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The Weitzenböck connection and time reparameterization in nonholonomic mechanics

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We show that the torsion of the Weitzenböck connection is responsible for the fictitious pseudogyroscopic force experienced by a general mechanical system in a noncoordinate moving frame. In particular, we show that for the class of mechanical systems subjected to nonintegrable constraints known as non-abelian nonholonomic Chaplygin systems, the constraint reaction force directly depends on this torsion. For these Chaplygin systems, we show how this torsional force can in some cases be removed by an appropriate choice of frame depending on a multiplier $f(q)$, linking these results to the process of Chaplygin Hamiltonization through time reparameterization. Lastly, we show that the cyclic symmetries of f in some cases lead to the existence of momentum conservation laws for the original nonholonomic system and illustrate the results through several examples. © *2011 American Institute of Physics*. [doi:10.1063/1.3525798]

I. INTRODUCTION

Elie Cartan's method of moving frames[10] has a long history of applicability in mechanics,[3,34] nonholonomic mechanics,[3,11] and general relativity.[36] As such, it represents a type of central node in a network linking these fields together. However, historical divergences exist which have prevented new results and interpretations based on the method from traveling between fields. With respect to our purposes here, the most interesting divergence is centered around Einstein's search for a unified field theory in the late 1920s and early 1930s.

Around this time period (overlapping with the development of Cartan's method of moving frames), Einstein began[19] an attempt to develop a unified field theory of gravitation and electromagnetism modeled on what is now called a *Weitzenböck spacetime*[26,41] (for a comprehensive overview of the period of interest to us here, see Ref. 42). These are Riemann–Cartan manifolds[26,41] equipped with a zero curvature yet nonzero torsion *Weitzenböck connection*[46] defined in terms of a moving frame. Such spacetimes possess *absolute parallelism, teleparallelism*, or *distant parallelism* (see Box 15.8.4 of Ref. 20 for a discussion), allowing the path independent parallel transport of vectors. Although Einstein abandoned his attempt at using moving frames to develop his theory, the fruitful discussions surrounding the method of moving frames and the associated Weitzenböck connection[12,42] led to the discovery that general relativity itself can be reformulated in terms of the Weitzenböck connection. In this formulation, the Einstein–Hilbert action is comprised of the Weitzenböck *torsion* square instead of the scalar *curvature*.[41] The reformulation is now called the *teleparallel equivalent* of general relativity[1] and has some advantages[41] over the conventional formulation.

The components of the Weitzenböck torsion are equivalent to the *objects of anholonomity*,[41,43] which are well known from the study of mechanical systems in a moving basis.[43] (If the configuration

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space of the mechanical system is a Lie group and the basis consists of the left-invariant vector fields, then the objects of anholonomity are the same as the *structure constants* of the Lie algebra.) With respect to Hamel's formulation of mechanics,[5,28,32] they are equivalent to Hamel's *transpositional symbols*.[30] Thus, although the objects of anholonomity, as well as their other counterparts above, are known widely throughout the mechanics literature, it seems that their connection to the Weitzenböck torsion has largely been ignored. In fact, several recent works[27,31,45] have either explicitly used the objects of anholonomity or studied mechanics on a Riemann-Cartan manifold without making reference to the Weitzenböck connection.

Our goal in this paper is twofold: to show that the Weitzenböck connection is responsible for the fictitious pseudogyroscopic force (see Ref. 18, footnote 11, Ref. 15, p. 106, and Ref. 30) appearing in the noncoordinate basis treatment of mechanics, and to apply the insights gained from this to the study of Chaplygin Hamiltonization.[22,24] With regard to the first objective, we will show that the *torsion* of the Weitzenböck connection is equivalent to the *curvature* of the Ehresmann connection defined by the constraint one-forms of a typical nonholonomic system. This establishes the constraint reaction force as a fictitious force arising from the anholonomy of the basis in a precise way through the Weitzenböck connection. As for the second objective, we will show that in some cases it is possible to *remove* this torsion by exploiting the fact that it arises from the adapted nonholonomic basis. This process will be shown to have the same effect as the reparameterization of time encountered in our earlier results[22,24] on Chaplygin Hamiltonization. Within this context, we will also show how associated symmetries in the multiplier correspond directly to momentum conservation laws in special cases.

After recalling some basic facts about the geometry of a Riemann–Cartan manifold in Sec. II, we introduce the Weitzenböck connection in Sec. II D and use it to illustrate the *torsion force* arising in a noncoordinate moving frame in Sec. III. We then consider the mechanics of nonholonomic systems in Sec. IV and rephrase our earlier conditions for Hamiltonization[22,24] in terms of the Weitzenböck torsion in Sec. V. Making use of the moving frame approach, we then show how the symmetries of the Hamiltonizing multiplier can in part lead to momentum conservation laws for the original nonholonomic system. We illustrate this by examples in Sec. VI.

II. THE RIEMANNIAN–CARTAN GEOMETRY OF MECHANICS

Consider an N dimensional Riemannian manifold Q with metric g equipped with a g-metric compatible affine connection ∇ possessing (in general) nonzero torsion, sometimes called a *Riemann–Cartan* manifold (Q, g, ∇).[26,41] In this section we briefly recall the relevant definitions and results from Riemann–Cartan geometry which we will use throughout the remainder of the paper.

A. Noncoordinate bases

As noted in Sec. I, many problems in mechanics and general relativity make use of so-called *noncoordinate* bases, also known as *nonholonomic* and *anholonomic* bases. Let us now recall the definition.[37]

Definition 1: A ***noncoordinate basis*** $\{e_a = e^i_a(q)\partial_i\}$ *is a frame of basis vectors obtained from the coordinate basis* $\{\partial_i\}$ *of* T_qQ *by a* $GL(N,\mathbb{R})$*-rotation of the basis* $\{\partial_i\}$ *preserving the orientation (thus,* $\{e^i_a(q)\} \in GL(N,\mathbb{R})$ *and we take* $det(e^i_a) > 0$*) and such that* $[e_a, e_b] \neq 0$ *for at least one pair of basis vectors, where* $[\cdot,\cdot]$ *is the Lie bracket of vectors fields on* Q.

Moreover, by introducing the coframe $\theta^a := E^a_i(q)dq^i$ such that $\theta^a(e_b) = \delta^a_b$, we can express the components of g with respect to the coframe:

$$g = g_{ij}dq^i \otimes dq^j = g(\partial_i, \partial_j)dq^i \otimes dq^j = g(E^a_i e_a, E^b_j e_b)\theta^a \otimes \theta^b = G_{ab}\theta^a \otimes \theta^b, \tag{2.1}$$

where $G_{ab} = g_{ij}e^i_a e^j_b$ are the components of the metric with respect to the noncoordinate dual basis and where hereafter we shall use the Latin indices $a, b, c, \ldots$ to index quantities in the nonholonomic basis and reserve the later Latin indices $i, j, k, \ldots$ for the coordinate basis, with both indices ranging from 1 to N.

Now, the Lie brackets of a nonholonomic basis define a new frame-dependent object:

$$[e_a, e_b] = -\Omega^c{}_{ab} e_c, \tag{2.2}$$

$$\text{with } \Omega^c{}_{ab} := -E^c_j\big(e^i_a \partial_i e^j_b - e^i_b \partial_i e^j_a\big) = -e^i_a e^j_b\big(\partial_j E^c_i - \partial_i E^c_j\big) = 2 e^i_a e^j_b \partial_{[i} E^c_{j]}, \tag{2.3}$$

where we have used the orthogonality of the basis and its dual to arrive at the equivalent expressions in (2.3).

The $\Omega^c{}_{ab}$ are known as the *objects of anholonomity*[43] and encode information about the anholonomy of the moving basis. For each fixed frame, they are the components of a type (1,2) tensor. In fact, from (2.3) one can show[20,44] that if $[e_a, e_b] = 0$, then the $\{e_a\}$ are in fact a *coordinate* or *holonomic basis* and the components e^i_a are the entries of the Jacobian matrix transferring the coordinate basis $\{e_i\}$ into another coordinate basis $\{e_a\}$. The objects of anholonomity are perhaps most familiar in the case when $Q = G$ is a Lie group and the e_a's are left-invariant vector fields. In this case, the $-\Omega^c{}_{ab}$ become the *structure constants* $C^c{}_{ab}$ of the Lie group.[3] We also note in passing that Ω is antisymmetric in its two lower indices.

