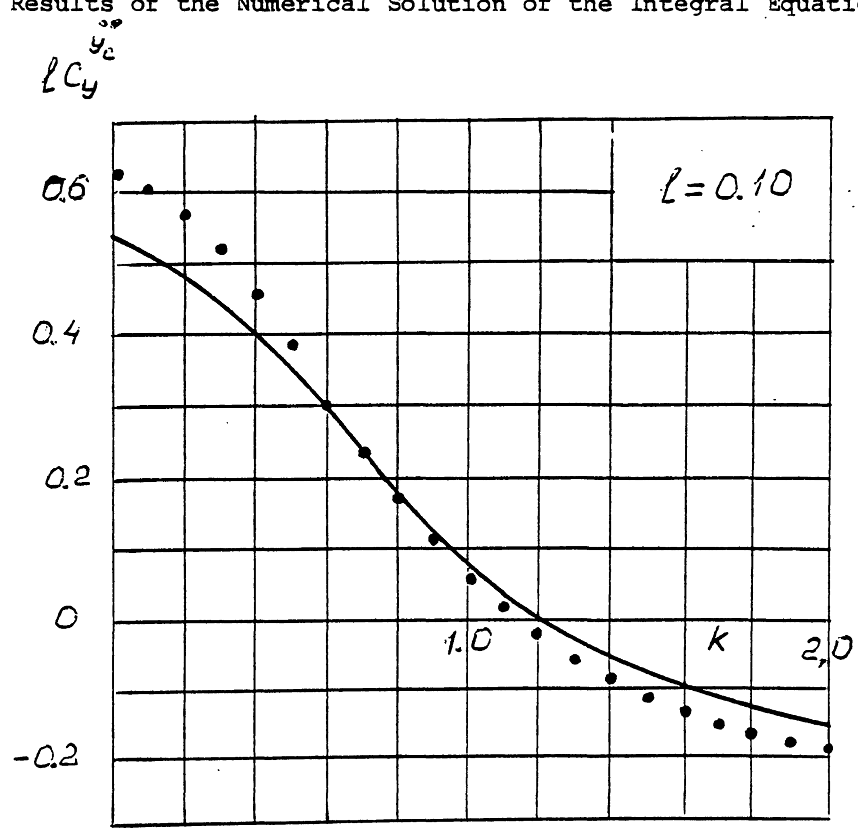

Figure 18. Unsteady Sway Derivative $\ell C_y^{\ddot{y}_c}$ (to the Order of $O(\ell)$) Versus Strouhal Number for the Case of Complete Blockage. (Comparison with the Results of the Numerical Solution of the Integral Equation)

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APPENDIX A. DETERMINATION OF THE BOW AND STERN POTENTIALS FOR THE PROBLEM OF A SHIP MOVING CLOSE TO A BANK

To find the homogeneous solution ϕ_h in the vicinity of the bow, map the bow-flow region onto an auxiliary upper halfplane $Im\{\zeta\} = \eta > 0$ using the Schwarz-Christoffel transformation. The correspondence of points in the bow-flow plane $\bar{z} = \bar{\nu} + i\bar{y}$ and auxiliary complex plane $\zeta = \xi + i\eta$ is shown in Figure A1. The mapping function is

$$\bar{z} = \frac{\bar{y}_r(1)}{\pi}(1 + \zeta + \ln\zeta) \quad . \tag{A.1}$$

For the case of a purely circulatory motion of the fluid (see Figure A2) around the bow, the flow with the unit velocity at the left infinity is represented by a band of a width $\bar{y}_r(1)$ in the plane of the complex potential,

$$F_h = \phi_h + i\psi_h \quad .$$

The mapping of this band onto the half plane $Im\{\zeta\} > 0$ is fulfilled with the help of the function

$$F_h = \phi_h + i\psi_h = \frac{\bar{y}_r(1)}{\pi}\ln\zeta \quad . \tag{A.2}$$

Thus, the solution of the homogeneous problem is given by the formulae (A.1) and (A.2). On the hull we have

$$\bar{z} = \bar{\nu} + i\bar{y}_r(1) \quad , \quad F_h = \phi_h + i\bar{y}_r(1) \quad ,$$

and so

$$\pi\bar{\nu} = y_r(1)\left(1 - e^{f_h} + f_h\right) \quad , \tag{A.3}$$

where
$$f_h = \frac{\pi}{\bar{y}_r(1)}\phi_h \quad .$$

It can be demonstrated that the velocity corresponding to the homogeneous solution has square root singularity at the bow. In fact, as $\bar{\nu} \to 0$, $\phi_h \to 0$ ($f_h \to 0$) , and (A.3) yields

$$\pi\bar{\nu} \simeq \bar{y}_r(1)[1 - (1 + f_h + \tfrac{1}{2}f_h^2 + \ldots) + f_h] \sim -\frac{1}{2}\bar{y}_r(1)f_h^2 \quad ,$$

