Supplementary Information - A numerical investigation of surface crevasse propagation in glaciers using nonlocal continuum damage mechanics

R. Duddu,[1*] J. N. Bassis,[2] H. Waisman.[3]

1. Damage mechanics model for ice under low tensile stress

Damage mechanics simulates the evolution of distributed 'flaws' within the ice and their effect on the bulk rheology of ice. In this approach, the usual continuum equations are augmented with an additional internal state variable D representing material deterioration or damage, ranging from $D = 0$ (ice is completely intact) to $D = 1$ (ice is entirely fractured). The evolution of the damage variable D accounts for the progressive accumulation and coalescence of micro-cracks (and also micro-voids) within the ice as it deforms under low loading rates. Typically, a damage mechanics model is formulated using the effective stress concept in conjunction with either the principle of strain equivalence [*Lemaitre*, 1971] or the principle of strain energy equivalence [*Sidoroff*, 1980]. For isotropic damage, this introduces a mapping between the effective stress $\tilde{\sigma}_{ij}$ and actual stress σ_{ij}:

$$\tilde{\sigma}_{ij} = \frac{1}{1-D}\sigma_{ij}, \tag{1}$$

where σ_{ij} represents the force per unit damaged area (including voids or cracks) in the physical space and $\tilde{\sigma}_{ij}$ represents the effective stress representing the force per unit undamaged area (not including voids or cracks) in the effective space. Note that throughout this article we use the Einstein's indicial notation for tensors and the standard summation convention for repeated indices.

Assuming small elastic strains, the total strain tensor can be additively decomposed as,

$$\varepsilon_{ij} = \varepsilon^{\mathrm{e}}_{ij} + \varepsilon^{\mathrm{v}}_{ij}, \tag{2}$$

where the superscripts e and v denote the elastic (time-independent and recoverable component) and viscous (time-dependent and irrecoverable component) components, respectively. In this study, we neglect the delayed elastic (time-dependent and recoverable) component of the strain since it is much smaller compared to the viscous strain component at low loading rates. The effect of delayed elasticity is only important at high strain-rates and can be neglected for the low strain-rates observed in ice shelves. We assume the elastic behavior of undamaged polycrystalline ice to be isotropic owing to the random orientation of its crystalline structure; so, we need only two elastic constants, namely, the Young's modulus, E and the Poisson's ratio ν to describe the elastic behavior. The multi-axial form of the damage-modified Hooke's law can be expressed as,

$$\sigma_{ij} = \frac{E(1-D)}{3(1-2\nu)}\varepsilon^{\mathrm{e}}_{kk}\delta_{ij} + \frac{E(1-D)}{2(1+\nu)}\left(\varepsilon^{\mathrm{e}}_{ij} - \frac{1}{3}\varepsilon^{\mathrm{e}}_{kk}\delta_{ij}\right). \tag{3}$$

The relation for the viscous strain component is given by the damage modified multi-axial Glen's flow law,

$$\dot{\varepsilon}^{\mathrm{v}}_{ij} = K\tilde{\sigma}_{\mathrm{e}}^{n-1}\tilde{\sigma}^{\mathrm{dev}}_{ij}, \tag{4}$$

where K is a temperature dependent viscosity parameter, n is the flow law exponent ($n \sim 3$), $\tilde{\sigma}_e = \sqrt{\frac{3}{2}\tilde{\sigma}^{\mathrm{dev}}_{kl}\tilde{\sigma}^{\mathrm{dev}}_{kl}}$ is the effective Von Mises stress and the effective deviatoric stress tensor $\tilde{\sigma}^{\mathrm{dev}}_{ij} = \tilde{\sigma}_{ij} - \frac{\tilde{\sigma}_{kk}}{3}\delta_{ij}$.