B. Affine connections

In this section we focus on the local components of the affine connection ∇ with respect to the nonholonomic basis $\{e_a\}$ (for a modern introduction to the theory of connections, see Ref. 16). The connection ∇ takes two vector fields X, Y on Q to the vector field $\nabla_X Y$, the *covariant derivative* of Y with respect to X. In components, in a nonholonomic basis we have

$$\nabla_X Y = X^b\big(e_b(Y^a) + \Gamma^a{}_{bc} Y^c\big) e_a =: X^b \nabla_b Y^c e_c, \tag{2.4}$$

where $\nabla_a Y := \nabla_{e_a} Y$ and where the $X^b = E^b_i X^i$ are the components of X with respect to the nonholonomic basis. The Γ here are the *Christoffel symbols of the second kind* defined by $\nabla_b e_c = \Gamma^d{}_{bc} e_d$ or equivalently by $\Gamma^a{}_{bc} = G^{ad}\Gamma_{dbc}$, where $\Gamma_{abc} = g(e_a, \nabla_b e_c)$ are the *Christoffel symbols of the first kind*. In the special case where $\alpha : I \to Q$ is a curve on Q, if $\nabla_{\dot\alpha(t)}\dot\alpha(t) = 0$ then $\alpha(t)$ is called a *geodesic*. Locally, this yields the geodesic equation (A7) in Appendix.

The *torsion* and *curvature* tensors of ∇ are defined by

$$\begin{aligned} T(X, Y) &= \nabla_X Y - \nabla_Y X - [X, Y], \\ R(X, Y)Z &= \nabla_X \nabla_Y Z - \nabla_y \nabla_X Z - \nabla_{[X,Y]} Z, \end{aligned} \tag{2.5}$$

with local components given by (A1) and (A2) in Appendix, respectively. Specializing to the unique torsion-free and g-metric compatible *Levi-Civita connection*,[16] its covariant derivative ∇^g is defined by[20]

$$\begin{aligned} g(W, \nabla^g_U V) : &= \frac{1}{2}[U(g(V, W) + V(g(U, W)) - W(g(U, V))] \\ &+ \frac{1}{2}[g([U, V], W) - g([U, W], V) - g(U, [V, W])], \end{aligned} \tag{2.6}$$

which should be regarded as a definition of $\nabla^g_U V$ in terms of the right-hand side. The components Γ_{abc} and $\Gamma^a{}_{bc}$ can then be extracted by letting $W = e_a$, $U = e_b$, and$V = e_c$ in (2.6) and are given by (A3) and (A5) of Appendix, respectively.

In an anholonomic basis, where $\Omega \neq 0$ from (2.2), anywhere the Lie brackets of the basis appear there will now be a correction term to geometric objects associated with our affine connection. Indeed, this correction term depends on the objects of anholonomity (see (A1)–(A3)). The presence of these $\Omega^a{}_{bc}$ as an additional contribution can be understood physically as emerging from the fact that the $e^i_a(q)$, being dependent on q, vary from point to point and are obtained at each $q \in Q$ by a $\mathrm{GL}(N, \mathbb{R})$ *rotation* of the coordinate basis preserving the orientation. Such rotation generally *twists* the nonholonomic basis as q varies throughout Q and thus contributes, for example, to the torsion (for example, recall the *Frenet–Serret frame* carried by a curve in $\mathbb{R}^3$). Indeed, the "rotation" in the so-called *Ricci rotation coefficients* (A4) can be similarly understood; with respect to the Levi-Civita connection, (A3) tells us that they contribute to the rotation that a parallel transported basis vector in an anholonomic basis experiences relative to the original basis vector.[25,29]

C. Ehresmann connections

Suppose now that Q has a fiber bundle structure[3] with projection map $\pi : Q \to M$ and define the *vertical space* $V_q := \ker T_q\pi$.

Definition 2: The vector-valued one-form A on Q such that

(a) $A_q : T_q Q \to V_q$ is a linear map for each $q \in Q$ and
(b) $A(v_q) = v_q$ for all $v_q \in V_q$,

is known as an ***Ehresmann connection****.*

If we take as bundle coordinates $q^i = (r^\alpha, s^C)$, where hereafter $A, B, C, \ldots$ range from 1 to $K < N$ and $\alpha, \beta, \gamma, \ldots$ range from 1 to $\sigma = N - K$, then $\pi : (r^\alpha, s^C) \mapsto r$ and locally we can take A to be of the form

$$A = \theta^C(q)\frac{\partial}{\partial s^C}, \quad \text{where} \quad \theta^C(q) = ds^C + A^C_\alpha(r, s)dr^\alpha. \tag{2.7}$$

The reason for the special form for θ^C we have taken will become apparent when we consider the mechanics of nonholonomic systems in Sec. II A.

The associated *horizontal space* $\mathcal{D} := \ker A$ defines a σ-dimensional distribution on Q, from which we obtain the decomposition $T_q Q = \mathcal{D} \oplus V_q$. In fact, for any $v_q = v^i\partial_i \in T_q Q$ we have

$$\begin{aligned} v_q &= v^\alpha\left(\partial_\alpha - A^C_\alpha(r, s)\partial_C\right) + \omega^C\partial_C, \quad \text{where} \quad \omega^C = v^C + A^C_\alpha(r, s)v^\alpha, \\ &=: v^\alpha e_\alpha + \omega^C e_C, \end{aligned} \tag{2.8}$$

from which it follows that $v \in \mathcal{D}$ if and only if $v = v^\alpha e_\alpha$ or equivalently $\omega^C = 0$. Thus, the Ehresmann connection decomposes a vector $v_q \in T_q Q$ into the sum of its *horizontal part* v^h_q and its *vertical part* v^v_q, where

$$\begin{aligned} v^h_q &= v_q - A(v_q) = v^\alpha e_\alpha, \\ v^v_q &= A(v_q) = \omega^C e_C. \end{aligned} \tag{2.9}$$

Moreover, given a tangent vector $v_r \in T_r M$, where $r = \pi(q) \in M$, we define the *horizontal lift* of v_r to be the unique vector $v^h_r \in \mathcal{D}_q$ that projects to v_r under $T_q\pi$, $v^h_r = v^\alpha_r e_\alpha \mapsto v^\alpha\partial_\alpha = v_r$.

The *curvature* of A is the vertical-vector-valued two-form B on Q defined by its action on two vector fields X, Y on Q through

$$B(X, Y) = -A([X^h, Y^h]) \tag{2.10}$$

and has local components $B(X, Y)^C = -\theta^C([X^h, Y^h]) = B^C_{\alpha\beta}X^\alpha Y^\beta$ given by (A8) in Appendix. We can see clearly from (2.10) that the curvature of A is zero if and only if the horizontal distribution $\mathcal{D}$ is integrable (in the sense of Frobenius),[3] which is equivalent to the requirement that the moving basis defined in (2.8) be holonomic. However, if the curvature is nonzero, then (2.8) defines a nonholonomic frame according to our Definition 1. Since

$$[X^h, Y^h] = [X^\alpha e_\alpha, Y^\beta e_\beta] = -\Omega^C_{\alpha\beta}X^\alpha Y^\beta e_C \Rightarrow -\theta^C([X^h, Y^h]) = \Omega^C_{\alpha\beta}X^\alpha Y^\beta, \tag{2.11}$$

a straightforward computation of the Ω's in (2.11) based on the nonholonomic frame defined in (2.8) shows that $\Omega^C_{\alpha\beta} = -B^C_{\alpha\beta}$, i.e., *the curvature coefficients of A are the negatives of the objects of anholonomity*. We will return to this crucial observation below. We note in passing that although comparing (2.10) to (2.11) would seem to imply that the components should be the same, due to the difference in the signs of A^C_α in (2.7) and (2.8) they instead have the aforementioned relationship.

Lastly, we note that if we take $\pi : TQ \to Q$ to be the tangent bundle, then $\pi : (q, \dot{q}) \mapsto q$, so that the fibers (the tangent spaces) are coordinatized by $s = \dot{q}$. If we now require that the sum of two (local) horizontal sections be horizontal, then (see Ref. 3, p. 109)

$$A^c_b(q, \dot{q}) = \Gamma^a{}_{bc}(q)\dot{q}^c, \tag{2.12}$$

where all indices now run from 1 to N since Q and $T_q Q$ have the same dimension (N). We then define *geodesic motion* along a curve $\alpha(t) \in Q$ by *parallel transport* of its tangent vector $\dot{\alpha}(t)$ along the curve, i.e., $\dot{\alpha}^h(t) \in \mathcal{D}$ or $\dot{\alpha}^v(t) = 0$. We then see from (2.9) and (2.12) that $\omega^a = 0$ gives back the geodesic equations (A7).