or

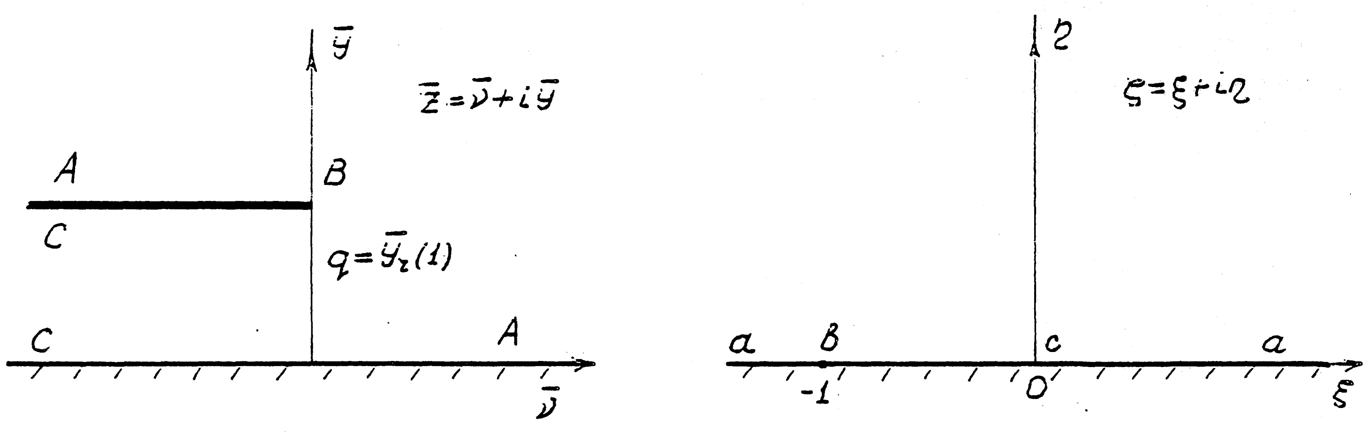

Figure A1. The Regions of the Flow in Physical and Auxiliary Planes

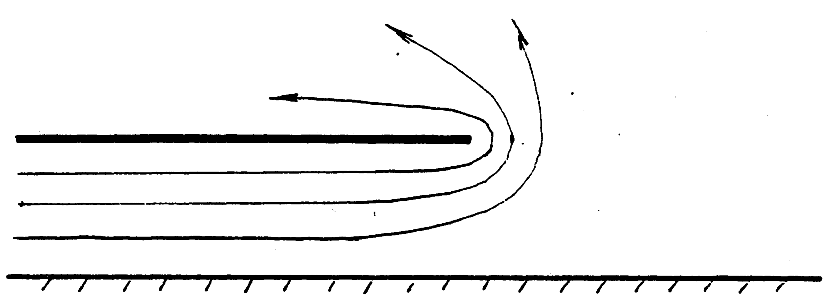

Figure A2. Picture of the Flow Corresponding to the Homogeneous Solution

$$\frac{df}{d\bar{\nu}} = \frac{\text{constant}}{\sqrt{-\bar{\nu}}} \quad , \quad \bar{\nu} < 0 \quad .$$

For the matching with the outer and channel flows we shall need the asymptotic expression for ϕ_h as $\bar{\nu} \to -\infty$, $\bar{y} = \bar{y}_r(1) \pm 0$:

On the upper side of the slit (right side of the ship), $\bar{\nu} \to -\infty$, $\bar{y} = \bar{y}_r(1) + 0$, $\phi_h \to \infty$ ($f_h \to \infty$) , and we have from (A.3)

$$f_h \sim \ln\left|\frac{\pi\bar{\nu}}{\bar{y}_r(1)}\right| \quad \text{or} \quad \phi_h = \phi_{hr} \sim \frac{\bar{y}_r(1)}{\pi} \ln\left|\frac{\pi\bar{\nu}}{\bar{y}_r(1)}\right| \; .$$

On the lower side of the slit (left side of the ship), $\bar{\nu} \to -\infty$, $\bar{y} = \bar{y}_r(1) - 0$, $\phi_h \to -\infty$ ($f_h \to \infty$) , and we can obtain from (A.3)

$$f_h \sim \pi \frac{\bar{\nu}}{\bar{y}_r(1)} - 1 \quad \text{or} \quad \phi_h = \phi_{h\ell} \sim \frac{\bar{y}_r(1)}{\pi} \left[\pi \frac{\bar{\nu}}{\bar{y}_r(1)} - 1 \right] .$$

Now we pass over to the determination of the nonhomogeneous solution ϕ_n . Using the same mapping function (A.1), we arrive at the following problem for the complex velocity $w_n = u_n - iv_n$ in the auxiliary plane ζ : To find the analytic function $w_n(\zeta)$ in the upper halfplane $Im\{\zeta\} = \eta > 0$, given the imaginary part, $Im\{w_n\} = -v_n$, on the ξ-axis (see Figure A3).

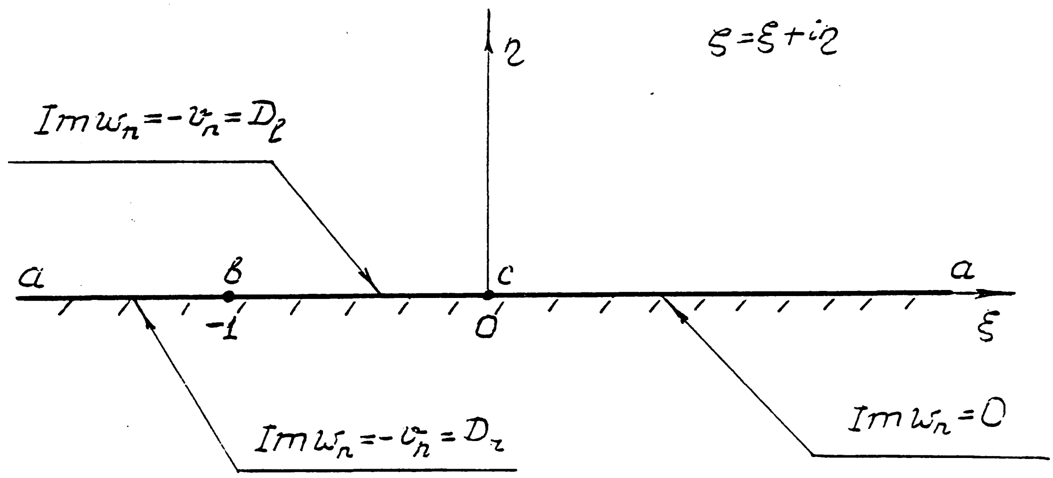

Figure A3. Nonhomogeneous Problem in Auxiliary Plane

It can be verified that the following solution satisfies the above formulated problem

$$w_n = \frac{1}{\pi} [(D_r - D_\ell)\ln(1 + \zeta) + D_\ell \ln\zeta] \quad .$$

On the hull, $\zeta = \xi < 0$, we have

$$u_n = Re\{w_n\} = \frac{1}{\pi} [(D_r - D_\ell)\ln|1 + \xi| + D_\ell \ln|\xi|] \quad . \tag{A.4}$$

The corresponding nonhomogeneous potential can be found with the help of the formula

$$\phi_n = \int_0^{\bar{\nu}} u_n d\bar{\nu} \quad . \tag{A.5}$$

The variable $\xi = Re\{\zeta\}$ is related to $\bar{\nu}$ on the hull in the following way:

$$\pi\,\bar{\nu} = \bar{y}_r(1)(1 + \xi + \ln|\xi|) \quad , \quad \xi < 0 \quad . \tag{A.6}$$

The flow picture corresponding to the nonhomogeneous solution is presented in Figure A4.

