Teaching Mathematical Knowledge for Teaching: Curriculum and Challenges

by

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ABSTRACT

Over the last decade, researchers have investigated the nature and scope of mathematical knowledge for teaching (MKT) and have provided evidence that teachers' MKT plays a critical role in the effectiveness of teaching and learning. This dissertation focuses on the problem of helping teachers develop MKT. Two goals shape the research: (1) to specify the tasks of and challenges involved in teaching MKT and (2) to build knowledge about how to organize a curriculum for teaching MKT. The data sources are curriculum materials designed to focus on MKT and video recordings collected in classrooms where teacher educators and teachers were working with the curriculum materials. The analysis focuses on not only the tasks for teacher educators, especially in managing instruction where teachers as learners worked on developing MKT, but also on MKT as content in teacher education. This study then combines findings from the data with the analytic literature review. Five central tasks of teacher educators are illustrated to identify what might make the teaching of MKT difficult and what might be helpful to teacher educators. Moreover, mathematical work of teaching and knowledge about mathematics are specified as the focus in teaching MKT by developing a framework for a curriculum of MKT as well as identifying what components are involved in teaching MKT in teacher education and how they fit within a larger terrain of MKT. Results contribute to the pedagogical considerations that underlie the teaching of MKT and to the design and implementation of a curriculum to teach MKT. Moreover, this research expands into teacher education as another place to study MKT and provides the groundwork for a shared curriculum in mathematics teacher education.

CHAPTER 1
THE RESEARCH PROBLEM

1.1 Introduction

In the last decade, progress has been made in specifying the mathematical knowledge needed for teaching elementary school mathematics. It is increasingly clear that the work of teaching mathematics demands a kind of depth and detail that goes well beyond what most adults need on a regular basis (e.g., Ball, Hill, & Bass, 2005). As Ball, Thames, and Phelps (2008) claim, although simply knowing mathematics for oneself means being able to calculate 307 – 168, for example, teaching a simple subtraction computation involves not only carrying out this computation but also analyzing students' responses, both correct and incorrect, interpreting students' imprecise mathematical language, and bringing errors to the surface for specific pedagogical purposes. They define mathematical knowledge for teaching (MKT) to be the mathematical knowledge, skill, and habits of mind entailed by the work of teaching. Teachers' MKT is crucial to the improvement of teaching and learning (National Mathematics Advisory Panel, 2008) and has been shown to be associated with achievement gains (Hill, Rowan, & Ball, 2005; Rockoff, Jacob, Kane, & Staiger, 2011).

Teaching MKT to teachers, however, is not straightforward. Several challenges exist. MKT may be relatively unfamiliar to those responsible for the mathematical education of teachers. In addition, the teaching of MKT may require different pedagogical approaches from those currently used in most mathematics courses because of the different goals of courses and the complexity of the task in teaching MKT. Teaching MKT requires sensibility about both the knowledge demands of teaching and the actual activities involved in the work (Ball & Forzani, 2009).

Thus, there is important progress yet to be made in studying MKT in the context of teacher education — that is, with respect to teachers' development of MKT. Questions that still need to be explored include: What is the potential content of MKT for a curriculum in teacher education? What should be emphasized in teaching MKT? How does one manage activities in teaching MKT? In the face of such challenges, teacher educators might neither plan nor teach MKT despite being aware of the need. Moreover, even if they were to use appropriate activities to teach MKT, few theoretical and practical foundations are in place to help teacher educators in terms of designing and implementing a curriculum for teaching MKT or in managing to teach MKT effectively. Teacher educators might think that MKT is still too difficult to try to teach in teacher education. They might fail to maintain a firm and consistent sense of the instructional purpose for an activity or discussion for MKT because the focus can slip easily. When discussing MKT, the focus might slide into thinking and talking about mathematics in ways that are remote from teaching or emphasize pedagogical issues, such as how children think about the topic or how to teach it. In brief, research on MKT with implications for mathematics teacher education still needs to be developed and specified. This research explores these issues by investigating MKT as it arises in the instruction of mathematics teacher education. There are two reasons why studying these issues is important.

First, investigating MKT as it plays out in the context of mathematics teacher education illustrates the kinds of attention needed for teaching MKT in mathematics teacher education in terms of both teaching practice and mathematics (Ball, Sleep, Boerst, & Bass, 2009). Although teacher educators aim to help teachers understand the kinds of skills, dispositions, and knowledge that enable them to engage in effective instructional practice, their work toward this goal currently depends on each individual teacher educator's effort because there is little established pedagogy for their teaching (Ball et al., 2009). Teacher educators undoubtedly have significant roles in mathematics teacher education. Nevertheless, support of teacher educators' work has received little emphasis even though every such teacher educator is responsible for designing and developing teachers' learning experiences (Zaslavsky & Leikin, 2004). Studying MKT as it is actually manifested in the instruction of mathematics teacher education can reveal teacher educators' challenges in the teaching of MKT and identify the kinds of attention needed for teaching MKT in order to support the work of teacher educators.

Second, exploring MKT as it arises in the instruction of mathematics teacher education provides a basis for building a curriculum in terms of MKT for mathematics teacher education. Teachers use MKT in teaching mathematics, and teacher educators use MKT in teaching MKT. MKT can be used for different purposes. Previous research on MKT has illustrated what mathematical knowledge teachers use to teach mathematics and has suggested what teachers ultimately need to learn and be able to do. However, these suggestions are different from what MKT is enacted in the context of mathematics teacher education. The pedagogical purpose in teacher education requires a theoretical support or a framework for MKT that shows its boundary and territory for teacher education. In other words, when teacher educators attempt to teach MKT, they should recognize what the focus is, and how it fits within a larger terrain of MKT. An investigation of MKT worked on in teacher education classrooms would elaborate an MKT that is feasible for the context of mathematics teacher education. Findings from the proposed study seek to offer a foundation to build a curriculum for the mathematical preparation of teachers with the specialized knowledge and skills needed for teaching mathematics. This work also provides a way to talk about MKT in the instruction of mathematics teacher education. It strives to identify ways to support to teacher educators and to contribute, in the long run, a shared curriculum in mathematics teacher education. A shared curriculum provides the common experiences designed to support teachers in having the specialized knowledge and skills needed for practice (Ball et al., 2009) and can help to develop the practice of teacher educators.

1.1.1 Purpose of the study

To further develop support for teacher educators, this study investigates how teacher educators approach helping teachers develop MKT. Two goals shape the research: (1) to specify the tasks of and challenges involved in teaching MKT; and (2) to build knowledge about how to organize the curriculum for teaching MKT. This is a study of the MKT viewed as the content formulated and emphasized, as shown in *Figure 1.1*, which illustrates the situation in teacher education.

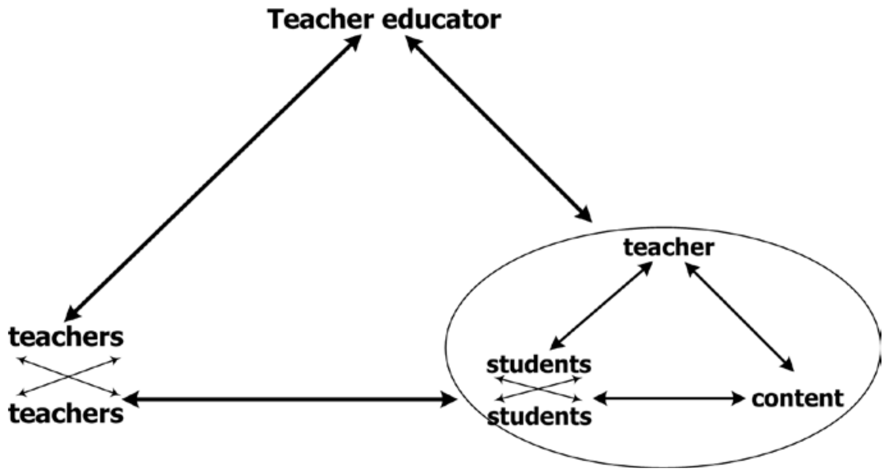

Figure 1.1 Instructional triangle in teacher education (Ball, 2012)

There are diverse expectations for mathematics teachers to meet: they need to know mathematics, lead discussions, use technologies, know the overall structure of the mathematics curriculum, and monitor and manage all students. These are just a few examples when considering the numerous kinds of the work in teaching. All of them are related to MKT. Some researchers argue specifically that, for the work of teaching, mathematics teacher educators should teach a sound basis of subject matter knowledge and foster teachers' practice in the ways of thinking and judging (Bransford, Darling-Hammond, & LePage, 2005; Grossman, 1990; Neubrand et al., 2009). Given the high expectations for the work of skillful teaching, teacher educators might feel a strong sense of responsibility towards teachers' preparation for teaching. The current research concentrates on MKT in the context of teacher education. MKT combines disciplinary attention with the work of teaching in mathematics teacher education (Ball et al., 2009). It still remains a major concern in mathematics teacher education to engage teachers in situations that require them to develop mathematical awareness and skill for the work of teaching (Ball, 2000; Wilcox, Lanier, Schram, & Lappan, 1992).

To address this concern, this study examines MKT as it plays out in mathematics teacher education. The research focuses on gaining a better understanding of the dynamics of teacher educators' instruction and the challenges that arise in so as to clarify what teacher educators need to attend to when teaching MKT. It aims to offer teacher educators guidance for the instruction of MKT. The current research also explores MKT worked on in teacher education classrooms, and specifies how teacher educators approach such goals in their teaching. The aim of this study is to develop a conceptual structure that can inform a curriculum for teaching MKT in mathematics teacher education. These two goals seek to contribute the expansion of the study of MKT to teacher education and the pedagogical support to teacher educators who attempt to teach MKT.

1.2 Overview of Study

This study investigates the teaching MKT in mathematics teacher education, and develops an analytic framework for conceiving of the work of teaching MKT to teachers. To do this, this research probes the tasks of teaching MKT in the case of a particular mathematical focus: when teacher educators teach teachers to reason about mathematical definitions in and for teaching. This study also explores MKT for mathematics teacher education. This section provides an overview of the study. First, the premises underlying this study are clarified. Then, the research questions that frame the investigation are presented. This section closes the contribution of this study.

1.2.1 Premise of the study

Teaching mathematics can be taught: The work of teaching includes broad cultural competence and relational sensitivity, communication skills, and the combination of rigor and imagination fundamental to effective practice (Ball & Forzani, 2009, p. 497). For example, the work of teaching includes learning about what individual students know, care about and can do, establishing the environment to manage behavior, teaching intellectual habit, and choosing the specific problem. Moreover, the work of teaching involves using diverse resources, managing and using environment, and coordinating groups of students to accomplish specific goals (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001; Lee, 2007), as shown in *Figure 1.2*.

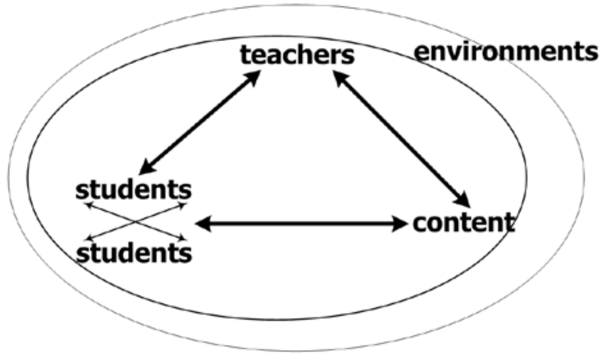

Figure 1.2 Instructional triangle

Researchers have identified forms of mathematical problem solving and ways of understanding mathematics that are special to the work of teaching and not involved in other forms of mathematical work. Therefore, teaching mathematics is intricate work. Since teaching mathematics is intricate and unnatural, Ball and Forzani (2009) assert that teaching needs to be learned and taught. Teaching mathematics requires to know and use mathematics in ways that are distinct from simply doing mathematics oneself, such as unpacking mathematical ideas and scaffolding them for students' learning, and figuring out what students are doing mathematically and how it makes sense to them (Ball et al., 2008). The knowledge and skills that teachers need for teaching can be taught by giving them opportunities to develop a flexible understanding of mathematical ideas central to the school curriculum and opportunities to engage in mathematical practices central to teaching (Suzuka et al., 2009) and by having them rehearse and develop discrete components of a complex practice in settings of reduced complexity (Grossman et al., 2005, April).

Mathematical preparation and skilled teaching are the goals of teachers' learning: The ultimate purpose of mathematics education is that all students learn mathematics and develop mathematical proficiency (National Research Council, 2001). Students' mathematics achievement cannot significantly improve without qualified teachers, and teachers' abilities cannot develop considerably if teacher educators do not provide opportunities for teachers to appreciate and practice knowledge and skills entailed in teaching mathematics in order to be mathematically ready to teach. The proposed study assumes that the purpose of mathematics teacher education is both the mathematical preparation (McDiarmid & Ball, 1989; S. M. Wilson, Floden, & Ferrini-

Mundy, 2002) and skilled teaching (Ball et al., 2009; Diezmann, English, & Watters, 2002; Hiebert & Stigler, 2000; Kilpatrick, 2006) that ensures students' mathematical proficiency. These entail the knowledge of and ability for core concepts and skills, culturally and linguistically sensitive interactions, active and equitable engagement, attention to mathematical language and reasoning, and careful diagnosis and response to students' difficulties. This dissertation assumes that the purpose of teachers' learning in mathematics teacher education is to become mathematically prepared for teaching and to develop skilled teaching practice.

A broad multidimensional approach to MKT is appropriate given current ideas: To be mathematically well-prepared, teachers need to know mathematics for teaching that both originates deep in disciplinary ideas and is flexible enough to associate with students' thinking, be able to hear and see mathematics from students' perspectives and to make instructional judgments, and have relational skills for all tasks (Suzuka et al., 2009). This dissertation is based on the hypothesis that, in the context of mathematics teacher education, teachers can develop specialized knowledge to be used in teaching (Ball et al., 2008). To do this, teachers must have opportunities to examine and unpack mathematics in order to articulate the task of instruction, understand students' ideas, and steer the instruction (Ball et al., 2009; Suzuka et al., 2009). They must also have practice to develop mathematical sense and reasoning for wise in-the-moment decisions in the practice of teaching mathematics (Ball, 1993; Lampert, 1990).

Mathematical definitions create activities in teacher education that include both main features of mathematics and critical tasks of teaching: Mathematical definitions provide efficient and valuable ways for concentrating on both subject matter knowledge and pedagogy in the context of teachers' work (Ball, 2000). In terms of subject matter knowledge, although the scope of mathematics is extensive and teacher educators can consider and use all of its many parts, mathematical definitions as topics have the advantage of being able to penetrate all fields of mathematics (Tappenden, 2008). Borel (2004) defines mathematics as the science that studies the relations between certain abstract objects defined in an arbitrary manner. Although a mathematical definition consists of a handful of terms, it serves to indicate the purported status and function of various elements of written mathematics (Pimm, 1993, p. 262). In terms of pedagogy in the context of teachers' work, Zaslavsky and Shir (2005) also demonstrate that mathematical definitions (1) introduce the objects of a theory and the essence of concepts, (2) constitute fundamental components for concept formation, (3) establish the foundation for proofs and problem solving, and (4) create uniformity in the meanings of concepts.

Mathematical definitions have particular features and significant roles in disciplinary mathematics and are compelling topics for encouraging teachers to recognize and practice the knowledge, skills and habits of mind entailed in the work of teaching. Mathematical definitions as topics for teacher education initiate discussions about concentrating on mathematical precision and related pedagogical concerns (Lampert, 1990). Moreover, a sense for mathematical definitions can lead to recognizing language issues in the practice of teaching and improving skills for appropriate language use (Ball & Bass, 2000b). Mathematical definitions also provide content in teacher education for learning the main practices of mathematicians through refuting and justifying proofs and constructing mathematically general explanations with uniformity in meaning (Lakatos, 1976a; Vinner, 1991). Mathematical definitions, as topics in mathematics teacher education, create opportunities by which teacher educators can help teachers understand and appreciate knowledge in and for teaching. Even though mathematical definitions do not contain everything in mathematics, it is clear that mathematical definitions are considered fundamental in disciplinary mathematics and mathematics education.

1.2.2 Research questions

Although this research is grounded in records of practice, it is primarily conceptual; its aim is to develop a conceptual framework for teaching MKT. This study use the cases from the tasks and activities related to the role and use of mathematical definitions in and for teaching. The overall question of this research is:

> *What MKT might be worked on, and in what ways, in the instruction of mathematics teacher education when teacher educators aim to help teachers develop MKT?*

This research focuses on MKT as content approached in the context of mathematics teacher education. This study investigates the dynamics and challenges in teaching MKT in order to clarify the kinds of attention that may support teacher educators in their classes. This study also probes MKT that emerges as the content in teacher education classrooms and clarifies how teacher educators might address such knowledge. In other words, this research aims at developing a conceptual framework that can inform a curriculum for teaching MKT in teacher education. This conceptualization includes what would be hard about teaching MKT and what would help teacher educators avoid and manage their challenges as well as what might make up the content and their structure to teach MKT in teacher education. It may be helpful to note that this study is not an examination of the MKT teachers learn and understand, or how teachers work together in this process of learning. This research also does not aim to explore individual teacher educators' understanding about and of MKT, or what teacher educators want to teach, or attempt to teach, regarding MKT in their classes as the intended curriculum. Instead, this study aims at both investigating the dynamics involved in teaching MKT and researching MKT arisen in teacher education classrooms.

To facilitate the investigation, the current research focuses on three supporting questions to explore the data. Although this research is conceptual, the investigation of the data helps develop a reasonable conceptualization. The first question regards the tasks of and challenges involved in teaching MKT. The other two questions are related to MKT as it is shaped by teacher educators and teachers in the instruction of mathematics teacher education. Following the recommendation of Ball (2000), this research looks at MKT as content in mathematics teacher education through both a disciplinary and pedagogical lens. The second supporting question regards the mathematical work of teaching, and the third concerns the disciplinary approach of mathematics. These three supporting questions are each followed by a brief description of the approach taken.

As teacher educators and teachers work on MKT in teacher education classrooms, using curriculum materials that address mathematical definitions in teaching:

1. *What are some of the challenges that arise?*

This research investigates what components teacher educators consider in order to manage the tasks of the teaching of MKT. The combination of roles of teachers as learners, teacher educators as instructors, and the use and roles of mathematical definitions as tasks and activities in order to teach MKT creates the distinctive dynamics of instruction of which teacher educators need to be aware. This study also probes the challenges teacher educators face because of the fact that these dynamics demand the attention of teacher educators. However, this study neither explores the specific challenges teacher educators recognize in their classes nor examines whether teacher educators acknowledge their challenges. Rather, this research will illustrate the challenges that teacher educators confront in managing tasks and activities to teach teachers MKT with curriculum materials to reason about mathematical definitions in and for teaching. This question, thus, contributes to clarify the kinds of attention needed for teaching MKT in a curriculum of mathematics teacher education.

2. *What mathematical work of teaching is prominent?*

   This research investigates the content that teacher educators use in their instruction in terms of the mathematical work of teaching. In other words, the purpose of this question is to explore MKT that teacher educators emphasize in order to provide support to teachers to acknowledge and practice the mathematical work of teaching when teacher educators teach teachers MKT with curriculum materials to reason about mathematical definitions in and for teaching. This question aims to clarify mathematical reasoning, skill, habits of mind and insight that are crucial in and for teaching (Ball et al., 2008) as content in mathematics teacher education. The findings here contribute to identify the mathematical work of teaching to teach MKT in mathematics teacher education.

3. *What kinds of mathematical issues are revealed?*

This research explores how teacher educators mathematically approach MKT in their instruction. Specifically, this question aims at examining what disciplinary objects of mathematics teacher educators unearth and make salient in teaching MKT. This question seeks to investigate a sense of the mathematical environment, major disciplinary ideas and structures, and core mathematical values and aesthetics of mathematics in and for teaching (Ball, 1993) involved in teaching MKT. This investigation contributes to clarify the mathematical issues in MKT that can serve a curriculum in mathematics teacher education.

Findings from the three sub-questions support the overall question empirically and theoretically. In other words, findings derived from the three supporting questions are conceptualized with the literature and through the process of structuring so as to respond to the overall question of this research. The first supporting question establishes pedagogical concern and attention involved in teaching MKT. The second and third questions also elucidate MKT in the instruction of mathematics teacher education, with regard to mathematical work of teaching and knowledge about mathematics.

1.2.3 MKT in teacher education

This research approaches MKT from two different perspectives: examining tasks of teacher educators and elaborating MKT for teacher education. The first perspective, examining tasks of teacher educators, concentrates on clarifying what would be hard in teaching MKT, what challenges that teacher educators might face, what strategy teacher educators could use in those situations and moments, and what teacher educators would need to pay attention to in teaching MKT. This research particularly concentrates on the tasks of teacher educators during the implemented phase of MKT instruction.

The second perspective focuses on creating a framework for teaching MKT that can work in teacher education and on identifying each component of the framework. In particular, the framework comprises two interrelated types of objects: mathematical work of teaching and knowledge about mathematics. The mathematical work of teaching entails the tasks of teaching that teachers perform with mathematics in and for teaching mathematics. Moreover, knowledge about mathematics is about the nature of knowledge in the discipline, such as where it comes from, how it changes, and how truth is established (Ball, 1990). This research holds that both the mathematical work of teaching and knowledge about mathematics are indispensable components of planning and implementing MKT in mathematics teacher education. The mathematical work of teaching considers specific tasks of teaching that teachers act out in teaching practice. However, knowledge about mathematics includes facts that are used in disciplinary mathematics and awareness and value that mathematicians generally consider in their research. Both the mathematical work of teaching and knowledge about mathematics function as two parts of a framework for a curriculum of MKT in teacher education. The result of the framework for curriculum in which to teach MKT is an articulation of the work of mathematics teaching and an understanding of mathematical issues of MKT as shown in the context of teacher education. The components of MKT that can work in teacher education and their relationship are depicted in *Figure 1.3*.

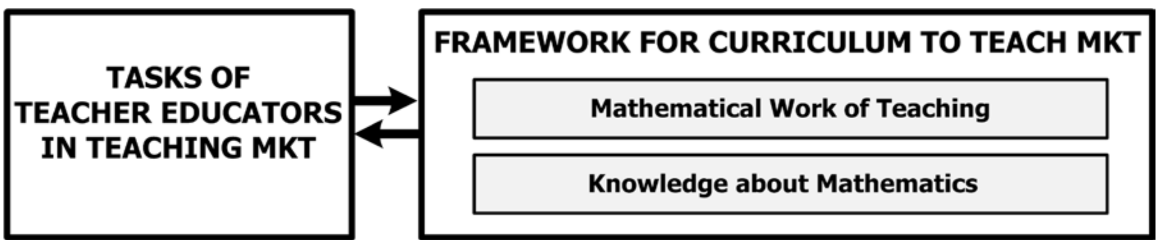

Figure 1.3 Conceptualization to teach MKT in mathematics teacher education

1.2.4 Contribution to the field

Mathematics teacher education must prepare teachers with the knowledge, skills and habits of mind necessary to do skilled teaching and to succeed at supporting students to master challenging mathematics. For this purpose, teacher educators need to develop their understanding of content for teachers' mathematical preparation and their practice for teaching such knowledge. This research will contribute to the pedagogical considerations that underlie the teaching of MKT and will help teacher educators plan and enact their curriculum to teach MKT. The major contribution of this research is building the groundwork for a shared curriculum in mathematics teacher education.

1.3 Organization of the Dissertation

The dissertation consists of seven chapters. The current chapter specifies the purposes of the study and provides an overview. Chapter 2 reviews the literature for dynamics in teaching, MKT, mathematical definitions, and mathematical work of teaching. Chapter 3 describes the data and methods of analysis used in this research. Chapter 4 provides three detailed examples of the curriculum materials as well as excerpts from transcriptions of instruction by two teacher educators. These are used to investigate the complexity of teaching MKT and to begin developing a framework for a curriculum to teach MKT in mathematics teacher education. Chapter 5 investigates what would be hard in teaching MKT and specifies what teacher educators need to attend to in teaching MKT. Chapter 6 presents the conceptual framework that can be used to teach MKT in mathematics teacher education. Chapter 7 considers this study’s potential contributions to mathematics education and teacher education. It lays out the next steps arising from the current study.

CHAPTER 2
FOUNDATIONS IN LITERATURE

2.1 Introduction

This study investigates how teacher educators approach supporting teachers to acquire MKT. Two goals guide this research study: enumerating the tasks of and challenges involved in teaching; and building knowledge about how to design a curriculum for teaching MKT. The aim of the literature review is to build a theoretical foundation for the present research study as well as to have an initial coding scheme for analyzing the empirical data. This chapter is organized into four main sections. First, research on attention in educational studies is reviewed. I then review studies on the dynamics of teaching, teacher educators as instructors, and teachers as learners in order to identify the factors of the dynamics of teaching MKT from the perspective of teacher educators in mathematics teacher education. This section is to conceptualize the ways MKT is worked on in the instruction of mathematics teacher education through investigating the dynamics and challenges in teaching MKT. Second, what is taught in mathematics teacher education is reviewed, and MKT, which is mainly investigated in this study, is explained to analyze the instruction of MKT. Third, attributes of mathematical definitions are reviewed both from educational studies and from disciplinary mathematics. Finally, mathematical work in the practice of teaching is collected from the literature and classified according to their similar features. The second through fourth sections are for the conceptualization of what MKT worked on in teacher education by investigating knowledge about mathematics and the mathematical work of teaching in teaching MKT. In other words, these three sections aim to offer a scholarly basis for investigating what is taught in teacher education in terms of MKT as well as for characterizing and unpacking MKT as content in teacher education regarding the nature of knowledge in the disciplinary mathematics and the tasks of teaching that teachers perform with mathematics.

2.2 Dynamics in Teaching Phenomena and Attention

Any kind of teaching phenomena is complex and messy (Freeman, 1996). Therefore, it is difficult to explain it in an elegant and systematic way (Doyle, 1986). This statement about teaching implies that an honest understanding of teaching needs to take account for what instructors and learners do in classrooms. The classroom is a place where various participants meet and interact, which gives rise to a dynamic tension between their differing actions of the nature and goals of mathematics teaching (Tudor, 2001). In the dynamic situation of teaching, instructors are most immediately confronted with diversity among their students, such as prior knowledge, cultural background, and so on. Given that there is scant research on the dynamics involved in the teacher education classroom, I review the broader body of literature that investigates the dynamics in teaching, which literature can provide insight to identify elements to which teacher educators might need to pay attention when they teach MKT. I first define *attention* in this study, and then elaborate elements of the dynamics in teaching phenomena from the diverse kinds of teaching phenomena.

2.2.1 What instructors attend to in the classroom

William James (1890), who was a pioneering psychologist, defined attention as follows:

> Every one knows what attention is. It is taking possession by the mind, in clear and vivid form, of one out of what seems several simultaneously possible objects or trains of thought. Focalization, concentration, of consciousness are of its essence. It implies withdrawal from some things in order to deal better with others (pp. 403-404).

His definition has been generally interpreted to mean that attention can be spontaneously reactive or intentionally responsive (Mason, 2011). Whether teacher's attention is spontaneous or intentional, many educational researchers have been interested in the influence of teacher attention to students. Teacher attention that entails positive praise or negative criticism of appropriate or inappropriate behavior is a powerful reinforcement of students' behavior (e.g., Broden, Bruce, Mitchel, Carter, & Hall, 1970; Kazdin & Klock, 1973; Thomas, Becker, & Armstrong, 1968). Researchers also have found positive relations between teacher attention and student on-task behavior (e.g., Martens, 1990), students' appropriate behavior (e.g., Cooper et al., 1992), and students' higher engagement time (e.g., Zanolli, Daggett, & Pestine, 1995). Some researchers have further suggested ways of getting better student attention or maintaining student attention in classroom (e.g., Corno, 1981; Payne & Hustler, 1980).

While most of the research has assumed instructor's attention as an individual skill, the present research goes beyond this approach. I view *what instructors attend to* as a social property of teaching practice in order to add a focus on shared practices that make sense to instructors as they are situated in institutional and social systems (Levin, 2008). Because I assume teacher educators to be reflective practitioners (Ball & Cohen, 1999; Schön, 1983; Tzur, 2001), practices to teach MKT include more than the application of specialized knowledge to well defined tasks, and there are forms of practical rationality. According to Herbst and Chazan (2003), this rationality often is at work where people perform the same job, and thus the rationality is not reduced to individual wisdom, talents, sensibilities or skills. The present research does not investigate teacher educators' attention in the classroom in terms of what they focus on while they teach MKT, but aims instead at developing conceptualized factors that might be attended to by teacher educators in and for teaching MKT. This approach is consistent with Lederman and Gess-Newsome's (1992) notion of identifying "the multitude of factors which compete for the classroom teacher's attention (p.18)," such as individual differences among learners, curricula, instructional preferences, classroom management, availability of time, and availability of materials (e.g., Borko & Livingston, 1989; Hollingsworth, 1989; Lederman & Gess-Newsome, 1991). In order to carry out a fundamental exploration of this subject, the teaching of MKT needs to be addressed, in its totality.

I now turn to the dynamics of instruction in order to elaborate elements that instructors might need to attend to in instruction.

2.2.2 Instruction and its components

Cohen et al. (2003) explained instruction and resources with *Figure 1.2*:

> Instruction consists of interactions among teachers and students around contents in environment. … "Interaction" refers to no particular form of discourse but to teachers' and students' connected work, extending through, days, weeks, and months. Instruction evolves as tasks develop and lead to others, as students' engagement and understanding waxes and wanes, and organization changes (Lampert, 2001). Instruction is a stream, not an event, and it flows in and draws on environments – including other teachers and students, school leaders, parents, professions, local districts, state agencies, and test and text publishers (p.122).

Instruction depends on interactions and evolves as the content progresses with the growth of the learners' understanding. Ball (2012) extended this idea to teacher education in order to emphasize teaching practice as content of teacher education, as shown in *Figure 1.1*. Although three agents – teacher educators as instructors, teachers as learners, and teaching practice as content – are not always distinct in practice, I treated them separately to analyze the dynamics of teaching MKT in this study.

Next, I briefly identify the components of the dynamics of teaching based on the studies that investigate them in various teaching situations. This literature has a broader focus than that of the dynamics of teacher education alone; nevertheless, this investigation offers a peek into the dynamics of teaching that teacher educators need to recognize. The research on dynamics of teaching in various situations informs factors with regard to the instructional triangle from the perspective of instructors, as shown in *Figure 2.1* below. The purpose of this summary from the literature is to provide a foundation about possible elements that function in any kind of instruction.

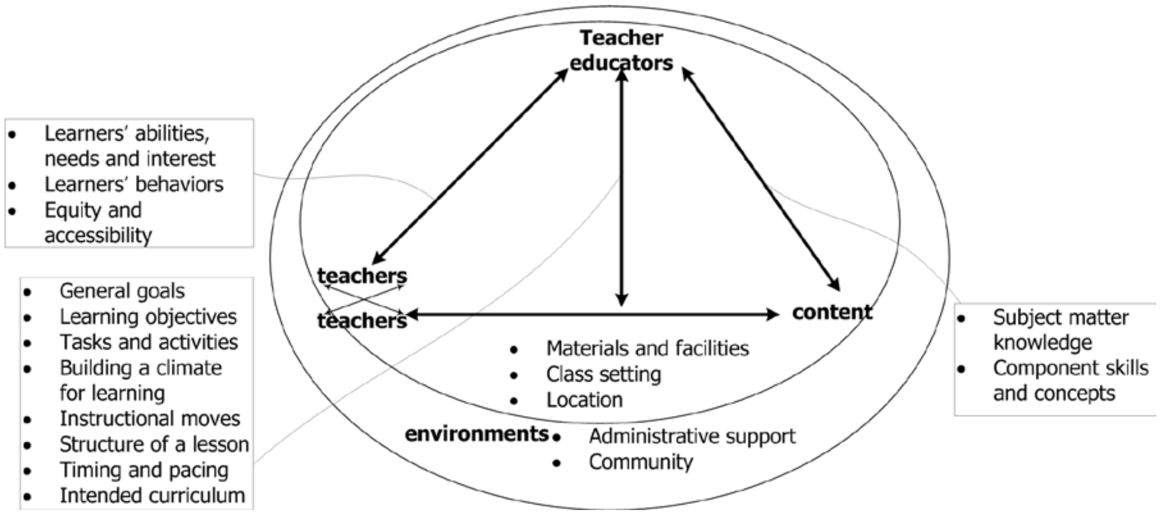

Figure 2.1 Dynamics of teaching and its factors from instructor's perspective

In terms of interaction with learners, instructors consider information about students (Cooney, Davis, & Henderson, 1975; Hillocks, McCabe, & McCampbell, 1971; Kapuscinski, 1982; Shavelson, 1983; Wilen, Hutchison, & Ishler, 2008). Specifically, Shavelson (1983) suggested what abilities, needs, and interests learners have. Instructors monitor learners' behaviors in a class and decide whether they are appropriate (Hillocks et al., 1971; Pizzini & Shepardson, 1992; Yinger, 1979), and oversee whether all learners approach the tasks in a class (Hiebert et al., 1997).

To help learners interact with content in a class, instructors focus on general goals and learning objectives of a class (Cooney et al., 1975; Gorham, 1999; Shavelson, 1983). Second, instructors consider tasks and activities that are used in instruction in terms of their nature (Hiebert et al., 1997) and their organization (Tudor, 2001). Third, instructors think about the academic climate in a class (Shavelson, 1983) and the standards and expectations for success in a class (Cooney et al., 1975; Wilen et al., 2008). Lampert (1990) particularly specified what is in conflict with the academic climate that Lakatos and Polya consider to be appropriate to doing mathematics, such as establishing the validity of results, keeping thinking implicit, exerting power over peers, and refusing to expose ideas. Fourth, instructors concentrate on instructional moves, such as giving instructions, demonstrating, questioning, presenting information, monitoring, reviewing, evaluating student performance, offering feedback, expressing values and opinions, and correcting (Beder & Medina, 2001; Hillocks et al., 1971; M. Stein, Kinder, Silbert, &

Carnine, 2006; Wilen et al., 2008; Yinger, 1979). Lampert (2001) focused more on the mathematics classroom through her teaching of a whole class over an entire academic year. She specified several instructional moves, such as observing and making sense of the range of student performance, using students' ideas to broach new topics, supporting students' work in individual and collaborative situations, giving chances for students to demonstrate acquired knowledge, and so forth. Fifth, instructors consider the structure of a lesson (Pizzini & Shepardson, 1992). Sixth, they think of timing and pacing (Shavelson, 1983). And, seventh, instructors consider the curriculum (Doyle, 1988; Kapuscinski, 1982; Wilen et al., 2008). In particular, instructors are concerned about what learners are expected to do and produce and how they are expected to accomplish it (Doyle, 1988).

Regarding the interaction with content, instructors attend to the subject matter knowledge that is involved in the tasks and activities (Hillocks et al., 1971; Kapuscinski, 1982; Shavelson, 1983), and skills and concepts that are demanded by the tasks and activities (M. Stein et al., 2006). Instructors also consider the materials and facilities that are used in a class (Kapuscinski, 1982; Shavelson, 1983; Yinger, 1979), instruction and individualized instruction as types of class setting (Beder & Medina, 2001; M. Stein et al., 2006), and the location of a class as basic environmental components (Yinger, 1979). As secondary components that influence the dynamics of teaching, instructors consider administrative support (Kapuscinski, 1982) and attributes of community (Kapuscinski, 1982).

The phenomena of teaching MKT consist of three agents: teachers as learners, teacher educators as instructors, and MKT as content. In the next section, found factors in this section, as shown in *Figure 2.1*, are revised for mathematics teacher education after the literature review in terms of teacher educators as instructors and teachers as learners for mathematics teacher education.

2.2.3 Dynamics of teaching in mathematics teacher education

I review studies on teacher educators as instructors and teachers as learners in mathematics teacher education in order to elaborate the dynamics of the instruction in mathematics teacher education as a steppingstone for the dynamics of teaching MKT. Because teacher educators and teachers are the main agents in teacher education as shown in *Figure 1.1*, adding features of teacher educators and teachers to the instructional triangle makes it function as a model for mathematics teacher education. Below is a brief list of what researchers found about teacher educators and teachers. At the end of this section, I synthesize the features of teacher educators as instructors and teachers as learners in mathematics teacher education, including the factors that are summarized in the previous section as shown in *Figure 2.1*.

***Teacher educators as instructors in mathematics teacher education*:** Despite the relatively small number of studies about teacher educators in mathematics teacher education, two types of studies emerge: empirical reports on what teacher educators did for their teaching and theoretical perspectives for mathematics teacher educators. In the first, teacher educators have reported their efforts to improve their teaching. Many of these studies are conducted by beginning teacher educators incorporating a self-study method of their teaching (e.g., Zeichner, 2005). Tzur (2001) identified teacher educators as reflective practitioners who can see their own ways of thinking and differentiate mathematics and mathematics for teachers because the way one thinks mathematically as a learner of mathematics is very different from the way one thinks mathematically as a teacher of mathematics, or as a teacher educator of teaching mathematics also has different ways of mathematical thinking.

Teacher educators, however, have three kinds of tensions for their work of teaching: selecting tasks in terms of mathematical and pedagogical aspects for both teachers as learners and their prospective students, helping teachers for expected purposes, and having better skills for teaching in mathematics teacher education (Nicol, 1997). They also have constraints on ways to give teachers initial routines and strategies to manage their students' behaviors and learning in the classroom (Geddis & Wood, 1997); and teachers' mathematical knowledge and beliefs about mathematics teaching and learning, curriculum and assessment requirements by external authorities, limited technology resources, reduction of hours for methods courses, limitation to find appropriate practicum placements, and prejudice that teacher education is low status work (Goos, 2008).

Teacher educators, moreover, need to emphasize the following in their instruction: debriefing and unpacking teaching; differentiating between the kinds of teaching decisions and the impact of thinking on subsequent actions; seeing in teaching practice what is intended to be taught, what is taught, and how it is taught; and cooperating and sharing ideas among teacher educators and preservice teachers (Loughran & Berry, 2005). For this work, teacher educators need to have subject matter knowledge, pedagogical content knowledge, curricular knowledge, knowledge of context, and research knowledge, all of which serve as organizing tools and provide a language to discuss what teacher educator-researchers in different arenas undertaking different roles might need (Chauvot, 2009).

A few studies within this first group of studies show the collective efforts of teacher educators as a group, specifically what they do and consider and how they work. Ball and her colleagues (2009) have collectively worked on teaching to teach mathematics at the University of Michigan. To decide content, they articulate the work of teaching mathematics, identify and choose high-leverage practices, and consider MKT. Moreover, they have developed curriculum materials that are elaborate and detailed, useful for practice in teaching teachers, accessible and useable by everyone who teaches the courses, and revisable. Similar to Ball's group work, Hiebert and Morris (2009) have also worked together with their teacher educators at the University of Delaware. They have shared goals across the mathematics courses that are part of the teacher preparation program, settled on learning goals in lessons, and developed daily and detailed lesson plans. Moreover, they identified four kinds of knowledge that they believe are embedded in lesson plans: knowledge of the lesson's purposes and why particular activities are included; knowledge of learners' thinking; knowledge of the curriculum as a connected set of ideas and materials; and knowledge of strategies and representations for teaching toward particular learning goals.

Zaslavsky and Leikin (2004) described one teacher educator's case in terms of his growth as a teacher educator in professional development for secondary mathematics teachers that was achieved by making plans individually and collaboratively with colleagues, consulting about his plan with the leader of teacher educators, implementing his plan in a class, listening to his observer's comments, discussing his teaching and reflecting on his thoughts and actions during a conversation with his colleagues after the class. Heaton and Lewis (2011) have worked together for ten years, across department lines to benefit both content course in Department of Mathematics and method course in the College of Education and Human Sciences. They shared and followed the recommendations for forming interdisciplinary partnerships given by the Conference Board of the Mathematical Sciences and the National Research Council. They have closely worked with teachers at a local public school to connect their courses to real classrooms. They asserted that their partnership has been the key to helping teachers acquire a deep understanding of school mathematics with the habits of mind of a mathematical thinker as well as to develop productive habits of pedagogy and to understand mathematics from the child's point of view.

In the second group of studies, some researchers probe certain theoretical perspectives for mathematics teacher educators, such as constructivism (Jaworski, 2001), realistic mathematics education (Dolk, den Hertog, & Gravemeijer, 2002), and a sociocultural framework (Goos, 2008). Some studies explore what teacher educators need to consider, such as subject-matter preparation of teachers (McDiarmid & Ball, 1989), contextual constraints and the limits of teachers' knowledge and skills (Ball & Feiman-Nemser, 1988), and sociomathematical norms and orchestrating discussions (Elliott et al., 2009). Other studies discuss the education for teacher educators (Cochran-Smith, 2003; Dolk et al., 2002; Even, 1999). In particular, Smith (2005) specified characteristics of good teacher educators through the results of questionnaires based on the literature.

Teachers as learners in mathematics teacher education**:** Many researchers have been interested in what teachers know about various concepts, such as number and operation (e.g., Adams, 1998; Ball, 1988; Crespo & Nicol, 2006; Graeber, Tirosh, & Glober, 1989; Lubinski, Fox, & Thomason, 1998; Ma, 1999; Newton, 2008; Rizvi & Lawson, 2007; Simon, 1993; Zazkis & Leikin, 2008), measurement (e.g., Baturo & Nason, 1996), statistics (e.g., Groth & Bergner, 2006), geometry (e.g., Tsamir, 2007), and function (e.g., Even & Markovits, 1993; M. R. Wilson, 1994). Many of these studies focus on prospective teachers' knowledge rather than that of inservice teachers.' As would be expected, most research on teachers' knowledge showed what they know and do not know, recognized teachers' lack of conceptual understanding for the work of teaching, and discussed what should be highlighted in mathematics teacher education.

These studies contribute to content in mathematics teacher education rather than what teachers learn, interact, and do in teacher education as learners in mathematics teacher education.

Implication for the dynamics of teaching MKT**:** Even though teacher educators are reflective practitioners with respect to teaching to teach mathematics in a class (Tzur, 2001), there is no study that examines what teacher educators actually *do* and *attend to* in teaching to teach mathematics in a classroom. However, the literature informs factors that can function to see the dynamics of teaching in mathematics teacher education. Based on the literature review about the dynamics of teaching (see *Figure 2.1*) with teacher educators as instructors and teachers as learners, *Figure 2.2* shows factors in the dynamics of teaching from the perspective of teacher educators in mathematics teacher education. Now, the purpose of this summary of the literature is to establish a foundation of the possible elements at work in the instruction of mathematics teacher education.

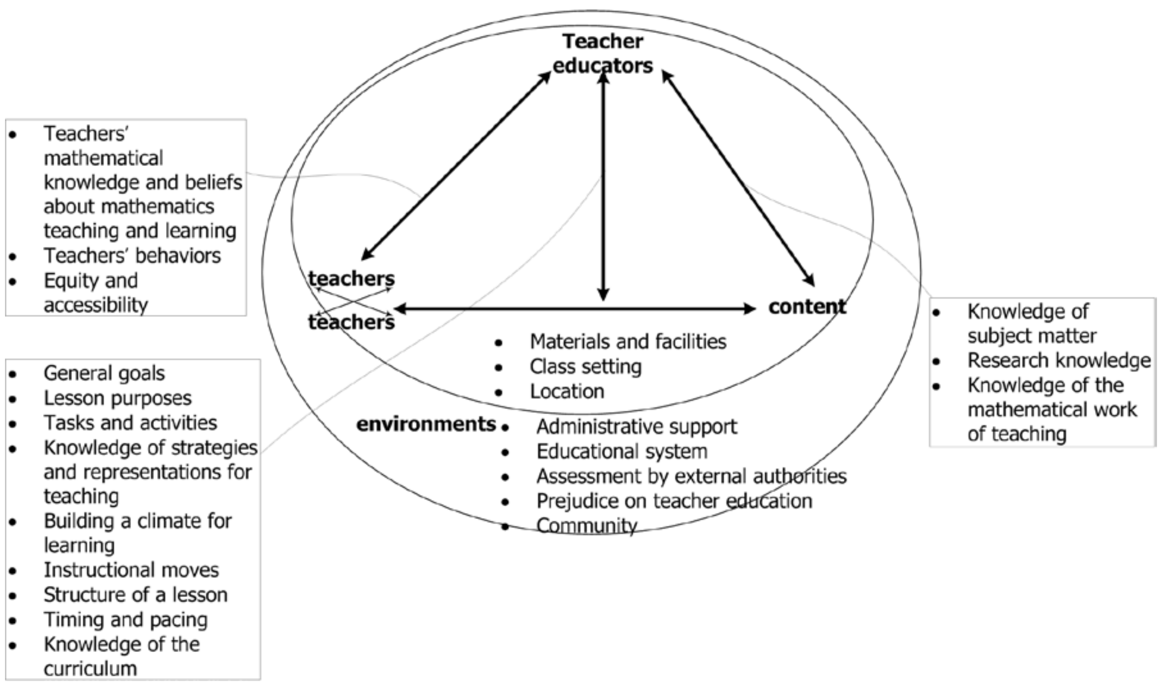

Figure 2.2 Dynamics of teaching in mathematics teacher education and its factors from the perspective of teacher educators

In interaction with teachers, teacher educators take into account teachers' characteristics that are their mathematical knowledge and their beliefs about mathematics teaching and learning (Ball & Feiman-Nemser, 1988; Goos, 2008; Hiebert & Morris, 2009), monitor teachers' behaviors in a class, deciding whether they are appropriate, and oversee whether all teachers work with the tasks in a class.

To help teachers interact with content, teacher educators pay attention to the lesson's purposes and why particular activities are included (Hiebert & Morris, 2009), tasks and activities that are related to the work of teaching and high-leverage practices (Ball et al., 2009; Geddis & Wood, 1997; Nicol, 1997), and strategies and representations for teaching in light of particular learning goals for a class (Hiebert & Morris, 2009). Moreover, teacher educators might consider how to create a climate for learning work of teaching and to lead a discussion (Elliott et al., 2009), what to do in the instruction (K. Smith, 2005), timing and pacing (Goos, 2008), and the knowledge of the curriculum as a connected set of ideas and materials and its implementation (Hiebert & Morris, 2009; Timmerman, 2003).

To interact with content, teacher educators might attend to the knowledge of subject matter (Chauvot, 2009; McDiarmid & Ball, 1989; K. Smith, 2005), research knowledge for organizing tools and a language of discussion (Chauvot, 2009; K. Smith, 2005), and knowledge of the mathematical work of teaching (Ball et al., 2009; Loughran & Berry, 2005; K. Smith, 2005). Teacher educators also might consider materials and facilities that are used in a class, instruction and individualized instruction as types of class setting, and location of a class as basic environmental components. As the secondary components that have influence on instruction of mathematics teacher education, teacher educators might consider administrative support, understand the educational system (K. Smith, 2005) and recognize assessment requirements by external authorities and prejudice that teacher education is low status work (Goos, 2008), and attributes of community.

The all factors are used to begin analyzing the data to elaborate the dynamics of teaching MKT and identify and elaborate factors that teacher educators might attend to during a lesson to teach MKT as shown in Chapter 5.

2.3 Curriculum in Mathematics Teacher Education

The key objective of this study is to lay a foundation for a curriculum for mathematics teacher education through an analysis of curriculum materials and video recordings that show enacted scenes of the curriculum materials. Therefore, I must define the term *curriculum* in this study. Before defining this term, I review the literature with regard to curriculum.

The term curriculum has multiple meanings. Johnson (1967) defined it as "a structured series of intended learning outcomes" (p.131). Egan (1978), using a more detailed specification, argued that a curriculum shows "what should children learn, in what sequence, and by what methods" (p.70). Kerr (1968) emphasized the role of the school for a notion of curriculum, defining it as "all the learning which is planned and guided by the school, whether it is carried on in groups or individually, inside or outside the school" (p.16). Unlike these three researchers, Kelly (2009) concentrated on learning results in the definition of curriculum, specifically, "the curriculum is the totality of the experiences the pupil has as a result of the provision made" (p.13). The present study follows Remillard's understanding of curriculum: "It (the term curriculum) is used to refer to overarching frameworks that specify what should be taught or to guides or other resources that teachers use when designing instruction and deciding what will be enacted in the classroom. For clarity, I use the term only to refer to the resources and guides used by teachers" (Remillard, 2005, p. 213).

Specifically, this research focuses on content for curriculum in teacher education. Content is significant for high quality and effective teacher education (Kennedy, 1998). Content includes not only that which is denoted or implied in the curriculum but also a large body of content selected by an instructor, which is not to be learned but exists to facilitate the desired learning (Johnson, 1967). In this section, I draw on research that aims at identifying what teachers need to know or learn for teaching mathematics, and a theory of MKT that is considered as content for mathematics teacher education in the present research.

2.3.1 Content for mathematics teacher education

There have been five approaches to the investigation of the content for mathematics teacher education. First, several researchers have made efforts to clarify a different version of mathematical knowledge needed for teaching (e.g., Ball et al., 2008; Baumert et al., 2010; B. Davis & Simmt, 2006; Lamon, 1999; Leinhardt & Smith, 1985; Ma, 1999; Rowland, Huckstep, & Thwaites, 2005; Shulman, 1986; Simon & Blume, 1994; Tatto et al., 2008; Thompson & Thompson, 1994). These researchers used different theoretical and practical approaches. Among them, the MKT that Ball and her colleagues have studied is reviewed in the next section.

Second, some researchers have emphasized the work of teaching, that is, the core tasks that teachers execute to help students learn (e.g., Ball et al., 2008; Ferrini-Mundy & Findell, 2010; Haertel, 1991; Reynolds, 1992; Rowland et al., 2005). These researchers highlighted teaching practice to identify the mathematical entailments of engaging in the work of teaching while they focused on different mathematical demands for teaching mathematics.

Third, some researchers examined the relationships between teachers' knowledge of mathematics and their instructional practice and asserted certain kinds of knowledge as content in mathematics teacher education, such as using inquiry tasks that emphasize cooperation (Chapman, 2007), connecting basic mathematical ideas and knowing main student misconceptions related to the topics (Charalambous, 2008), articulating ideas and features of various relationships in different representations (Lloyd & Wilson, 1998), and having key mathematical ideas and organizing them for easy accessibility (M. K. Stein, Baxter, & Leinhardt, 1990). This approach provides teacher educators with ideas about tasks and activities for their instruction.

Fourth, several studies explore the effect of certain activities in mathematics teacher education for teachers' knowledge. Geddis and Wood (1997) focused on lesson planning in order to have teachers transform their subject matter knowledge into knowledge that teachers use for their teaching, specifically, developing their ways to represent mathematical knowledge for teaching and evaluating various representations. For example, Ben-Chaim, Keret, and Ilany (2007) asserted that proportional reasoning authentic investigative tasks lead to a significant positive change in the preservice teachers' mathematical content and pedagogical knowledge.

Fifth, a few researchers in particular made comments on content in mathematics teacher education. Hill and Ball (2004) suggested exploring and linking alternative representations, providing and interpreting explanations, and delving into meanings and connections among ideas as content in mathematics teacher education in order to help teachers acquire more flexible and developed knowledge. Sowder, Phillip, Armstrong, and Schappelle (1998) emphasized subject matter preparation, specifically relational and deep investigation of rational numbers, such as parts of wholes, a measure, quotients, and an operator in order to have a meaningful impact on the development of teachers' mathematical knowledge for teaching.

***Implications for the curriculum of teaching MKT*:** While all of these studies on content appear to offer teacher educators references for their decisions about what they teach in their instruction, a more overall and general approach to capture content in mathematics teacher education is lacking.

Theoretical and practical progress about the mathematical knowledge entailed in teaching and the work of teaching have initiated various studies for more extensive teacher learning (Cohen & Barnes, 1993). Wilson, Floden and Ferrini-Mundy (2002) emphasized subject matter preparation and argued that teachers should have a deep conceptual understanding in order to respond to student questions and extend lessons beyond the basics. Darling-Hammond and her colleagues (2009) asserted that the concrete tasks of teaching, assessment, observation and reflection were content. The effects of the content that teacher educators use in mathematics teacher education are apparent in the teachers' knowledge for skilled teaching as I noted previously. This indicates the need to carefully identify what teacher educators need to teach as content in order to support the mathematical knowledge that is entailed in teaching. Drawing on Dewey (1904/1964), Ball (2000) argued for intertwining subject matter knowledge and pedagogy in the context of teachers' work for a curriculum of mathematics teacher education. Ten years later, she elaborated on how to integrate the two as the fundamental work that teacher educators use to build a curriculum: shifting from what teachers know and believe to what teachers do, breaking down practice into parts to make it visible and learnable, and creating settings for learning practice (Ball & Forzani, 2009). She also claimed the content that both aims at teaching practice and centers in mathematics for mathematics teacher education (Ball et al., 2009). Adler and Davis (2006) also investigated tasks of teaching mathematics involved in teaching at eleven professional developments in South Africa; they identified the features of tasks that were used. They also found that it rare and difficult to have a task that unpacked the knowledge about teaching. Through examining two different examples from secondary school classrooms in South Africa, (Adler, 2010) highlighted (1) designing, adapting or selecting tasks, and managing processes and objects and (2) valuing and evaluating diverse learner productions as tasks of teaching. However, the specification of content in mathematics teacher education is still insufficiently developed to advance the professionalism about teaching mathematics that people need to learn.

Next, I turn to the literature aimed at demonstrating the nature and scope of MKT that is the foundation to articulate the content of teaching practice using disciplinary mathematics for mathematics teacher education.

2.3.2 Mathematical knowledge for teaching (MKT)

Teaching mathematics requires mathematics for teachers' own purposes. To mathematicians, it is clear that 16÷8=2, and nothing further is needed regarding that calculation. However, the work of teaching this mathematics sentence asks for different kinds of knowledge. In particular, there are the grouping and sharing models in the division within whole numbers: if grouping by eight apples, two bags are necessary; and, if sharing sixteen apples with eight people, each person gets two apples. The grouping model shows 16-8-8=0, but the sharing model is not explained with this subtraction sentence. The grouping model is more accessible to early elementary students who are familiar with subtraction. However, this can create epistemological obstacles (Bachelard as cited in Sierpinska, 1994) for early elementary students, for example, a dividend should be larger than a divisor in division, and it is not appropriate to make the relationship between division and fraction clear. The sharing model is free from the problematic issues of the grouping model, but the sharing model does not show the continuity and density of fractions as rational numbers. Therefore, a number line can be the other model. To teach a problem of long division, such as an example in *Figure 2.3*, teachers might use mathematically unclear terms such as “draw a short segment and a-short-curved line to attach the left point of the segment…” In fact, since there are no public or agreed-upon terms to call and describe this line, teachers might use their own terms when teaching it. These are subject matter knowledge areas that teachers need to know for teaching division within whole numbers.

$$\begin{array}{r} 2 \\ 8\overline{)16} \\ \underline{16} \\ 0 \end{array}$$

Figure 2.3 A long division of 16÷8=2

Teachers do not just possess knowledge; they perform actions with it. In this case, teachers are expected to mathematically interpret students’ responses, recognize phenomena related to knowledge, such as fractions and multiplication of this mathematics sentence, consider diverse representations, create activities, decide efficient manipulatives, and investigate knowledge that students have related to this topic. All of these constitute the epistemology of practice that teachers do before, during and after mathematics classes. This epistemology is based on the possibility of a deeper level of explanation regarding the nature of all activities related to mathematics in the mathematics classrooms, as Rasche and Chia (2009) emphasize what people do in practice. We refer to this as knowledge embedded in the practice of teaching mathematics. Ball and Bass (2003b) have developed a practice-based and discipline-grounded approach to examine what mathematical knowledge is used in practice and how it is used in order to develop notions of MKT. MKT is defined as the mathematical knowledge, skills and, habits of mind needed to carry out the work of teaching mathematics (Ball et al., 2008). It should be noted that teaching MKT in mathematics teacher education means to teachers to possess knowledge and practice ways of knowing that Cook and Brown (1999) identified. This theory for mathematics teacher education addresses both possessing knowledge and practicing ways of thinking and judging that practitioners use in the work of teaching. The interplay of possessing knowledge and practicing ways of thinking and judging can support teachers to improve their practice.

The development of MKT builds on and refines Shulman's work. Shulman (1986) identified *pedagogical content knowledge* as a special domain of teacher's knowledge that intertwines aspects of teaching and learning with content from subject matter knowledge. Even though this identification provides the theoretical basis for why education is a special area of research, his idea has remained undeveloped in analytic clarification and empirical testing. The work of teaching mathematics entails MKT, as illustrated in *Figure 2.4* below.

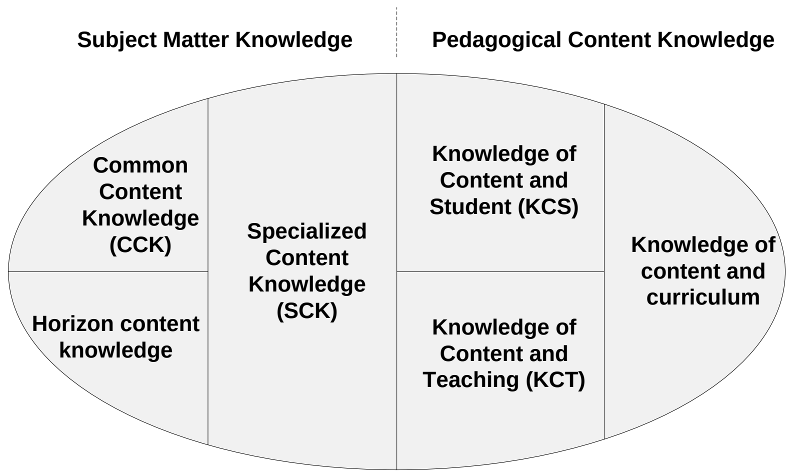

Figure 2.4 Domains of Mathematical Knowledge for Teaching (MKT)

Figure 2.4 shows the subdomains of MKT. The left three subdomains consist of *subject matter knowledge*, and the right three subdomains comprise pedagogical content knowledge, as introduced by Shulman. MKT expands the empirical foundation of Shulman's study.

In subject matter knowledge, *common content knowledge* refers to the mathematical knowledge and skill possessed by any well-educated adult, while *specialized content knowledge* is the mathematical knowledge and skill used by teachers in their work but not generally possessed by well-educated adults, such as how to accurately represent mathematical ideas, provide mathematical explanations for common rules and procedures, and examine and understand unusual solution methods to problems (Ball et al., 2005). In pedagogical content knowledge, *knowledge of content and students* contains knowing about both mathematics and students, that is, as content knowledge intertwined with knowledge of what students think about, and how they know or learn a particular content (Hill, Ball, & Schilling, 2008). *Knowledge of content and teaching* includes knowing about both mathematics and teaching (Delaney, Ball, Hill, Schilling, & Zopf, 2008), as content knowledge intertwined with knowledge of how best to build student mathematical thinking or how to remedy student errors. Although knowledge and reasoning in the two domains of common content knowledge and specialized content knowledge are used in teaching, knowledge of students or knowledge of pedagogy are not needed. On the other hand, knowledge of content and students and knowledge of content and teaching are amalgams of subject matter knowledge and pedagogical knowledge, and are thus types of pedagogical content knowledge (Shulman, 1986).

As unconceptualized domains, horizon content knowledge refers to an orientation to and familiarity with the discipline that contributes to the school subject at hand, providing teachers with a sense of how the content being taught is situated in and connected to the broader disciplinary territory. This knowledge is a resource for balancing the fundamental tasks of connecting learners to a vast and highly developed field, and also includes an awareness of core disciplinary orientations and values, and of big ideas and major structures of the discipline. Knowledge of curriculum denotes all knowledge interconnected with curriculum, pedagogy, and psychology for teaching and students.

MKT is grounded in the discipline of mathematics while also grounded in teaching practice. Each domain works together in the mathematical work of teaching. For example, articulating the mathematical point for one lesson in an elementary classroom demands all the four main domains (Sleep, 2009). Moreover, teacher educators generally do not plan or focus on a certain domain of MKT in their classes but consider what knowledge teachers learn and what teachers are able to do after a class or a semester. This means that teacher educators concentrate on a certain mathematical work of teaching as content in their classes. Thus, this research also investigates the mathematical work of teaching regarding the domains of MKT. This should help teacher educators recognize which aspects of MKT are worked on in their instruction, as shown in Chapter 6 and *Figure 6.16*.