The model is fully specified by a damage evolution equation, which for creep damage takes on the power-law form [*Kachanov*, 1958; *Rabotnov*, 1963]:

$$\dot{D} = \frac{B\langle\chi\rangle^r}{(1-D)^{k_\sigma}}, \tag{5}$$

where the dot decoration denotes differentiation with respect to time (in a Lagrangian sense), $\langle\ \rangle$ denote the Macaulay brackets, B is a temperature dependent material parameter and the exponents r, k_σ are material parameters that can be determined using laboratory data. In the above equation, χ is the Hayhurst equivalent stress given by a linear combination of the largest effective principal stress $\tilde{\sigma}^{(1)}$, the effective Von Mises stress $\tilde{\sigma}_{\mathrm{e}}$ and the effective pressure $\tilde{P}$:

$$\chi = \begin{cases} \alpha\tilde{\sigma}^{(1)} + \beta\tilde{\sigma}_{\mathrm{e}} - 3(1-\alpha-\beta)\tilde{P}, & \text{if } \sigma_{ii} \geq 0, \\ 0, & \text{if } \sigma_{ii} < 0. \end{cases} \tag{6}$$

where α and β material parameters calibrated with laboratory data *Pralong et al.* [2005]. In this study we mainly focus on tensile failure behavior and so χ is set to zero under compression, that is, we assume ice behaves like an undamaged viscoelastic material under compression. The presence of an existing microcrack or microvoid leads to a stress concentration in its vicinity and so new damage (i.e. microcrack or microvoid growth and nucleation) usually occurs near an existing defect. Thus, existing damage increases the damage rate in tension and this effect is described by the stress dependent parameter k_σ. For a general multiaxial state of stress, the dependence of k_σ on the stress tensor shall be assumed as,

$$k_\sigma = k_1 + k_2|\sigma_{ii}|, \ \text{if } 0 \leq \sigma_{ii} \leq 1\text{MPa}, \tag{7}$$

where k_1 and k_2 are parameters determined using a linear fit.

[1]Department of Civil and Environmental Engineering, Vanderbilt University; 400 24th Avenue South, 274 Jacobs Hall; Nashville, TN 37212. Corresponding author email: rduddu@gmail.com (Ravindra Duddu)

[2]Department of Atmospheric, Oceanic and Space Sciences, University of Michigan; 2529 Space Research Building; 2455 Hayward St.; Ann Arbor, MI 48109-2143.

[3]Department of Civil Engineering and Engineering Mechanics, Columbia University; 610 Seeley W. Mudd Building; 500 West 120th Street, Mail Code 4709; New York, NY 10027.

0094-8276/13/$5.00

2. Model Calibration

The model physics is fully specified by Equations (1)–(7). To apply the model to glacier ice we need to calibrate the model parameters and unfortunately, there is a dirth of laboratory measurements available to constrain parameters, especially under multi-axial loading. We tentatively use the laboratory experiments *Mahrenholtz and Wu* [1992] on lab grown polycrystaline ice to calibrate the eight parameters $K, n, B, r, k_1, k_2, \alpha$ and β are introduced to describe the visco-elastic constitutive damage model. The sudden fracture or rupture of ice under tensile loading that is observed experimentally is modeled by prescribing a critical damage D_{cr}. Additionally, we also prescribe a maximum damage $D_{\max}$ so that we can continue the computation until complete structural failure occurs. The parameter values used in this study and their sources are listed in Table 1.

Better laboratory data, preferably using glacier ice or field observations could be used to improve the calibration. However, we note that our focus is on determining the maximum depth to which crevasses penetrate. Re-calibrating the damage mechanics model will cause damage to accumulate faster (or slower) and will thus alter the speed of crevasse propagation, but not the depth of penetration, which is controlled by the location where the tensile stress vanishes. The advantage is that our damage model can be useful in identifying the interplay between macro-scale processes such as surface, basal and hydrofracture-induced crevasse propagation. The field data from remote sensing or satellite imagery can be used to validate model predictions and to discover the role of various processes in full depth fracture propagation. Alternatively, the construction and calibration of models may be performed at the large scale using field data, however, in this case one could lose the ability to decipher the various mechanisms affecting glacial calving.