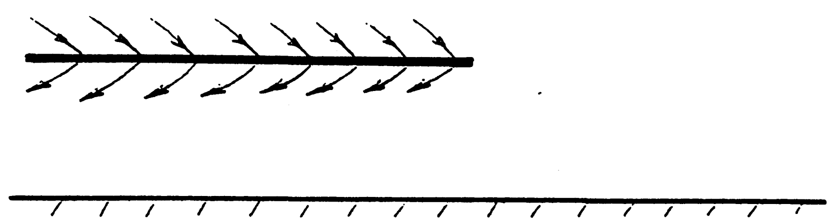

Figure A4. Picture of the Flow Corresponding to the Nonhomogeneous Solution

For matching with the channel and outer flows, asymptotic estimates of u_n and ϕ_n will be needed. For u_n, the desired results can be obtained directly from (A.4) and (A.6). For the potential, we must carry out the integration indicated in (A.5). This is done most easily in the ζ plane. We obtain the following results:

On the upper side of the slit (right side of the ship), where $\bar{\nu} \to -\infty$, $\bar{y} = \bar{y}_r(1) + 0$, $\xi \to -\infty$, we have

$$\bar{\nu} \sim \frac{\bar{y}_r(1)}{\pi}\,\xi \quad , \quad u_{n_r} \sim \frac{D_r}{\pi} \ln|\xi| \quad ,$$

and so

$$u_n = u_{n_r} \sim \frac{D_r}{\pi} \ln\left|\frac{\pi\bar{\nu}}{\bar{y}_r(1)}\right| \quad . \tag{A.7}$$

Substituting (A.4) and (A.6) into (A.5), integrating, and transforming back to the $\bar{z}$ plane, we obtain

$$\phi_n = \phi_{n_r} \sim \frac{D_r}{\pi} \bar{\nu} \left[\ln \left| \frac{\pi \bar{\nu}}{\bar{y}_r(1)} \right| - 1 \right] .$$

On the lower side of the slit (left side of the ship), where $\bar{\nu} \to -\infty$, $\bar{y} = \bar{y}_r(1) - 0$, $\xi \to 0$, we have

$$\bar{\nu} = \frac{\bar{y}_r(1)}{\pi} (\ln|\xi| + 1) \tag{A.8}$$

and

$$u_n = u_{n_\ell} \sim \frac{D_\ell}{\pi} \ln|\xi| , \tag{A.9}$$

and so

$$u_n = u_{n_\ell} \sim D_\ell \left[\frac{\bar{\nu}}{\bar{y}_r(\ell)} - \frac{1}{\pi} \right] . \tag{A.10}$$

From (A.5), we obtain

$$\phi_n = \phi_{n_\ell} \sim \frac{D_\ell}{\pi} \left[\frac{\pi \bar{\nu}^2}{2\bar{y}_r(1)} - \bar{\nu} \right] .$$

Finally we can write down, in terms of the variable $\nu = \ell\bar{\nu}$, the following asymptotic formulae for the bow-flow potential ϕ_b in the overlap regions:

On the right side of the ship, the one-term outer expansion of the two-term bow expansion (2.29) is

$$\Phi_b = \Phi_{br} \sim \frac{a_1 \ell}{\pi} \bar{y}_r(1) \ln \left| \frac{\pi \nu}{\ell \bar{y}_r(1)} \right| + \frac{\ell D_r}{\pi} \nu \left[\ln \left| \frac{\pi \nu}{\ell \bar{y}_r(1)} \right| - 1 \right] + a_3 \ell \nu + a_4 \ell .$$

On the left side of the ship, the two-term channel expansion of the two-term bow expansion (2.29) is

$$\Phi_b = \Phi_{b\ell} \sim \frac{a_1 \bar{y}_r(1)}{\pi} \left[\frac{\pi \nu}{\bar{y}_r(1)} - \ell \right] + \frac{D_\ell}{\pi} \left[\frac{\pi \nu^2}{2\bar{y}_r(1)} - \ell \nu \right] + a_3 \ell \nu + a_4 \ell .$$

In similar fashion we can derive the nonhomogeneous solution for the stern flow. The homogeneous solution must be excluded from the expression for the stern potential because it gives a square-root singularity for the velocity at the edge and hence does not satisfy the Kutta-Joukowsky condition. The following asymptotics are obtained for the matching:

On the right side of the ship, the one-term outer expansion of the two-term stern expansion has the following form:

$$\Phi_s = \Phi_{sr} \sim \frac{\ell E_r^+}{\pi} \nu \left[\ln \left| \frac{\pi \nu}{\ell} \right| - 1 \right] + b_2 \ell \nu + \ell b_3^+ \quad .$$

On the left side of the ship, the two-term channel expansions of the two-term stern expansions are obtained both for the potential and velocity:

$$\Phi_s = \Phi_{s\ell} \sim \frac{E_\ell}{\pi} \left[\frac{\pi}{2} \nu^2 - \ell \nu \right] + b_2 \ell \nu + \ell b_3^- \quad ,$$

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{s\ell}}{dx} = - \frac{d\Phi_{s\ell}}{d\nu} \sim - \frac{E_\ell}{\pi} (\pi\nu - \ell) - \ell b_2 \quad .$$

APPENDIX B. DETERMINATION OF THE BOW AND STERN POTENTIALS FOR THE PROBLEM OF A SHIP MOVING IN A NARROW CANAL

To find the homogeneous solution for the flow around the bow, map the bow-flow region onto the upper half of the auxiliary plane $\zeta = \xi + i\eta$ using the following Schwarz-Christoffel transformation:

$$\tilde{z} = \tilde{\nu} + i\tilde{y} = \frac{1}{\pi}[q\ln\zeta - \ln(q\zeta+q-1)] + i \quad , \tag{B.1}$$

where $q = \tilde{y}_r(1)$. Correspondence of the points in $\tilde{z}$ and ζ planes is given in Figure B1. For the purely circulatory flow shown in Figure B2, the homogeneous complex potential $F_h = \phi_h + i\psi_h$ (ψ_h = stream function) in the ζ-plane takes the form

