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Internal heating driven convection at infinite Prandtl number

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We derive an improved rigorous lower bound on the space and time averaged temperature $\langle T \rangle$ of an infinite Prandtl number Boussinesq fluid contained between isothermal no-slip boundaries driven by uniform internal heating. A singular stable stratification is introduced as a perturbation to a non-singular background profile yielding $\langle T \rangle \geq 0.419[\mathrm{R}\ \log \mathrm{R}]^{-1/4}$ where R is the heat Rayleigh number. The analysis relies on a generalized Hardy-Rellich inequality that is proved in the Appendix. © *2011 American Institute of Physics*. [doi:10.1063/1.3637032]

I. INTRODUCTION

Thermal convection is the buoyancy-driven flow of a fluid heated from below and/or cooled from above. An ongoing challenge for analysis, theory, computation, and experiment is to ascertain how the heat transport depends on the thermal forcing as gauged by a nondimensional Rayleigh number and the fluid's material properties, typically characterized by the dimensionless Prandtl number, the ratio of kinematic viscosity to thermal diffusivity.[1] Bounds on heat transport within the Boussinesq approximation were pioneered by Howard[2] and elaborated by Busse.[3] Later, following the motivational work of Hopf,[4] an alternative variational framework for bounds on turbulent transport of momentum, mass, and in the case of convection, heat, known as the "background method" was formulated.[5] This is the approach we adopt here.

In this paper, we consider an infinite Prandtl number Boussinesq fluid contained between two rigid isothermal boundaries thermally driven by constant internal heating. This model is inspired by convection in the Earth's mantle where the Prandtl number is $\mathcal{O}(10^{24})$ and the motion is predominantly driven by a semi-uniform heating from radioactive decay. For definiteness, we consider an idealization of the actual geophysical conditions: an isoviscous fluid subject to no-slip isothermal vertical boundary conditions—without loss of generality the temperature equals 0 on the boundaries—and uniform heating at volumetric heat rate $\mathcal{H}\rho c$ where ρ is the density of the fluid and c is the specific heat.

Bulk heating is measured in terms of the dimensionless "heat Rayleigh number" $\mathrm{R} = \frac{g\alpha\mathcal{H}h^5}{\nu\kappa^2}$ where g is the acceleration of gravity, α is the fluid's thermal expansion coefficient, h is the thickness of the layer, ν is the fluid's viscosity, and κ is its thermal diffusivity.[6] At low heating rates, i.e., for R below a finite critical value, the fluid remains at rest and heat is transported to the boundaries by conduction within a parabolic temperature profile across the layer shown in Fig. 1 (the solid curves). At higher heating rates convection sets in to actively transport heat, predominantly toward the upper boundary. Figure 1 also contains a sketch (the dashed curve) of the expected form of the horizontally averaged temperature profile for internal heat driven convection with fixed temperature boundaries:

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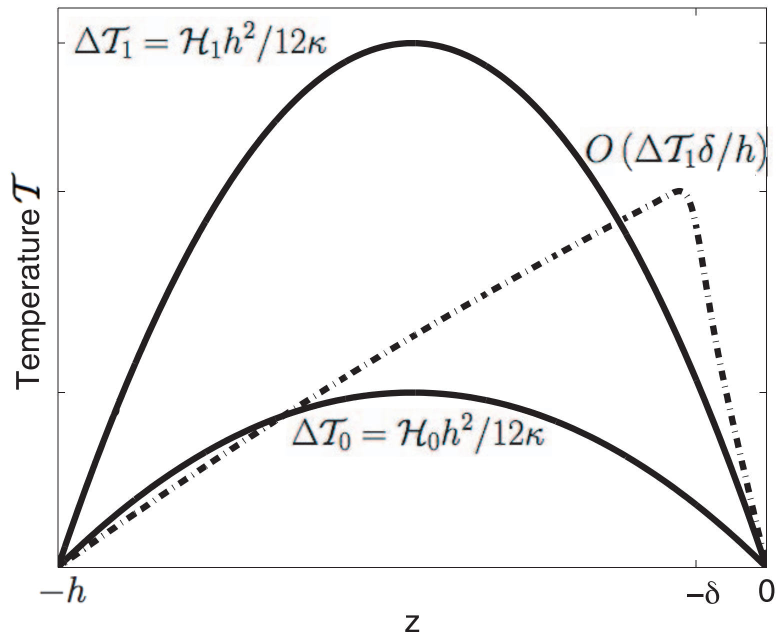

FIG. 1. Sketches of horizontally and temporally averaged temperature profiles. The parabolic conduction profiles (solid curves) are stable at low R (say, for heating rate $\mathcal{H}_0$) and unstable for large R (say, for $\mathcal{H}_1 > \mathcal{H}_0$). In the turbulent convection state at the higher heating rates $\mathcal{H}_1$, the heat is preferentially transported upward and a thermal boundary layer of thickness $\delta \ll h$ appears. The natural temperature scale is $\Delta\mathcal{T} = \mathcal{H}h^2/\kappa$, proportional to the maximum of the quadratic conduction profiles, but the amplitude of convection profiles is reduced by a factor of δ/h.

convection decreases the temperature relative to the purely conductive values throughout most of the layer, preferentially transporting heat upward and producing a boundary layer of thickness $\delta \ll h$ near the top to satisfy the temperature boundary condition.