D. The Weitzenböck connection

Let us now return to the discussion surrounding affine connections (see Sec. II B). As we discussed, the Levi-Civita connection "senses" the presence of an anholonomic basis through the objects of anholonomity. However, is there a connection which *produces* the torsional effects that Ω induces, yet has *zero* curvature? The answer is yes, and the connection is known as the *Weitzenböck connection* or, in the context of Riemann–Finsler geometry, the *crystallographic connection*.[2]

The Weitzenböck connection ∇^w arises by instead taking ∇ to have zero curvature and nonzero torsion (in contrast to the Levi-Civita connection). With respect to such a flat connection, the parallel transport of vectors would now be path *independent*. Thus, this condition is equivalent to the existence of n vector fields covariantly constant with respect to the connection ∇^w,[41]

$$0 = \nabla^w_i e^j_c = \partial_i e^j_c + e^k_c W^j{}_{ik}, \tag{2.13}$$

which gives the connection coefficients of the Weitzenböck connection,

$$W^i{}_{jk} = e^i_a \partial_j E^a_k. \tag{2.14}$$

The connection ∇^w is g-metric compatible[41] and also parallel transports the dual basis. This follows from the fact that since $\partial_i \left(e^j_a E^a_k\right) = \partial_i \delta^j_k = 0$, we have that

$$\nabla^w_j E^a_k = \partial_j E^a_k - E^a_i W^i{}_{jk} = \partial_j E^a_k - E^a_i \left(e^i_b \partial_j E^b_k\right) = \partial_j E^a_k - \delta^a_b \partial_j E^b_k = 0.$$

Moreover, using the standard transformation law for connections[16,37] shows that the components of the Weitzenböck connection in the moving frame, $W^a{}_{bc}$, vanish. From (A1) we then arrive at the components of the torsion of the Weitzenböck connection,

$$\overset{w}{T}{}^a{}_{bc} = \Omega^a{}_{bc}, \tag{2.15}$$

which shows that the *objects of anholonomity can be thought of as the components of the torsion of the Weitzenböck connection relative to the nonholonomic moving frame* $\{e_a\}$. Moreover, if we combine this with the analysis of the curvature of the Ehresmann connection in the bundle picture from Sec. II C, we see that *the torsion of the Weitzenböck connection W defined by a nonholonomic frame is the negative of the curvature of the Ehresmann connection A defined by its coframe*.

From (2.4) and using $W^a{}_{bc} = 0$, we see that the covariant derivative of a vector field Y in the direction of another vector field X associated with the Weitzenböck connection is particularly simple in the nonholonomic frame,

$$\nabla^w_X Y = X^a e_a(Y^b) e_b, \tag{2.16}$$

which is simply the directional derivative of Y in the direction of the vector field X, resembling the covariant derivative of flat space (the resemblance is not accidental, since, after all, W has zero curvature). Continuing the similarity, since W has zero curvature, the Weitzenböck connection makes possible path independent parallel transport, leading to teleparallelism (as discussed in Sec. I), as in flat space. However, the difference with a flat connection is of course that W possesses torsion.

III. GEOMETRIC MECHANICS

The torsion of W creates a closure failure in infinitesimal parallelograms and is intimately related to the *transpositional relations*[31,38,39] of mechanics. Let us now describe this and its relevance to our goals in this paper.

A. The transpositional relations in mechanics

We begin by first introducing the notion of a *virtual displacement.*[3]

Definition 3: Consider a trajectory $q(t) \in Q$ with fixed endpoints $q(a) = q_a$ and $q(b) = q_b$. A ***variation*** *of the trajectory is a smooth (C^2) mapping $w : [a, b] \times [-\delta, \delta] \mapsto Q$ such that*

(i) $w(t, 0) = q(t)$, $\forall\, t \in [a, b]$,
(ii) $w(a, \epsilon) = q_a$, $w(b, \epsilon) = q_b$.

A ***virtual displacement*** *corresponding to the variation of $q(t)$ is defined as $\delta q(t) = (\partial/\partial\epsilon) w(t, \epsilon)|_{\epsilon=0}$, with $\delta q(a) = \delta q(b) = 0$.*

The virtual displacement is thus a vector field defined along the curve $q(t)$. As such, it depends on the choice of frame as

$$\delta q(t) = \delta q^i(t)\partial_i = \delta q^i(t) e_i^a(q(t)) e_a =: \Sigma^a(q(t)) e_a, \tag{3.1}$$

where $\Sigma^a(q(t))$ are the images of $\delta q^\mu(t)$ in the moving frame (and thus they satisfy $\Sigma^c(q(b)) = \Sigma^c(q(a)) = 0$).

Denoting $d_t := d/dt$, we have $d_t \delta q^i(t) = \delta d_t q^i(t)$ from the definition of a virtual displacement. Thus, with respect to a torsionless connection, from (2.5) we have the identity $0 = \nabla_{\dot q}\delta q - \nabla_{\delta q}\dot q - [\dot q, \delta q]$. In other words, the parallelogram formed by the parallel transport of the tangent and virtual displacement vector fields closes (as it should for a torsionless connection). However, in a nonholonomic frame (which possesses torsion by (2.15), one would expect that such a parallelogram would not close. Indeed, this fact was recognized at least as early as Ref. 33. Using the results of the previous sections, along with the notation $\omega^a = E_i^a \dot q^i$ for the *quasivelocities* (the components of $\dot q$ with respect to the moving frame $\{e_a\}$), for the Weitzenböck connection we get

$$\begin{aligned} [\dot q, \delta q] &= \overset{w}{\nabla}_{\dot q}\delta q - \overset{w}{\nabla}_{\delta q}\dot q - \overset{w}{T}(\dot q, \delta q) \\ &= \omega^a e_a(\Sigma^b) e_b - \Sigma^a e_a(\omega^b) e_b - \overset{w}{T}{}^b{}_{cd}\, \omega^c \Sigma^d e_b \\ &= (d_t \Sigma^b) e_b - (\delta\omega^b) e_b - \overset{w}{T}{}^b{}_{cd}\, \omega^c \Sigma^d e_b. \end{aligned} \tag{3.2}$$

The expression (both left-hand side and right-hand side) in (3.2) is called a *transpositional relation* in mechanics.[39] Such relations were studied early on in the modern history of analytical mechanics (see Ref. 38 and references therein), but not from the viewpoint of Riemann–Cartan geometry. Early derivations of different forms of the equations of motion of nonholonomic systems created controversy, since the first line of (3.2) involves the "variation velocities" (the velocity vectors tangent to the variation curves), which are *a priori* undefined.[38,39] However, a proper definition was given in Ref. 38, along with a good discussion of the historically different choices for these variation velocities (see also Refs. 27, 35, and 45). For our purposes, we shall follow Hamel[28] and define these variation velocities to be $\dot q_w := \partial w(t, \epsilon)/\partial t$. Then, $[\dot q, \delta q] = 0$ and we arrive at the relevant transpositional relation in our case,

$$(d_t \Sigma^b) - \delta\omega^b = \overset{w}{T}{}^b{}_{cd}\, \omega^c \Sigma^d = -[\omega, \Sigma]^b, \tag{3.3}$$

where the last equality follows from the definition (2.2).

Equation (3.3) relates the transpositional relation used by Hamel to the torsion of W (through (2.15)). It should thus be no surprise that Hamel's own equations involve the Weitzenböck connection.

B. The Hamel equations

Consider the regular mechanical Lagrangian $L : TQ \to \mathbb{R}$ given by $L = T - V$, where $T(q, \dot{q}) = (1/2)g(\dot{q}, \dot{q}) = (1/2)g_{ij}\dot{q}^i\dot{q}^j$ and $V : Q \to \mathbb{R}$, and define the *Weitzenböck Lagrangian* $L^w(q, \omega) := L(q, \dot{q}^i = e^i_a(q)\omega^a)$. Then a straightforward computation using (3.3) (see Ref. 5) proves the following equivalence of critical action principles.