$$F_h = \frac{q}{\pi} \ln\zeta \quad . \tag{B.2}$$

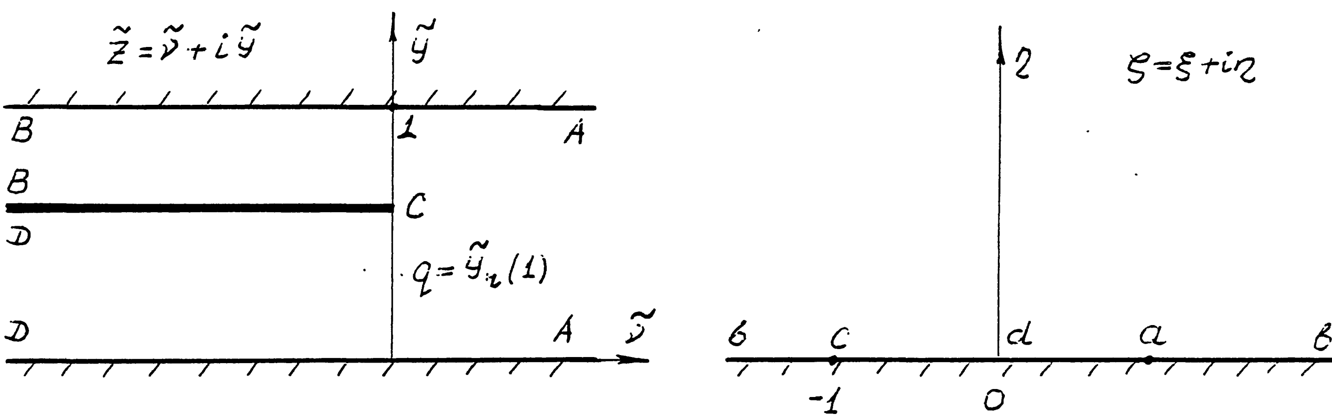

Figure B1. The Regions of the Flow in Physical and Auxiliary Planes

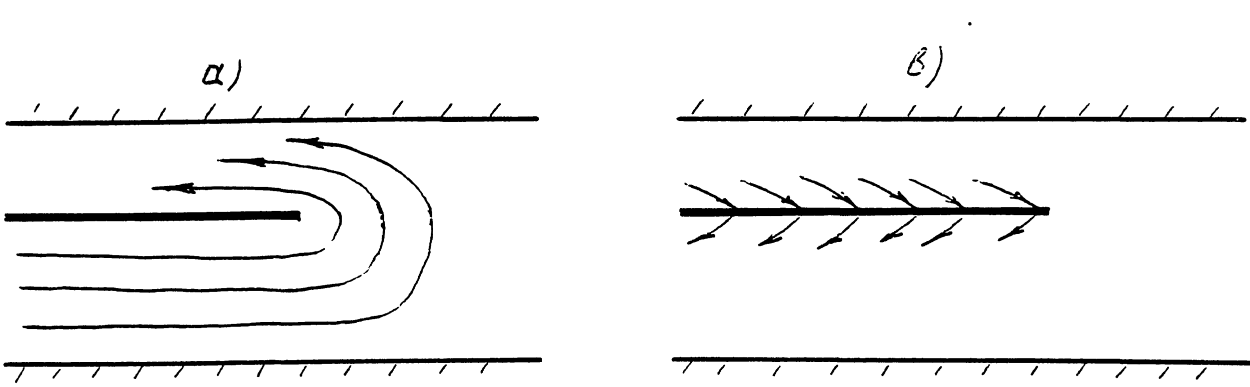

Figure B2. The Picture of the Flow Corresponding to a) Homogeneous and b) Nonhomogeneous Solutions

Note that on the walls $(\zeta = \xi > 0)$ $\psi_h = 0$ and on the hull $(\zeta = \xi < 0)$ $\psi_h = q$. Thus the final though implicit expression for F_h is

$$\tilde{z} = F_h - \frac{1}{\pi} \ln\left[q \exp \frac{\pi F_h}{q} + q - 1\right] + i \quad . \tag{B.3}$$

On the hull, $F_h = \phi_h + i\psi_h = \phi_h + iq$, $\tilde{z} = \tilde{\nu} + iq$, and (B.3) yields

$$\tilde{\nu} = \phi_h - \frac{1}{\pi} \ln \left|1 - q + q \exp\left(\frac{\pi\phi_h}{q}\right)\right| \quad . \tag{B.4}$$

It can be easily verified that this solution has a square-root singularity for the velocity at $\bar{\nu} = 0$ (i.e., at the edge).

For matching with the channel-flow solutions, we shall need the asymptotic expression for ϕ_h as $\tilde{\nu} \to -\infty$, $\tilde{y} = q \pm 0$. <u>On the upper side of the slit</u> (right side of the ship), as $\tilde{\nu} \to -\infty$, $\phi_h \to \infty$, the exponential inside of the logarithm prevails and we obtain

$$\phi_h = \phi_{hr} \sim -\frac{q}{1-q}\left[\tilde{\nu} + \frac{\ln q}{\pi}\right].$$

<u>On the lower side of the slit</u> (left side of the ship), as $\tilde{\nu} \to -\infty$, $\tilde{y} = q - 0$, $\phi_h \to -\infty$, the exponential vanishes, so that

$$\phi_h = \phi_{h\ell} \sim \tilde{\nu} + \frac{1}{\pi} \ln|1-q| \quad .$$

To determine the <u>nonhomogeneous solution</u> ϕ_n we use the same function (B.1) to map the $\tilde{z}$ plane onto the upper ζ halfplane and formulate the following problem for the complex velocity $w_n = u_n - iv_n$: Find the analytic function w_n in the upper halfplane, $Im\{\zeta\} = \eta > 0$, given the imaginary part $Im\{w_n\} = -v_n$ on the ξ axis (see Figure B3). It can be shown that the following function is a solution of the problem formulated above:

$$w_n = \frac{D_r - D_\ell}{\pi} \ln(1 + \zeta) + \frac{D_\ell}{\pi} \ln \zeta \quad . \tag{B.5}$$

On the hull, $\zeta = \xi < 0$,

$$u_n = Re\{w_n\} = \frac{1}{\pi} \left[(D_r - D_\ell)\ln|1+\xi| + D_\ell \ln \xi\right] \quad . \tag{B.6}$$

The corresponding nonhomogeneous potential can be found as follows:

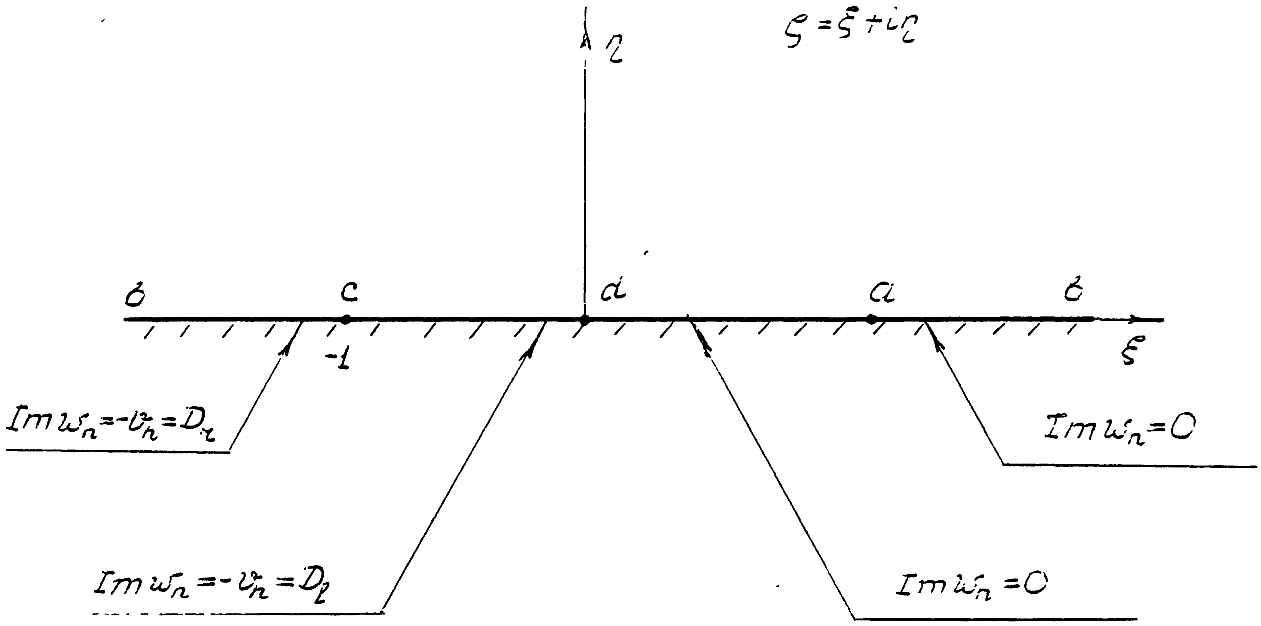

Figure B3. Nonhomogeneous Problem in Auxiliary Plane

$$\phi_n = \int_0^{\tilde{\nu}} u_n d\tilde{\nu} \quad . \tag{B.7}$$

The variable $\xi = Re\{\zeta\}$ is related to $\tilde{\nu}$ on the hull through the mapping function

$$\pi\tilde{\nu} = q\ln|\xi| - \ln|1-q-q\xi| \quad , \quad \xi < 0 \quad . \tag{B.8}$$

For matching with channel-flow solutions, the following asymptotic expansions of u_n and ϕ_n can be obtained: <u>On the upper side of the slit</u> (right channel), $\tilde{\nu} \to -\infty$, $\tilde{y} = q + 0$, $\xi \to -\infty$:

$$\tilde{\nu} \sim \frac{1}{\pi}[q\ln|\xi| - \ln|q\xi|] + O\left(\frac{1}{\xi}\right) = \frac{1}{\pi}\,[(q-1)\ln|\xi| - \ln q] \quad ,$$

$$u_n = u_{nr} \sim \frac{1}{\pi}\, D_r \ln|\xi| \quad ,$$

wherefrom

$$u_n = u_{nr} \sim \frac{D_r}{\pi(q-1)}\,(\pi\tilde{\nu} + \ln q) \quad ,$$

$$\phi_n = \phi_{nr} \sim \frac{D_r}{\pi(q-1)}\left(\frac{\pi}{2}\tilde{\nu}^2 + \tilde{\nu}\ln q\right) \quad .$$

On the lower side of the slit (left channel), $\tilde{\nu} \to -\infty$, $\tilde{y} = q - 0$, $\xi \to 0 \pm 0$:

$$\tilde{\nu} \sim \frac{1}{\pi} [q \ln|\xi| - \ln|1-q|] + O(q\xi) \quad ,$$

$$u_n = u_{n\ell} \sim \frac{1}{\pi} D_\ell \ln|\xi| \quad .$$

Combining the last two expressions, we obtain

$$u_n = u_{n\ell} \sim \frac{D_\ell}{\pi q} (\pi\tilde{\nu} + \ln|1-q|) \quad ,$$

$$\phi_n = \phi_{n\ell} \sim \frac{D_\ell}{\pi q} \left(\frac{\pi\tilde{\nu}^2}{2} + \tilde{\nu}\ln|1-q|\right) \quad .$$

Finally, the asymptotic formulae for Φ_b (bow-flow potential) in the overlap regions take the forms: On the upper side of the slit ($\tilde{y} = q + 0$) , the two-term right-channel expansion of the two-term bow expansion (2.78) is

$$\Phi_b = \Phi_{br} \sim -\frac{a_1 q}{1-q}\left(\nu + \varepsilon\frac{\ln q}{\pi}\right) + \frac{D_r}{\pi}\left[\frac{\pi\nu^2}{2(q-1)} + \frac{\varepsilon\nu\ln q}{(q-1)}\right] + a_3\varepsilon\nu + a_4\varepsilon \quad . \quad (B.9)$$

On the lower side of the slit ($\tilde{y} = q - 0$) , the two-term left-channel expansion of the two-term bow expansion (2.78) is

$$\Phi_b = \Phi_{b\ell} \sim a_1\left(\nu + \frac{\varepsilon}{\pi}\ln|1-q|\right) + \frac{D_\ell}{\pi q}\left(\frac{\pi\nu^2}{2} + \varepsilon\nu\ln|1-q|\right) + a_3\varepsilon\nu + a_4\varepsilon \quad . \quad (B.10)$$

The flow corresponding to the nonhomogeneous solution is sketched in Figure B2.

