Equivariant Complex Cobordism

by

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CHAPTER I

Foundations: Equivariant Stable Homotopy Theory

1.1 Introduction

Our goal is to advance understanding of the homotopical equivariant complex cobordism spectrum MU_G of a finite abelian group G. This will be accomplished on the one hand by concrete computation of the coefficients of MU_G over the non-equivariant unitary cobordism ring MU_*, generalizing results of Kriz [21], and on the other hand, by concrete computation of the equivariant formal group law arising from MU_G, insofar as MU_G^* is a complex oriented cohomology theory. The computation of $(MU_G)_*$ introduces a new Isotropy Separation Spectral Sequence, which tool has broader application to the computation of the coefficients of equivariant spectra. We will at times use cohomological grading MU_G^* and at other times use homological grading $(MU_G)_*$ as convenient, since in any case these graded rings are just re-indexings of each other.

The interest in equivariant formal group laws follows the non-equivariant case, where Quillen's Theorem establishes the universality of the formal group law corresponding to cobordism MU^* among formal group laws (cf. Ravenel [29]).

Theorem I.1. *(Quillen) For any formal group law F over any commutative ring* *with unit R there is a unique ring homomorphism $\theta : MU^* \to R$ such that*

$$F(x,y) = \theta F_U(x,y),$$

where F_U is the formal group law $\mu^*(x) = F_U(x \otimes 1, 1 \otimes x) \in MU^*[[x \otimes 1, 1 \otimes x]]$, and $\mu : \mathbb{C}P^\infty \times \mathbb{C}P^\infty \to \mathbb{C}P^\infty$ is the tensor product map.

That such a universal formal group law exists is trivial. Simply let

$$F(x,y) = x + y + \sum_{i,j} a_{i,j} x^i y^j$$

over the ring L generated by the $a_{i,j}$ subject to the minimal set of relations so that F is a formal group law. Lazard [22] was the first to study the structure of this formal group law in detail, and for this reason the ring of the universal formal group law, which we know by Quillen's theorem to be isomorphic to the complex cobordism ring, is denoted by L and is called the Lazard ring. Quillen's proof of Theorem I.1 in [28] exhibited the isomorphism between the formal group law corresponding to the complex cobordism ring MU^* and the universal formal group law over the Lazard ring L. Formal group laws in the non-equivariant sense play a crucial role in building a lexicon between algebraic and topological information. Quillen's Theorem represents a key piece of this correspondence, allowing us to study the complex cobordism ring using the algebra of formal group laws.

There is a corresponding notion of equivariant formal group laws. Any complex-oriented equivariant cohomology theory has a naturally associated equivariant formal group law. Since MU_G^* admits a natural complex orientation, there is an associated equivariant formal group law. Conjecture 2.4 of Greenlees [13] suggests that this equivariant formal group law is algebraically universal for equivariant formal group laws, and this is the proposed equivariant analog to Quillen's Theorem. If proved, Greenlees' Conjecture will allow similar algebraic methods to those developed in the non-equivariant case to be carried over to the study of equivariant spectra.

Before we can proceed to compute the equivariant complex cobordism ring, we must develop a sufficient scaffold of knowledge in equivariant stable homotopy theory. One may ask, why should I care about equivariant stable homotopy theory? Beside its inherent interest and anology with nonequivariant homotopy theory, Hill, Hopkins, and Ravenel's solution of the Kervaire Invariant One problem [16] demonstrated the applicability of the structures of equivariant stable homotopy theory to important problems in mathematics. Hill, Hopkins, and Ravenel show that elements of Kervaire invariant one in $\pi_{2^{j+1}-2}S^0$ exist only for $j \leq 6$, so that smooth framed manifolds of Kervaire invariant one exist only in dimensions 2, 6, 14, 30, 62, and possibly 126. Their proof relies heavily upon the structures of equivariant stable homotopy theory, and the study of a $\mathbb{Z}/8$-spectrum Ω.