3. Discussion of damage model

The model employed for this study is a simplified version of the one presented in *Duddu and Waisman* [2012a]. A few important points to note are:

1. We do not use any threshold criterion for the initiation of damage because it not clear from the experiments of *Mahrenholtz and Wu* [1992] whether a stress or strain threshold exists under tensile loading. Moreover, using a threshold for the initiation of damage, whose value is usually small, will only decrease the rate of crack propagation and, possibly, slightly decrease the predicted crevasse depth. Therefore, this will not affect the main conclusion of the study that surface crevasses alone are not responsible for calving events in marine terminating and thin glaciers..

2. A more general model that takes into account the damage-induced anisotropy arising from the microcrack damage can be implemented by defining damage as a second order tensor [*Murakami*, 1983; *Duddu and Waisman*, 2012a]; however, due to limited experimental data and simplicity of numerical implementation it is practicable to assume a scalar isotropic damage.

3. The material behavior of ice under compression at temperatures close to its melting point is complex and the kinetics of deformation are highly temperature- and loading rate- dependent. In this study, we neglect damage accumulation due to microcracking and/or softening due to the microstructure evolution of ice under compression due to dynamic recrystallization. Since our objective is to investigate tensile crevasse propagation, we believe it is reasonable to approximate the compression behavior of ice as visco-elastic without any damage or recrystallization effects.

4. Damage initialization using notches

Observations indicate that the surface of tidewater glaciers is visibly covered with crevasses. Hence, we believe it is reasonable to assume at least a few defects in ice that serve as seeds for future crack propagation. The worst-case scenario is when the location where ice is the weakest (with zero or very little strength idealized as a notch) coincides with the location of maximum tensile stress. To determine the location of the initial notch in our simulations, we first analyzed an ice slab with no notches assuming homogeneous material behavior. From this analysis result, shown in Figure I, we found that the maximum tensile stress develops at the top surface at a distance approximately one-ice thickness away from the slab terminus (right edge) and the tensile stress is almost a constant beyond this point away from the slab terminus. In all the simulation studies we conducted we initiated crevasses using notches placed a distance greater than one-ice thickness away from the terminus, as shown in Figure II. We found that our results are insensitive to the precise position of the notch when placed beyond one-ice thickness away from the terminus.

In a fracture mechanics approach it is necessary to prescribe a starter crack to perform crack growth analysis. In contrast, in a damage mechanics approach it is not necessary to prescribe a starter crack; however, usually a stress concentration feature such as a notch or hole or an inhomegeneity is considered to study localized effects that typically lead to eventual structural collapse [*Murakami et al.*, 1988; *Ando et al.*, 1990; *Hall and Hayhurst*, 1991; *de Borst*, 1997; *Becker et al.*, 2002; *Desmorat et al.*, 2007; *Jirasek and Grassl*, 2008; *Chow et al.*, 2011; *Verhoosel et al.*, 2011]. Alternatively, one may initiate cracks by considering a random distribution of damage, however, in this case the crack paths would be dependent on the choice of the statistical distribution of damage. Further, to predict crevasse growth the analysis should be conducted by considering several realizations of damage patterns and statistical averages need to be determined. Such an exercise is not pursued herein since this study is focused on investigating the mechanics of crevasse propagation rather than on the prediction of glacial calving rates. However, our numerical implementation is robust and works well even when damage is not initialized. Figure III shows the results from a simulation study without considering an initial notch, wherein multiple crevasses nucleate and propagate at locations of maximum tensile stress that are at least one ice-thickness away from the terminus. In this simulation, we considered a coarse mesh in order to reduce the computational cost associated with a finer resolution finite element mesh. Our study shows that considering a notch does not affect the predicted final crevasse depth but it affects only the crevasse propagation rate.

Investigating the mechanics of crevasse propagation using an isolated crevasse assuming idealized boundary conditions and glacier geometries is definitely not realistic. However, our simulation studies are designed to estimate the maximum surface crevasse penetration depth. Let us consider a simulation wherein an initial notch is specified at mid-length of the slab. The simulation result show in Figure IV indicates that surface crevasses can only penetrate a fraction of the entire ice slab thickness. This is because crevasses are driven by tensile stress generated by the cryostatic stress induced creep flow that varies with the depth. As the first crevasse, originating from the notch, propagates deeper the variation of the horizontal creep flow velocity decreases and so the tensile stress driving crevasse propagation decreases. Similar to the Nye zero stress model, the crevasse penetrates to the depth where the tensile stress at the crevasse