The nonhomogeneous solution for the stern flow is constructed in a similar way. (The homogeneous solution in the vicinity of the stern is not included as it does not satisfy Kutta-Joukowsky condition.) The following asymptotic expressions are obtained for matching the stern-flow with the channel flows: On the upper side of the slit ($\tilde{y} = q_1 + 0$, $q_1 = \tilde{y}_r(0) = \tilde{\ell}$) , the two-term right-channel expansions of the two-term expansions of the stern potential and velocity are

$$\Phi_s = \Phi_{sr} \sim \frac{E_r}{\pi(q_1-1)}\left(\frac{\pi}{2}\nu^2 + \varepsilon\nu\ln q_1\right) + b_2\varepsilon\nu + \varepsilon b_3^+ \quad , \qquad (B.11)$$

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{sr}}{dx} = -\frac{d\Phi_{sr}}{d\nu} \sim \frac{E_r}{\pi(1-q_1)}(\pi\nu + \varepsilon\ln q_1) - b_2\varepsilon \quad . \qquad (B.12)$$

On the lower side of the slit $(\tilde{y} = q_1 - 0)$, the two-term left-channel expansions of the two-term expansions of the stern potential and velocity are

$$\Phi_s = \Phi_{s\ell} \sim \frac{E_\ell}{\pi q_1}\left(\frac{\pi}{2}\nu^2 + \varepsilon\nu\ln|1-q_1|\right) + b_2\varepsilon\nu + \bar{b}_3\varepsilon \quad , \tag{B.13}$$

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{s\ell}}{dx} = - \frac{d\Phi_{s\ell}}{d\nu} \sim - \frac{E_\ell}{\pi q_1}(\pi\nu + \varepsilon\ln|1-q_1|) - b_2\varepsilon \quad . \tag{B.14}$$

Let us now consider the asymptotic behavior of the edge solutions far upstream and downstream. Far from the edge in the canal,

$$\tilde{z} \to \infty + i\tilde{y} \quad , \quad 1 > \tilde{y} > 0 \quad .$$

In the auxiliary plane, $\zeta \to a$ (see Figure B1). Put

$$\zeta = a + \zeta_1 \quad , \quad |\zeta_1| \to 0 \quad , \quad a = \frac{1-q}{q} \quad .$$

Then the expression for the mapping function yields

$$\tilde{z} \sim - \frac{1}{\pi}\ln\zeta_1 + O(1) \quad ,$$

or

$$\zeta_1 = \exp(-\pi\tilde{z}) \quad . \tag{B.15}$$

It follows from formulae (B.2) and (B.15) that, when $\zeta \to a$,

$$F_h = F_{hc} \sim \frac{q\ln a}{\pi} + \frac{q}{a\pi}\exp(-\pi\tilde{z}) \quad ,$$

$$w_{hc} = \frac{dF_{hc}}{d\tilde{z}} = u_{hc} - iv_{hc} = - \frac{q}{a}\exp(-\pi\tilde{z}) \quad .$$

The formulae for the nonhomogeneous complex velocity w_n and potential take the forms

$$w_n = w_{nc} \sim \frac{D_r - D_\ell}{\pi}\ln\frac{1}{q} + \frac{D_\ell}{\pi}\ln\frac{1-q}{q} + O(\exp(-\pi\tilde{z})) \quad ,$$

$$F_n = F_{nc} \sim w_{nc}\,\tilde{\nu} \quad .$$

The one-term upstream expansions of the two-term bow potential and velocity expansions are, finally,

$$\Phi_b = \Phi_{bc} \sim [D_\ell \ln(1-q) - D_r \ln q]\varepsilon\nu + a_3\varepsilon\nu + a_4\varepsilon \quad ,$$

$$\frac{d\Phi_b}{dx} = \frac{d\Phi_{bc}}{dx} = + \frac{d\Phi_{bc}}{d\nu} \sim \frac{1}{\pi}\,[D_\ell \ln(1-q) - D_r \ln q]\varepsilon + a_3\varepsilon \quad , \qquad (B.16)$$

with $\nu = x - 1$ and $q = \tilde{y}_r(1)$. The one-term downstream expansion of the two-term stern velocity expansion can be obtained in a similar way:

$$\frac{d\Phi_s}{dx} = \frac{d\Phi_{sc}}{dx} = - \frac{d\Phi_{sc}}{d\nu} \sim - \frac{1}{\pi}\,[E_\ell \ln(1-q_1) - E_r \ln q_1]\varepsilon - b_2\varepsilon \quad , \qquad (B.17)$$

with $q_1 = \tilde{\ell}$.

APPENDIX C. SOME INTERMEDIATE RESULTS FOR THE PROBLEM OF UNSTEADY MOTION OF A SHIP CLOSE TO A BANK IN SHALLOW WATER

We use the solution techniques developed previously, subdividing the flow into the outer flow, channel flow and edge flow regions. The outer solution ($x = O(1)$, $y = O(1)$, $\ell \to 0$) should satisfy the equations:

$$\frac{\partial^2 \Phi_r}{\partial x^2} + \frac{\partial^2 \Phi_r}{\partial y^2} = 0 \quad ,$$

$$\frac{\partial \Phi_r}{\partial y} = \kappa(x)\Gamma(x,t) - \frac{\partial y_c}{\partial x} + \frac{\partial y_c}{\partial t} - \frac{1}{2}\,\sigma(x,t) \equiv f_r \quad , \quad 1 > x > 0 \quad ,$$

$$\frac{\partial \Phi_r}{\partial y} = -\,\alpha_{w_1}(x,t) \quad , \quad 0 > x > -\infty \quad , \quad y = 0 + 0 \quad ,$$

$$\frac{\partial \Phi_r}{\partial y} = 0 \quad , \quad x > 1 \quad , \quad y = 0 + 0 \quad ,$$

$$\nabla \Phi \to 0 \quad \text{as} \quad x^2 + y^2 \to \infty \quad ,$$

where α_{w_1} is the induced lateral velocity in the wake defined as in Rozhdestvensky (1974, 1977) (by consideration of the channel flow between the wake and the bank):

$$\alpha_{w_1} = \frac{\partial^2 \phi_{1\ell}}{\partial x^2} = \frac{\partial^2 \phi_{1\ell}(0,t^*)}{\partial t^{*2}} \quad , \quad t^* = t + x \quad .$$

This problem has the following solution

$$\Phi_r = \frac{Q}{2\pi}\ln|(1-x)| + \frac{1}{2\pi}\int_{-\infty}^{1} q(\xi,t)\ln|x-\xi|\,d\xi \quad ,$$

where

$$q(\xi,t) = \begin{cases} -2f_r(x,t) \ , & 1 > x > 0 \ , \\ -2\alpha_{w_1}(x,t) \ , & 0 > x > -\infty \ . \end{cases}$$

For the case of harmonic oscillations,

$$\alpha_{w_1} = -k^2\phi_1(0,t^*) = -k^2(g_1 \cos kt^* + g_2 \sin kt^*) \quad .$$

During the calculations of Φ_r , the following integrals have to be dealt with:

$$I_1 = \int_0^\infty \cos k\xi \,\ln\xi \,d\xi \quad ; \qquad I_2 = \int_0^\infty \sin k\xi \,\ln\xi \,d\xi \quad .$$