The challenge is to determine how the space-time averaged temperature $\langle\mathcal{T}\rangle$ varies with $\mathcal{H}$. Equivalently, the enhancement of heat transport is gauged nondimensionally by the space-time averaged temperature measured in units of $\mathcal{H}h^2/\kappa$ and the challenge is to determine how dimensionless $\langle T\rangle = \kappa\langle\mathcal{T}\rangle/\mathcal{H}h^2$ varies with R. The no-convection parabolic conduction solution exists for all values of $\mathcal{H}$ (or R) and, even though it is unstable at high heating rates, realizes the upper limit on the bulk averaged temperature for all values of $\mathcal{H}$ (or R). That is, $\langle\mathcal{T}\rangle \le \frac{\mathcal{H}h^2}{12\kappa}$ (or $\langle T\rangle \le \frac{1}{12}$). The goal is to derive lower bounds on $\langle\mathcal{T}\rangle$ as a function of $\mathcal{H}$. In nondimensional terms, at high R the question is how low can $\langle T\rangle$ go?

A heuristic marginally stable boundary layer argument[7] predicts the sublinear scaling estimate $\langle\mathcal{T}\rangle \sim \mathcal{H}^{3/4}$ in the presence of convection, corresponding to $\langle T\rangle \sim \mathrm{R}^{-1/4}$. The basic idea is that as the heating increases and convection sets in, the upper thermal boundary layer forms where the fluid is pinned at rest by the no-slip boundary. The constant heating produces a fixed heat flux across the upper boundary layer and the peak temperature of the averaged profile must be $\sim \frac{\delta}{h} \times \frac{\mathcal{H}h^2}{\kappa}$ so that the top of the boundary layer has a slope $\sim \frac{\mathcal{H}h}{\kappa}$ to conduct the majority of the heat out of the layer. Then $\langle\mathcal{T}\rangle \sim \frac{\mathcal{H}h\delta}{\kappa}$, so if we can infer how δ varies with the control parameters we can infer the scaling. The fundamental hypothesis is that the boundary layer thickness is precisely what it needs to be so that as a convection system unto itself, the boundary layer is marginally stable: if it were any thinner diffusion would cause it to grow, and if it were any thicker fluid motion would commence and it would not exist as a static layer.

So consider the boundary layer from $-\delta$ to 0 as a distinct convective layer. Although we do not strictly speaking have boundary conditions on the bottom of the boundary layer, we presume that the velocity satisfies free-slip boundary conditions there; it certainly satisfies no-slip conditions at the top. Meanwhile the temperature satisfies a fixed temperature $\mathcal{T} = 0$ at the top and a fixed heat flux condition at the bottom of the boundary layer. The volumetric heating constant $\mathcal{H}$ is the same in the boundary layer as in the bulk, but there is also an imposed temperature gradient due to the incoming flux of heat from below. There are thus two driving forces that can each be described through nondimensional numbers.[8] The first is a measure of the strength of the internal heating in the layer,

$$\mathrm{N}_\delta = \frac{\mathcal{H}\delta^2}{\Delta\mathcal{T}_\delta\kappa} \sim \frac{\delta}{h}, \tag{1}$$

where $\Delta\mathcal{T}_\delta \sim \frac{\mathcal{H}h\delta}{\kappa}$ is the temperature drop across the layer. This vanishes rapidly as the boundary layer decreases in size, i.e., for $\delta/h \ll 1$. The second non-dimensional number reflects the influence of the temperature gradient, namely the traditionally defined Rayleigh number

$$\mathrm{Ra}_\delta = \frac{g\alpha\Delta\mathcal{T}_\delta\delta^3}{\nu\kappa}. \tag{2}$$

Because $\mathrm{N}_s = O\left(\frac{\delta}{h}\right)$ we can neglect the effect of internal heating in the boundary layer and consider that in the turbulent regime, the boundary layer is a marginally stable conductive solution driven by the temperature gradient imposed by the fixed flux from below. Marginal stability means that Ra_δ assumes the relevant critical value 647, the critical Rayleigh number for convection with free-slip, fixed-flux on bottom, and no-slip, fixed temperature upper boundary conditions. Thus

$$647 \approx Ra_\delta = \frac{g\alpha\mathcal{H}\delta^4 h}{\nu\kappa^2} = \frac{\delta^4}{h^4}\mathrm{R}. \tag{3}$$

It follows that $\delta \approx 5h\mathrm{R}^{-1/4}$ suggesting that

$$\langle\mathcal{T}\rangle \approx \frac{\mathcal{H}h\delta}{2\kappa} \sim \frac{5\mathcal{H}h^2}{2\mathrm{R}^{1/4}\kappa} \sim \mathcal{H}^{3/4} \tag{4}$$

and, nondimensionally, that

$$\langle T\rangle \approx 2.5\,R^{-1/4}. \tag{5}$$

In this paper, we prove

$$\langle T\rangle \geq 0.419\,\mathrm{R}^{-1/4}\,(\log\mathrm{R})^{-1/4} \quad \Rightarrow \quad \langle\mathcal{T}\rangle \gtrsim \mathcal{H}^{3/4}\,(\log\mathcal{H})^{-1/4}. \tag{6}$$

The rigorous lower bound is, modulo a logarithmic correction, consistent with the predictions of the marginally stable boundary layer argument. Moreover, it is not inconsistent with a scaling law measured from direct numerical simulations[9] implying that $\langle T\rangle \sim 1.65\,\mathrm{R}^{-0.234}$. We note, however, that those computations employed free-slip velocity boundary conditions rather than the no-slip conditions employed in the analysis here. Boundary conditions can drastically affect the fluid dynamics (and the bounds[10,11]) for Rayleigh-Bénard convection so the comparison must be taken with a degree of caution.