I now turn to review educational research related to mathematical definitions and three mathematicians' thoughts about mathematical definitions for disciplinary ground to uphold MKT as content in mathematics teacher education.

2.4 Attributes of Mathematical Definitions

Mathematical definitions are the main objects in this current research. In this section, literature about mathematical definitions is reviewed in terms of the practice of teaching and different perspectives of disciplinary mathematics in order to lay a solid mathematical foundation with regard to discipline-grounded approach (Ball & Bass, 2000b).

2.4.1 Mathematical definitions in teaching practice

Mathematical definitions create important aspects in and for teaching. For example, in disciplinary mathematics, the definition of a circle is a set of points at a fixed distance from a fixed point (Coolidge, 1916; C. G. Gibson, 2003; Lachlan, 1893). It does not include a flat and solid disc. However, in the real life that students experience, the term circle presents all of (a), (b) and (c) from *Figure 2.5.*

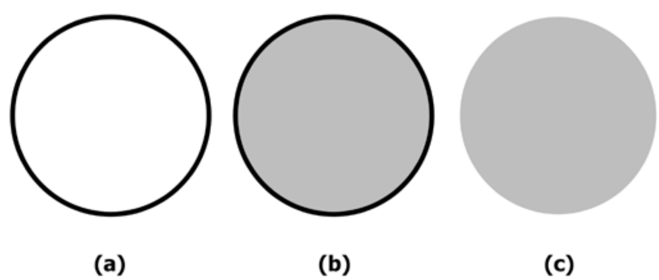

Figure 2.5 Examples of pictures of circles

In (b) and (c) of *Figure 2.5*, someone, including students, might think that a circle is the area or portion inside the circumference. These two shapes do not correctly indicate circle as disciplinary mathematics terms, where a point on a circle is a point on the boundary, not a point in the interior. Without a clear definition of circle, the meaning of the term and its use can be varied. When introducing the circle as a figure in early elementary school, many teachers and developers of curriculum materials use colored paper with a circle shape or introduce the circle through an activity of drawing the boundary using circle-shaped objects on colored paper and then cutting them out. These activities are closer to that of (b) and (c) in *Figure 2.5*. Rather than criticizing that these activities are mathematically incorrect for teaching the circle to students, it is important to note that they provide students with the experience of recognizing the properties of circle.

When they use these activities, teachers need to know the mathematically correct definition of circle; acknowledge the differences between the precise mathematical definition of circle, and the purpose and characteristics of these activities; understand their students' subsequent learning related to the circle; consider possible students' epistemological hurdles to study the circle; and think about how to support students. Teachers who teach the circle through an activity using a compass for drawing circles or Cartesian coordinates in analytical geometry in later levels also need to recognize students' prior and subsequent learning.

In addition, mathematical definitions have a crucial role for mathematical reasoning in both disciplinary mathematics and mathematics education. While mathematical definitions are simply delivered names to be memorized, mathematical definitions originate in and emerge from new ideas and concepts (Ball & Bass, 2003a, p. 33) and are negotiated in the process of investigating what is true (Lampert, 1990, p. 42). Moreover, mathematical definitions can function like axioms in mathematical reasoning (Vinner, 1991), and mathematical definitions can be conjectures that always ask for refutations and justification through proofs because the demands for rigor are increased (Lakatos, 1976a). Therefore, mathematical definitions facilitate mathematical reasoning about both the new and known ideas by specification.

Furthermore, mathematical definitions initiate concentrating on the role of mathematical language in teaching mathematics. Ball and Bass (2000b) explain salient issues involving mathematical language:

> [*Mathematical*] *language is used here expansively, comprising all of the linguistic infrastructure that supports mathematical communication, with its requirements for precision, clarify, and economy of expression.* … Mathematical language is not simply an inert canon, inherited and learned from a distant past. *It is also a medium in which mathematics is enacted, used, and created.* … Decisions about what to name, when to name it, and how to specify that which is being named is an important component of mathematical sensibility and discrimination central to the construction of mathematical knowledge (p.205, emphasis added).

Therefore, teaching mathematics asks the careful use of mathematical language. Mathematical definitions particularly require a sensitivity to the nature and role of language in mathematics.

2.4.2 Attributes of mathematical definitions in disciplinary mathematics

Seeing mathematical definitions in particular philosophies of mathematics evidences that mathematical definitions have diverse roles and attributes. This section discusses three mathematical views about mathematical definitions: Aristotle's view of Idealism; Hilbert's view of Formalism; and Lakatos' view of quasi-empiricism. The investigation of these different perspectives and analyses of Aristotle, Hilbert, and Lakatos about mathematical definitions provides a mathematical groundwork for analyzing records of practice and building a conceptual framework of knowledge about mathematics as content in mathematics teacher education.

Real definition to state an essence of a mathematical object by the intellect—Aristotle's conclusions about mathematical definitions: Even though Aristotle is most famous as a philosopher, his statements also influenced the philosophy of mathematics (Cleary, 2001; Jesseph, 1993; Lear, 1982). In his writing on mathematics (e.g., Physics and Metaphysics), Aristotle emphasized that only a small group of people can be intellectual and that those people play a role in mathematics and defining mathematical objects. The intellect means only a few people who have a sound state of understanding cannot be false and can recognize what is intelligible (Cleary, 1995). Thus, only they are capable of abstractions that refer to precise and extraordinary understanding and touch on what is absolutely intelligible (Jesseph, 1993).[1] Aristotle's particular view is significant because mathematical objects are built from ordinary experiences by removing the sensible attributes, and mathematics does not rely on specific features of the sensible world. He considered mathematics the seminal groundwork in terms of both accuracy and truth in order to achieve scientific truth and realize the eternal truth as theoretical science (Apostle, 1952). Thus, mathematical definitions are known by the intellectual, and all other individuals, according to Aristotle, should avoid defining mathematical objects because each individual has a different state of knowledge and thus could easily have erroneous understanding.

How did Aristotle specify definitions? He thought that mathematical definitions are explanatory of what makes mathematical objects in order to articulate the essence of

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[1] In Aristotelian thought, the meaning of intelligible refers only to that which is apprehensible by the intellect, rather than meaning understandable.

mathematical objects. For example, a definition of a circle is equivalent to the response to the question of what causes a circle. In fact, Aristotle recognized two kinds of definitions in Posterior Analytics: real definition and nominal definition. Real definition is an account revealing why a mathematical object is, and nominal definition is an account of what a name or some other name-like account signifies (Robinson, 1954). Aristotle considered that real definition can work in mathematics because real definition can articulate the essential attributes of a mathematical object, called the essence. In other words, a real definition is a formula that states an essence, and the essence of a mathematical object is known through its definition. Moreover, Aristotle asserted that a mathematical object has only one essence and, therefore, only one definition (Apostle, 1952, p. 99).

How can a mathematical definition be stated as a formula that shows an essence? Aristotle specified the existence of two parts, genus and differentia (differentiation), that compose a mathematical definition. Genus is both a broad and invariant category to which the object belongs, and differentia is the distinctive features that set the defined object apart from all the other objects in the category. As the genus takes on each differentia, a mathematical definition becomes more specified. For example, "a quadrilateral is a four-sided polygon" identifies "a polygon" as the genus and specifies "four-sided" as the differentia that distinguishes all quadrilaterals from other polygons. Moreover, "a pentagon is a five-sided polygon" has "a polygon" as the genus, such as a quadrilateral, but has "five-sided" as the differentia, which differentiate all pentagons from other polygons. Therefore, to define a mathematical object, clear knowledge is required because defining a mathematical object needs an identifying genus and differentia.

There are two points that are needed in defining mathematical objects. First, all terms that are used in a mathematical definition should be clarified before the mathematical definition is known. This means that knowledge of a defined mathematical object requires knowledge of the undefinables because a mathematical definition cannot consist of an infinite number of words. For example, a polygon is the genus of a quadrilateral, and a figure is the genus of a polygon. But a figure is a kind of a magnitude, and a magnitude is a continuous quantity, and a quantity is indefinable

(Apostle, 1952, p. 39). Thus, some definitions are immediate, such as point and quantity, but the other definitions are examined by identifying the causes of mathematical objects. Definitions, like points, are assumed by their existence. Without prior definitions, mathematicians would have to identify all mathematical objects necessary for defining.

Second, various descriptions or propria (properties) of a mathematical object cannot be definitions (W. R. B. Gibson & Klein, 1908). For example, roundness is one proprium of circle. While this can be used to differentiate circles from diverse polygons, such as quadrilateral and pentagon, it cannot be a mathematical definition because roundness applies not only to circles but also to some non-circles such as ovals and cylinders. This means that what is more obvious and easily recognized is not enough to define a mathematical object even if descriptions or propria are more familiar. What become more apparent and more discernible are distinctions, which are farther away from feeling and closer to being absolutely intelligible. Moreover, descriptions or propria would assume that a mathematical object had different essences, and this is in conflict with the assumption that there is a unique answer to the question of what makes a mathematical object (Cleary, 1995).

Mathematical definitions that are consistent in a system—Hilbert's conclusions about mathematical definitions: An understanding of Hilbert's mathematical background helps clarify his ideas about mathematical definitions. While Aristotle was associated with idealism, Hilbert was one of the early proponents of formalism, which assumes that mathematics is a solid and complete logical structure and a consistent building entity based on systems of axioms that do not need to be interpreted (Bostock, 2009).

Hilbert believed that mathematics should be a consistent system. According to his belief, he formalized mathematics in axiomatic form. Therefore, formalization is a method and a tool for studying the properties of theories in preexistent mathematics (Murawski, 2004). For example, let's assume three axioms.

Axiom1: if $a \neq b$ then aRb or bRa.

Axiom2: if aRb then $a \neq b$.

Axiom3: if aRb and bRc then aRc.

I would like to deduce one theorem.

Theorem1: if aRb then $b \not R a$.

Proof: Suppose that aRb and bRa exist at the same time. Because of Axiom3, aRa exists. However, because of Axiom 2 if aRa then $a \neq a$. This is a contradiction. Therefore, the supposition, aRb and bRa exist at the same time, is incorrect. Thus, Theorem1 is proved.

There are only formal statements and symbols without any meanings. Hilbert asserted that all theorems are proved and systemized from axioms in this way.

Each symbol can be changed into an appropriate word. However, this should not result in a contradiction in the framework. For example, let a, b and c be integers, and R given the meaning of bigger.

Axiom1: if $a \neq b$ then $a < b$ or $a > b$.

Axiom2: if $a < b$ then $a \neq b$.

Axiom3: if $a < b$ and $b < c$ then $a < c$.

Theorem1: if $a < b$ then $b \not< a$.

In this case, there is no contradiction in this framework. This is referred to as an interpretation. There is not only one interpretation, such as R can be less. However, Hilbert was not interested in interpretation and does not concentrate on whether any particular interpretation is intended or not (Bostock, 2009). In this regard, he stated, "If I think of my points as some system or other of things, for example the system of love, of law, or of chimney sweeps… and then conceive of all my axioms as relations between these things, then my theorems, for example the Pythagorean one, will hold of these things as well" (Hilbert as cited in Frege, 1971, p. 13). In other words, formalization removes all meaning from mathematics and reduces it to symbol manipulation. As a result, mathematics becomes a formal symbolic system. Hilbert emphasized axioms and deductions from these axioms and believed that formal systems are appropriate objects for mathematics research.[2]

Consistency is also a key in mathematical definitions. Hilbert thought that a mathematical definition is a new symbol or a combination of symbols by presenting

[2] If he had succeeded theoretically, mathematics would have been the study of uninterpreted systems (Brown, 1999).

another combination of symbols that is already clarified. The point is that mathematical objects cannot be explicitly and independently defined but can be defined by selecting their meaning in the axioms and maintaining consistency in the system. He thought that all theories are frameworks or schemas of concepts that have their necessary relations to one another in the system. He clarified that "If the arbitrarily posited axioms together with all their consequences do not contradict one another, then they are true and the things defined by these axioms exist. For me, this is the criterion of truth and existence" (Hilbert as cited in Frege, 1971, p. 12). He thought that the truth of mathematical definitions depends on consistency. Therefore, if consistency is solidly maintained in the system, changing a term of a defined object does not matter because a mathematical definition does not show an intuitive basis of the term but indicates symbols or combination of symbols that consistently work well in the system. Thus, he argued that, "It must be possible to replace in all geometric statements the words, point, line, plane by table, chair, mug" (Hilbert as cited in Ewald, 1996, p. 4). Finally, mathematical definitions identify the relations in the system and work to differentiate each other, but interpretations or meanings of mathematical definitions are not critical.

On the other hand, in his time, Cantor's theory of infinite numbers is the only part of mathematics in which consistency was not apparent because a set theory was not yet clearly formulated at that time. However, Hilbert admired Cantor's theory, and declared that "no one shall drive us out of the paradise which Cantor has created for us" (Hilbert, 1926/1983, p. 191). Hilbert believed that Cantor's theory had to be released from the threat of inconsistency and tried to prove the consistency of various branches of mathematics.[3] Thus, Hilbert needed to show that the various parts of infinite mathematics correspond with one another in infinite mathematics and with finite mathematics without any inconsistency. He thought that if he successfully reaches his goal, mathematicians could freely conduct their research without the hurdles that originate from different systems. He also recognized that different frameworks influence mathematical definitions. He said "[E]ach axiom contributes something to the definition, and therefore each new axiom alters the concept. 'Point' is always something different in Euclidean,

[3] In fact, Hilbert never did any serious work on this question because there are too many issues to resolve before approaching this problem (Bostock, 2009). Furthermore, Gödel shows that Hilbert's success is impossible.

non-Euclidean, Archimedean, and non-Archimedean geometry respectively" (Hilbert as cited in Frege, 1971, p. 13). Hilbert believed that a mathematical object can be differently defined because of different axioms in different systems. As a result, if different mathematical systems are taken into account, a mathematical object might not have an identical definition in different systems, and a mathematical definition might not result in the same object in different systems. In other words, Hilbert recognized incommensurability as a characteristic of mathematical definitions in mathematics (Brown, 1999).

In summary, Hilbert believed that a mathematical definition is a new symbol or a combination of symbols and is shown as other combinations of symbols which are already known. Thus, mathematical definitions show relations written in symbols. Rather than concentrating on extensive interpretations or meanings of mathematical definitions, Hilbert highlighted that mathematical definitions should be non-contradictory, consistently work well, and clarify the relations in the systems. Hilbert also recognized the influence of systems on mathematical definitions.

A proof-generated definition—Lakatos' conclusions about mathematical definitions: While Aristotle and Hilbert did not doubt the truth of mathematics, Lakatos was a twentieth century mathematician who believed that mathematics is fallible because mathematics has been developed by the interplay between conjecture, proof, counterexample, and refinement of conjecture (Kleiner, 1991). Thus, Lakatos squarely refuted a deductive system in mathematics which considers mathematics to be a priori as Aristotle asserted. Lakatos (1967) described this system: axioms as absolute truth are injected into the top of a deductive system and flow down through the safe truth-preserving channels of valid inferences; he then claimed that mathematics as a quasi-empirical system is for upward retransmission of falsity from the basic statements to the axioms. Lakatos also disagreed with formalism because he thought it is dogmatic philosophy of mathematics (Davis & Hersh, 1981). In Lakatos' view, formalism just attempted a perfect reorganization of classical mathematics based on the classical deductive system as a kernel and tried to stretch the scope of this system to infinite mathematics (Lakatos, 1976b).

Lakatos (1967) asserted that mathematics is a quasi-empirical theory as previously mentioned because he believed in the methodological science-likeness of mathematics (Glas, 1995). Quasi-empirical theories are founded on proofs to implant conjectured theorems in clearly true or already proven statements. He thought that mathematical theorems are indeterminate and cannot be guaranteed absolute certainty because knowledge can be fallible, a view that is influenced by Popper (Koetsier, 1991). Popper (1959) argued the importance of falsification for scientific advancement through the analysis of competing theories, specifically conjectures and refutations. Theorems are constantly examined and can be rejected through counterexamples. Proofs are also tools of discovery rather than instruments of justification in order to develop concepts and refine conjectures. While the axioms ascertain the theorems in Euclidean theory, Lakatos believed that the axioms merely explain the underlying principles of the theorems. In brief, Lakatos (1976a) acknowledged the possibility of fallible mathematics and emphasized the importance of social aspects of mathematical inquiry like discussions in mathematics.

Lakatos' view of mathematical definitions is revealed in a dialogue in Proofs and Refutations. In this dialogue, mathematical definitions have been proposed and revised during a long process of discussion, and mathematical definitions ultimately have evolved because the demands for rigor are increased (Khait, 2005). In this regard, Lakatos (1976a) classified two different kinds of definitions: a zero-definition and a proof-generated definition. On the one hand, a zero-definition is used for practical purposes at the beginning of the research process or discussion to denominate a certain mathematical object before a proper mathematical term is defined (Ouvrier-Buffet, 2006). A zero-definition can be spontaneously uncovered because mathematics is heuristic. It is not important, however, whether or not a zero-definition captures preliminarily analytic or intuitive findings about a defined object (Brown, 1999). On the other hand, a proof-generated definition is a definition that is needed in order to prove a specific conjecture regarded as valuable (Werndl, 2009). A proof-generated definition is originated and developed by proofs from a zero-definition and may seem the end product of a zero-definition (Ouvrier-Buffet, 2006, p. 266). In the dialogue in Proofs and Refutation, participants formulate a zero-definition and explore counterexamples to refute the proposed definition. This process continuously guides one to include or exclude counterexamples and to reformulate a mathematical definition. Lakatos emphasized that mathematical definitions can be fallible and are not fixed but evolving: mathematical definitions should be modified through proof. In this respect, mathematical definitions are conjectures, which always ask for refutations and justification through proofs. Moreover, a new proof-generated definition can have a role in providing knowledge because it answers the question of which notion is needed to prove a specific conjecture, which might be a previously provided definition (Werndl, 2009). This process is exactly the procedure of concept formation.

The evolution from a zero-definition to a proof-generated definition is not a matter of labeling mathematical objects, but rather one of trying to find the essence of those objects through proofs. Through this process, Lakatos first explored the invariants in a given mathematical object and then investigated variants distinguished from other objects. This is similar to Aristotle's recognition about genus and differentia (differentiation) to investigate real definition. As a result, neither of them considers mathematical definitions as nominal or stipulative.

Features of mathematical definitions in disciplinary mathematics: Aristotle, Hilbert and Lakatos had different perspectives and ideas on mathematical definitions. Mathematical definitions have significant roles in mathematics: they can state the essence of mathematical object (Apostle, 1952); they can be answers to the questions of what causes mathematical objects (Robinson, 1954); they work for differentiation (Hibert as cited in Ewald, 1996); they have incommensurability (Hilbert as cited in Frege, 1971); they are recursive for the consistent mathematical system (Curry, 1951); they can be generated, evolved, and changed by proving in the group (Lakatos, 1976a); they are derived in ways which will be explained in connection with classes and relations (Russell, 1903); and, they remove ambiguity (W. R. B. Gibson & Klein, 1908). Therefore, mathematical definitions are fundamental and critical in disciplinary mathematics.

Aristotle, Hilbert, and Lakatos represented different perspectives in mathematics. Their different ideas about and awareness of mathematics have developed the discipline of mathematics. Their ideas about mathematical definitions provide what mathematicians take for granted to develop their research, what they are aware of in their research, what they think profound in the discipline of mathematics, and what make mathematics valuable. I used their ideas to examine mathematical issues and conceptualize knowledge about mathematics in teaching MKT. The findings can be found in Chapter 6.

I now turn to review research with regard to the mathematical work of teaching and to develop the initial coding scheme from the literature.

2.5 Mathematical Work in Teaching

To articulate the knowledge for the content of mathematics teacher education, I considered the specifications of Ball and her colleagues (2009). They asserted that both articulating the work of teaching mathematics and identifying and choosing high-leverage practices are necessary in order to develop an approach to preparing teachers that is both aimed at practice and centered in content. Moreover, MKT has been investigated through a practice-based and discipline-grounded approach (Ball & Bass, 2003b), as previously reviewed. There are two main purposes in this section: to lay out the theoretical foundation about the mathematical work of teaching and to delineate the initial categorization schemes. MKT is embedded in the mathematical work of teaching (Ball et al., 2008) and teaching MKT requires articulation of the mathematical work of teaching (Ball et al., 2009). Therefore, specifying what has been investigated with regard to the work of teaching provides an initial foundation on which I can build a framework of the mathematical work of teaching. According to the views of Ball and her colleagues (2009) about exploring notions of MKT for mathematics teacher education, I first searched the literature in order to gather works of teaching that are typical and particular in the practice of teaching. I also reviewed the literature of mathematics education about definitions because the present research focuses on mathematical definitions as content to teach MKT in mathematics teacher education and this investigation helps gather cases of the work of teaching related to mathematical definitions. Through this literature review, I developed the initial categorization schemes that specify and differentiate the mathematical work of teaching.[4] I used these categorization schemes to begin analyzing

[4] See Appendix A for the initial categorization for mathematical work of teaching.

records of practice for explicating MKT as the content in mathematics teacher education. Throughout the analyses of the data, I returned to findings from the literature to examine and improve the evolving conceptual framework.

2.5.1 Work of teaching in educational research

Articulating the work of teaching mathematics is the first step in developing a curriculum for teaching the practice of teaching (Ball et al., 2009). While educational researchers do not have a shared taxonomy of and language for the practices of teaching (Grossman & McDonald, 2008), several researchers have specified the work of teaching in the classroom or illustrated examples to highlight the importance and focus of practice. This section collects all work of teaching that they highlighted.

While each researcher in this section had different theoretical background, all of them focused on the practice of teaching in the classroom and specified the work of teaching. I gathered their specification of the work of teaching in this section. It is significant because this specification provides language to describe, analyze, and conceptualize MKT from the empirical data. While some of them did not include mathematical issues or attributes, their work were still helpful in elaborating the mathematical work of teaching. Several researchers elaborated the work of teaching and enumerated it extensively.

The Connecticut Competency Instrument (CCI) is a classroom observation system used with beginning teachers (Haertel, 1991). This instrument focuses on ten indicators that illustrate work of teaching in instruction.[5] The purpose of these indicators is to assess teachers' lessons. Reynolds (1992) reviewed the literature about effective teaching in order to illustrate what beginning teachers are expected to do.[6] This list shows what

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[5] Ten indicators are promoting a positive learning environment; maintaining appropriate standards of behavior; engaging students in activities of the lesson; effectively managing routines and transitions; creating a structure for learning; presenting appropriate lesson content; developing a lesson to promote achievement of lesson objectives; using appropriate questioning techniques; communicating clearly; and, monitoring student understanding and adjusting teaching (Haertel, 1991, p.21)

[6] The list includes planning lessons that enable students to relate new learning to prior understanding and experience; developing a rapport and personal interactions with students; establishing and maintaining rules and routines that are fair and appropriate in the classroom; arranging the physical and social classroom environment in ways that are conducive to learning and that fit the academic task; representing and presenting subject matter in ways that enable students to relate new learning to prior understanding and that help students develop metacognitive strategies; assessing student learning using a variety of measurement

should be assessed in performance based assessments for teacher licensure. Both the specification of CCI and Reynolds' list includes the work of teaching for not only mathematics but also any kind of subject matter knowledge.

There are researchers who focus on teaching in mathematics classroom in terms of the work of teaching. Ball and Bass (2003b) addressed what mathematical work teachers have to do to teach effectively and gave the examples to provide a glimpse of how centrally mathematical reasoning and problem solving figure in the work of teaching.[7] These scholars claimed that teaching is mathematically intensive work: knowing mathematics in and for teaching includes both elements of mathematics as found in the student curriculum and aspects of knowing and doing mathematics that are less visible in the textbook (p.8). The current research follows their notion about the mathematical work of teaching. In particular, their emphasis on such work extended to their research on MKT. Ball, Thames and Phelps (2008) conceptually investigated what MKT is. They described their approach to develop a practice-based theory of content knowledge for teaching and explored subdomains built on Shulman's notion of subject matter knowledge and pedagogical content knowledge. They presented examples of routine and distinctive tasks of teaching which demand unique mathematical understanding and reasoning.[8] Through their examples, they showed that there is a domain of content knowledge unique to the work of teaching.

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tools and adapting instruction according to the results; and, reflecting on their own actions and students' responses in order to improve their teaching (Reynolds, 1992, p.26)

[7] Their examples are choosing a task to assess student understanding; interpreting and evaluating students' non-standard mathematical ideas; making and evaluating explanations; designing mathematically accurate explanations that are comprehensible and useful for students; using mathematically appropriate and comprehensible definitions; representing ideas carefully; mapping between a physical or graphical model, the symbolic notation, and the operation or process; interpreting and making mathematical and pedagogical judgments about students' questions, solutions, problems, and insights; being able to respond productively to students' mathematical questions and curiosities; making judgments about the mathematical quality of instructional materials and modifying as necessary; being able to pose good mathematical questions and problems that are productive for students' learning; assessing students' mathematics learning and choosing the appropriate next steps; unpacking of ideas; making connections across mathematical domains; helping students build links and coherence in their knowledge; anticipating how mathematical ideas change and grow; examining correspondences among representations; and, investigating whether or not representations are equivalent (Ball & Bass, 2003b).

[8] Their examples are presenting mathematical ideas; responding to students' "why" questions; finding an example to make a specific mathematical point; recognizing what is involved in using a particular representation; linking representations to underlying ideas and to other representations; connecting a topic being taught to topics from prior or future years; explaining mathematical goals and purposes to parents; appraising and adapting the mathematical content of textbooks; modifying tasks to be either easier or

Ball and Forzani (2009) highlighted the tasks of teaching for a teacher education curriculum. They argued that tasks of teaching should be made the core of teachers' professional preparation, given teaching is hard work that many people need to learn to do well. Their assertion provides a theoretical foundation about the importance of the mathematical work of teaching as content in teacher education. They also offered examples of what might be involved in teaching practice.[9]

There are two studies about the mathematical work of teaching in teacher education. Zopf (2010) focused on teacher education for elementary school teachers. She investigated tasks of teaching MKT and asserted there were three key ones, selecting interpretations and representations of mathematical ideas, choosing examples to support these interpretations and representations, and managing the enactment of mathematical tasks. This study focuses on MKT as content in teacher education, but this research is an empirical investigation of classes of the two teacher educators. Ferrini-Mundy and Findell (2010) addressed what mathematics prospective secondary school mathematics teachers need to know and specified six activities that teachers need to be able to draw on and do.[10] They argued that these kinds of mathematical activities are essential in teaching (p.35).

With regard to mathematical work of teaching, the current study is in line with Ball and Bass (2003b), Ball, Thames and Phelps (2008), Ball and Forzani (2009), Zopf (2010), and Ferrini-Mundy and Findell (2010). However, the current study concentrates

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harder; evaluating, often under time constraints, the plausibility of students' claims; giving or evaluating mathematical explanations; choosing and developing usable definitions; using mathematical notation and language and critiquing its use; asking productive mathematical questions; selecting representations for particular purposes; and inspecting equivalencies (Ball, Thames, & Phelps, 2008, p.400)

[9] Their examples are identifying ways in which a learner is thinking about the topic or problem at hand; structuring the next steps in the learner's development; overseeing and assessing the learner's progress; listening to and watching students as closely as is required to probe their ideas carefully; identifying key understandings and misunderstandings; intentionally provoking discordant thinking or errors in logic and argument; asking questions to which the answer is known or partially known by the teacher; probing others' ideas; not presuming a shared identity; seeking to learn others' experiences and perspectives; sizing up a student's incorrect mathematical response; considering these mathematical affordances; anticipating what students might do; phrasing to use in asking student's question; creating a respectful learning environment; assessing students' mathematics skills; and, reviewing homework (Ball & Forzani, 2009).

[10] Their six activities are finding the logic in someone else's argument or the meaning in someone else's representation; deciding which of several mathematical ideas has the most promise, and what to emphasize; making and explaining connections among mathematical ideas; situating a mathematical idea in a broader mathematical context; choosing representations that are mathematically profitable; and maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea (Ferrini-Mundy & Findell, 2010).

on the mathematical work of teaching in teacher education for elementary school teachers and creates a conceptual framework for teaching MKT.

2.5.2 Work of teaching related to mathematical definitions

Because mathematical definitions are objects of teaching and learning mathematics in schools, there are educational studies about mathematical definitions. While there is no study that concentrates on the mathematical work of teaching for teaching mathematical definitions, the results of the studies contribute to knowledge entailed in teaching mathematics. I searched out what mathematical work of teaching is described in educational research in terms of mathematical definitions. It is important because a key objective of this research is to develop a conceptual framework to teach MKT in mathematics teacher education. I then classified all of them with similar features of the mathematical work of teaching with work of teaching that are found in the previous section.[11]

Pimm (1987, 1993) addressed unpacking mathematical definitions for teaching and identifying features of representations. Wilson (1990) talked about using precise language, recognizing that certain mathematical definitions contain other mathematical definitions, and acknowledging undefined terms to compose mathematical definitions and roles of mathematical definitions in proofs and logical arguments. Sierpinska (1994) provided examples to show how a concept has been developed for a mathematical definition. Van Dormolen and Zaslavsky (2003) explained investigating the logical aspects of mathematical definitions, and Keiser (2004) demonstrated presenting and using multiple representations for mathematical definitions. Shield (2004) focused on clarifying relations between provided mathematical definitions and other definitions as well as the structural knowledge of mathematical definitions. Euler and Sadek (2005) examined using alternative mathematical definitions and unpacking them, and Burn (2005) suggested a genetic approach to the mathematical definitions. Dobbs (2005) addressed unpacking ideas in mathematical definitions, and Knapp (2006) clarified that mathematical definitions depend on the choice of the mathematical universe. Ouvrier-Buffet (2006) examined preparing for responses as appropriate feedbacks to students, and

[11] See Appendix A for the initial categorization for mathematical work of teaching.

Berge (2006) conducted an historical investigation around mathematical definitions. Goldenberg and Mason (2008) discussed using counterexamples for revising mathematical definitions, and Semadeni (2008) stated roles of the stability of meanings. Davis (2008) analyzed informal language used outside the mathematics classroom and demonstrated what terminology and representations are used in the real world for formal language in disciplinary mathematics. Usiskin and colleagues (2008) explored the roles of mathematical definitions, such as classifying objects, identifying a category, and specifying how an object is distinguished from others in that category.

Again, all this research does not aim at denoting the mathematical work of teaching but is interested in investigating mathematical definitions as topics. Each study has different theoretical and practical background and does not focus on specifying the mathematical work of teaching. However, all researchers straightforwardly study mathematical definitions and state the mathematical work of teaching in brief illustrations or comments.

CHAPTER 3
METHODS OF DATA COLLECTION AND ANALYSIS

3.1 Introduction

Chapter 1 explained that this research focuses on MKT in the context of mathematics teacher education. There are two purposes in the current research: exploring the tasks of and challenges involved in teaching MKT and developing a framework to teach MKT that lays the foundation for a curriculum in mathematics teacher education. For these purposes, I used two types of records of practice. The first one was curriculum materials specifically designed to focus on MKT, and the second one was video recordings of twenty-five classes where nine teacher educators used the curriculum materials. Because the curriculum materials were used as lesson plans in the twenty-five classes, it obviously confirmed that users of the materials intended to teach MKT and their lessons provided the instruction of MKT. Therefore, it was important to examine what the teacher educators intended. Moreover, the analysis of the curriculum materials helped analyze the video recordings more efficiently. Among users of the materials the data set collected consisted of twenty-five classes by nine teacher educators. These video recordings provided a live form of instruction of MKT for this research. As regarding the purposes of this research, the video recordings showed the dynamics and challenges in teaching MKT, what MKT the teacher educators highlighted, and how the teacher educators approached such knowledge in their teaching. Because the teacher educators had a variety of backgrounds, institutions, instructional aims, and topics covered in their courses, the video recordings were different from one another. Although I analyzed the data from nine teacher educators, I do not make any claims about what they should have emphasized or whether they taught efficiently.

As such, this study examines MKT as it arises in instruction of MKT. I did an iterative analysis of the literature and the records of practice. I focused on clarifying what teacher educators need to attend to when they teach MKT and building knowledge about MKT for teacher education. This research is primarily conceptual: developing a framework that can inform a curriculum to teach MKT in mathematics teacher education. For these purposes, I analyzed the data through methods described by Erickson (1986, 2006).

This chapter describes records of practice and the methods of analysis that I used. I begin by describing two kinds of records of practice used in this dissertation, and then I explain how I analyzed them. I close the chapter with a discussion of the limitations of the study.

3.2 Data Collection

Two kinds of records of practice were collected for this study: curriculum materials and video recordings. First, I explain the curriculum materials that teacher educators used and how I selected teacher educators among users of the curriculum materials and what classes were recorded.

3.2.1 Curriculum materials

I selected *Using Definitions in Learning and Teaching Mathematics* as the curriculum materials for this study, developed by the *mod4* project at the University of Michigan (Mathematics Teaching and Learning to Teach, 2008, 2009). There are a couple of reasons to select these curriculum materials. First, these practice-based materials were designed to focus on MKT. In particular, the materials focus on helping teachers learn mathematical knowledge and skills for the work of teaching. The materials use records of classroom practice as contexts for learning, target mathematical knowledge and skills needed for the work of teaching, and situate instructional activities in mathematical problems and tasks of teaching. Second, the materials specifically focus on mathematical definitions as mathematical content. The materials concretely consider various features of mathematics through definitions, such as mathematical precision, the careful use of language, and mathematical reasoning and proving. Moreover, the materials are for content and methods courses at universities and for professional development. The feature of the use in the multi-situations helped me gather video recordings from a variety of situations.

There are five lessons in the materials. Because all episodes in the following chapters are related to the activities of the materials, a detailed explanation of each lesson is shown in Table 3.1.

Table 3.1 *Lessons in the Curriculum Material, Using Definitions in Learning and Teaching Mathematics*

Lesson	Main Activity	Explanation
Lesson 1	Why mathematical definitions matter	This lesson provides an opportunity for teachers to develop a curiosity and an understanding of the role of mathematical definitions in both mathematics and mathematics teaching. With a given definition, teachers choose numbers and apply the mathematical definition to identify which ones are even. This work gives teachers an opportunity to reflect on mathematics as a discipline in which meanings are precise and to begin to see the need for this precision. Exploring the question about why ½ is not considered an even number leads to two features of mathematical definitions: the requirement that a mathematical definition makes significant distinctions and the "implied universe" of a mathematical definition.
Lesson 2	Hearing definitions in children's talk	This lesson focuses on practicing interpreting children's mathematical talk and to recognize the mathematical definitions implied in children's reasoning. This lesson uses focused observation and analysis of a video segment from a third grade class to provide teachers with opportunities to (1) hear, interpret, and analyze (with observation-based evidence) children's mathematical thinking and their use of implicit definitions to support their reasoning about even and odd numbers and (2) gain an appreciation for the role that definitions play – as well as their significance – in providing precision and supporting reasoning in both mathematics and mathematics teaching.
Lesson 3	Evaluating definitions	This lesson provides teachers with explicit criteria for evaluating the quality of mathematical definitions and set the stage for more conscious use of mathematical definitions in mathematical communication and reasoning. This session develops criteria for "good" mathematical definitions and then has teachers use them to generate and evaluate definitions for an "even number." After introducing two criteria for mathematical definitions (i.e., precision and usability), teachers apply these criteria in a teaching context — either writing and then evaluating each other's definitions, or analyzing definitions that one might find in an elementary textbook. Teachers apply the criteria, practice testing examples and generating counterexamples, and learn to attend carefully to the use of mathematical language.
Lesson 4	Reasoning with definitions	This lesson offers experience with constructing and evaluating proofs of the mathematical

	for proof	claim “odd + odd = even.” It is designed to develop not only teachers’ skills with the language and methods of mathematical proof but also their understanding of the elements of a proof and the standards of proof for a community. In proving claims about even and odd numbers, the teachers further encounter the use of mathematical definitions to support mathematical reasoning.
Lesson 5	Reasoning with definitions for explanations	This lesson is designed to expand teachers’ experience with explaining in mathematics and to develop their skill in producing and evaluating mathematical explanation. This lesson focuses on learning to construct and evaluate mathematical explanations for a familiar “rule.” Using the units digit rule as the context for the work, teachers explain something that they already “know,” drawing upon important foundational knowledge – such as the definition of an even number and an understanding of the base ten number system. Working in this way on a familiar idea can highlight the difference between “knowing” something in mathematics (i.e., being able to recite it and even use it) and actually understanding why it works. This can also help to show the power of a more general explanation because explaining the units digit rule supports being able to explain other divisibility rules. This work on mathematical explanation is central to learning mathematics for teaching because explaining is such a major part of the work teachers do. It includes practices such as providing explanations, helping children construct explanations, and evaluating explanations.

3.2.2 Video recordings

Teacher educators' major and teaching experience: Because this study is about discovering the dynamics involved in teaching MKT as well as developing a conceptual framework for teaching MKT in mathematics teacher education, I collected the data by theoretical sampling. According to Strauss and Corbin (2008), theoretical sampling enables researchers to discover and explore in depth the concepts and themes that are relevant to research questions. The variation in conditions maximizes the opportunity to discover new properties and dimensions (Strauss & Corbin, 2008). Therefore, I needed teacher educators with diverse backgrounds who teach in a variety of settings.

I considered teacher educators' disciplinary backgrounds in order to provide sufficient variation in their teaching. I anticipated that this variation might have an influence on their behavior and decisions in their practice of teaching. Moreover, for the same reason, as with the approach taken with disciplinary background, I also considered teacher educators' teaching experience in mathematics teacher education because even if teacher educators are experienced in K-12 classes, teaching teaching is different from teaching subject matters (Ball et al., 2009). However, I did not make these choices based on an assumption that a certain disciplinary background or longer teaching experience would imply better teaching of teaching mathematics, but rather that these two factors would provide variation in conditions. Among all possible teacher educators considered with these criteria in mind, I selected teacher educators who could consecutively teach three or four classes in which they teach teachers to reason about mathematical definitions in and for teaching. Ultimately, I chose nine teacher educators, and analyzed their twenty-five classes in which they taught MKT based on the curriculum materials. Table 3.2 shows teacher educators' major and teaching experience in mathematics teacher education.

Table 3.2 *Teacher Educators' Major and Teaching Experience*[12]

Major (Undergraduate)		
Mathematics	Education	etc. (Business administration)
Julie, Daniel, Emily, Kellie, Nellie	Matthew, Cate, Betty	Sandy

Teaching experience in mathematics teacher education		
Beyond 20 yrs.	10 yrs. to 20 yrs.	Under 10 yrs.
Julie, Sandy, Betty	Daniel, Kellie, Nellie	Emily, Matthew, Cate

Because the materials were developed for different contexts of teacher education, I selected teacher educators who had worked with the materials in various situations, as shown in Table 3.3. Learners in all contexts were elementary teachers. However, this research did not perform any other analysis, such as comparing and contracting differences among the three settings of teacher education.

Table 3.3 *Teacher Educators' Classes*

Method course	Content course	Professional development
Kellie, Cate, Betty, Nellie	Daniel, Emily, Matthew	Julie, Sandy

Another possible issue was teacher educators' experience in teaching MKT. As demonstrated by the number of times teacher educators had used the curriculum materials developed by *mod4* as shown in Table 3.4 below, they have diverse teaching experience of MKT. However, I did not anticipate that longer teaching experience of MKT would correspond to better teaching of MKT. Instead, the wide spectrum of the experience of teaching MKT was chosen because it would enhance the diversity of the data. Similarly, this research conducted no other analysis, such as comparing and contrasting differences among teacher educators' teaching experience of MKT because this research does not aim to explore individual teacher educators' understanding about and of MKT or what teacher educators attempt to teach regarding MKT in their classes.

[12] All teacher educators' names are pseudonym.

Table 3.4 *Teacher Educators' Experience of Teaching MKT*

First time	Second time	Third time
Julie, Sandy, Emily, Matthew	Kellie, Cate, Nellie	Betty, Daniel

Video recordings that show MKT teaching: I used video recordings collected in classrooms where teacher educators teach MKT. Video recordings as records of practice can be used to inquire into *practice* (Ball & Cohen, 1999). This assumption is appropriate to probe the kind of phenomena that is the purpose of this research because this study is based on what teacher educators actually do in their instruction rather than on examining individual teacher educators' understanding about MKT or probing what they attempt to teach concerning MKT in their classes. Teacher educators who teach MKT might have different levels of understanding of MKT. However, those with a lower understanding of MKT might be able to adequately talk about the main ideas of MKT because many teacher educators have worked in academic jobs and are readily capable of capturing and noting the main terms without having a full understanding of MKT.[13] Moreover, in interviews, teacher educators might potentially exaggerate what they did or did not do in their classes, or they might hide their difficulties and shortcomings in their teaching. This would prejudice the analysis of data from observation.

I observed and video-recorded, nine-teacher-educators' classes between 2007 and 2010. As previously noted, they taught three or four consecutive classes using various activities related to mathematical definitions. I recorded at least two classes of each teacher educator, using a digital video camera. I situated the tripod at the back or side of the room in order to clearly capture teacher educators' talk and movements, teachers' responses in the class, and work done on the board. I also used a digital audio recorder to capture the words of teacher educators and teachers. However, when it was not reasonable to visit the classes myself because of time or location constraints, teacher educators recorded their classes and sent me video recordings. To diminish the limitation

[13] I experienced this phenomenon before doing this research. In the meeting with teacher educators who used diverse tasks and activities to teach MKT, all of them talked fluently about the main ideas about MKT and they seemed to efficiently teach MKT to their teachers. In fact, I was very impressed by their comments. However, when I observed their classes, I found that some teacher educators' teaching was different from what they said in the meeting.

of using video recordings that teacher educators recorded, I asked what was not captured in the video recordings when the video cameras were stationary. Fortunately, almost all of the video recordings that the teacher educators sent included teacher educators' movements, teachers' responses, and the use of the blackboard very well. Moreover, because I am not a participant observer, but rather an observer describing and analyze the scenes from the third person point of view, using video recordings that teacher educators sent seems reasonable in this research (Erickson, 2006). In one instance, I was unable to visit a second class of one teacher educator in this study. Thus, I observed and recorded only one class in her site. Table 3.5 shows the name of teacher educators' lessons that I observed.

Table 3.5 *Participants and Lessons that were Observed for This Research*

Teacher educator	Lesson
Julie	• Exploring why mathematical definitions matter • Hearing mathematical definitions in students' talk • Evaluating mathematical definitions
Sandy	• Exploring why mathematical definitions matter • Hearing mathematical definitions in students' talk • Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Daniel	• Hearing mathematical definitions in students' talk • Evaluating mathematical definitions • Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Emily	• Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Matthew	• Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Kellie	• Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Cate	• Hearing mathematical definitions in students' talk • Evaluating mathematical definitions
Betty	• Hearing mathematical definitions in students' talk • Evaluating mathematical definitions • Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Nellie	• Reasoning with mathematical definitions for proof

All courses were for elementary teachers. Table 3.6 shows all information about teacher educators and video recordings. Again, the curriculum materials were particularly designed for teaching MKT and the video recordings were selected from users of these curriculum materials. Therefore, the collected video recordings are valuable for studying curriculum and dynamics in mathematics teacher education.

The data of this study are the curriculum materials and the video recordings that show the implementation of the curriculum materials. The design principles of curriculum materials may influence the use of the materials, and this influence may be seen in the video recordings. This linkage between the design of the curriculum and its usage might make this study seem to be only a study of this curriculum. However, in using this curriculum in their classes, users of the curriculum materials freely analyzed, interpreted, and modified them. This study is not about investigating how users implement a written curriculum or exploring the process leading from written curriculum to enacted curriculum. Instead, this research is about teaching MKT by using the topic of mathematical definitions, where the *mod4* materials are worked. The curriculum materials validated that selected teacher educators tried to teach MKT.

Table 3.6 *Participants and Video Recordings*

Teacher educator	Major	Teaching experience	Experience of teaching MKT	Classes	Lessons of video recordings
Julie	Mathematics	Beyond 20 yrs.	First time	Professional Development	• Exploring why mathematical definitions matter • Hearing mathematical definitions in students' talk • Evaluating mathematical definitions
Sandy	Business administration	Beyond 20 yrs.	First time	Professional Development	• Exploring why mathematical definitions matter • Hearing mathematical definitions in students' talk • Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Daniel	Mathematics	10 yrs. to 20 yrs.	Third time	Content course	• Hearing mathematical definitions in students' talk • Evaluating mathematical definitions • Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Emily	Mathematics	Under 10 yrs.	First time	Content course	• Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Matthew	Education	Under 10 yrs.	First time	Content course	• Evaluating mathematical definitions • Reasoning with mathematical definitions for explanations
Kellie	Mathematics	10 yrs. to 20 yrs.	Second time	Method course	• Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Cate	Education	Under 10 yrs.	Second time	Method course	• Hearing mathematical definitions in students' talk • Evaluating mathematical definitions
Betty	Education	Beyond 20 yrs.	Third time	Method	• Hearing mathematical definitions in students' talk

				course	• Evaluating mathematical definitions • Reasoning with mathematical definitions for proof • Reasoning with mathematical definitions for explanations
Nellie	Mathematics	10 yrs. to 20 yrs.	First time	Method course	• Reasoning with mathematical definitions for proof

3.3 Data Analysis

The analysis considered three primary contributors to classroom interactions and the dynamics of instruction, namely, as shown in *Figure 1.1*, teachers as learners, teacher educators as instructors, and MKT as the content. I used this analytical differentiation to investigate the dynamics of teacher educators' practice, the MKT being taught, and ways of teaching MKT being used. Each category was investigated by methods described by Erickson (1986, 2006) to work progressively between particular description and general claims. As set forth in Chapter 1, the overall question of this research is:

> *What MKT might be worked on, and in what ways, in the instruction of mathematics teacher education when teacher educators aim to help teachers develop MKT?*

To facilitate the investigation, this research used supporting research questions as follows:

As teacher educators and teachers work on MKT in teacher education classrooms, using curriculum materials that address mathematical definitions in teaching,

1. *What are some of the challenges that arise?*
2. *What mathematical work of teaching is prominent?*
3. *What kinds of mathematical issues are revealed?*

I explain here how I analyzed the data to investigate each of the supporting research questions and how I conceptualized the finding from the empirical data for the overall question. For the efficient explanation, I start with the second supporting question.

3.3.1 Investigating research question: Developing a framework for mathematical work of teaching

This research question concentrates on illustrating the mathematical work of teaching that teacher educators specify and highlight to teach MKT in teacher education instruction. I conducted this research through an iterative analysis of the literature and the data, between particular descriptions and general claims (Erickson, 1986, 2006), in order to develop a conceptual framework that can inform a curriculum to teach MKT in mathematics teacher education. I began by identifying general themes about what mathematical work of teaching was involved as content in teaching MKT. These ideas were based on my reading of the literature and its synthesis as summarized in Chapter 2 and the observation memos made during the collection of data (Ryan & Bernard, 2000). In particular, I organized the ideas from the literature review into preliminary categories as shown in Appendix A. A detailed explanation of my analysis is as follows.

Watching video recordings and writing detailed observation notes. Before the serious analysis, I had to be familiar with events in the video recordings. Thus, I repeatedly watched them, but there were four different levels to my viewing. The first viewing was to have a broad outline, and the second was to have notes according to the time line. The third was to identify each event in the video recordings, and the fourth was to have detailed observation notes and to select episodes. Through the four levels, I could recognize all the events in the video recordings and have detailed observation notes that made me ready to analyze the data. What follows is a detailed explanation of my watching.

I first watched all the video recordings to recognize and understand what happened in each class and have a story line of each class. When I watched the video recordings for the second time, I took observation notes. Even though I wrote observation memos when I recorded classes, those memos consisted of short sentences that show what I found and thought in the actual moments of the lessons. In fact, when I observed and recorded the lessons, I also focused on managing the video cameras and audio recorders, and I was concerned about whether the recordings were going well. Thus, I needed to revise and improve my observation memo and have observation notes. I watched the video recordings very carefully for observation notes. I wrote the notes using the time code as I noticed activities that teacher educators offered, topics that the class discussed, particular demonstrations that teacher educators gave, any examples that teacher educators provided, questions that teacher educators asked, teaching moves to manage activities to teach MKT, and wrote about the ideas that occurred to me while watching and my interpretations, and comments.

During the third time of watching, I focused on each activity, stopping and replaying them this time, and finalized detailed observation notes. These detailed observation notes also contain information about teacher educators, teachers, time, and place, diagrams of classrooms, problems that were being discussed, writing on the public space by teacher educators and teachers, and my comments about my observations. These notes, at this point, served to capture all the events in the video recordings in detail.

When I watched the video recordings for the fourth time, I aimed at having better transcripts. I received a grant to transcribe all the video recordings, but I needed to polish the transcripts for use in my analysis. I also differentiated episodes related to tasks of teaching MKT, any mathematical issues, attributes of mathematical definitions, mathematical demands of teaching, and tasks of teaching. The reason for separating episodes in watching video recordings with observation notes was that transposing the spoken word into a text might have an influence on judgment and interpretation (Marshall & Rossman, 2006). Even if my analysis heavily depended on verbal statements rather than non-verbal behavior, selecting episodes in the video recordings reduces the risk of missing any nonverbal behavior during all events (Erickson, 2006). In fact, because I already had the observation notes at this point, I referred to it. I then formatted the transcript into a table with each row corresponding to a turn of talk, by the teacher educator or teachers. Then, I divided the observation notes and inserted them into the reviewed transcripts. In the end, I had one document for each lesson that includes detailed information about the class, descriptions and brief interpretations, marked episodes, and transcripts with a tabulated form.

I also created description notes about the curriculum materials that paralleled what I wrote about the video recordings. I described each part of the curriculum materials for each lesson. Moreover, as with my approach to the observation notes of the video recordings, I added what I noticed in the curriculum materials, such as tasks that teacher educators are expected to offer, topics that the class would discuss, demonstrations that teacher educators might give, possible examples that teacher educators use and possible questions that teacher educators ask, and any activities and comments that seemed to be related to the research questions. I then captured the images of all parts of the curriculum materials, inserted them into the description notes, and tabulated them. Finally, I made one document for each lesson of the materials that included descriptions and brief interpretations and images of the materials with the tabulated form.

At this point, I had the documents from the video recordings and the materials for use in my analysis.

Coding the data. Through watching the video recordings at least four times and describing the materials, I engaged in an initial coding of the data in order to revise and improve my preliminary categorization scheme from the literature review. The categorization scheme from the literature review, as shown in Appendix A, showed several kinds of distinctive work of teaching with examples, but the scheme revealed none of their components. I looked for the entire set of events that teacher educators engaged in that seemed related to working on MKT in instruction, such as types of mathematical work of teaching and types of mathematical issues that teacher educators provided and the class discussed and tasks of teacher educators that seemed intended to manage the activity to teach MKT. I also looked for moments that the lesson lost the track of MKT and identified tasks that might have helped avoid these issues in the literature, the materials, and ways that were found in another teacher educator's classes.

Through this process, I reorganized, revised, and added categories and subcategories of the episodes. Episodes which are utterances from the classes were coded using a constant comparative method of coding (Strauss & Corbin, 2008). I had to frequently rename the categories and subcategories as I gained more examples and as the mathematical work of teaching and the mathematical issues being described in each became more apparent over time. From this initial pass through the data, the categorization scheme evolved and I formulated clearer notions of my analytic distinctions. From the analysis of the data, I pulled out the main components and an overall structure of the mathematical work of teaching and knowledge about mathematics as well as major features of teacher educators' attention and their challenges in teaching MKT and placed them into their own particular category. I also had an extensive list of episodes for each category and subcategory.

Clarifying interpretive commentary. I then needed to confirm that my categories were sufficient to capture the data and further scrutinize the content which had been worked on in the instruction. This allowed me to go over each category with a fine-tooth comb and identify its features. I did this through a process of *focused coding* of the subset of the data. Focused coding means using the most significant and/or frequent earlier codes to sift through large amounts of data (Charmaz, 2006, p. 57). Focus coding helps determine what initial codes make the most analytic sense in order to categorize the data incisively and completely. In terms of focused coding, I analyzed one lesson of the materials and two lessons of the two teacher educators from the video recordings: a lesson of hearing definitions in children's talk from the materials, Matthew's lesson on evaluating definitions and Emily's lesson on explaining why the units digit rule works.[14] I chose the following lessons because they included different activities with regard to the mathematical work of teaching. They helped assure that my coding procedure would apply across all of my data. Specifically, Matthew's and Emily's lessons relied on the materials. However, they had different styles of whole group discussion, managed different challenges in instruction, and modified the materials for their own instruction. Moreover, I thought these two lessons showed the largest spectrum of the MKT as worked on in instruction in the data set. In the observation notes, I added columns and coded the three lessons corresponding to the categories and subcategories in the coding scheme. I analyzed them turn by turn, identifying examples of mathematical work of teaching and mathematical issues in each category and tasks of teacher educators in terms of challenges and attentions. I also made notes about how the examples are related to one another. After coding the entire lessons, I went back through the table, column by column, and tried to sort each example I had identified into my categorization scheme. If an example did not fit, I revised and reorganized the categories and subcategories to accommodate it. My analysis of the lessons in terms of MKT as content in mathematics teacher education was facilitated by the following questions: (1) What might teacher educators be doing and what are they doing at this moment to teach MKT?; and (2) What MKT, with respect to the mathematical work of teaching and mathematical issues, is being worked on at this moment? I also wrote down any ideas related to MKT and any possible tasks of teacher educators to meet their challenges. I worked row by row to fit the example moves into the categorization scheme. Over time, my categorization scheme became stable.

[14] All teacher educators' names are pseudonym.

I then coded the remaining lessons with the categorization scheme that I found from the focused coding procedure. In fact, I proceeded to code the lessons in a similar way, but I could speed up because at this point I was very familiar with coding the data and my categorization scheme had a stable form. I added some categories and subcategories through the analysis, but there was no dramatic transformation. Finally, I elucidated the interpretation that precedes and follows an instance of particular description and identified theoretical discussion that points to the more general significance of the patterns identified in the episodes that are reported (Erickson, 1986).

During this stage, I also indicated whether teacher educators' instructional moves went well in terms of MKT. My research is not evaluating who teaches MKT well or how well it is done. However, I used this record to specify the issues and the intentional attention in managing tasks to teach MKT, which is related to the research question about the task of teacher educators, discussed in more detail later. Furthermore, after I watched the entire lessons several times, I concentrated on coding the parts of the lesson, specifically in the whole group because I did not seem to be identifying any new types of MKT as content in the interactions with individual teachers that I had not already captured in the whole group segments. However, I coded the teacher educators' instructional moves during the individual or small group segments when they were clearly observed (e.g., walking around the classroom, taking notes, etc.). This is because instructional moves are extended within one lesson and influence subsequent moves.

Evolving organization and representation of framework. My analysis resulted in hierarchically organized categories and subcategories with specific examples from the data. As its size grew increasingly larger, I needed to have a reasonable structure to represent my findings to both the audience and myself. In fact, I had continued to reorganize the lists of categories into a conceptual framework both during and after coding. To respond to the overall question, I here account for the several important events in the evolution of the framework from the findings of the analysis of the data.

A first and major event to evolve findings from the data for conceptualization was capitalizing on both the analysis of the empirical data and the literature review. The basic structure and main elements of the overall framework were found in the empirical data. I will elaborate the structure and its elements in Chapter 6. However, the analysis of the data revealed two different layers to the mathematical work of teaching as content to teach MKT, as shown in *Figure 6.2*. Moreover, findings from the data showed three kinds of zoomed-out mathematical work of teaching, as shown in Table 6.1, and two dimensions in zoomed-in mathematical work of teaching, as shown in *Figure 6.3*. However, it was insufficient for a refined framework. The most critical inclusion based on solely the literature concerned the long periods in the continuum of curriculum. In their teaching practice, teachers are expected to consider units or chapters of instruction or an entire school year (Lampert, 2001). Finally, *making a decision for a long period* was included as a component of the mathematical work of teaching. Then, details were extrapolated from the details of *making a decision for one lesson* that were analyzed and elaborated from the data. For example, from the data, I found *making mathematical and pedagogical judgments about students learning throughout a lesson* as one component in *making a decision for one lesson*. It was applied to *making a decision for long a period*, such as *making mathematical and pedagogical judgments about students learning throughout a unit or a chapter of instruction, one semester, and one school year*.

Moreover, as I set up the basic structure of the framework, based on the data, the logic was formulated in elaborating the framework itself. For example, from the data, I found “constructing proofs based on certain definitions or axioms” and “creating representations with particular limitations” to be components of the mathematical work of teaching. However, I found nothing related to “creating algorithms, rules, and procedures within a certain limitation,” such as creating an explanation of how to subtract with whole numbers. The logic of combining of “creating” and “a certain limitation” with definitions or axioms and proofs confirmed the existence of “creating algorithms, rules, and procedures under a certain mathematical limitation.” Thus, the framework included “creating statements and examples of algorithms, rules, and procedures under a certain mathematical limitation.” Logic that was formulated in putting together the basic structure of the framework elaborated the findings from the data and finally played a critical role in developing the conceptualization for the overall research question.

A second major change related to the structure of the mathematical work of teaching. When I made the preliminary categorization scheme about the mathematical work of teaching from the literature review as shown in Appendix A, I conceived of it as having six main and independent categories. During subsequent analyses, I found different types of the mathematical work of teaching were worked on together in instruction within the context of one lesson. In other words, in the middle of the main activity of a lesson, there appeared to be a concentration of one or several kinds of mathematical work of teaching. At the end of the lesson, however, they were nested within a bigger mathematical work of teaching. This meant that the mathematical work of teaching as content has different layers and work together as content in instruction. Ultimately, I differentiated the mathematical works of teaching into the two layers and reorganized it.[15]

A third major change had to do with the structure of mathematical work of teaching. In the preliminary categorization scheme about the mathematical work of teaching from the literature review and observation memos as well as in the initial analysis of the data, even after recognizing the existence of the two layers of the mathematical work of teaching, my coding scheme of mathematical work of teaching was just the list, including such items as recognizing, interpreting, evaluating, selecting, modifying, and constructing. During subsequent analyses, I acknowledged that mathematical objects in teaching are one axis to determine the mathematical work of teaching. The curriculum materials that teacher educators used in this study focused on mathematical definitions, but, in their instruction, various mathematical objects were worked on. Moreover, the curriculum materials included and used various mathematical objects. Therefore, I extended my list into the two dimensional table.

A fourth major change related to domains of MKT. For a more concise and consistent framework I had to keep comparing and contrasting contents of the framework from the data and the literature review and differentiating them in terms of the domains of MKT. I applied the definitions of each domain of MKT by Ball et al. (2008), summarized in Chapter 2. This examination helps identify the features of mathematical work of teaching and relations between the mathematical work of teaching and domains of MKT. However, I recognized certain patterns of domains of MKT in the mathematical work of teaching. For example, I found CCK, SCK, and KCS in the group of recognizing

[15] One layer is called "zoomed-in mathematical work of teaching," and the other is called "zoomed-out mathematical work of teaching."

and articulating as one kind of zoomed-in mathematical work of teaching, but I was undecided as to whether KCT, KCC and HCK were included here. As another example, any of the zoomed-out mathematical work of teaching seemed not to involve CCK. I eventually realized that not every domain of MKT is equally associated with a task of teaching and I removed some domains of MKT in several categories of the mathematical work of teaching.

A fifth major change had to do with the relationship between the mathematical work of teaching and knowledge about mathematics. I believed at the time of the initial analyses that the mathematical work of teaching as activity to teach MKT *induces* learning knowledge about mathematics at the end of a lesson. This was because I often observed that teacher educators explained some features of practice in mathematics at the end of a lesson. However, even though instructional moves to explain features of knowledge about mathematics occurred at the end of a lesson, actually these features were planned and worked on throughout a lesson. Therefore, the relationship between the mathematical work of teaching and knowledge about mathematics existed independently and functioned simultaneously as content in instruction of MKT. In the end I realized they functioned as the two parts to build a foundation that can inform a curriculum of MKT in teacher education.