Proposition 1: The following statements are equivalent:

(i) The curve $q(t)$ is a critical point of the action functional

$$\int_a^b L(q, \dot{q})\, dt, \tag{3.4}$$

where we choose variations of $q(t)$ that satisfy $\delta q(a) = \delta q(b) = 0$.
(ii) The curve $(q(t), \omega(t))$ is a critical point of the action functional

$$\int_a^b L^w(q, \omega)\, dt, \tag{3.5}$$

with respect to the variations $\delta\omega$ induced by the variations $\delta q = \Sigma^a e_a$ through (3.3) given by

$$\delta\omega = d_t\Sigma + [\omega, \Sigma]. \tag{3.6}$$

Now, given the equivalence of the action principles in Proposition 1, we can derive the equivalent equations of motion in both the coordinate and nonholonomic frames. The latter are called the *Hamel equations,*[5,32] and we have the following analog of Proposition 1 (we remind the reader of the index conventions of Sec. II A).

Proposition 2: The following statements are equivalent to (i) and (ii) of Proposition 1:

(a) The curve $q(t)$ satisfies the Euler–Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \tag{3.7}$$

or written with respect to the Levi-Civita connection ∇^g in the coordinate basis $\{\partial_i\}$,

$$\nabla^g_{\dot{q}}\dot{q} = -grad(V) \quad or \quad \frac{d\dot{q}^i}{dt} + \left\{ {i \atop jk} \right\}\dot{q}^j\dot{q}^k = -g^{il}\partial_l V, \tag{3.8}$$

where $grad(V) = (g^{jk}\partial_k V)\partial_j$ is the gradient of V in the coordinate basis and the $\left\{ {i \atop jk} \right\}$ are the Christoffel symbols of the metric g.

(b) The curve $(q(t), \omega(t))$ satisfies the Hamel equations

$$\frac{d}{dt}\frac{\partial L^w}{\partial \omega^d} - e_d(L^w) = \frac{\partial L^w}{\partial \omega^a}\, \overset{w}{T}{}^a{}_{dc}\, \omega^c \tag{3.9}$$

or written in terms of the Levi-Civita connection ∇^g in the moving basis $\{e_a\}$,

$$\nabla^g_{\dot{q}}\dot{q} = -grad(V) \quad or \quad \frac{d\omega^a}{dt} + \Gamma^a{}_{bc}\omega^b\omega^c = -G^{ab}e_b(V), \tag{3.10}$$

where $grad(V) = (G^{ab}e_b(V))e_a$ is the gradient of V in the moving basis and the $\Gamma^a{}_{bc}$ are the components of the Levi-Civita connection from (A6).

Proof: The equivalence of (i) and (a) is a standard computation of the variational derivative of the action functional in (3.4),[3] along with a straightforward expansion of (3.7) in a coordinate basis (see Ref. 9). The equivalence of (ii) and (b) follows by once again varying (3.4) but instead with respect to the variations (3.6),

$$\begin{aligned}
\delta \int_a^b L(q,\dot q)\,dt &= \int_a^b \delta L^w(q,\omega)\,dt \\
&= \int_a^b \left[\frac{\partial L^w}{\partial q^i}\delta q^i + \frac{\partial L^w}{\partial \omega^a}\delta\omega^a\right] dt \\
&= \int_a^b \left[\frac{\partial L^w}{\partial q^i}e_d^i - \frac{d}{dt}\frac{\partial L^w}{\partial \omega^d} - \frac{\partial L^w}{\partial \omega^a}\overset{w}{T}{}^a{}_{cd}\,\omega^c\right]\Sigma^d\,dt,
\end{aligned} \tag{3.11}$$

where we have used (3.1) and (3.3). Upon assuming the independence of the variations Σ, this variational derivative vanishes if and only if the Hamel equations (3.9) are satisfied (we have used the antisymmetry of Ω in (3.9)).

Lastly, by using the fact that $L^w(q,\omega) = (1/2)G_{ab}\omega^a\omega^b - V(q)$ to compute (3.9) we arrive at

$$G_{da}\frac{d\omega^a}{dt} + G_{da}\begin{Bmatrix} a \\ bc \end{Bmatrix}\omega^b\omega^c - G_{be}\overset{w}{T}{}^e{}_{dc}\,\omega^b\omega^c = -e_d(V), \tag{3.12}$$

where the $\left\{{a \atop bc}\right\}$ are the Christoffel symbols of the moving basis given by (A6). Multiplying (3.12) by the inverse G^{fd}, by (A3), and the antisymmetry of T it then follows that $\Gamma^a{}_{bc}\omega^b\omega^c = \left(\left\{{a \atop bc}\right\} - T_b{}^a{}_c\right)\omega^b\omega^c$, where $T_b{}^a{}_c = G_{bd}G^{ae}\overset{w}{T}{}^d{}_{ec}$. Using this in (3.12) then yields (3.10). □

Let us now discuss the insights gained from (3.9). To better illustrate our point, let us specialize to the $V = 0$ case and assume we have chosen a *g-orthonormal* moving basis. Then from (A3) we have $\Gamma^a{}_{bc} = \gamma^a{}_{bc}$, so that $\Gamma^a{}_{bc}\omega^b\omega^c = -\Omega_b{}^a{}_c\omega^b\omega^c$. Since $\left\{{a \atop bc}\right\}$ vanishes by orthogonality, it then follows from (3.12) that

$$\dot p_b = \overset{w}{T}{}^a{}_{bc}\,p_a\omega^c, \tag{3.13}$$

where $p_a = \partial L^w/\partial\omega^a = \delta_{ab}\omega^b$. In a coordinate basis, the time rate-of-change of the momenta of an unconstrained mechanical system is determined by the metric, as (3.7) shows. However, in a noncoordinate basis, as (3.13) shows, they are here determined by the torsion of the Weitzenböck connection. This is precisely the situation encountered in, for example, the Euler equations on $Q = SO(3)$.

In general, Proposition 2 shows that *the geometric origin of the fictitious force arising in the Hamel equations (3.9) is precisely the torsion of the Weitzenböck connection*. Physically, this should not be surprising, since by the very definition of the nonholonomic frame $\{e_a\}$ from Sec. II A, an observer utilizing the moving frame employed in (3.10) would experience the additional *rotational* forces from the point-to-point rotation of the frame $\{e_a\}$ quantified by the Ω term in (3.12). Moreover, as we shall see in the context of nonholonomic mechanics in Section IV B, the Weitzenböck torsion will also be responsible for the "pseudogyroscopic" constraint reaction force (see footnote 11 of Ref. 18, p. 106 of Ref. 15, and Ref. 30) present in the reduced equations of motion.

IV. NONHOLONOMIC MECHANICS

Suppose now that we impose linear, homogeneous nonholonomic constraints on our mechanical system and let $\mathcal{D}$ be the nonintegrable distribution describing these constraints. Locally, $\mathcal{D}$ is given by

$$\mathcal{D} = \{\dot q \in TQ | \theta^C(\dot q) = 0\},$$

where the θ^C are the constraint one-forms and where we refer the reader to the index conventions of Sec. II C.

Now, let $\{\tilde{e}_\alpha\}$ be a basis for $\mathcal{D}$ and extend this to a basis $\{\tilde{e}_\alpha, \tilde{e}_C\}$ for TQ such that the $\{\tilde{e}_C\}$ spans the g-orthogonal complement to $\mathcal{D}$. We can then decompose $\dot{q} \in TQ$ into $\dot{q} = \omega^\alpha \tilde{e}_\alpha + \omega^C \tilde{e}_C$. The nonholonomic mechanics is then derived by projecting the Hamel equations from Proposition 2 onto $\mathcal{D}$, resulting in a *nonholonomic (affine) connection*.

A. The nonholonomic connection

Let $P : TQ \to \mathcal{D}$ and $Q : TQ \to \mathcal{D}^\perp$ be complementary g-orthogonal projectors. Then the affine connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y = \nabla^g_X Y + \left(\nabla^g_X Q\right)(Y) \tag{4.1}$$

is called the *constrained affine connection*[7,8,15] or the *nonholonomic connection*.[17] It can be verified that this is indeed an affine connection[15,17] and that it is metric with respect to the metric $\overline{g}$ on $\mathcal{D}$ induced from g (Ref. 17) (we will denote the components of this induced metric by $\overline{G}_{\alpha\beta}$).