Bounds for this internal heating problem were previously considered by Lu *et al.*[12] who used estimates originally derived for boundary driven Rayleigh-Bénard convection[13] and a simple piecewise linear background profile to produce a lower bound. That result was $\langle T\rangle \geq 0.81\,\mathrm{R}^{-2/7}$, or in dimensional units $\langle\mathcal{T}\rangle \gtrsim \mathcal{H}^{5/7}$. Subsequent developments[10,14] indicated that optimal background profiles for infinite Prandtl number convection may include some stable stratification suggesting there was room for improvement. In particular, a singular integral analysis produced a key estimate that was then utilized in the background method to establish an upper bound on the Nusselt number Nu, the dimensionless measure of the enhancement of heat transport in boundary-driven Rayleigh-Bénard convection, in terms of the traditional Rayleigh number Ra of the form $\mathrm{Nu} \lesssim [\mathrm{Ra}\log(\mathrm{Ra})]^{1/3}$.[14] In this paper, we show that that key estimate is a modified Hardy-Rellich inequality and we derive the sharp prefactor. The newly derived inequality, along with some additional considerations, is then applied to the internal heating problem via the background method to obtain the improved result.

The rest of this paper is organized as follows. Section II describes the Boussinesq equations of motion with internal heating and provides an outline of the background method applied to the problem. Section III introduces the particular background temperature field as a logarithmic perturbation of a quadratic profile and applies the modified Hardy-Rellich estimate to obtain the bound (6). Section IV discusses these results and briefly remarks on the parallels between the internal heating and boundary driven convection problems. The new derivation of the Hardy-Rellich inequality is described in the Appendix.

II. INTERNAL HEATING AND THE BACKGROUND METHOD

The equations of motion in the Boussinesq approximation are

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + g\alpha \mathcal{T} \hat{\boldsymbol{k}} \tag{7}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{8}$$

$$\frac{\partial \mathcal{T}}{\partial t} + \boldsymbol{u} \cdot \nabla \mathcal{T} = \kappa \nabla^2 \mathcal{T} + \mathcal{H} \tag{9}$$

where p is the pressure field, ν is the kinematic viscosity, g is the acceleration of gravity, α is the thermal expansion coefficient, and κ is the thermal diffusivity. The volumetric heat flux pumped into the system is $\mathcal{H}\rho c$ where ρ and c are the fluid's density and specific heat. We consider a fluid layer of height h with no-slip boundary conditions in the vertical (z) direction, i.e., $\boldsymbol{u}|_{z=-h} = 0 = \boldsymbol{u}|_{z=0}$. The temperature satisfies $\mathcal{T}|_{z=-h} = 0 = \mathcal{T}|_{z=0}$, and all variables are periodic in the horizontal (x and y) directions with periods L_x and L_y.

The conventional dimensionless formulation[12] uses the time-scale h^2/κ, the length scale is h, and the temperature is measured in units of $\mathcal{H}h^2/\kappa$. Then

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u}\right) + \nabla p = \nabla^2 \boldsymbol{u} + \mathrm{R}\, T\, \hat{\boldsymbol{k}} \tag{10}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{u} \cdot \nabla T = \nabla^2 T + 1 \tag{11}$$

$$\nabla \cdot \boldsymbol{u} = 0 \tag{12}$$

where the heat Rayleigh number is $\mathrm{R} = \frac{g\alpha\mathcal{H}h^5}{\nu\kappa^2}$ and $\mathrm{Pr} = \frac{\nu}{\kappa}$ is the Prandtl number. Combined with the divergence-free condition (12), the no-slip velocity boundary condition implies that $\partial w/\partial z = 0$ at the top and bottom boundaries, where $w = \hat{k} \cdot \boldsymbol{u}$ is the vertical component of velocity.

Defining the space-time average of a function $f(x, y, z, t)$ as

$$\langle f \rangle = \lim_{t\to\infty} \frac{1}{t} \int_0^t ds\, \frac{h}{L_x} \int_0^{L_x/h} dx\, \frac{h}{L_y} \int_0^{L_y/h} dy \int_{-1}^0 dz\, f(x, y, z, s) \tag{13}$$

(assuming that the limits exist) we are interested in obtaining a lower bound on the average temperature $\langle T \rangle$ in terms of the Rayleigh heat number R. From this point on we focus on the infinite Pr limit of (10), the validity of which has recently been established,[15] so that the Navier-Stokes momentum equations become the Stokes equations

$$\nabla p = \nabla^2 \boldsymbol{u} + \mathrm{R} T \hat{\boldsymbol{k}}. \tag{14}$$