Through the framework's evolutions, I reviewed the coding schemes by applying the research question. As categories of meaning emerged, I searched for those that have internal convergence and external divergence so the categories would be internally consistent but distinct from one another in overall episodes (Marshall & Rossman, 2006). Through this process, I renamed, reorganized, consolidated and reduced categories and subcategories and reconstructed the framework as well. Finally, I confirmed my categorization scheme and made a conceptual framework. Appendix B explains how I analyzed the data and how I built the conceptual framework out of episodes captured in the data. The final representation of this conceptual framework is presented in Chapter 6.

3.3.2 Investigating research question: Developing a framework for knowledge about mathematics

Another research question asks: as teacher educators and teachers work on MKT in teacher education classrooms, using curriculum materials that address mathematical definitions in teaching, what kinds of mathematical issues are revealed? I concentrated my investigation on the MKT that teacher educators emphasized in terms of disciplinary mathematics in order to encourage teachers to thoroughly understand and judge mathematical definitions and recognize the roles and use of mathematical definitions in mathematics and teaching mathematics.

As described above, my analyses on identifying what mathematical issues were highlighted in instruction of MKT began with the research question about the mathematical work of teaching; in other words, my investigation on the mathematical work of teaching and mathematical issues went together because I thought that they were the foundations necessary to build a conceptualization that contributes curriculum for teaching MKT in teacher education.

After the literature review, I made the observation notes of the video recordings and the description notes of the curriculum materials with some ideas about the teacher educators' performances in terms of knowledge about mathematics. When I coded the data, I looked for the mathematical issues in the full set of lessons that teacher educators provided. Through the process of coding, I realized that mathematical issues that I noticed in the data and the literature review go together with Ball's (1990) notion of "knowledge about mathematics" that is about the nature of knowledge in the discipline. I reorganized, revised, and added categories and subcategories of the episodes, pulling out the main components of knowledge about mathematics involved in teaching MKT and placing them into their own particular category. Through a process of focused coding, I found the categories that were internally consistent but distinct from one another across all episodes. Moreover, the literature review, in particular the attributes of mathematical definitions in disciplinary mathematics as well as mathematicians' writings about their research, added reasonable explanations of the components of the framework related to knowledge about mathematics in teaching MKT. The literature review also compensated for what the findings from the analysis of the data had not illustrated. Finally, I created structures and conceptual categories, which are discussed as knowledge about mathematics in Chapter 6.

3.3.3 Investigating research question: Conceptualizing what challenges teacher educators face and what teacher educators need to pay attention to in teaching MKT

The other research question asks: as teacher educators and teachers work on MKT in teacher education classrooms, using curriculum materials that address mathematical definitions in teaching, what are some of the challenges that arise? I focused my investigation on identifying what challenges teacher educators face in teaching MKT and on illustrating what teacher educators need to attend to in order to carry out such teaching.

As described above, my analyses on identifying how teacher educators manage their tasks of teaching MKT and what teacher educators pay attention to in teaching MKT began as writing observation notes in terms of teaching moves to manage activities to teach MKT. I concentrated on specifying the tasks of and the dynamics involved in teaching MKT so that the framework evolved, and I pulled out features of teacher educators' attention and their challenges in teaching MKT and placed them into their own particular category.

I coded these problems in the lessons of video recordings using the processes described above. In the analyses, I used factors of the dynamics of teaching in mathematics teacher education from the literature review, shown in detail in *Figure 2.2*. They showed elements that teacher educators might consider and perform in terms of an instructional triangle in teacher education. It also offered language to conceptualize that which would be difficult in teaching MKT and what teacher educators need to pay attention to in teaching MKT. Moreover, the threshold for deciding teacher educators' challenges in teaching MKT was set in Ball and Bass's (2003b) notion about mathematics in and for teaching. That is, mathematics as it is found in the student curriculum and aspects of knowing and doing mathematics that are less visible in the textbook (p.8). That threshold was also defined in Ball, Thames, and Phelps' (2008) definition of MKT—mathematical knowledge, skill, and habits of mind entailed by the work of teaching (p.398). In other words, I used these authors' notions and definitions to empirically analyze the data. I both decided whether teacher educators worked with MKT or non-MKT and identified how they managed interactions between teachers as learners and MKT as content. Furthermore, I used these authors' notions and definitions to theoretically conceptualize tasks of teacher educators in terms of their challenges and attention in teaching MKT.

The problems overlap and happen simultaneously in instruction. On the other hand, managing a particular instructional move could be used to handle various problems. In some cases, a particular move could be interpreted as an approach to various problems. I identified any instructional moves which maintained instruction in terms of MKT through the coding process. I focused on capturing all the tasks of teacher educators in terms of their challenges. I also concentrated on denoting whether any instructional move was toward MKT or not. After I had coded all the data, I gathered tasks that the teacher educators performed to sustain the track of MKT in their instruction and also gathered tasks that resulted in a non-MKT track. I looked through these and identified the patterns that became challenges teacher educators might face in managing the tasks of teaching MKT. I noted the factors that would be attended to by teacher educators in and for teaching MKT. I discuss them in Chapter 5.

3.4 Limitations of Study

There are a number of limitations in this study. Because of the data in this study, there are several limitations. Within the context of mathematics teacher education in the United States, I seek to specify challenges involved in teaching MKT and build knowledge about how to organize the curriculum for teaching MKT based on a literature review and records of practice. Therefore, the tasks of teaching MKT may be different in the context of mathematics teacher education in other countries. This dissertation is also limited by the video recordings of nine teacher educators' lessons, which I used within this context of mathematics teacher education in the United States. I did not gather data about teachers as learners in the classrooms, and I, thus, did not know about the ethnic diversity of the classrooms. While teaching is a complex phenomenon and can be affected by diverse economic, social and ethnic issues of participants in classes, this kind of perspective is outside the scope of my analysis because I did not ask and use teachers' demographic or social and economic information. Therefore, my framework might miss some aspects of challenges involved in teaching MKT or some elements of MKT as content in teacher education that were not visible in the settings I observed. There may be, moreover, differences in teaching MKT in contexts other than those I studied, such as secondary teacher education or even elementary teacher education using different curriculum materials.

Other limitations are caused by the materials that focus on mathematical definitions as topics to teach MKT. While I have and clarify rationales of focusing on mathematical definitions for teaching MKT in this dissertation, mathematical definitions are just one possible topic that teacher educators can use for teaching MKT. This fact might influence my conceptualization for the teaching of MKT that can be enacted in the context of mathematics teacher education. Even focusing on mathematical definitions, my conceptualization and particularization of MKT implemented within this context is limited because of the features of definitions of an even number, which is the main topic in the materials that the teacher educators use. However, I attempted to use the data to add the detailed examination of the phenomena for teaching MKT and explore substantial components of MKT as the content in mathematics teacher education.

In regard to methods of analysis, this research is based on my observations, description and analysis of the literature and records of practice. Even though I clarified my rationales and reasons at each stage, this dissertation is limited to the threads I recognized in the literature and the data. While limited in scope, this study is an important step. It could be used for providing the recommendations for developing a curriculum for teaching MKT based on results of research proposed here and, in future studies, elaborating pedagogical supports for teacher educators who teach MKT.

I present the results of my analyses in the next three chapters. Within these chapters, I articulate the tasks of and challenges involved in teaching MKT and build knowledge about how to organize the curriculum for teaching MKT. In Chapter 4, I closely look at instruction of MKT from three lessons of the materials and the video recordings to investigate the complexity of teaching MKT and to begin to develop the MKT framework that can inform a curriculum in mathematics teacher education. In

Chapter 5, I use the data to illustrate challenges teacher educators confront in managing activities and to clarify the kinds of attention needed for teaching MKT. In Chapter 6, I investigate the data to identify components of MKT as content in mathematics teacher education.

CHAPTER 4
THE COMPLEXITY OF TEACHING MATHEMATICAL KNOWLEDGE FOR TEACHING

4.1 Introduction

As discussed in Chapter 1, this research is developing a conceptualization that can inform a curriculum for teaching MKT in mathematics teacher education. The conceptualization is, first, to identify tasks of teacher educators in and for teaching MKT in order to explain in what ways MKT might be worked on and, second, to specify the framework for curriculum to teach MKT in order to demonstrate what MKT might be worked on when teacher educators help teachers develop MKT. Teaching MKT includes more than isolated mathematical or pedagogical facts or even sets of isolated mathematical ideas or cases of teaching. MKT as content in mathematics teacher education is at least as complex in nature as both the work of teaching and the discipline of mathematics and the management of various mathematical issues that teachers have as learners.

This chapter provides detailed instances from the data of materials and scenes of instruction by two teacher educators: hearing definitions in children's talk from the materials, Matthew's lesson on evaluating definitions, and Emily's lesson on explaining why the units digit rule works. There are two purposes for using these examples: first, using examples stresses the complexity of MKT as content as there are multiple and intricate tasks which have diverse grain sizes and various emphases that teacher educators simultaneously, continuously, and immediately consider MKT as content in teacher education; second, this use of examples establishes a more analytic conceptualization for teaching MKT with respect to tasks of teacher educators and in terms of the mathematical work of teaching and knowledge about mathematics that will be presented in Chapters 5 and 6. Coding a certain moment of instruction tends to cut it into pieces. The issue is that codes of moments should not be distorted within instruction but harmonized. The current chapter plays a role in creating, from the data, coding schemes and, finally, in conceptualizing them so that they go together and become entrenched in the instruction of MKT. Consequently, these examples are referenced throughout the results section that follows. Methodologically, as described in Chapter 3, these three lessons were used in the process of focused coding (Charmaz, 2006). These three lessons involve different activities in terms of the mathematical work of teaching and different mathematical issues as MKT.

The first example of a lesson plan from the materials addresses hearing definitions in children's talk (Lesson 1). This lesson uses a short video clip as a record of practice that shows a third grade students' discussion about definitions of an even number. The second example is Matthew's lesson on evaluating definitions (Lesson 3), and the third example is Emily's lesson on explaining why the units digit rule works (Lesson 5). Both teacher educators closely follow the materials.

4.2 Lesson from Curriculum Materials on Hearing Definitions in Children's Talk

As mentioned in Chapter 3, the materials of *mod4* have six lessons. Each lesson is similar to a lesson plan in the teacher's manual of a curriculum. Like other lessons, this lesson—hearing definitions in children's talk—starts with an overview that contains goals and the sequence of activities as shown in *Figure 4.1*.

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mod4

using definitions in learning and teaching mathematics

session 1: hearing definitions in children's talk

Overview

Length: 65 minutes (with optional 15-minute activity)

Description: This session provides teachers[1] with opportunities to practice interpreting children's mathematical talk and recognizing the definitions implied in children's reasoning through viewing and analyzing a video episode from a third-grade classroom.

Goals:

- To learn about the nature and roles of definitions in mathematics and mathematics teaching
  - To understand and appreciate that definitions add precision to mathematical communication and are basic building blocks for mathematical reasoning
  - To compare, and reconcile if necessary, alternative definitions
  - To move from the common view that there is a single, correct, delivered mathematical definition to be memorized toward a more organic view that includes the development of mathematical definitions as they emerge from genuine need and continue to evolve through mathematical inquiry and are collectively fashioned over time
- To learn to hear and interpret children's (sometimes implicit) uses of mathematical definitions in their reasoning
  - To develop skills that help children use definitions to ground their mathematical arguments
  - To attend to precision and clarity in mathematical talk
  - To see that definitions play an important role even in elementary mathematics
- To develop a desire to learn more about definitions in mathematics as well as their use and development in teaching

Sketch of activities:

1. Brief overview: *Why Work on Mathematical Definitions?* (~5 minutes)
2. Set up the video background and viewing focus (~10 minutes)
3. Video viewing: *SeanNumbers-Ofala* (~10 minutes) (+ optional 15-minute activity)
4. Reflection (~10 minutes)
5. Whole group discussion of video (~25 minutes)
6. Wrap up (~5 minutes)

[1] The term "teachers" is used throughout to refer to both prospective and practicing teachers (i.e., students in preparatory courses as well as participants in professional development contexts).

Figure 4.1 Lesson 1- Overview

After showing the estimated time of the lesson, 65 minutes, the description specifies the lesson's two main mathematical work of teaching–interpreting children's mathematical talk and recognizing the definitions that are implicit in their reasoning–using a record of practice from a third-grade classroom. The goals of this lesson are depicted in detail.

The goals involve possessing mathematical sense and skills entailed in mathematics teaching. The first main goal is gaining an appreciation about the role of definitions. They provide precision and support reasoning in both mathematics and mathematics teaching. Ultimately, this goal concerns developing mathematical awareness related to definitions both in disciplinary mathematics and in and for teaching. It also includes a couple of specific tasks, such as comparing and revising definitions. The second main goal is practicing hearing and interpreting children's definitions of an even number in their reasoning. This goal deals with the moment-to-moment work that is quite distinct from work that is relatively large in scope. Such would include planning a lesson or a semester, and determining what mathematics is involved in using a particular representation. In this study, the relatively small-sized work is called the zoomed-in mathematical work of teaching, and the relatively large-sized work is called the zoomed-out mathematical work of teaching. These two goals show, on the one hand, that MKT in the context of teacher education would have two features: (1) objects that both are generally accepted by researchers in disciplinary mathematics and work in teaching and (2) specific mathematical work that teachers carry out in teaching. In terms of the first feature, mathematical researchers do not often say them explicitly but follow general practice. For example, definitions should be precise, emphasized by Aristotle, and definitions, far from fixed, should evolve over time, highlighted by Lakatos. This awareness exists in disciplinary mathematics, and MKT includes mathematical awareness that would critically influence teachers' mathematical decisions in and for teaching. The second feature—specific mathematical tasks in teaching—is what teachers actually perform in teaching. Hearing and interpreting mathematical issues from children's talks is specific work that teachers are supposed to carry out in teaching. MKT as content in teacher education involves the particular mathematical work of teaching. On the other hand, goals play a significant role in making instructional decisions (Gorham, 1999). Because of this, teacher educators will use them to manage and determine the direction of instruction concerning teacher educators' help for interaction between teachers and content teacher education. How to manage interaction in teaching MKT is an important issue to teacher educators who actually use MKT as objects for teaching. The overall structure of this lesson is briefly outlined.

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session 1: hearing definitions in children's talk

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Preparation

To do:

• Test the animation of the slides	Talk through the slides to get a sense of whether the animation fits with your delivery.
• Make copies of the transcript	The transcript will be used to aid the viewing and discussion of the video. It would be best for each person to have his/her own copy on which to take notes during and after the video viewing.
• Test the video on your computer/DVD player	Make sure the SeanNumbers-Ofala video will play on your computer or DVD player. You may need to install software or arrange to borrow a different computer if the video does not open or play smoothly (a common problem with computers that have older processors or not enough memory). Please see the DVD instructions guide on the *fall2009 files* disc. NOTE: The mod4 project can also make other video formats available to you; please contact us at *mod4@umich.edu* to request assistance.
• Test the video set up in the room you will be using	The video image will need to be displayed with a projector or a large-screen television. Make sure the video and subtitles can be seen from any seat in the room. ***The sound for the video will need to be amplified with speakers.*** Make sure you can get the sound loud enough to be heard by everyone. It is important to note that the children in the video are more difficult to hear than the teacher. Therefore, when testing your audio, you should use a sample of the video where the children are talking for your tests. Also, please note that the sound will need to be slightly louder when the room is filled with people than when it is empty.
• Set up a public note-taking space that can be used to record teachers' ideas during the whole group discussion	During this session, the group will be discussing the video and sharing their observations of the different ways the children are thinking about even and odd numbers. It will be useful to create a public record of this discussion – on a projected computer, pad of chart paper, overhead projector, or blackboard/whiteboard – so that comments can be revisited, revised, and connected. One advantage of using a computer or chart paper in this case is that it might be useful to save these notes for later reference when doing other work on definitions of even/odd numbers.

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Figure 4.2 Lesson 1- Preparation

Figure 4.2 shows what teacher educators need to technically prepare before a class in terms of materials and facilities. Materials and facilities are not directly involved in interactions between teacher educators and teachers or between teacher educators and teachers. Nevertheless, they are important elements because the efficient use of materials and facilities improves teacher educators' teaching and enhances discussions in a lesson. As materials, the transcript of the video clip is expected to be copied and distributed to facilitate teachers' note taking and enhance the discussion of the video clip. Moreover, teacher educators will, on their computers, test the animations of the slides provided in the *mod4* materials. In terms of playing the SeanNumbers-Ofala video, teacher educators will try out whether the video clip plays on the computer or DVD player with proper sound and subtitles while being projected onto a large screen. To record and share teachers' ideas and revisit, revise, and connect them through a whole group discussion, it is important to have a public note-taking space. This could be a projected computer, pad of chart paper, overhead projector, blackboard or whiteboard. When and how teacher educators use a public note-taking space is indicated specifically in the description of the activity.

Optional resources for facilitator:

• Template for taking notes	A notes template for Microsoft Word is provided for taking publicly shared notes on the discussion of the SeanNumbers-Ofala video.
• Sample discussion notes on the SeanNumbers-Ofala video	The sample discussion notes document is intended to provide facilitators/instructors with a concrete image of what the discussion notes might look like as well as the types of comments that might be raised as teachers discuss the SeanNumbers-Ofala video.
• Mathematical observations on the SeanNumbers-Ofala video	The mathematical observations document contains commentaries on some of the mathematical issues that can be seen in the SeanNumbers-Ofala video and might be useful for facilitators/instructors to consider when preparing for this session.
• Teaching observations on the SeanNumbers-Ofala video	The teaching observations document contains commentaries on different facets of the work of teaching that can be seen in the SeanNumbers-Ofala video and might be useful for facilitators/instructors to consider when preparing for this session.

Figure 4.3 Lesson 1- Optional resources

Figure 4.3 briefly introduces what optional resources the *mod4* materials offer in this lesson to teacher educators. Although all these resources are placed in the last part of the materials, I raise and describe each of them here.

sessionname
sessiondate

Discussion of the SeanNumbers-Ofala Video

What do children in this class understand about what an even or an odd number is?

- **What definitions do they have for "even number"?**
- **What other observations can you make about their knowledge of even and odd numbers?**

Notes

Figure 4.4 Lesson 1- Template for taking notes

The first resource is a template for a public note-taking space, which as shown in *Figure 4.4* can be used on a projected computer. This template includes the focus questions that would be suggested and used in the discussion. The first question asks teachers to recognize and articulate what definitions students use in the video clip. In fact, this is a task teachers do very often and in a moment of teaching. Again, MKT, as content in teacher education, includes the mathematical work of teaching, and this question asks one of it. The second question asks teachers to gather their observations, rather than interpretations, related to the definitions of even and odd numbers. This activity obviously assumes that teachers know a fact of an even number, too. In other words, facts of an even number would be identified in the enacted phase, and this identification is a part of MKT.

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Sample Discussion Notes
This document portrays the kind of discussion notes that might be generated during a class discussion of the SeanNumbers-Ofala video. Additionally, it captures the types of comments that may come up when viewers discuss this video excerpt. We provide these notes to help the facilitator anticipate the kinds of responses that the video may evoke.

Discussion of SeanNumbers-Ofala Video

What do students in this class understand about what an even or an odd number is?

- **What definitions do they have for "even number"?**
- **What other observations can you make about their knowledge of even and odd numbers?**

Notes

- Sean seems to understand even and odd numbers at least somewhat because he says that 3 is odd and 2 is even (#15-16, 34-37).
- Sean is talking about groups of 2 (#106-107, #115-117), but the working definition seems to deal with two equal groups (#64-66).

Figure 4.5 Lesson 1- Sample discussion notes (parts)

The second resource is the sample discussion notes, as shown in *Figure 4.5*. It illustrates the various comments that teachers might generate in response to the focus questions during a class discussion of the SeanNumbers-Ofala video. This is to help teacher educators anticipate possible responses and mathematical issues and have a concrete image of the notes that might be recorded in the discussion. These notes also include what students used which definitions with line numbers of the transcript, which for this lesson comprises the main mathematical work of teaching. This resource would support teacher educators both in their understanding of the discussion of the video clip and in their helping teachers interact with content.

Mathematical Observations on SeanNumbers-Ofala Video
There are many noticeable mathematical issues at play in the SeanNumbers-Ofala video segment. This document offers some commentaries on particular mathematical issues as a resource to support and develop instructors'/facilitators' own insights into the video. This information is not intended to be presented to teachers but to serve as background material for the teacher educator/professional developer.

Mathematical Observations on the SeanNumbers-Ofala Video

Mathematical reasoning
On the surface, this classroom episode depicts some rather unusual teaching and learning about even and odd numbers. What is perhaps most distinctive about the lesson, however, are the practices of mathematical communication and collective reasoning around the topic and the resources (notably mathematical definitions) that are mobilized to support this collective reasoning. If one were to ask, "What is the mathematics that these students are learning in this lesson?" an important part of the answer must be these practices of mathematical reasoning.

The focus of the lesson (though not planned by the teacher) is the provocative claim by Sean that the number 6 can be both even (it is *two* 3s) and also odd (it is *three* 2s). Essentially the whole class challenges his claim that 6 is odd, yet Sean holds tenaciously to his idea. The teacher guides the class as they collectively use mathematical reasoning to try to resolve this conflict.

The norms of classroom mathematical discourse
The teacher orchestrates the students' discourse but carefully reserves the responsibility of mathematically resolving disagreements for them. Students are expected to articulate mathematical claims and to propose mathematical justifications for them (lines #5-13, 90-96, 115-117, 165-185). Student pronouncements are often followed by words like "because" (#8, 25, 106, 132). Students are

Figure 4.6 Lesson 1- Mathematical observations on the SeanNumbers-Ofala video (parts)

The third resource is the commentary of the mathematical observations on the SeanNumbers-Ofala video, the start of which is shown in *Figure 4.6*. Because of the complexity of the phenomena of teaching and learning mathematics, it is very hard to immediately and concisely recognize and specify mathematical issues in the mathematical discussion from the video. This resource helps teacher educators identify the SCK, which is one of domains in MKT, involved in the video: mathematical reasoning, the norms of classroom mathematical discourse, and mathematical analysis of the claims and arguments. Moreover, it explains which mathematical facts, mathematical awareness, and mathematical values are included in the SeanNumbers-Ofala video. For example, to understand Sean's idea in the video clip, it is critical to be able to differentiate two 3s and three 2s. Moreover, it explains how children investigate an even number, which is included in the awareness of exploring an object in disciplinary mathematics. The norms of classroom mathematical discourse explain how mathematics has been built on substantial construction and extend its solid foundation, which makes disciplinary mathematics valuable.

Teaching Observations on the SeanNumbers-Ofala Video
Some viewers of the SeanNumbers-Ofala video segment have difficulty seeing the work of teaching captured in this clip. This document offers some commentary on the teacher's work that can serve as a resource for instructors'/facilitators' analyses and provide insight into the video segment. It is not intended to be presented to teachers but to serve as background material for teacher educators/professional developers.

Teaching Observations on the SeanNumbers-Ofala Video

There are many lenses through which one might view the classroom activity in this video. A focus on *students* could support understanding the ways in which third grade students can learn to reason and represent mathematical ideas. A focus on *mathematics* could support deeper understanding of fundamental aspects of mathematical content and practices. In this commentary, *the work of mathematics teaching* is the focus. Of course, the work of teaching interacts with students and mathematics. By attending to what the teacher is doing in the episode, viewers can gain a sense of what it might look like or sound like to teach while paying serious attention to students' ideas and showing rigorous respect for the discipline. The main focus here is on three aspects of the work of teaching:

1. Using students' ideas as the basis for mathematical discussion
2. Establishing and justifying mathematical ideas
3. Using and supporting the use of representations in mathematical discourse

These aspects are worthy of further consideration in this episode because there is a strong temptation to attend to the effects of the teaching – namely the vivid engagement of the children or the interesting mathematics that is in play – and lose sight of the work of the teacher that produced them. Furthermore, these three aspects also relate to other important parts of teaching such as attending to the curriculum and establishing a generative classroom environment.

Using students' ideas as the basis for mathematical discussion
Capitalizing on "teachable moments" requires a teacher to recognize that a student's interest or confusion is worthy of instructional attention and also to have the skills and resources to effectively utilize the student's idea or question. The discussion in the video is initiated by Sean's statement that the number 6 can be both even (it is *two* 3s) and also odd (it is *three* 2s) (line #11-12). This idea is put forward

Figure 4.7 Lesson 1- Teaching observations on the SeanNumbers-Ofala video (parts)

The final resource is the commentary of the teaching observations on the SeanNumbers-Ofala video the start of which is shown in *Figure 4.7*. It serves teacher educators to acknowledge the work of teaching contained in the video clip. The focus questions are about identifying how children define an even number. Nevertheless, all the viewers would recognize or be curious about the teacher's work in the video clip. Moreover, because it is not easy for teacher educators to clearly acknowledge tasks of teaching that the teacher performs in the video clip, this commentary also assists teacher educators. The resource specifically elaborates on the tasks of teaching involved in the video clip: using students' ideas as the basis for mathematical discussion, establishing and justifying mathematical ideas, and using and supporting the use of representations in mathematical discourse. The commentary specifies which mathematical work of teaching is mainly carried out in the video clip with detailed explanations and evidence.

Finally, the two kinds of commentaries, mathematical issues and the teacher's work shown in the SeanNumbers-Ofala video as shown in *Figure 4.6* and *Figure 4.7*, are intended to develop teacher educators' insights into the video clip. Again, using a video clip in the context of teacher education requires teacher educators' careful analysis and full understanding of it. These resources support teacher educators to recognize various issues that teachers might initiate and be curious about. Furthermore, it helps teacher educators differentiate between mathematical issues and pedagogical issues involved in the SeanNumbers-Ofala video, which eventually demonstrate various aspects of MKT. I move to the activity section.

Activities

These activities are organized around the focused viewing and analysis of a video segment of a third grade class in which the children are reasoning about even and odd numbers. The video provides a context for immersing the teachers in one of the fundamental tasks of teaching – hearing and interpreting children's mathematical thinking – in this case, focusing particularly on their mathematical language and the definitions implicit in their mathematical arguments. Although there is a wide range of interesting things that can be observed in the video, we have found that unless the discussion is carefully directed and shaped by the facilitator, viewers often miss the mathematics or do not engage in the kinds of *mathematical* work that teaching requires – such as listening to children's mathematical explanations, probing and analyzing their ways of reasoning, assessing their uses of language, etc – that is the aim of this session.

Figure 4.8 Lesson 1- Brief sketch of activities

Before presenting the specific activity, the materials briefly sketch the central activities, their aims, and the record of practice, as shown in *Figure 4.8*. It also discloses the challenges that teacher educators might confront in doing this lesson. This sketch helps teacher educators have a pseudo-image of a class and anticipate what they need to pay attention to in the teaching. Thus, this sketch serves teacher educators' knowledge for teaching MKT.

Now, the activity detail section, which can be read as a lesson plan, starts. It provides a detailed description of the activities in a two-column table. The left column – detailed description of activity – provides the sequence of activities and details about types of class setting, examples, and elaboration of the class materials. This description particularly explains activities, ways for representation in a class, pacing, instructional moves, and knowledge, which are applicable to all the lessons of mathematical definitions. The right column – comments & other resources – includes notes about the activities, such as common learner conceptions or misconceptions, links to later parts of the lesson or to other lessons or resources, demonstrations of mathematical fact, and so forth. These comments help teacher educators recognize mathematical knowledge and knowledge in terms of the practice of teaching that are involved in the provided tasks and activities. Knowledge about the task of teaching MKT is also described. The two-column plans also show where activities in the lesson shift. Times are provided to give a sense of the length of a particular activity and also its relative importance.

1. Brief overview: *Why Work on Mathematical Definitions*? (~5 minutes)

Detailed description of activity	Comments & other resources
These central questions guide the work for the Using Definitions stand. This session will touch on the first three bullets; other sessions will focus on other combinations of these questions. You may want to start this work on definitions by posing these questions to teachers and having them begin to think about their initial responses as a way to orient them to the work they will be doing. (This is particularly useful if you will be doing all or most of the sessions but may not be appropriate if you will only be working on Session 1.) Guiding questions for our work on definitions • What do definitions *do* in mathematics, and why do they matter? • Why do definitions matter in teaching elementary school? • How do definitions matter for mathematical reasoning? • What makes a good mathematical definition? You might say something like, "Today we will begin with four key questions that will set up and motivate our work over the next few sessions." Read (or have the teachers read) the questions and give them five minutes to write individually about their initial responses to these questions. The purpose of this is to simply capture their initial thoughts about these questions; do not discuss them at this point. If you plan on doing the next session, *Session 2: Evaluating Definitions*, the teachers will have an opportunity to look back on their initial ideas and discuss them. Launch Launch the next task by connecting these guiding questions to the mathematical work that teachers are about to see in the video. You might want to say something like, "In order to help us think about why definitions matter for teaching, we're first going to think about the role of definitions in math."	*The four guiding questions will be revisited throughout the Using Definitions strand. As teachers work on the different sessions, they will have opportunities to learn that mathematics is a discipline in which meanings are precise – a condition necessary for effective mathematical communication. Teachers will also have opportunities to consider the role of definitions in teaching (for example, scrutinizing the definitions that are used in curriculum materials or defining terms carefully for and with children). In addition, sensitivity to the importance of definition helps teachers listen more carefully to their pupils' talk, better enabling them to notice ambiguous talk and to analyze disagreements for their possible roots in different or imprecise uses of terms.*

Figure 4.9 Lesson 1- Guiding questions through lessons of mathematical definitions

A teacher educator is expected to begin a class by showing four questions, shown in the slide of *Figure 4.9*. The work through all the other lessons of the materials: What do definitions do in mathematics and why do they matter? Why do definitions matter in teaching elementary school? How do definitions matter for mathematical reasoning? And, what makes a good mathematical definition? The questions are directed at the goals through all the lessons of the *mod4* materials that teacher educators are expected to always contemplate. The right column gives specific comments, regarding mathematical awareness in disciplinary mathematics, that meanings should be precise and precision improves the efficiency of communication in mathematics. Moreover, in terms of mathematical work of teaching, the right column explains that various tasks of teaching are related to mathematical definitions and teachers should take care to listen to what students say, to recognize the implicitness in their saying, and to examine what they say for deciding their mathematical root. Therefore, comments in the right column show two features of MKT: what awareness mathematicians generally accept and share in disciplinary mathematics and what specific mathematical work teachers perform in teaching. A teacher educator would read the questions or ask teachers to read them. Then, the teacher educator asks teachers to take five minutes to write their responses to the questions. This is for them to capture their initial ideas rather than to initiate discussion. Whatever answers the teachers put down on paper, they will be the ones to review their response, in later lessons. Each question is associated with various tasks of teaching and seeks to teach how to use language carefully and to demonstrate how to make mathematical decisions. In other words, these two tasks are based on and connected to several moment-to-moment acts of work.

2. Set up the video background and viewing focus (~10 minutes)

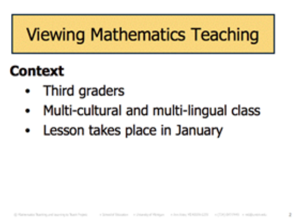
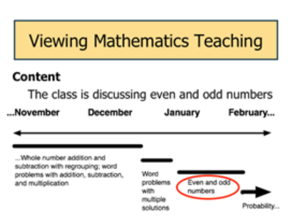
Detailed description of activity		Comments & other resources
	Background: Class context • The video captures a third grade mathematics lesson in a public elementary school where the teacher has been teaching for more than fifteen years. • The nineteen children come from diverse cultural and linguistic backgrounds. • The video segment is taken from a lesson in January, approximately halfway through the school year.	*See the table on the last page of the transcript for more information about the children's cultural and linguistic backgrounds.*
	Background: Mathematical content • Since September, the class has worked on a variety of elementary mathematics topics, including integers and whole number addition and subtraction. • In this video segment, the children are discussing even and odd numbers. • This topic arose the previous week in the context of solving word problems where the solutions relied heavily upon even numbers.	*Have a sample problem ready in case teachers ask about word problems that would lead to discussions of even and/or odd numbers. For example:* *Pencils cost 7¢ and erasers cost 2¢. If you want to spend exactly 30¢, how many pencils and erasers is it possible to buy?* *(The number of pencils must be even.)*

Figure 4.10 Lesson 1- Set up the video background

A teacher educator secondly introduces background information about the SeanNumbers-Ofala video that they will watch. Slides in *Figure 4.10* show grades, school, the number of students, time to record the class, and mathematical content that the students discussed. The teacher educator will see more information in the transcript of the video. Using records of practice situates teachers in the middle of a classroom with observers' perspectives. Providing information about the video smoothly guides teachers to get absorbed into the practice of teaching. The teacher educator hands out transcripts to the teachers. The teacher educator might use a sample problem, one suggested in the right column, in case a teacher asks a question about word problems of even and odd numbers.

<u>Optional preliminary fifteen-minute activity before introducing the viewing focus (i.e. the activity below):</u> If you have extra time available (approximately fifteen minutes), you might want to do a preliminary activity before introducing the viewing focus for the video. (See the notes on the right for the rationale and suggested work.) • Rather than presenting the next slide containing the viewing focus, say that you will be showing the ten-minute video once all the way through before turning to the real focus of the work. Explain that people often have strong reactions to or questions regarding the teaching shown in the video and you want to provide them with a little time to make general comments before returning to your work on definitions. • Show the video and spend just a few minutes having the teachers talk in small groups and/or collecting comments in whole group about their observations and the things that stood out to them. • When you are ready to move on, you might want to mention that one notable thing about the teacher in the video is that her teaching practice makes the children's reasoning and ways of thinking more visible and available for examination. (This is true regardless of how the teachers felt about the teaching and the students' interactions). At this point, you can say that the group will move on to watching for and discussing the children's ideas and reasoning about even and odd numbers.	*This video can provoke strong reactions among viewers who sometimes find the teaching quite exciting or rather troubling. It can be helpful to allow time for general reactions to be aired so that these first impressions are given some attention. Often this preliminary activity enables viewers to then turn to the viewing focus more directly.*

Figure 4.11 Lesson 1- Optional activity

The materials suggest that if around fifteen minutes are available, a teacher educator can do the optional activity shown in *Figure 4.11*. The purpose of this activity is to gather teachers' general reactions. Indeed, the SeanNumbers-Ofala video can provoke in teachers strong reactions. In fact, it is not easy to observe a class and immediately seize and describe mathematical issues. Therefore, the teacher educator can use this optional activity to scaffold the mathematical observation, which is offered as the main topic of this session. After the teacher educator plays the video clip, the running time of which is around ten minutes, teachers talk in small groups or the teacher educator takes five minute to gather their observations. All comments are acceptable, but the teacher educator should emphasize that the teacher's work in the video clip is making the children's reasoning and ways of thinking more visible and available for examination. This part seems to have no critical mathematical issue, but apparently it works pedagogically to make teachers move to the mathematical investigation involved in the video clip, with the focus questions. The teacher educator suggests focusing on the children's ideas about even and odd numbers and then presents the slide shown in *Figure 4.12*.

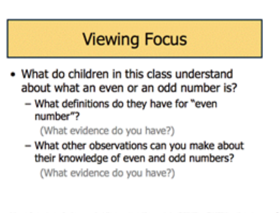
	<u>Viewing focus</u> • While watching this video, try to figure out what the children understand about what an even or an odd number is. • What definitions – or implicit definitions – of "even number" do you hear? • Take notes on the transcript; be prepared to give specific references to important moments in the video that shed light on the children's ideas about even and odd numbers. • There will be time immediately after watching the video to look over and add to your notes.	

Figure 4.12 Lesson 1- Focus questions

A teacher educator shows key questions: What do children in this class understand about what an even or an odd number is? What definition do they have for "even number"? What other observation can you make about their knowledge of even and odd numbers? The focus questions prompt teachers to use their mathematical lens to figure out what the children understand about what an even or an odd number is. The questions prompt teachers to practice recognizing and specifying mathematical definitions that children use, zoomed-in work in teaching. The teacher educator explains the feature of teaching practice: children sometimes explicitly state definitions or children's definitions are implicit in their reasoning and statements. Moreover, the teacher educator asks teachers to consider the focus questions when they watch the video again. It drives teachers to immerse themselves in using a mathematical lens rather than in other issues. Teachers take notes on the transcript to highlight the children's ideas. Here they also identify the evidence with regard to the focus questions. This task aims at practicing to reason mathematically with evidence in teaching practice. It is with this that the teacher educator would finally highlight the mathematical work of teaching in this lesson.

How would you define even number?	Before watching the video, ask teachers to write down a definition for even number. You might comment that, although you are not going to discuss their definitions now, thinking about the definition themselves is a way to help them hear students' ideas in the video.	*If teachers ask whether they should write a definition for themselves or for children, comment that either one is fine at this point, but highlight that the question of who the definition is for is an extremely important one. (This issue is discussed in Session 2 with the "usability" criterion for definitions.)* *Do not discuss their definitions now. There will be some reflection on their definition in this session after watching the video, but the work of writing and analyzing definitions is the focus of the next session. If you do not plan to do Session 2, you might want to discuss teachers' definitions in more detail after discussing the video. See Session 2, Option A for suggestions for discussing written definitions.*

Figure 4.13 Lesson 1- Writing a definition for an even number

Before showing the clip, a teacher educator asks teachers to write down a definition of an even number; this will help them as they listen to the children's ideas in the video, as shown in *Figure 4.13*. Definitions are not gathered, discussed, or judged. Thus, the point is not to write definitions that are mathematically precise or accessible to children. However, the teacher educator must differentiate between precision and accessibility. Therefore, this part does not focus on the mathematical fact.

3. Video viewing: *SeanNumbers-Ofala* (~10 minutes) (+ optional 15-minute activity)

Detailed description of activity	Comments & other resources
Play the video straight through. 	*To aid viewing, make sure the subtitles are displayed and that video is being played at full screen. Also be sure to use speakers to amplify the sound.*

Figure 4.14 Lesson 1- Watching the video

A teacher educator plays the SeanNumbers-Ofala video with proper sound with subtitles as guided in *Figure 4.14*.

4. Reflection (~10 minutes)

Detailed description of activity	Comments & other resources
Individual reflection Have teachers spend approximately five minutes writing notes about the children's understandings of even/odd numbers and the definitions implicit in their reasoning. Encourage them to refer to specific things said or done as much as possible. Discuss in pairs Have teachers spend approximately five minutes in pairs, sharing and discussing the things they observed and noted.	*During this time, it would useful to walk among the teachers, looking at their notes and listening in on their conversations to see who is picking up on the children's definitions and which definitions are being seen. You may want to:* • *Make notes about which teachers are picking up on which definitions (see below for the children's definitions to be discussed during the whole group discussion); you could use this information to move the whole group discussion forward later.* • *Redirect individuals and pairs to consider the children's definitions and understandings of even/odd if need be.* • *Watch for comments that are expressions of values, assumptions, or beliefs that are not necessarily supported by evidence (e.g., Shea is completely confused; Shea is just acting up; the teacher is not teaching). In such cases, you might try asking where in the video or transcript they see evidence that would support their description or analysis. Is there other evidence that could lead one to a different conclusion? (For example, is there evidence that Shea understands something about even and odd numbers? What?)*

Figure 4.15 Lesson 1- Reflection

Teachers write their finding in the transcript or their notes about the children's definitions and understandings of even and odd numbers for five minutes, as shown in *Figure 4.15*. Then, they share and discuss the findings with their partners. As part of the mathematical work of teaching, teachers are expected to figure out and interpret children's mathematical speech and articulate children's definitions according to the children's reasoning. In the meantime, a teacher educator moves around the classroom gathering teachers who found different definitions so as to use collected information for a whole group discussion. Having evidence for children's mathematical ideas leads to a more concise and reasonable discussion, much more effective than hastily and carelessly judging children and losing track of the discussion. The right column explains that if teachers seem to focus on value or beliefs unsupported by evidence, which are non-MKT topics, the teacher educator assists them by reminding them of the focus questions and that finding evidence for their responses is the task of a teacher educator in teaching

MKT. In other words, the comments in the right column specify what the teacher educator is expected to pay attention to in leading this discussion.

5. Whole group discussion of video (~25 minutes)

Detailed description of activity	Comments & other resources
Launch Bring the group together to discuss *the children's understandings about even and odd numbers, and the definitions implicit in their reasoning.* Refer back to the last slide, titled "Viewing Focus," to orient the discussion. Take notes that will be displayed during the discussion. If this is not something you routinely do, explain that taking shared notes will not only provide a record of the conversation, but will also allow the group to refer back to earlier comments – to elaborate and refine the ideas – as the discussion progresses. Note that the children seemed to be drawing upon several different definitions of – or ways of thinking about – "even number" in the episode. Ask if someone is willing to share one way the children seem to be thinking about even numbers.	*Alternatively, you might want to call on someone who had started to articulate one of the children's definitions during the paired sharing.*
Eliciting comments and facilitating the discussion **Grounded in evidence:** As teachers share their comments, consistently press them to identify what in the transcript or video they saw or heard that led to their claim. Where can this be found? This is an important form of rigor for making observations. It also lessens the risk of having personal beliefs/positions dominate the discussion. **Generating notes/making comments hearable:** Write brief notes in the public space about people's observations (or have one of the teachers help you). Use the note-taking process as an opportunity to restate ideas clearly and concisely (have the person who made the comment assist with this if it is helpful to you/him/her). Also, record the transcript location (line numbers) that served as the basis for each comment. **Making connections:** As the notes are generated, ask teachers to look for connections or contradictions among the comments. If it seems important, you may want to briefly comment about interesting or helpful connections you see or press people to resolve apparent contradictions, refine their statements, etc.	*There is a notes template available for taking discussion notes that can be projected from a computer.* *See sample discussion notes to see how the template can be used and the types of comments that often arise during this discussion.*

Figure 4.16 Lesson 1- Whole group discussion 1

A teacher educator gains the teachers' attention and launches a discussion of the children's understanding of even and odd numbers. The materials suggest four steps for the whole group, but *Figure 4.16* shows only the first two. The materials specifically illustrate the teacher educator's instructional moves. First, the teacher educator presents again the focus questions shown in *Figure 4.12*. The teacher educator then prepares a public note-taking space to record teachers' ideas and spurs the discussion through refining and elaborating the ideas. The teacher educator states that the children in the video clip seem to be drawing upon several different definitions of – or ways of thinking about – an "even number." The teacher educator then motivates teachers to share one way that children are thinking about even numbers or articulate one of the children's definitions. Or, the teacher educator calls on someone who articulated one of the children's definitions during the pair discussion.

Second, the teacher educator collects the teachers' comments, which are supported with evidence, such as transcript line numbers for rigorous observation rather than personal beliefs about the observed class. Therefore, the teacher educator consistently asks, "Where can this be found?" to ascertain what in the transcript led to the teachers' claims. All teachers' comments and their supporting evidence are recorded, restated concisely and clearly, in the note-taking space. Moreover, the teacher educator traces the recorded ideas and prods teachers to connect or contradict them so as to resolve contradictions or to refine their statements.

Examining children's definitions Three significant definitions (or ways of thinking about even numbers) that the children used in the episode should surface in this discussion: • **Fair share** – a number is even if it can be made of two equal groups with none left over, using only whole numbers (Sean) • **Pairs** – a number is even if it can be grouped in pairs, with nothing left over (Ofala) • **Alternating** – even and odd numbers alternate on the number line, starting with zero being even (Cassandra) If one or more of these definitions does not surface, you might consider: • Calling on someone you noticed had picked up on the missing definition(s) during the individual reflection time or pair discussions • Asking about the definition or understanding of "even number" that one of the specific children used (e.g., How was Tina determining whether 6 was even or odd in lines 28-33 of the transcript?) • Asking the group if there is evidence that indicates the children had another way to tell whether a number – like 1, 2, 3, 4, or 10 – is even or odd?	*You could also ask about definitions of or assumptions about odd numbers.* *If the teachers seem to cover this ground with some confidence, you might ask them to consider how they might try to reconcile the definitions. For example,* • *Are the children's definitions compatible? How could you show this?* • *If the definitions are all correct, then how could you prove that the fair share and pairs definitions are equivalent; in other words, that a number is even by one definition if and only if it is even by the other?* • *Of these two definitions, which would be easier to use if you wanted to show that it is equivalent to the alternating definition?*
Transition to wrap up Summarize the three different definitions the children used and identify the one or two children most clearly aligned with each definition. Refer to the discussion notes if they seem helpful. To transition to the wrap up, briefly discuss the children's use of definitions. Ask: • Why are definitions showing up in the children's talk? • What use are the children making of them? • How do the children's definitions compare with the definition you wrote before watching the video?	*Some sample responses:* • *Definitions are being used to identify which numbers are even and odd* • *Children don't need to turn to the teacher to resolve the conflict; definitions enable them to use mathematical reasoning* • *Children are using definitions both explicitly and implicitly to make mathematical arguments and resolve conflict* • *Using different definitions (without realizing it) makes it hard for students to understand and respond to each other's arguments*

Figure 4.17 Lesson 1- Whole group discussion 2

Third, as shown in *Figure 4.17*, the teacher educator attempts to identify from the video clip three significant definitions that the children use explicitly or implicitly: (1) fair share (used by Sean) – a number is even if it is can be made of two equal groups with none left over, using only whole numbers; (2) pairs (used by Ofala) – a number is even if it can be grouped in pairs, with nothing left over; and (3) alternating (used by Cassandra) – even and odd numbers alternate on the number line, starting with zero being even. If any of the definitions go unmentioned, the teacher educator might ask the teacher who worked the definition during the individual or pair work. Or, the teacher educator might ask teachers to find the definition of an even number that the specific children used in the video clip. In this activity, teachers as learners are expected to interpret the children's talk and figure out and articulate their definitions as the mathematical work of teaching. If, on the other hand, the class finds the three definitions and has sufficient time, the teacher educator can extend the discussion: the teacher educator then asks about definitions or assumptions about odd numbers; or, the teacher educator would initiate a discussion to reconcile the different definitions. This task opens up the opportunity for teachers to make mathematical and pedagogical decisions based on their observations. Apparently, in this step, making mathematical and pedagogical decisions is based on the accumulated tasks of teaching, such as articulating and interpreting definitions that children use in the video clip. The work of teaching that teachers would carry out is both making mathematical and pedagogical decisions and figuring out and interpreting children's definitions. The latter ones are nested in the former ones.

Through the moves, the discussion is expected to focus entirely on mathematical issues, specifically definitions children come up with in the video clip. In other words, a teacher educator is expected to establish a mathematical focus, recognize mathematical issues that teachers raise, focus on mathematical articulation and interpretation, and manage mathematical ideas that teachers have. These are important tasks a teacher educator performs in teaching MKT.

Finally, a teacher educator summarizes the three definitions and identifies what children used which definition, by reviewing the public notes. Beginning to the wrap up, the teacher educator asks: Why are definitions showing up in the children's talk? What use are the children making of them? How do the children's definitions compare with the definition you wrote before watching the video? as shown in *Figure 4.17*. The teacher educator specifies the mathematical and pedagogical decision through the video and through their discussion: The use of different definitions creates conflict; students use definitions explicitly or implicitly to argue and refute their ideas and to solve their conflict; different definitions are based on a different mathematical foundation; figuring out, articulating and examining students' definitions; and being sensitive to language are the critical tasks in teaching. The first three are about the mathematical awareness of definition and the last one is tasks of teaching that teachers learn and practice in the lesson.

6. Wrap up (~5 minutes)

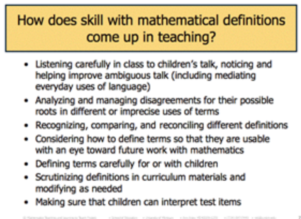
Detailed description of activity	Comments & other resources
Summary I: Examples of the role of definitions in <u>mathematics</u> Display the title of this slide and pose the question, "Why are definitions necessary in mathematics?" Have teachers suggest a few ideas about the role that definitions play in mathematics. After collecting a few ideas, display the four bullets on this slide and have teachers look at this partial list. Talk through the slide, highlighting where the ideas that teachers raised can be seen on the list. Say a few words about things that are on the list that were not mentioned (see comments on the right for a few ideas), and reiterate ideas that the teachers may have raised that are not captured on this slide. **Why are definitions necessary in mathematics?** • Mathematics is a domain in which meanings are precise and not connotative (e.g., "*rectangle*") • Most mathematical terms are also used in other ways more generally (e.g., *"even"*) • Being clear about mathematical meanings makes effective communication possible (e.g., *"fairly"*) • Mathematical reasoning depends on definitions	*<u>Mathematics insists on precision for effective communication</u>* *Definitions function more strictly in mathematics than they do in other disciplines. Multiplicity of meanings and implicit meanings are commonplace in other disciplines, whereas in mathematics, definitions are specific and explicit. For example, mathematically, rectangles have more in common than just general appearance; they satisfy very specific conditions. In this sense, terms have precise meanings in mathematics and are not merely connotative.* *<u>Everyday vs. mathematical usage</u>* *The terms "even" and "odd" have everyday meanings. If something is "odd" it may be strange or unfamiliar; calling something "even" may mean that it is smooth or it is the same size as something else. Mathematical work requires that terms be defined unambiguously and with shared usage. Similarly, the idea of "sharing fairly" can mean different things for different people. It may be based on equality of portions, but it could just as easily be based on distributing differing portions by how much each person would need or deserve. Establishing a shared understanding of what is meant by "fairly" is a precondition for mathematical work using this term.* *<u>Basis for mathematical reasoning</u>* *Definitions, along with basic logical principles, are one of the foundations of mathematical reasoning. In the video, Tina uses a definition in her argument. She claims that 6 <u>is</u> even because zero is even, and even and odd numbers alternate, so when you start from zero, you get: 1 is odd; 2 is even, 3 is odd; 4 is even; 5 is odd; and so 6 must be even.*

Figure 4.18 Lesson 1- Wrap up 1

This lesson closes with an emphasis on gaining an appreciation for the roles of definitions in both mathematics and mathematics teaching. *Figure 4.18* shows the first summary for recognizing the roles of definitions in mathematics. A teacher educator displays the title of the slide shown in *Figure 4.18*, and gathers teachers' ideas about the role of definitions in mathematics. Then, the teacher educator shows the four bullets on the slide and connects them with the ideas teachers raised. For example, the teacher educator points out the bullets showing ideas that the teachers presented or reiterating the teachers' ideas not shown on the slide. According to the teacher educator's knowledge of practices in mathematics, which is related to the role of definitions in mathematics, mathematical definitions play three particular roles: (1) They are specific and explicit; terms in mathematics are not connotative but have precise meanings; (2) unlike everyday life, mathematical work requires terms unambiguously defined and publicly shared; and (3) mathematical definitions are one of the foundations of mathematical reasoning. This list merely reflects the implicit mathematical awareness that mathematicians generally possess. MKT as content includes their awareness.

Summary II: Relating definitions to the work of *teaching*

Having discussed the implicit definitions used in the video and the variety of ways definitions play a role in instruction, consider the ways in which skill with mathematical definitions may also be important in teaching.



Again, you may want to display just the title of this slide and ask teachers to generate responses to the question based upon their experiences with the video and the preceding discussion. Once a few responses have been offered, relate those ideas to the ones listed on the slide and spend time highlighting any that weren't yet touched upon.

Alternatively, you could display all the bullets on the slide at once and use examples from the video or from the discussion to illuminate or expand upon some of the points on the slide. For example, in the video, definitions were used to do the following tasks of teaching:

- Interpret Shea's reasons for saying that 6 is even and his reasons for saying that 6 is odd
- Recognize that the class does not have an explicit definition for odd numbers
- Help Tina and Shea productively discuss their reasoning by first recognizing that they are reasoning from different definitions

The summary to the right contains other points that might be useful to make.

Teachers must consider definitions given to students and the students' ability to interpret and use them
Definitions given in textbooks are sometimes incorrect and sometimes hard for students to understand or use. Teachers need to scrutinize definitions, decide among definitions, modify definitions, and help students learn to use them.

Teachers must manage mathematical reasoning and disagreement
Because definitions are essential building blocks for mathematical reasoning, teachers need to understand their role in order to support student reasoning, help students appraise arguments, and reconcile disagreements that result from reasoning based on different definitions.

Teachers must be aware of and be able to hear students' definitions and use them in instruction
As students learn and reason about mathematics, teachers must be able to hear the implicit and often underspecified definitions in students' thinking and the potential confusion that arises between the mathematical meaning and the everyday use of terms. They must be able to listen generously to students' definitions and have language available to identify what is ambiguous and help make meanings precise.

Figure 4.19 Lesson 1- Wrap up 2

As for the second summary as shown in *Figure 4.19*, work of teaching related to definitions are highlighted. A teacher educator displays the title of the slide in *Figure 4.19* and collects the teachers' ideas. Then, the teacher educator shows all bullets of the slide and relates the teachers' comments to the list. Or, the teacher educator displays all the bullets on the slide and then talks about examples from their observation of the video that relate to the bullets. The materials articulate three particular mathematical work of teaching related to definitions: Teachers need to consider definitions given to students, such as textbooks, probe them, and decide whether students can understand and use them and, if necessary, revise them; teachers need to recognize the importance of definitions in mathematical reasoning and help students appraise arguments and reconcile disagreements because of different definitions; finally, teachers are expected to hear and recognize definitions in students' reasoning and consider possible points of confusion because of the difference between the mathematical meaning and the everyday use of the terms. This articulation assists the teacher educator in explaining the knowledge of the mathematical work of teaching. Such is the task of teaching that teachers perform with mathematical objects in mathematics classrooms.

Summary

This session used focused observation and analysis of a video segment from a third grade class to provide teachers with opportunities to:

1. Hear, interpret, and analyze (with observation-based evidence) children's mathematical thinking and their use of implicit definitions to support their reasoning about even and odd numbers.
2. Gain an appreciation for the role that definitions play – as well as their significance – in providing precision and supporting reasoning in both mathematics and mathematics teaching.

Figure 4.20 Lesson 1- Summary

Figure 4.20 illustrates the summary of this lesson. First, a teacher educator uses a video clip from a third grade class as a record of practice and prompts teachers to practice hearing, interpreting, and analyzing what children say and what definition they use in their reasoning about even and odd numbers. This summary specifies the mathematical work of teaching that a lesson is expected to emphasize. Second, the teacher educator develops teachers' mathematical knowledge, their skills and habits of mind that go into the work of teaching, related specially to mathematical definitions. This summary focuses on mathematical awareness related to definitions.

4.3 Matthew's Lesson on Evaluating Definitions

This section shows the full story of one lesson where Matthew taught MKT to preservice teachers in the content course. The main activity of this lesson was to evaluate four definitions of an even number. The length of the lesson was more than seventy minutes. As noted in Chapter 3, Matthew majored in education and had less than ten years teaching experience in teacher education. In the description of the lesson, commentaries have been added concerning the purposes of the current research.

When the lesson starts, twenty teachers are seated. Matthew stands in the middle of the classroom and looks around at the teachers, as shown in *Figure 4.21*. He introduces the agenda of this class: he will comment on the teachers' notebooks, and then the class will learn the criteria for evaluating definitions that the teachers were curious about in the last class and finally practice evaluating definitions used in textbooks.

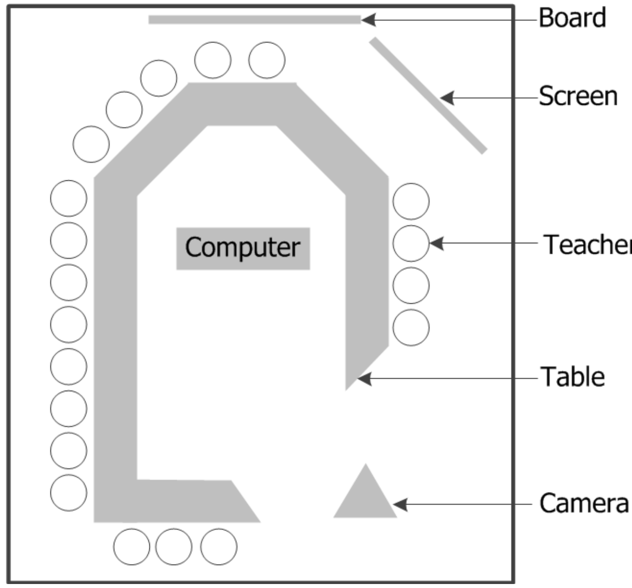


Figure 4.21 Matthew's classroom

From the questions that the teachers answered in their notebooks, Matthew comments about two of them: what number or numbers is a multiple of one, and what number or numbers is a multiple of zero. He informs the teachers that these two questions seemed to give them the most trouble. He says that he does not grade their notebooks and that they should feel comfortable about developing their ideas related to the questions. Then, he encourages the teachers to think further about these questions over the weekend. He gives them the hint of trying to understand what the definition of a multiple is and using the definition of factor. He then briefly explains that they will discuss whether splitting into groups of two is the same as splitting into two groups and whether negative numbers and zero can be split into groups of two. Matthew then shows the slide shown in *Figure 4.22*, asking the teachers what region D represents. One teacher responds that D indicates even numbers and multiples of three and offers twelve and negative twelve as examples. However, when Matthew asks why these two numbers can be included in D, she cannot explain. The other two teachers demonstrate that the two numbers are integers that are multiples of two and three. In this discussion, Matthew's questions include the following: "Can you give us an example?" "Why should negative twelve go into D?" "How do you know that it's a multiple of three?" "Why is it an even number?" and "Can somebody prove to us that it's a multiple of three?" Finally, he closes the discussion by clarifying that twelve and negative twelve are multiples of two *and* three, and, therefore, they cannot be in only A or only G. He advises teachers to revise if they need to.

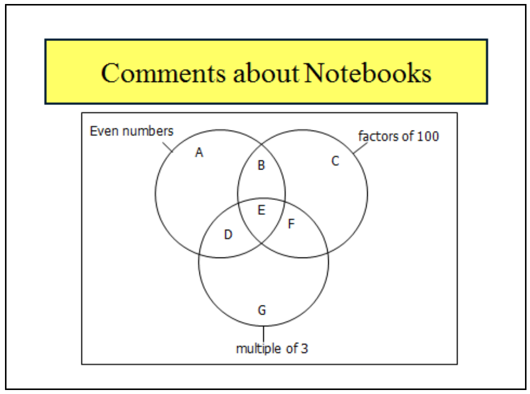


Figure 4.22 Regions of numbers

In this episode, two questions—what number or numbers is a multiple of one and what number or numbers is a multiple of zero—were discussed. At this moment, Matthew simply did not offer answers to the two questions. He suggested investigating the definitions of multiples and factors that are terms used in the questions. This suggestion related to a way of thinking about mathematical issues that mathematicians generally use in their work, especially for definitions. In other words, he worked with his teachers about how to approach their mathematical issues related to definitions, through a style of thinking that is a function for mathematics. This study refers to that as *mathematical awareness*. Furthermore, his comments showed one mathematical task of teaching—figuring out concepts of multiples and factors in the provided questions. That is, the teacher educator and teachers worked this specific task of teaching as content. Moreover, the two questions were related to mathematical facts that are the objects of disciplinary mathematics included in MKT. In other words, teaching MKT includes mathematical facts that are generally uncontroversial. The discussion about region D was also related to mathematical facts. He threw out such questions as "Can you give us an example?" "Why should negative twelve go into D?" "How do you know that it's a multiple of three?" "Why is it an even number?" and "Can somebody prove to us that it's a multiple of three?" His questions showed what mathematical issues Matthew recognized in the class. He managed teachers' ideas with questions. The questions made the class stick to the mathematical point of the discussion. He closed the discussion with a mathematical explanation of facts, which was reasonable. Finally, this scene shows that there are different features of MKT that work in teacher education, such as mathematical awareness, tasks of teaching, and mathematical facts.

Teachers also ask several questions about the assignment – thinking of three definitions of an even number. One teacher is unsure whether ¼ is not an even number even though it can be broken into two equal groups; other teacher points out that if an even number is the number when multiplying numbers by two, it is unclear what numbers mean, such as integers, whole numbers, and so on; other teacher asserts that the definitions in the assignment are not accurate. Matthew seems very glad to hear these questions. He neither agrees nor disagrees with the teachers' ideas, but he verifies that the three definitions are inaccurate. He states that all of these issues are discussed in this class. This portion of the class takes about 12 minutes.

In this episode, the teachers seemed to construct, probe and evaluate definitions and recognize different meanings of number in the definition as tasks of teaching. Matthew did not answer immediately but managed the teachers' interests and kept them to the main activity of the lesson.