Table 1. Model parameters for ice at temperature $T = -10°C$

Parameter	Value in Tension	Value in Compression	Units	Source
E	9500	9500	MPa	*Karr and Choi* [1989]
ν	0.35	0.35	-	*Karr and Choi* [1989]
n	3.1	3.1	-	*Karr and Choi* [1989]
K	2.56×10^{-7}	9.40×10^{-8}	MPa^{-n} s^{-1}	*Duddu and Waisman* [2012a]
B	5.23×10^{-7}	-	MPa^{-r} s^{-1}	*Duddu and Waisman* [2012a]
r	0.43	-	-	*Pralong et al.* [2005]
k_1	-2.63	-	-	*Duddu and Waisman* [2012a]
k_2	7.24	-	MPa^{-1}	*Duddu and Waisman* [2012a]
α	0.21	-	-	*Pralong et al.* [2005]
β	0.63	-	-	*Pralong et al.* [2005]
D_{cr}	0.6	-	-	*Duddu and Waisman* [2012b]
D_{max}	0.97	-	-	*Duddu and Waisman* [2012b]

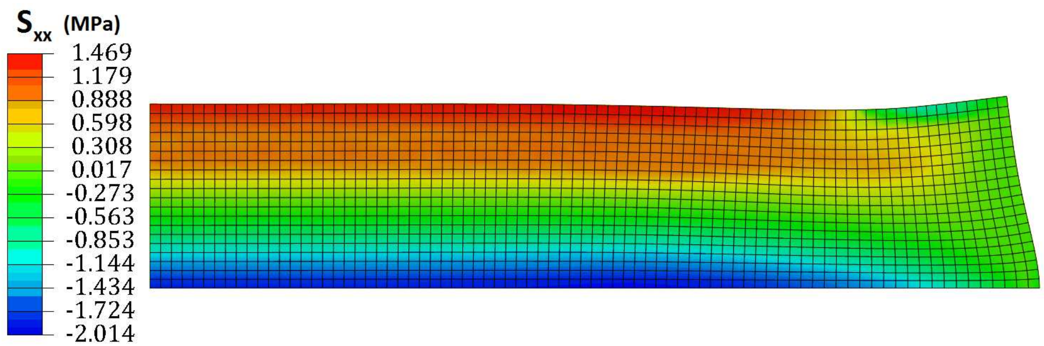

Figure I. Longitudinal stress variation in the ice slab due to creeping flow of the slab after 1 day. Note that in these simulations ice is considered to be homogeneous and undamaged.

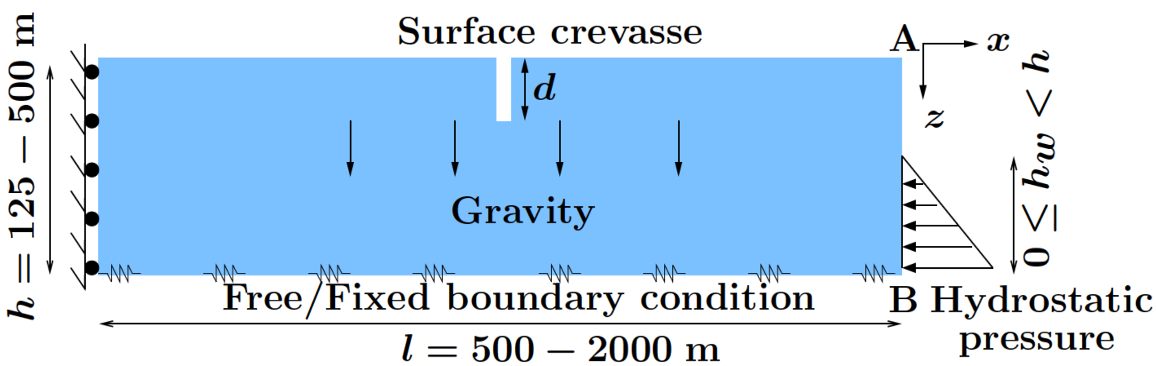

Figure II. Schematic representation of a notched rectangular ice slab extending under the action of gravity and hydrostatic seawater pressure. The basal boundary condition can be free slip or no slip (fixed). Points A and B denote the top right corner and the bottom right corner, respectively.