To apply the background method, write the temperature field as the sum of a stationary background profile and fluctuations according to $T(x, y, z, t) = \tau(z) + \theta(x, y, z, t)$ where $\tau(-1) = \tau(0) = 0$ so the fluctuation $\theta(x, y, z, t)$ satisfies homogeneous Dirichlet conditions at the top and bottom boundaries. Applying this decomposition to the equations of motion and considering the space time average of the momentum and properly weighted temperature equations, it is straightforward to show that (see Ref. 12 for details of the derivation):

$$\langle T\rangle \geq 2\langle\tau\rangle - \langle(\tau')^2\rangle \tag{15}$$

as long as the quadratic (in θ) functional

$$Q = \langle|\nabla\theta|^2\rangle + \langle 2\tau' w\theta\rangle \tag{16}$$

is positive semidefinite among temperature fluctuations and velocity fields satisfying the boundary conditions. Q is quadratic in θ because, applying the curl operator twice to (14), it is evident that there is an instantaneous linear albeit nonlocal slaving of the vertical velocity w to θ:

$$\Delta^2 w = -\mathrm{R}\Delta_H\theta \tag{17}$$

where Δ is the full Laplacian and Δ_H is the horizontal Laplacian $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$.

For calculational convenience, we apply the Fourier transform in the horizontal directions to obtain the relation for each wave number $k = |\mathbf{k}|$,

$$\left(\frac{d^2}{dz^2} - k^2\right)^2 \hat{w}_{\mathbf{k}} = \mathrm{R}k^2\hat{\theta}_{\mathbf{k}}$$

where now for all $\mathbf{k}$ the single-wavenumber quadratic forms

$$Q_{\mathbf{k}} := \int_{-1}^{0}\left[\left|\frac{d\hat{\theta}_{\mathbf{k}}}{dz}\right|^2 + k^2|\hat{\theta}_{\mathbf{k}}|^2 + 2\tau' Re[\hat{\theta}_{\mathbf{k}}\hat{w}_{\mathbf{k}}]\right]dz$$

must all remain positive semidefinite. In the following, we consider $Q_{\mathbf{k}}$ wavenumber by wavenumber so we drop the $\hat{}$ and subscript $\mathbf{k}$. In other words, we seek to maximize $2\langle\tau\rangle - \langle(\tau')^2\rangle$ while maintaining positivite-semidefiniteness of

$$Q := \int_{-1}^{0}\left[\left|\frac{d\theta}{dz}\right|^2 + k^2|\theta|^2 + 2\tau' Re[\theta w]\right]dz \tag{18}$$

uniformly in k, where $\theta(z)$ satisfies homogeneous Dirichlet boundary conditions and $w(z)$ solves

$$w'''' - 2k^2 w'' + k^4 w = \mathrm{R}k^2\theta, \tag{19}$$

and satisfies both homogeneous Dirichlet and Neumann boundary conditions on $[-1, 0]$.

Previous analysis of this problem and boundary driven convection considered background profiles $\tau(z)$ constant in the bulk of the layer so the only possible negative contribution to Q relied on the product of $w(z)$ and $\theta(z)$ in boundary layers where both are constrained to be relatively small in magnitude. As was discovered for boundary driven convection, however, a stably stratified (i.e. $\tau'(z) > 0$) profile in the bulk can be exploited to utilize the positive weighted correlation between w and θ resulting from the slaving and improve the positivity of Q, allowing for sharper estimates.[14]

III. SINGULAR PERTURBATION OF A STABLY STRATIFIED PROFILE

Consider the family of background profiles illustrated in Figure 2,

$$\tau(z) = \begin{cases} a\log\left(\frac{-1}{z}\right) + b(1-z^2) & -1 \leq z \leq -\delta \\ \frac{-z}{\delta}[a\log(1/\delta) + b(1-\delta^2)] & -\delta \leq z \leq 0, \end{cases} \tag{20}$$

where the positive parameters $\delta < 1$, a, and b will be chosen to optimize the bound. The logarithmic term enhances the positivity of Q, and hence leads to an improved scaling of the boundary layer with R, while the quadratic term is meant to increase the integral of $\tau(z)$ sufficiently to offset the slow logarithmic growth near $z = -1$ and lessen the negative impact of the Dirichlet integral in (15).

It is easily verified that

$$\int_{-1}^{0}\tau(z)dz = a\left[1 - \delta - \frac{\delta\log(1/\delta)}{2}\right] + b\left[\frac{2}{3} - \frac{\delta}{2} - \frac{\delta^3}{6}\right], \tag{21}$$

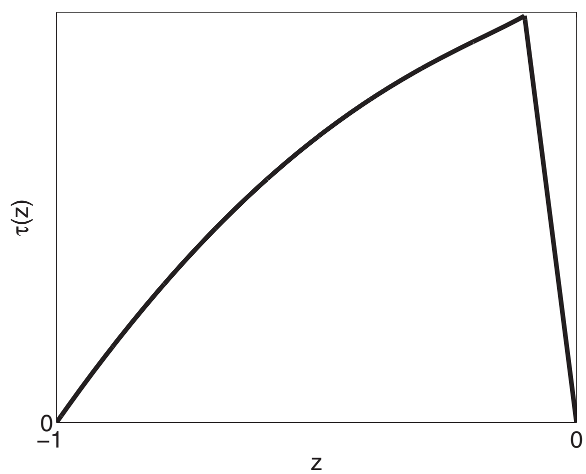

FIG. 2. Background profile (20).

$$\int_{-1}^{0} (\tau'(z))^2 dz = a^2 \left(\frac{1}{\delta} - 1 + \frac{[\log(1/\delta)]^2}{\delta} \right) \tag{22}$$

$$+ ab \left(2\frac{\log(1/\delta)}{\delta} + 4 - 2\delta \log(1/\delta) - 4\delta \right) \tag{23}$$

$$+ b^2 \left(\frac{1}{\delta} + \frac{4}{3} - 2\delta - \frac{\delta^3}{3} \right), \tag{24}$$

producing the lower bound on the average temperature $\langle T \rangle$ given by (15) when the positivity of Q is maintained. An appropriate choice of the scaling of the parameters a and b with respect to δ will allow us to determine both the "correct" boundary layer scaling, and to maximize the lower bound on $\langle \mathrm{T} \rangle$.