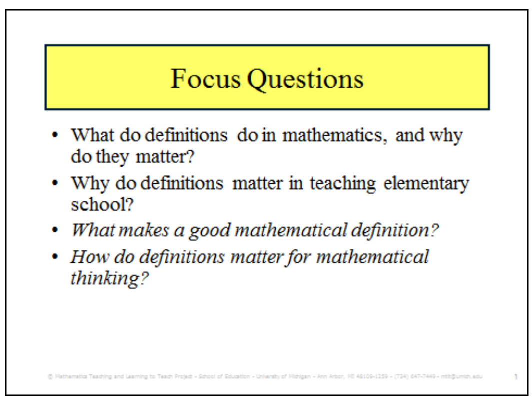


Figure 4.23 Focus questions

Matthew shows the slide shown in *Figure 4.23* that he introduced in the class before. He reads the four questions on the slide and states that this class will focus on the last two questions: what makes a good mathematical definition; and, how do definitions matter for mathematical thinking.

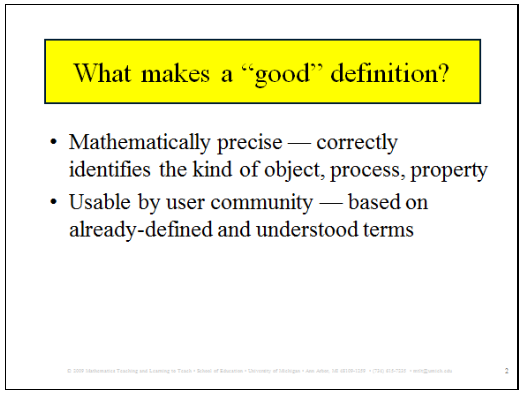


Figure 4.24 The two criteria for a good mathematical definition

Matthew introduces the two criteria for a good mathematical definition with the slide as shown in *Figure 4.24*. Stating that a good mathematical definition should be precise, Matthew explains what precision means:

> When we talk about mathematical precision, we need a definition that will allow us to say whether something is included in the set that we define, or is excluded. So, for instance the definitions that you were given about even numbers – did they help you decide that all even numbers are indeed even numbers? And did you… did they help you exclude all the non-even numbers? If they didn't, they were not precise.

He also demonstrates usability as a criterion. He reminds the class of what a teacher said:

> We can give your students very elaborate definitions, but will they be able to use them? Will they be able to understand them? Remember when we started working on definitions, and we introduced that definition of even numbers, Teacher17 said "Well, we need to understand the terms: what does 'integer' mean, and what do we mean by 'integer' and 'multiple'?" And we had this very nice discussion about the integer and multiple. So we need to make sure that our students understand what the terms are in order to be able to use these definitions. OK?

He explains that although there might be more criteria for a mathematical definition, the class would use these in this lesson.

In the episode, the two criteria reflect mathematical awareness that mathematicians have about definitions: definitions should be precise and statements of them should be acceptable by the mathematics audience. While mathematicians might emphasize precision rather than accessibility in their work, teachers should consider both. A teacher's main job is helping others learn mathematics. This job requires considering others' perspectives on mathematics. In other words, teaching requires mathematical awareness of definition as well as an ability to use language that students find accessible.

A good mathematical definition in the practice of teaching might not be the same as in disciplinary mathematics. Using a good definition in teaching practice is related to the careful use of language that is mathematically precise and that students can use. Being able to do so is critical in teaching.

When Matthew asks teachers' ideas about the criteria, one teacher wonders about using an imprecise definition that students can understand or tailoring a definition so as not to offer an imprecise definition that students later recognize as being incorrect. Matthew smiles. He remarks that teachers need to know well both their students and mathematics and make a balance between precision and usability:

> We need to keep a balance. But, in trying to keep this balance, we shouldn't be – you know – making some changes in the definition in order to make it usable, but then it won't be precise. So we need to have both in mind – both the precision and the usability. OK?

Matthew then emphasizes that examining the definition that appears in textbooks is a task that teachers should be able to do. He reminds teachers to think about problems from when they worked imprecise or unusable definitions into their assignment.

In this scene, the teacher asked about using language mathematically and accessibly as being part of the mathematical work of teaching. Matthew highlighted making the balance between precision and usability in community. Teachers need the ability to offer mathematically precise definitions. However, if the definitions include language incomprehensible to students, teachers need to consider the use of those definitions again. Inversely, teachers need an ability to provide definitions that students can comprehend. If the definitions are not precise, however, teachers need to think again about their use of language. This process requires diverse kinds of small tasks, such as recognizing possible concerns in language and judging definitions mathematically. Matthew clarified one of the zoomed-out types of work that teachers carry out in teaching. Zoomed-out mathematical work of teaching captures each lesson and long periods in the continuum of curriculum as well as the mathematics that students learn later and the overall territory of mathematics. Zoomed-out work stands in contrast to zoomed-in work, which captures the moment-by-moment interactions and activity in the continuum of curriculum as well as a small domain of mathematics. The terms zoomed-out and zoomed-in are borrowed from a metaphor of a photograph for instruction found in

Lampert (2001). In particular, as one of the zoomed-out mathematical work of teaching, using language mathematically and accessibly requires mathematical rigor while still allowing for comprehension and usability by students. Imprecise language will become an obstacle to students' developing their mathematical ideas; moreover, language that is not geared toward students will delay the appropriate learning of mathematics.

Matthew shows the slide as shown in *Figure 4.25* and asks teachers to evaluate the four definitions. He also hands out copies of these four definitions. He highlights precision and usability as two criteria: does the definition include all even numbers and exclude all non-even numbers; and, what concerns might students generally have.

> Precision means that all the numbers that need to be included are included in this set by the criteria that we are using, and all the numbers that need to be excluded are excluded. For instance, *does the first definition help us identify all the even numbers and exclude all the non-even numbers?* … So when we talk about usability, you might raise this question "Well, we don't know our students yet, we don't know how third graders think." And this is work that we'll be doing next semester. So to consider this criterion of usability, just *think of concepts or ideas in every definition that might be problematic to your students, whatever grade the students are.* So, if you are given a definition, what might cause problems for elementary students in general? OK? (emphasis added)

In this activity, teachers are to decide whether they are being precise and what might be their concerns about using definitions; they are to clarify why they are or are not being precise. This activity includes probing and evaluating definitions and using language mathematically and accessibly as in the mathematical work of teaching.

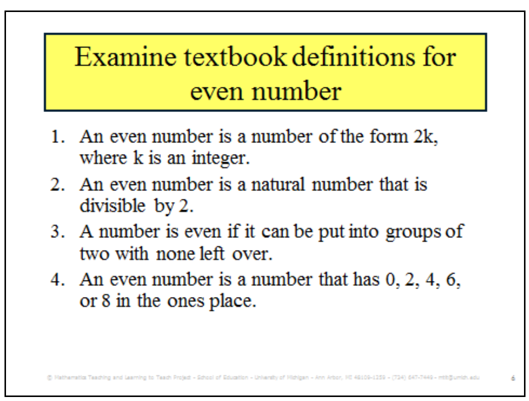


Figure 4.25 Definitions of an even number from textbooks

For around twelve minutes, teachers, in pairs, work at the task. Matthew goes around each pair to listen to their ideas and give comments. In the middle of this activity, after three minutes of the pair work, he calls the teachers' attention, saying that one teacher asked, in reference to definition 2, "What are natural numbers?" Matthew explains that natural numbers are counting numbers and whole numbers includes natural numbers and zero:

Teacher1: They're not imaginary numbers. They don't have the little "i" next to it. So, if they're numbers on the number line, anywhere from zero to infinity, like zero, point one, point two, point three, one, two, three, four, five…

Matthew: Wait, you said "everything from zero to infinity," so…

Teacher1: Negative infinity to positive infinity along the number line.

Matthew: So, that it includes fractions? That, it includes irrational numbers like pi?

Teacher1: That's a great question. Doesn't it include pi?

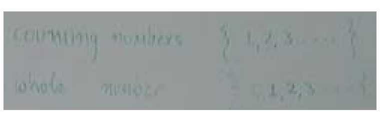
Matthew: No, so when we're talking about natural numbers… Does anyone want to answer this question? [Pause] It's the counting numbers. It's only the counting numbers, although people do not agree on this definition. There are some people who also include zero. So, for those people who include zero, it's the whole numbers. Zero and the counting numbers are the whole numbers. What we discussed last class. You can decide with your partner whether you want to use it with the counting numbers or the whole numbers.

…

Teacher1: Does that also include the negative numbers?

Matthew: No, because the definition says counting. To count, we don't use negative numbers.

Teachers resume their work in pairs with Matthew going around to each pair. He also writes down sets of counting numbers and whole numbers on the board as shown in *Figure 4.26*.



Counting numbers {1, 2, 3, …}
Whole number {0, 1, 2, 3, …}

Figure 4.26 Sets of counting numbers and whole numbers

This episode shows that the mathematical work of teaching goes together with mathematical issues. Specifically, an issue arises concerning what a natural number is in a process of probing and evaluating definitions. The teacher's question indicated both recognizing "a natural number" in the definition as the mathematical work of teaching and mathematical awareness of definition in the work of teaching. It is the need for knowledge about a mathematical fact that critically functions through mathematical research as well as the work of teaching.

This scene also shows that Matthew carefully observed teachers' small group discussions and recognized mathematical issues that the teacher faced in probing definitions. He immediately opened the whole group discussion and clearly identified what natural numbers, counting numbers, and whole numbers are and what numbers are not. This identification is related to the types of numbers, one of the important mathematical structures in disciplinary mathematics. Therefore, mathematical facts and structures are objects of disciplinary mathematics that play within MKT as content in teacher education. The tasks that Matthew performed in this moment helped teachers keep probing and evaluating the definitions. This was introduced as the main activity for teaching MKT. Furthermore, even though he helped teachers to identify what natural numbers are, in the practice of teaching, teachers have to articulate what they are.

Matthew sits at the computer in the middle of the class and shows the empty table on the screen to record whole group discussion as shown in *Figure 4.27*. Matthew types the teachers' comments on the laptop and teachers could see them in the screen.

Proposed Definition	Mathematically Precise	Usable by User Community
An even number is a number of the form 2k, where k is an integer.		
An even number is a natural number that is divisible by 2.		
A number is even if it can be put into groups of two with none left over.		
An even number is a number that has 0, 2, 4, 6, or 8 in the ones place.		

Figure 4.27 Table to record the whole group discussion

The class starts discussing the first definition: An even number is a number of the form 2k, where k is an integer.

Teacher2: I only thought number one was like precise, but I might be wrong.
Matthew: OK, Teacher2 says that this is precise. *Why do you think so?*
Teacher2: Um, because through trial and error, yeah, through trial and error, and it also defines k and an integer and…
Matthew: *Does it include all the even numbers we want to have in our set?*
Teacher2: I believe it does, yeah.
Matthew: *Can you convince us?*
Teacher2: No. (Laughs)

Matthew: Can somebody else help us? You said trial and error, so *you were trying some examples.*
Teacher3: We did some like, you know, if you think, any number. Two times three is six, and two times minus three is minus six, still an even number.
Matthew: So notice what Teacher3 is saying. So they tried six and negative six. So they tried a positive number and a negative number. What else do we need to try to be convinced that…
Teacher4: Maybe have zero
Matthew: Zero. So remember that we had these discussions? Negative numbers, zero, and positive numbers…
Teacher1: All integers, not all numbers.
Matthew: Only integers. That's a very good observation. It includes… See how easy it is to have these ambiguities? So *we really need to be careful about the terms that we are using.* (emphasis added)

Matthew asks whether the definition is precise and by what reason. He also asks whether the definition includes all the even numbers and identifies kinds of numbers to test the definition: negative numbers, zero, and positive numbers. In terms of the teacher's specification of integer rather than number, Matthew praises Teacher1's comment and emphasizes the careful use of language. When Matthew asks concerns related to usability, teachers points out "form," "2k," and using variables. Teachers assert that children might not understand "the form 2k" and how to use a variable. Moreover, another teacher presents the possible error of using variables, such as 21 when k is 1 in 2k.

This episode shows two kinds of mathematical work of teaching: evaluating a definition and probing concerns in the provided definition. Matthew also accepted another possible error in reading expressions containing variables, though he chose not to focus on it in this moment.

The class next discusses the second definition: An even number is a natural number that is divisible by 2. A pair of teachers are curious about the meaning of *divisible*. Does it refer to divisor (no remainder) or separating something into two groups? The class tries to clarify and understand this point. In this talk, Matthew repeatedly asks whether this definition is precise, prompting a lively discussion. He decides that the obscurity of the meaning of divisible shows that this definition is not, at first glance, precise. He also comments that if they clarify the meaning of divisible and this definition includes positive numbers, zero, and negative numbers, this definition might be precise:

> So you had this conversation. If you have this many comments, I think it's fair to say that it's not precise. And you're right that what causes a problem is this idea of "divisible." So if we define "divisible," and we all agree what "divisible" is, and if the way that we define "divisible" allows us to have negative integers, zero, and positive, positive integers in this set, then the definition is OK. But as it stands right now, isn't precise, because of all this comment that we had right now.

In fact, this definition is mathematically imprecise because this definition includes only {2, 4, 6, …} and does not include {… -4, -2, 0}, not because the class had various ideas on divisible or that the meaning of divisible is not clear. The class just left a chance to investigate the meaning of divisible, which is often used in teaching practice. On the other hand, his repeated question, "is the definition precise?" worked as highlighting the importance of precision in mathematics.

Matthew asks about usability. Teachers respond that students need to know what a natural number is and what divisible means:

Matthew:	How about its usability, then?
Teacher5:	You need to know what a natural number is.
Matthew:	Is it usable or not?
Teacher2:	*If it's not precise, it's not usable. You could use it, but it would be wrong.*
Matthew:	Yeah, you can use it? A student might understand it, but still it might not be precise. This is what Teacher2 mentioned before. It can be usable. *Students might understand it, but still it might be wrong.* It's not usable and again you said having problems with natural numbers. And we also had problems with-
Teachers:	divisible
Matthew:	divisible (emphasis added)

Another teacher acknowledges that students might understand the definition, despite it not being precise. Matthew reaffirms that using imprecise definitions is problematic.

This episode shows how to use language mathematically and accessibly as the part of the mathematical work of teaching. The class decided that using imprecise definitions was problematic with students despite their being able to use or understand them. Their decision in terms of using language mathematically and accessibly was based on the other task, evaluating a definition. Moreover, Matthew managed to keep teachers' mathematical ideas on the track of MKT without sliding toward mathematics or situations of a K-9 classroom to the exclusion of the other.

Before moving to the next definition, another teacher asks about the definition of *divisible*. Matthew says, “So the definition that we could use here to make it more precise is that we can divide something into, let’s say, groups of two with no remainder.” He explains divisible by two.

This scene shows recognizing a term in a statement as both a mathematical task of teaching and a mathematical fact as a disciplinary object. In everyday usage, divisible often means simply that something can be divided. However, in school mathematics, it is typically agreed that if a number, *n*, is said to be divisible by a number, *d*, it means that *n* and *d* are integers such that *n* can be divided by *d* with zero remainder. Specifically, a number *n* is said to be divisible by *d* if *d* is a divisor of *n* (Weisstein, 2011b). However, divisible is different from divided. For example, an even number is divisible by two, and any number can be divided by two, such as $7 \div 2 = 3.5$, $\pi \div 2 = \pi/2$. It is important for teachers to understand the special mathematical meaning of divisible and differentiate between divisible and divided. It is a mathematical fact. Therefore, Matthew specified the meaning of divisible as grouping by two with no remainder, which works within whole numbers. His explanation might not work well with negative integers.

The class then discusses the third definition: A number is even if it can be put into groups of two with none left over. Another teacher asserts that this definition could not exclude fractions and decimals because fractions can be divided into groups of two. She presents that one-half can be two groups of two one-eighths as a counterexample. Matthew goes to the board and writes down her ideas, shown in *Figure 4.28*.

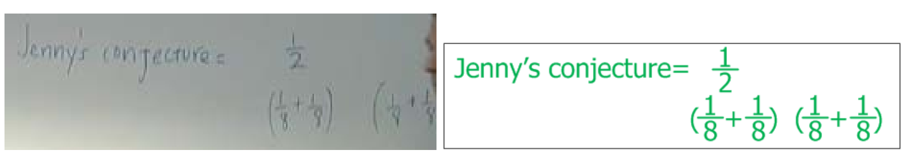


Figure 4.28 A teacher’s conjecture: Fractions can be groups of two

Matthew takes candies in his hand and gives them to the teacher. He asks her to make groups of two. She makes pairs of candies, and he asks the teachers to see what she is doing. Then, he goes back to the board and asks the teachers whether one-half is even. He clarifies and writes that one-half cannot have any groups of two, but one-half is left over as shown in *Figure 4.29*. He emphasizes that this definition is different from dividing into two equal groups and one-half cannot be split into groups of two.

Figure 4.29 None group of two and one-half

Matthew tries to examine why the teacher decides this definition is not precise:

Matthew: Why do you think that it's problematic?

Teacher6: For kids?

Matthew: Or for you?

Teacher6: *I don't think it's problematic for me*, I think it's – I mean, I interpreted it that way, so I don't have a problem with interpreting it that way at all. *I mean for kids of course it would be problematic*. I wasn't interpreting it as a ten-year-old. (emphasis added)

Matthew: Why do you think it would be problematic for kids then?

Teacher6: … If they're really good with fractions maybe they could handle it, it's certainly not the simplest definition.

Matthew: Why do they need to work with fractions?

Teacher6: For my interpretation, I chose to use some arithmetic functions with fractions. But for-

Matthew: But wait, was this accurate? Would what you did – you had this thing – one eighth plus one eighth, one eighth plus one eighth. [He writes (1/8+1/8)+(1/8+1/8) on the board.] Was this what the definition is saying? Dividing into groups of two?

Teacher6: That was my interpretation, yes. I'm not suggesting that that would be a good approach for children. Obviously it would not. But I just- I'm asking as adults is that an acceptable interpretation of this definition.

Matthew: What do other people think?

In this discussion, Teacher6 used the two subjects to evaluate the definition, herself and children. In point of fact, mathematical precision works for the whole; it is not applied differently depending on the audience. Therefore, her differentiating the two subjects to evaluate the definition was mathematically unreasonable. Furthermore, she was still sure that this definition is not precise for students because if students are able to do arithmetic operations with fractions, they would interpret it inappropriately. While she did not clearly say, she seemed to be suspicious of units of "two" in this definition. In other words, Teacher6 recognized mathematical issues in the definition, but her recognition was not reasonable because mathematical precision does not change according to the subjects. Matthew identified the meaning of the definition with pieces of candy and clarified that the definition differed from dividing into two groups. The teachers, however, appeared not to be persuaded by his comment. The discussion continues:

Teacher7:	Well, right here, two is not defined.
Matthew:	Why?
Teacher7:	Because it doesn't say it has to be two wholes.
Matthew:	Ok, but why do you think that this is true?
Teacher6:	Well just like Teacher7 said, it depends on your interpretation of the word "two" – is two an integer or are you simply trying to group things in a group where the number of items is two?
	…
Teacher8:	… When we talk about that do we mean, is the total of the elements two? Or do we mean the number of elements is two?
Matthew:	So instead of size "two." So groups of two. The number of elements is two.
Teacher8:	Does it mean number of elements, or does it mean the total of the elements?
Matthew:	How do you think that these two are different?
Teacher8:	Because the way Teacher6's example – the one eighth plus one eighth – there are two elements in that group.
Matthew:	*I think that this is harder for a kid to consider than say two wholes.* So that's why what I did is- I gave you these (He indicates candies.) to divide, because this is what students do. I mean at the early grades. *Two, two, two- so they do not consider elements. They consider things, whole numbers.*
Teacher9:	You put either of those in the definition, and this is no longer an issue.
	…
Matthew:	OK, let me remind you though, the conversation we had with Teacher2. Why not just introducing the term "integers"? *What are we going to lose if we introduce the term "integers" though?*
Teacher10:	Meaning
Matthew:	Meaning? What else?
Teacher11:	Simplicity.
	…
Matthew:	OK, what if I introduce the word "wholes"? If you think that this will make problems to kids. Although I would guard you that for kids, given their experiences, *they're not going to get to this point.* (emphasis added)

In this discussion, teachers claimed that "two" is ambiguous in this definition and suggested revising the definition, such as adding the terms "whole number" or "integer." However, Matthew consistently pointed out that students are not in trouble when it comes to understanding the definition. He asserted that students would rather be in confusion if the definition is "A number is even if it can be put into groups of integer two with none left over" or "A number is even if it can be put into groups of whole number two with none left over." He said "Two, two, two. So, they (students) do not consider elements. They consider things, whole numbers." In fact, two is a number. It is specifically a whole number and an integer. To assert that two should have a unit, we also have to call three "integer three" and it is awkward. Although teachers should think about possible interpretations as they can, excessive and unnecessary doubt can make teachers miss the point. Their argument about "two" got more serious because of negative integers. The discussion continues:

Teacher12: I think even before that is how you deal with negative numbers. You can't really put negative numbers into groups of two.
Matthew: Another very good question. So and this is something that you had in your homework. The way that we introduce negative numbers to kids- they're using two different colors. Let's say, we will make an agreement, and we will use red for positive numbers and green for negative numbers. So I'm going to give Teacher13 all negatives, and I want him to group them into groups of two.
Teacher13: The value two?
Matthew: Into groups of two. Remember what we are doing, again like the candy-
Teacher13: Put them in groups of two elements? How can you put them into a group that has a value of two?
Matthew: Into groups of two things.
Teacher8: Just seems like you switched. When you're dealing with negatives, you use one definition of "groups of two" but when you were dealing with fractions-
Matthew: No, I can't work- let's see what I was doing. So we started with the candies. OK, we said if we have these (He brings candies.).
Teacher8: What if you were using pieces of pie, and you have a pie cut into eight pieces.
Matthew: So you're getting into a different region. You are using fraction numbers. Notice that I didn't use fraction numbers, so I'm using the *whole pieces*, and I'm dividing into groups of two pieces. These are my groups. OK? And if I am going to use negative numbers, one group – this is minus two – another group – minus two.
Teacher14: You just went to positive. As soon as you held up something, that is an item; it's not a negative.
Matthew: Wait, we defined greens to be negative. OK, what is this, if we define it?
Teacher14: It's an object. It's one object. I see positive one object.
Matthew: Well, but this is a convention that we're making. How are you going to introduce negative numbers to kids? There are different ways, and this is one way we introduce negative to kids. So we say let's make a

convention. If you have reds, these are positive. If you have greens, these are negatives. So one, two, three, four – negative four – and I'm dividing these into groups of two. One group of two, two groups of two. Teacher8?

Teacher8: I don't know this is an important issue, but, if you're just going to say by convention, these stand for negative one, what's the difference between making a convention that a green stands for negative one and a convention that green stands for an eighth, or some other numbers?

Matthew: Yeah, you can make this convention, I agree with you. It's just, this is the convention that we are using.

Teacher6: Are there actually, I know there's a lot of manipulatives out there. So are there manipulatives that teachers can use that specifically designate negative numbers?

Matthew: Well, this is one model that teachers use, the green and-

Teacher1: Use the two different-

Matthew: the red counters. They also use the number line.

Teacher6: OK, cool.

Matthew: But it's a really hard concept to define, the negative numbers, so that's why we are using some conventions. (emphasis added)

In this episode, another teacher asked whether negative numbers can be divided into groups of two, and the teachers thought that it is not possible because "two" is a positive number, not a negative number. It looked reasonable because of Matthew's explanation with the pieces of candy. In this time, he introduced chips of two colors as didactical devices in order to represent integers, red ones for positive numbers and green ones for negative numbers. He explained that candies just represented whole numbers and the two colored chips could represent integers. Moreover, using these kinds of didactical devices could confirm whether the definitions can stand. However, the teachers believed that using different colored chips is equivalent to switching the unit of "two," specifically from positive two to negative two, and, if so, this would be equivalent to using fractions as the unit of two just as the class had discussed previously. Matthew identified "whole pieces," not fractional pieces as groups of two. In point of fact, the materials expect this kind of argument to evaluate the definition as shown *Figure 4.30*. In fact, all these issues are from the definition. In other words, the definition itself can be problematic. However, Matthew presented different meanings of "two," 2 and -2.

Proposed Definition	Mathematically Precise	Usable by User Community
A number is even if it can be put into groups of two with none left over	Precise – however, it might be debated as to whether negative numbers can be put into groups of two	The language is understandable by most elementary students; however, students might not be sure as to whether 0 or negative numbers can be split into groups of two

Figure 4.30 Commentary of the definition in the materials

In the middle of this discussion, another teacher claims that because green chips are objects, they cannot represent negative numbers. Matthew demonstrates that different colored chips are generally used to introduce negative numbers in schools and various representations and manipulatives as didactical devices are used under mathematically contracted situations that Brousseau (1997) asserts. Finally, teachers revise this definition:

Teacher7: I think that if you start with a whole number, then it doesn't matter. I think solves the problem if you start with a whole number.
Matthew: If you say what?
Teacher7: If you say *a whole number is even if it can be put into groups of two with none left over*.
Matthew: Nice. Can you say why?
Teacher7: Except that, that doesn't handle negative numbers?
Teachers: Oh. Yes. OK. Whole numbers are zero, one, two, … Oh, you're right. Integers are negative… Oh!
Matthew: So it limits—wait. It limits the set to only whole numbers.
Teacher8: Right.
Matthew: Although it won't be accurate because for kids, let's say of third grade, if we say a whole number is even, we're not defining the whole set. We're saying, for the whole numbers we're considering right now, we are going to say that these are even if they can be put into groups of two.
Teacher9: And then you would modify it once they started learning negative numbers.
Matthew: Yes. And this actually addresses for both of your questions. We can make it usable and still be honest to the mathematics. We are not saying something that is inaccurate. Notice what we are saying: we are defining only the whole numbers. And then we can modify it and say to kids, "Well right now we're going to consider negative numbers as well." So let's see how we can switch this, or tweak it a little bit to make it include this whole set. (He is typing.) So you might include the phrase "the whole number." And actually this goes to reduce any ambiguities. So, with this modification, what do you think about the precision then? Teacher13?
Teacher13: I would say if you modify it, then yeah.

Matthew: (Typing) OK, more precise if we include the term, whole number.
Teacher13: People – I think people might still have a problem with the negative concept of it, though.
Teacher16: But it's not speaking to negatives, right?
Matthew: So right now, no. Right now we can use it, let's say with third graders, who know nothing about negative numbers, and still be accurate and honest to the mathematics. OK? … Our intention was not to do all possible definitions, but just to give you a sense of all these subtle differences and all these tiny details that make a difference. And what we did right now – notice that we criticized definitions, but also tried to improve them, having in mind both the kids and the mathematics. (emphasis added)

Despite their long discussion, the teachers and Matthew failed to reconcile what "two" means in the definition and whether the definition is precise. However, in this episode, Teacher7 revised the definition, using a whole number instead of a number: A whole number is even if it can be put into groups of two with none left over. The class evaluated the revised definition and recognized that it precisely specifies when a whole number is even without excluding negative numbers because negative numbers are not part of the domain of the revised definition. It is mathematically correct and generally usable in the elementary grades. The class made a decision through modifying, evaluating, and probing the definition. These mathematical tasks are nested in using language mathematically and accessibly as zoomed-out work. Furthermore, the series of mathematical work of teaching entailed mathematical fact, mathematical awareness of definition, and mathematical value for a sound foundation, which are objects in disciplinary mathematics. Matthew pointed out that this is a way to manage the tension between precision and usability in the classroom.

The class turns to the last definition: An even number is a number that has 0, 2, 4, 6, or 8 in the ones place. Matthew offers that this definition is often used in textbooks. The class discusses whether this definition is precise:

Teacher17: I think it works fine if they haven't learned about decimal places.
Matthew: OK, so is this precise?
Teacher17: Precise to second-graders.
Matthew: Again, notice that term precision. *We need to decide if it includes everything that needs to be included, and if it excludes everything that needs to be excluded*.
Teacher18: It can't be precise.
Teacher7: It's precise to a whole number.
Matthew: Yes. So we say more precise if we consider the whole numbers.

Teacher19: Then according to that – like what Teacher17 said, he said that was precise to second-graders.

Matthew: So it won't include—so this, if we want to modify it in this way, even this one is not that precise.

Teacher20: Correct.

Matthew: But as Teacher7 said- you were right. That if we're going to limit it to let's say whole numbers, we might say that it's ok, but it won't be that precise. Because *precise needs to include everything that needs to included, and exclude everything that needs to be excluded.* … so I could give you a decimal—

Teacher21: Well decimals, if it was 2.25, that's not an even number.

Matthew: OK, this is another one, 2.25. I have a two in the ones' place, but still this is not an even number.

Teacher1: Yeah, so again modify—my suggestion is to modify the definition so that it only includes whole numbers or integers. Yeah, or that has no decimal following the number.

Matthew: So it needs modification. It definitely needs modification. (emphasis added)

In this discussion, a couple of teachers asserted that the definition is precise for a certain group of people, students who have not yet learned decimals. Matthew asked whether this definition is precise, reiterating that precision means whether a mathematical definition identifies the concept or class of objects to which it does and does not apply. It should be consistent with accepted mathematical usage, rather than with different usage by different subjects. The class refuted the definition with a counterexample and recognized that this definition should be revised so that it stands for whole numbers or integers.

Matthew then explains that this definition focuses on the units digit but does not explain why a number is even. Matthew states:

> But I would like to point your attention to another thing here. This definition obscures one of the basic ideas of evenness, this idea of 'twoness', of two things. I mean, by focusing that much on the units place, it doesn't really help kids understands why we have this two. The 'twoness', which is really basic in considering even numbers. … This definition actually places more emphasis on the units place. … You cannot find this (groups of two) anywhere in this definition. Or being able to split it in half. Where if you consider other definitions, you can see in '2k', you can see it in the 'groups of two', we can see it in here, 'divisible by two'. We can see it in all other definitions. We cannot see it in the last definition. And this is a definition that is used very widely in elementary schools. OK, nice.

His comment was not about evaluating the definition but describing how this definition is different from the others, focusing on the units digit rather than the concept of twoness.

Another teacher, however, responds that this definition would help students understand intuitively which numbers are even and initiate students to investigate an even number. Matthew is glad to hear this idea and states that the class will work more this definition in the next class. Moreover, another teacher asks why this definition is used widely even though it is not precise, and Matthew answers that it is easy for students. Lastly, Matthew shows the slide as shown in *Figure 4.23* again and asks teachers to answer for the last two questions in their notebooks. The class is closed here.

In this class, Matthew and his teachers evaluated four definitions of an even number and probed possible concerns using them. He introduced the two criteria for a good definition. In evaluating the definitions, Matthew emphasized what precise means regarding a mathematical definition. Through the activity, the teachers decided whether the definition was mathematically precise, tried to identify mathematical objects as pedagogical concerns, and revised the definitions. Finally, the class could practice to use language mathematically and coordinate both mathematical rigor and students' accessibility. In the lesson, using language mathematically and accessibly was the zoomed-out work that the teachers studied. Furthermore, this zoomed-out work nests the aforementioned series of tasks, such as probing and evaluating definitions that the teachers repeatedly practiced with the four definitions. The class also discussed mathematical facts and structures from the definitions and practiced mathematical awareness of definitions. Mathematical facts and structures and mathematical awareness are objects required in mathematical research. As for instructional moves, Matthew clearly specified the activity and carefully observed what teachers were thinking regarding the recognition of mathematical issues; he also helped teachers perform the activity. He repeatedly asked whether the definition was precise or whether it fell into such pedagogical issues as who can understand a definition or whether students can understand a definition. However, it seemed challengeable for the teacher educator to maintain a mathematical consistency within the lesson.

4.4 Emily's Lesson on Explaining Why the Units Digit Rule Works

This section shows the full story of one lesson of Emily teaching MKT to preservice teachers. The main activity of this lesson was constructing an explanation of why the units digit rule works. The length of the lesson was around forty minutes. As introduced in Chapter 3, Emily majored in mathematics and had less than ten years teaching experience in teacher education. In the description of the lesson, commentaries are also added in terms of the purposes of the current research.

When the lesson starts, in the middle of the class shown as *Figure 4.31*, Emily stands and asks her teachers, "So who can tell me what the units digit rule is?" Twenty-six teachers are getting back to their seats. She asks again "If I give you a number, honestly, how do you know if it's even? Do you split it into two equal groups? No." One teacher says that a number is even if it ends in an even number. Emily restates "If you want to know if it's even or odd, you look at the last digit, right?" In this manner, she motivates her teachers about the units digit rule. As yet, however, she has not elicited the units digit rule explicitly.

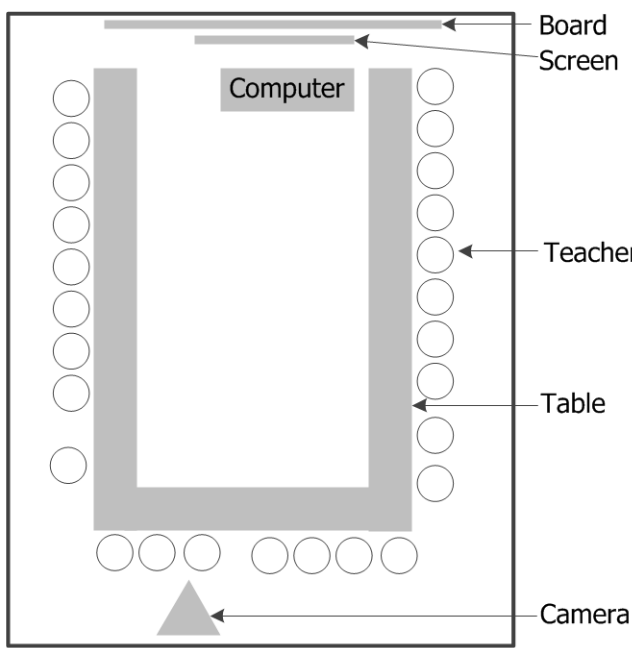


Figure 4.31 Emily' classroom

Emily writes 12,327 on the board and asks the teachers whether this number is even or odd. They answer simply, "Seven is odd." Emily elaborates, "It ends in seven, which is odd, so it must be odd." Her revoicing confirms the teachers' response. She then explains that the definitions – splitting into two equal groups, pairing out or dividing by two, which were discussed before as the definitions of an even number – are not generally used to determine whether a number is even. She says "You can just look at the last number and decide if it's even." The class seems to concur. Hence, the teachers used the units digit rule and considered it valid. Moreover, her comment briefly specifies what the units digit rule is. She then introduces that this lesson's task – explaining why this rule works.

At this point, a teacher suggests "the last whole digit" rather than "the last digit" because of decimal numbers. Emily writes 12.327 on the board and says, "Evenness and oddness – or sometimes we'll call it parity – is actually not defined for things like this." She asks whether her number is even. The teachers respond that it is not even because it is not a whole number and an integer. Another teacher wonders about the different notation of 12,327.0, claiming that the units digit rule fails in this case. Emily points out that, in this number, the units digit is 7 not the last digit of 0. Emily explains that is why the rule is called the units digit rule. She clarifies that this rule does not hold for decimals or fractions. Emily, at last, spells out that the rule holds only for integers. "If you have an integer, you look at the units of it and that tells you if it's even or odd." She clarifies that this rule is not for decimals or fractions. She finally identifies that the units digit rule works for only integers.

In the four-minute introduction of the lesson, Emily motivated teachers' interest in the units digit rule. After she specified what the units digit rule was, the teachers tried to investigate terms used in the rule and identify an accurate term for it. To clarify the mathematical domain of the units digit rule, the teachers probed when it is in effect. In doing so, the teachers confronted the mathematical foundation of the units digit rule: the rule works with integers rather than with real numbers or other types of number. This process was an important mathematical issue – finding precise language and looking for the mathematical domain of a given conjecture or rule. Emily elicited teachers' mathematical ideas and responded to all their comments and questions. In carrying out a non-mathematical task, Emily managed teacher engagement, emboldening the teachers to volunteer their ideas.

Emily then gives the task – proving why the units digit rule works – and asks teachers to work individually. Later, she asks the teachers to revoice the task. One teacher says, “We’re working on a proof for why the units digit rule works?” Emily responds, “Right, so why is it we can look at the last digit – or the units digit of a number to decide if it’s even or odd.” The teacher’s revoice and Emily’s repeating of the task ensured that all teachers plainly recognize what they must investigate and consciously keep the track of the lesson’s progress.

Emily shows the slide shown in *Figure 4.32* and explains that the lesson’s purposes are to learn how to explain and prove a proposition in mathematics teaching and to know, through a mathematically substantial approach, the units digit rule.

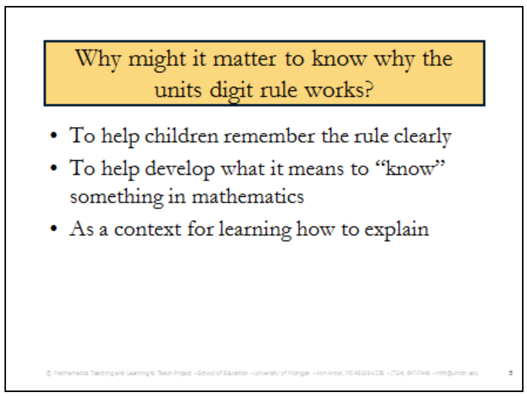


Figure 4.32 Why it matters to know why the rule works

She then shows the slide as shown in *Figure 4.33* which specifies the units digit rule and one teacher reads it. She tells the teachers to think about why this rule works.

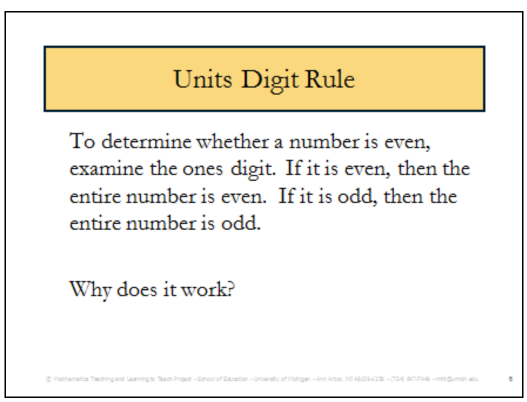


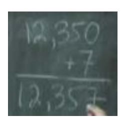
Figure 4.33 Units digit rule

The materials of *mod4* offer some information about this slide. Digit refers to a symbol used in a numeral to represent a number. There are ten symbols or digits, 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. For example, the digit 3 is used in the numeral 12,357 which represents the number twelve thousand three hundred fifty seven and 3 itself represents three hundred. In another case, the ones digit of -12,357 is 7 even though its value is -7. Therefore, the units digit rule determines that -12,357 is an odd number. Moreover, units digit and ones digit are synonyms, indicating the digit in the ones place of an integer in the base ten number system. The class discussed this term previously and found "last digit" to be improper.

After about a minute of teachers individually examining why the units digit rule, Emily asks a question "What are some tools that you could be using to think about this problem?" She wants the teachers to unpack the units digit rule and identify its mathematical foundation. A couple of teachers suggest using the number line and creating even numbers by just adding two to an even number. However, many teachers consider place value. One teacher says, "The tens place will always be even no matter what the number is." The teacher asserts that 2 placed in the tens place represents twenty and is an even number. Emily asks the teacher what happens if she changes the number in the tens place, making 12,327 to 12,357. The teacher replies that 5 in 12,357 means fifty and, thus, still an even number. Another teacher says, "That's like all of them, not just the tens place, but everything except the ones will always be even." Another teacher agrees with this claim and partially explains how the units digit rule works for the number 12,357. She also says "You just pretend the seven doesn't exist, and what that five represents is a fifty. And if you take out the five and the seven, what that three represents is actually three hundreds. And those numbers will always be divisible by two." In other words, in the number 12,357, ten thousand represented as 1 in the ten thousands place, two thousand represented as 2 in the thousands place, three hundred represented as 3 in the hundreds place, and fifty represented as 5 in the tens place are all even numbers when the ones digit is disregarded.

In this episode, teachers began probing the units digit rule and recognizing several mathematical issues. The class ultimately recognized that the place value of the base ten number system is a significant resource in unpacking the units digit rule. The teachers acknowledged that tens, hundreds, and thousands are always even numbers. Place value is an important mathematical structure in this task in terms of knowledge about mathematics. The class found its existence and importance but was still in the process of identifying how the place value functions in the units digit rule. The class maintained mathematical awareness of exploring mathematical objects. Moreover, Emily attempted to figure out the teachers' ideas and remarkably managed the mathematical discussion. When teachers presented pertinent ideas to unpack the units digit rule, she asked questions to clarify their remarks: "What is it?" "What do you mean?" She gave strongly affirmative comments, such as "Right!" Throughout the lesson, she avoided immediately correcting or passing over irrelevant ideas. She accepted their comments and tried to relate them to the activity. She asked questions to redirect teachers to the main issue. Her management helped position teachers as competent mathematical thinkers, fully engaging them in the discussion.

Emily continues to discuss place value. She asks Teacher22, who found the tens place will always be even, to explain her idea. Teacher22 goes to the board and writes 12,357. With her right hand, she hides the 7. "If we just cover this up and imagine it as a zero, this number is going to be even because the five represents fifty, the three represents three hundred, etc." Teacher22 and the other teachers find out that 12,357 is 10,000 + 2,000 + 300 + 50 + 7 and that the first four numbers are even. Teacher22 rewrites 12,357 as a mathematics sentence shown in *Figure 4.34*. The class discusses it:



$$\begin{array}{r} 12{,}350 \\ +\,7 \\ \hline 12{,}357 \end{array}$$

Figure 4.34 12,350+7=12,357

Emily: What's the reason for pulling out the seven, and not the five and the three?

Teacher22: Because the last number's what matters, and (pointing at 12,350 in *Figure 4.34*) this is already even as it stands, so when we add an odd number to an even number, we get an odd number.

Emily: Hmm.

…

Teacher23: But how can you say that twelve three fifty is an even number? Because it has a zero? Because I thought that's what we're trying to prove.

Teacher24: Then you could divide it by two or groups of two.

Teacher22: Because any time … if something's in the tens place, it's going to be even, whether this is a (pointing at the 5 in *Figure 4.34*) six, six or a nine, it's always even.

Teacher23: How would you say it's even?
Emily: Right, so what… yeah, so…
Teacher25: We have to prove that. We understand that it is. But, how would you prove it?
Emily: Right, so what definition of even are you using to be able to say that that number is even? Because Teacher23 was saying if you say that it's even because it ends in zero- well, that's not a very good reason. Because that's what we're trying to show – if something ends in an even number then it's even.

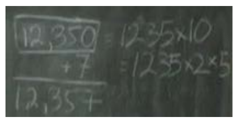
In this episode, Emily asked Teacher22 a key question, why did she separate seven. Teacher22 responded that 12,350 is even and the units digit number, 7, determines whether 12,357 is even. After all, an even number plus an even number is an even number and an even number plus an odd number is an odd number. She highlighted again that whatever number is replaced in the tens place is even. The teachers wondered how they knew whether 12,350 was even. They concluded that, despite knowing 12,350 was even, they needed to prove it. This exchange represents mathematical awareness that is reminiscent of mathematical explanation in the discipline. While Hanna tried explaining why the units digit rule works in 12,357, other teachers heard, figured out, and probed her explanation, and articulated what they needed to investigate. These tasks are the mathematical work of teaching that teachers carried out. Emily confirmed that the issue is why 12,350 is even. She stressed that they had to find a good reason for why an integer ending in zero is even without using the units digit rule as Teacher23 demonstrated. It was critical to help teachers identify what they needed to investigate. She also gave a mathematical clue – using a definition of an even number to reason why the units digit rule works.

Emily then gathers teachers' ideas for why 12,350 is even. One teacher assumes that since ten is an even number then multiples of ten, which all end in zero, are even:

Teacher26: I don't know how I can show this, but I think that you're—you're defining even numbers differently works for this group because—wait, before you do that step you have to get people to agree with you that ten is an even number, which you could do in a number place. I don't know what the best explanation is.
Emily: But, it's OK. But, why does it matter that ten is even?
Teacher26: Because the next step would be to say … multiples of ten ending in zero would be even. So you can knock all of those out and look at the units digit.
Emily: Do people follow what Teacher26's saying?

Teacher: No.

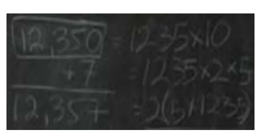
Teacher26 cannot explain why her ideas are valid. Teachers cannot follow her explanation well, though her ideas are significant in explaining why 12,350 is even and why the units digit rule works. Then some teachers seem to lose track of the discussion. They suggest splitting up into two groups and try defining zero. Emily asks again why 12,350 is even in order to keep the focus of the discussion. One teacher explains that 12,350 equals ten times 1,235 and two times five times 1,235. Emily represents the teacher's explanation on the board as shown in *Figure 4.35*.



$$\begin{array}{r l} 12{,}350 & =1235\text{X}10 \\ \underline{+7} & =1235\text{x}2\text{x}5 \\ 12{,}357 & \end{array}$$

Figure 4.35 12,350=1235 X 2 X 5

Emily asks "Why does rewriting it this way matter and why can you do it?" Some teachers think that the mathematical sentences complicate the topic. Another teacher demonstrates that multiples of two are even. Emily writes on the board one more mathematical sentence, 2(5 x 1,235), as shown in *Figure 4.36*. She asks the teachers to synthesize and summarize the discussion. Another teacher explains that the tens, hundreds and thousands are multiples of two and thus even. The teacher then complains about the need to probe and explain the rule because such elaboration would be complicated for students. Emily identifies that the purpose of this work is to understand why the units digit rule works. To practice explaining it, she says, is a teacher's task. A couple of teachers expand on Emily's explanation. Teachers, they say, have to understand mathematical reasons of cases or rules because students will ask teachers why they work. Teachers need to know and be capable of giving various explanations, not just one. Emily again encourages the teachers to present diverse explanations whenever possible.



$$\begin{array}{r l} 12{,}350 & =1235\text{X}10 \\ \underline{+7} & =1235\text{x}2\text{x}5 \\ 12{,}357 & =2(5\text{x}1235) \end{array}$$

Figure 4.36 12,350=2(5 X 1235)

Until now, the class discovered five things: (1) the units digit rule itself, (2) that it works only with integers, (3) the importance of the place value of the base ten number system, (4) that to understand why the rule works, they needed to prove why 12,350 is even, and (5) that it is even because it is a multiple of two. However, it is still difficult for all the teachers to understand and explain why 12,350 is even. One teacher stated that 0, 2, 4, 6, and 8 are even numbers and 1, 3, 5, 7 and 9 are odd numbers between zero and nine and claimed that ten is an even number because it ends with zero. However, another teacher criticized it for being a circular argument. Emily pointed out that it does not explain why 12,350 is an even number and why the units digit rule works. She suggested again using the definitions that the class discussed in the previous lesson in order to have a complete explanation.[16] Emily helped teachers to understand an intended activity and she acknowledged teachers' mathematical issues to explain why the rule works.

Emily specifies what they need to explore with *Figure 4.36*:

> So the unit digit rule tells us that if we have this number (She is pointing out 12,357), that, so we can break apart a number like this (She is pointing out 12350+7), and we only have to look at this number (She is pointing out 7). So the question that remains is "what can we say about this number (She is pointing out 12,350)." And actually what- so we want to be able to say that this number (She is pointing out 12,350) is even. … I think that's what this example starts to do- for an example. … If we work through this one example, we can think about why is it true for any other case. And what can we generalize from this example into a proof.

In this explanation, she proposed anticipating how and why the units digit rule works in the number 12,357 and emphasized that this work serves to have the generalized explanation of why the units digit rule works. She explains it:

> (She is point out the 2(5 X 1235) in *Figure 4.36*) So looking at this last line, this two times five times this number- so we can see that this number is actually even because this is two times an integer. … So does that make sense to everyone? So just looking at this last line. Whatever number is, we know it's even, because it's two times an integer. And so to get to here, we're just kind of rearranging. We knew that- that we could always pull out whatever our number was times- pull out a ten, and from the ten we could pull out a two, so we always get this even number. … So starting- starting with our original number, we kind of- we break off the units digit. … We don't worry about (She is pointing out 12,350 in *Figure 4.36*) this one. This (She is pointing out 7 in *Figure 4.36*) is what's going to tell us

[16] As Matthew's class, Emily' class also analyzed and evaluated several definitions of an even number in the previous lesson, such as an even number is a multiple of two; an even number is a number of the form 2k, where k is an integer; an even number is a natural number that is divisible by two; and an even number is a number that has 0, 2, 4, 6, or 8 in the ones place.

> what's even or odd. … That way if this is even plus an odd is going to give us an odd.

Emily explained concisely why 12,350 is even and how the units digit rule works in 12,357. First, she told her class, consider a new number, 12,350. Its ones place is zero. It is even because all multiples of ten, having two as a factor, are even; depending on the addend, an even number becomes odd or even; because 7 is an odd number, 12,357 is an odd number. Indispensable to her explanation, are the facts that an even number plus an even number equals an even number and an even number plus an odd number equals an odd number. Unlike its importance, she did not give any comment on it because the class already proved the conjecture in the previous lesson, an odd number plus an odd number equals an even number. According to their proof, the class seemed to assume that they could prove, and know well, that an even number plus an even number equals an even number and that an even number plus an odd number equals an odd number.

Next, Emily proposes explaining why the units digit rule works in a general case. She writes "a-b-c-d" on the board:

> If I had a number, and I didn't know what the number was, but I knew that it's digits were a, b, c, d, (writing "a-b-c-d" on the board) and I wanted to look at… and I wanted to argue why the units digit rule worked, who could use this kind of general example to walk through the reasoning?

In this episode, Emily proposed an abstract approach, using variables, for a more generalized explanation. She declined, however, to clarify the mathematical assumption. In "a-b-c-d," each variable represents each digit in each place and is one of nine numbers, 0, 1, 2 … 9. For example, "a" is a symbol placed in the thousands place. Therefore, "a-b-c-d" is an integer between 0 to 9,999. According to this assumption, "a-b-c" is an integer between 0 to 999. While the units digit rule also works for negative integers, the assumption of using "a-b-c-d" fails to consider the application of the units digit rule for negative integers. It is unclear whether Emily was considering negative integers in her statement. She might need to prove negative "a-b-c-d" with the same assumption. However, Emily's suggestion initiated the teachers' investigation.

One teacher, Teacher27, goes to the board and explains why the units digit rule is correct in the general case of "a-b-c-d." What she writes on the board is shown in *Figure*

4.37. Emily asks the teachers to listen and try to understand Teacher27's explanation and to work out how a teacher would explain the units digit rule:

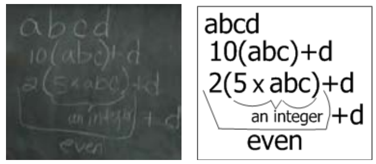


Figure 4.37 Why d decides whether abcd is even or odd

Teacher27: OK. So we're saying this (pointing at "c") is the tens digit, this (pointing at "d") is the ones digit. So we're separating it and trying to figure out "d." Now, we just throw out what you're saying because you could have 10 times "a-b" plus "d" (writing '10ab+d' on the board).

Emily: What happened to the "c"?

Teacher27: The "c" is (pointing at 10) – I screwed up.

Teacher28: You can write c.

(Teacher27 rewrites, "10(abc) + d")

Emily: OK, so that's splitting up, so can someone… so why can you do that? Can anyone talk about that? Actually… so why can you break it up like that?

Teacher27: Because the… I was just looking at the example over there. They have the 10 times all the places except for the ones digit.

Emily: Hmm. Does that make sense to people… have questions about that step?

Teacher29: Can you repeat that again? I didn't hear you.

Teacher27: OK, I'm sorry, I'll be louder. For… In that example we have 10 times all the digits including up to the tens digit, here and plus the units digit. Does that make sense?

Teacher29: OK.

Teacher27: So we have 10 times all the digits up to the unit digit plus the unit digit or the ones digit.

Emily: Units or ones, either one is fine.

Teacher27: OK, so then that's the same as (writing "2(5 x abc) + d") 2 times 5 times abc plus d. So then I have…

Emily: Do people understand what she's doing?

Teachers: 2 times 5 times abc. She is multiplying …

Teacher27: So this is 2 times an integer (writing "integer"). So, this here, 2 times an integer is an even number, so an even number – we have an even number here (pointing at "2(5 x abc)"), always, no matter what these numbers are – plus whatever the units digit is, will determine whether is. Because an even plus an even, it would be even, or an even plus… if this is odd, it would be an odd.

In this discussion, Teacher27 elicited that "abcd = 10abc + d = 2·5·abc + d." Teacher27 concluded that '2·5·abc' is always an even number. Therefore, "d" determines whether “abcd” is an even number or an odd number because an augend, 2·5·abc, is always an even number and an addend determines whether a sum is an even number or an odd number. Emily figured out what Teacher27 stated. Emily also examined what Teacher27 missed to reason why the units digit rule works. Emily helped Teacher27 present a mathematically reasonable explanation. Emily gave a question to help Emily correct her reasoning, for example, “What happened to the ‘c’?” Emily encouraged Teacher27 with favorable comments, such as “Units or ones, either one is fine.” She asked key questions to focus teachers’ attention on the important moment of reasoning as well as keep Teacher27 on track: “Why can you do that? … Why can you break it up like that?” Emily also asked questions to check whether the other teachers were following Teacher27’s explanation: “Does that make sense to people… have questions about that step?” “Do people understand what she’s doing?” Emily nudged teachers into figuring out and probing Teacher27’s explanation, into evaluating whether her reasoning was valid. Of course, Teacher27 practiced explaining why the units digit rule worked in “abcd.” The teachers finally unpacked the units digit rule as the mathematical work of teaching. Mathematically, this episode included four elements: (1) multiples of two defining an even number, (2) all tens, hundreds, and thousands are multiples of ten because of the place values in the base ten number system, (3) proofs of (an even number) + (an even number) = (an even number) and (an even number) + (an odd number) = (an odd number), and (4) mathematical reasoning to weave all these mathematical concepts. Emily also guided the class to see that each step moved mathematically and carefully and each moment was mathematically valid. It is similar to the process of mathematicians’ work, specifically what they are aware of and take for granted. This study calls it *mathematical awareness of exploring mathematical objects*. While their discussion was stopped to investigate a specific case, Emily steered their discussion to a general case. It is mathematical value that builds substantial construction.

Another teacher then asks Emily whether Teacher27’s presentation is a proof that explains why the units digit rule works. Teacher27 answers “You can put any number there.” Emily points out that Teacher27’s demonstration is for the general case of any four-digit number but that it is, nonetheless, not a proof.

In fact, a proof is an unequivocal demonstration of the truth based on definitions by Aristotle. As Emily' comments, Teacher27 provided important reasons to justify the units digit rule, and all participants of this class already knew that the units digit rule is correct. However, the discussion ended with justification only for a general case of a four-digit number rather than for any integer, despite the group having found a key idea to constructing a proof. Emily clearly differentiated between a proof for all cases and a proof applicable within a narrow scope.

Another teacher emphasizes that Teacher27's explanation is based on an assumption that is already known, namely that an even number plus an even number equals to an even number and an even number plus an odd number equals an odd number. Again, this class proved, in the previous lesson, the conjecture that an odd number plus an odd number equals an even number. All the teachers, through the discussion, acknowledge that they are correct. Another teacher also says that students might think 12,357 is "highly odd" because there are four odd numbers, 1, 3, 5, and 7 in this number and, inversely, students might have their own ideas of determining whether a number is even or odd. Emily comments that teachers can be more powerful when they know not just what the rules are but why they work. She asks the teachers whether they have any more questions and then asks them to summarize their discussions in their notebooks and reminds them that they will work divisibility rules later.

The class began with an incomplete work. Working together, the teachers uncovered the key elements of the units digit rule and constructed the explanation of the rule for a general case of a four-digit-number. They practiced explaining and reasoning what they already knew and used as mathematical work of teaching. They also had a sensibility for well-presented explanations, which are concise and mathematically solid. In Emily's class, the teachers carried out various mathematical tasks of teaching. They specified and probed the units digit rule; they created an explanation of why the rule worked and selected numbers to demonstrate why, and they evaluated diverse explanations of why the rule works. Moreover, the class recognized place value as a critical foundation for explaining why the rule works and then investigated it. Finally, using the various tasks of teaching, they unpacked the units digit rule for an even number. Moreover, in terms of knowledge about mathematics, Emily helped teachers learn "being aware of" in exploring mathematical objects through a class. Moreover, she led the teachers to participate in creating valuable construction in mathematics. Mathematical facts and structure are also significant to making mathematical explanations.

At the end of the class, they discussed elements of the explanations. This discussion looked like wisdom about teaching that was far removed from the tasks of teaching. The teachers worked to specify what mathematical sense was necessary for the mathematical work of teaching.

CHAPTER 5
CHALLENGES OF TEACHING MATHEMATICAL KNOWLEDGE FOR TEACHING

5.1 Introduction

In this chapter, I use examples from the data to investigate what components teacher educators consider in order to manage the tasks of teaching MKT. Teaching MKT creates distinctive dynamics of instruction of which teacher educators need to be aware. In this study, what teacher educators attend to is a social property because teacher educators could make sense of it as shared practices in the context of teacher education (Levin, 2008). Moreover, teaching MKT requires practical rationality (Herbst & Chazan, 2003). This rationality is at work where people perform the same job and it cannot be reduced to individual wisdom, talents, sensibilities, or skills (Herbst & Chazan, 2003). Even though teacher educators are knowledgeable about mathematics and familiar with working with teachers, they do not find teaching MKT to be easy. This is because of the different goals of instruction of MKT and the complexity of the task in teaching MKT. I focus on conceptualizing what would be hard in teaching MKT and what would help teacher educators avoid obstacles and manage challenges. This has a direct influence on teachers' engagement in learning MKT. What MKT teacher educators plan and consider is important and influences on teachers' learning. In other words, how teacher educators manage the task of teaching MKT and what teacher educators pay attention to in teaching MKT play critical roles in the front line of mathematics teacher education.

The analyses of the data started with the factors shown in *Figure 2.2*. The factors are from the literature review about the dynamics of teaching from the perspective of teacher educators in mathematics teacher education. The overall purpose of the analyses, specifically for this chapter, is to identify in what ways MKT is worked in the instruction of teacher education. I looked at this in terms of what is likely to be hard about teaching

MKT for any teacher educator in any situation and what might be helpful to teacher educators. Finally, to respond the overall question, I decided to focus on a more specific illustration about the tasks teacher educators face in those moments and situations. Even though the findings in this chapter are not illustrated with an instructional triangle in teacher education, they are from the analyses in terms of an instructional triangle in teacher education. Findings in this chapter heavily depend on the analyses of the video recordings, which show the implemented phase of the curriculum materials. Nevertheless, this chapter is also based on both the literature review and an analysis of the curriculum materials. This is to develop a conceptualization of what might be attended to by teacher educators in and for the teaching of MKT. Finally, this research identified the following tasks of the teaching of MKT that teacher educators need to be aware of in mathematics teacher education:[17]

- Establishing the focus of an activity
- Recognizing mathematical issues
- Focusing on mathematical articulation and interpretation
- Emphasizing ways of mathematical thinking
- Managing mathematical ideas

Throughout the descriptions in the previous chapter of Matthew's and Emily's lessons, I tried to highlight examples of these tasks and challenges in which I thought the teacher educators were trying to manage them. My intent in situating this discussion in the context of a lesson was to explain the complexity of teaching MKT. Specifically, the explanations focused on how tasks could be managed and how the challenges would be attended to during instruction.

In this chapter, this study now concentrates on each task and challenge that teacher educators need to address. This chapter elaborates each one independently using examples from across the set of lessons in the study. Taking the cases out of the context of a particular lesson allows them to be unpacked in more detail. In particular, looking across a range of teacher educators' practice elucidates a wider variety of strategies for managing the task of teaching MKT and a broader collection of challenges that teacher

[17] As described in Chapter 3, these tasks were identified when coding the data, in particular, the video recordings.

educators may confront in teaching MKT. Even though this research focuses on each task and challenge independently, it should be kept in mind that the challenges overlap and may occur simultaneously in instruction. Moreover, the management of a particular task and challenge is not associated with a specific set of teaching practices for overcoming that challenge. At any time, in fact, there are a variety of teaching moves that could address each one. Moreover, a particular teaching move could be used to manage multiple tasks and challenges.