tip becomes very small or vanishes. If the simulation is continued forward in time we observe that new crevasses nucleate at the surface and begin to propagate downward as shown in Figure IV. We have run the simulations for longer times with no notch, with a single notch and with multiple notches. In all cases, our simulations indicate that eventually the whole slab will get cracked with several surface crevasses but none of the crevasses will not propagate the full depth but rather approach the full depth asymptotically. From these studies we arrive at the conclusion that only surface crevasses are not responsible for glacial calving events.

5. Comparison with Nye-zero stress model

In Section 3.2 of the paper, we compared the equilibrium crevasse depths estimated directly from crack growth simulations with those from the Nye zero-stress model. The Nye model depths were determined by evaluating the depth at which tensile stress vanishes from the stress contour plots of simulations conducted using the viscoelastic model without any damage. These simulations illustrate the importance of creep damage evolution which causes tensile stresses to be established at greater depths as the top surface fractures. An important point to note is that the Nye model is most appropriate for a field of closely-spaced crevasses, where the stress concentration at the bottom of the each fracture is reduced by the presence of other fractures nearly [*Mottram and Benn*, 2009]. However, the simulations presented in the paper only examined the behavior of a single crevasse in which stress concentrations are maximized.

To understand the role of creep damage evolution in the case when multiple closely spaced crevasses can propagate, we conduct the following simulation study. We consider a

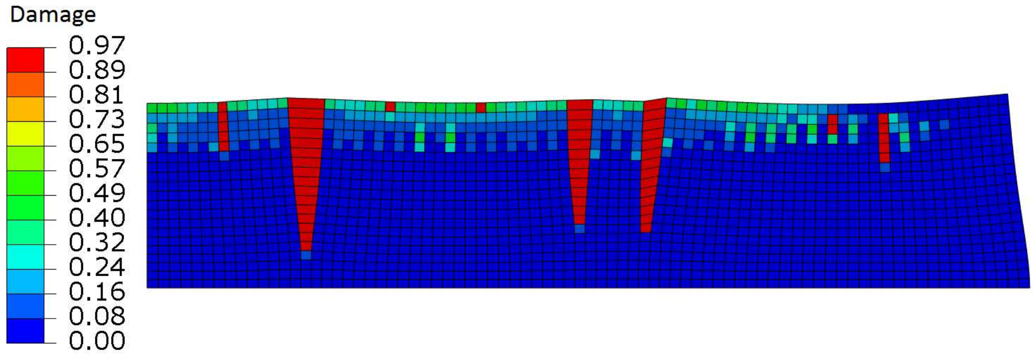

Figure III. Damage in the ice slab after 9 days without any initial notches. The crevasses nucleate and propagate at locations of maximum stress. The red colored elements show completely damage elements indicating the crack path. The blue elements have negligible damage.

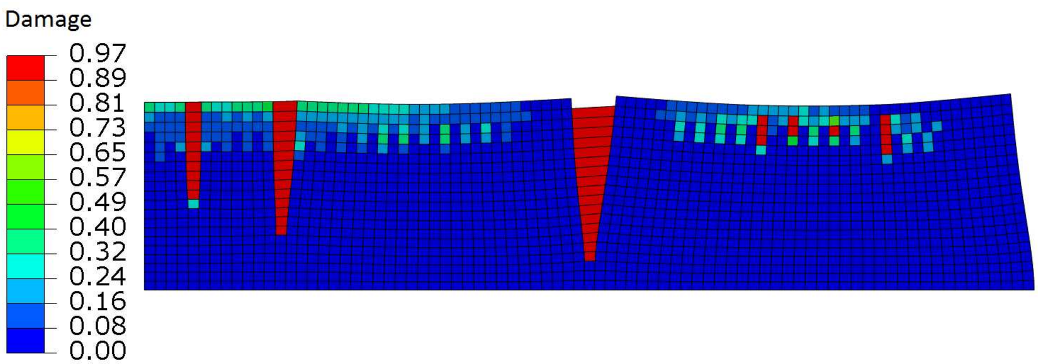

Figure IV. Damage in the ice slab after 8 days without one initial notch. At longer times multiple crevasses are found to nucleate and propagate near the top surface of the ice slab.