The two key inequalities required for the analysis are

$$\int_{-1}^{0} \theta(z) w^*(z) z dz \leq 0 \tag{25}$$

and

$$\int_{-1}^{0} \frac{\theta(z) w^*(z)}{z} dz \;\leq\; \frac{4}{\mathrm{R}} \int_{-1}^{0} \frac{|w(z)|^2}{z^3} dz \;\leq\; 0. \tag{26}$$

The first inequality (26) is an exercise in integration by parts the details of which are left to the reader. The second inequality (26) is a restatement—and slight improvement in the prefactor—of the key result previously derived for Rayleigh-Bénard convection.[14] While a prefactor improvement may be considered minor, our approach significantly simplifies the proof and embeds the problem in the context of generalized Hardy-Rellich inequalities. The proof, provided in the Appendix, also establishes that the estimate with this prefactor is sharp.

To determine conditions guaranteeing the positivity of Q we reformulate it neglecting much of the L^2 norm of $\frac{d\theta}{dz}$ as well as the $k^2|\theta|^2$ term and use (25) to observe

$$Q \geq \int_{-\delta}^{0} \left| \frac{d\theta}{dz} \right|^2 dz - 2a \int_{-1}^{0} \frac{Re[\theta w^*]}{z} dz \tag{27}$$

$$- \int_{-\delta}^{0} \left(\frac{2a \log(1/\delta)}{\delta} - \frac{2a}{z} - 4bz + \frac{2b(1-\delta^2)}{\delta} \right) Re[\theta w^*] dz. \tag{28}$$

In the above, we added the bulk terms to the boundary layer in order to apply (25) and (26) to the entire interval. Applying (26) then implies

$$Q \geq \int_{-\delta}^{0} \left|\frac{d\theta}{dz}\right|^2 dz - \frac{8a}{\mathrm{R}} \int_{-1}^{0} \frac{|w|^2}{z^3} dz - \int_{-\delta}^{0} \left(\frac{2a\log(1/\delta)}{\delta} - \frac{2a}{z} - 4bz + \frac{2b(1-\delta^2)}{\delta} \right) Re[\theta w^*] dz. \tag{29}$$

Bound the magnitude of the last integral in (29) as follows:

$$\left| \int_{-\delta}^{0} \left(\frac{2a\log(1/\delta)}{\delta} - \frac{2a}{z} - 4bz + \frac{2b(1-\delta^2)}{\delta} \right) Re[\theta w^*] dz \right| \tag{30}$$

$$\leq \int_{-\delta}^{0} \left(\frac{2a\log(1/\delta)}{\delta} - \frac{2a}{z} - 4bz + \frac{2b(1-\delta^2)}{\delta} \right) z^2 \frac{|\theta|}{|z|^{1/2}} \frac{|w|}{|z|^{3/2}} dz \tag{31}$$

$$\leq 2 \left(\sup_{-\delta \leq z \leq 0} \frac{|\theta(z)|}{|z|^{1/2}} \right) \left(\int_{-\delta}^{0} z^4 \left[\frac{a\log(1/\delta)}{\delta} - \frac{a}{z} - 2bz + \frac{b(1-\delta^2)}{\delta} \right]^2 dz \right)^{1/2} \tag{32}$$

$$\times \left(\int_{-1}^{0} \frac{|w|^2}{|z|^3} dz \right)^{1/2}. \tag{33}$$

The homogeneous boundary conditions on $\theta(z)$ mean that for $z \in (-\delta, 0)$,

$$|\theta(z)| = \left| \int_{z}^{0} \frac{d\theta}{d\tilde{z}} d\tilde{z} \right| \leq |z|^{1/2} \left(\int_{z}^{0} \left| \frac{d\theta}{d\tilde{z}} \right|^2 d\tilde{z} \right)^{1/2} \leq |z|^{1/2} \left(\int_{-\delta}^{0} \left| \frac{d\theta}{dz} \right|^2 dz \right)^{1/2}. \tag{34}$$

Hence we can bound the supremum in (32) and apply Young's inequality to see that

$$\left| \int_{-\delta}^{0} \left(\frac{2a\log(1/\delta)}{\delta} - \frac{2a}{z} - 4bz + \frac{2b(1-\delta^2)}{\delta} \right) Re[\theta w^*] dz \right| \tag{35}$$

$$\leq \int_{-\delta}^{0} \left| \frac{d\theta}{dz} \right|^2 dz + \int_{-\delta}^{0} z^4 \left[\frac{a\log(1/\delta)}{\delta} - \frac{a}{z} - 2bz + \frac{b(1-\delta^2)}{\delta} \right]^2 dz \times \int_{-1}^{0} \frac{|w|^2}{|z|^3} dz. \tag{36}$$