The nonholonomic equations of motion are given by projecting (3.9) or (3.10) onto $\mathcal{D}$ through P (or equivalently by setting $\omega^C = 0$) and are given (in our notation) by

$$\overline{\nabla}_{\dot{q}}\dot{q} = -P(grad(V)), \quad \text{or} \quad \frac{d\omega^\alpha}{dt} + \Gamma^\alpha{}_{\beta\gamma}\omega^\beta\omega^\gamma = -\overline{G}^{\alpha\beta}\tilde{e}_\beta(V), \tag{4.2}$$

where $\Gamma^\alpha{}_{\beta\gamma} = \overline{G}^{\alpha\delta}\Gamma_{\delta\beta\gamma}$ (recall (A3)), with $\overline{G}^{\alpha\beta}$ the matrix inverse of the submatrix $G_{\alpha\beta}$ of G. These equations appear in Ref. 27, and if we orthogonalize the basis $\{\tilde{e}_\alpha\}$, then (4.2) reproduces the equations in Ref. 8. Moreover, in analogy with (3.13), if we further assume that $V = 0$, then we can express (4.2) as

$$\dot{p}_\beta = \overset{w}{T}{}^a{}_{\beta\gamma}\ p_a\omega^\gamma. \tag{4.3}$$

The preceding equations show that whereas physically the main force generating the nonholonomic dynamics is the constraint reaction force arising from the nonholonomic constraints, *geometrically the constraint force is nothing but the (projected) torsion force arising from the Weitzenböck connection defined by the nonholonomic moving (and rotating) basis* $\{\tilde{e}_\alpha\}$. Incidentally, Eq. (4.3) are a special case of the generalized Suslov equations (see Ref. 3, Section 8.6).

The insight gained from (4.3) can now be used to recharacterize the process of *Chaplygin Hamiltonization.*[22,24] Before doing so, let us now define the particular class of nonholonomic systems known as *non-abelian Chaplygin* systems.[3,15,30]

B. Non-abelian Chaplygin systems

Non-abelian Chaplygin systems on Q are described by the triple $(L, G, \mathcal{D})$, where L is a (regular) mechanical Lagrangian (see Sec. III B), and G is a Lie group acting freely and properly on Q which leaves L and the constraints described by the nonintegrable distribution $\mathcal{D}$ invariant. They are characterized by the splitting of the tangent spaces according to $T_qQ = T_q\text{Orb}(q) \oplus \mathcal{D}$, where $\text{Orb}(q)$ is the group orbit through q. Therefore, there is a unique principal connection $\mathcal{A} : TQ \to \mathfrak{g}$ on the bundle $\pi : Q \to M := Q/G$ whose horizontal space is $\mathcal{D}$.[3] In the special case when $G = \mathbb{R}^k$ or $G = \mathbb{S}^k$, we call the system *abelian* Chaplygin. This case corresponds to the classical Chaplygin systems considered by Chaplygin himself.[15]

Denote $r = \pi(q)$ and $\xi = g^{-1}\dot{g} \in \mathfrak{g}$ (with $g \in G$), so that $q \in Q$ can be written as $q = (r^\alpha, g^A)$, with the index conventions of Sec. II C. We can then decompose $\dot{q} \in TQ$ into its horizontal and vertical parts as in (2.8) and (2.9), where the moving frame $\{\tilde{e}_a\} = \{\tilde{e}_\alpha, \tilde{e}_A\}$ is now given by

$$\tilde{e}_a = \tilde{e}^i_a\partial_i = \begin{pmatrix} I_{\sigma\times\sigma} & -\mathcal{A}^A_\alpha g^B_A \\ 0_{K\times\sigma} & g^B_A \end{pmatrix} \begin{pmatrix} \partial_\alpha \\ \partial_B \end{pmatrix}. \tag{4.4}$$

The $\{\tilde{e}_\alpha\}$ form a nonholonomic basis for $\mathcal{D}$, and the nonholonomic constraints are given simply by $\omega^B = 0$, where $\omega^B = \xi^B + \mathcal{A}^B_\alpha(r)\dot{r}^\alpha$ in analogy to (2.8). Then, as in Sec. IV A, we arrive at the equations of motion by projecting the Eq. (3.9) or (3.10) onto $\mathcal{D}$ through P.[3] Anticipating the appearance of the $\Omega^{a}{}_{\beta\gamma}$ as in (4.3), a straightforward computation shows that $\Omega^{\alpha}{}_{\beta\gamma} = 0$, while

$$[\tilde{e}_\alpha, \tilde{e}_\beta] = -\Omega^D_{\alpha\beta}\tilde{e}_D =: \mathcal{B}^D_{\alpha\beta}\tilde{e}_D, \tag{4.5}$$

$$\text{where} \quad \mathcal{B}^D_{\alpha\beta} = \frac{\partial \mathcal{A}^D_\alpha}{\partial r^\beta} - \frac{\partial \mathcal{A}^D_\beta}{\partial r^\alpha} - C^D_{AB}\mathcal{A}^A_\alpha\mathcal{A}^B_\beta, \tag{4.6}$$

and where the C^D_{AB} are the structure constants of $\mathfrak{g}$, $[\tilde{e}_A, \tilde{e}_B] = C^D_{AB}\tilde{e}_D = -\Omega^D{}_{AB}\tilde{e}_D$.

In agreement with the relationship we found in Sec. II C, the $-\Omega^C{}_{\alpha\beta}$ are the local components of the curvature of the principal connection $\mathcal{A}$.[3] In fact, a quick glance at Ref. 5, Eqs. (2.19)–(2.21) shows that in the general case of a mechanical system with symmetry (nonholonomic or not), the nontrivial forces arising in the equations of motion are just the various components of the torsion of the Weitzenböck connection. We note that in our notation the function $P(L^w) = L^w_c$ is the *constrained reduced* Lagrangian $l_c : TM \to \mathbb{R}$ of Ref. 5.

V. TIME REPARAMETERIZATION AND CHAPLYGIN HAMILTONIZATION

Given the fact that one can choose any nonholonomic frame to express the dynamics of a mechanical system, one may wonder if there are particularly useful choices which simplify the equations of motion. A popular avenue is to try and *Hamiltonize* the nonholonomic system through Chaplygin's reducibility theorem.[6,13,14,21]

As noted in Refs. 22 and 24, one can view Chaplygin's time reparameterization $d\tau = f(q)dt$ from Sec. I in a different way as follows: we have $\dot{q} = dq/dt = f(q)(dq/d\tau) =: f(q)\omega$, which defines the quasivelocities $\omega = q'$ on Q (for a recent discussion of quasivelocities in nonholonomic mechanics, see Ref. 5), where we will henceforth denote differentiation against τ with a prime. These quasivelocities define a moving frame, as discussed in Sec. II A. Thus, we can now study Chaplygin Hamiltonization within the framework of the nonholonomic moving frames discussed in the earlier sections.

A. Chaplygin Hamiltonization

In view of this interpretation of time reparameterization, consider the non-abelian Chaplygin nonholonomic systems from Sec. IV B and choose the nonholonomic basis $\{e_a = f\tilde{e}_a\}$, where $f = f(r)$ is a nowhere zero smooth function, with $\{\tilde{e}_a\}$ the nonholonomic basis from Sec. II A. Since Ω now depends on f, we can decompose it as follows:

$$\begin{aligned} -\Omega^c{}_{ab}e_c &= [e_a, e_b] \\ &= [f\tilde{e}_a, f\tilde{e}_b] = \frac{1}{f}\big(e_a(f)\delta^c_b - e_b(f)\delta^c_a\big)e_c - f\tilde{\Omega}^c{}_{ab}e_c, \end{aligned} \tag{5.1}$$

which relates the $\Omega^c{}_{ab}$ to the $\tilde{\Omega}^c{}_{ab}$ (here $\tilde{\Omega}$ is the object of anholonomity corresponding to the original basis $\{\tilde{e}\}$), which do not depend on f.

To obtain the reduced nonholonomic equations of motion, we now project the equations (3.9) onto $\mathcal{D}$ using the projector P from Sec. IV A. Rather than repeating the calculation, we note that the result is given simply by setting $\omega^C = 0$ in (3.12), to obtain

$$G_{\alpha\rho}\frac{dr'^\rho}{dt} + \Gamma_{\alpha\beta\gamma}r'^\beta r'^\gamma = G_{\beta C}\,\overset{w}{T}{}^C{}_{\alpha\gamma}\,r'^\beta r'^\gamma - e_\alpha(V), \tag{5.2}$$

where $G_{\alpha\beta} = f^2\overline{G}_{\alpha\beta}$. Then, we have the following.