For each task, this research provides a general description of each task of the teaching of MKT as well as discusses what teacher educators need to pay attention to and challenges that teacher educators need to consider. This research illustrates these points by referring back to examples from Matthew's and Emily's lessons, as well as to other lessons in the data.[18] Because some of these examples are used to show what is difficult about the tasks of teaching MKT, this research sometimes describes an aspect of a lesson that did not go well, such as an episode that was not clear. The purpose of such an example is not to claim that a particular teacher educator was not able to teach MKT well or to highlight the quality of the people who do that work. Here in the dissertation, this research uses examples from the data to characterize the tasks of teacher educators in terms of what teacher educators paid attention to and the challenges they faced in teaching MKT. Moreover, a particular teacher educator's tasks and management of the challenges most likely varies within and across lessons. Therefore, it would not be warranted to make those sorts of broad claims about an individual teacher educator.

It is also important to bear in mind from the outset that some issues that arise with respect to the different attentions and challenges might be attributed to the nature of the curriculum materials and to the details of an activity that were taken directly from them. For example, focusing on mathematical articulation and interpretation might be more evident in a curriculum which uses records of practice such as the materials. The distinction is not critical at this time. However, the role of the curriculum materials and the influence of other factors such as teacher educator's knowledge are very important in considering implications in teaching practice and helping to manage the challenges.

[18] It might be helpful to refer back to Table 3.2 and Table 3.3, which list each teacher educator's name, background, and lesson topic.

Furthermore, the objects in tasks of teaching addressed in this chapter are elaborated in Chapter 6 in terms of the mathematical work of teaching and knowledge about mathematics. For example, in recognizing mathematical issues, mathematical issues are related to mathematical facts and structures that are considered as knowledge about mathematics in Chapter 6. While teacher educators perform the tasks of the teaching of MKT described in this chapter, the mathematical work of teaching and knowledge about mathematics that are unpacked for teaching MKT in Chapter 6 are worked as content in mathematics teacher education.

This chapter turns now to a discussion of the first task of the teaching of MKT that emerged in my analysis: establishing a mathematical focus of a task.

5.2 Establishing the Focus of an Activity

Establishing goals of a lesson is central to any kind of teaching practice. The clear specification of an activity is critical in deciding whether instruction moves on track properly. Therefore, it is not surprising that offering and specifying activities appropriately and providing key questions to launch activities were critical for the work of teaching MKT by the teacher educators in this study. However, at times the teacher educators lost track of this critical component by inappropriately interpreting the activity. For example, focusing on which grade level students can learn a certain concept is different from investigating concerns that students might have in understanding the concept. Being clear about the focus of an activity helps to clarify what is exactly supposed to be done in a lesson and focuses attention on MKT. Having a clear MKT learning goal can help to prevent sliding into pedagogical issues or talking about disciplinary mathematics. It also explains what a lesson has as a directing point, which is developing teachers' MKT. Based on the analyses of the data, three tasks of the teaching of MKT recurred across the work of the teacher educators in the study: giving and specifying an activity, helping teachers understand an intended activity, and identifying the purposes of an activity.

5.2.1 Giving and specifying an activity

The teacher educators in this study generally developed their classes through four steps: (1) giving and specifying an activity; (2) having teachers engage in individual, pair or group work; (3) having a whole group discussion; and (4) closing a class. Giving and specifying an activity was carried out at the beginning of classes and set the direction of the work. However, giving and specifying an activity involved a couple of issues: (1) whether and how well teacher educators specified an activity, and (2) whether a particular activity or a topic was proper for teaching MKT. These issues seemed typical, but they were not simple in practice. In teaching MKT specifying exactly what an activity is is no easy task. For example, after mentioning precision and usability as the two criteria for a good definition, Betty, one of the teacher educators, gave an activity – writing a precise definition of an even number.[19]

Betty:	OK. So, your task, right now. I asked you write about even number on Wednesday, thinking about these things. … I want you to write down a precise definition of what even number is. OK? Take a couple of minutes to think about it and write down a precise definition.
Teacher30:	*Do we need to think age level or grade?* Or–
Betty:	*You have to decide that.*
Teacher30:	OK. (emphasis added)

In this episode, Betty clearly said "Write down a precise definition of what an even number is." However, the activity was not obviously identified because she did not explain what a precise definition is and did not provide a guiding question to help teachers appraise their definitions to carry out the activity. Mathematical precision is of course critical in mathematics. However, this episode shows that precision could be disregarded without clear assumptions. It also shows how difficult it can be to focus attention on maintaining mathematical precision. Again, the activity was "Write down a precise definition of what even number is." Being precise was the main goal of this activity. Moreover, this episode illustrates what a challenge it can be to properly deal with one teacher's concern about considering students' levels of understanding to write a precise definition. Mathematical precision is not related to the level of the audience, and regarding students' grade is associated to students' school curriculum rather than

[19] All teacher educators' names are pseudonym.

mathematics entailed in teaching. In teaching MKT, effectively articulating what an activity is in a lesson is an arduous task.

One way to specify an activity is using examples for the activity and giving questions that help doing an activity. Here is one example that illustrates how Emily, one of the teacher educators, presented an activity. After showing three definitions of an even number on the slide, Emily detailed what the activity was:

> Think about each of the definitions and how usable they are and how precise they are. And so when you think about usability, don't try to think about what age of student they're usable for, but what's the knowledge that they have to have to be usable? So, for example in the first one, 'An even number is a number in the form of 2k where k is an integer.' So maybe the first thing you would notice is, well, 'integer' is something that needs to be unpacked a little bit more. And so to use this definition you need to understand that. So … think about what are the terms you kind of need to set the foundation for to make them usable. And then how precise are they? How well do they actually separate the even numbers from every other number? So does that task make sense? So choose your partner and try to go through each one and use these criteria to think about it.

In this episode, Emily asked the teachers to evaluate the definitions with the two criteria of precision and usability. In terms of usability, the teachers were expected to find concerns for using the definitions. It is noteworthy that she differentiated the two questions of what knowledge is essential to use a definition and what age of students can use a definition. She then emphasized that the former one had to be investigated. Emily referred to one definition and identified one of the concerns. In other words, giving a strategic example for the activity could help teachers not be distracted by other issues. Moreover, providing a specific question, such as Emily's question for the other criterion – precision, is also critical. This episode shows what teacher educators would attend to for the clear illustration of what teachers are supposed to do and how teachers conduct a provided activity.

A major problem that might arise in giving an activity is the topic of discussion falling into a non-MKT topic, such as talking about what teachers are impressed by or how teachers go about mathematics. For example, the teachers in Julie's class watched the SeanNumbers-Ofala video that shows third grade students' discussion about an even number. In the video clip, one student asserts that six is even and odd and his classmates try to refute his assertion. Julie and her teachers investigated the definitions used in students' reasoning. Then, Julie turned to the topic – reactions to the teacher and the students in the video clip.

Julie:	Let's talk about the classroom for a minute. We don't have to take notes on this but, what was your reaction to this class?
Teachers:	Wow.
Julie:	Wow. OK, OK. What made it 'wow'?
Teacher31:	Participation.
Teacher32:	The very rich discussion.
Teacher33:	The intellectual community that was there.
Julie:	Question: just these special kids? No.
Teacher34:	The teacher had to facilitate that, and I bet it took her a lot of time to get it to that point.
Teacher35:	It was January. She had that fourth beginning part of the school year. I imagine in September they didn't have those kind of rich discussions. She didn't have that kind of classroom to start out with. That's something that you have to build.
Teacher36:	But we also don't know what the other seventeen kids that were not talking in that group were thinking. And you can't analyze what they're doing and what they're thinking unless you get something out of them.
Julie:	Ok, of the nineteen students how many did you see- because, remember, we only saw the front of the classroom. How many do you think were really engaged?

This episode shows just the first part of their talking about their impressions of the teacher and the students in the video clip. This class spent approximately eleven minutes to share their impression about the video clip. Most of their talking was about what they liked and what they were doubtful about in the video clip: teachers were impressed that students could discuss mathematics, curious about how many students participated in the discussion, loved that students freely use a blackboard, among other reactions. Obviously, this would seem a natural topic. When an audience views a video clip and feels surprised, they would like to share their impressions. If this class had this discussion before investigating definitions used by the students in the video clip, the discussion would work to move on to discussing the students' ideas and reasoning about even and odd numbers. However, giving the topic to share the teachers' impressions in the middle of the whole discussion does not deal with mathematical knowledge, skill, and habits of mind entailed by the work of teaching. Furthermore, spending more time on their impression than mathematical knowledge for teaching seemed to confirm what the teacher educator concentrated on in the lesson. Therefore, when a teacher educator uses a record of practice, such as a video recording of a classroom or student work, it is difficult to choose an activity and have a discussion that leads to MKT rather away from it.

One of the difficult tasks of teaching MKT by the teacher educators is curriculum issues. For example, some teacher educators in this study often initiated a discussion about which grade level students can understand definitions or how students understand definitions. If the teacher educators and the teachers open the guides that explains curriculum, they could easily figure it out. But, it is not an object in teaching MKT. Therefore, another difficult task in teaching MKT is giving an activity that focuses on MKT.

In terms of topics, it is also hard to distinguish a topic for teaching MKT in teacher education from a topic for teaching mathematics in school. Topics for teaching MKT do not aim at being used as topics in school. In other words, although definitions of an even numbers are used as main topics to teach MKT, it is not reasonable for teacher educators to emphasize or recommend that teachers teach definitions of even numbers to their students. For example, Sandy asked teachers whether they tried to teach even numbers in their school before wrapping up a class.

Sandy: *Have you ever done that (Having this discussion about even and odd numbers)?*

Teacher37: Integer works …

Sandy: So even and odd numbers where we extend to the integers. And use some proofs that way, too. *Because this is a rich environment for working on proofs in a manageable way with kids.*

Teacher37: I actually did the first even definition in class. I went right into my advanced class and did it with them, and it was a wonderful class.

Sandy: So *tell me what happened.*

Teacher37: ... Because I asked them to write down the definition of an even number – and here I'm teaching the definition of slope. And they're like 'Even number, where are you coming from?' So it took them a minute to kind of focus, and then they wrote out their definitions, and then I had them share their definitions in a group, talk to each other about it. … You could just sort of see all their little heads going sideways like this because when you started talking about [inaudible]. They were questioning their own thinking. … So it was that good, deep thought that you like to do with students.

Sandy: … Well *I'm so glad you tried that. Did anybody else try it? Yes, what happened?*

Teacher38: … I said 'Well what is an even?' And they got in the hugest argument. I mean, this was my high class in fourth grade, and one side of the students, "No, no, no... this has got to be it." I said "Well let's try it." … And I mean, it was constant, and when they finally figured it out.

…

Sandy: *You worked on that piece of the definition there as well?*

Teacher38: Yeah. (emphasis added)

In this episode, the class shared a couple of teachers' experiences where they taught definitions of an even number. Teachers can teach a topic that they discussed in professional development. However, it is required for teacher educators to differentiate between a topic that the teachers perform to learn MKT and a topic that students carry out to learn mathematics. Careful epistemology about different instructional phenomena is required of teacher educators when they give activities and topics to teach MKT. Moreover, sharing teaching experiences is not a MKT topic.

5.2.2 Helping teachers understand an intended activity

Even if teacher educators specify an activity quite well, teachers might not fully understand what the activity is and what they are supposed to do. Ultimately, they might fail to engage in the activity that teacher educators intended. Therefore, teacher educators need to be concerned about whether their teachers get started on the right foot.[20] Teachers' revoicing of what the activity is would be a strategy that teacher educators use when they present the activity. For example, after Emily introduced the units digit rule, she asked teachers to explain why the rule works.[21]

Emily: Spend a couple minutes, just by yourself first. See if you can think about how you could prove that this rule works. Why would it make sense that you could do this? So kind of the big question is, 'if I have a number like this (12,327), why is it that this is the only digit that matters to me?' So, *can someone try to say again what it is I'm asking you to work on?*

Teacher39: We're working on a proof for why the units digit rule works.

Emily: Right, so why is it we can look at the last digit or the units digit of a number, to decide if it's even or odd. So that's the question. So

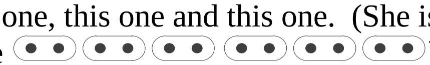
[20] Having teachers be engaged in the activity that teacher educators intend does not depend on only understanding the activity. Teacher educators are responsible for managing teachers' staying on track through a class. This management is discussed in recognizing mathematical issues and managing mathematical ideas later.

[21] The units digit rule is "To determine whether an integer is even, examine the ones digit. If it is even, then the entire number is even. If it is odd, then the entire number is odd."

spend a couple minutes by yourselves independently and looking at this rule. (emphasis added)

In this episode, Emily gave the activity with a supporting question to help teachers perform the activity. Moreover, she asked teachers to revoice what the activity was and one teacher gave her understanding of the activity. Revoicing by teachers could help teacher educators know whether their teachers understand the activity.

Even though teachers initially understand what an activity is, they are prone to losing track of it throughout a class. Therefore, teacher educators need to attend to how teachers' understanding shifts. Teacher educators would help elucidate those moments when the teachers get lost in discussions. One way to keep teachers' understanding on track is to ask them questions about whether they are following the discussion and whether they recognize representations. For example, in Nellie's class, to prove the conjecture, odd + odd = even, the class investigated one of their proofs and tried to find a definition of an odd number used in the proof.

Teacher40: An odd number into groups of two, there's going to be a remainder. There's going to be one that doesn't have a group.
Nellie: (Nellie is pointing out the proof) Does this go with the definition? Is that what you guys meant? Any integer that cannot be divided equally by two – is this the same thing as this? *Teacher41, what do you think?*
Teacher41: (pause) Yeah.
Nellie: *Yes? Does this make sense to all of you?* (Nellie is pointing out the written definition.) *Do you understand the definition for odd numbers?* When you break up an odd number into groups of two, there is going to be a remainder, one. *Can you picture that?*
Teachers: Yeah.
Nellie: *Does this go with their picture that they have here?*
Teacher42: (One teacher comes to the blackboard) They are… have to be circled together. And, this one, this one and this one. (She is making groups of pairs, like )
Teacher43: Then you wouldn't have the one leftover from each group to add together.
Nellie: So where do you see an odd number in this drawing? Dan?
Teacher44: Well ok, you have five on this side and you have seven on the other side.
Nellie: So you have five here, and here's seven.
Teacher44: Yes.
Nellie: How does this definition show that they are odd?
Teacher44: When you group them together. Well, when you group one number and then the other number, there's two and two and then one left

over for the five. On the other side there's two, two, two, and one left over, which is seven. And then-

Nellie: *Before you continue, does everybody see that this definition goes with that? Those locations?*

Teachers: Yes.

Nellie: So we see that five is an odd number because when we break number into groups of twos, or into pairs, we have one left over. So five is –

Teacher 45: An odd number.

Nellie: An odd number. And seven is an odd number, because you see the pairs and there's one left over. (emphasis added)

In this discussion, without hindering their mathematical discussion, Nellie asked the questions to decide whether her class understood and shared their findings well. The one teacher's representation worked as evidence for both Nellie's decision and reinforcement of their finding to all teachers in the classroom. Because of her effort to stay aware of teachers' understanding, she could easily recognize when the teachers could not understand the activity. After the class discussed the proof which was based on a definition of an odd number, "When you break up an odd number into groups of 2, there is going to be a remainder 1," they acknowledged another definition of an odd number, "Odd numbers can be split into 2 equal groups, with 1 left out" in another proof. However, they found the proof was not consistent. Nellie asked the teachers to prove the conjecture with the latter definition.

Nellie: So what I want you to do right now is to write a proof for yourself to show that this conjecture is true, but this time you are using the second definition. The definition is up here. … (Nellie is pointing out "Odd #s can be split into 2 equal groups, with 1 left out" on the poster.) … I want you to write a proof for the same conjecture by using a different definition.

Teachers: (Teachers speak simultaneously. What they say is not audible.)

Nellie: *Is this task clear to everyone? Is it clear, Teacher46? Is this task clear?*

Teacher46: I'm confused.

Nellie: *Are you confused?* OK. I want you to write a proof for the same conjecture, meaning we want to prove that it works for all odd numbers. But this time, we want to use a different definition for odd numbers than the one that we talked about. … And this is the second definition. (Nellie is pointing out the definition in the poster.) I want you to use the second definition.

Teachers: (Teachers work and discuss.) (emphasis added)

In this episode, Nellie tried to see whether the teachers understood the activity and found Teacher46 had difficulty understanding it. Nellie clarified what the activity is –

proving the conjecture with the second definition. This episode shows that even though teachers are adults, they need some guidance in carrying out activities, and teacher educators need to help teachers stay on top of the intended MKT.

Evaluating the teachers' first presentation is also critical to helping them recognize an activity. If teacher educators fail to manage it well, the whole class could slip into a non-MKT focus. For example, Betty gave the activity to explain why the units digit rule works, and the teachers had a group discussion about it. In a whole group discussion, each group presented a couple of examples that they worked on with transparency. This activity was about having a general explanation about the rule rather than having a few examples that shows how the rule works in particular instances. Although there was no explanation of why the rule works, Betty's response for each group presentation was in most cases "Questions? OK. Another table." Therefore, it would be hard to evaluate and clarify the first presentation in teaching MKT.

A possible strategy is, thus, evaluating the teachers' first presentation and helping them not deviate from the intended MKT. If teacher educators briefly evaluate whether teachers work appropriately on an activity during the first presentation, the teachers could get a feel for what they are supposed to concentrate on. For example, Sandy gave an activity to explain why the units digit rule works, and the teachers had group discussions about that rule. The first group presented their discussion to the whole group.

Teacher 47: We talked about- she had done this just today with her class of fourth graders. … In second grade when we did it … And we had a discussion about fractions and whole numbers, and that a fraction is not actually part of the whole number. … So we got into the discussion about even and equal as separate words. … (Teacher12 speaks for a minute and a half.)

Sandy: *So did you have any kind of conclusion about why the 0, 2, 4, 6, 8 rule works as a rule that we use?*

Teacher 47: Well, when we talked about it, we looked at numbers like thirty-four and talked about that- a little girl in my class said 'Well, because it ends with the four'. I said 'Well what about the thirty? The three is- that's most of the number, so shouldn't it be even or something?' And she said 'No, because it would actually be thirty plus four, and thirty you can divide into two equal groups.' You can show the cubes and separate them. Even, excuse me, even.

Sandy: Two groups that were even.

Teacher 48: (A teacher from the same group of Teacher1): That was our definition – it ends in 0, 2, 4, 6, or 8, a whole number.

> Sandy: *That was our rule. … They couldn't quite give an argument about why the ending digits would make it justify.* (emphasis added)

In this episode, unlike the activity that Sandy gave teachers, Teacher47's group shared their experience in their schools. They fell into pedagogical issues, and did not do the provided activity for MKT. Sandy directly asked a key question of the activity and obviously clarified that their presentation was not related to the activity. Therefore, teacher educators' clarification can work to assure that presenters and other teachers do not distort the intended MKT. Without teacher educators' evaluation and clarification for the first presentation, the classes might waste time on a non-MKT focus.

5.2.3 Identifying purposes of an activity

When people know the purposes of what they are doing, they do not easily lose their direction. There are two kinds of strategies in terms of identifying the mathematical purpose of an activity so as to help teachers be involved in the intended MKT activity: specifying purposes of an activity in the beginning of a class and in the middle a class. Each of them works differently. For example, Nellie clarified purposes of a lesson in order to motivate teachers in the beginning of a class.

> Today … *We will be thinking about how we can use those definitions to provide proofs for our mathematical conjectures*. And in mathematics, we see a lot of patterns around us, and we make conjectures about patterns that we notice. And then we try to prove those conjectures or sometimes try to disprove those conjectures. And *to get a better sense of what proof- the process of proof looks like in mathematics, and what kind of role definitions play in that process*. We'll practice proving some conjectures today. (emphasis added)

Although the purposes were offered with other comments, she apparently identified the purposes which were constructing proofs of a mathematical conjecture and experiencing roles that definitions play in constructing mathematical proof. Identification of purposes could work to show what teacher educators are expecting of teachers in a class.

Another strategy would be to articulate the purpose in the middle of the class. For example, Emily' class evaluated definitions from textbooks. After discussing a couple of definitions, she explained the purpose of their activity.

> I just want to highlight, so *this whole conversation is kind of pointing toward the idea of why definitions matter in schools.* Why is this something that as a teacher, it's worth thinking about? And I think *there are these important systematic ways*

to think about teaching or testing definitions for if they're precise or not, and then also what are the considerations that you have to have, especially in a classroom.

She explained developing skills for using definitions in the context of instruction and developing appreciation for the role of mathematical language in teaching. Therefore, how a teacher educator identifies the purposes of an activity depends on his or her style.

Table 5.1 summarizes the above discussion of the challenges of establishing the focus of an activity. A similar summary table will be included after the discussion of each challenge.

Table 5.1 *Summary of the Establishing of the Focus of an Activity*

What teacher educators need to attend to for establishing a mathematical focus of an activity:

- Giving and specifying an activity
    - Giving a proper activity for teaching MKT
    - Specifying an activity in detail
    - Giving key questions that help doing an activity
    - Offering an example to show how to perform an activity
- Helping teachers understand an intended activity
    - Asking teachers to revoice what an activity is
    - Ensuring that teachers understand an activity
    - Evaluating whether the first teachers' presentation is on the right track of intended MKT
- Identifying purposes of an activity
    - Specifying purposes of an activity in the beginning of a class to identify what a teacher educator expects teachers to do in a class
    - Specifying purposes of an activity in the middle of a class to illustrate skill and appreciation for teaching

Challenges that teacher educators might face:

- Lack of explanation about an activity
- Falling into or offering non-MKT topics
- Confusing an activity to teach MKT in teacher education with an activity to teach mathematics in school
- Not giving an evaluation of the first teacher's presentation according to the learning goals

5.3 Recognizing Mathematical Issues

In teaching MKT, teacher educators recognize mathematical issues from teachers' statements and ideas and help teachers use and learn mathematically accurate knowledge. For example, in the middle of a pair activity, Matthew, one of the teacher educators in the data, went around his classroom and found issues that the teachers had. He briefly opened a whole group discussion and identified types of numbers for the activity. Teacher educators open their eyes and ears wide to observe and hear what teachers say and show in lessons and to seize opportunities to clarify what mathematical issues and ideas they have. Because the instruction of MKT is based on interaction as shown in *Figure 1.1*, recognizing mathematical issues would help teachers' active involvement in a lesson and their development of MKT from their ideas and experiences. However, it might be also easy to pass over or overlook mathematical issues even though teachers' explanations were not accurate. Based on my analyses of the data and the literature, three tasks of the teaching of MKT recurred across the work of the teacher educators, with regard to recognizing mathematical issues: identifying and acknowledging teachers' mathematical ideas, responding to help teachers have appropriate ideas, and identifying mathematical features of teachers' statements.

5.3.1 Identifying and acknowledging teachers' ideas

Teacher educators' articulation of the teachers' ideas shows what mathematical issues the teacher educators acknowledge. Seen throughout the data, the teacher educators repeated the whole or part of the teachers' comments. Moreover, the teacher educators rephrased the teachers' ideas, interpretations, and observations and represented them in the public space. For example, Matthew heard one teacher's conjecture, in particular one-half can be two groups of two one-eighths, when the class discussed, whether "A number is even if it can be put into groups of two with none left over." He went to the board and wrote out her ideas, shown in *Figure 4.28*. It seemed to work for Matthew's understanding of teachers' comments as well as to help the teachers reason and discuss topics clearly in classes. However, it would be sometimes hard for teacher educators to correctly articulate teachers' comments. Moreover, it might be easy for them to ignore teachers' ideas that are essential to performing activities.

A more critical challenge would be that it is very easy to pass over mathematical issues even though the teachers' ideas, interpretations, and observations are not accurate. It might be hard to acknowledge teachers' inappropriate and incomplete reasoning, wrong uses of language, and improper interpretations that intermittently surface in a lesson. For example, while one teacher assumed that the complementary set of even numbers was equivalent to the set of odd numbers in Emily's class, this idea was passed over and not discussed in the class. One teacher had an incorrect observation for Sean's definition of an even number in Daniel's class; however, this observation was not recognized and focused on by the teacher educator and other teachers. Although one teacher used an imprecise definition in Cate's class, that definition never became an issue. Another challenge might be to acknowledge incomplete mathematical explanations but to not comment on them. For example, the teachers in Betty's class presented how they proved the conjecture, odd number + odd number = even number. Even though most presentations included mathematically imprecise explanations and representations, she did not give comments or have discussions about these issues. As a result, her class ended without any instruction on the mathematics entailed in teaching mathematics. Therefore, if teacher educators overlook incorrect reasoning or ideas and do not offer appropriate comments on them, this lack of response might act as a tacit agreement and transmission of an affirmative message and teacher might regard their ideas and reasoning as precise. Because instruction occurs in a very fluid context in a lesson, if teacher educators missed proper moments, it might be very hard to help teachers improve their knowledge.

5.3.2 Responding to help teachers have appropriate ideas

Teacher educators generally give responses when teachers present their ideas. In the data, the teacher educators' responses were sometimes short, such as "Right" and "Definitely," and sometimes brief comments, such as "There is more," and "I think it would." These responses show that the teacher educators recognized some issues in the teachers' statements and presented their opinions. Furthermore, the teacher educators' responses worked for making mathematical decisions whether the teachers' statements are correct or incorrect, and it helped teachers have appropriate mathematical ideas.

Giving a response, however, is hard. One of the challenges might be praising teachers and agreeing with their assertions despite their not being mathematically proper. For example, when one teacher chose an imprecise definition that he would use in school, Cate, one of the teacher educators in this study, agreed with his choice. Giving inappropriate comments on improper ideas might also be a challenge. For example, a few teacher educators in the data decided that imprecise definitions were mathematically precise ones. Moreover, when the teacher fell into a discussion of curriculum issues, Betty gathered teachers' attention to the teacher's statement and inappropriately emphasized grade levels until the end of the lesson. It would be also easy to overlook mathematical issues even though teachers' explanations are not accurate. It might also be hard to critically respond to teachers' statements that are mathematically unsound and disprovable. Not commenting at all could be interpreted as not judging one way. However, it seems that no comment on the part of the teacher educator might generally be interpreted as approval that a statement is correct.

5.3.3 Identifying mathematical features of teachers' statements

In responses to teachers' ideas, teacher educators identify the mathematical features of them. For example, when Matthew collected teachers' examples of even numbers to test the units digit rule, all teachers' examples were three-digit numbers. He pointed out that if teachers proposed only three-digit numbers, students might think the rule works for only three digit numbers. This interaction shows not only mathematical features of the examples but also mathematical concerns from the examples for teaching. Furthermore, teacher educators mathematically would interpret the teachers' ideas. For example, Emily specified the parity of evenness and oddness for integers as a response to the teacher's identification of decimals that are not regarded for even and odd numbers. Emily continued her explanation to identify an assumption of the rule. Teacher educators' brief identification could offer starting points for more major mathematical issues of activities. More fundamental approaches to mathematical issues are classified as following section, focusing on mathematical articulation and interpretation.

Table 5.2 summarizes the above discussion of the recognizing mathematical issues.

Table 5.2 *Summary of the Recognizing Mathematical Issues*

What teacher educators need to attend to for recognizing mathematical issues:
• Identifying and acknowledging teachers' ideas
  o Repeating or rephrasing teachers' statements
  o Representing teachers' ideas in the public space
• Responding to help teachers have appropriate ideas
  o Giving brief comments about teachers' ideas
• Identifying mathematical features of teachers' statement
  o Pointing out features of examples that teachers made
  o Mathematically interpreting teachers' ideas

Challenges that teacher educators might face: • Incorrectly articulating teachers' comments • Ignoring a teacher's idea that is essential to perform an activity • Overlooking mathematical issues that are embedded in teachers' mathematically inaccurate, improper, or incomplete statements • Praising or agreeing with mathematically improper ideas • Offering inappropriate comments for improper ideas

5.4 Focusing on Mathematical Articulation and Interpretation

Giving precise mathematical interpretation is one of the most important and also most frequent tasks throughout teaching MKT across the cases in this study, such as giving an explanation of a term and using counterexamples to refute definitions. This task greatly helps the instruction of MKT have a mathematically sound foundation. However, it does not mean that discussing MKT is talking about mathematics in way that is remote from teaching, such as typical courses in mathematics department. Rather, it is important that the teacher educators should make efforts not to weaken mathematical soundness in the instruction of MKT. Nevertheless, the teacher educators in this study sometimes had challenges offering mathematically precise comments and interpretation. From the analysis of the data, four tasks of teaching MKT recurred in terms of focusing on mathematical articulation and interpretation: (1) specifying mathematical terms, facts, and relations; (2) identifying mathematical objects; (3) identifying mathematically appropriate ideas and interpretation; and (4) using examples for mathematical investigation.

5.4.1 Specifying mathematical terms, facts, and relations

Mathematically sound knowledge is an essential prerequisite for any kind of mathematical discussion. Since instruction for MKT is also grounded in mathematics, teacher educators occasionally explain meanings of terms and specified properties and relations in mathematics in order to use accurate mathematical knowledge. For example, Daniel, one of the teacher educators, showed the sets of integers and even numbers before his teachers evaluated definitions of even numbers because the activity was concentrated on evaluating definitions rather than knowing what integers are. When he wrote the set of even numbers on the board, he said “So realize that we’re counting negative six as an even number, zero as an even number, negative two as an even number.” He pointed out features of the set, and he seemed to expect that teachers would consider the set of even numbers and the features when they evaluated definitions of an even number. Specifically, zero and negative even numbers are even numbers, and a definition should include them. His explanation helped teachers reduce the mathematical burden. Moreover, his specification supported teachers to recognize the mathematical demands for the task of teaching mathematics. Like Daniel’s case, teacher educators would sometimes spend time identifying mathematical facts, such as types of numbers and their relations.

Whether or not the teacher educators’ explanations about mathematical terms, fact, and relations are short or lengthy, they are very critical in the instruction for MKT. When teacher educators give inaccurate mathematical facts, their discussions could be mathematically unreasonable. A challenge might be that while teacher educators recognize the need to explain a term, they might provide no such explanation. For example, when Daniel’s class evaluated “An even number is a natural number that is divisible by 2,” he recognized the need to explain the meaning of divisible, specifically to say “we need to know what divisible means,” but he did not. Moreover, Julie confused divisible and divided with a couple of examples, such as “Is 24.6 divisible by two? Yes.” In everyday usage, “divisible” often means simply that something can be divided. However, in school mathematics, it is typically agreed that if a number (n) is said to be “divisible by” a number (m) it means that n and m are integers such that n can be divided by m with zero remainder. It is important for students to understand this special mathematical meaning of divisible (Mathematics Teaching and Learning to Teach, 2008, 2009). The classes of these two teacher educators ended without differentiation between divisible and divided.

Moreover, teacher educators would pay attention to specifying mathematical properties and relations. For example, Sandy, one of the teacher educators, specified the structure for the divisibility of six after her class investigated the divisibility of two and three. Julie, one of the teacher educators, also identified integers as the implied universe of even and odd numbers. She also pointed out that “number” in an even number indicates integers rather than fractions or irrational numbers. The mathematical specifications of these two teacher educators helped their teachers move to cores of mathematics that are not often explicitly considered. In contrast, inappropriate articulation might hinder teachers’ performing an activity correctly. For example, when specifying the units digit rule, Kellie, one of the teacher educators in the data, limited the scope of numbers as whole numbers, and the teachers, therefore, investigated why the rule works within whole numbers unlike what other teachers did within integers in other classes. Moreover, Betty, one of the teacher educators, introduced the rule as “If the ones digit is 0, 2, 4, 6, or 8, it’s an even quantity.” This articulation was the same as the definition that her class criticized with counterexamples in the previous class. Because of the absence of detailed explanation of the rule, nobody seriously tried to explain why the rule works in Betty’s class.

How well teacher educators explain mathematical objects would make the instruction of MKT mathematically sound. MKT is not general mathematics but special mathematics knowledge for teaching. Nevertheless, it is also important that MKT as content is based on mathematical soundness.

5.4.2 Identifying mathematical objects

Teacher educators would pay attention to mathematical objects for making pedagogical decisions and articulating students’ statements in a record of practice. Identifying mathematical objects for pedagogical decisions aims at illuminating mathematical objects that teachers need to consider as pedagogical concerns. For example, in the class to evaluate definitions, Matthew found form, variables, and 2k as mathematical concerns to understand the definition that an even number is a number of the form 2k, where k is an integer. Identifying mathematical objects that teachers need to be concerned about shows a specific way of thinking for teaching. It also emphasizes the importance of the careful use of language in teaching.

Teacher educators would also pay attention to articulating mathematical ideas that students have in records of practice that teacher educators observed with their teachers. Even though several strong points attend using records of practice in teacher education, it also calls on teacher educators to carefully identify mathematical ideas of students' from the records of practice. If teacher educators are unable to accurately capture what mathematical ideas students assume from the records of practice, using them would not work well in teaching MKT. Finding definitions that students implicitly and explicitly used in a video clip can be a main activity itself for teachers to probe students' definitions. However, since a school classroom situation is complicated, it might be also hard for teacher educators to specify definitions used by students from the record of practice. [22]

5.4.3 Identifying mathematically appropriate ideas and interpretation

One of the main tasks that the teacher educators generally carry out is interpretation. Seen through the data, the teacher educators gave explanations for definitions, rules, and proofs. For example, Kellie, one of the teacher educators in the data, interpreted definitions of an even number with pictures. Matthew compared two different concepts to define an even number: groups of two which is called pairing and two equal groups which is called fair sharing. In the class to explain why the units digit rule works, Emily explained why 12,350 is even based on long teachers' discussions in order to finally clarify why 7 is critical to decide whether 12,357 is even or odd. She said:

> So why is this number (Emily is pointing out 12,350 written on the board.) even? Well this number is even partly because it has a ten. We can always pull out a ten. And from this ten- we can break this ten into a two and a five, and we could take this two and pull it to the front. And that means that this whole number is even, because it's two times and integer.

[22] Analyzing students' definitions is described in the following section.

Teacher educators also interpret teachers' mathematical comments or representations. For example, Nellie interpreted a teacher's comments about the incoherent use of definitions in proof:

> What Lily is saying is she was thinking that grouping- matching those two together, but then she realized that what she would end up having wouldn't go with the definition of even numbers that we started with.

Teacher educators' explanations would work well when they are mathematically accurate in appropriate moments. However, giving accurate explanations when they are necessary is not easy in teaching MKT. For example, in Betty's class, one teacher complained that her partner did not understand that zero is an even number because zero is nothing. Although several teachers presented their ideas about zero, Betty closed their talks without any clarification about zero. Moreover, she gave inappropriate explanations about incomplete proofs for the units digit rule rather than giving mathematical explanations to refine their proofs.

When teacher educators use a record of practice, they also focus on analyzing mathematical comments from it. Although analyzing the students' statements from the video clip would be one of the activities that teacher educators could give their teachers, the teacher educators also have to work together with the teachers. For example, after articulating definitions shown in the SeanNumbers-Ofala video, the teacher educators in the data probed definitions with their teachers, compared the definitions and identified how they are mathematically different. However, it would be hard for teacher educators to analyze the record of practice correctly. For example, Daniel analyzed the definition that Sean used for an even number as equivalent to Ofala's, which in fact is not equivalent. Julie also did not well articulate why Sean thought six can be an odd number while her class spent time discussing his ideas. Therefore, when teacher educators use records of practice in their instruction, it requires their careful attention to interpreting them.

5.4.4 Using examples for mathematical investigation

Using examples is one strategy to focus on mathematical investigation in classes. Teacher educators would use examples as preludes for moving toward considering generalization. For example, Emily suggested using an example to explain why the units digit rule works before her class created explanations that work in any cases, not only in one case. Matthew also pointed out three types of numbers – positive integers, negative integers, and zero – and used numbers from the different types to test definitions of an even number. Moreover, teacher educators would refute imprecise definitions with counterexamples. Yet, when teacher educators use examples, they need to consider their features. Using examples is not easy. For example, unlike Matthew, Betty did not use negative integers in her lesson, and her class seemed not to include negative even numbers as even numbers and all proofs that her class discussed were vague about negative integers. Some examples did not work well:

Betty: Is seven divisible by two?
Teacher49: Yes. But, the result is not an integer.
Betty: Oh
Teacher49: You know what I mean. Any odd number divided by two is going to have a point five. So, that's not an integer. Any even number divided by two is going to have a whole number. Look, it will be integer.
Betty: Well, if I had seven cookies, and you and I would want to share them, and I gave you three, and I kept three and break the last one into two pieces and share.
Teacher49: It sounds seven is still an even number then, because that half cookies are point five.
Betty: But, divided it by two.
Teacher50: You didn't get an integer.
Teacher51: It's not an integer though.
Betty: OK. So, do first graders know what integers are?

In this discussion, Teacher49 did not know what divisible means and could not differentiate between divisible and divided. All numbers, including seven, are divided by two, but only even numbers are divisible by two. Her idea, which is seven is divisible by two but the result is no an integer, is not mathematically correct. The problem here is Betty's use of an example, seven cookies. Her example did not function to help Teacher49 recognize what divisible is. Betty's example would be appropriate to emphasize the careful use of manipulatives or experience that students might have related to divisibility of two which are not appropriate. Instead, her example reinforced that Teacher49's idea was correct, when it was in fact incorrect. Using examples that can help teachers is challenging. Teacher educators need to pay attention to features of examples and carefully decide whether examples work well.

Table 5.3 summarizes the above the discussion of the recognition of mathematical issues.

Table 5.3 *Summary of the Focusing on Mathematical Articulation and Interpretation*

What teacher educators need to attend to for focusing on mathematical articulation and interpretation:
• Specifying mathematical terms, facts, and relations
  o Explaining mathematical facts and enumerating their mathematical features that teachers need to appreciate in performing an activity
  o Demonstrating mathematical properties and relations that are cores of mathematics which are not explicitly considered
• Identifying mathematical objects
  o Illuminating mathematical objects that teachers need to consider as pedagogical concerns
  o Enumerating mathematical ideas used by students in records of practice
• Identifying mathematically appropriate ideas and interpretations
  o Giving mathematical explanations
  o Articulating mathematical reasons
  o Mathematically interpreting teachers' comments
• Using examples for mathematical investigation
  o Using examples to proceed generalization
  o Using special examples for test

Challenges that teacher educators might face: • Not giving mathematical explanations about what teachers are concerned about • Inappropriately articulating mathematical facts and relations • Not explaining mathematical facts that some teachers do not appropriately understand • Incorrectly interpreting mathematical ideas used by students in records of practice • Not using examples skillfully to support teachers

5.5 Emphasizing Ways of Mathematical Thinking

Teacher educators consider emphasizing ways of thinking as the task of the teaching MKT. Ways of mathematical thinking are sometimes specifically illustrated by teacher educators and attributed to a certain intentional task, such as asking questions repeatedly. Although this is a task that is relatively smaller than other work of teaching MKT by the teacher educators in the data, it is still distinctive and it is worthwhile to specify how the teacher educators emphasize ways of mathematical thinking in their instruction. Based on my analyses of the data, three tasks of teaching MKT recurred across the work of the teacher educators in terms of emphasizing ways of mathematical thinking: asking key questions repeatedly and intentionally, introducing mathematical practice, and giving comments for teaching practice.

5.5.1 Asking key questions repeatedly and intentionally

The teacher educators in this study used the same questions intentionally throughout discussions to concentrate on how the teachers approached mathematical issues and made decisions. Several examples were discovered in the data. For example, after watching the SeanNumbers-Ofala video, the teacher educators repeatedly asked "What definitions for even number did they have?" until the teachers found all definitions used by the students in the video. When the teachers did not recognize a certain definition, the teacher educators specifically asked "What definition did Sean use?" When the teachers found definitions, the teacher educators also asked where the teachers found the definitions in the transcript, through questions such as "What line are we looking at?" These two kinds of questions for both articulating students' definitions and identifying evidence helped the teachers have logical decisions to see what mathematically happened in the discussions rather than listing vague impressions. In the lessons to evaluate definitions, Sandy and Matthew intentionally used the same pattern of questions to evaluate four different definitions of an even number, such as "Does it include all the even numbers?" to determine whether a definition is precise, and "What things do you have to know?" or "What do you need to understand" to find concerns for understanding definitions. Their patterned use of the questions clearly let teachers know what they needed to consider for evaluating definitions. Moreover, Nellie repeatedly asked "Is this convincing you?" when her class presented proofs in the lesson to prove a conjecture, and Kellie always asked "Does this explanation work for all cases?" after each group presented their explanation of why the units digit rule works.

One hard task for teacher educators would be to decide whether a question works well in terms of repeating it as a question. For example, in the lesson to evaluate definitions, Emily asked "Would you give this definition to a first grader?" to the teachers for all definitions whether they were mathematically appropriate or not. In other words, she asked this question for imprecise definitions but did not investigate mathematical issues around those definitions even when the teachers were interested in whether the definitions were precise.

The intentional and repeated use of the same questions is one of the strategies to emphasize ways of mathematical thinking. However, if an inappropriate question is used, it does not work as a way for mathematical thinking.

5.5.2 Explaining mathematical practice

The teacher educators in this study often explained features of mathematical practice, such as roles of definitions, nature and elements of proof, and roles of counterexamples and conjectures that are objects in disciplinary mathematics.[23] For example, at the end of a class of reasoning with definitions for proof, Daniel explained what constitutes a proof with the slide as shown in *Figure 5.1*:

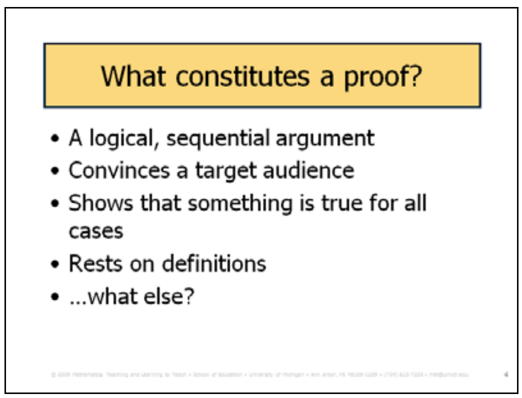


Figure 5.1 What constitutes a proof

> This would be our ending slide if I would show it to you. (Daniel shows the slide as shown in *Figure 5.1*.) Okay. So you approve it. And proving's a lot harder than just verifying something is true, or stating that it's true. We were looking for a logical sequential argument, right? Convincing a target audience. Today the target audience was pre-service teachers rather than kids, but when kids are the target audience, we have even more to say about the usability issue, as Libby was mentioning. We're trying to show that it works every time. Not just in the examples we looked at. And that's maybe the difference between the first examples you were testing out and actually doing a proof. And then what are- what is this activity doing in the definitions? You look how this kind of mathematical thinking, called proving, depends on definitions. We need to know what definition we're using to know whether we've proven it or not. And when the definitions are blurry, the whole proof gets blurry. So one thing definitions do is

[23] These objects are elaborated in Chapter 6 in terms of disciplinary objects in mathematics.

> give us a common language for doing this kind of mathematical reasoning. And I guess on that thought, I'm going to let you go and see you Wednesday.

In fact, not only Daniel's explanation but also those of most teacher educators in the data were similar to those introduced in materials. Therefore, explaining mathematical practice might not be easy without the support of materials, and teacher educators need to specify what mathematical practice would work in an activity. Moreover, when a discussion goes poorly, it might be challenging to give clear explanations about mathematical practice. For example, Betty evaluated the teachers' proofs as logical arguments and explained sequential arguments as a component of proofs even though all teachers' works were not proofs nor mathematically sound.

5.5.3 Giving comments for teaching practice

Teacher educators often comment on what teachers are expected to do mathematically in K-9 classrooms.[24] For example, the teacher educators in the data explained the different uses of the same terms in mathematics and everyday context and demonstrated why definitions matter in teaching mathematics. They also explained a way to evaluate definitions: deciding whether a definition includes the set of the defined and excludes its complementary set; and, finding possible concerns for understanding a definition. Moreover, they gave comments about the careful use of language, the mathematical territory that students would learn later, the effect of using manipulatives, and skills to select examples. In fact, what teacher educators give comments on for teaching practice depends on and connects with the activities they offer. In other words, it would be necessary for teacher educators' attention to be on comments for teaching practice that are related to the activities that they perform with their teachers. However, it is challenging to give comments that function in teaching practice in terms of MKT. For example, when using precise definitions is in conflict with using usable definitions, some teacher educators in the data recommended sacrificing one of them, rather than emphasizing making a balance, specifically the use of mathematically precise definitions that students could understand. Therefore, teacher educators need to pay attention to teaching practice that is mathematically reasonable and pedagogically manageable rather

[24] Objects of teacher educators' comments are elaborated in Chapter 6 in terms of the mathematical work of teaching and disciplinary objects in mathematics.

than adhering to easily or conventionally used ways to solve conflict in the K-9 classroom.

Table 5.4 summarizes the above the discussion of recognizing mathematical issues.

Table 5.4 *Summary of the Emphasizing Ways of Mathematical Thinking*

What teacher educators need to attend to for emphasizing ways of mathematical thinking: • Asking key questions repeatedly and intentionally through a lesson • Explaining mathematical practice • Giving comments for teaching practice
Challenges that teacher educators might face: • Repeating a question that makes slide into non-MKT discussion • Specifying what mathematical practice would work in an activity • Giving clear explanations about mathematical practice when a discussion goes poorly

5.6 Managing Mathematical Ideas

Teacher educators make efforts to manage teachers' ideas in instruction. There are various strategies for this task. Teacher educators often select certain topics among teachers' diverse ideas for moving forward to an activity. For example, after showing the results of group discussions to prove the conjecture, even number + even number = odd number, Nellie's class investigated one of them and extended their discussion to generalize the explanation for the proof. Moreover, teacher educators sometimes intentionally lead teachers away from teaching practice in order to concentrate on mathematical issues. Managing mathematical ideas is a task that requires keeping within bounds, specifically not to fall into talking about pedagogical issues and not to slide into thinking about mathematics in ways that are remote from teaching. It is the task that aims to have a meaningful instruction of MKT. However, it might be difficult to manage teachers' active participation in discussions or having a whole group discussion about teachers' mathematical ideas at the appropriate moment. Based on the analyses of the data, six tasks of the teaching of MKT recurred with regard to managing mathematical ideas: (1) situating teachers in or away from the context of K-9 classrooms; (2) staying on a mathematical topic; (3) giving mathematical assistance; (4) engineering engagement in mathematical practice; (5) having mathematical consistency; and (6) doing manipulative preparation.

5.6.1 Situating teachers on or away from the context of K-9 classrooms

There are two ways of managing the context of classrooms for teaching MKT: moving teachers away from teaching practice and situating teachers in the context of mathematics classrooms. First, the intentional detachment from teaching practice aims at helping teachers concentrate on mathematical issues. It could help teachers not to be distracted by whether and how students can use mathematical objects. Although the focus on students is a good instinct, it sometimes hinders investigation of the mathematics entailed in teaching. Being removed from teaching practice also contributes to teachers' appreciation of the mathematics that student may encounter further on. In fact, the teacher educators' statements are not long to make teachers away from the context of school classroom. For example, Daniel said when he introduced the activity, proving the conjecture:

> So, that happens, and as we think about this conjecture today, we're not trying to do it at a level that elementary school kids would understand. Okay, we're just here we are college students in this class, thinking of it on our own terms, this idea that an odd number plus an odd number equals an even number.

Daniel's comment worked to caution teachers about making assumptions about what students can or cannot do, and his class focused on mathematical issues in proving the conjecture.

Interest in mathematics that students can approach, however, is not easy to manage. For example, Sandy gave a comment telling teachers to be removed from the teaching practice. However, when she asked a meaning of 'multiple of two,' one teacher defined it as 'groups of two.' The teacher's response was not totally wrong, but was mathematically rudimentary, so that students could understand it. Thus, teachers are prone to consider the mathematics that students can use, and it seems challenging to focus on mathematics that works in the disciplinary mathematics when the teachers are involved in the competent performance of tasks of teaching. Lack of attention to the situation that teachers consider in an activity might make discussion mathematically valueless. For example, Betty did not efficiently handle the context of a classroom when she led the discussion. As a result, her teachers presented mathematically careless proofs.

Second, locating teachers in the context of mathematics classrooms aims at facilitating teachers to experience performing the mathematical work of teaching in the setting of mathematics classroom. For example, the teacher educators in the data played the SeanNumbers-Ofala video as a record of practice to situate teachers in the third grade mathematics classroom. Because of various elements and complexity in instruction which is shown in the video clip, it was much harder for the teacher educators to help teachers concentrate on the mathematical work of teaching. The teachers were frequently likely to talk about their impressions of the students and the teacher in the video clip. Therefore, some teacher educators gave opportunities where teachers presented general reactions after watching the video first. These opportunities made the teachers turn to directly carry out the task.

Although using an activity embedded in the context of a K-9 classroom is important to teach MKT, how to situate teachers in and away from that context is critical to manage teachers' learning of both mathematics for teaching and the work of teaching. Teacher educators need to intentionally manage the context of the classroom for teaching MKT.

5.6.2 Staying on a mathematical topic

As previously mentioned in Chapter 1, MKT is a slippery concept. The data repeatedly evidenced showed that it was easy for the teacher educators to change a topic into pedagogical issues. For example, the following conversation occurred in Betty's class when they evaluated "An even number is a number that can be divided by 2."

Teacher52: Any number can be divided by 2.
Betty: Any number can be divided by 2. But, that might be appropriate for a certain grade level. For a younger grade level, maybe?

This episode shows a broken discussion about MKT by the teacher educator. Even though the teacher explained why this definition is problematic, Betty did not make a decision whether the definition is mathematically precise or not and changed the topic to which grade level students might understand the definition. While "primary" means to substitute incomplete or incorrect concepts in mathematics, it refers to the rudiments of many important concepts in more advanced branches of the discipline (Ma, 1999, p. 116). Thus, it might be challenging to emphasize using mathematically precise language that students can approach, rather than allowing mathematically imprecise definitions for any grade-level students. Preventing the discussion from falling into non-MKT issues could help teachers stay on the mathematical topic in teaching MKT. The following episode from the data shows how Sandy managed the discussion of MKT to keep on track.

Sandy:	Let's just start by taking the first definition (An even number is a number of the form 2k, where k is an integer.) and, when you looked at that, what could you say about that definition in terms of precision?
Teacher53:	Students wouldn't understand that definition.
Sandy:	Well, I want to go back to precision.
Teacher53:	OK, precision.
Sandy:	You think it's precise, right?
Teachers:	Yes

In this episode, while teacher53 was also prone to talk about whether students could understand the definition, Sandy maintained the focus on mathematics entailed in teaching rather than curriculum. In the above two episodes, the teacher educators' statements were not long. However, they determined whether their classes could work profoundly on the tasks for MKT or not.

It is, however, hard to handle the flow of the discussion of MKT because teachers sometimes would be interested in non-MKT issues. Not deeply probing teachers' non-MKT comments could help teacher educators. It seems hard, however, as some teacher educators in the data took the time to explain. For example, a teacher in Julie's class asked which grade level is taught integer, which was used as a term in one definition of an even number. The following episode shows the beginning of the discussion.

Teacher54:	What grade is "integer" first taught?
Julie:	That's a good question.
Teacher54:	Because we don't do much with it in sixth grade. We do the very end of the year.
Teachers:	fourth grade as well.
Julie:	It is introduced into fourth grade. …

Julie took up Teacher54's comment and talked about curriculum issues related to integer approximately five minutes. Julie took care of all teachers' questions and comments, but some of them were about curriculum issues, and finally slid into non-MKT discussion.

5.6.3 Giving mathematical assistance

Providing a lot of assistance removes learners' learning opportunities, but, at the same time, learners may have difficulties carrying out an activity without any assistance (Brousseau, 1997). Therefore, teacher educators need to assist teachers properly and strike the right balance, avoiding too much assistance and too little. In the data, it was generally seen that the teacher educators represented teachers' assertions on the public space and gave explanations with statements or pictures by the teacher educators and the teachers. It helped the teacher educators and the teachers to reflect presenters' comments and compare and synthesize them. Therefore, using the public space is one of the typical elements in any kind of instruction, and it is also important in instruction for MKT.

More extensive assistance would be giving comments or questions to initiate teachers to consider critical structures or performance. For example, Cate, one of the teacher educators in the data, told the teachers to "test some of your definitions, test number seven for example, or, a half" when the teachers carried out the activity to write precise definitions of an even number. She suggested using counter examples to test their definitions. Moreover, in Sandy's class, when the teachers did not find what a divisibility test for nine and why it works, Sandy gave comments:

Sandy:	Is there another way to rename 100, so it's not 100? What's the biggest number less than a hundred?
Teachers:	99.
Sandy:	Ah, it's divisible by three. I wonder if that would help you.
Teacher55:	Three plus three plus three plus one.
Sandy:	Do some work at your table.

Before this discussion, Sandy's class found the divisibility rule of 3. Teacher55 recognized that ten is a multiple of three plus one as well as a multiple of nine plus one. Therefore, the teachers found a hint that the divisibility rule of 9 would work like the divisibility rule of 3. In fact, Sandy did not explain what the divisibility rule of 9 is and why it worked at that point and asked the teachers to investigate it. Finally, her assistance helped the teachers acknowledge how to approach the issue but serious investigation was left to the teachers.

Teacher educators could give their assistance to confirm teachers' ideas. For example, one of the issues that the teachers found in evaluating definitions was the conflict between mathematical rigor and students' accessibility. Some teacher educators pushed the teachers to find how to solve it.

Teacher7:	If you say a whole number is even if it can be put into groups of two with none left over.
Matthew:	Nice. Can you say why?
Teacher7:	Except that, that doesn't handle negative numbers?
Teachers:	Oh... Yes. OK. Whole numbers are zero, one, two, … Oh, you're right. Integers are negative… Oh!
Matthew:	We can make it usable and still be honest to the mathematics. … And actually this is going to reduce any ambiguities. Right now we can use it, let's say with third graders, who know nothing about negative numbers, and still be accurate and honest to the mathematics. OK?

When the teachers in Matthew's class realized the conflict between mathematical rigor and students' accessibility, Teacher7 showed her solution to other teachers. Matthew assisted her to explain why her suggestion could work. Therefore, teacher educators would pay attention to teachers' statements and assist them to develop mathematically valid explanations or assertions.

5.6.4 Engineering engagement in mathematical practice

Learners' participation is critical in any kind of instruction. Teacher educators make efforts to foster teachers' engagement in mathematical practice, which is created through discussions of intended MKT tasks. There are several strategies.

Moving around a class to gather teachers' ideas would help teacher educators garner more engagement. For example, Daniel, one of the teacher educators, always walked around the classroom when the teachers had individual or pair work. He listened to their assertions and reasoning and also gave questions or comments to support their work. Moreover, he made notes about who had which ideas. In fact, he actively used his notes when he opened whole group discussions. He glanced at his notes, called teachers' names and asked them to present their ideas to other teachers. It worked to offer a certain idea for launching a whole group discussion. He seemed to carefully engineer the conversation to discuss ideas as diverse as possible because all presenters had different mathematical assertions and, as a result, his class had discussions with various issues. He employed this strategy of using notes across all his classes.

Teacher educators would give attention to resources to gather teachers' ideas. For example, several teacher educators in the data distributed chart papers or transparencies in each group and the teachers were expected to draw or write a conclusion from their group discussions. Therefore, each teacher could have opportunities to play a role in creating a mathematical conclusion. The teacher educators gathered the chart papers and transparencies where the teachers wrote down their findings, or helped the teachers present their group discussions to the whole group. Both the teacher educators and the teachers could figure out, compare, and evaluate presentations and have discussions.

Other strategies for teachers' engagement in instruction are to gather their attention and give opportunities to those not presenting. For example, Nellie sometimes said "Just a second – could you guys pay attention to this for a second?" or "Let's see what Sara was thinking about in terms of proving the conjecture by using that." She also tried to distribute mathematical talk to as many teachers as possible, by making comments such as "Let's try to hear from people who haven't said much today. Caroline, how do you feel about that?" In fact, her efforts worked well because the teachers in her class seemed to be engaged in mathematics practice throughout her class.

Controlling the tempo of a discussion is also one of the strategies for teachers' engagement, such as reviewing and synthesizing their discussions. For example, Sandy's class watched the SeanNumbers-Ofala video and examined what definitions students used in their discussion. Before wrapping up the class, Sandy briefly reviewed what definitions they found, such as "Let's just see if we can summarize what we think the major definitions are. … Ofala's definition which was what? …". Synthesizing what the class found showed the results of the teachers' engagement and it worked like putting a period at the end of the discussion.

Teacher educators also would try to identify the relations between current lessons and previous lessons in order to show an extension of teachers' learning. For example, when Nellie started a lesson, she briefly mentioned the NCTM standards that teachers read and the SeanNumbers-Ofala video that they watched:

> Today we are going to work on reasoning with definitions. In fact, for your first assignments you should already have been reading reasoning and proof standard for grades three to five, right? And your job was to try to understand what those standards mean and how we can understand it. And so far we have been working

> on definitions that we use in mathematics – in doing mathematics and teaching mathematics. And last Tuesday, you did watch a video clip that came from a third grade classroom, and we did see how elementary school students deal with definitions and how they try to convince each other about those definitions. Right? And today we are going to be moving beyond just thinking about definitions themselves.

Her talk was brief, but it worked to offer a story line across lessons to the teachers and draw their engagement efficiently: what they did, what they do, and their relationship between the two.

Teacher educators, however, might have difficulties managing teachers' engagement in discussion. For example, Betty tried to motivate teachers to think about "Why work on mathematical definitions?" and rephrased the question several times, such as asking "Why was it important to think about definitions in the video," "What do you need to know about mathematical definitions?" and "What are some of the things you need to think about?" However, the teachers seemed not to be fully engaged in the discussion of the question. They could not answer very well. The teachers might not be familiar with the question because they did not have a lot of experience in the kind of teaching reflected in Betty's comment. The teacher might not understand the meaning of the question well, or the practice of this classroom might not good for having discussion. The reason was not clear, but the teachers were not very engaged in this discussion.

5.6.5 Having mathematical consistency

Not having mathematical consistency within a class could result in mathematical confusion to teachers in instruction. For example, one of the teachers in Matthew's class was curious about the unit, "two" in the definition; a number is even if it can be put into groups of "two" with none left over. He explained that two is a whole number. However, when another teacher asked how this definition works for negative integers, he explained that two indicates two negative integers, and the teachers were confused when examining this definition. "Two" generally means 2 rather than two of one-third. The teacher made the excessive interpretation about "two." Later, Matthew said "two" for a negative integer indicates -2. In fact, all these issues are from the definition, a number is even if it can be put into groups of two with none left over. In other words, the definition itself can be problematic in terms of mathematical consistency. However, it seemed very hard for Matthew to make different meanings of “two,” 2 and -2.

5.6.6 Doing manipulative preparation

The teacher educators’ specific challenges in terms of manipulative preparation for teachers are not found in the data, such as distributing materials, using various ways to record discussions in public note-taking spaces and gathering teachers’ group discussions, and using the animation of the slides and a computer or DVD player. All teacher educators in the data managed well for organizing materials before and during the lesson. In fact, the curriculum materials offer details of what should be prepared before a class. The details would help the teacher educators prepare for technical or manipulative needs. Therefore, considering and using technical and manipulative preparation beforehand might not be easy. However, if any of their preparation for slides or DVD players is disorganized, a class might perform a task poorly or teachers might not be successfully engaged with an intended MKT task. Thus, it seems important that teacher educators pay attention to work to establish and maintain an intended MKT task.

Table 5.5 summarizes the above discussion of the recognizing of mathematical issues.