Inserting (36) into (29), we see that

$$Q \geq \left\{ \frac{8a}{\mathrm{R}} - \int_{-\delta}^{0} z^4 \left[\frac{a\log(1/\delta)}{\delta} - \frac{a}{z} - 2bz + \frac{b(1-\delta^2)}{\delta} \right]^2 dz \right\} \int_{-1}^{0} \frac{|w|^2}{|z|^3} dz. \tag{37}$$

The integral about the boundary layer in (37) can be computed exactly. At this point we choose $a = a'\delta/\log(1/\delta)$ and $b = b'\delta$ where a' and b' are $\mathcal{O}(1)$ absolute constants. Then

$$\int_{-\delta}^{0} z^4 \left[\frac{a\log(1/\delta)}{\delta} - \frac{a}{z} - 2bz + \frac{b(1-\delta^2)}{\delta} \right]^2 dz \tag{38}$$

$$= a'^2 \frac{\delta^5}{5} + 2a'b' \frac{\delta^5}{5} + b'^2 \frac{\delta^5}{5} + O\left(\delta^5 \log(1/\delta)\right) \tag{39}$$

as $\delta \to 0$. Comparing this with (37) we see that the minimal requirement for Q to remain positive in the $\delta \to 0$ or $\mathrm{R} \to \infty$ limit is

$$\frac{8}{\mathrm{R}} \sim a' \frac{\delta^4[\log(1/\delta)]}{5} + 2b' \frac{\delta^4 \log(1/\delta)}{5} + \frac{b'^2}{a'} \frac{\delta^4 \log(1/\delta)}{5} \tag{40}$$

$$\Rightarrow \frac{1}{\mathrm{R}} \sim \frac{\xi(a', b')}{4} \delta^4 \log(1/\delta) \tag{41}$$

where

$$\xi(a', b') = \frac{(a'+b')^2}{10a'}. \tag{42}$$

This yields the scaling of the boundary layer thickness as

$$\delta \sim [\xi(a', b')\mathrm{R}\log(\mathrm{R})]^{-1/4}. \tag{43}$$

The average temperature is bounded by two times (21) minus (22) implying that, asymptotically,

$$\langle T\rangle \geq \frac{4}{3}b'\delta - a'^2\delta - 2a'b'\delta - b'^2\delta \tag{44}$$

$$\sim \left(\frac{4}{3}b' - a'^2 - 2a'b' - b'^2\right)\xi(a', b')^{-1/4}\left(\mathrm{R}\log(\mathrm{R})\right)^{-1/4}. \tag{45}$$

To obtain the "best" prefactor, we maximize over a' and b' to achieve

$$\langle T\rangle \geq \frac{2^{3/4}5^{1/4}}{6}\left(\mathrm{R}\log(\mathrm{R})\right)^{-1/4} \sim 0.419\left(\mathrm{R}\log(\mathrm{R})\right)^{-1/4} \tag{46}$$

where the optimal prefactor is obtained for $a' = \frac{1}{16}$ and $b' = \frac{7}{16}$.

IV. DISCUSSION AND CONCLUSIONS

The background profile (20) can be considered the sum of a singular logarithmic profile and a smooth conduction-like quadratic profile. If the logarithmic term only is considered, i.e., $b = 0$, then the profile would be

$$\tau_0(z) = \begin{cases} a\log\left(\frac{-1}{z}\right) & -1 \leq z \leq -\delta \\ -\frac{a\log(1/\delta)z}{\delta} & -\delta \leq z \leq 0. \end{cases} \tag{47}$$

This would be analogous to the approach taken in Ref. 14 for boundary driven convection. However if the same steps are followed, the optimal estimate occurs for $a \sim \frac{\delta}{[\log(1/\delta)]^2}$ in which case the bound becomes

$$\langle T\rangle \geq \frac{\delta}{[\log(1/\delta)]^2}. \tag{48}$$

This is a weaker result than that derived above. However, the same analysis performed to ensure the positivity of Q, but with (47) as the background profile yields the "pure" boundary layer thickness scaling $\delta \sim \mathrm{R}^{-1/4}$.

Figure 3 yields further insight. The purely logarithmic profile (47) (the dashed line) yields a thicker boundary layer at high R because the steep gradient near $z = 0$ enhances the positivity of Q sufficiently to maintain the increased size of the boundary layer. But this costs dearly in the Dirichlet integral that negatively affects the estimate of $\langle \mathrm{T}\rangle$ while adding very little to the computation of $\langle\tau_0(z)\rangle$. That is, while the quadratic term ($b > 0$) thins the boundary layer, it also contributes significantly to $\langle\tau\rangle$ and raises the lower bound. In previous applications of the background method,[13,14,16] the scaling of the boundary layer dictates the bound: typically the heat transport is bounded by $\frac{1}{\delta}$ where δ is the size of the boundary layer. Bounding the average temperature from below for the internal heating problem creates a different situation where the "optimal" boundary layer scaling in terms of δ yields an apparently sub-optimal bound in terms of R.