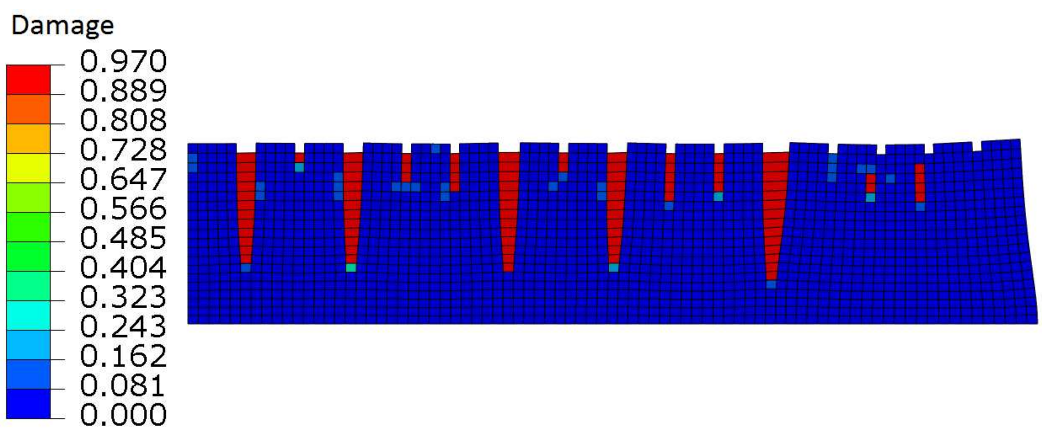

Figure V. Damage in the ice slab showing the development of a field closely spaced crevasses. The deepest crevasse after 9 days penetrates about 75% of the ice slab thickness. Note that no crevasses appear near the notches that are located at a distance less than one ice slab thickness from the terminus.

2000 m x 500 m ice slab with several initial notches placed 100 m apart in order to simulate a closely spaced field of crevasses using a coarse resolution mesh (see Figure V). In this case, we found that the stress at the crack tip is smaller and the crevasse propagation rate was slower due to the shadowing effect of nearby fractures. An interesting point to observe in Figure V is the existence of a periodic length scale for deeper crevasses, wherein one or two shallow crevasses can exist in between two deep crevasses. This periodic length scale (distance between deep crevasses) depends on the thickness of the ice slab that determines the flow velocity variation across its depth. The simulation predicts that after 9 days the deepest crevasse penetrated only 75% of the ice slab thickness whereas with one single crevasse the 9-day penetration depth was 85% and the Nye depth is estimated to be about 60% ice thickness. However, we still find that the Nye zero stress model predicted shallower crevasses compared to the direct crack growth simulations. This is because in reality only some crevasses will grow deeper while other crevasses are stunted due to crack shielding effects.

Therefore, a closely spaced field of crevasses all growing at the same rate and having the same depth is impossible to simulate.

6. Favorable location of crevasses

In all the simulations presented in this supplementary article we assume free slip at the ice slab base. There results indicate that surface crevasses propagate to greater depths when they are located at a distance greater than one ice slab thickness. This is because the tensile stress, induced by the depth variation in ice flow velocity, requires this length scale for attaining the maximum value as shown in Figure I. Beyond a distance of one ice slab thickness from the terminus the stress is nearly constant with some minor variation. In Section 3.3 of the paper, we estimated the favorable location for a crevasse that is closest to the terminus based on the location of the maximum tensile stress, whose distance $l_{\max}$ is measured from the terminus. The idea, therein, was to demonstrate that the boundary conditions at the glacier base and terminus can alter the stress distribution in ice slab and can not only affect the crevasse depth but also the crevasse location. Since damage evolution is dictated by tensile stress, we assumed the location of the maximum tensile to be the location of the first crevasse and eventually, this crevasse leads to a calving event. Thus, determining $l_{\max}$ can give us an estimate for the size of iceberg that is likely to calve from the glacier.