Table 5.5 *Summary of the Managing Mathematical Ideas*

What teacher educators need to attend to for managing mathematical ideas:

- Intentionally situating teachers on or away from the context of K-9 classrooms
    - Considering mathematics that teachers know rather than mathematics that students use
    - Offering records of practice
- Staying on a mathematical topic
    - Not opening up discussion to non-MKT issues
    - Not probing teachers' non-MKT comments
- Giving mathematical assistance
    - Representing teachers' assertions on the public board or interpreting them
    - Giving comments or questions to initiate teachers to consider critical structures or performance
    - Explaining to create a balance between mathematical rigor and students' accessibility
- Engineering engagement in mathematical practice
    - Moving around a class to gather teachers' ideas
    - Gathering teachers' ideas by using chart papers or transparencies for each group
    - Helping have small group discussions before whole group discussions
    - Discussing diverse assertions
    - Gathering teachers' attention
    - Giving opportunities to teachers who do not present
    - Reviewing or synthesizing the discussions
    - Identifying the relationship between a current lesson and previous lessons
- Having mathematical consistency
- Doing manipulative preparation
    - Distributing materials
    - Preparing for public note-taking spaces
    - Using equipment, such as a projector, a computer, or a DVD player.

Challenges that teacher educators might face:

- Teachers' tendency to consider mathematics that students can use
- Teachers' tendency to present impressions about students or teachers rather than focusing on MKT
- Inefficiently managing the context of K-9 classrooms
- Turning to a non-MKT discussion during an investigation of a MKT issue
- Digging into issues related to curriculum
- Difficulties achieving teachers' engagement with a discussion
- Considering and using technical and manipulative preparation beforehand

5.7 Conclusion

Any kind of teaching phenomena is complex and messy, and it is difficult to explain it in an elegant and systematic way (Doyle, 1986). An honest understanding of teaching phenomena needs to take account of the various factors that assume prominence for instructors and learners present in classrooms. To attempt to gain this understanding, the present research investigates the phenomena of teaching MKT and identifies challenges of teaching teacher educators attend to in order to manage the tasks of the teaching of MKT. Teacher educators encourage teachers to learn knowledge and skills for teachers' mathematical preparation and their practice for teaching mathematics. This educational purpose is implemented in the mathematics teacher education classroom by the combination of roles of teachers as learners, teacher educators as instructors, and the tasks and activities involved in teaching MKT. This combination creates the distinctive dynamics of instruction of which teacher educators need to be aware. Findings from this study outline the pedagogical considerations that underlie the teaching of MKT.

CHAPTER 6
FRAMEWORK FOR CURRICULUM TO TEACH MATHEMATICAL KNOWLEDGE FOR TEACHING

6.1 Introduction

Two assumptions underlie this dissertation. The first is that the ultimate goal of mathematics teacher education is to develop teachers' mathematical preparation and skilled teaching. The second is that mathematics for teaching originates deep within disciplinary ideas and is flexible enough to be associated with students' thinking. This research has conceptualized the framework that can inform a curriculum for teaching MKT in mathematics teacher education. Chapter 4 provided three detailed examples of one lesson of the curriculum materials and two teacher educators' teaching of MKT.[25] Through the commentary and the excerpts from the materials and the video recordings, Chapter 4 tried to highlight the complexity around teaching MKT and to establish a more analytic conceptualization for teaching MKT. Chapter 5 identified the challenges teacher educators might face and the kinds of attention needed for teaching MKT. In the current chapter, the two different components for teaching MKT are focused – the mathematical work of teaching and knowledge about mathematics, which are critical in teaching MKT as content in mathematics teacher education. The aim of this chapter is to explain the structure of the framework and walk through each of the sections.

As Chapter 3 described, the framework and its components emerged from the conceptual analytic work of this study. In particular, the findings for conceptualization have been completed through both the analysis of the empirical data and the literature review. The basic structure and main elements of the overall framework were found

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[25] As explained in Chapter 3, this study used the curriculum materials as designed to focus on MKT, specifically, " Using Definitions in Learning and Teaching Mathematics" developed by the *mod4* project at the University of Michigan. Because the curriculum materials were used as lesson plans in the twenty-five classes for this study, it obviously confirmed that users of the materials intended to teach MKT and their lessons provided the instruction of MKT.

from the findings of the empirical data. This is despite my starting to analyze the data with a preliminary categorization scheme as shown in Appendix A and Chapter 2's literature review. The analysis of the data also provided many examples of teaching MKT in terms of both different mathematical work of teaching and knowledge about mathematics. Because of this, it helped the conceptualization come alive in instruction and have rich explanations of it. However, it was not enough for a refined framework. For example, the most critical inclusion based on solely the literature concerned the long periods, such as a unit or a chapter of instruction, one semester, and a school year, in the continuum of curriculum in the framework. Teachers are expected to consider long periods in teaching practice (Lampert, 2001). Finally, *making a decision for a long period* was included as one component of the mathematical work of teaching. Then, the details of making a decision for long periods were induced from the details of *making a decision for one lesson* that were analyzed and elaborated from the data. This induction could work because setting up the basic structure of the framework formulated the logic to elaborate the framework itself. For example, I found "constructing proofs based on certain definitions or axioms" and "creating representations with particular limitations" as the mathematical work of teaching from the data. Specifically, several teacher educators in this study discussed having a proof of (odd number) + (odd number) = (even number) with a fair share or pairs definitions. This motivated the teachers to construct presentations about their proofs.[26] However, I found nothing related to "creating algorithms, rules, and procedures within a certain limitation" from the data, such as creating an explanation of how to subtract with whole numbers. The logic of combination of "creating" and "a certain limitation" with definitions or axioms and proofs confirmed the existence of "creating algorithms, rules, and procedures under a certain limitation." Thus, the framework included "creating statements and examples of algorithms, rules, and procedures under a certain limitation." Logic that was formulated in putting together the basic structure of framework finally played a critical role in developing the conceptualization for the overall research question. Moreover, for a more concise and consistent framework, I had to keep comparing and contrasting contents of

[26] Fair share definition means that a number is even if it is can be made of two equal groups with none left over, using only whole numbers, and pairs definition means that a number is even if it can be grouped in pairs, with nothing left over.

the framework with the data and the literature review and differentiated them in terms of the domains of MKT. I applied the definitions of each domain of MKT by Ball et al. (2008), summarized in Chapter 2. This examination helps identify features of the mathematical work of teaching and relations between the mathematical work of teaching and domains of MKT.

MKT in mathematics teacher education is content that teacher educators help teachers obtain to improve knowledge entailed in teaching mathematics to students. In a sense, MKT as content in mathematics teacher education has two main components: *mathematical work of teaching* and *knowledge about mathematics*. The mathematical work of teaching is the tasks of teaching that teachers perform with mathematics in classrooms. MKT is the mathematical knowledge needed to carry out the work of teaching mathematics (Ball et al., 2008, p. 395). MKT is imbedded in the tasks of teaching and it is concerned with the mathematical demands inherent in these tasks. Therefore, the mathematical work of teaching is based on aspects of the work as well as guides to the content demands of teaching. This research suggests the framework of the mathematical work of teaching and identifies what MKT, specifically domains of MKT, is entailed in the mathematical work of teaching. This is done to help teacher educators teaching MKT. On the other hand, knowledge about mathematics concerns the nature of knowledge in the discipline that Ball (1990) introduced. It includes what counts as an "answer" in mathematics? What establishes the validity of an answer? What is involved in doing mathematics? What do mathematicians do? Which ideas are arbitrary or conventional and which are logical? What is the origin of some of the mathematics we use today and how does mathematics change? (Ball, 1990, p. 458). Because they function as two parts of the framework for curriculum of MKT in mathematics teacher education, the two are ranked in no specified order. Thus, both the mathematical work of teaching and knowledge about mathematics are indispensable components to planning and implementing of MKT in mathematics teacher education.

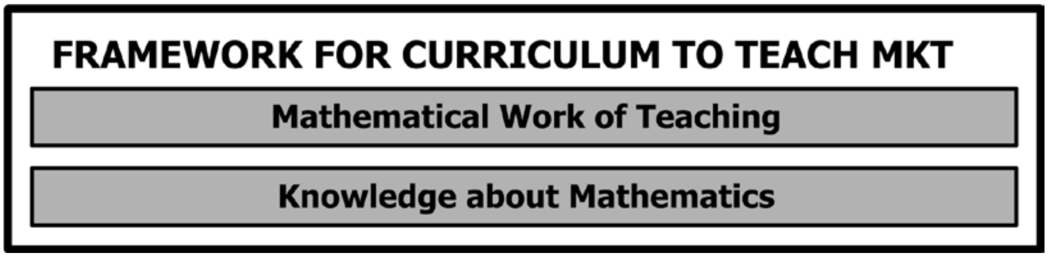

Figure 6.1 The two components of the framework for curriculum to teach MKT

The next two sections unpack the ramifications of the mathematical work of teaching and knowledge about mathematics, which are main two parts for curriculum to teach MKT in teacher education. Each section introduces their structures first and then specifies their elements. The clarification of the structures and the specification of their elements help systemically identify what can be content of MKT in teacher education, such as the table of content for MKT. This unpacking provides a detailed decomposition of MKT as content in mathematics teacher education. After particularizing the insides of the framework, this section summarizes it with an elaborate figure that incorporates the main ideas from each component of the framework.[27]

6.2 Conceptualization of Mathematical Work of Teaching as Content in Mathematics Teacher Education

The mathematical work of teaching consists of those tasks of teaching that call for mathematical knowledge in mathematics classrooms. When describing its features, Lampert (2001) uses the metaphor of a photograph.

> To do the work of teaching, the teacher in the classroom also needs to do something akin to zooming in and zooming out, acting simultaneously in both "the big picture," across time and relationships, and in the moment-by-moment interactions with individual students. … These actions must be, at the same time, both narrowly convergent and widely panoramic, and everything in between. And, they must often converge on more than on focal point. The importance of the "zooming" metaphor here is that actions in narrow contexts are embedded in broader actions taken across time and across students; practice in the moment is not carried out separately from larger exploits. … As the teacher "zooms in" and "zooms out" across multiple dimensions, she can make use of different units of time and social networks as resources to make more kinds of teaching possible, and the units of time and interaction in which the work occurs overlap. (p. 430)

[27] The elaborate figure is introduced at the end of Chapter 6 in *Figure 6.16*.

Instruction occurs across two continuums: (1) across each moment, each activity, each lesson, and each school year and (2) across a small domain of mathematics, mathematics that students learn later, and the overall territory of mathematics. The analogy of zooming in and out is found to be apt; indeed, in mathematics teacher education it is a main feature of mathematical work of teaching as content. Zooming-in, the microscopic perspective on instruction, captures the moment-to-moment and activity in the continuum of curriculum as well as a small domain of mathematics. Zooming-out, the macroscopic perspective on instruction, captures each lesson and each school year in the continuum of curriculum; it also captures the mathematics that students learn later and the overall territory of mathematics. Using these two different lenses offers the two layers to the mathematical work of teaching as a curriculum for teaching MKT. The layers are referred to in this dissertation as *zoomed-in* and *zoomed-out* mathematical work. The former is through the microscopic approach; the latter is through the macroscopic approach. What one sees in the mathematical work of teaching depends on what lens one uses. Both perspectives—the microscopic and macroscopic —are integral to the work of teaching. Therefore, in mathematics teacher education the topics of the mathematical work of teaching as content encompass both the zoomed-in and zoomed-out mathematical work of teaching.

In carrying out the work of teaching, the two are closely connected in performing the work of teaching. The zoomed-in mathematical work of teaching is nested at several levels within the zoomed-out mathematical work of teaching. This is because what is observed in the microscopic approach is nested in what is observed in the macroscopic perspective. That is, teachers and teacher educators work on several kinds of zoomed-in mathematical work of teaching that converge with a single zoomed-out mathematical work of teaching. *Figure 6.2* shows the structure of mathematical work of teaching as content in mathematics teacher education.

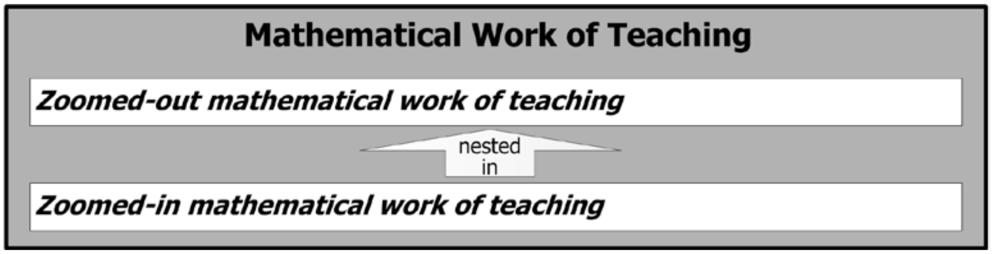

Figure 6.2 Two categories in terms of mathematical work of teaching as content in mathematics teacher education

Each zoomed-in mathematical work of teaching and zoomed-out mathematical work of teaching are examined regarding domains of MKT. This examination helps identify the following: features of each mathematical work of teaching, relations between domains of MKT and the mathematical work of teaching, and the identification of each domain of MKT. The next two sections elaborate on each category.

6.2.1 Zoomed-out mathematical work of teaching

The zoomed-out mathematical work of teaching adopts a more macroscopic lens, rather than the microscopic of the zoomed-in work, to identify the mathematical work of teaching as content for teaching MKT. As shown in Table 6.1, this study revealed from the data three kinds of zoomed-out mathematical work of teaching: providing and justifying mathematical and pedagogical decisions, situating and unpacking mathematics ideas and practice, and using language mathematically and accessibly. One kind of zoomed-out mathematical work is generally focused on each class of teaching MKT. In the data, classes for exploring why mathematical definitions matter and hearing definitions in children's talk aimed at providing and justifying mathematical and pedagogical decisions; classes for reasoning with definitions for proof and reasoning with definitions for explanations concentrated on situating and unpacking mathematics ideas and practices; and a class for evaluating definitions focused on using language mathematically and accessibly.

Table 6.1 *The Basic Structure of Zoomed-Out Mathematical Work of Teaching*

Providing and justifying mathematical and pedagogical decisions		Situating and unpacking mathematics ideas and practices		Using language mathematically and accessibly
Making a decision regarding one lesson	Making a decision regarding a long period	Unpacking mathematics ideas	Situating in mathematics	

There are two features in zoomed-out mathematical work of teaching in terms of relationships with zoomed-in mathematical work of teaching and domains of MKT. First, each zoomed-out mathematical work of teaching nests several zoomed-in mathematical work of teaching. Second, each zoomed-out mathematical work of teaching is classified into domains of MKT. To elaborate on the first, the series of tasks that follow, for example, is ultimately subordinate to unpacking features of even numbers: (1) specifying a set of even numbers and identifying its complementary set as non-even numbers, (2) creating, revising, and evaluating definitions of an even number, (3) probing mathematical terms and concepts related to a definition of an even number, and (4) identifying types of numbers. However, this research does not assert here that all zoomed-in mathematical work of teaching should be so nested. It would be quite possible, and valuable, to emphasize each zoomed-in mathematical work of teaching on its own in one lesson of mathematics teacher education. This study is simply trying to highlight that the mathematical work of teaching are strongly related to one another. This section is introducing a lens here to see intensively and extensively the mathematical work of teaching as content in mathematics teacher education.

The aforementioned second feature is again that each zoomed-out mathematical work of teaching is classified into domains of MKT. For example, appraising and adapting the mathematical content of textbooks is one of the tasks in providing and justifying mathematical and pedagogical decisions. It is included in SCK and is based on several kinds of zoomed-in mathematical work of teaching: (1) specifying what mathematical objects are – related to CCK, (2) playing with statements and examples of the mathematical objects – related to SCK, and (3) evaluating whether statements and examples of the mathematical objects are instructionally advantages or not – related to KCT. Therefore, one kind of zoomed-out mathematical work of teaching can be involved in one or several MKT domains. It is also based on several kinds of zoomed-in mathematical work of teaching, all of which involve various domains of MKT. Because CCK is the knowledge for solving mathematics problems with no consideration of teaching, such as simply calculating an answer, CCK is not emphasized in the zoomed-out mathematical work of teaching that requires deep and consistent insight about teaching. Now, let's see each zoomed-out mathematical work of teaching in detail.

Providing and justifying mathematical and pedagogical decisions. This includes the work involved in anticipating, planning, observing, and judging how mathematical aspects change and develop in an activity of mathematics teacher education which considers one lesson as well as the span of a long period of school mathematics. Here, a long period indicates a unit or a chapter of instruction, one semester, and a school year, in the continuum of curriculum. Because one lesson and a long period for school mathematics entail different mathematical work of teaching, providing and justifying mathematical pedagogical decisions is divided into making decisions for one lesson and making decisions for a long period.

One lesson constitutes a unit that teachers generally prepare and handle in and for teaching. In other words, making a decision for one lesson as content in mathematics teacher education indicates specifying, practicing, and talking about the skills and reasoning to see one lesson as a unit. Each lesson is an authentic and tangible time and place for teaching and learning mathematics. Each lesson is a place where students have opportunities to improve their mathematical thinking and knowledge and students will likely not have another chance to experience it again. Hence, a teacher's responsibility in each lesson should be considered as immense if he or she considers only one chance in a student's life.

Making decisions for one lesson entails various kinds of mathematical work of teaching that can be classified into domains of MKT. First of all, the mathematical work of teaching, that is SCK in making decisions for one lesson, is about the synthetically anticipating and evaluating the activities of a lesson. In particular, SCK in making decisions for one lesson includes explaining mathematical goals and purposes and appreciating and adapting topics that are presented to and worked with students in order to have knowledge and skills beyond that necessary to teach students a lesson. It also involves anticipating the change and development of mathematical ideas with the topics being taught in order to assume and have a mathematical route that could be traversed with students in a lesson. It contains efforts to judge mathematical qualities of diverse resources used in a lesson and, if necessary, to adjust them.

The mathematical work of teaching that is KCS in making decisions for one lesson, is anticipating common student conceptions and misconceptions with activities and problems in a lesson and observing of students understanding throughout a lesson. KCS in making decisions for one lesson involves knowing and anticipating what students are most likely going to appropriately or inappropriately think. It also involves conducting offered activities and solving problems in a lesson to be well up on students' mathematical ideas and reasoning. It also includes monitoring how students' understanding moves throughout a lesson and diagnosing students' emerging and inchoate thinking. Finally, it includes judging students learning, mathematically and pedagogically, throughout a lesson. These mathematical skills and knowledge are based on being familiar with students and their mathematical thinking.

The mathematical work of teaching that is KCT in making decisions for one lesson is about instructional decisions to plan a lesson and manage classroom discussions and activities. This indicates, in other words, the ability to design and manage instruction with specific mathematical understanding in and for one lesson. It includes elaborating ways to pose problems and activities and to accomplish mathematical goals in a lesson and actually conducting them. It also involves decisions for the sequence of activities, representations, examples, explanations, and questions in a lesson for the smooth and reasonable flow of a lesson. It contains decisions about when to ask questions and pause for explanations and to remark on students' ideas in a lesson so as to control the dynamics of a lesson. It embraces asking questions to motivate students' mathematical learning and enhance their participation in classroom discussions. It involves the adjustment of instruction after monitoring students learning in a lesson for a mathematically valuable and pedagogically efficient lesson. It also includes the consideration, based on results of evaluation, of the next steps for students' mathematical development. Such mathematical knowledge and reasoning go together in an understanding of pedagogical issues that influence student learning in a lesson. It also includes selecting an activity or a way to assess students understanding in a lesson.

The mathematical work of teaching that is KCC in making decisions for one lesson is about identification of the purposes for teaching mathematics and relationships in the curriculum. KCC in making decisions for one lesson includes deciding what students are supposed to learn with provided activities and problems in a lesson as well as sizing up what are associated with the offered activities and problems in the curriculum. This means, in particular, what are the previous and next lessons in the curriculum and where topics are related to the lesson in the curriculum. Such mathematical knowledge and skills are based on curricular knowledge.

The mathematical work of teaching that is HCK in making decisions for one lesson is about the identification of the connection in terms of the broader disciplinary territory. HCK in making decisions for one lesson includes clarifying mathematically important notions of activities and problems offered in a lesson. It also includes characterizing which topics being taught in a lesson work foundationally with topics to be offered later. It also involves identifying connections, with respect to mathematical domains, between topics of activities and problems offered in a lesson and topics that were offered before or will be offered later. It contains judging the maintainability of the way to portray a notion in a lesson when mathematically more sophisticated ideas are coming up. It includes thinking of how a current topic, in terms of the discipline with integrity, is an instantiation of something that was taught or will be taught later.

Table 6.2 summarizes the above discussion of making decisions for one lesson in terms of domains of MKT. A similar summary table will be included after the discussion of each zoomed-out mathematical work of teaching.

Table 6.2 *Specification of Making Decisions regarding One Lesson*

SCK: synthetic anticipation and evaluation about contents and activities of a lesson • explaining mathematical goals and purposes • appraising and adapting the mathematical topics of textbooks • anticipating how mathematical ideas change and grow in one lesson • making judgments about the mathematical quality of instructional materials and modifying as necessary KCS: anticipation of students' common conceptions and misconceptions and the observation of students' understanding throughout a lesson

- anticipating, with given activities and problems in a lesson, what students are likely to do and get confused about
- monitoring students understanding throughout a lesson
- making mathematical and pedagogical judgments about students learning throughout a lesson

KCT: instructional decisions to plan a lesson and manage a classroom discussion and activities

- elaborating on how problems and tasks are posed
- elaborating on how mathematical goals are accomplished
- deciding the sequence of activities, representations, examples, explanations, questions, etc.
- deciding when to pause and ask questions and offer explanations and when to use students' ideas during a lesson
- posing mathematical questions that are productive for students' learning
- adjusting teaching based on the monitoring students learning
- taking next steps according to results of evaluations of a lesson
- structuring the next steps in the students' development
- choosing a task to assess students understanding

KCC: identification of the purposes for teaching mathematics and relationships in the curriculum

- deciding what is most important for students to know and understand about the provided tasks and problems in a lesson
- grasping where a lesson is situated in the curriculum – what was the previous and will be the next lessons in the curriculum and where topics related to the lesson are in the curriculum

HCK: identification of the connection in terms of the broader disciplinary territory

- investigating mathematically significant notions that underlie tasks and problems of a lesson
- understanding which topics being taught in a lesson are foundational to later topics
- connecting a topic being taught to topics from prior and future years and across mathematical domains
- considering whether the way a notion is currently portrayed will maintain its mathematical integrity as more sophisticated mathematical ideas are introduced
- recognizing how a current topic is an instantiation of something that was taught before or will be taught later

Teaching also requires a broad view on developing mathematical ideas throughout a longer period, such as a unit or a chapter of instruction, one semester, and one school year. Teaching over a long period, in terms of mathematical tasks of teaching, requires knowledge, skill, and reasoning that ensure teachers can make far-sighted decisions in order to manage students mathematical learning responsibly and responsively. Therefore, making decisions for a long period as content in mathematics teacher education means specifying, practicing, and talking about skills and reasoning to see a long period as a unit. Each period is a time unit in which teachers can plan and support students to produce a mathematically enormous harvest.

The mathematical work of teaching that is SCK in making decisions for a long period concerns the synthetic anticipation and evaluation of mathematical ideas and understanding throughout a unit or a chapter of instruction, one semester, and a year. SCK in making decisions for a long period includes anticipating and judging on how mathematical ideas change and develop throughout a long period. It also involves the choice of activities to assess how students' understandings move and grow throughout a year.

The mathematical work of teaching that is KCS in making decisions for a long period involves anticipating common students' conceptions and misconceptions in a unit or a chapter of instruction, one semester, and a grade level and the observation of students understanding throughout a long period. KCS in making decisions for a long period also involves anticipating what students are most likely to have conceptions and misconceptions in a long period in order to mathematically know them backwards and forwards. It also includes continuously monitoring students' understanding throughout a long period so as to diagnose how they mathematically develop and decide how to mathematically and pedagogically support them. It takes in assessing students' mathematical learning with a variety of measurement tools through a long period to clarify what they do and do not understand. Such mathematical knowledge and skills are based on the combination of knowing students and knowing about specific mathematics.

The mathematical work of teaching that is KCT in making decisions for a long period involves instructional decisions to plan and manage a unit or a chapter, a semester, and a year. KCT in making decisions for a long period includes decisions about the sequence of contents and topics of mathematics in a long period for an efficient and systematic development of mathematics throughout a long period. It also involves adjustment of types and qualities for teaching based on the results of assessments of student learning during a long period for a mathematically authentic and pedagogically desirable a long period. Such mathematical skill and reasoning demands both mathematical understanding and an understanding of pedagogical issues that makes an instructional influence on student learning throughout a long period.

The mathematical work of teaching that is KCC in making decisions for a long period is about identifying purposes of mathematics education for a unit or a chapter, a semester, and a grade level and for a structure of a curriculum. KCC in making decisions for a long period includes deciding what is most important for students to know and understand throughout a long period of mathematics education. This is needed in order to identify what teachers should be oriented in and for mathematics teaching throughout a long period. It also involves clarifying mathematical goals of mathematics education in a long period that teachers and students must strive to attain. Moreover, it encompasses clarifying mathematical topics in a long period mathematics curriculum and their sequence and relationships in order to see each topic within a map known as the curriculum and to have a relational understanding of the topics. It also contains the structure of the curriculum, specifically the vertical and horizontal structure of a curriculum within a school year and across grade levels. Such mathematical knowledge and reasoning are based on curricular knowledge of mathematics.

The mathematical work of teaching that is HCK in making decisions for a long period is about identifying the connections among topics provided at a unit or a chapter, a semester, and a grade level with respect to the broader disciplinary territory. HCK in making decisions for one long period includes connecting topics across mathematical domains at a given period as well as topics across time as mathematical ideas develop and extend toward mathematical integrity. It also involves investigating mathematically significant notions that underlie topics offered and emphasized through a long period. Such knowledge and reasoning are based on the broader disciplinary territory.

Table 6.3 summarizes the above discussion of making decisions for a long period regarding domains of MKT.

Table 6.3 *Specification of Making Decisions regarding a Long Period*

SCK: synthetic anticipation and evaluation of mathematical ideas and understanding throughout a long period

- anticipating and judging how mathematical ideas change and grow in a long period
- choosing activities to assess students understanding through a long period

KCS: anticipation of students' common conceptions and misconceptions at a particular long period and the observation of students understanding through a long period

- anticipating what students are likely to have conceptions and misconceptions in a long period
- monitoring students understanding throughout a long period
- assessing students' mathematical learning using a variety of measurement tools throughout a long period
- making mathematical and pedagogical judgments about students learning throughout a long period

KCT: instructional decisions to plan and manage a long period

- deciding the sequence of contents and topics in a long period
- adjusting types and qualities of teaching according to results of assessment during a long period

KCC: identification of purposes of mathematics education in a long periodl and the structure of curriculum

- deciding what is most important for students to know and understand throughout a long period mathematics education
- clarifying mathematical goals of mathematics education in a long period
- grasping topics in the curriculum and the interrelationship and sequence of topics
- identifying the structure of the curriculum within a grade level and across grade levels

HCK: identification of the connections, with respect to the broader disciplinary territory, among topics provided in a long period

- connecting across mathematical domains both at a given long period and across time as mathematical ideas develop and extend
- investigating mathematically significant notions that underlie topics emphasized throughout a long period

Providing and justifying mathematical and pedagogical decisions is the main mathematical work that teachers first face in their classrooms. The two kinds of work in this zoomed-out mathematical work of teaching are related to each other: a long period is made up of many lessons. Furthermore, they require at least several and, sometimes, a great deal of zoomed-in mathematical work of teaching and final decisions for each different time period.

Situating and unpacking mathematics ideas and practices. The field of mathematics is immense and tremendous and mathematical objects are connected to one another. If a certain mathematical object becomes the focus without the consideration of the related mathematical domain, mathematical teaching can deteriorate and become distorted. Situating and unpacking mathematics ideas and practices is about using decompressed mathematical knowledge and dealing with advanced mathematics that is presented in the mathematics students learn. It is classified into two kinds of work: unpacking mathematics ideas and situating teachers in mathematics.

Unpacking mathematics ideas refers to investigating the mathematical ideas behind what students use and are likely to think and topics given in curriculum materials. For example, teachers in Emily's lesson on why the unit digit rule works explored one example, 12,357. They pointed out that the 1 in the ten-thousand digit means 10,000, the 2 in the thousands digit means 2,000, the 3 in the hundreds digit means 300, and the 5 in the tens digit means 50. 10,000, 2,000, 300 and 40 are even numbers. All of them are multiples of 10 because of the base-ten number system. The sum of them is always an even number because 10 has 2 as a factor. Here are the two key ideas of the units digit rule: the base ten number system and the fact that the sum of even numbers is an even number. When the teachers realized these two concepts, they could generalize their examples. In other words, unpacking mathematics idea includes exploring how and why certain mathematical rules, representations, and algorithms work. Such exploring includes representing and mapping across a long multiplication and area model, investigating why the long division algorithm works, or investigating why zero cannot be a divisor. It helps teachers have a more fundamental understanding about mathematics and provides mathematical groundwork in and for teaching. All this kind of work also requires various zoomed-in mathematical work through diverse mathematical objects. More details follow.

The mathematical work of teaching that is SCK in unpacking mathematics ideas is about the investigation of what underlies mathematical knowledge that is taught directly to students in order to have knowledge of why mathematical objects can exist and work in and for teaching. It includes presenting and justifying mathematical ideas and clarifying how they are connected to have knowledge and skill to make particular content visible to students. The logical aspects of mathematical content are investigated, too. These tasks make clear what mathematics is involved in a particular representation, example and explanation and the mathematical structure of a task. Furthermore, it includes explaining how to choose, create, and use representations, examples, and explanations that clarify mathematical decisions in and for teaching.

Representations, in particular, are often used in and for teaching, and the use of representation requires various tasks of teaching, such as finding representations that are mathematically logical, work well, or capture mathematical core, linking representations to underlying ideas and to other representations, and evaluating whether or not representations were equivalent. SCK in unpacking mathematics idea also includes anticipating what students might do with a topic and explaining why, and deciding which mathematical ideas have the most promise among mathematical ideas. It also entails determining what to emphasize and understate in teaching to help students approach core concepts of the content. It involves modifying tasks to be either easier or harder as well as responding to students' "why" questions.

The mathematical work of teaching that is KCS in unpacking mathematics idea is about making mathematical decisions according to familiarity with students' typical ideas. KCS in unpacking mathematics idea includes selecting, constructing, and using explanations, examples, and representations so that students can understand why and how mathematical objects exist and work as well as anticipating what students are likely to know and do with explanations, examples, and representations and with what they typically have difficulties. These tasks require being familiar with the students.

The mathematical work of teaching that is KCT in unpacking mathematics ideas is about mathematical decisions that have instructional effects on students learning. KCT in unpacking mathematics ideas includes using different aspects of a topic that make a difference at different points in students' learning and considering different care in the use of representations, explanations, and contexts in order to make mathematical issues salient and usable by students. It also involves intentionally asking questions, which might be mathematically correct or incorrect, to motivate students thinking about why and how mathematical objects exist and work as well as giving careful demonstrations about them. These tasks influence what students learn and how they learn them.

KCC in unpacking mathematics idea is about the identification of the sequences through the curricular of grade levels, and HCK in unpacking mathematics ideas is about the identification of the connection in mathematical territory. Finally, unpacking mathematics ideas develops fluency to work with students while using compressed mathematical knowledge.

Table 6.4 summarizes the above discussion of unpacking mathematics ideas in terms of domains of MKT.

Table 6.4 *Specification of Unpacking Mathematics ideas*

SCK: investigation of why mathematical objects can exist and work in and for teaching

- presenting and justifying mathematical ideas
- clarifying and explaining connections among mathematical ideas
- investigating the logical aspects
- recognizing what mathematics is involved in using a particular representation, example, and explanation
- identifying and specifying the mathematical structure of a task
- explaining how to choose, create, and use representations, examples, and explanations
- finding representations that are mathematically logical, work well, or capture mathematical core
- linking representations to underlying ideas and to other representations
- evaluating whether or not representations were equivalent
- anticipating what students might do with a topic and explaining why
- deciding which of several mathematical ideas has the most promise and what to emphasize
- modifying tasks to be either easier or harder
- responding to students' "why" questions

KCS: mathematical decisions according to familiarity with students' typical ideas

- selecting, constructing, and using explanations, examples, and representations so that students can understand why and how mathematical objects exist and work
- anticipating what students are likely to know and do with explanations, examples, and representations and with what they typically have difficulties

KCT: mathematical decisions to make instructional effects on student learning

- using different aspects of the content that make a difference at different points in students' learning
- using different care in the use of representations, explanations and contexts in

order to make the mathematical issues salient and usable by students
- asking questions to motivate students thinking about why and how mathematical objects exist and work
- demonstrating reasons for students' "why" questions

KCC: identification of the sequences through the curricula of grade levels
- sizing up the sequences related to the topic within a school year and across grade levels

HCK: identification of the connection in mathematical territory
- clarifying important links with others and structural knowledge

Situating teachers in mathematics is a more fundamental investigation of the territory of mathematics. This aspect indicates exploring a number of things: the mathematical environment surrounding the disciplinary location that students currently stand in; the major disciplinary ideas and structures and key mathematical practices for illuminating critical dimensions of that content; it also indicates anticipating the mathematics of what the student may encounter further along the path (Ball, 1993). For example, in the task of representing a fraction on the number line, exploring ideas of the density of the rational numbers and recognizing that all the numbers of K-8 mathematics "live" on the number line are included here.

The mathematical work of teaching that is SCK in situating in mathematics is about having the intellectual honesty of mathematics in and for teaching. SCK in situating in mathematics includes inspecting equivalencies among mathematical ideas in mathematics and maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea. These tasks ask mathematical decomposition in terms of disciplinary mathematics.

The mathematical work of teaching that is KCT in situating in mathematics is about mathematical and pedagogical decisions for mathematically authentic learning. KCT in situating in mathematics involves leading discussions that are honest to mathematics and respectful of students and deciding whether and how teachers respond to students' ideas and questions that entail higher levels of mathematics. Mathematical discussions with students should not only be mathematically sound but also respects students. In the discussion, it is necessary that teachers recognize the mathematics in teaching that will be studied by students in years to come and to make fast and efficient decisions regarding students questions, specifically whether teachers answer immediately or not, what responses or explanations teachers give, and how teachers respond.

The mathematical work of teaching that is KCS in situating in mathematics is about observing student learning with respect to mathematics. It includes understanding what will make sense to students and monitoring how students take hold of and transform different situations and models. These tasks need to be deeply located in disciplinary mathematics rather than partially function or distort mathematics.

As part of unpacking mathematics ideas, KCC in situating in mathematics includes seizing sequences related to the topic within a school year and across grade levels. However, the mathematical work of teaching that is HCK in situating in mathematics is extended. HCK in situating in mathematics is about recognizing the advanced mathematics that is demonstratively related to the work of teaching in school. It includes knowing how mathematics being taught is situated in the broader mathematical territory and understanding relationships between specific advanced mathematics and specific ideas arising in the topics being taught and learned in school. It also involves understanding the disciplinary motivation for given topics and comprehending how they have developed. Moreover, it contains having an intuitive grasp of core ideas involved and being familiar with basic techniques developed to contend with the ideas. It includes grasping important mathematical structures of the discipline, specifically ones that are structurally related to content in the school curriculum. It also involves appreciating structure, both in the sense of gaining familiarity with important mathematical structures of the discipline and with understanding them and being able to use them as structures. Finally, it includes having and considering mathematical affordances for intellectual honesty in and for teaching.

Table 6.5 summarizes the above discussion of situating in mathematics in terms of domains of MKT.

Table 6.5 *Specification of Situating in Mathematics*

SCK: having the intellectual honesty of mathematics in and for teaching

- inspecting equivalencies
- maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea

KCS: observation of students learning with respect to mathematical territory

- understanding what will make sense to students
- monitoring how students take hold of and transform different situations and models

KCT: mathematical and pedagogical decisions for mathematically authentic learning

- leading discussions that are honest to mathematics and honors students' thinking
- deciding whether and how teachers respond to students' ideas and questions which entails for them higher levels of mathematics

KCC: identification of the sequences through the curricula of grade levels

- sizing up the sequences related to the topic within a school year and across grade levels

HCK: recognition of the advanced mathematics that is demonstratively related to the work of teaching in school

- knowing how the mathematics being taught is situated in the broader mathematical territory
- understanding relationships between specific advanced mathematics and specific ideas arising in the content being taught and learned in school
- understanding the disciplinary motivation for given topics and how they developed
- having an intuitive grasp of the core ideas involved
- being familiar with the basic techniques developed to contend with the ideas
- grasping important mathematical structures of the discipline, specifically ones that are structurally related to content in the school curriculum
- appreciating structure, both in the sense of gaining familiarity with important mathematical structures of the discipline and with understanding them and being able to use them as structures
- having and considering mathematical affordances for intellectual honesty in and for teaching

Situating and unpacking mathematics ideas and practices drives a generic approach to mathematics and facilitates having a profound understanding of fundamental mathematics (Ma, 1999). Identifying a mathematically fundamental ground assists having mathematical principles, diverse and flexible reasoning, and relations toward teaching mathematics. Ultimately, teaching mathematics can become a mathematically honest form (Bruner, 1960).

Using language mathematically and accessibly. This category includes tasks related to the use of language in and for teaching. The use of language in and for teaching requires mathematical rigor while still allowing for comprehension and usability by students. Neither can be sacrificed. Imprecise language will become an obstacle for students' developing their mathematical ideas; moreover, language that is not geared toward students will delay the learning of mathematics appropriately. For example, after probing possible concerns for understanding a definition and deciding whether a definition is mathematically precise, teachers determine conflicts between them and choose a definition that is mathematically precise and, simultaneously, one that students can approach as needed in the flow of instruction for third graders. Consider a definition of an even number: an even number is a natural number that is divisible by 2. It is imprecise because it does not include negative even numbers. Moreover, if "divisible" is not obviously defined as dividing with no remainder, then all natural numbers would be considered even. To offer it to students, "natural number" and "divisible by 2" could be concerns if they understood them. More mathematically precise is, "A natural number is even if it is two times a natural number." While it does not function to differentiate between all even numbers and non-even numbers, it clearly specifies which natural numbers are even. It does not mention negative even numbers, but it does not inappropriately separate between all even numbers and all non-even numbers. However, it still has accessibility concerns by students, about a "natural number" and "two times." If students know both of them, this definition is proper for students.

In particular, SCK in using language mathematically and accessibly entails mathematically accurate and precise notation and language. For example, specifying, probing, evaluating, representing, revising, and constructing definitions calls for the careful use of language in order to have mathematically clear communication and guide the practice in mathematics about the value of accurate and exact representations and statements. Using unclear and imprecise language hinders proceeding to the territory of mathematics. For example, secondary school students sometimes rely on their learning about certain concepts when they were in elementary school, and this impedes their mathematical development. Moreover, SCK in using language mathematically and accessibly includes talking about how mathematical language is used in and for teaching and considering what mathematical universe is assumed in statements or examples. It also involves mapping informal language and terminology and representations in real world to formal language in mathematics and specifying difficulties to use mathematical words rather than everyday words.

KCT in using language mathematically and accessibly includes using mathematically precise language and identifying how language and metaphors can assist or confound student learning. KCS in using language mathematically and accessibly involves coordinating both mathematical rigor and comprehension by students. This refers to explicitly addressing conflicts to achieve a balance between disciplinary rigor and structure and students' accessibility as well as focusing on language issues between mathematical details and what students use and know. Specifically, it involves choosing and constructing a mathematically appropriate definition that is comprehensible to students and designing mathematically accurate explanations that are comprehensible to and useful for students. It further includes interpreting whether students' language comprises mathematics words or everyday words and making real world contexts accessible.

Like situating and unpacking mathematics ideas and practices, KCC in using language mathematically and accessibly includes sizing up the sequences related to a topic within a school year as well as across grade levels. Moreover, HCK in using language mathematically and accessibly involves acquiring a command of mathematical language.

Using mathematical language mathematically and accessibly is supported by having a linguistic infrastructure. Teaching mathematics happens through the use of language. The kind of language used in and for teaching determines how much mathematical sensibility is reflected and that mathematical knowledge has a central role in teaching (Ball & Bass, 2000b).

Table 6.6 summarizes, regarding domains of MKT, the above discussion of using language mathematically.

Table 6.6 *Specification of Using Language Mathematically*

SCK:

- using mathematically precise notation and language of mathematical tasks and being critical of its use
- talking about how mathematical language is used
- considering kinds of universes for definitions
- mapping informal language and terminology and representations from the real world to formal language in mathematics
- specifying difficulties in using mathematics words rather than everyday words

KCS:

- choosing and constructing a mathematically appropriate definition that is comprehensible to students
- designing mathematically accurate explanations that are comprehensible to and useful for students
- interpreting whether students' language consists of mathematics words or everyday words
- making real world contexts accessible

KCT:

- employing mathematically precise language
- identifying how language and metaphors can assist or confound student learning

KCC:

- sizing up the sequences related to the topic within a school year and across grade levels

HCK:

- acquiring a command of mathematical language

This section has conceptualized the zoomed-out mathematical work of teaching and illustrated each in detail. The zoomed-out mathematical work of teaching involves three kinds of work. Tables from 6.7 to 6.11 show the overall structure of the zoomed-out mathematical work of teaching with domains of MKT. Comparing each mathematical work in these tables reveals several features. First, all the zoomed-out mathematical work of teaching is encompassed by five domains of MKT, specifically SCK, KCS, KCT, KCC, and HCK. Second, unpacking mathematics ideas and using language mathematically and accessibly involve a relatively great deal more of SCK than other mathematical work of teaching. Third, making a decision for one lesson and unpacking mathematics ideas are closely related to KCT. Indeed, these tasks greatly influence the design of instruction. Fourth, situating in mathematics does not include much KCS because this task is closer to investigation in disciplinary mathematics. Fifth, making a decision for one lesson and making a decision for one school year are relevant to KCC because these tasks entail knowledge of curriculum. Finally, making a decision regarding one lesson and situating in mathematics are relatively prevalent in HCK because they ask to clarify apparent relations between activities in a lesson and their places in disciplinary mathematics.

6.2.2 Zoomed-in mathematical work of teaching

The zoomed-in mathematical work of teaching requires an in-depth analysis of both the mathematical objects in teaching and the work of mathematics teaching. The framework accomplishes this detailed analysis. Mathematical objects in teaching refer to the kinds of mathematical objects that teachers teach. For example, properties of an even number involve mathematical aspects that differ from the proof of a conjecture, such as the sum of two odd numbers is equal to an even number. Depending on the lesson, different mathematical objects are highlighted. Therefore, mathematical objects in teaching are sorted into four objects: *concept, property, and definition*; *procedure, algorithm, and rule*; *representation and tools*; and *proof*. Furthermore, various tasks of mathematics teaching from the data were gathered and sorted into five kinds of tasks: *recognizing and articulating*; *probing, interpreting, and comparing*; *evaluating*; *selecting and modifying*; and *constructing*. All of them consist of work that teachers carry out in and for teaching. In other words, teachers perform the work of mathematics teaching when they prepare for, have, and evaluate mathematics lessons with or without their students. These objects and tasks are used to construct the zoomed-in mathematical work of teaching as shown in Table 6.7

Table 6.7 *The Basic Structure of Zoomed-In Mathematical Work of Teaching*

		Work of mathematics teaching				
		Recognizing Articulating	Probing Interpreting Comparing	Evaluating Judging	Selecting Modifying	Constructing
Mathematical objects in teaching	Concept Property Definition					
	Algorithm Rule Procedure					
	Representation Tools					
	Proof					

The combination of five kinds of work of mathematics teaching and four kinds of mathematical objects in teaching creates twenty kinds of zoomed-in mathematical work of teaching. Depending on the mathematical region, such as number and operations, geometry, measurement, data analysis, and probability, the zoomed-in mathematical work of teaching can include different disciplinary content. Here, what should be highlighted is the seeing of the curriculum of mathematics teacher education from the perspective of the zoomed-in mathematical work of teaching. The zoomed-in mathematical work of teaching operates in the moment of teaching practice and a couple of them often are at play immediately, simultaneously, and continuously. For example, teachers figure out, interpret, and evaluate students' definitions of an even number as such definitions are made and used implicit or explicit through student reasoning, and think of how to refine them. Teachers generate and revise representations that they will use for teaching and select examples for illustrations. All of the work happens immediately, simultaneously, and continuously. Therefore, each zoomed-in mathematical work of teaching is fundamental and direct in and for teaching.

Table 6.7 shows all combinations of work of mathematics teaching and mathematical objects. For example, a cell which intersects recognizing and definition indicates recognizing a definition as a zoomed-in mathematical work of teaching. This section now describes each of the zoomed-in mathematical work of teaching. There are several considerations in this specification. First, for convenient descriptions, this section explains the zoomed-in mathematical work of teaching in terms of five types. In other words, it would also be the same if this section explained zoomed-in mathematical work of teaching in terms of the four mathematical objects in teaching. However, like the overall architecture of the framework, there is no particular order in which the analyses occur. Second, in the structure, mathematical objects in teaching involve explanations, examples, and reasoning related to mathematical objects. For example, definition in the framework, as shown in Table 6.7 indicates an explanation of a definition, an example of a definition, or reasoning related to a definition. Thus, the combination between probing and definition indicates probing an explanation of a definition, probing an example of a definition, or probing reasoning related to a definition.

Third, two approaches for mathematical objects in teaching are taken specifically for SCK: mathematical ideas from discipline or textbooks and mathematical ideas from and for students. While the work of mathematical ideas from discipline or textbooks entails mathematical completion and accuracy, mathematical ideas from and for students entail well-captured specification of students' ideas and applicability for students. Both of them, moreover, engage a different type of work. Related to mathematical ideas from and for students, their features are specified here for efficient statements in the following sections. Students' ideas are mathematically standard, conventional, and predictable. Or, their ideas are non-standard, alternative, and unusual. For example, students can solve a multiplication problem with a standard algorithm or alternative methods as shown in *Figure 6.3*.

Ⓐ	Ⓑ	Ⓒ
35	35	35
x 25	x 25	x 25
125	175	25
+75	+700	150
875	875	100
		+600
		875

Figure 6.3 Three alternative methods for multiplying 35 x 25 (Ball & Bass, 2003b, p. 7)

Their ideas are sometimes complete and logical but, because students are on the process of learning, sometimes incomplete and illogical (Roh, 2008; Tall & Vinner, 1981). Moreover, students sometimes explicitly express their ideas by generally accepted ways and language or sometimes implicitly use their own ways and language (J. D. Davis, 2008; Kazima, 2007; Lampert & Cobb, 2003). Such facets are considered whenever "students' ideas" are mentioned, though they are not always spelled out. Specifically, the differentiation between mathematical ideas from discipline or textbooks and mathematical ideas from and for students lies in the investigation of SCK as one domain of MKT through the zoomed-in mathematical work of teaching. This is because SCK is a domain to work differently in these two approaches although the other domains require them.[28] This differentiation helps detail the mathematical work of teaching.

Fourth, the consideration of the specification of the zoomed-in mathematical work of teaching, the five kinds of tasks—recognizing and articulating, probing and comparing, evaluating, selecting and modifying, and constructing—have different features. Recognizing and articulating explanations, examples, and reasoning of mathematical objects is to acknowledge a certain existence and to clarify what they are exactly. Probing and comparing explanations, examples, and reasoning of mathematical objects is for identifying their features. These two tasks require seeing the phenomena in instruction and capturing it exactly. The last three kinds of tasks of teaching make certain decisions based on the first two tasks of teaching. Evaluating makes what explanations, examples, and reasoning of mathematical objects are correct, appropriate, and better. Selecting makes some choices among explanations, examples, and reasoning of mathematical objects, and modifying makes revisions of explanations, examples, and reasoning of mathematical objects. Moreover, constructing creates necessary explanations, examples, and reasoning of mathematical objects. KCT is closely related to making decisions about teaching rather than understanding and identifying the phenomena in instruction. Because of this KCT is related to evaluating, selecting and modifying, and constructing as tasks of teaching, unlike recognizing and articulating, and probing and comparing.

Now, next section describes each zoomed-in mathematical work of teaching.

Recognizing and articulating mathematical objects**.** Mathematical objects related to recognizing and articulating can be differentiated into knowledge about mathematics in mathematics and students' ideas. In terms of objects in mathematics, recognizing and articulating are basically figuring out and identifying what mathematical objects mean exactly in disciplinary mathematics without interpretation or evaluation. If teachers teach

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[28] CCK does not need these differentiations, KCS focuses on students, and KCT is related to decisions for helping students.

an even number, they should first know what an even number is. Second, with regard to students' ideas, recognizing and articulating are being aware of and clarifying what students think in classrooms and what they might think. Listening to students' ideas and observing what students are performing are expected in order to recognize what students think. Moreover, questioning so as to further clarify and make explicit students' ideas is expected because students' ideas are sometimes implicit. In other words, teachers need to clearly know students' ideas and development. Here, it is not important whether their ideas are mathematically plausible or implausible. Revoicing students' questions and statements is also required to ensure that all classmates are engaged in an activity. Furthermore, considering and specifying what students might know and think related to mathematical objects have roles in reasoning for teaching. These tasks contribute to provide important evidence and, as a result, to size up their mathematical understanding. Recognizing and articulating mathematical objects are related to CCK, SCK, and KCS that are domains of MKT. The tasks are detailed in terms of each mathematical object.

In terms of CCK, recognizing and articulating concepts, properties, and definitions calls for mathematically complete knowledge. For example, if a set of even numbers is shown as {2, 4, 6, 8…}, it negatively influences to construct or evaluate definitions of an even number. This is because this set fails to correctly show the set of even numbers. Hence, decisions made based on the provided set should be improper. Having exact meanings of mathematical terms is also important. For example, zero is a whole number and an integer, but not a natural number. The exact illustration of these different types contributes to a consideration of what happens with particular numbers or examples. Recognizing and explaining how concepts, properties, and definitions function is involved in SCK.

Furthermore, being aware of concepts, properties, and definitions calls for carefully watching and listening to students' statements and articulating them. Students' ideas related to concepts and definitions are inclined to be personal, partial, intuitive, and experiential rather than formal (Tall & Vinner, 1981). However, this is not intended to devaluate the importance of specifying students' ideas. Without the specificity of students' ideas, basic interaction of instruction is impossible. To motivate students to actively disclose their ideas, which are mathematically correct or incorrect, questioning is expected with closed- or open-answer questions. If teachers fail to recognize how students reason with different definitions related to a provided task, their discussions are mathematically unstable and make it hard to draw sound conclusions. Therefore, recognizing definitions that students use in discussions is fundamental to leading discussions. Teachers should, through observation, recognize and determine which of concept, property, and definition students use or have. Revoicing students' ideas helps all students comprehend their classmates' ideas, too. Furthermore, articulating to students what concept, property, and definition are and figuring out what experience students have related to concept, property, and definition are critical tasks in teaching. These tasks are included in KCS as an MKT domain.

Recognizing and articulating algorithm, rule, and procedure includes specifying the general forms of algorithms, rules, and procedure, and making sure what understanding of them students have. Specifying the general forms of algorithms, rules, and procedures requires specific and exact knowledge. It is involved in CCK. Moreover, teachers look for and explain what they are and how they function from textbooks, guides, or other recourses and articulate them explicitly. For example, the exact specification of the units digit rule provides a sound foundation for any kind of discussions related to the rule. The units digit rule focuses on a number in an integer's ones digit, rather than the number as the last digit of decimals. In other words, if the units digit rule is assumed to concentrate on a number in the last digit of any number, then discussions of the units digit rule should float freely and, not be anchored properly. This is related to SCK.

Making sure of students' understanding of algorithm, rule, and procedure requires careful observation of student use. Teachers observe what students say in whole group discussions and circulate around the classroom while students work either individually or in small groups. In this fashion teachers gather information about what and whom to concentrate on and how students are doing the work. Oftentimes, teachers ask questions to motivate students to clarify how they use algorithm, rule, and procedure, such as how they solve problems of computations of multi-digit additions or divisions of fractions. Moreover, listening to students' ideas and interpretations provides some resources for teachers to have a developed discussion or to create strategies to help students. Articulating what algorithm, rule, and procedure are to students and figuring out what experience students have related to algorithm, rule, and procedure are also tasks of teaching involving KCS. Particularly, some students tend to use algorithm, rule, and procedure with their own ways.   Teachers, for their own purposes in classes, summarize and revoice students' understanding and ways to approach algorithms, rules, and procedures.

Recognizing and articulating representation and tools includes recognizing the thing to be represented and the representation as well as discerning students' representation and uses of the tools. Just searching out, specifying, and articulating what representation entails CCK. In terms of SCK, recognizing the thing to be represented and the representation calls for having various representations and tools related to the thing to be represented. While teachers generally use representations and manipulatives suggested in textbooks and teachers' guides, they would search for more diverse representations from other resources. Moreover, teachers introduce and show representations and manipulatives to students with explanations of their meanings. For example, teachers might articulate that 8 is even by using the representations shown in *Figure 6.4*.

Figure 6.4 8 is even

Discerning students' representations and uses of tools is also observing what representations students use and how they interpret them. Representation is necessary for students' understanding of mathematical concepts and relationships (Kaput, 1998; Wang & Paine, 2003). Moreover, representations provide information as to how students think about mathematical concepts and serve as tools for students and teachers to learn and teach mathematics in their classrooms (Kalathil & Sheril, 2000). Therefore, identifying different parts of students' representations and interpretations are significant tasks in teaching mathematics. With regard to KCS, teachers ask questions to clarify students' ideas related to their own or other classmates' representation and uses of manipulatives. Articulating what representations and tools are to students and figuring out what experience students have related to representations and tools are also tasks of teaching which is involved in KCS. Teachers also draw images or rehearse using manipulatives based on students' statements. KCS also functions: regarding recognizing students' presentations, articulating ways in which students are thinking of representation and tools, watching students to recognize and articulate which of representation and tools students use or have, and revoicing students' statements and questions related to representation and what uses of manipulatives represent.

Recognizing and articulating proof are differentiated into articulating proofs in disciplinary mathematics and perceiving proofs that students use. While mathematics in elementary textbooks looks primary, it contains the rudiments of many important concepts from more advanced branches of the discipline (Ma, 1999). Therefore, contents in elementary mathematics textbooks are comprised with proofs even though the proofs are not introduced to students. Finding and searching for proofs related to the contents of textbooks strengthen mathematical ground. This is related to CCK. On the other hand, teachers ask questions about what proof students construct and how they interpret proofs in order to identify students' ideas about proofs. This task offers evidence to help decide how to steer discussions and help students learning mathematics. Teachers also articulate what a proof is to students and figure out what experience students have related to proofs. Such tasks are included in KCS. Furthermore, teachers articulate students' proof from their listening. Revoicing students' statements and questions related to proof are tasks of recognizing proofs in the practice of teaching. Articulating and explaining how proofs work, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others is involved in SCK.

The main components of the analyses and their examples are summarized in Table 6.8.

Table 6.8 *Specification of Recognizing and Articulating Mathematical Objects in Teaching and Its Examples*

		Recognizing / Articulating
Mathematical objects in teaching	Concept Property Definition	CCK • searching out, specifying, and articulating what concept, property, and definition are SCK • recognizing and explaining how concepts, properties, and definitions function, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others KCS • questioning, listening to, and articulating students' ideas and interpretations of concept, property, and definition • revoicing students' statements and questions related to concept, property, and definition in order to make students involved in discussions • recognizing and articulating which of concept, property, and definition students use or have • articulating to students what concept, property, and definition are • figuring out what experience students have related to concept, property, and definition
	Algorithm Rule Procedure	CCK • searching out, specifying, and articulating what algorithm, rule, and procedure are SCK • articulating and explaining how algorithms, rules, and procedures work and function, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others KCS • questioning, listening to, and articulating students' ideas and interpretations of algorithms, rules, and procedures • revoicing students' statements and questions related to algorithms, rules, and procedures and students' uses • recognizing and articulating which of algorithms, rules, and procedures students use or have • articulating how students use algorithms, rules, and procedures • articulating what algorithms, rules, and procedures are to students • figuring out what experience students have related to algorithms, rules, and procedures

	Representation Tools	CCK • seeking out, specifying, and articulating what representation is SCK • Related to representation and tools, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o searching out and having diverse representation related to the thing to be represented o looking for manipulatives that students can use o showing representation and tools and articulating their meanings o introducing manipulatives to students KCS • questioning, listening to, and articulating students' ideas and interpretations of representation and tools • recognizing students' presentations • observing how students use manipulatives • watching students to recognize and articulate which of representation and tools students use or have • revoicing students' statements and questions related to representation and what uses of manipulative represent • articulating what representation and tools are to students • drawing images or rehearsing using manipulatives based on students' statements • figuring out what experience students have related to representation and tools
	Proof	CCK • seeking out, specifying, and articulating a proof that is mathematically correct SCK • articulating and explaining how proofs work, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others KCS • questioning, listening to, and articulating students' ideas and interpretations of proofs • articulating students' proofs • revoicing students' statements and questions related to proofs • articulating to students what a proof is • figuring out what experience students have related to proofs

Probing, interpreting and comparing mathematical objects. Probing, interpreting, and comparing work differently between mathematical facts and structures and students' ideas. First, probing, interpreting, and comparing mean that teachers examine mathematical objects, describe the meanings of mathematical objects, and look for the differences among mathematical objects thoroughly. They do this to find features of mathematical objects. Second, probing, interpreting, and comparing examine students' ideas. Teachers should carry out these tasks immediately in classes and review students' assignments. Analyzing students' ideas strongly impacts teachers' decision making, such as evaluation, judgment, selection, and modification. The tasks are detailed in terms of each mathematical object. Probing, interpreting, and comparing mathematical objects are related to CCK, SCK, and KCS, which are domains of MKT.

Probing concepts, properties, and definitions and explaining their meanings requires mathematically thorough investigations. For example, consider one definition of an even number, an integer that is divisible by two. Reading and making sense of it involves CCK. The practice of teaching, however, calls for more investigation. This definition indicates that all even numbers are integers. Moreover, this definition enjoins one to examine "divisible." In school mathematics, it is generally agreed that if a number (n) is said to be "divisible by" a number (m) it means that n and m are integers such that n can be divided by m with zero remainder. Therefore, this definition tells one that 23.6 is not an even number because it is not an integer. Moreover, to use this definition, one must know what integer and divisible mean.

Consider another definition: A number is an even number if it can be divided by two. This definition seems similar to the previous because of "divided." They differ, however, because any number can be divided by two, such as $7 \div 2 = 3.5$; $\frac{1}{2} \div 2 = \frac{1}{4}$; $23.6 \div 2 = 11.8$; and $\pi \div 2 = \pi/2$. Examining concepts, properties and definitions that students explicitly or implicitly use necessitates creatively playing with students' concepts, properties, and definitions related to a provided topic and providing meaningful interpretations. Students, while possessing certain ideas, sometimes fall short of clarifying them. Their ideas can sometimes be predictable but also unusual. Thus, teachers must be able to ask appropriate questions to get a clear analysis of students' ideas.

Suppose a student asserts that six is an even number because it can be split fairly, but that it can also be odd because of three twos.[29] What are the main tasks to understanding the student's way of thinking? They are reviewing the student's statements and questions, figuring all the possible reasons and ways behind the student's

[29] This is Sean's assertion in the SeanNumbers-Ofala video clip.

coming up with such an assertion, and finding meaningful interpretation of it. It is also critical to have good interaction in instruction. Interpretation, with evidence, is clarified when considering what students understand or do not understand. Moreover, the interpretations are expanded when they are compared with other students' ideas. These situations involve various tasks of teaching involving SCK. In fact, they involve a long list of SCK tasks: playing with statements and examples of definition, looking for aspects of definition that are not represented by examples and statements, considering and running through sensible and special cases that might not be captured by examples and statements, considering possible counterexamples and their features in definition, considering the features of or what happens with examples and statements, examining terms used in statements of definition, comparing several of definition to identify its features, looking for possible relationships or similarities among examples and statements that might be focused on definition, sizing up mathematical concerns for understanding, sizing up the nature of errors that students have, having and providing meaningful interpretation about students' ideas and responses that might be non-standard, and playing with ways in which students' statements and questions can be produced.

Moreover, the situation includes several tasks of teaching related to KCS, such as questioning and hearing to have clear analysis of students' ideas, considering what might be understood and not be understood related to definition and identifying evidence, interpreting students' thinking about definition, and clarifying what common errors students make related to definition. Identifying ways in which a student is thinking about concept, property, and definition encourages teachers to grasp students understanding and provide a next step to helping them learn mathematics. All of these tasks also work with concept and property.