Theorem 3: *The reduced nonholonomic equations of motion of a non-abelian Chaplygin nonholonomic system are Lagrangian after the time reparameterization $d\tau = f(r)dt$, with*

$L_c^\tau(r, r') = (1/2)G_{\alpha\beta}r'^\alpha r'^\beta - V(r)$, *if and only if there exists a nowhere zero smooth* $f(r)$ *such that*

$$G_{\beta C} \overset{w}{T}{}^C{}_{\alpha\gamma} + G_{\gamma C} \overset{w}{T}{}^C{}_{\alpha\beta} = 0, \quad \textit{for all } \alpha, \beta, \gamma. \tag{5.3}$$

Moreover, the sufficient condition is simply $G_{\beta C} \overset{w}{T}{}^C{}_{\alpha\gamma} = 0$.

Proof: We can rewrite (5.2) more suggestively as

$$G_{\alpha\rho}\frac{dr'^\rho}{dt} + \frac{f}{2}\big[\partial_\beta(G_{\gamma\alpha}) + \partial_\gamma(G_{\beta\alpha}) - \partial_\alpha(G_{\beta\gamma})\big]r'^\beta r'^\gamma = G_{\beta C} \overset{w}{T}{}^C{}_{\alpha\gamma}\, r'^\beta r'^\gamma - f\partial_\alpha(V), \tag{5.4}$$

where we have used the fact that $e_\alpha(f) = f\tilde{e}_\alpha(f) = f\partial_\alpha(f)$ due to the fact that f only depends on r, and similarly for V and G. On the other hand, the Euler–Lagrange equations of L_c^τ are

$$G_{\alpha\rho}\frac{dr'^\rho}{dt} + \frac{f}{2}\big[\partial_\beta(G_{\gamma\alpha}) + \partial_\gamma(G_{\beta\alpha}) - \partial_\alpha(G_{\beta\gamma})\big]r'^\beta r'^\gamma = -f\partial_\alpha(V). \tag{5.5}$$

Clearly, (5.4) and (5.5) are equivalent if and only if (5.3) is satisfied, although a sufficient condition is that $G_{\beta C} \overset{w}{T}{}^C{}_{\alpha\gamma} = 0$. □

Written out explicitly, the condition (5.4) reads

$$\frac{\partial f}{\partial r^\gamma}G_{\beta\alpha} + \frac{\partial f}{\partial r^\beta}G_{\gamma\alpha} - 2\frac{\partial f}{\partial r^\alpha}G_{\beta\gamma} = f\big(G_{\beta C}\mathcal{B}^C_{\alpha\gamma} + G_{\gamma C}\mathcal{B}^C_{\alpha\beta}\big), \tag{5.6}$$

where we note that one can replace G by $\overline{G}$ everywhere (since they differ by a multiplicative factor of f^2). Equation (5.6) matches our earlier necessary and sufficient conditions for Chaplygin Hamiltonization,[24] and we also note that their solubility implies that the original nonholonomic system conserves measure with density $f^{\sigma-1}$.[22,24] In the special case that $f = const.$ solves (5.6), then the nonholonomic system is called *conditionally variational*.[23]

Now, given the discussion surrounding (4.3), we can now geometrically interpret the condition (5.3). Chaplygin Hamiltonizable systems are ones for which we can eliminate the fictitious Weitzenböck torsion force arising from the anholonomy of the basis by inducing a counter force (arising from (5.1)) through a reparameterization of time. This reparameterization is also physically interesting, since it relates time measured between the nonholonomic and Hamiltonized frames (see Example 5.3 below), loosely analogous to the distinction between "proper time" and "universal time" in general relativity.

1. Conservation laws through moving frames

Let us first begin with a simple observation. Theorem 3 provides us with a Hamiltonian form of the constrained nonholonomic mechanics for a non-abelian Chaplygin system (provided a solution f to (5.3) exists). As such, one may then apply any of the well-known results from the unconstrained theory of Hamiltonian mechanics to this Hamiltonized system. In particular, one can investigate the integrability of the original nonholonomic system by applying Hamilton–Jacobi theory to the Hamiltonized system (this is currently work in progress[40]). In addition, if this Hamiltonized system is invariant under the action of a Lie group G (note that we no longer need to worry about the constraints), then one may perform a Marsden–Weinstein reduction[3] to a lower-dimensional system and in the process acquire momentum conservation laws resulting from the corresponding momentum equations (we will see this below in Example 5.1).

The process of finding the symmetry groups which leave L_c^τ invariant can become complicated if the Hamiltonized form of the restriction of the metric g to $\mathcal{D}$ (the $G_{\alpha\beta}$ in our notation) is complicated. However, since according to (2.1) these components depend on the choice of our original moving frame, we can eliminate this extra layer of complexity by g-orthonormalizing the basis $\{\tilde{e}\}$ from (4.4). After doing so, the Hamiltonized constrained reduced Lagrangian L_c^τ would only depend on r through f and V, making the search for symmetry groups easier. In the special case when there is no potential ($V = 0$), one could then explicitly relate the symmetries of f to the existence of momentum conservation laws for the original nonholonomic system.

Let us make our point more explicit by considering the special case of cyclic symmetries. Suppose now that we g-orthonormalize the moving frame $\{\tilde{e}\}$ from (4.4). Then, assuming there exists a new f satisfying the corresponding (5.3), we have the following result.

Proposition 4: Consider a non-abelian Chaplygin nonholonomic system for which we have g-orthonormalized the basis (4.4), and suppose that there exists f satisfying the corresponding Hamiltonization conditions (5.3). Then, if $u_\alpha(f) = u_\alpha(V) = 0$, we have the conservation law

$$\chi^\alpha = k^\alpha / f^2, \tag{5.7}$$

where $k^\alpha \in \mathbb{R}$ and χ^α is the α-component of the velocity in the g-orthonormalized basis.

Proof: Let $\{u_\alpha\}$ denote the g-orthogonalized $\{\tilde{e}_\alpha\}$. The kinetic energy metric is now $f^2\delta_{\alpha\beta}$ due to the orthonormalization. Thus, the nonholonomic equations of motion are then given by (5.2), with $G_{\alpha\rho}$ replaced by $f^2\delta_{\alpha\rho}$ and $G_{\beta C}\Omega^C{}_{\alpha\gamma}$ replaced by $f^2\delta_{\beta C}\hat{\Omega}^C{}_{\alpha\gamma}$ (we will denote the objects of anholonomity of the g-orthonormalized basis by $\hat{\Omega}$). Now, since we assume that there exists an f satisfying the new Hamiltonization conditions (the conditions (5.3) with the aforementioned replacements of G and Ω), then the reduced dynamics of the nonholonomic system are Lagrangian after the time reparameterization $d\tau = f(r)\, dt$. Since the αth Hamiltonized nonholonomic equation is given by

$$\frac{d}{d\tau}(f^2\chi^\alpha) - u_\alpha\left(\frac{f^2}{2}\delta_{\beta\gamma}\chi^\beta\chi^\gamma - V\right) = 0, \tag{5.8}$$

this leads directly to the conservation law (5.7) if $u_\alpha(f) = u_\alpha(V) = 0$. □

We remark that since the new quasivelocity $\chi^\alpha = \chi^\alpha_\beta r'^\beta = (1/f)\chi^\alpha_\beta \dot{r}^\beta$, the conservation law (5.7) can easily be written in terms of $\dot{r}$ (or the nonholonomic momentum for that matter). Moreover, if $V = 0$, then Proposition 4 provides a direct link between the symmetries of the Hamiltonizing multiplier f and the momentum conservation laws of the original nonholonomic system.

For reference purposes, we note that applying Proposition 4 requires the existence of a solution $f(r)$ to the g-orthonormalized version of (5.3), given by:

$$\left(u^\nu_\gamma\delta^\beta_\alpha + u^\nu_\beta\delta^\gamma_\alpha - 2u^\nu_\alpha\delta^\beta_\gamma\right)\frac{\partial f}{\partial r^\nu} = -f\left(\hat{\Omega}^\beta{}_{\alpha\gamma} + \hat{\Omega}^\gamma{}_{\alpha\beta}\right), \tag{5.9}$$

where the u^α_β are the components of the g-orthonormalized basis $\{u\}$ and the $\hat{\Omega}$ are its associated objects of anholonomity, computed from (2.3).