References

Ando, K., Y. Takeda, and K. Takezoe, Brittle and ductile creep rupture life prediction of 1 crmov steel notched thick plates, *Journal of Pressure Vessel Technology - ASME*, *12*(3), 225–232, 1990.

Becker, A. A., T. H. Hyde, W. Sun, and P. Andersson, Benchmarks for finite element analysis of creep continuum damage mechanics, *Computational Materials Science*, *25*(1-2), 34–41, 2002.

Chow, C., J. Mao, and J. Shen, Nonlocal damage gradient model for fracture characterization of aluminum alloy, *International Journal of Damage Mechanics*, *20*(7), 1073–1093, 2011.

de Borst, R., Some recent developments in computational modeling of concrete fracture, *International Journal of Fracture*, *86*(1-2), 5–36, 1997.

Desmorat, R., F. Gatuingt, and F. Ragueneau, Nonlocal anisotropic damage model and related computational aspects for quasi-brittle materials, *Engineering Fracture Mechanics*, *74*, 1539–1560, 2007.

Duddu, R., and H. Waisman, A temperature dependent creep damage model for polycrystalline ice, *Mechanics of Materials*, *46*, 23–41, 2012a.

Duddu, R., and H. Waisman, A nonlocal continuum damage mechanics approach to simulation of creep fracture in ice-sheets, *Computational Mechanics*, p. online first, 2012b.

Hall, F. R., and D. R. Hayhurst, Modelling of grain size effects in creep crack growth using a non-local continuum damage, *Proceedings: Mathematical and Physical Sciences*, *433*(1888), 405–421, 1991.

Jirasek, M., and P. Grassl, Evaluation of directional mesh bias in concrete fracture simulations using continuum damage models, *Engineering Fracture Mechanics*, *75*, 1921–1943, 2008.

Kachanov, L. M., Time of the rupture process under creep conditions, *Izv. Akad. Nauk., USSR. Otd. Tekhn. Nauk.*, *8*(1), 26–31, 1958.

Karr, D. G., and K. Choi, A three-dimensional constitutive damage model for polycrystalline ice, *Mechanics of Materials*, *8*(1), 55–66, 1989.

Lemaitre, J., Evaluation of dissipation and damage in metals, *In: proceedings of the International Congress on the Mechanical Behavior of Materials (I.C.M.), Kyoto, Japan*, *1*, 1971.

Mahrenholtz, O., and Z. Wu, Determination of creep damage parameters for polycrystalline ice, *paper presented at Third International Conference on Ice Technology, Advances in Ice Technology, Massachusetts Institute of Technology, Cambridge, MA*, 1992.

Mottram, R., and D. Benn, Testing crevasse-depth models: a field study at Breiethamerkurjokull, Iceland, *Journal of Glaciology*, *55*(192), 746–752, 2009.

Murakami, S., Notion of continuum damage mechanics and its application to anisotropic creep damage theory, *Journal of Engineering Materials and Technology - Transactions of the ASME*, *105*(2), 99–105, 1983.

Murakami, S., M. Kawai, and H. Rong, Finite-element analysis of creep crack-growth by a local approach, *International Journal of Mechanical Sciences*, *30*(7), 491–502, 1988.

Pralong, A., A., and M. Funk, Dynamic damage model of crevasse opening and application to glacier calving, *Journal of Geophysical Research*, *110*(B1), B01,309, 2005.

Rabotnov, I. N., On the equations of state for creep, *Progress in Applied Mechanics, the Prager Anniversary, North-Holland, Amsterdam.*, *8*, 307315, 1963.

Sidoroff, F., Description of anisotiopic damage application to elasticity, *IUTAM Colloqium on Physical Nonlinearities in Structural Analysis, Berlin, Germany*, p. 237244, 1980.

Verhoosel, C. V., M. A. Scott, T. J. R. Hughes, and R. de Borst, An isogeometric analysis approach to gradient damage models, *International Journal for Numerical Methods in Engineering*, *86*(1), 115–134, 2011.