Examining and accounting for algorithms, rules, and procedures aims at identifying how algorithms, rules, and procedures work. Solving and calculating problems involving algorithms, rules, and procedures is related to CCK. The practice of teaching requires more exploration. For example, the units digit rule determines whether an integer is even or odd by examining the ones digit. If the ones digit is even, then the entire number is even; if odd, the entire number is odd. Generally, probing algorithms and rules begins with using them, such as solving problems or trying examples that include them. Trying the units digit rule with several examples initiates working through the rule and probing how the rule works. If we have 4,694, we know it is even because 4 in the ones digit is even. This means that regardless the number place in other digits' places, the number in the ones digit determines whether an entire number is even or not. Teachers would give this kind of explanation to students, too.

It is also critical to investigate possible issues to understand a rule and, thus, examine terms used in the statements of rule. Inquiring and illuminating students understanding of algorithms, rules, and procedures is attempting to spell out how students use and approach algorithms, rules and procedures. Regardless of whether students use any algorithms or rules properly, improperly, or in different ways, students have their own mathematical reasons. Or, students might have partial understandings of them. Understanding them supports mathematically sound instruction. Playing with the various ways in which students comprehend and use algorithms, rules, and procedures opens up all sorts of possibilities of how students use them. Then, walking through and checking each step of how students operate algorithms, rules, and procedures would clarify why they use them in such ways. Teachers also ask questions to have accurate analysis of students' ideas.

These situations involve various tasks of teaching related to SCK. In particular, they involve the following: working with and explaining each step of a rule, playing with statements and examples of rule, probing how a rule works, looking for aspects of rule that are is represented by examples and statements, considering and running through sensible and special cases that might not be captured by examples and statements, considering possible counterexamples and their features in rule, considering the features of or what happens with examples and statements, examining terms used in statements of rule, comparing several rules to identify their features, looking for possible relationships or similarities among examples and statements that might be focused on rule, sizing up mathematical concerns for understanding, sizing up the nature of errors that students have, having and providing meaningful interpretations of students' ideas and responses that might be non-standard, and playing with ways in which students' statements and questions can be produced.

Moreover, the situation includes several tasks of teaching related to KCS. Sometimes it is not so easy to identify their ways of using algorithms, rules, and procedures in written forms. There are more: questioning and listening to have a clear understanding of students' ideas, considering what might be understood and not be understood related to rules and identifying evidence, interpreting students' thinking about rules, clarifying what common errors students have related to rules. These tasks also work with algorithms and procedures.

In terms of CCK, probing representation indicates just probing its parts in representation. In teaching practice, investigating and elucidating representation and tools calls for elucidating apparent specifications between a representation and the represented and identifying features and relationships prominent in representations and tools. For example, an even number can be defined in a couple of ways: a number that can be divided into two equal parts with none left over and a number that can be divided into groups of two with none left over. Both are special cases of commutativity of multiplication by two, but they are represented differently. The former one is represented as 2 x N = N + N, or (a) in *Figure 6.5*. There are two groups and the number of circles in one group is N. The latter one is represented as N x 2 = 2 + 2 + 2 + … + 2 (a sum of N twos) or (b) in *Figure 6.5*. There are N groups of two. Or, both can be represented using a 2 x N rectangular array as shown (c) in *Figure 6.5*.

Figure 6.5 Representing definitions of an even numbers

Here is another example. "Even and odd numbers alternate on the number line, starting with zero being even" assumes a number line, rather than diagrams in *Figure 6.6*, which includes marks only for integers or whole numbers.

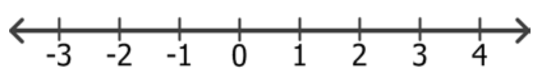

Figure 6.6 Number line with integers

Because the number line includes all rational numbers, this kind of use of the number line would be mathematically problematic even though it would work for a certain audience that knows only whole numbers as numbers. It should consider that when students generally learn integers, they have already learned fractions. Therefore, depending on purposes, different representations and tools would work differently. Exploring and construing the use of representations and tools by students is prompting to explicate how students use representations and tools and understand them. Students' drawings are sometimes partial and unusual compared to representations in discipline (S. P. Smith, 2003).

Identifying different parts of students' representations and, simultaneously, considering their interpretations of them from their statements and questions work for mathematically valuable instruction. Ultimately, such efforts bring about students being fully involved in mathematical discussion. Teachers ask questions to examine exactly students' ideas as well as to provide other students meaningful explanations of a fellow student's representations. Finally, teachers identify what the students understand and do not understand. Moreover, comparing students' representations throws into relief the varied ideas students have on a given topic.

These situations involve various tasks of teaching related to SCK. In particular, these include the following: working through and mapping both the thing to be represented and the presentation, identifying representation and its parts in representation, identifying different parts of representation and considering diverse interpretations of it, playing with ways in which the representation might be explained to fit with the thing to be represented or in which students' ideas can be produced related to representation and tools, looking for features and relationships prominent in the design of the objects being considered, considering and running through sensible cases of representation and special cases that might not be captured in representation, considering possible counterexamples and their features in representation, considering the features of or what happens with examples and statements; comparing representation, looking for possible relationships or similarities among examples and statements that might be focused on representation, sizing up mathematical concerns for understanding and using representation, sizing up the nature of errors that students have, identifying strong and weak points of representation for a provided purpose, playing out how representation would be used for purposes, having and providing meaningful interpretation about students' ideas, questions, and responses which might be non-standard, and playing with ways in which students' statements and questions can be produced.

Moreover, the situation includes several tasks of teaching related to KCS. These include questioning and hearing to have clear analysis of students' ideas, identifying how students use manipulatives, considering what might be understood and not be understood related to the represented and identifying evidence, interpreting students' thinking about them, and identifying what difficulties students typically have related to presentation. These tasks also function with the uses of tools.

With regard to CCK, probing a proof means probing each step of a proof. In the teaching practice, examining proofs from textbooks and in disciplinary mathematics requires going through each step of a proof to clarify what logic, definitions, and axioms were used and how they work and identifying what would be key clues in proofs. Working with examples is often used to explain proofs. Comparing proofs also initiates and emphasizes what mathematical features of proofs have been uncovered. In this process, teachers would analyze mathematical concerns when students understand proofs. On the other hand, analyzing students' ideas related to proofs is both investigating students' thoughts related to proofs, which are generally provided from textbooks by teachers, and examining proofs that students create and which might be non-standard.

In terms of students' understanding and questions related to a proof, teachers play with the diverse possible ways that students could consider related to a proof, offering and explaining pertinent meanings to them. Regarding students' proofs, teachers work through each step of students' proofs and identify logic, conjecture, definitions, and axioms used in the proofs. Students' proofs might be mathematically partial or non-standard. It is, in any case, important to elucidate conjectures, definitions, and axioms that students implicitly use in their proofs and to consider what might and might not be understood about the proofs. This series of tasks of teaching provides mathematical indicators on how to guide students, on what questions, tasks, and comments should be posed.

These situations involve various tasks of teaching related to SCK. That diversity includes the following: working through each step of proof and identifying logic, conjecture, definitions, and axioms used in proof; playing with statements and examples of proofs; looking for aspects of a proof that are not represented by examples and statements; considering and running through sensible and special cases that might not be captured by examples and statements; considering possible counterexamples and their features in a proof; considering the features of or what happens with examples and statements; comparing several proofs to identify their features; looking for possible relationships or similarities among examples and statements that might be focused on proof; examining terms used in proofs; sizing up mathematical concerns for understanding; sizing up the nature of errors that students have; having and providing meaningful interpretation about students' ideas, questions, and responses about a proof that might be non-standard; playing with the ways in which students' statements and questions can be produced; and playing with different proof that students construct.

The situation, moreover, includes several tasks of teaching related to KCS. These specifically include: specifically, questioning and hearing to have clear analysis of students' ideas, identifying ways in which students think about proofs considering what might be understood and not be understood related to proof and identifying evidence, interpreting students' thinking about a proof, and clarifying what common errors students have related to a proof.

The main components of the analyses and their examples are summarized in Table 6.9.

Table 6.9 *Specification of Probing, Interpreting and Comparing Mathematical Objects in Teaching and Its Examples*

		Probing / Interpreting / Comparing
Mathematical objects in teaching	Concept Property Definition	CCK • interpreting statements and examples of concept, property, and definition SCK • Related to concept, property, and definition, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o playing with statements and examples of concept, property, and definition o looking for aspects of concept, property, and definition that are not represented by examples and statements o considering and running through sensible and special cases that might not be captured by examples and statements o considering possible counterexamples and their features in concept, property, and definition o considering the features of or what happens with examples and statements o examining terms used in statements of concept, property, and definition o comparing several concepts, properties, and definitions to identify their features o looking for possible relationships or similarities among examples and statements that might be focused on concept, property, and definition o sizing up mathematical concerns for understanding o sizing up the nature of errors that students have • Related to students' ideas of concept, property, and definition o having and providing meaningful interpretation about students' ideas and responses that might be non-standard o playing with ways in which students' statements and questions can be produced KCS • questioning and listening to get clear understanding of students' ideas • identifying ways in which a student is thinking about concept, property, and definition • considering what might and might not be understood about the concept, property, and definition and identifying evidence • interpreting students' thinking about concept, property, and definition • clarifying what common errors students make related to concept, property, and definition
	Algorithm Rule Procedure	CCK • solving or calculating problems involved in algorithms, rules, and procedures SCK • Related to algorithms, rules, and procedures, which are offered from textbooks, guides, and other resources, or by other students, teachers,

<table>
<tr><td></td><td></td><td>and others
o working with and explaining each step of algorithms, rules, and procedures
o playing with statements and examples of algorithm, rule, and procedure
o probing how algorithms, rules, and procedures work
o looking for aspects of algorithms, rules, and procedures that are not represented by examples and statements
o considering and running through sensible and special cases that might not be captured by examples and statements
o considering possible counterexamples and their features in algorithm, rule, and procedure
o considering the features of or what happens with examples and statements
o examining terms used in statements of algorithm, rule, and procedure
o comparing several algorithms, rules, and procedures to identify their features
o looking for possible relationships or similarities among examples and statements that might be focused on algorithm, rule, and procedure
o sizing up mathematical concerns for understanding
o sizing up the nature of errors that students have
• having and providing meaningful interpretations of students’ ideas and responses of algorithm, rule, and procedure that might be non-standard
• playing with ways in which students’ statements and questions of algorithm, rule, and procedure can be produced

KCS
• questioning and listening to gain clear analysis of students’ ideas
• identify their ways of using algorithms, rules, and procedures in written forms
• considering what might be understood and not be understood related to algorithms, rules, and procedures and identifying evidence
• interpreting students’ thinking about algorithms, rules, and procedures
• clarifying what common errors students have related to algorithm, rule, and procedure</td></tr>
<tr><td></td><td>Representation Tool</td><td>CCK
• probing its parts in representation simply

SCK
• Related to representation and tool, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others
o working through and mapping both the thing to be represented and the presentation
o identifying representation and its parts in representation
o identifying different parts of representation and considering diverse interpretations of it
o playing with ways in which the representation might be explained to fit with the thing to be represented or in which students’ ideas can be produced related to representation and tool
o looking for features and relationships prominent in the design of the objects being considered</td></tr>
</table>

		o considering and running through sensible cases of representation and uses of tools and special cases that might not be captured in representation and tools o considering possible counterexamples and their features in representation o considering the features of or what happens with examples and statements o comparing representation and tools o looking for possible relationships or similarities among examples and statements that might be focused on representation and tools o sizing up mathematical concerns for understanding and using representation and tools o sizing up the nature of errors that students have o identifying strong and weak points of representation and tools for a provided purpose o playing out how representation and tools would be used for purposes • having and providing meaningful interpretation of students' ideas, questions, and responses of representation and tools that might be non-standard • playing with ways in which students' statements and questions of representation and tools can be produced KCS • questioning and listening to get clear understanding of students' ideas • articulating ways in which students are thinking of representation and tools • considering what might and might not be understood related to the represented and identifying evidence • interpreting students' thinking regarding presentation and tools • identifying what difficulties students typically have related to presentation and tools
	Proof	CCK • probing each step of a proof SCK • Related to proofs, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o working through each step of a proof and identifying logic, conjecture, definitions, and axioms used in the proof o playing with statements and examples of proofs o looking for aspects of proofs that are not represented by examples and statements o considering and running through sensible and special cases that might not be captured by examples and statements o considering possible counterexamples and their features in a proof o considering the features of or what happens with examples and statements o comparing several proofs to identify a proof's features o looking for possible relationships or similarities among examples and statements that might be focused on a proof o examining terms used in a proof

		o sizing up mathematical concerns for understanding o sizing up the nature of errors that students have • having and providing meaningful interpretations of students' ideas, questions, and responses about proofs that might be non-standard • playing with ways in which students' statements and questions of proof can be produced • playing with different proofs that students construct KCS • questioning and listening to get clear understanding of students' ideas • identifying ways in which students think about proofs • considering what might and might not be understood related to proofs and identifying evidence • interpreting students' thinking about proofs • clarifying what common errors students have related to proofs

Evaluating and judging mathematical objects. Evaluating and judging refers to teachers' determining, after thinking carefully, the mathematical significance or condition of mathematical objects. Generally, it means deciding whether statements, explanations, examples, and reasoning of mathematical objects are mathematically appropriate. For example, teachers often evaluate topics and statements from textbooks and students' work, such as tests and assignments. Evaluating and judging mathematical objects are related to CCK, SCK, KCT, and KCS, which are domains of MKT.

Deciding whether statements, explanations, and examples of concept, property, and definition are mathematically appropriate requires mathematical decisions. Evaluating and judging concept, property, and definition involves basically deciding whether answers to problems related to concept, property, and definition are correct. Generally, here, problems are closed questions. It is included in CCK.

The practice of teaching calls for more investigation. For example, many textbooks define an even number as a number having a 0, 2, 4, 6, or 8 in the ones place. Although this definition is correct as far as it goes, but it is imprecise for allowing such numbers as 12.8 and 0.7 to be included. Therefore, this is not appropriate definition of an even number. Recognizing "a number" in this definition is critical to making a mathematical decision. This definition works when a number is a whole number, but not when a number is a rational number. A definition that holds only within a limited mathematical domain is not mathematically acceptable. This situation include the following: appreciating whether a concept or a property is correctly identified; judging whether explanations of a concept, a property, and a definition are appropriately stated; specifying why a concept, a property, and a definition are unacceptable; sizing up what is important and potentially hard to understand; appraising whether a definition includes the things of the defined and excludes the things of the complementary set of the defined; and determining whether examples works well with a concept, a property and a definition and justifying why. These tasks relate to SCK.

Assessing students' ideas about a concept, a property, and a definition is attempting to evaluate his or her statements and examples related to a concept, a property, and a definition in terms of mathematical precision, understanding, and difficulty. Teaching one lesson or reviewing students' homework requires continual decision making. In the SeanNumbers-Ofala video clip, for example, one student says, "If you have a number that you can split up evenly without having to split one in half, then it's an even number." This definition excludes odd whole numbers by stipulating, "without having to split on in half." However, this definition is not mathematically precise because it does not exclude all that it should, such as rational numbers. ½ can be split into two groups of ¼. ¼ is not included in a case of "without having to split one in half." The students would have an understanding of numbers within whole numbers. It would be hard for students to recognize or know well division of rational numbers or types of numbers related to the definition. This situation includes evaluating the plausibility of students' claims about concepts, properties, and definitions; judging whether students' statements of a concept, a property, and a definition are mathematically precise; identifying why students' ideas are not mathematically reasonable; and sizing up what there is to understand in students' ideas about a concept, a property, and a definition. These are related to SCK. This phenomenon of the SeanNumbers-Ofala video clip also involves as well as deciding how the different statements and examples of concepts, properties, and definitions are easier or more difficult for students. These are included in KCS. KCT contains judging whether activities are appropriate and evaluating whether statements and examples of concept, property, and definition are instructionally advantageous.

Evaluating and judging algorithm, rule, and procedure are appreciating statements, explanations, and examples of algorithm, rule, and procedure in terms of whether they mathematically go well together and penetrate to grasp their principles. Deciding whether answers to problems related to algorithm, rule, and procedure are correct is a plain task which is included in CCK. The practice of teaching requires more exploration. For example, the unit digit rule may be introduced thus "If you want to know if a number is even or odd, look at the last digit." Because of decimal number, however, this definition is imprecise. The imprecise rule makes a mathematically unreasonable decision. For example, the last digit in 12.326 is 6, thus making the number even, according to the aforementioned definition. Or, the last digit in 12,327.0 is 0, and, thus, the number is even. The unit digit rule works only for integers. Therefore, statements about the unit digit rule needs to clarify types of numbers that can work in the unit digit rule, such as, "If you have an integer, look at its units and that tells you if it's even or odd." When students know only natural numbers or whole numbers, "an integer" can be replaced with natural number or whole numbers. The statements of the rule still work. This process involves the following: judging whether explanations and uses of a rule are mathematically appropriate; explaining why rules can be inadequate; deciding whether an explanation focuses on what is most needed for rules; and judging whether rules work well with various definitions or axioms. These tasks are related to SCK. On the other hand, determining what is different mathematically about the problems and how these differences might impact students' approaches to solve the problem relate to KCS.

Appraising students' explanations and examples of algorithms, rules and procedures prompts teachers to judge students' ideas of algorithms, rules and procedures. Using mathematically reasonable or unreasonable bases, students might have different understanding of algorithms, rules, and procedures. Saying, for example, that 12,350 is even because it ends in 0 reflects circular logic. At that point, the units digit rule becomes an object of discussion rather than an object of use. Neither is it appropriate to split it here into two groups because that fails to explain why the rule works. This series of tasks includes the following: evaluating whether or not students have the same approach or idea for algorithms and rules; determining whether students' explanations of algorithms, rules, and procedures are mathematically reasonable; sizing up what there is to understand in students' ideas about the rule; and clarifying why students' uses of algorithms, rules, and procedures are not precise. These are related to SCK. KCT contains judging whether activities are appropriate and evaluating whether statements and examples of algorithms, rules, and procedures are instructional advantages.

Representation and tools as mathematical objects here mean presented images, explanations about representations, the represented of using tools, and its explanations. Evaluating and judging representations and tools are deciding whether explanations and examples of representations and tools are mathematically acceptable. Simply, deciding whether answers to problems related to representation and tools are correct is part of CCK. The practice of teaching calls for more tasks. Because representations create diverse interpretations, careful decisions are required: representations and the use of tools have to be mathematically coherent with what is represented. This series of tasks includes the following: evaluating whether representations and tools from textbooks or other resources are mathematically accurate, considering and running through sensible cases or demonstrations for representations and tools and judging them, deciding whether representations go with other mathematical objects, and specifying why representations and tools do not mathematically work well. These tasks are included in SCK.

As another example, let's suppose students say, "Odd numbers can be split into two equal groups with one left out. Adding this to another odd number will have the same result. You will get an even number." with *Figure 6.7*. Their definition of an odd number does not square with their picture because the 3 in the figure is represented by another definition – groups of two with a remainder of one. This series of tasks include appreciating representations and tools that students create or use and mathematically judging them, sizing up what there is to understand in students' ideas about representations and uses of tools, and identifying why students' representations and their use of tools are not mathematically accurate.

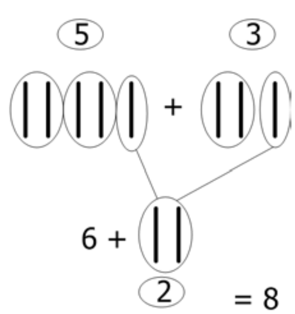

Figure 6.7 Sum of two odd numbers

KCT contains evaluating whether activities appropriately include representations and manipulatives and evaluating whether representations are instructionally advantageous. Furthermore, KCS includes determining how different representations might impact students' approaches to solving problems.

Whether or not proofs are constructed by mathematicians or students, they are generic creature to build mathematics. It requires mathematical consistency and reason to show a conjecture always works. Therefore, CCK includes three things: deciding whether answers to problems related to proofs are correct, making judgments and responses about closed-questions related to proofs, and validating whether proofs are reasonable. To investigate tasks of teaching related to proofs, note the examples of a proof in *Figure 6.8*.

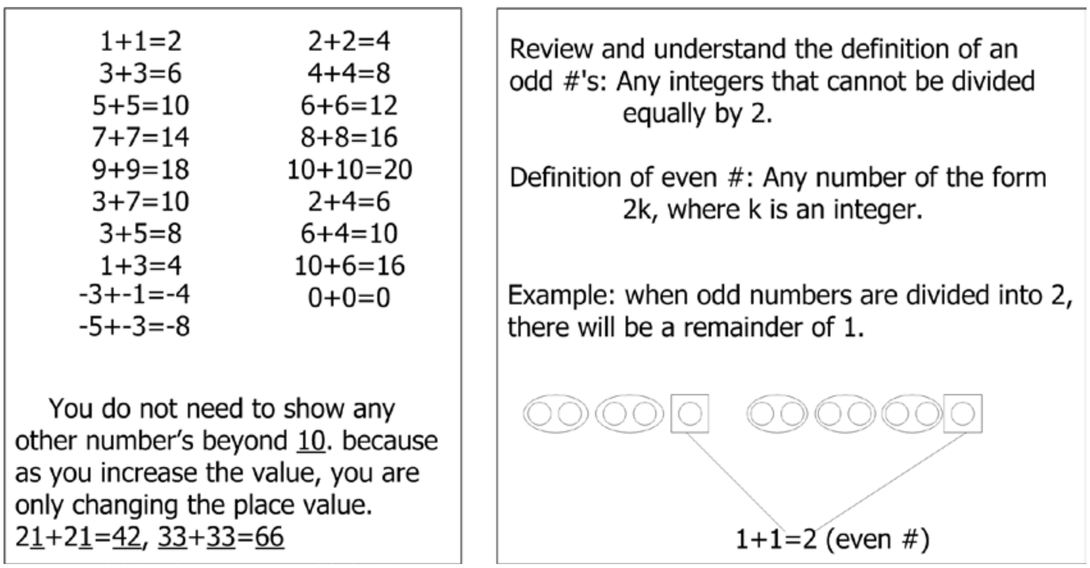

Figure 6.8 Proofs of (odd number) + (odd number) = (even number)

Neither example qualifies as a proof of showing why (odd number) + (odd number) = (even number). The first one does not show why the conjecture always works for all cases. It mentions the place value, but fails to clarify how the place value functions to prove the conjecture. The second one is inconsistent in how it defines an odd number and an even number. Based on its definition of an odd number, an even number should be any integers that can be divided equally by 2. Or, based on the definition of an even number, an odd number should be any number of the form 2k+1, where k is an integer. Moreover, it does not have logical statements to be a proof. It shows just a picture of one case, 5+7.

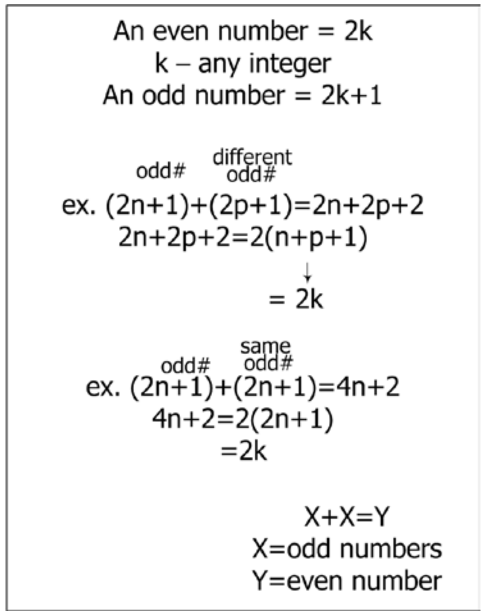

Figure 6.9 Proof of (odd number) + (odd number) = (even number)

Figure 6.9 shows one more example. This proof uses mathematically consistent definitions for even and odd numbers. The statements of the proof seem mathematically fine, but it is not obvious why it differentiates odd numbers as different and the same. The last portion of the statements does not work well with the proof. This series of tasks, which are involved in SCK, include the following three: evaluating whether proofs are mathematically reasonable, considering sensible cases for proofs and judging them, deciding whether students' conjectures and proofs are mathematically acceptable, and explaining why proofs are unreasonable. KCS contains determining how the different conjectures and proofs in the problems are easier or more difficult for students in their learning. KCT contains judging whether activities are appropriately embedded in proofs, from textbooks, guides, and other resources, or by teachers and evaluating whether statements and examples of proof are instructionally advantageous or not.

The main components of the analyses and their examples are summarized in Table 6.10.

Table 6.10 *Specification of Evaluating and Judging Mathematical Objects in teaching*

		Evaluating / Judging
Mathematical objects in teaching	Concept Property Definition	CCK • deciding whether answers to problems related to concept, property, and definition are correct • making judgments and responses about closed-questions related to concept, property, and definition SCK • Related to concepts, properties, and definitions, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o judging whether explanations of concept, property, and definition are mathematically precise, and specifying why o appraising whether a definition includes the things defined and excludes those of the complementary set of the defined • evaluating the plausibility of students' claims about concept, property, and definition • sizing up what there is to understand in students' ideas about concept, property, and definition • sizing up what is important to understand concept, property, and definition • deciding whether examples work well with concept, property, and definition and justifying why KCT • judging whether activities are appropriately embedded in concept, property, and definition, from textbooks, guides, and other resources, or by teachers • evaluating whether statements and examples of concept, property, and definition are instructionally advantageous or not KCS • determining how the different statements and examples of concept, property, and definition are easier or more difficult for students to understand
	Algorithm Rule Procedure	CCK • deciding whether answers to problems related to algorithms, rules, and procedures are correct • making judgments and responses about closed-questions related to algorithms, rules, and procedures SCK • Related to algorithms, rules, and procedures, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o judging whether explanations and uses of algorithms, rules, and procedures are mathematically appropriate, and specifying why o deciding whether explanations focus on what most needed for algorithms, rules, and procedures

		• evaluating the plausibility of students' claims about algorithms, rules, and procedures • sizing up what there is to understand in students' ideas about algorithms, rules, and procedures • sizing up what is important to understand algorithms, rules, and procedures • deciding whether algorithms, rules, and procedures work well with various definitions or axioms or in diverse situations • Related to students' ideas of algorithms, rules, and procedures o evaluating whether or not students have the same approaches or ideas for algorithms, rules, and procedures o clarifying why students' uses of algorithms, rules, and procedures are mathematically reasonable or not KCT • judging whether activities are appropriately embedded in algorithms, rules, and procedures, from textbooks, guides, and other resources, or by teachers • evaluating whether statements and examples of algorithms, rules, and procedures are instructionally advantageous or not KCS • determining what is different mathematically about problems related to algorithms, rules, and procedures and how these differences might impact students' approaches to solve the problems
	Representation Tools	CCK • deciding whether answers to problems related to representation and tools are correct • making judgments and responses about closed-questions related to representation and tools SCK • Related to representations and tools, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o evaluating whether representations and uses of tools are mathematically accurate o deciding whether representations focus on what is most needed for the represented • evaluating the plausibility of students' claims about representations and tools • sizing up what there is to understand in students' ideas about representations and uses of tools • deciding whether representations go with other mathematical objects • considering and running through sensible cases or demonstrations for representations and tools and judging them KCT • evaluating whether activities appropriately include representations and manipulatives • evaluating whether representations are instructionally advantageous or not

		KCS • determining how different representations might impact students' approaches to solving problems
	Proof	CCK • deciding whether answers to problems related to proofs are correct • making judgments and responses about closed-questions related to proofs • validating whether proofs are reasonable SCK • evaluating whether proofs are mathematically reasonable, from textbooks, guides, and other resources, or by students, teachers, and others, and explaining why • evaluating the plausibility of students' claims about proofs • sizing up what there is to understand in students' ideas about proofs • sizing up what is important to understand about proofs KCT • judging whether activities are appropriately embedded in proofs, from textbooks, guides, and other resources, or by teachers • evaluating whether statements and examples of proofs are instructionally advantageous or not KCS • determining how the different proofs in problems are easier or more difficult for students in their learning

Selecting and modifying mathematical objects. Selecting and modifying encompasses teachers' choice of examples, explanations, and reasoning related to mathematical objects, and changing and improving mathematical objects to be more precise and acceptable or less extreme. Depending on their evaluation of mathematical objects, teachers determine which examples and explanations they would offer about mathematical objects in lessons or they revise examples and explanations to make them easier or more difficult. In particular, students' examples and statements would be partially precise or partially include certain key ideas. Teachers intentionally choose appropriate or inappropriate statements and examples related to mathematical objects. Moreover, their selection and revision takes into consideration the understanding of the students. Their selection and revision should always be supported with mathematical clarification and appropriate reasons. Selecting and modifying mathematical objects are tasks related to CCK, SCK, KCT, and KCS, which are domains of MKT.

Choosing and modifying examples and statements related to concept, property, and definition is deciding which examples or statements of concept, property, and definition are used or revised. It is related to CCK. Because of the practice of teaching, choosing and modifying requires more investigation. With regard to SCK, before making a decision, it is necessary to consider relationships and features of examples or statements, and special cases and counterexamples. For example, -4, -6, and 0 can work as significant examples to test definitions of an even number because all of them are non-natural numbers and integers. Moreover, 4.6 or 1/6 work to test them as counterexamples. Therefore, one definition, an even number is a natural number that is divisible by 2, would be revised, such that a natural number that is divisible by 2 is even, or an even number is a number that is divisible by 2. With the same point, selecting or finding statements and examples of concept, property, and definition to make specific mathematical points is involved in SCK. With respect to KCT, the imprecise definition could be selected by a teacher who would like to motivate students to investigate meanings of an even number, a natural number, and divisible. It also applies to students' examples and statements related to concept, property, and definition. Intentional revision of statements and examples of concept, property, and definition is also included in KCT. Selecting and revising statements and examples of concept, property, and definition that enable students to understand is involved in KCS.

Algorithms, rules, and procedures are specific sets of instructions for solving problems, and, in particular, a process that applies an algorithm to an input to obtain an output makes a computation. Therefore, selecting correct statements and examples of algorithms, rules, and procedures and revising statements or answers to problems related to them are included in CCK. In a pedagogical situation, statements and examples of algorithms, rules, and procedures work to show how to use and apply them or why they work. Therefore, a teacher's decision would be diverse according to their purposes. Again, their decisions should be intentional and based on mathematical determinations.

For example, a teacher selects 5,694 and 78 as examples that explain how the units digit rule works. They have different digits and different numbers in the ones digit. If a teacher chooses 5,694 and 7,404, these examples might lead to misapprehensions, such as the rule works only for four digit numbers or when a number has 4 in the ones digit. Moreover, the statement, 5,694 is an even number because 4 is in the ones digit, shows how to use the units digit rule, but does not indicate why it works. If a teacher's purpose is the latter, this explanation would not be selected because it does not capture its core idea. For example, "5,690 is an even number and 4 is an even number. Because the sum of even numbers is always an even number and 5,690 + 4 = 5,694, then 5,694 is an even number." This explanation uses the example, 5, 694 and aims at identifying why the rule works. Similarly, students' examples and statements would include different purposes and a teacher would selectively use them to emphasize different points. These are related to SCK.

To teach, selecting and revising statements and examples of algorithms, rules, and procedures from textbooks and students' ideas is relevant to KCT. It is based on teachers' purposes and intentions, and, thus, selection and revision could be mathematically reasonable or unreasonable. Selecting and revising statements and examples of algorithms, rules, and procedures that enable students to understand is part of KCS.

In terms of CCK, selecting and modifying representations indicates simple selection among representations and revision of mathematically wrong representations. For the practice of teaching, it calls for deeper exploration. Choosing and modifying representation and tools is deciding about selection and revision for pedagogical purposes or according to features between the represented and representations. Selection and revision depends on (1) how well things are mapped between each element of the represented and each element of representation and (2) the overall logic of how both the represented and representation go together. Representation, which has over or under emphasized elements, might lead students to misunderstand the represented. Therefore, representation itself is significant and representation works together with the represented. Selection and revision would be based on these kinds of decisions. For example, *Figure 6.7* would not be chosen because it does not go well with students' explanations. Or, the representation would be revised based on the explanation, odd numbers can be split into two equal groups with one left out, as shown in *Figure 6.10*.

Figure 6.10 Revised representation 1

Or, both the definition and the representation would be revised: an odd number can be separated groups of two with one left over and representation as shown in *Figure 6.11*. These are related to SCK.

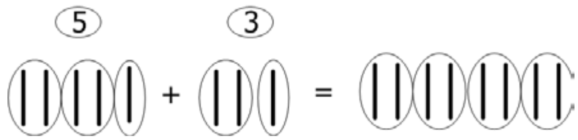

Figure 6.11 Revised representation 2

To help students grasp common mathematical ideas, what is important, regarding KCT, are different situations with diverse representations and tools (Dienes, 1963, 1969). Specifically, selecting statements and examples of algorithms, rules, and procedures with certain teachers' purposes is pertinent to KCT. Also related to KCT is the intentional revising of representations, reasonable or unreasonable, so as to launch discussions. It is also critical and relevant to KCS to consider whether students can understand representation.

Concerning proofs, students can use empirical arguments as their proofs. Teachers would use explanations for general cases rather than specific cases and apply thorough proofs. Selecting and modifying proofs requires first that determine whether statements are proofs. It is critical, moreover, to determine whether proofs are mathematically logical and include core ideas. Selecting a correct proof or revising it is involved in CCK. It relates to SCK when selection and modification of a proof happen through textbooks or ideas from students. It relates to KCT when one selects a proof from among textbook proofs or those from students that is reasonable or unreasonable and intentionally revises it. It relates to KCS when one selects and revises a proof so as to make it understandable to students.

The main components of the analyses and their examples are summarized in Table 6.11.

Table 6.11 *Specification of Selecting and Modifying Mathematical Objects in teaching*

		Selecting / Modifying
Mathematical objects in teaching	Concept Property Definition	CCK • selecting correct statements and examples of concept, property, and definition • revising statements about concept, property, and definition • revising answers of problems related to concept, property, and definition SCK • Related to concept, property, and definition, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o revising statements and examples of concept, property, and definition to be mathematically precise o choosing or finding statements and examples of concept, property, and definition to make specific mathematical points KCT • Related to concept, property, and definitions, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o choosing statements and examples of concept, property, and definition that are mathematically correct or incorrect so as to launch discussions for teachers' purposes o selecting examples based on students' ideas of concept, property, and definition o intentionally choosing some concept, property, and definition that students have o intentionally revising statements and examples of concept, property, and definition to be mathematically reasonable or unreasonable KCS • selecting and revising statements and examples of concepts, properties, and definitions to be understandable by students
	Algorithm Rule Procedure	CCK • selecting correct statements and examples of algorithms, rules, and procedures • revising statements about algorithms, rules, and procedures • revising answers of problems related to algorithms, rules, and procedures SCK • Related to algorithms, rules, and procedures, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o revising statements and examples of algorithms, rules, and procedures to be mathematically precise o choosing or finding statements and examples of algorithms, rules, and procedures to make specific mathematical points KCT

		• Related to algorithms, rules, and procedures, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o choosing statements and examples of algorithm, rule, and procedure that are mathematically correct or incorrect in order to launch discussions of how they work or for teachers' purposes o selecting examples based on students' ideas of algorithms, rules, and procedures o choosing some of algorithms, rules, and procedures that students have with intentional purposes o intentionally revising statements and examples of algorithms, rules, and procedures to be mathematically reasonable or unreasonable for purposes KCS • selecting statements and examples of algorithms, rules, and procedures that students can understand • changing statements and examples of algorithms, rules, and procedures, so that students can comprehend them
	Representation Tools	CCK • selecting correct statements and examples of representations and tools • revising statements about representations and tools • revising answers of problems related to representations and tools SCK • Related to representations and tools, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o playing out how to use each presentation for a purpose and revising them to be mathematically precise o choosing or finding representations and tools to make specific mathematical points KCT • Related to representations and tools, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o choosing representations and tools that are mathematically correct or incorrect in order to launch discussions of how they work or for teachers' purposes o selecting representations and tools based on students' ideas o intentionally choosing some representations and tools that students have o intentionally revising representations and tools to be mathematically reasonable or unreasonable for teachers' purposes KCS • selecting representations that students can understand and tools that students can use • changing representations and suggesting other tools, so that students can comprehend the thing being represented
	Proof	CCK • selecting correct statements and examples of proofs

		• revising statements about proofs • revising answers of problems related to proofs SCK • revising proofs to be mathematically reasonable, proofs that are offered from textbooks, guides, and other resources, or by other students, teachers, and others • choosing or finding proofs to make specific mathematical points KCT • Related to proofs, which are offered from textbooks, guides, and other resources, or by other students, teachers, and others o choosing a proof that is mathematically logical and works well o choosing a proof that is mathematically correct or incorrect so as to spark discussions of how they work or for teachers' purposes o choosing some of students' proofs with intentional purposes o intentionally revising proof to be mathematically unreasonable for teachers' purposes KCS • selecting proofs that students can understand • changing proof so that students can comprehend them

Constructing mathematical objects. Constructing mathematical objects is a task of teaching where teachers create, build, and generate statements, representations, and examples of mathematical objects in and for teaching. Since mathematical objects in and for teaching are already found and proved by mathematicians, constructing mathematical objects here is not equivalent to mathematicians' usual job that is finding new proofs. However, teachers in some cases need to generate sentences, examples, and representations for their own purpose even though mathematicians have already found and announced them. The task of teaching in those moments is the constructing of mathematical objects in and for teaching. To define an even number, for example, teachers write out definitions and draw diagrams for "groups of two" and "two groups," both of which have mathematically different origins. Constructing mathematical objects is related to CCK, SCK, KCT, and KCS, which are domains of MKT.

Constructing statements of concept, property, and definition requires considering mathematical rigor, particular situations, diverse versions, pedagogical purposes, and possible ideas. In terms of CCK, precise statements of concept, property, and definition provide a mathematical foundation in and for teaching. While it means that teachers provide students precise statements, it also means that teachers have a foundation that should help them maintain the mathematically correct track in and for teaching. Concerning SCK, features of the pedagogical setting create a mathematically distinct situation. If students' previous learning is only whole numbers and they are capable of computing only addition and subtraction, then whole numbers and the ability to add and subtract create a particular situation. In this case, a mathematically precise definition – an even number is an integer divisible by two – is not appropriate. What can work instead is identifying a limitation of a mathematical structure and using partial definitions. For example, an even number can be defined as a whole number that can be divided into two equal groups of whole numbers. It also has a different version, such as a whole number is even if it ends in 0, 2, 4, 6, and 8. Creating mathematically productive questions related to concept, property, and definition is also relevant to SCK. In accordance with KCT, teacher educators can be tasked with provide teachers with concocting statements and examples of concept, property, and definition that are mathematically accurate or inaccurate. The intentional fabrication should be based on a valid reason and purpose. With respect to KCS, teachers consider possible definitions that students might think of: a number is even if it can be divided into two equal groups of whole numbers or a number is even if it ends in 0, 2, 4, 6, and 8. Although these definitions are mathematically imprecise, teachers could assume and prepare for what students would think and use such ideas to encourage students to investigate an even number, a whole number, and the base ten number system.

Constructing statements of algorithms, rules, and procedures should clarify concisely and exactly how they are work. Because algorithms, rules, and procedures are used for computation or repeated operation and construction, they are efficient and mathematically well-established knacks. For example, the units digit rule is as follows: "To determine whether an integer is even, examine the ones digit. If it is even, then the entire number is even. If it is odd, then the entire number is odd." Making statements and examples that are mathematically thorough is related to CCK. The aforementioned rule is limited to integers and explains how to make a decision. First, a number should be an integer; second, just focus on the number in the ones digit; and third, the number in the ones digit decides whether the entire number is even or odd. More investigation for pedagogical context is relevant to SCK; in particular, how algorithms, rules, and procedures work and why do they work? Of course, depending on what students can understand, terms used in statements will differ. Statements of how they work have to emphasize what is supposed to happen in each step and what kinds of results are expected to be made in each step. Therefore, statements should be mathematically correct. Because of pedagogical setting, statements sometimes need to consider certain limitations according to students' knowledge. Finally, using algorithms, rules, and procedures is being automatized after repeated practice, and, finally, being executed automatically without thinking (Hiebert, 1990). However, if statements of algorithms, rules, and procedures act to concentrate on a logical presentation of formulated knowledge, to minimize the metaphorical use of knowledge, and to neglect a meta-cognitive strategy, it can cause formal abidance (Brousseau, 1986, November; Kang, 1990), as shown in *Figure 6.12*.

The Associative Principle for Addition [APA]
(____ + _ _ _) + ... = ____ + (_ _ _ + ...)
The Twist Principle for Addition [TPA]
(____ + _ _ _) + (. . . . + ~~~~) = (____ +. . . .) (_ _ _ + ~~~~)
The Left Distributive Principle for Multiplication over Addition [LDPMA]
. . . . · (____ + _ _ _) = (. . . . · ____) + (. . . . · _ _ _)
The Principle for Multiplying by +1 [PM +1]
_ _ _ · +1 = _ _ _

Figure 6.12 A list of the principles for real numbers in a textbook by the University of Illinois Committee on School Mathematics (UICSM)

The following tasks of teaching also pertain to SCK, KCT, and KCS: generating various versions of statements with special forms about algorithms, rules, and procedures; fabricating statements with specific intentions about algorithms, rules, and procedures; forming statements and cases that students might consider about algorithm, rule, and procedure; and creating mathematically productive questions related to algorithms, rules, and procedures. Conscious thoughts and statements about why algorithms, rules, and procedures occur just that way are related to unpacking core ideas of algorithms, rules, and procedures. It is not just creating texts, but it calls for various kinds of zoomed-in work. This was investigated in the previous section—unpacking mathematics ideas and practices.

Constructing representations and creating tools includes drawing images or making manipulatives. In terms of CCK, constructing representations is simply drawing images for certain reasons. While representation seems to be convenient or efficient for introducing and explaining a mathematical concept, it in fact entails mathematically hard tasks. Representation should show the whole logic of the represented illuminated critical dimensions. However, no representations capture all aspects of an idea, nor are all equally useful for particular students (Ball, 1993, p. 384). Therefore, careful work focus is required, such as considering mathematical rigor and particular limitation. In terms of SCK, teachers sometimes make manipulatives, such as bean sticks, to highlight the base ten number system (e.g., Ball & Bass, 2000a). Mapping between the represented and the representations and tools is also considered; this might avoid students taking a quite different form from the mathematician's mathematics, a so-called matacognitive shift (Brousseau, 1997; Kang & Kilpatrick, 1992). Having diverse versions of representations, creating representations with teachers' purposes, creating mathematically productive questions related to representations and tools, and drawing representations that students might think of are also tasks of teaching that are related to SCK, KCT, and KCS.

Constructing proofs is the closest teachers come to a mathematician's work because to construct a proof is a way to build a castle. Therefore, proving conjectures are CCK. Concerning SCK, however, constructing a proof as a task of teaching happens in quite different circumstances than it happens for pure mathematicians. There is the fact that a teacher is limited, due to students' breadth of knowledge and their constrained ability to reason, in his or her use of definitions or axioms and logic. For example, *Figure 6.13* shows a proof that mathematicians generally use to prove the conjecture, (an odd number) + (an odd number) = (an even number). Mathematicians' work ends here. In a pedagogical setting, this proof is based on several assumptions: students can use variables and the deductive method. In other words, if any of these assumptions are not satisfied, this proof cannot be used in teaching practice. However, this does not mean that this proof is useless. While this proof would not be used with students, teachers could be aware of what the mathematical core is in the proof: one from each an odd number creates one group of two.

Conjecture: (an odd number) + (an odd number) = (an even number)

An odd number is 2k+1 when k is an integer.
An even number is 2k when k is an integer.

(an odd number) + (an odd number) =2k'+1+2k"+1
=2k'+2k"+2
=2(k'+2k"+1)
=2k
=(an even number)

Therefore, (an odd number) + (an odd number) = (an even number) is correct.

Figure 6.13 Proof of a conjecture: (an odd number) + (an odd number) = (an even number)

Having another version of a proof is a critical task in teaching. *Figure 6.14* is one of them. It uses different definitions from *Figure 6.13*. While the logic of the proof is based on deductive reasoning, representations help to visualize the proof and representations go well with the definitions introduced. However, this proof also includes a limitation because one might contend over whether negative numbers can be put into groups of two and cases for the sum of odd negative and positive integers.

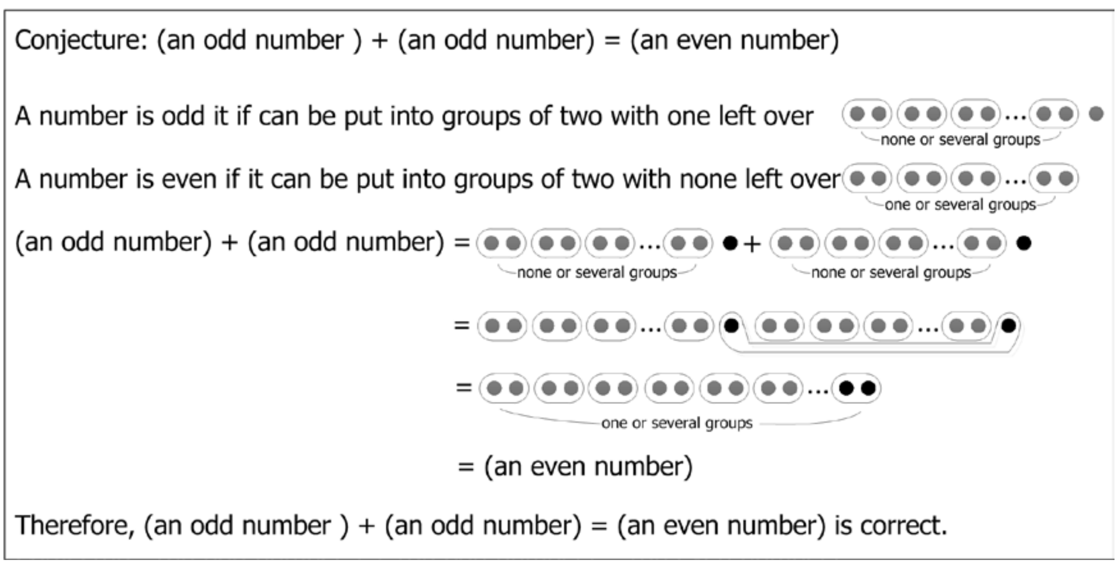

Figure 6.14 Another proof of a conjecture: (an odd number) + (an odd number) = (an even number)

Creating mathematically productive questions related to proofs is also involved in SCK. Moreover, using obvious reasons to build proofs that are rational or irrational is included in KCT. Creating proofs that students might come up with is regarded as KCS.

The main components of the analyses and their examples are summarized in Table 6.12.

Table 6.12 *Specification of Constructing Mathematical Objects in Teaching*

		Constructing
Mathematical objects in teaching	Concept Property Definition	CCK • making statements and examples of concept, property, and definition that are mathematically rigorous SCK • creating statements and examples of concept, property, and definition, sometimes in a particular situation • asking productive questions of concept, property, and definition • generating diverse versions of statements and examples of concept, property, and definition with special forms KCT • fabricating statements and examples of concept, property, and definition that are mathematically accurate or inaccurate with specific intentions and reasons for teachers' purposes KCS • forming statements and cases of concept, property, and definition that students might think of
	Algorithm Rule Procedure	CCK • making statements and examples of algorithms, rules, and procedures that are mathematically rigorous SCK • creating statements and examples to show how algorithms, rules, and procedures work • creating statements and examples of algorithms, rules, and procedures under a certain limitation • asking productive questions of algorithms, rules, and procedures • generating diverse versions of statements of algorithms, rules, and procedures with special forms KCT • fabricating statements of algorithms, rules, and procedures that are mathematically accurate or inaccurate with specific intentions and reasons for teachers' purposes KCS • forming statements and cases of algorithms, rules, and procedures that students might think of
	Representation Tools	CCK • creating representations SCK • forming representations that work well mathematically

		• creating representations with particular limitations • asking productive questions of representations and tools • having diverse versions of representations to indicate a certain idea KCT • creating representations that are mathematically accurate or inaccurate with specific intentions and reasons for teachers' purposes KCS • drawing representations that students might think of
	Proof	CCK • proving conjectures SCK • constructing proofs based on certain definitions or axioms • asking productive questions of proofs KCT • building proofs that are rational or irrational for obvious reasons KCS • creating proofs that students might come up with

This section has conceptualized zoomed-in mathematical work of teaching and illustrated each of them in detail as shown in Tables from 6.1 to 6.5. Comparing each mathematical work in these tables uncovers several features. First, all work of mathematics teaching and all mathematical objects in teaching are encompassed in four main domains of MKT, specifically CCK, SCK, KCS, and KCT. Second, two kinds of tasks, the first being figuring out, recognizing, and articulating and the second being probing, interpreting, and comparing, do not include KCT. These tasks require seeing the phenomena in and for instruction and capturing it exactly rather than making certain decisions. KCT is related to making decisions about teaching, such as evaluating, selecting, modifying, and constructing. Third, tasks of probing, interpreting, and comparing involve relatively much SCK with any mathematical object in teaching. It is because these tasks demand mathematical understanding and reasoning to teach. Fourth, selecting and modifying requires relatively much KCT with all mathematical objects in teaching. These tasks call for decisions that influence design of instruction. Fifth, generally the two kinds of work, evaluating and judging, and selecting and modifying require more CCK than do the other kinds of work. This is because CCK makes various contributions to these two kinds of work for teaching.

The zoomed-in mathematical work of teaching is analytic work, which involves applying four kinds of mathematical objects in teaching and five kinds of work of mathematics teaching at different grain sizes. Through the double focal analysis, zoomed-in mathematical work of teaching is uncompressed and made specific.

Now, this chapter turns to the second main component of MKT as content in mathematics teacher education: knowledge about mathematics.

6.3 Conceptualization of Knowledge about Mathematics as Content in Mathematics Teacher Education

Teaching MKT requires mathematics. However, as previously mentioned, mathematics in instruction for MKT has a different role. For example, the teacher educators in this study often explained features of mathematical practice, such as roles of definitions, the nature and elements of proofs, and roles of counterexamples and conjectures that are objects in disciplinary mathematics. There explanations are generally followed after activities about several kinds of mathematical work of teaching. Moreover, teacher educators sometimes directly offer mathematical knowledge. For example, some teacher educators from the data showed the sets of integers and even numbers in order to reduce the mathematical burden in evaluating definitions of even numbers. Herein the sets of integers are not the focus of instruction but critical to teaching MKT.

This section focuses on MKT with regard to mathematical issues in disciplinary mathematics, specifically what knowledge about mathematics is prominent in teaching MKT. In particular, this study conceptualizes it based on Ball's (1990) notion of "knowledge about mathematics." This regards the nature of knowledge in the discipline. It includes what counts as an "answer" in mathematics? What establishes the validity of an answer? What is involved in doing mathematics? What do mathematicians do? Which ideas are arbitrary or conventional and which are logical? What is the origin of some of the mathematics we use today and how does mathematics change? (Ball, 1990, p. 458). Specifically, this research classifies knowledge about mathematics in teaching MKT into three categories: facts and structures that mathematicians have used and developed; what they are aware of and take for granted in their research; and what they think valuable and profound in mathematics. Facts and structures have stabilized forms because they are generally published in mathematicians' writings. Moreover, disciplinary mathematics is involved in teaching MKT and works as the foundation. For example, in a discussion about an even number, it is crucial to have a common definition of what an even number is. If such a point is not made clear or shared, any discussion may easily be mathematically incorrect and, ultimately, a waste of time. The second and third categories do not have stabilized forms because mathematicians feel it rather unskillful to specify what they do and think about mathematics (Hardy, 1992). What mathematicians are aware of and what they take for granted in and through their study can often be observed through data and several mathematicians' writings. What they believe makes mathematics valuable can also be gathered this way. This research illustrates each of them in the following.

6.3.1 Disciplinary facts and structures

Mathematical facts. Facts are the very clearest objects of disciplinary mathematics in teaching MKT. For example, when teachers ask which numbers are integers, or when teacher educators recognize those moments when teachers are unsure, then the teacher educators state or write a set of integers. Here, a set of integers is a fact that explains what numbers are included in integers. Mathematical facts are generally uncontroversial, and mathematicians accept them from a variety of philosophical positions (Easwaran, 2008). In other words, when teacher educators teach MKT to teachers, the teacher educators should depend on a certain topic, such as division of fractions, an area and a perimeter of a rectangle, or tables and graphs. Facts related to each topic function as a mathematical foundation in teaching MKT.

Mathematical facts are independent of what individuals think (Hersh, 1995). Therefore, mathematics is often considered as objective rather than subjective. Moreover, because mathematical facts are unanimously accepted, each individual takes himself or herself to be justified in accepting them (Easwaran, 2008). Historically, however, mathematical facts have changed and developed through mathematicians' studies. Goodman (1991) says "The classical theorem that the sum of the angles of a triangle is two right angles only holds approximately because it was also refuted by Einstein (p.386)."

This research emphasizes that mathematical facts are not central topics that teacher educators and teachers should focus on throughout a lesson but have a critical role to make mathematically sound progress in teaching MKT. In this point, this study differentiates three kinds of mathematical facts in teaching MKT: *facts about the selected topic*, *facts about terms and parts used in statements and representations of the selected* *topic*, and *critical ideas to use the selected topic*. For example, to hear what students say about an even number through a video clip, it is critical to know what an even number is and it is a fact that even numbers are { … -4, -2, 0, 2, 4, …}. *Facts about the selected topic* mean concepts and properties that are a focused-topic in teaching MKT. Again, while knowing what properties an even number has is not generally a main topic for teaching MKT, teacher educators generally provide or clarify a set of even numbers or some properties of it.

Facts about terms and parts used in statements and representations of the selected topic are typically talked through teaching MKT. Any topic should be stated and represented in conversations about the topic. The point is that each term should be mathematically obvious in teaching MKT. For example, when a teacher states an even number is a multiple of 2, the meaning of "multiple" has to be mathematically apparent and shared with all participants. If not, the discussion achieves nothing but mathematical distortion. Finally, *critical ideas to use the selected topic* indicate facts expanded from the selected topic. For example, a teacher educator might expand the discussion of an even number to touch on an odd number, multiplication, division into equal parts, division by equal part, or types of numbers. With regard to representations, the discussion might consider discrete quantity model and continuous quantity model. Then, the discussion would approach the critical ideas of an even number. These kinds of extensions are related to Skemp's (2006) relational understanding and Ma's (1999) profound understanding of fundamental mathematics in terms of extended understanding.

On certain occasions, one of the three kinds of mathematical facts would work in teaching MKT. Generally, however, teaching MKT depends on all of them.

Mathematical structures. Structure is a term often used in mathematical discussions without being precisely defined. It seems to exist as more of a shared understanding among mathematicians. Mathematical structure has been highlighted in structuralism, such as "Structures are primary; mathematical objects are nothing but places in a structure (Brown, 2008, p. 62)." In this section, a mathematical structure does not mean structuralism or any arguments, though some of those ideas might be similar to how it is used here.

P. J. Davis and Hersh (1981) explain that mathematical structures consist of mathematical objects linked together by certain relationships or laws of combination. A structure on a set consists of additional mathematical objects that in some manner attach or are related to the set, making it easier to visualize or work with, or endowing the collection with meaning or significance (Mathematical structure, n.d.). It makes a selected topic exist and work. For example, in the number line, the number 1 is the first whole number after 0 and all the other number are defined by their own places. In the base-ten number system (or the decimal number system), each place has a different value, so a 9 in the tens place means ninety, but a 9 after the decimal point means nine-tenths. The base-twelve number system (or the duodecimal number system) also has different place values. A mathematical structure has its own manner in relating mathematical objects and endowing the collection with meaning or significance.

In teaching MKT teacher educators often emphasize mathematical structure as they drive lessons toward mathematically plain discussions. A typical example is a "number." A number among rational numbers can be a different type of number from whole numbers. A decimal is neither a whole number, an integer, nor an irrational number. It is a rational number and a real number. In a discussion, a teacher educator must identify which types of numbers are being established as a structure. In other words, what mathematical structure is based on a discussion determines the possible examples, meanings, relations, and way something works. For example, Euclidean geometry and non-Euclidean geometry are different structures that show different ways of understanding the world (Hegedus & Moreno-Armella, 2011). Seeing topics with respect to mathematical structures is again relevant to Skemp's (2006) relational understanding and Ma's (1999) profound understanding of fundamental mathematics because mathematical structure requires understanding that can capture the broad area of mathematics rather than small domains of mathematics.

Disciplinary mathematics arrives at general agreement about facts and structures either through deductive verification or through convention. Each topic of mathematics teacher education is dependent on diverse facts and structures. MKT embraces mathematical facts and structures to lay mathematically sound foundation rather than embracing the learning, practicing, or memorizing them.

6.3.2 Mathematical awareness

MKT encompasses appreciation, sense, skills, and ways of thinking and reasoning that mathematicians generally take for granted as they use them in their research. For example, mathematicians believe that a proof should show that something is true for all cases not limited cases; they sometimes use counterexamples to test or refute conjectures. Obviously, there is a style of thinking that is a function of operations, processes, and dynamics only for mathematics (Burton, 1984). In other words, mathematics emerges through mathematicians' own styles of thinking that enables them to think and act in certain ways, and Gattegno (1970, 1987) calls it *awareness*. MKT includes specific illustrations and explicit discussion of the awareness that mathematicians have. It helps teachers recognize features of mathematical reasoning that they need. This research calls it mathematical awareness and arbitrarily defines it as having knowledge of and being in the state of consciousness about a system of interactive manipulation with mathematical objects. There are three specifications: awareness of definitions and axioms, awareness of exploring and proving, and aesthetic sensitivity in mathematics.

Awareness of axiom and definition. Mathematics is the scientific study of hypotheses which it first frames and then traces to their consequences (Peirce, 2010, p. 4). In such tracing, mathematicians use axioms and definitions to make a controversial result uncontroversial (Easwaran, 2008). Thus, there are certain roles, features, and uses of axioms and definitions in mathematics. Mathematicians are aware of these in their research.

An axiom is a statement accepted as true without proof (R. C. James & James, 1992; Weisstein, 2012). It is any mathematical statement that serves as a starting point from which other statements are logically derived. Easwaran (2008) asserts that the role of axioms is to allow mathematicians to stay away from philosophical debates and thus focus on proving theorems that are their primary goals. Regarding definitions, the attributes of mathematical definitions in disciplinary mathematics can be found in Chapter 2. Briefly, a definition assigns properties to some sort of mathematical object (Weisstein, 2011a). It should correctly identify the kinds of object, process, and properties.

Axioms and definitions should be uncontroversial in a particular domain or the overall domains with their clarification. P. J. Davis and Hersh (1981) characterize them as “rock bottom self-evident facts … held by the bolts of logic” (p. 149). Sentences containing them should be stated with care. They make mathematics a domain in which meanings are precise and not connotative; communication is based on clear meanings; and mathematical reasoning can be sound. If mathematicians accept and believe that they are mathematically uncontroversial or precise, they can proceed, implicitly or explicitly using them, with specific discussions or proofs.

Any mathematical discussion is implicitly or explicitly based on certain axioms and definitions. This means that such a discussion is tacitly going and sharing what axioms and definitions should be and how they work in mathematics. Awareness of axiom and definition as MKT in mathematics teacher education generally comes about when a mathematical topic is examined and discussed with teachers. It also happens when teacher educators illustrate what mathematicians are aware of related to axioms and definitions in mathematics, such as their roles, features, and values in mathematics. Awareness of axiom and definition is not memorizing and reciting them or solving certain problems but having a sense about axioms and definition in mathematics.

Awareness of axiom and definition is apparently a habit of mind in conducting mathematical research. Hence, such awareness can arise through activities that use and investigate axioms and definitions. Following those kinds of activities, it can also be aroused as content in teaching MKT by specific illustrations of roles, features, and uses of axioms and definitions. Axioms and definitions provide clear meanings, support reasoning implicitly and explicitly, make mathematics precise not connotative, and make effective communication possible. Moreover, axioms and definitions have relations with adjacent concepts and can have different meanings because of the different mathematical roots. In particular, definitions use mathematical terms that are also used in other ways more generally. Definitions should also have the set of the defined and its complementary set.