VI. EXAMPLES

A. The vertical rolling disk

Consider the nonholonomic vertically rolling disk pictured in Fig. 1 with configuration space $Q = \mathbb{R}^2 \times S^1 \times S^1$ and parameterized by the coordinates (x, y, θ, φ), where (x, y) is the position of the center of mass of the disk, θ is the angle that a point fixed on the disk makes with respect to the vertical, and φ is measured from the positive x-axis. This system has Lagrangian and constraints given by

$$\begin{aligned} L &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2, \\ &\dot{x} - R\cos\varphi\,\dot{\theta} = 0, \\ &\dot{y} - R\sin\varphi\,\dot{\theta} = 0, \end{aligned} \tag{6.1}$$

where m is the mass of the disk, R is its radius, and I, J are the moments of inertia about the axis perpendicular to the plane of the disk and about the axis in the plane of the disk, respectively.

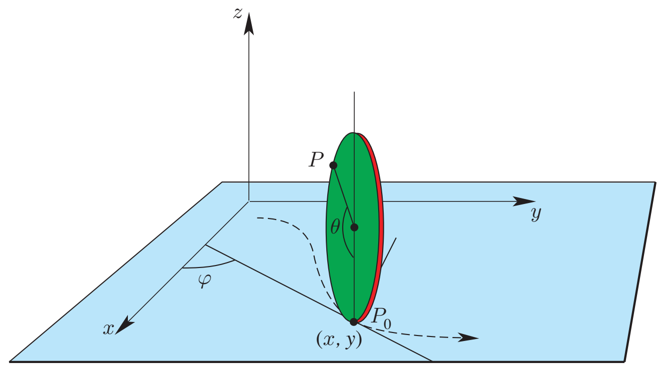

FIG. 1. (Color online) The vertically rolling disk.

For simplicity, let us take $m = I = J = 1$. Then, since (6.1) is invariant under the additive action of $G = \mathbb{R}^2_{xy}$, the system is abelian Chaplygin according to Sec. IV B. The Ehresmann connection A from (2.7) describing the constraints is given by

$$A = (dx - \cos\varphi\, d\theta)\partial_x + (dy - \sin\varphi\, d\theta)\partial_y, \tag{6.2}$$

and its kernel is $\mathcal{D} = \mathrm{span}\{\partial_\varphi, \partial_\theta + \cos\varphi\partial_x + \sin\varphi\partial_y\}$. Thus, the moving frame (4.4) is given explicitly by $\{\tilde{e}\} = \{\tilde{e}_\alpha, \tilde{e}_C\}$, where

$$\begin{aligned} \{\tilde{e}_\alpha\} &= \{\tilde{e}_\varphi = \partial_\varphi, \tilde{e}_\theta = \partial_\theta + \cos\varphi\,\partial_x + \sin\varphi\,\partial_y\}, \\ \{\tilde{e}_C\} &= \{\tilde{e}_x = \partial_x, \tilde{e}_y = \partial_y\}. \end{aligned} \tag{6.3}$$

The nonzero components of the associated Weitzenböck torsion are

$$\overset{w}{T}{}^x{}_{\theta\varphi} = -\sin\varphi, \qquad \overset{w}{T}{}^y{}_{\varphi\theta} = -\cos\varphi,$$

and from this, a straightforward calculation shows that the right-hand side of (5.6) vanishes, implying that $f = const.$ is a solution. Thus, Theorem 3 tells us that the reduced nonholonomic equations are Lagrangian after the "reparameterization" by f with $L_c^\tau = (1/2) f^2(\theta'^2 + \varphi'^2)$. However, since in this case $f = const.$, one need not reparameterize at all (thus the vertical disk belongs to the special class of nonholonomic systems studied in Ref. 23). Moreover, since L_c^τ is cyclic in *both* θ and φ, then we immediately have the conservation laws $\dot{\theta} = k_\theta$ and $\dot{\varphi} = k_\varphi$. It is also interesting to note that in the standard literature (see Ref. 3, Section 5.6.1), the θ conservation law is induced by a horizontal symmetry while the φ is not, whereas here we have found both conservation laws directly.

With respect to Proposition 4, a MAPLE verification shows that the conditions (5.9) also have the solution $f = const.$ However, no new insight is gained since we have already arrived at the two conservation laws describing the reduced constrained dynamics.

B. The nonholonomic free particle

Consider a nonholonomically constrained free particle with unit mass (more details can be found in Ref. 3), and Lagrangian and constraint given by

$$\begin{aligned} L &= \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \\ \dot{z} + x\dot{y} &= 0. \end{aligned} \tag{6.4}$$

The system is Chaplygin Hamiltonizable with $f(x) = (1 + x^2)^{-1/2}$ solving (5.3) (see also Ref. 24), which are given in this case by

$$\frac{\partial f}{\partial x}(x, y) = -\frac{x}{1 + x^2} f(x, y), \qquad \frac{\partial f}{\partial y}(x, y) = 0.$$

It then follows that $L_c^\tau = (1/2) f^2(x'^2 + (1 + x^2) y'^2)$. From this, we again see that since L_c^τ is cyclic in y, then we have the associated conservation law $y' = k_y$, where k_y is a constant. Using the fact that $\dot{y} = f y'$ from Sec. II A, we can rewrite this conservation law as $\sqrt{1 + x^2}\,\dot{y} = k_y$.

With respect to Proposition 4, a straightforward calculation of the g-orthonormalized basis $\{u\}$ gives

$$\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+x^2}} & -\frac{x}{\sqrt{1+x^2}} \\ 0 & \frac{x}{\sqrt{1+x^2}} & \frac{1}{\sqrt{1+x^2}} \end{pmatrix} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix},$$

and the only non-zero objects of anholonomity (or equivalently Weitzenböck torsion components) are $\hat{\Omega}^y{}_{xz} = -1/(1 + x^2)$ and $\hat{\Omega}^z{}_{xy} = 1/(1 + x^2)$. Moreover, since only the objects of anholonomity with all Greek indices (x and y in our case) enter into the right-hand sides of (3.9) and (5.9), it follows immediately that $f = const.$ is a solution to (5.9). Thus, the g-orthonormalizing basis has removed the need for Hamiltonization (or, equivalently, our system is Hamiltonian in the g-orthonormalized basis).

As in the example above, since f is independent of *both* x and y and the potential $V = 0$, then by Proposition 4 it follows that, with $\chi^x = \dot{x}$ and $\chi^y = \sqrt{1 + x^2}\,\dot{y}$, we have the conserved quantities $\chi^x = k_x$ and $\chi^y = k_y$ where k_x and k_y are constants. Thus, we not only recover the y conservation law already extracted, but in addition the x one as well.

Now, since the reduced dynamics in this case are completely integrable by using the aforementioned conservation laws, we can explicitly investigate the time reparameterization $d\tau = f(x)dt$. Let us assume that $\alpha_x := \dot{x}(0) \neq 0$ and $\alpha_y := \dot{y}(0) \neq 0$. Then the solutions to the reduced dynamics are given by

$$x(t) = \alpha_x t, \qquad y(t) = \frac{\alpha_y}{\alpha_x} \ln(x(t) + \sqrt{1 + (x(t))^2}), \tag{6.5}$$

where we have chosen, without loss of generality, the initial conditions $x_0 = 0$ and $y_0 = 0$. We can now explicitly find the reparameterization by integrating $d\tau = (1 + (x(t))^2)^{-1/2}\, dt$. We arrive at

$$\tau(t) = \frac{1}{\alpha_x} \ln\left(x(t) + \sqrt{1 + (x(t))^2}\right) + \tau_0, \tag{6.6}$$

where $\tau_0 = \tau(t = 0)$, but we shall set $\tau_0 = 0$ henceforth for simplicity. Using $x(t)$ from (6.5) in (6.6) allows us to invert (6.6) and find $t(\tau)$,

$$t(\tau) = \frac{1}{\alpha_x} \sinh(\alpha_x \tau). \tag{6.7}$$

Using (6.6) and (6.7) in (6.5) then allows us to express the reduced dynamics in terms of τ:

$$x(\tau) = \sinh(\alpha_x \tau), \qquad y(\tau) = \alpha_y \tau. \tag{6.8}$$

Although we could have found the explicit reparameterizations (6.8) directly from the Euler–Lagrange equations of L_c^τ (in fact, the linearity of $y(\tau)$ follows directly from the quadrature of the conservation law $y' = k_y$ from above), by directly employing the reparameterization in (6.5), we have managed to find the explicit relationship between the constants of motion for the dynamics expressed in the two times.