Another key difference between this problem and the traditional application of the background method to boundary driven Rayleigh-Bénard convection is the symmetry of the problem. In purely boundary driven convection in the Boussinesq approximation, the system is naturally symmetric in the vertical (across the mid-plane of the layer) so the optimal choice of background profile is symmetric as well. Indeed, in that case thermal boundary layers near both the bottom and top boundaries are unstably stratified. In the case of an internal heat source there is an inherent

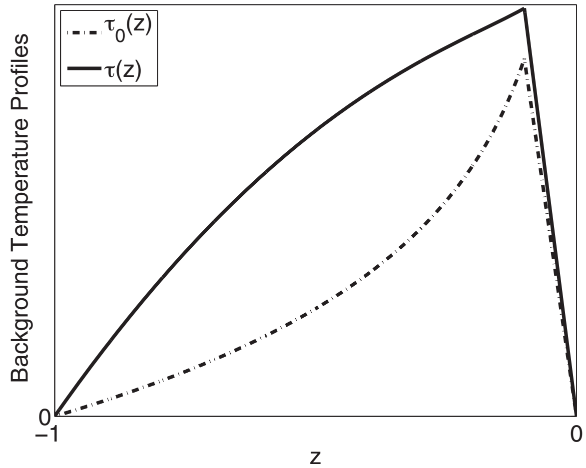

FIG. 3. The background profile (20) where a and b scale in the optimal sense compared to the logarithmic profile given by (47).

asymmetry: the (horizontally and time averaged) vertical temperature profile throughout the bulk is stably stratified toward the bottom of the layer and unstably stratified only near the top.

The lower bound on the mean temperature (46) is remarkably close to the scaling derived from the marginally stable boundary layer argument, and it is not inconsistent with numerical simulations[9] although we reiterate that the simulations employed stress-free (a.k.a. free-slip) boundary conditions on the velocity, as opposed to the no-slip conditions employed here, that may affect the scaling behavior. The stress-free internal heating problem is addressed in Ref. 17, where it is shown that $\langle T \rangle \gtrsim \mathrm{R}^{-5/17}$ or dimensionally $\langle \mathcal{T} \rangle \gtrsim \mathcal{H}^{12/17}$. It will also be of interest to examine numerical solutions of the Euler-Lagrange equations for this problem, as has been done for boundary driven Rayleigh-Bénard convection for finite[18] and infinite[10] Prandtl numbers. Their solution would indicate what the true optimal background profile is, and may provide additional insight into the pursuit of further rigorous bounds.

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APPENDIX: A GENERALIZED HARDY-RELLICH INEQUALITY

We will establish (26) for all functions $w(z)$ and $\theta(z)$ that satisfy (19) with the prescribed boundary conditions. Note that with the change of variables $z \to -z$ this is equivalent to casting the problem on the positive unit interval as

$$Re \int_0^1 \frac{\theta w^*}{z} dz \geq \frac{4}{\mathrm{R}} \int_0^1 \frac{|w|^2}{z^3} dz, \tag{A1}$$

where (19) is satisfied for $z \in [0, 1]$ and $w(0) = w(1) = w'(0) = w'(1) = \theta(0) = \theta(1) = 0$. In this context, (A1) is recognized as a factor of two improvement on the original proof [14]. As in the original proof we will prove the following proposition:

Proposition A.1: If $0 < c \leq d \leq \infty$, the smooth function $w(z)$ satisfies

$$w(c) = 0 = w(d),\ w'(c) = 0 = w'(d), \tag{A2}$$

and $\theta(z)$ is defined by $w'''' - 2k^2 w'' + k^4 w = Rk^2\theta$, then

$$Re \int_c^d \frac{\theta w^*}{z} dz \geq \frac{4}{R} \int_c^d \frac{|w|^2}{z^3} dz. \tag{A3}$$

In order to see the connection between (A3) and Hardy-Rellich inequalities, make the change of variables $w(z) = z^{1/2}\zeta(z)$. It follows that $\zeta(z)$ also satisfies (A2). Inserting this change of variables into the fourth order term that results from the definition of $\theta(z)$, we see that

$$\int_c^d \frac{w'''' w^*}{z} dz = \int_c^d |\zeta''|^2 dz - \frac{3}{2}\int_c^d \frac{|\zeta'|^2}{z^2} dz + \frac{45}{16}\int_c^d \frac{|\zeta|^2}{z^4} dz. \tag{A4}$$

A similar calculation leads to

$$\int_c^d \frac{w'' w^*}{z} dz = -\int_c^d |\zeta'|^2 dz + \frac{1}{4}\int_c^d \frac{|\zeta|^2}{z^2} dz. \tag{A5}$$

Putting (A4) and (A5) together, we see that (A3) can be restated as

Lemma A.1: For smooth functions $\zeta(z)$ satisfying the boundary conditions (A2),

$$\int_c^d \left(|\zeta''|^2 - \frac{3}{2}\frac{|\zeta'|^2}{z^2} + \frac{45}{16}\frac{|\zeta|^2}{z^4} \right) dz + k^2 \int_c^d \left(2|\zeta'|^2 - \frac{1}{2}\frac{|\zeta|^2}{z^2} \right) dz + k^4 \int_c^d |\zeta|^2 dz \geq$$
$$\geq\ 4k^2 \int_c^d \frac{|\zeta|^2}{z^2} dz. \tag{A6}$$

Traditionally a Hardy-Rellich inequality is formulated in terms of the L^p norms of the operator $D^q = \frac{d^q}{dz^q}$ where $q = 1, 2$ and possibly higher orders (see Ref. 19 for example). (A6) is, with the appropriate integrations by parts, nothing else than the L^2 norm of the differential operator $D^2 - k^2$ acting on $\zeta(z)$. The inclusion of the wave number k here causes us to refer to this inequality as a generalized Hardy-Rellich inequality.