As content of MKT in mathematics teacher education, awareness of axiom and definition is unique. It is generally embedded in activities that explicitly explore and use axioms and definitions. That is to say, it does not exist independently as content in mathematics teacher education. While this in no way diminishes the importance of awareness of axiom and definition, it highlights awareness of axiom and definition as skills and habits of mind that should always operate in mathematical research.

Awareness of exploring and proving. Mason (1998) asserts that effective teaching demands knowing the process of conducting the mathematics. This process reveals what mathematicians are aware of and take for granted in their exploring and proving; indeed, mathematicians believe that their main function is proving new theorems (Easwaran, 2008; Hardy, 1992). With regard to awareness in a process, Endsley (1995, 2000) suggests *situation awareness*, which involves three factors: perception of meaningful elements in an environment, comprehension of their meaning, and projection of their status in the near future. Mathematicians perceive meaningful ideas to explore and prove theorems, comprehend their meanings, and apply them. Because of this, mathematicians tacitly use the three factors of situation awareness that Endsley suggests. Zbiek and Conner (2006) more specifically illustrate that exploring and proving in mathematics entails seeing a mathematical structure within a situation. Moreover, mathematicians zoom in on any of the conditions, assumptions, properties, or parameters and combine properties and parameters into more manageable pieces (Zbiek & Conner, 2006). Mathematicians keep going until they have desirable or useful results.

Poincaré (1952) and Polya (1954) identify what mathematicians do in their exploring and proving. Mathematicians find a conjecture by observations and particular instances, a process called induction. They examine and combine the collected observations to find any hidden clues. A profound analysis results in resemblances and differences and mathematicians perceive the possibility of a generalization, which is a conjecture. A conjecture is a general statement suggested by certain particular instances that are found to be true. After having conceived a conjecture, mathematicians try to find out whether it is true or false. The conjecture being found true in all instances examined increases mathematicians' confidence. Mathematicians replace a constant by a variable and remove a restriction and finally to discover the demonstrative argument, a proof. In summary, abstraction, formalization, axiomatization, and deduction are ingredients of a mathematician's exploration and proof (P. J. Davis & Hersh, 1981).

What mathematicians are aware of and take for granted in the process of their work are ways of thinking and skills in MKT. This research specifically calls it awareness about exploring and proving. As with awareness of axiom and definition, awareness of exploring and proving is embedded in activities that teacher educators offer to create proofs and mathematical explanations and discuss them with others. While awareness of exploring and proving depends on specific facts or structures, it is indispensable to perform these kinds of activities as well as mathematical studies. Teacher educators can give comments about what teachers are aware of to perform their activities for mathematically scrupulous work. Teacher educators also illustrate roles and features of proof and what mathematicians care about related to exploring and proving. They share the specification with their teachers who have experience exploring and proving. Proving and explaining general cases should move toward making logical and sequential arguments, being true for all cases, and convincing proofs and explanations to the audience. Looking for the mathematical domains of provided conjectures, considering sensitive cases, using counterexamples, and comparing possible proofs and explanations are significant skills in exploring and proving in mathematics. Again, these illustrations can powerfully function as content of MKT when the activities of creating proofs and mathematical explanations are given. Mathematical awareness can be educated (Gattegno, 1987).

Aesthetic sensibility. Poincaré (1956) proclaims that all real mathematicians know a true emotional feeling, specifically "the feeling of mathematical beauty, of the harmony of numbers and forms, of geometric elegance." It is the aesthetic sensibility in mathematics. Generally, mathematicians consider mathematics as aesthetic when they find harmony, symmetry, unity, (unexpected) simplicity, elegance, conciseness, clarity, and patterns (Dreyfus & Eisenberg, 1986; Hardy, 1992; Huntley, 1970; Penrose, 1974; Poincaré, 1952). Aristotle long ago pointed out that order, symmetry, and limitation are the greatest forms of the beautiful in mathematics. Although some who have a failed experience in mathematics might disagree with the existence of any aesthetics, mathematicians generally believe the aesthetics in mathematics has its roles.

When mathematicians inquire mathematics, Sinclair (2009) asserts that aesthetics has three roles: evaluation, generation, and motivation. Mathematicians consider aesthetics when judging the values of ideas (Tymoczko, 1993) and prefer proofs and theorems that are connected to others (Burton, 1999; Penrose, 1974). Moreover, aesthetics guides mathematicians to actually make actions or choice. One of the distinguishing features in mathematicians' mind is not the logical but the aesthetics of constructing possible combinations of ideas and selecting the fruitful one (Dreyfus & Eisenberg, 1986; Hofstadter, 1997; Papert, 1978; Poincaré, 1952). Moreover, Huntley (1970) and Penrose (1974) claim that mathematicians are motivated in their research because of aesthetics. The aesthetic sensibility of mathematicians enables them to make a decision for the most beautiful combination (Featherstone, 2000).

Dreyfus and Eisenberg (1986) and Penrose (1974) exactly identify the main issues in the discussion about aesthetic sensibility in mathematics. Aesthetic judgments vary so much from person to person, and perhaps one's aesthetic judgments will change as well even among mathematicians. Moreover, mathematicians themselves often focus on the usefulness of mathematics rather than the aesthetics. The more critical issue in terms of the pedagogical setting, very little formal description exists on how to cultivate aesthetic sensibility (P. J. Davis & Hersh, 1981). Children cannot be expected to have an appreciation for the aesthetic sensibility of mathematics (von Glasersfeld, July 1985) even though aesthetic sensibility can be achieved serendipitously (Feynman, 1985).

In fact, this research found nothing related to the aesthetic sensibility in the data. It does have, however, a critical role in mathematics. For example, decisions to be more easily understood, more simply stated, and easier to use are not purely mathematical; they call for a certain aesthetic judgment (Bass, 2012). Finally, this research included it as part of mathematical awareness that mathematicians have. This research arbitrarily defines the aesthetic sensibility as a mode of cognition that is perceived in part as being intuitive and recognized at an emotional level as being pleasurable. As with awareness of axiom and definition and awareness of exploring and proving, aesthetic sensibility is embedded in activities of creating and evaluating definitions, proving conjectures, and creating explanations about general cases. It does not exist independently as content in mathematics teacher education. While this does not lessen its importance, it emphasizes aesthetic sensibility as skills and habits of mind that perform a significant role in mathematical studies. Teacher educators would illustrate and comment on how aesthetic sensibility functions for easier understanding, simpler statements, and easier using.

This section illustrated three kinds of mathematical awareness: awareness of axiom and definition, awareness of exploring and proving, and aesthetic sensibility. Serious investigation and performance of various mathematical work of teaching guides teachers to become aware of mathematical sense and skills for the mathematical work of teaching. Furthermore, when teacher educators specify explicitly what teachers need to consider to improve the mathematical work of teaching, mathematical awareness is obviously revealed and shared with their teachers.

6.3.3 Mathematical value

A generally accepted truism is that, "mathematics is invaluable (Wren, 1933, p. 105)." One reason for this is that its concepts and techniques are essential in a wide variety of theoretical and practical disciplines. It plays a central role in modern culture and some knowledge of mathematics is requisite for scientific literacy. It is about applicability to other fields. However, in terms of disciplinary mathematics, mathematical value is always attached to proofs, which provide mathematics with a *solid foundation* and *substantial construction*. Manin points out both of these features. "Axioms, definitions and theorems are spots in a mathscape, local attractions and crossroads. Proofs are the roads themselves, the paths and highways. Every itinerary has its own sightseeing qualities, which may be more important than the fact that it leads from A to B (Manin as cited in Hanna, 2000, p. 7)." Related to proofs, here are just several comments that mathematicians have made. Kline (1979) explains that "the supreme value of mathematics … is that it reveals order and law where mere observation shows chaos (p.1)." Proofs are mathematicians' ways of building order and law from observation. Corfield (2003) says "Human mathematicians pride themselves on producing beautiful, clear, explanatory proofs, and devote much of their effort to reworking results in conceptually illuminating ways (pp. 38~39)." Hersh (1993) says "In mathematical practice, in the real life of living mathematician, proof is convincing arguments judged by qualified judges. … the essential mathematical activity is finding the proof, not checking after the fact that it is indeed a proof (pp. 389~390)." Since proofs show that a theorem is true and why it is true, mathematics can have a more solid foundation as well as a substantial construction.

Teaching MKT includes creating, having, and checking proofs. It expects that teachers recognize how to prove a conjecture or a theorem and understand how mathematics has developed a solid foundation and substantial construction. Proving is offering mathematical justification for conjectures whose truth or validity are uncertain. On the other hand, MKT as content in teacher education also includes explanations for general cases. Explaining is closely related to proving because explanations sometimes make proofs or good approximations of them. However, explanations for general cases offer mathematical justification for mathematical objects that are already known to be valid by mathematicians. For example, “invert and multiply rule” works for division of fractions. Obviously, it was already validated by mathematicians. Explaining general cases is coming up with an explanation for why it works. In fact, mathematicians make and use “explanations” in their work for themselves or with others and their communications. Therefore, proving and explaining for general cases are included in MKT as content in mathematics teacher education.

In terms of a solid foundation, all definitions and axioms are reviewed, and all logical deductions are built and appraised through cause and effect, contradiction, or negation. In particular, theorems and definitions that are already proved function as a solid foundation in mathematics, and mathematicians’ studies have been performed to generate more theorems and definitions. Mathematicians develop their research based on the theorems and definitions that are robust and performed and authorized by mathematicians. Using theorems and definitions to make mathematical issues indisputable is also part of that solid foundation. In a class of teaching MKT, the solid foundation is planned and implemented when teacher educators offer activities that establish and use theorems and definitions. However, the solid foundation is not a main focus in teaching MKT unlike in typical mathematics courses. In other words, the solid foundation offers mathematical soundness in teaching MKT; it is generally associated with the mathematical work of teaching. For example, writing mathematically precise definitions and evaluating whether definitions are mathematically precise are pertinent to a solid foundation. They are equivalent to mathematicians’ work of using, reviewing, checking, and confirming theorems and definitions in the process of proving. Furthermore, teacher educators specifically illustrate features and roles of theorems and definitions in mathematics.

In a class focused on teaching MKT, substantial attention is paid to creating, probing, comparing, evaluating, and refining proofs and creating general explanations about mathematical objects rather than writing down in their notebooks concise proofs provided by teacher educators or by textbooks and then memorizing and regurgitating. Proofs, as previously mentioned, are rigorous mathematical arguments that demonstrate unequivocally the truth of given conjectures (Weisstein, 2011c). Proving conjectures and creating general explanations are not demonstrations of disparate cases but universal and logical demonstrations that hold for all cases. Because of generality and the pursuit for efficiency, expressions of demonstrations for general cases are often formal and abstract. In a class for MKT, rather than forms of proofs and general explanations, substantial construction can be implemented through teacher educators' emphasis on mathematical soundness, logical completeness, and persuasiveness to the audience in activities for creating proofs and general explanations. Moreover, teacher educators explain in particular features and roles of proofs and offer general explanations in mathematics.

Solid foundation and substantial construction can, therefore, live in both activities related to proofs and general explanations and teacher educators' specific comments about their roles and features in mathematics. Here, it is important to emphasize that a text can be a mathematical proof when it is recognized as valid not by incontrovertible means but by mathematicians (Arsac, 2007). In other words, although mathematics has a solid foundation as well as substantial construction, it is produced by fallible mathematicians and so cannot establish absolute truth (Bloor, 1983; P. Davis, 1972; Lakatos, 1976a). Thurston (1994) explains concisely:

> Mathematicians can and do fill in gaps, correct errors, and supply more detail and more careful scholarship when they are called on or motivated to do so. Our system is quite good at producing reliable theorems that can be solidly backed up. It's just that the reliability does not primarily come from mathematicians formally checking formal arguments; it comes from mathematicians thinking carefully and critically about mathematical ideas. … Once a theorem has been proven, the mathematical community depends on the social network to distribute the ideas to people who might use them further—the print medium is far too obscure and cumbersome (pp. 170~171).

Therefore, teacher educators are not the sole or final arbiter of validity about solid foundation and substantial constructions in classes of MKT. Mathematical value consists of all the efforts to build a solid foundation and substantial construction with the mathematics community. MKT in teacher education includes and pursues the mathematical value, solid foundation, and substantial construction.

6.4 Conclusion

The framework presented in this chapter begins to deconstruct what is involved in the mathematical work of teaching and in knowledge about MKT as content in mathematics teacher education. The main categories included in the framework are summarized in *Figure 6.15*. It elaborates the basic architecture described at the beginning of the chapter (*Figure 6.1*). This framework is based on the analysis of the curriculum materials developed by the *mod4* project. Therefore, the developed framework is consistent with the implicit framework that developers of the materials might have considered when they developed the materials. However, the developed framework is more extended and elaborated than their implicit framework. The developed framework is also based on the analysis of the implemented phase of the materials through the video recordings of the classes that teacher educators worked with them. The data about the implementation helps the framework include what the curriculum materials themselves do not anticipate. The mathematical work of teaching and knowledge about mathematics that are obviously illustrated in the materials are generally enacted in the classrooms with teacher educators and teachers in this study. However, the implementation was extended to others, which is not suggested in the curriculum materials. The curriculum materials focus on mathematical definitions as topics, but the developed framework seeks to make claims about other mathematical objects in teaching and the work of mathematics teaching.

FRAMEWORK FOR CURRICULUM TO TEACH MKT

Mathematical Work of Teaching

Zoomed-out mathematical work of teaching

Providing and justifying mathematical and pedagogical decisions		**Situating and unpacking mathematics ideas and practices**		**Using language mathematically and accessibly**
• Making a decision regarding one lesson ○ SCK ○ KCS ○ KCT ○ KCC ○ HCK	• Making a decision regarding a long period ○ SCK ○ KCS ○ KCT ○ KCC ○ HCK	• Unpacking mathematics ideas ○ SCK ○ KCS ○ KCT ○ KCC ○ HCK	• Situating in mathematics ○ SCK ○ KCS ○ KCT ○ KCC ○ HCK	○ SCK ○ KCS ○ KCT ○ KCC ○ HCK

nested in

Zoomed-in mathematical work of teaching

		Work of mathematics teaching				
		Recognizing Articulating	Probing Interpreting Comparing	Evaluating Judging	Selecting Modifying	Constructing
Mathematical objects in teaching	Concept Property Definition	CCK SCK KCS	CCK SCK KCS	CCK SCK KCS KCT	CCK SCK KCS KCT	CCK SCK KCS KCT
	Algorithm Rule Procedure	CCK SCK KCS	CCK SCK KCS	CCK SCK KCS KCT	CCK SCK KCS KCT	CCK SCK KCS KCT
	Representation Tool	CCK SCK KCS	CCK SCK KCS	CCK SCK KCS KCT	CCK SCK KCS KCT	CCK SCK KCS KCT
	Proof	CCK SCK KCS	CCK SCK KCS	CCK SCK KCS KCT	CCK SCK KCS KCT	CCK SCK KCS KCT

Knowledge about Mathematics

Mathematical facts and structures	Mathematical facts
	Mathematical structure
Mathematical awareness	Awareness of definition and axioms
	Awareness of exploring and proving
	Aesthetic sensibility
Mathematical value	Solid foundation
	Substantial construction

Figure 6.15 The main components for curriculum to teach MKT

Mathematics teacher education must equip teachers with the knowledge, skill and habit of mind necessary to do skilled teaching and to succeed in supporting students to master challenging mathematics. For this purpose, teacher educators need to develop their understanding of content for teachers' mathematical preparation and their practice of conveying such knowledge. The instruction was carefully observed and analyzed, as shown in *Figure 1.1*, in order to capture the mathematical work of teaching and knowledge about mathematics as the agents for interaction among teacher educators, teachers, and content. Then, the findings from the data were conceptualized to support the effective teaching of MKT.

A new lens was introduced here to consider the curriculum of mathematics teacher education. Both teaching practice and mathematical aspects were contemplated, rather than an exclusive emphasis on only one of them (Ball et al., 2009). With regard to balance, equilibrium among the mathematical work of teaching and knowledge about mathematics was also emphasized. Teacher educators must strive to give opportunities to teachers to learn and practice the various kinds of mathematical work of teaching and knowledge about mathematics, rather than be inclined to focus on certain work of teaching or mathematical objects or certain mathematical facts or values. Routines that teachers face in mathematics classrooms are embedded in a teaching practice that is complex as well as in mathematical objects, which are multifarious.

Before closing this chapter, it should step back to make some general comments about the framework. The findings are summarized and some potential contributions and uses are described, and the limitations of this work are discussed.

6.4.1 Potential contributions and uses

The conceptualization for the teaching of MKT in mathematics teacher education contributes to teacher educators' work and curriculum in mathematics teacher education. The major contribution of the framework is the conceptualization of a central aspect for teaching MKT in mathematics teacher education: identifying the main MKT of an activity and steering a lesson toward those components. Despite the effects of teachers' MKT on the students' achievement and teacher educators' interests about MKT for their teaching, it is left implicit how a curriculum for MKT is designed to help teachers have mathematical preparation. Therefore, what the framework does is bring MKT as content into the foreground of mathematics teacher education.

An overall conceptualization of the mathematical work of teaching and knowledge about mathematics to teach MKT provides language for the practice of teaching that teacher educators can share with teachers. Even if the language used here does not take purchase beyond this dissertation, it has helped identify aspects of teaching practice and knowledge about mathematics that warrant naming and further research in mathematics teacher education. Just as mathematical terms and symbols compress mathematical concepts into objects that can be manipulated and operated upon more easily, naming the mathematical work of teaching and knowledge about mathematics compresses a set of important ideas and practices into an object that can be more easily discussed and studied.

Another contribution is that my conceptualization offers teacher educators a lens for designing and organizing a curriculum to teach MKT. Moreover, the conceptual framework provides a way to analyze and evaluate the curriculum of mathematics teacher education in terms of MKT. This contribution is one of the substantial ways teacher educators can use the framework. For example, activities planned in teacher educators' courses are probed in order to identify what kinds of mathematical work of teaching and knowledge about mathematics will be performed with the teachers they teach. Furthermore, the framework could be used to analyze the overall curriculum of the mathematics teacher education program. The framework could also be used to evaluate teacher's knowledge, skill, and performance of teaching by providing additional insight.

Another contribution and another way to use the framework is letting it help teacher educators recognize and articulate how their instruction goes and what teachers think about MKT in instruction and to manage and steer their teaching toward MKT. The developed framework is also straightforward to the implementation phase. Teacher educators in the implementation phase are required to make quick and accurate decisions on what teachers think about MKT, how the instruction moves on, how their teaching should be changed or kept toward the MKT, and what should be emphasized or understated in terms of MKT. The framework could serve as the basis of teacher educators' decisions in the implementation.

Ultimately, this research expects that findings from this study will offer a foundation for building a shared curriculum for the mathematical preparation of teachers with the specialized knowledge and skills needed to teach mathematics. Diverse attempts and discussions about designing, managing, and accomplishing curriculum for skilled teaching will develop a shared curriculum of mathematics teacher education. Ultimately, this should lead to improvement in teaching and learning and thus gains in achievement.

6.4.2 Limitations

One of major limitation is the manageability. Each section of the framework was analyzed, and it became long lists of things that teacher educators consider as content. How useful that format will be for teacher educators is still unclear. Moreover, in spite of the length of the lists, the framework is still not complete. Therefore, this research tried to make the framework meaningful, on that could be acted upon in the practice of teacher education. To help mediate this issue, this study tried to make visible an overarching architecture for the framework as shown in *Figure 6.15*. Some coherency would support an understanding and remembering of the details. Even though it helps, the architecture itself has its limitations. For example, the matrix representation does not depict the interactions across cells.

Another limitation is that this framework makes no attempt to characterize the quality of the content even though the framework aims to describe MKT as content thorough decomposition. For example, simply considering everything in the framework would be insufficient. The mathematical work of teaching and knowledge about mathematics can be analyzed with different perspectives and different degrees of sophistication and understanding. A specified topic can be more or less central as content. Connections across lessons can be variously compelling. Obviously an articulation of the curriculum is important, but there is much more to be done and there are other ways to describe what it means. Being able to describe the quality of the content has implications for future research that might try to evaluate MKT as content to study whether there is a relationship between the mathematical work of teaching and knowledge about mathematics and other aspects of teachers' learning and teaching, such as teacher achievement or their mathematical quality of instruction.

Another limitation is that while the framework shows what can be content as MKT in mathematics teacher education, it does not show how that can be planned and enacted. However, *how* is an important issue in the practice of teacher education. Generally, teacher educators are responsible for coordinating the available resources to support teacher's mathematical preparation. Therefore, teacher educator should have some understanding about MKT. The specific aspects of the mathematical work of teaching and knowledge about mathematics as contents of MKT will need a particular context. Moreover, there are likely some aspects of the mathematical work of teaching and knowledge about mathematics that teacher educators have to do regardless the context and the resources they have. Even though these issues are important, they are beyond the scope of the current study.

CHAPTER 7

MATHEMATICAL KNOWLEDGE FOR TEACHING IN TEACHER EDUCATION: CONCLUSIONS AND THE NEXT STEP

7.1 Summary of Dissertation

This research yields a conceptualization that can inform a curriculum for the teaching of MKT in mathematics teacher education. The conceptualization includes both identifying tasks of teacher educators and elaborating content into the framework that can be used to design and enact curriculum to teach MKT in the context of teacher education. In particular, the framework for curriculum to teach MKT has two main components, the mathematical work of teaching and knowledge about mathematics. The mathematical work of teaching is the tasks of teaching that teachers perform related to mathematics in classrooms. Moreover, the knowledge about mathematics is what are considered through mathematicians' research to scaffold Ball's (1990) notion of "knowledge about mathematics." In other words, this research asserts that both the mathematical work of teaching and knowledge about mathematics are indispensable components to planning and implementing of MKT in mathematics teacher education.

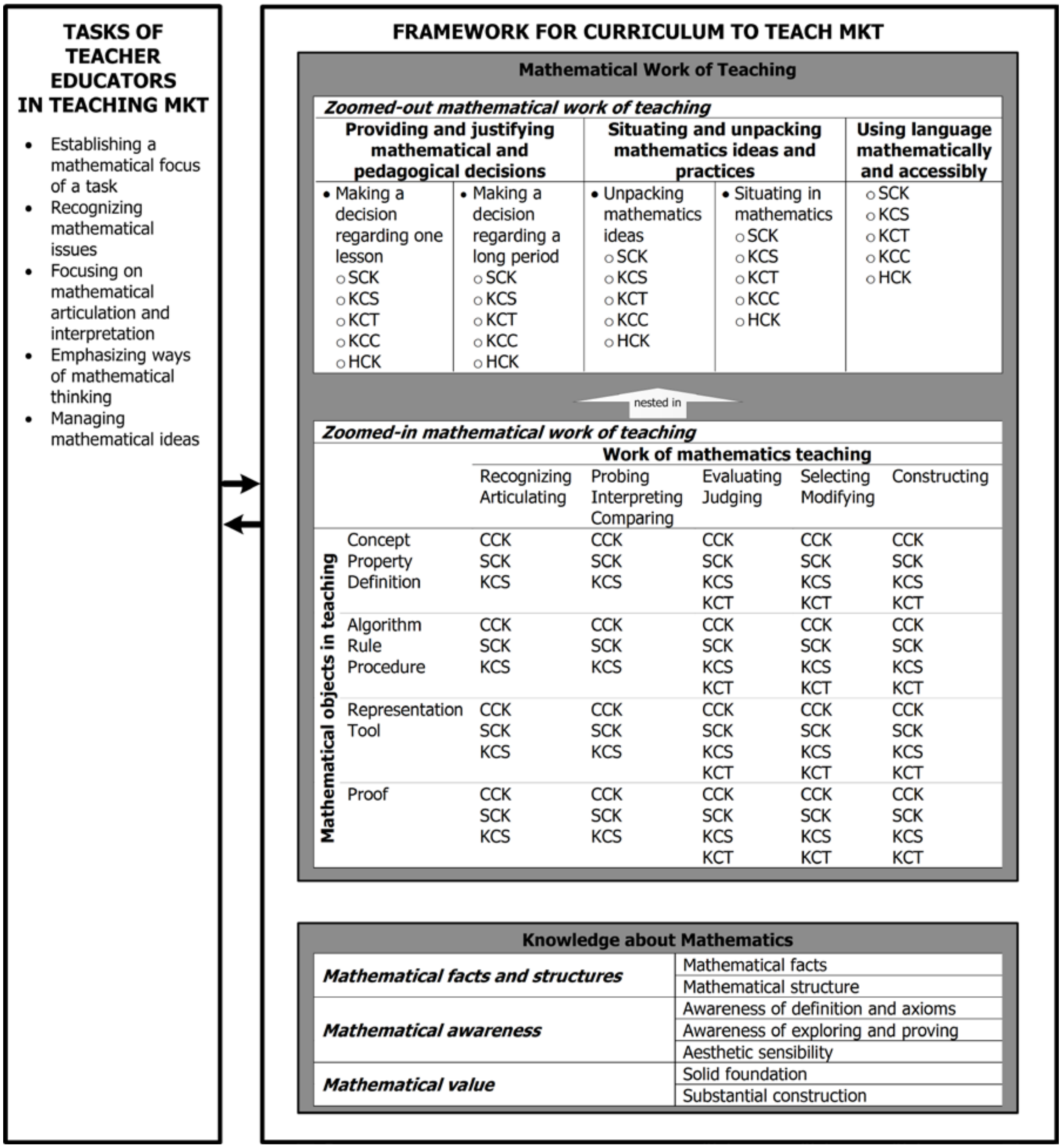

Figure 7.1 Elaboration of the conceptualization to teach MKT in mathematics teacher education

A cyclical relationship exists between the tasks of teacher educators in teaching MKT and the framework for the curriculum to teach MKT, as shown in *Figure 7.1*. Tasks of teacher educators in teaching MKT require knowing well what mathematical work of teaching and what knowledge about mathematics are being planned and enacted. Moreover, the framework for curriculum to teach MKT is intended to help teacher educators perform their tasks in teaching MKT and help teachers develop MKT. In other words, specifying the framework for curriculum to teach MKT can help manage challenges and stay on track for teaching MKT. Tasks of teacher educators then inform both which mathematical work of teaching and knowledge about mathematics are shared and discussed or not in mathematics teacher education. The cyclical relationship occurs simultaneously during an activity's enactment. For example, a teacher educator would consider a certain mathematical work of teaching and a certain knowledge about mathematics which are both involved in an activity as well as issues that teachers might have in the activity.

To help clarify the aspect of tasks of teaching MKT to demonstrate in what ways MKT is worked on and objects of MKT to explain what MKT is worked on being illustrated by the lessons, this research offers an detailed diagram of the conceptualization for teaching MKT than can inform a curriculum in mathematics teacher education as shown in *Figure 7.1*. This section highlights several key features of the conceptualization that the diagram is trying to reflect.

Figure 7.1 shows several tasks that teacher educators conduct in lessons of teaching MKT. Even though they are separately listed, attentions and challenges are overlapping. Chapter 5 examined each challenge with examples from across the range of lessons from the data in order to specify what attention teacher educators need to pay and what issues can arise when teaching MKT.

The mathematical work of teaching as content of MKT was conceptualized with two layers. It is subdivided into zoomed-in and zoomed-out mathematical work of teaching. The former is nested at several levels within the latter. That is, teachers and teacher educators work on several kinds of zoomed-in mathematical work of teaching that converge into a single zoomed-out mathematical work of teaching. This research argues that this multi-layered feature of the mathematical work of teaching is a characteristic of the practice of teaching mathematics. Instruction occurs across two continuums: (1) across each moment, each activity, each lesson, and each school year, and (2) across a small domain of mathematics, mathematics that students will learn later, and the overall territory of mathematics. Each smaller unit is nested within a larger unit. It is because of these varying levels of instruction that this research classifies the various mathematical work of teaching into zoomed-in and zoomed-out mathematical work of teaching. Zoomed-in mathematical work of teaching captures the moment-to-moment and activity in the continuum of curriculum as well as a small domain of mathematics. Zoomed-out mathematical work of teaching captures each lesson and each school year in the continuum of curriculum as well as the mathematics that students learn later and the overall territory of mathematics. Both perspectives are integral to the work of teaching.

Knowledge about mathematics as content of MKT is explicit specification about the content of MKT in mathematics teacher education in terms of disciplinary perspective of mathematics. There are three components in knowledge about mathematics as content of MKT: mathematical facts and structures, mathematical awareness, and mathematical value. Disciplinary facts mean abstract or general ideas inferred or derived from specific instances in mathematics, and disciplinary structure in mathematics as a conceptual organization of mathematical objects and proxies. Second, mathematical awareness includes ways of thinking and reasoning, appreciation, sense, and skills that mathematicians generally use in their research. Such a demonstration ensures teachers can recognize the features of mathematical reasoning needed to perceive qualities of teaching. Third, mathematical value refers to a set of principles concerned with the nature of mathematics: what makes it desirable, valuable or useful. That is, that MKT includes features of work that mathematicians do. The details of the mathematical work of teaching and knowledge about mathematics were presented and further explored in Chapter 6.

Figure 7.1 is based on the conceptualization of the lessons that considered various factors about teacher educators and the materials that they used in diverse situations. Therefore, the framework developed in the current study does not intend that every single teacher educator is expected to do consider one or the other all aspects of the framework for every activity. Rather, it anticipates objects that can be content of MKT. For example, a lesson related to an addition of one digit numbers would not consider the volume of a sphere. Depending on the context, the object will be different, such as courses, grades of teachers, teachers' experiences, materials, standards for teachers, and curriculum schedules. The lessons in this chapter are from the data, and thus they show cases in particular situations. The original goal is to use these specific examples to generate a general description of objects as content of MKT rather than analyzing who is doing what in a certain situation of teaching MKT. In any situation, teacher educators have significant roles in teaching MKT for planning and teaching a lesson and managing teachers.

7.2 Potential Contributions to Research in Education

More than ten years ago, Ball (2000) identified three problems about integrating subject matter knowledge and pedagogy in mathematics teacher education: what content knowledge matters for good teaching; how subject matter must be understood to be usable in teaching; and how to create opportunities for learning subject matter that would enable teachers not only to know but to learn to use what they know in the varied contexts of practice. This section uses Ball's three questions to explore this research's potential contributions to mathematics education and teacher education. The chapter concludes with ideas for specific next steps arising from the current study.

7.2.1 Talking about MKT in terms of mathematics teacher education

Ball et al. (2008) define MKT as the mathematical knowledge needed to carry out the work of teaching mathematics (p.395). This content knowledge is established based on the analysis of what teachers actually do on in and for mathematics classroom. Its effect on students achievement is critical (Hill et al., 2005; Rockoff, Jacob, Kane, & Staiger, 2008). However, an understanding of what can be content to teach MKT in teacher education was not clear. Grossman and McDonald (2008) and Ball et al. (2009) emphasize articulating the work of teaching mathematics which is decomposed and, thus, becomes a collection of smaller practices that can be identified, taught, and rehearsed in teacher education, and integrated in the actual work of teaching. Furthermore, a shared taxonomy of and language for the practices of teaching is lacking (Grossman & McDonald, 2008). The results of this dissertation name the actual work that teachers do and teacher educators use in their lessons as well as labeling subject matter as it is used in practice. Moreover, the detailed description of each element represented in the framework makes visible aspects of MKT that might be missed by other perspectives.

The current study is taking another step forward to specify MKT for mathematics teacher education and how MKT is approached in the context of teacher education. Therefore, teacher educators as practitioners in mathematics teacher education can use a practice-based theory of content knowledge (Ball & Bass, 2003b) because the work in this dissertation theorizes about the content for teaching MKT. Furthermore, the conceptualization of the mathematical work of teaching and knowledge about mathematics in this dissertation and the resulting framework for teaching MKT provide a different lens for viewing MKT. The focus of the framework is identification of elements to teach MKT in mathematics teacher education, and this has a number of implications for helping research MKT and teacher education. The framework shows the mathematics entailed in teaching for the use of content in mathematics teacher education. In other words, this research verified mathematics teacher education as another place to research MKT. It enriches areas of MKT.

7.2.2 Managing MKT in mathematics teacher education

According to Ball, teachers need "the capacity to deconstruct one's own knowledge into a less polished and final form, where critical components are accessible and visible" (Ball, 2000, p.245). The conceptualization of the mathematical work of teaching as content for mathematics teacher education developed in this dissertation assumes features of students with content as its growing and unfinished state and, then, identifies tasks of teaching in harmony with mathematical objects. In addition, the conceptualization of the knowledge about mathematics embedded in teaching MKT provides what sort of content understanding and insight matters in practice, such as mathematical awareness, aesthetic sense, and mathematical value.

The results of this dissertation also provide in what ways activities are managed in teaching MKT that helps teachers understand the mathematics usable in teaching. The conceptualization of teaching MKT contributes to the pedagogical considerations that underlie the teaching of MKT and has highlighted the importance of what teacher educators need to pay attention to. This research found that teaching MKT involves big challenges for any teacher educator through analyzing the data. It might be because teacher educators have different knowledge, skills, and reasoning about mathematics, K-9 mathematics classrooms, students, teachers, teacher education programs, or educational policies. While it might be important to explore these kinds of different factors and backgrounds that can influence teaching MKT, this research considers that it is more significant to observe teacher educators' teaching in practice and articulate what they do in teaching MKT. This research named and defined the tasks of teacher educators to teach, and this identification and definition helps mediate research on a variety of commonsense fundamental problems, such as exploring tasks of teaching in mathematics teacher education. For example, now that the tasks of teacher educators to teach MKT are objects, they can be designed to try to reflect or evaluate teacher educators' work. Such evaluation could then be used to support teacher educators in improving their teaching.

7.2.3 Having a foundation to set up a curriculum to teach MKT

When proposing the question of what it takes to learn to use the content knowledge that matters for teaching, Ball suggests that teacher educators "design and explore opportunities to learn content that are situated in the contexts in which subjects matter is used" (p. 246) as a solution to help teachers learn to use knowledge for teaching. She provides several illustrative examples, such as using student work as a site to analyze and interpret what students know and are learning and work on the content itself and using cases of classroom episodes (Lampert & Ball, 1998; M. K. Stein, Smith, Henningsen, & Silver, 2000). She places a greater emphasis on the generality of teachers' learning of content and their capacity to use it in a variety of contexts rather than the use of specific contexts.

The results of this dissertation can be used to help teachers improve and use their knowledge and skills in general places rather than specific instances because this study encompasses diverse cases in tasks of teaching and mathematical objects in teaching. Furthermore, following Ball's suggestion, the work in this dissertation contributes to set up a curriculum to teach MKT for teachers' mathematical preparation that can function in any teaching place: identifying what MKT is worked on and building knowledge about how to organize the curriculum for teaching MKT. This conceptualization navigates the complexities of in-the-moment decisions in the practice of teaching mathematics that Ball (1993) and Lampert (1990) point out, as well as pivotal decisions for the pedagogical purposes of teaching MKT. Moreover, the work in this dissertation offers not only insights into which mathematical work of teaching can be content for teaching MKT but also which knowledge about mathematics is embedded in teaching MKT. Thus, teacher educators can see a broader area of MKT from the practice of teaching as well as disciplinary mathematics. These detailed descriptions help teacher educators plan and enact curriculum in mathematics teacher education to teach MKT. Furthermore, the major contribution of this research is building the groundwork for a shared curriculum in mathematics teacher education.

7.3 Next Steps: Beginning this Line of Work

In many ways, this dissertation is about setting the stage for future work. Conceptualizing teaching MKT in mathematics teacher education enables furthers study of the aspects of MKT and mathematics teacher education. The above discussion of this study's contributions to education scholarship points to a number of concrete next steps in this line of work. This section briefly discusses some of these below.

7.3.1 Using the framework to analyze the curriculum materials and teacher educators' lessons to teach MKT

One next step would be to reanalyze the curriculum materials and the data from teacher educators' lessons using the framework for teaching MKT. For this dissertation, the data were analyzed to develop the framework. The curriculum materials developed by *mod4* and teacher educators' uses of them were used to examine MKT as content in mathematics teacher education and the tasks of teacher educators to teach MKT. This analysis did not focus on determining whether the curriculum materials well support teacher educators. Therefore, a possible next step would be to develop a way to use the framework to code in what ways the curriculum materials support teacher educators' efforts and in what ways issues come up for which the curriculum materials provide insufficient support in order to investigate differences between the written curriculum and enacted curriculum to teach MKT.

The results of this analysis could provide important features to developers of the curriculum materials designed to focus on MKT. Such descriptions of the phases of curriculum use could help developers of the curriculum materials to teach MKT learn more about what teacher educators bring to their lessons and what should be included to support teacher educators for teaching MKT. They also could suggest productive directions for future research in terms of *Mathematical Tasks Framework*, which M. K. Stein, Remillard, and Smith (2007) suggest with respect to teaching MKT. It would help improve the consistency from the written curriculum to the enacted one in teaching MKT in mathematics teacher education.

7.3.2 Using the framework to analyze different curriculum materials and diverse teacher educators' lessons to teach MKT

Another important next step would be to study MKT in different contexts of teacher education. This line of research would investigate whether the framework of teaching MKT is different in different contexts and how it is differently seen in different contexts, such as with different curriculum materials, in the differing contexts of professional development, content courses, or method courses, or in different countries. With regard to whether the framework of teaching MKT is different in different contexts, I assume that the structure of the framework presented in this dissertation is a general one that would be applicable across different contexts of teaching MKT in the United States. As previously explained, the data was from records of practice collected in classrooms where teacher educators and teachers are working with the curriculum materials specially designed to focus on MKT, focusing, in particular, on mathematical definitions as they arise in teaching. Moreover, the framework reflects the literature I reviewed, although it is likely that there may be new things to add to the framework that were not visible in the data or in the literature I reviewed.

This dissertation used the curriculum materials that focus on mathematical definitions to teach MKT and video recordings. These show classes where teacher educators worked with the materials in diverse contexts – professional development, content course in mathematics department, and method course in education department. The feature of the use in the multiple situations helped develop a framework that can universally function to teach MKT. However, differences across curriculum materials may be more apparent, and differences across diverse contexts may be more manifest. For example, teacher educators who work with the curriculum materials that focus on representations might place small value on proofs as mathematical objects in teaching but might emphasize instead unpacking the ideas involved in representations. Teacher educators in professional development may easily set up a scaffold to articulate and probe patterns of students' errors, but may be prone to fall into pedagogical issues with a relatively small mathematics lens. In comparison, teacher educators in content courses of mathematics department may attempt to slide into mathematics in ways that are comparatively remote from teaching. In any case, more fundamentally, teacher educators teach any part of MKT, but different features of MKT are seen and emphasized in different contexts.

More understanding about how the framework of teaching MKT is differently seen in different contexts would offer specific features of MKT in various contexts of teaching MKT. There might be certain aspects of MKT that teacher educators generally emphasize more, or there might be a similar tendency to teach MKT across contexts. If so, those areas of similarities might be more common in teaching MKT. In addition**,** areas of dissimilarities would need to be particularly highlighted in the curriculum materials so that teacher educators could teach harmoniously diverse MKT for teacher mathematical preparation.

7.3.3 Studying both MKT in terms of teacher education and MKT in the practice of teaching

Another next step would be to try to relate the MKT that was investigated and conceptualized for mathematics teacher education in this dissertation to the practice of teaching. This dissertation has considered the use of MKT by teacher educators in mathematics teacher education. In other words, this research used the contexts of mathematics teacher education as another place to study MKT and considered teacher educators as other users of MKT. This focus provides for interesting contrasts in terms of the differences in the contexts of mathematics teacher education and mathematics classrooms and differences between teacher educators and teachers as users. The framework that this research developed in this dissertation could function to observe mathematics classrooms through K-9 grade levels. However, because the purposes of the use of MKT are different – one is teaching MKT and the other is teaching mathematics – the framework could serve different additional function. For example, this research hypothesized two layers of the mathematical work of teaching. But the practice of teaching might include multi-layers rather than just two, or a continuous and simultaneous performance of diverse mathematical work of teaching might be emphasized because of its complexity in the practice of teaching. In any case, comparison of the different uses of MKT would identify features of MKT as content in mathematics teachers compared with MKT in the practice of teaching, and, ultimately, this will extend the research of MKT.

7.3.4 Developing a tool for teacher educators' reflection

The framework developed in this dissertation could be translated into a tool for helping teacher educators efficiently teach MKT. It could be check-list style or open-ended, involving the reflection about tasks of teacher educators after a lesson. During data analysis, it was often seen that the teacher educator seemed to have their own strengths and weaknesses that are repeatedly found throughout their lessons for teaching MKT even though this research did not aim to compare who generally did or did not do certain tasks in their teaching. A tool to help teacher educators review and reflect on what they carry out would signal important points to consider and put them in the position to do the analysis themselves in order to better teach MKT. Such a tool could be built from the framework regarding the tasks of teacher educators in teaching MKT.

Appendix A: Initial categorization of the mathematical work of teaching with examples

Introduction

This research reviewed tasks of teaching from the literature review, made groups with similar features to the mathematical work of teaching, and classified them. This research also reviewed examples in each element and renamed each element. This section reports the literature that contributed to create a pool of diverse mathematical work of teaching as well as clarifies examples in each mathematical work of teaching in each categorization.

List of the Literature to Gather Examples of the Mathematical Work of Teaching

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(3) Ball, D. L., & Forzani, F. (2009). The work of teaching and the challenge for teacher education. Journal of Teacher Education, 60(5), 497-511.

(4) Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389-407.

(5) Berge, A. (2006). Convergence of numerical sequences-A commentary on "The vice: Some historically inspired and proof generated steps to limits of sequences" by R. P. Burn. Educational Studies in Mathematics, 61(3), 395-402.

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(16) Herbst, P. G. (2003). Using novel tasks in teaching mathematics: Three tensions affecting the work of the teacher. American Educational Research Journal, 40(1), 197-238.

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(19) Lampert, M. (2001). Teaching problems and the problems in teaching. New Haven, CT: Yale University Press.
(20) Ouvrier-Buffet, C. (2006). Exploring mathematical definition construction processes. Educational Studies in Mathematics, 63(3), 259-282.
(21) Pimm, D. (1987). Speaking mathematically: Communication in mathematics classrooms. London: Routledge & Kegan Paul.
(22) Pimm, D. (1993). Just a matter of definition. Educational Studies in Mathematics, 25(3), 261-277.
(23) Reynolds, A. (1992). What is competent beginning teaching? A review of the literature. Review of Educational Research, 62(1), 1-35.
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(28) Van Dormolen, J., & Zaslavsky, O. (2003). The many facets of a definition: The case of periodicity. Journal of Mathematical Behavior, 22(1), 91-106.
(29) Wilson, P. S. (1990). Inconsistent ideas related to definitions and examples. Focus on Learning Problems in Mathematics, 12(3&4), 31-47.

Examples of Mathematical Work of Teaching in Each Categorization

There are six kinds of the mathematical work of teaching: solving mathematical problems in and for teaching; unpacking mathematics ideas and practices; developing horizon knowledge; investigating mathematical concepts or solving mathematical problems without the context of teaching; recognizing and developing ways of mathematical sense and reasoning in and for teaching; overarching instruction for mathematical understanding.

Solving mathematical problems in and for teaching indicates mathematical interpretation, evaluation, analysis, selection, creation and decision for solving mathematical problems that happen in and for teaching; and, expanding mathematical knowledge for this. For example, this includes hearing and interpreting children's (sometimes implicit) uses of mathematical definitions in their reasoning, evaluating definitions or specifications given in textbooks, finding concerns that students can have, making diagram to illustrate for students, or explaining and comparing two definitions. Providing reasons or meanings about interpretation, evaluation or analysis is also included here, but is not required. This is for addressing, considering or aiming at the practice of teaching: if not, *investigating mathematical concepts or solving mathematical problems outside of the context of teaching* is related. Moreover, if a problem asks to consider overall one lesson, this is related to *overarching instruction for mathematical understanding*.

Unpacking mathematics ideas and practices refers to investigation about mathematical ideas behind what students use and are likely to think or tasks and concepts given in curriculum materials. In other words, this includes exploring how and why certain mathematical rules, representations, algorithms work, for example, representing and mapping across a long multiplication and area model, investigating why the long division algorithm works, or investigating why zero cannot be the divisor. The practice of teaching or K-12 mathematics classrooms can be used for contextualization, but is not required.

Developing horizon knowledge indicates exploration about mathematical environment surrounding disciplinary location that students currently stand, major disciplinary ideas and structures, key mathematical practices, and core mathematical values and aesthetics for illuminating critical dimensions of that content and anticipating mathematics of what the student may encounter father along the path. For example, in the task that represents a fraction on the number line, which is included in *solving mathematical problems in and for teaching*, exploring ideas of density of the rational numbers and recognizing that all the numbers of K-8 mathematics "live" on the number line are included here.

Investigating mathematical concepts or solving mathematical problems outside of the context of teaching refers to construction and establishment of thoroughly mathematical knowledge and its fluency and speed without specific address to the practice of teaching or classroom context. In other words, if the contextualization about teaching practice is built on, this is related to *solving mathematical problems in and for teaching*. For example, solving 1 3/4÷1/2 is included here, but analyzing a student's error in 1 3/4÷1/2 is coded as *solving mathematical problems in and for teaching*. Moreover, knowing a short cut to find multiples of four is included here, but proving or investigating why the last digits are deciding multiples of four is coded as *unpacking mathematics ideas and practices*. Constructing and evaluating proofs of a mathematical conjecture or claim is included here if proving does not intend illustrating mathematical ideas behind a concept that *unpacking mathematics ideas and practices* is focusing on. For example, proving that opposite sides of a parallelogram are equal is included here.

Developing mathematical sense and recognizing ways of mathematical reasoning in and for teaching includes specific illustration and explicit talk about ways of thinking and reasoning, appreciation, sense and skills in and for teaching for ensuring teachers have recognized features of mathematical reasoning that teachers need. For example, this includes recognizing that using accurate language adds precision to communication and acknowledging that definitions is not delivered but continuously evolved by inquiry. Moreover, this code includes introducing precision and usability as criteria to evaluate definitions in the flow of instruction or given in textbooks, but evaluating a definition based on the criteria is included in *solving mathematical problems in and for teaching*.

Overarching instruction for mathematical understanding indicates specifying, practicing and talking about skills and reasoning to see one lesson as unit and plan and anticipate how mathematical ideas change and develop within a lesson, for example sequencing figures for developing the concepts of rectangle and explaining mathematical purposes of a lesson. This component is part of *solving mathematical problems in and for teaching*. The focus of this component is whether mathematical problems in and for teaching are for planning and appreciating a whole lesson.

The following table shows examples of each categorization. Each example includes a number that indicates the reference in the previous section. This categorization and examples are used to begin analyzing the data in order to build knowledge about how to organize the curriculum for teaching MKT.

Solving mathematical problems in and for teaching					
Making mathematical decisions in and for teaching					
Recognizing /Articulating	Probing /Interpreting/ Comparing	Evaluating/ Judging	Creating: Presenting	Creating: Selecting	Creating: Modifying/ Constructing
• Identifying key understandings and misunderstandings (3) • identifying ways in which a learner is thinking about the topic or problem at hand (3) • listening to and watching others as closely as is required to probe their ideas carefully (3) • monitoring student understanding (15) • not presuming shared identity; seeking to learn others' experiences and perspectives (3) • seeing people more descriptively (3) • observing the range of student performance (19) •	• deciding/ analyzing where to direct students' activity when a new idea comes up (16) • elaborating how interactions mathematically occur (13) • elaborating how learning develops (13) • elaborating how mathematical goals are accomplished (13) • elaborating how problems are posed (13) • elaborating how questions are asked (13) • eliciting students ' conceptual actions that are instrumental for the task (16) • finding the logic in someone else's argument or the meaning in someone else's representation (12) • interpreting and evaluating students' non-standard mathematical ideas (2) • interpreting students' questions, solutions, problems, and insights (both predictable and unusual) (2) • monitoring student understanding (15) • probing others' ideas (3) • reviewing homework (3) • using counterexamples for revising (14)	• evaluating diverse learner productions (1) • appraising the mathematical content of textbooks (4) • assessing student learning using a variety of measurement tools (23) • assessing students' math skills (3) • assessing students' mathematics learning (2) • deciding/ evaluating where to direct students' activity when a new idea comes up (16) • evaluating explanations: Multiplication (2) • evaluating mathematical explanations (4) • evaluating students' non-standard mathematical ideas (2) • evaluating the plausibility of students' claims (often quickly) (4) • making judgments about the mathematical quality of instructional materials (2) • making mathematical and pedagogical judgments about students' questions, solutions, problems, and insights (both predictable and unusual) (2) • overseeing and assessing the learner's progress (3) • reflecting on their own actions and students' responses in order to improve their teaching (23) • reviewing homework (3) • sizing up a pupil's incorrect mathematical response (3)	• a teacher's appropriate feedbacks (20) • asking productive mathematical questions (4) • asking questions to which you often do know (at least part of) the answers (3) • giving mathematical explanations (4) • making explanations: Multiplication (2) • phrasing to use in asking her question (3) • posing good mathematical questions and problems that are productive for students' learning (2) • presenting and using multiple representations (17) • presenting mathematical ideas (4) • representing ideas carefully (2) • representing the mathematical objects involved in a task (16) • responding productively to students' mathematical questions and curiosities (2)	• choosing a task to assess student understanding: Decimals (2) • choosing representations that are mathematically profitable (12) • finding an example to make a specific mathematical point (4) • selecting representations for particular purposes (4)	• modifying tasks to be either easier or harder (4)

Solving mathematical problems in and for teaching		Unpacking mathematics ideas and practices
Using language mathematically	*Coordinating both mathematical rigor and comprehension by students*	
• communicating clearly (15) • informal language/terminology and representations in real world to formal language in mathematics (9) • using mathematical notation and language and critiquing its use (4) • using precise language (28)	• choosing and developing useable definitions (4) • designing mathematically accurate explanations that are comprehensible and useful for students (2) • representing and presenting subject matter in ways that enable students to relate new learning to prior understanding and that help students develop metacognitive strategies (23) • using mathematically appropriate and comprehensible definitions (2)	• accessing knowledge of more complex geometric schemas, such as symmetry and congruence when the concept of square is the focus of discussion (8) • anticipating what students might do with it (3) • clarifying important links with others and structural knowledge (25) • Connecting a topic being taught to topics from prior or future years (4) • deciding which of several mathematical ideas has the most promise, and what to emphasize (12) • examining correspondences among representations (2) • having understanding of a concept/what teachers need for concept for the work of teaching (22) • identifying features of representations and numbers (21) • investigating the logical aspects (28) • investigating whether or not representations were equivalent (2) • linking representations to underlying ideas and to other representations (4) • making and explaining connections among mathematical ideas (12) • mapping between a physical or graphical model, the symbolic notation, and the operation or process (2) • recognizing that definitions contain other definitions (29) • recognizing what is involved in using a particular representation (4) • responding to students' "why" questions (4) • unpacking alternative definition (11) • unpacking ideas (10) • unpacking of ideas (2)

Developing horizon knowledge	Investigating mathematical concepts or solving mathematical problems without the context of teaching	Recognizing and developing ways of mathematical sense and reasoning in and for teaching	Overarching instruction for mathematical understanding
• a genetic approach to the notion (7) • connecting both across mathematical domains at a given level, and across time as mathematical ideas develop and extend (2) • considering these mathematical affordances (3) • historical investigation (5) • how a concept has developed (26) • Inspecting equivalencies (4) • maintaining essential features of a mathematical idea while simplifying other aspects to help students understand the idea (12) • making connections across mathematical domains, helping students build links and coherence in their knowledge (2) • situating a mathematical idea in a broader mathematical context (12) • Linking students with content across events (19)		• definitions are ultimately composed of undefined terms (29) • definitions-classify objects, identifying a category, identify how an object is distinguished from others in that category (27) • roles of definitions in proof and logical argument (18) • roles of stability of meanings (24) • teaching intellectual courage, intellectual honesty, and wise restraint (19)	• designing, adapting or selecting tasks (1) • adapting instruction according to the results (23) • preparing for a lesson (19) • adapting the mathematical content of textbooks (4) • adjusting teaching (15) • anticipate how mathematical ideas change and grow (2) • developing a lesson to promote achievement of lesson objectives (15) • explaining mathematical goals and purposes to parents (4) • making real world contexts accessible (6) • modify instructional materials as necessary (2) • plan lessons (23) • presenting appropriate lesson content (15) • provoking discordant thinking or errors in logic and argument intentionally (3) • provoking disequilibrium and error (3) • structuring the next steps in the learner’s development (3) • taking next steps (2) • anticipating the connections across lessons (19) • covering the curriculum (19)

Appendix B: Explanation with episodes for how I analyzed the data and how I built the conceptual framework

In this appendix, I briefly explain what I found from the data and how I built the conceptual framework based on the analysis of the data. I show two episodes from different classes of different teacher educators, which are elaborated in Chapter 4. In each episode, I explain my entering assumptions before I analyzed the data, and then show how my analysis affected the final outcome. Following this, I discuss these episodes together.

In Matthew's lesson on evaluating definitions, I assumed that probing and evaluating definitions would be the most important mathematical work of teaching. This was partly true. The following episode shows the first discussion that Matthew and his teachers had with the first definition: An even number is a number of the form 2k, where k is an integer.

Teacher2:	I only thought number one was like precise, but I might be wrong.
Matthew:	OK, Teacher2 says that this is precise. *Why do you think so?*
Teacher2:	Um, because through trial and error, yeah, through trial and error, and it also defines k and an integer and…
Matthew:	*Does it include all the even numbers we want to have in our set?*
Teacher2:	I believe it does, yeah.
Matthew:	*Can you convince us?*
Teacher2:	No. (Laughs)
Matthew:	Can somebody else help us? You said trial and error, so *you were trying some examples.*
Teacher3:	We did some like, you know, if you think, any number. Two times three is six, and two times minus three is minus six, still an even number.
Matthew:	So notice what Teacher3 is saying. So they tried six and negative six. So they tried a positive number and a negative number. What else do we need to try to be convinced that…
Teacher4:	Maybe have zero
Matthew:	Zero. So remember that we had these discussions? Negative numbers, zero, and positive numbers…
Teacher1:	All integers, not all numbers.
Matthew:	Only integers. That's a very good observation. It includes… See how easy it is to have these ambiguities? So *we really need to be careful about the terms that we are using.* (emphasis added)

My assumption was right because two kinds of mathematical work of teaching—evaluating a definition and probing concerns in the provided definition—were emphasized. But, to be more precise, what I did not anticipate was: the teacher educator and teachers worked with both evaluating a definition and probing concerns in the provided definition that are nested into using language mathematically and accessibly as in the mathematical work of teaching. In other words, while evaluating a definition, probing concerns in a definition, and using language mathematically and accessibly are mathematical work of teaching, in this lesson of teaching MKT, evaluating a definition and probing concerns in a definition were converged into using language mathematically and accessibly throughout one class of teaching MKT. I found the two different layers of mathematical work of teaching, and I differentiated then into zoomed-in and zoomed-out mathematical work of teaching. In fact, I found *nested relations* between these two in many classes. For example, in Sandy's lesson on hearing mathematical definitions, articulating and interpreting definitions were main mathematical work of teaching, which converged into providing and justifying mathematical and pedagogical decisions. Here is one more example.

In Emily's lesson on reasoning with mathematical definitions for explanations, I assumed that it would focus on probing a rule and creating statements to show how a rule works. The following episode happened in the close to the end of the lesson.

One teacher, Teacher27, goes to the board and explains why the units digit rule is correct in the general case of "a-b-c-d." What she writes on the board is shown in Figure 4.37. Emily asks the teachers to listen and try to understand Teacher27's explanation and to work out how a teacher would explain the units digit rule:

Teacher27: OK. So we're saying this (pointing at "c") is the tens digit, this (pointing at "d") is the ones digit. So we're separating it and trying to figure out "d." Now, we just throw out what you're saying because you could have 10 times "a-b" plus "d" (writing '10ab+d' on the board).

Emily: What happened to the "c"?

Teacher27: The "c" is (pointing at 10) – I screwed up.

Teacher28: You can write c.

(Teacher27 rewrites, "10(abc) + d")

Emily: OK, so that's splitting up, so can someone… so why can you do that? Can anyone talk about that? Actually… so why can you break it up like that?

Teacher27: Because the… I was just looking at the example over there. They have the 10 times all the places except for the ones digit.
Emily: Hmm. Does that make sense to people… have questions about that step?
Teacher29: Can you repeat that again? I didn't hear you.
Teacher27: OK, I'm sorry, I'll be louder. For… In that example we have 10 times all the digits including up to the tens digit, here and plus the units digit. Does that make sense?
Teacher29: OK.
Teacher27: So we have 10 times all the digits up to the unit digit plus the unit digit or the ones digit.
Emily: Units or ones, either one is fine.
Teacher27: OK, so then that's the same as (writing "2(5 x abc) + d") 2 times 5 times abc plus d. So then I have…
Emily: Do people understand what she's doing?
Teachers: 2 times 5 times abc. She is multiplying …
Teacher27: So this is 2 times an integer (writing "integer"). So, this here, 2 times an integer is an even number, so an even number – we have an even number here (pointing at "2(5 x abc)"), always, no matter what these numbers are – plus whatever the units digit is, will determine whether is. Because an even plus an even, it would be even, or an even plus… if this is odd, it would be an odd.

In this discussion, only one of my assumptions was correct. The class probed the rule, but they did not create statements to show how the rule works. Instead, they investigated why the rule works. In other words, the teacher educator and teachers unpacked the units digit rule as the mathematical work of teaching. Mathematically, this episode included four elements: (1) multiples of two defining an even number, (2) all tens, hundreds, and thousands are multiples of ten because of the place values in the base ten number system, (3) proofs of (an even number) + (an even number) = (an even number) and (an even number) + (an odd number) = (an odd number), and (4) mathematical reasoning to weave all these mathematical concepts. Furthermore, Emily also asked questions to check whether the other teachers were following Teacher27's explanation: "Does that make sense to people… have questions about that step?" "Do people understand what she's doing?" Emily nudged teachers into figuring out and probing Teacher27's explanation, into evaluating whether her reasoning was valid. Finally, the teacher educator and teachers worked with probing a rule, evaluating whether explanation about a rule is appropriate, and modifying and creating its explanation which are nested into unpacking mathematical ideas.

In these two episodes, I also found the same work of teaching mathematics: probing and evaluating. In the first episode, a definition was probed and evaluated, and in the second episode, an explanation about a rule was probed and evaluated. It shows logic that a same task of teaching mathematics can be applied into different objects of teaching mathematics. In this point, I realized it is a reasonable decision to identify probing and evaluating a representation as an important element of the work. In fact, later I found it in the Nellie's lesson on reasoning with mathematical definitions for proof. In other words, it means a two-dimensional-structure of zoomed-in mathematical work of teaching. It is a concise and logical approach to represent mathematical work of teaching, which I did not expect at all. Finally, from the data, I found various work of teaching mathematics and categorized it into five components: recognizing and articulating; probing, interpreting, and comparing; evaluating; selecting and modifying; and constructing. Moreover, I also found four mathematical objects in teaching: concept, property, and definition; algorithm, rule, and procedure; representation and tool; and proof. However, it does not mean that I found each work or each object separately. From the data, I found each example of the mathematical work of teaching, but I used a structure with both work of teaching mathematics and mathematical objects in teaching to represent mathematical work of teaching in the two dimensions.

In summary, the analysis of the data found the two different layers of mathematical work of teaching: zoomed-in and zoomed-out mathematical work of teaching. They have nested relations. Three kinds of zoomed-out mathematical work of teaching have been found through the data: providing and justifying mathematical and pedagogical decisions; situating and unpacking mathematics ideas and practices; and using language mathematically and accessibly. Furthermore, the analysis of the data revealed two dimensions of the zoomed-in mathematical work of teaching: mathematical objects in teaching and work of mathematics teaching. Mathematical objects in teaching include concept, property, and definition; algorithm, rule, and procedure; representation and tool; and proof. Work of teaching mathematics includes recognizing and articulating; probing, interpreting, and comparing; evaluating; selecting and modifying; and constructing.

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