These integrals are divergent in a conventional sense, but they can be treated in a generalized (Abel-Poisson) sense:

$$I_1 = \lim_{\delta\downarrow 0} \int_0^\infty e^{-\delta\xi}\cos k\xi \,\ln\xi \,d\xi$$

$$= \lim_{\delta\downarrow 0} \left\{ -\frac{1}{k^2+\delta^2}\left[\frac{\delta}{2}\ln(k^2+\delta^2) + k\arctan\frac{k}{\delta} + \delta C\right]\right\} = \frac{\pi}{k} \quad , \qquad (C.1)$$

$$I_2 = \lim_{\delta\downarrow 0} \int_0^\infty e^{-\delta\xi}\sin k\xi \,\ln\xi \,d\xi \; = \lim_{\delta\downarrow 0}\left\{ \delta\arctan\frac{k}{\delta} - kC + \frac{1}{2}k\ln(k^2+\delta^2)\right\}$$

$$= \begin{cases} \frac{1}{k}(\ln k - C) & \text{when } k > 0 \;, \\ 0 & \text{when } k = 0 \;, \end{cases} \qquad (C.2)$$

where $C \simeq 0.5772$ is the Euler constant. Physically, the generalized integration implies that we consider the oscillations with slightly decreasing amplitude which , in the limit, become constant amplitude oscillations. A similar approach was employed by Theodorsen for the calculation of integrals of the type $\int_0^\infty \sin k\xi d\xi$ and $\int_0^\infty \cos k\xi d\xi$ in his work on the problem of harmonic oscillations of a thin airfoil in unlimited fluid.

Taking into account (C.1) and (C.2), we obtain

$$\Phi_r = \frac{Q}{2\pi}\ln|1-x| - \frac{1}{\pi}\int_0^1 f_r(\xi,t)\ln|x-\xi|d\xi$$

$$+ \frac{k}{\pi}\left[(g_1T_1 + g_2T_2)\cos kt + (g_2T_1 - g_1T_2)\sin kt\right] \quad ,$$

where

$$T_1 = M\cos kx + N\sin kx \quad ,$$
$$T_2 = M\sin kx - N\cos kx \quad ,$$
$$M = si(kx) - \sin kx \ln x \quad ,$$
$$N = 2\ln k + \cos kx \ln x - ci(kx) \quad ,$$

and $si(kx)$ and $ci(kx)$ are sine and cosine integrals, defined as

$$si(kx) = -\int_{kx}^{\infty} \frac{\sin\xi}{\xi}\, d\xi \quad ,$$

$$ci(kx) = -\int_{kx}^{\infty} \frac{\cos k\xi}{\xi}\, d\xi \quad .$$

The asymptotic expansions for Φ_r near the extremities are obtained in the form:

<u>Near the bow,</u> $\nu = x - 1$:

$$\Phi_r = \Phi_{rb} \sim \frac{Q}{2\pi} \ln\nu + \frac{1}{\pi} f_r(1,t)\nu\ln\nu + \nu\frac{A_1}{\pi} + \frac{A_2}{\pi} \quad ,$$

where

$$A_1 = -f_r(1,t) - \int_0^1 [f_r(\xi,t) - f_r(1,t)]\frac{d\xi}{1-\xi}$$

$$- k^2[(g_1T_{21} - g_2T_{11})\cos kt + (g_1T_{11} + g_2T_{21})\sin kt] \quad ,$$

$$A_2 = -\int_0^1 f_r(\xi,t)\ln|1-\xi|\,d\xi + k[(g_1T_{11} + g_2T_{21})\cos kt + (g_2T_{11} - g_1T_{21})\sin kt] \quad ,$$

with $T_{11} = T_1(1)$, $T_{21} = T_2(1)$.

<u>Near the stern,</u> $\nu = -x$:

$$\Phi_r = \Phi_{rs} \sim \frac{\Delta\alpha}{\pi} \nu\ln\nu + \frac{B_1\nu}{\pi} + \frac{B_2}{\pi} \quad ,$$

where

$$\Delta\alpha = f_r(0,t) - \alpha_{w_1}(0,t) \quad ,$$

$$B_1 = \frac{1}{2}Q - f_r(0,t) - \int_0^1 [f_r(\xi,t) - f_r(0,t)]\frac{d\xi}{\xi} +$$

$$+ k\left\{[g_1(kT_{20} - 1) - g_2kT_{10}]\cos kt + \right.$$

$$\left. + [g_2(kT_{20} - 1) + g_1kT_{10}]\sin kt\right\} \quad ,$$

$$B_2 = -\int_0^1 f_r(\xi,t)\ln\xi d\xi + k[(g_1T_{10} + g_2T_{20})\cos kt + (g_2T_{10} - g_1T_{20})\sin kt] \quad ,$$

with

$$T_{10} = T_1(0) = -\frac{\pi}{2k} \quad , \quad T_{20} = T_2(0) = \frac{1}{k}(C - \ln k) \quad .$$

Matching of the potentials near the bow and in the region to the right of the stern gives

$$a_1 = \frac{1}{2}Q = \frac{\partial\phi_{1\ell}}{\partial x}(1,t) \quad ,$$

$$a_3 = \frac{1}{\pi}[A_1 + f_r(1,t)\left(1 - \ln\frac{\pi}{\ell}\right)] \quad ,$$

$$a_4 = \frac{1}{\pi}[A_2 - a_1\ln\frac{\pi}{h}] \quad ,$$

$$b_1 = \Delta\alpha = f_r(0,t) - \alpha_{w_1}(0,t) \quad ,$$

$$b_2^+ = \frac{1}{\pi}[B_1 + \Delta\alpha\left(1 - \ln\frac{\pi}{\ell}\right)] \quad ,$$

$$b_3^+ = \frac{1}{2}B_2 \quad .$$

Application of the Kutta condition (3.54) at the stern leads to the equation

$$b_2^- + \frac{\partial b_3^-}{\partial t} = b_2^+ + \frac{\partial b_3^+}{\partial t} = \frac{1}{\pi}\left[B_1 + \Delta\alpha\left(1 - \ln\frac{\pi}{\ell}\right) + \frac{\partial B_2}{\partial t}\right] \quad .$$

Boundary conditions for the channel flow potential ϕ_ℓ are obtained in the form

$$\phi_\ell(1,t) = \frac{\ell}{\pi}[A_2 - a_1\left(1 + \ln\frac{\pi}{\ell}\right)] \quad ,$$

$$\frac{\partial\phi_\ell}{\partial x} - \frac{\partial\phi_\ell}{\partial t} = \frac{\ell}{\pi}\left(\Delta\alpha\ln\frac{\pi}{\ell} + \frac{\partial B_2}{\partial t} - B_1\right) \quad \text{at} \quad x = 0 \quad .$$

The asymptotic structure of the channel flow potential is identical to (3.10).