Omitting some details, the $\mathbb{Z}/8$-spectrum Ω is constructed by considering the smash product

$$MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}} \wedge MU_{\mathbb{R}}$$

of four copies of the real cobordism spectrum as a $\mathbb{Z}/8$-spectrum with action

$$(x_1, x_2, x_3, x_4) \mapsto (\overline{x_4}, x_1, x_2, x_3).$$

Ω is then obtained by inverting an equivariant version of the Bott periodicity class and taking homotopy fixed points. The nonexistence of elements of Kervaire invariant one follows from three theorems proved by Hill, Hopkins, and Ravenel [16]:

Theorem I.2. *(Detection Theorem) Elements of Kervaire invariant one have nonzero images in the homotopy groups of* Ω*.*

Theorem I.3. *(Periodicity Theorem) The homotopy groups of Ω are periodic with period* 256.

Theorem I.4. *(Gap Theorem) The homotopy groups of Ω in degrees $-4 < i < 0$ are all zero.*

The result follows.

There have also been substantial recent developments within equivariant stable homotopy theory, many of which concern complex cobordism, which will be discussed in Section 1.5. Comezana and May [8] proved a completion theorem for complex cobordism, and Greenlees and May [15] proved localization and completion theorems for modules over complex cobordism. Kriz [21] computed the equivariant complex cobordism ring of the p-primary cyclic group, and Strickland [31] gave a description of generators and relations for the equivariant complex cobordism ring of $\mathbb{Z}/2$. Sinha [30] gave a non-constructive description of generators and relations for MU_G when G is a torus, and deduced generators for all abelian groups. Greenlees [13] proved that the equivariant formal group law associated with the equivariant complex cobordism spectrum classifies equivariant formal group laws over Noetherian rings. In light of this recent progress, there has been a resurgence of interest in the subject, and there is much to be gained from building up concrete algebraic descriptions of the important objects of equivariant stable homotopy theory, and perhaps chiefly from developing such an understanding of equivariant complex cobordism.

Thus, we proceed to develop equivariant stable homotopy theory at a level suitable for our applications, and in the mean time note a curious result obtained by the author and Kriz [3] pertaining to the foundations of the theory. In later chapters, we will provide the promised computations in equivariant complex cobordism.

1.2 The Equivariant Stable Homotopy Category

Our exposition most closely follows the standard source: Lewis, May, and Steinberger's foundational book *Equivariant Stable Homotopy Theory* [23]. Let G be a compact Lie group. A *G-universe* is a countably infinite dimensional real inner product space on which G acts by isometries, which is the direct sum of its finite dimensional G-invariant sub-inner product spaces, which contains infinitely many copies of all of its finite-dimensional sub-inner product spaces, and which contains a trivial representation of G. A *complete G-universe* $\mathcal{U}$ is a universe which contains every irreducible representation of G. Recall that the *regular representation* of a group G is the representation afforded by the action of G on itself by left translation. If G is finite, this is given as the free real vector space generated by the elements of G, and can be decomposed as the direct sum of the irreducible representations of G with multiplicity their dimensions. If G is finite, the countable sum of copies of the regular representation gives a canonical complete G-universe, and since this work concerns finite groups, we will usually refer to *the* complete G-universe. A *based G-space* is a based topological space equipped with a continuous G-action which acts trivially on the basepoint. Given a finite-dimensional subrepresentation V of $\mathcal{U}$, the one-point compactification of V with basepoint at ∞ is a based G-space, denoted S^V. For $V \subset W$, write $W - V$ for the orthogonal complement of V in W. For any based G-space X define the suspension and loop spaces by V as

$$\Sigma^V X = X \wedge S^V \text{ and } \Omega^V X = F(S^V, X),$$

where $X \wedge Y = (X \times Y)/(X \vee Y)$ and $F(X, Y)$ is the function space of based G-maps from X to Y.