To prove the Lemma, consider the following one-parameter family of integrals,

$$0 \leq \int_c^d z^{2\nu} \left[\left(D^2 - k^2\right) \frac{\zeta}{z^\nu} \right]^2 dz, \tag{A7}$$

where $\zeta(z)$ satisfies the homogeneous boundary conditions. Expanding (A7) and integrating by parts multiple times leads to the following identity:

$$\int_c^d |\zeta''|^2 dz + 2\nu(\nu - 2)\int_c^d \frac{|\zeta'|^2}{z^2} dz + 2k^2 \int_c^d |\zeta'|^2 dz + \nu(\nu + 6 + \nu^3 - 4\nu^2)\int_c^d \frac{|\zeta|^2}{z^4} dz$$
$$+ k^4 \int_c^d |\zeta|^2 dz \geq 2\nu^2 k^2 \int_c^d \frac{|\zeta|^2}{z^2} dz.$$

Setting $\nu = \frac{3}{2}$ produces

$$\int_c^d |\zeta''|^2 dz - \frac{3}{2}\int_c^d \frac{|\zeta'|^2}{z^2} dz + \frac{45}{16}\int_c^d \frac{|\zeta|^2}{z^4} dz + 2k^2 \int_c^d |\zeta'|^2 dz$$
$$+ k^4 \int_c^d |\zeta|^2 dz \geq \frac{9}{2} k^2 \int_c^d \frac{|\zeta|^2}{z^2} dz$$

which is easily rearranged to establish the Lemma.

The strictness of the inequality derived here can be verified by considering functions $\zeta(z)$ that saturate (A7), that is those functions satisfying the boundary conditions together with

$$\left(D^2 - k^2\right)\frac{\zeta(z)}{z^\nu} = 0. \tag{A8}$$

Solutions of (A8) are linear combinations of modified Bessel functions:

$$\zeta(z) = z^{1/2+\nu}\left[C_1 K_q(kz) + C_2 I_q(kz)\right] \tag{A9}$$

where $q = \sqrt{2\nu^2 + 2\nu + 1/4}$. Just as the original Hardy inequality[20] is not saturated for any nontrivial analytic functions, the functions (A9) cannot satisfy all the boundary conditions simultaneously so there is no analytic solution to (A8) that saturates (A7). However, regularizing (A9) appropriately at the boundaries will produce a sequence of functions that satisfy the boundary conditions and, in the unregularized limit, solve (A8). Hence while (A7) is never saturated, there can be no improvement on the prefactor derived by this method, i.e., the approach outlined here is not only robust and amenable to adaptation, but also produces sharp estimates.

This methodology lends itself immediately to extension to other operators, and possibly higher dimensions as well. The free parameter ν can be adjusted as desired, indicating a significant utility to this method of producing Hardy-Rellich type inequalities. Hardy-Rellich inequalities with remainder terms can also be computed by optimizing over the wave-number k (for an example of other Hardy-Rellich type inequalities with remainder terms see the work of Evans and Lewis[21]).

[1] G. Ahlers, S. Grossmann, and D. Lohse, Rev. Mod. Phys. **81**, 503 (2009).
[2] L. N. Howard, J. Fluid Mech. **17**, 405 (1963).
[3] F. H. Busse, J. Fluid Mech. **37**, 457 (1969).
[4] E. Hopf, Math. Ann. **117**, 764 (1940).
[5] C. R. Doering and P. Constantin, Phys. Rev. E **53**, 5957 (1996).
[6] P. H. Roberts, J. Fluid Mech. **30**, 33 (1967).
[7] W. V.R. Malkus, Proc. R. Soc. London, Ser. A **225**, 196 (1954).
[8] E. M. Sparrow, R. J. Goldstein, and V. K. Jonsson, J. Fluid Mech. **18**, 513 (1963).
[9] C. Sotin and S. Labrosse, Phys. Earth Planet. Interiors **112**, 171 (1999).
[10] G. R. Ierley, R. R. Kerswell, and S. C. Plasting, J. Fluid Mech. **560**, 159 (2006).
[11] J. P. Whitehead and C. R. Doering, Phys. Rev. Lett. **106**, 244501 (2011).
[12] L. Lu, C. R. Doering, and F. H. Busse, J. Math. Phys. **45**, 2967 (2004).
[13] C. R. Doering and P. Constantin, J. Math. Phys. **45**, 784 (2001).
[14] C. R. Doering, F. Otto, and M. G. Reznikoff, J. Fluid Mech. **560**, 229 (2006).
[15] X. Wang, Commun. Pure Appl. Math. **57**, 1265 (2004).
[16] J. Otero, R. W. Wittenberg, R. A. Worthing, and C. R. Doering, J. Fluid Mech. **473**, 191 (2002).
[17] J. Whitehead and C. R. Doering, "Rigid rigorous bounds on heat transport in a slippery container," (unpublished).
[18] S. C. Plasting and R. R. Kerswell, J. Fluid Mech. **477**, 363 (2003).
[19] A. Kufner and L.-E. Persson, *Weighted Inequalities of Hardy Type* (World Scientific, Singapore, 2003).
[20] G. H. Hardy, Math. Z. **6**, 314 (1920).
[21] W. D. Evans and R. T. Lewis, J. Math. Inequal. **1**, 473 (2007).