APPENDIX D. ANALYSIS OF THE LIMITING FORM OF BECK'S (1976) INTEGRAL EQUATION AS THE DISTANCE FROM ONE OF THE CANAL WALLS TENDS TO ZERO

In his report, "Forces and Moments on a Ship Moving in a Canal," R. Beck arrives at the following far-field integral equation of the "porous" airfoil in a canal (see Beck (1976), p. 14, eq. 2.13):

$$\int_{-\ell}^{\ell} d\xi\gamma(\xi)\left[\frac{(1-\cos 2\alpha)\coth k_0(x-\xi)}{\cosh 2k_0(x-\xi)-\cos 2\alpha}+\frac{2w}{C(x)}H(x-\xi)\right]$$

$$= -\frac{\pi U}{wh\beta^2}\sin 2\alpha \int_{-\ell}^{\ell} d\xi\; S(\xi)\frac{\sinh 2k_0(x-\xi)}{[\cosh 2k_0(x-\xi)-\cos 2\alpha]^2}\quad , \qquad \text{(D.1)}$$

where, in Beck's notations,

$\alpha = \pi a/w$,

a = distance to starboard wall of canal ,

$\beta = [1 - F_{r_h}^2]^{1/2}$,

$C(x)$ = blockage coefficient ,

w = canal width ,

$k_0 = \pi/2w\beta$,

S = sectional area ,

γ = strength of vorticity along the ship hull in the near field ,

U = ship speed ,

h = water depth .

Inspecting his equation, Beck concludes that "... as $a \to 0$, both the right-hand side and the part of the kernel identifiable with the downwash of the vortices and their image system go to zero. This is because mathematically when the ship is right at the wall the problem is again symmetrical..."

However, it can be shown that the problem does not become symmetrical in the limit $a \to 0$ because the limit is singular.

We shall demonstrate that in the limit $a \to 0$ the integral equation degenerates into a much simpler one. Further on, we assume that $w = O(1)$ and we put $\beta = 1$ for convenience. Performing an integration by parts in the right-hand side of the equation, we can rewrite it as

$$\int_{-\ell}^{\ell} \gamma(\xi)K(x-\xi)d\xi + \frac{2w}{C(x)}\int_{-\ell}^{x} \gamma(\xi)d\xi$$

$$= \int_{-\ell}^{\ell} \frac{\sigma(\xi)\ \sin 2\alpha\ d\xi}{\cosh 2k_0(x-\xi) - \cos 2\alpha} \quad , \tag{D.2}$$

where

$$\sigma = US'(x)/h \quad ,$$

$$K(x-\xi) = \frac{(1-\cos\alpha)\coth k_0(x-\xi)}{\cosh 2k_0(x-\xi) - \cos 2\alpha} \quad .$$

Now assume that $a \to 0$ and try to obtain an asymptotic solution of the integral equation (D.2). First, we have to expand the right and left-hand sides with respect to $a \to 0$. Expand the first term on the left-hand side of equation (D.2). It is obvious that the small parameter a is not always small compared to $(x-\xi)$. In order to single out the region $(x-\xi) = O(a)$ it is convenient to subdivide the interval of integration

$$T = \int_{-\ell}^{\ell} \gamma K d\xi = \int_{-\ell}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\ell} = T_1 + T_2 + T_3 \quad .$$

It can be shown (Rozhdestvensky (1977)) that the integrals T_1 and T_3 are of the order of $O(a^3)$ when $a \to 0$, $\delta \to 0$, $\delta/a \to \infty$, and the main contribution of the order $O(a)$ is provided by

$$T_2 = \int_{x-\delta}^{x+\delta} \gamma K d\xi \quad .$$

Considering T_2 with the new variable of integration,

$$\bar{\xi} = (\xi - x)/\alpha \quad , \quad \alpha = \pi a/w \quad ,$$

we obtain for $a \to 0$ ($\alpha \to 0$)

$$T_2 \simeq -\int_{-\delta/\alpha}^{\delta/\alpha} \gamma(x + \alpha\bar{\xi})\ \frac{\alpha d\bar{\xi}}{(k_0^2\bar{\xi}^2+1)k_0\alpha\bar{\xi}} \quad . \tag{D.3}$$

Outside of the vicinities of the points $x = 0$, $x = 1$ the function $\gamma(x+\alpha\bar{\xi})$ can be expanded in Taylor series,

$$\gamma(x+\alpha\bar{\xi}) = \gamma(x) + \alpha\bar{\xi}\gamma'(x) + O(\alpha^2) \quad . \tag{D.4}$$

Substituting (D.4) into (D.3), we obtain after integration, for $\alpha \to 0$, $\delta \to 0$, $\delta/\alpha \to \infty$,

$$T_2 = \frac{-2\alpha}{k_0^2}\gamma'(x) \arctan\frac{\delta}{\alpha} = \frac{-\alpha\pi}{k_0^2}\gamma'(x) \quad .$$

Performing similar operations with the right-hand side, we finally obtain the limiting form for the equation (D.2) as $a \to 0$:

$$-\frac{\alpha\pi}{k_0^2}\gamma'(x) + \frac{2w}{C(x)}\int_{-\ell}^{x}\gamma(\xi)d\xi = -\frac{\pi\sigma(x)}{k_0} \quad . \tag{D.5}$$

Recalling Beck's notations and introducing the circulation of the velocity as

$$\Gamma(x) = \int_{-\ell}^{x}\gamma(\xi)d\xi \quad ,$$

we arrive at the following equation with respect to Γ :

$$a(\Gamma'' - \bar{\kappa}(x)\Gamma) = \frac{1}{2}\sigma(x) \quad , \tag{D.6}$$

where

$$\bar{\kappa}(x) = 1/2aC(x) \quad ,$$

which, as seen from Chapter II, could have been obtained by the method of matched expansions.

Thus, we have proved that (i) Beck's integral equation is valid in the limiting case as $a \to 0$ and in this case it degenerates to take a much simpler form, (ii) if $w = O(1)$ and $a \to 0$, the influence of another wall vanishes in the limit (note that equation (D.6) does not contain the canal width w).