We are now in a position to define equivariant prespectra. An *indexing space* in a G-universe U is a finite dimensional G-invariant sub-inner product space. An *indexing sequence* is an increasing sequence $A = \{A_i | i > 0\}$ of indexing spaces such that $A_0 = \{0\}$ and $U = \cup A_i$. An *indexing set* $\mathcal{A}$ is a set of indexing spaces which contains an indexing sequence. A *G-prespectrum* D indexed on an indexing set $\mathcal{A}$ consists of based G-spaces D_V for each finite-dimensional subrepresentation $V \in \mathcal{A}$, and, for $V \subset W$ both in $\mathcal{A}$, structure maps

$$\sigma : \Sigma^{W-V} D_V \to D_W,$$

such that $\sigma : \Sigma^0 D_V \to D_V$ is the identity map, and for $V \subset W \subset Z$, the following diagram commutes:

(1.1)
$$\begin{tikzcd}
\Sigma^{Z-W}\Sigma^{W-V} D_V \arrow[r, "\Sigma^{Z-W}\sigma"] \arrow[d, "\cong"'] & \Sigma^{Z-W} D_W \arrow[d, "\sigma"] \\
\Sigma^{Z-V} D_V \arrow[r, "\sigma"'] & D_Z.
\end{tikzcd}$$

Thus, whereas nonequivariant spectra are indexed over the integers, G-prespectra are indexed over finite-dimensional representations of G. Let $G\mathcal{T}$ denote the category of based G-spaces. Because of the space-level adjunctions

$$G\mathcal{T}(\Sigma^V X, Y) \cong G\mathcal{T}(X, \Omega^V Y),$$

the maps $\sigma : \Sigma^{W-V} D_V \to D_W$ are equivalent data to based G-maps $\tilde{\sigma} : D_V \to \Omega^{W-V} D_W$ satisfying a similar commutative diagram, namely:

(1.2)
$$\begin{tikzcd}
D_V \arrow[r, "\tilde{\sigma}"] \arrow[d, "\tilde{\sigma}"'] & \Omega^{W-V} D_W \arrow[d, "\Omega^{W-V}\tilde{\sigma}"] \\
\Omega^{Z-V} D_Z \arrow[r, "\cong"'] & \Omega^{W-V}\Omega^{Z-W} D_Z.
\end{tikzcd}$$

A *map* $f : D \to E$ of G-prespectra indexed on $\mathcal{A}$ is then a collection of based G-maps $f_V : D_V \to E_V$ for $V \in \mathcal{A}$ finite-dimensional G-representations, such that for $V \subset W$

the following diagram commutes:

$$\begin{tikzcd} \Sigma^{W-V} D_V \arrow[r, "\Sigma^{W-V} f_V"] \arrow[d, "\sigma"'] & \Sigma^{W-V} E_V \arrow[d, "\sigma"] \\ D_W \arrow[r, "f_W"'] & E_W. \end{tikzcd} \tag{1.3}$$

While equivariant stable homotopy theory can be developed over arbitrary indexing sets, we will usually work with the *standard indexing set* consisting of all finite-dimensional subrepresentations of U. Moreover, we will usually work over the complete universe $\mathcal{U}$.

Following Lewis-May-Steinberger [23], the category of G-prespectra indexed over the complete universe $\mathcal{U}$ in this way is denoted $G\mathcal{PU}$. A G-prespectrum is an *inclusion prespectrum* if the structure maps $\tilde{\sigma}$ are all inclusions. A G-prespectrum is a *G-spectrum* if the maps $\tilde{\sigma}$ are all homeomorphisms, and the category of G-spectra indexed over $\mathcal{U}$ is denoted $G\mathcal{SU}$. The forgetful functor $l : G\mathcal{SU} \to G\mathcal{PU}$ has a left adjoint $L : G\mathcal{PU} \to G\mathcal{SU}$. In general, a construction of the functor L is complicated, but for *inclusion prespectra* D - prespectra whose structure maps $\widetilde{\sigma}$ are all inclusions - we have

$$LD \cong \mathrm{colim}\ \Lambda^V \Sigma^\infty D_V,$$

where $\Lambda^V \Sigma^\infty : G\mathcal{T} \to G\mathcal{SU}$ is thought of as "shift desuspension", and is left adjoint to the Vth space functor $\Omega^\infty \Lambda_V$, both of which will be defined later. Because G-prespectra were defined constructively and are well-understood, results about spectra are generally proved for prespectra first then mapped over using L. For instance, $G\mathcal{SU}$ has all colimits because $G\mathcal{PU}$ does, and left adjoints preserve colimits. $G\mathcal{SU}$ also has all limits.