C. The knife edge on an inclined plane

Consider a plane slanted at an angle α from the horizontal and let (x, y) denote the position of the point of contact of the knife edge with respect to a fixed Cartesian coordinate system on the plane (see Ref. 3, Section 1.6). Moreover, let φ represent the orientation of the knife edge with respect to the xy-axis. The Lagrangian and constraints are then given by

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{\varphi}^2\right) + x \sin\alpha,$$
$$\dot{y} - \tan\varphi\dot{x} = 0, \tag{6.9}$$

where we have set all the parameters (mass, moment of inertia, and the gravitational acceleration) equal to one for simplicity.

Now, the only nontrivial equations in (5.3) are

$$\frac{\partial f}{\partial \varphi}(x,\varphi) = -\tan\varphi f(x,\varphi), \quad \frac{\partial f}{\partial x}(x,\varphi) = 0, \tag{6.10}$$

whose solution is spanned by $f(\varphi) = \cos\varphi$. Now, although f has zeroes, we will nonetheless see that this will not prevent the application of the results obtained thus far. Indeed, we have that $L_c^\tau = (1/2) f^2(\sec^2\varphi x'^2 + \varphi'^2) + x\sin\alpha$. Although there are no cyclic symmetries here, the Euler–Lagrange equations for L_c^τ are given by

$$x'' = \sin\alpha, \qquad \varphi'' = \tan\varphi\varphi'^2, \tag{6.11}$$

from which it follows that $x' = \sin\alpha\tau + K_1$ and $\varphi' = K_2\sec\varphi$, where K_1, K_2 are arbitrary constants, and here we take $K_2 \neq 0$. Using $\dot{q} = fq'$ from Sec. V converts these into

$$\dot{x} = \sin\alpha\cos\varphi\tau + K_1\cos\varphi, \qquad \dot{\varphi} = K_2, \tag{6.12}$$

and without the loss of generality, taking $\varphi(0) = 0$ gives $\varphi(t) = \omega t$, where we have set $K_2 = \omega$. This is precisely the φ solution to the reduced nonholonomic dynamics, given by

$$\ddot{x} + \tan\varphi\dot{\varphi}\dot{x} = \sin\alpha\cos^2\varphi, \qquad \ddot{\varphi} = 0. \tag{6.13}$$

To make sense of the other first integral in (6.12), we need to find $\tau(t)$. We do this by using our solution $\varphi(t)$ to explicitly integrate $d\tau = f(\varphi)\,dt = \cos(\varphi(t))\,dt$, arriving at

$$\tau(t) = \frac{1}{\omega}\sin(\varphi(t)), \tag{6.14}$$

where we have again set $\tau(t=0) = 0$ for simplicity. Then, using (6.14) in (6.12) gives $\dot{x} = \frac{\sin\alpha}{\omega}\cos\varphi\sin\varphi + K_1\cos\varphi$, from which we identify K_1 as $\dot{x}(0) =: \kappa$. A simple quadrature of (6.12) then gives the solution to the reduced nonholonomic dynamics,

$$x(t) = \frac{\sin\alpha}{2\omega^2}\sin^2(\varphi(t)) + \frac{\kappa}{\omega}\sin\varphi + x_0, \qquad \varphi(t) = \omega t, \tag{6.15}$$

which agrees with that found in Ref. 3, Section 1.6. We also wish to note that since ω has units of inverse time, then it follows from (6.14) that τ has units of time, so that (6.14) does indeed represent a reparameterization of time. Hence, our system is Hamiltonian in τ-time, but nonholonomic in t-time.

VII. CONCLUSION

We have endeavored to show that the Weitzenböck connection W plays a central role in the mechanics of systems in general and is particularly fundamental to the interesting physical and geometric characteristics that nonholonomically constrained systems possess. Indeed, we have shown that the pseudogyroscopic force that arises when considering the equations of motion of such systems in a moving frame adapted to $\mathcal{D}$ (the projection by P of the equations (3.9)) is in fact a *torsional* force arising from the torsion of the Weitzenböck connection through its relationship (2.15) with the objects of anholonomity of the frame. Given the considerable interest in the so-called teleparallel equivalent of general relativity (see[1]), we believe this relationship could provide an interesting set of research questions. For example, loosely speaking, when studying the motion of a mechanical system in the teleparallel theory one begins in a Minkowski spacetime, chooses a moving frame, and then interprets the Weitzenböck torsion of this frame as a gravitational force acting on the system. In principle, this is precisely what we have done in the examples above (before enforcing the constraints), except that the relevant interpretation of the Weitzenböck torsion for us is as a constraint force which enforces the nonholonomic constraints.

In addition to the theoretical aspects associated with the Weitzenböck torsion, in Sec. II A we also showed how this torsional force can, in some cases, be removed via an appropriate time reparameterization. Aside from the interesting parallelism between this time reparameterization and the distinction between "proper" and "coordinate" time in general relativity (see the discussion at the end of Sec. VI A), this ability to remove the torsional force induced by a moving frame allowed us to better understand the process of Chaplygin Hamiltonization. In a nutshell, one is searching for an inertial frame in which the nonholonomic dynamics are not subjected to a pseudoforce, relative to the reparameterized time. Moreover, the freedom in the choice of a moving frame, in addition to making results like that of Proposition 4 possible, in theory allows one to investigate, for example, which moving frames give one the most conservation laws for a given nonholonomic system. Given the framework developed in Sec. II A, this is now theoretically possible (whereas those who have worked with moving frames before would tend to consider the choice of a "good" frame something of an art).

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APPENDIX

A simple calculation[37] of the components of the torsion $T^a{}_{bc} = \langle \theta^a, T(e_b, e_c)\rangle$ and the curvature $R^a{}_{bcd} = \langle \theta^a, R(e_c, e_d)e_b\rangle$ tensors yields

$$T^a{}_{bc} = \Gamma^a{}_{bc} - \Gamma^a{}_{cb} + \Omega^a{}_{bc}, \tag{A1}$$

$$R^a{}_{bcd} = 2e_{[b}\Gamma^a{}_{c]d} + 2\Gamma^a{}_{[b|e|}\Gamma^e{}_{c]d} + \Gamma^a{}_{eb}\Omega^e{}_{cd}. \tag{A2}$$

In addition, by letting $U = e_b$, $V = e_c$, and $W = e_a$ in (2.6) we have $\Gamma_{abc} = g(e_a, \nabla^g_b e_c)$ given by

$$\Gamma_{abc} = \frac{1}{2}\left[e_b(G_{ca}) + e_c(G_{ba}) - e_a(G_{bc})\right] + \gamma^a{}_{bc}, \tag{A3}$$

where the γ_{abc} are the components of the second bracketed term in (2.6) and are known as the *Ricci rotation coefficients,*[20,25]

$$\gamma_{abc} = -\frac{1}{2}\left[G_{ad}\Omega^d{}_{bc} + G_{bf}\Omega^f{}_{ac} + G_{cf}\Omega^f{}_{ab}\right]. \tag{A4}$$

From (A3) we can the define the Christoffel symbols of the second kind $\Gamma^a{}_{bc}$ by

$$\Gamma^a{}_{bc} = \begin{Bmatrix} a \\ bc \end{Bmatrix} + \gamma^a{}_{bc}, \tag{A5}$$

where we have introduced the notation for the well-known Christoffel symbols,

$$\begin{Bmatrix} a \\ bc \end{Bmatrix} = \frac{1}{2}G^{ad}\left[e_b(G_{cd}) + e_c(G_{bd}) - e_d(G_{bc})\right], \tag{A6}$$

noting that in the coordinate basis these reduce to the standard Christoffel symbols $\left\{\begin{smallmatrix} i \\ jk \end{smallmatrix}\right\}$. Locally, the geodesic equation is then given by

$$\ddot{\alpha}^a + \begin{Bmatrix} a \\ bc \end{Bmatrix}\dot{\alpha}^b\dot{\alpha}^c = 0. \tag{A7}$$

Lastly, the components of the curvature of the Ehresmann connection A of (2.7) are given by[3]

$$B^C_{\alpha\beta} = \partial_\beta A^C_\alpha - \partial_\alpha A^C_\beta + A^B_\alpha \partial_B A^C_\beta - A^B_\beta \partial_B A^C_\alpha, \tag{A8}$$

and by (4.6) in the case of a principal connection.

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