We will need to define a smash product

$$\wedge : G\mathcal{SU} \times G\mathcal{T} \to G\mathcal{SU}$$

of a spectrum with a space. This is defined as follows:

Definition I.5. Let $D \in G\mathcal{PU}$ be a G-prespectrum, and $X \in G\mathcal{T}$ be a based G-space. Then $D \wedge X$ is defined by $(D \wedge X)_V = D_V \wedge X$ for representations V, with structure maps

$$\sigma = \sigma \wedge 1 : \Sigma^{W-V}(D_V \wedge X) \cong (\Sigma^{W-V} D_V) \wedge X \to D_W \wedge X.$$

The spectrum level functor is then defined by letting $E \wedge X = L(lE \wedge X)$ for $E \in G\mathcal{SU}$ and $X \in G\mathcal{T}$. One can define $X \wedge D$ and $X \wedge E$ analogously, and they are naturally isomorphic to $D \wedge X$ and $E \wedge X$, respectively.

We can also define function spectra.

Definition I.6. For $D \in G\mathcal{PU}$ and $X \in G\mathcal{T}$, $F(X, D) \in \mathcal{GPU}$ is defined by $F(X, D)_V = F(X, D_V)$, with structure maps

$$\tilde{\sigma} : F(X, D_V) \to^{F(1,\tilde{\sigma})} F(X, \Omega^{W-V} D_W) \cong \Omega^{W-V} F(X, D_W).$$

Definitions I.5 and I.6 allow us to define cylinders $E \wedge I_+$, cones $CE = E \wedge I$, suspensions $\Sigma E = E \wedge S^1$, free path spectra $F(I_+, E)$, path spectra $PE = F(I, E)$, and loops $\Omega E = F(S^1, E)$, where S^1, I, and I_+ all have trivial G-action, and the subscript "+" denotes union with a disjoint basepoint. One also gets generalized suspensions $\Sigma^V E = E \wedge S^V$ and generalized loops $\Omega^V E = F(S^V, E)$ of spectra.

Several other functors are of interest. There is a "zero-th space functor"

$$G\mathcal{PU} \to G\mathcal{T}$$

which sends $D \mapsto D_0$. This gives by restriction a functor

$$\Omega^\infty : G\mathcal{SU} \to G\mathcal{T}.$$

Infinite loop G-spaces and G-maps are those spaces and maps, respectively, which lie in the image of Ω^∞. This is an important class of spaces, studied for instance by May in [25]. A good reference is the book [4] of Adams.

Conversely, for a space $X \in G\mathcal{T}$, there is associated a suspension G-prespectrum D such that $D_V = \Sigma^V X$, and the maps $\sigma : \Sigma^{W-V} D_V \to D_W$ are determined by the natural isomorphisms $\Sigma^{W-V}\Sigma^V X \cong \Sigma^W X$. We obtain a functor

$$\Sigma^\infty : G\mathcal{T} \to G\mathcal{SU}$$

by taking $\Sigma^\infty X = LD$, D as above. Σ^∞ is left adjoint to Ω^∞ ([23], Proposition I.2.3).

Analogous to the zero-th space functor, we have for any Z in our indexing set the Zth space functor

$$\Omega^\infty \Lambda_Z : G\mathcal{SU} \to G\mathcal{T},$$

i.e. $\Omega^\infty \Lambda_Z(E) = E_Z$. This functor also has a left adjoint, denoted

$$\Lambda^Z \Sigma^\infty : G\mathcal{T} \to G\mathcal{SU}.$$

It is constructed as follows: first map $G\mathcal{T}$ to $G\mathcal{PU}$ by sending X to the prespectrum D such that $D_V = \Sigma^{V-Z} X$ if $Z \subset V$, and $D_V = \{*\}$ otherwise, and whose maps are given by the obvious isomorphisms $\Sigma^{W-V}\Sigma^{V-Z} X \cong \Sigma^{W-Z} X$. Then apply the functor L.

Let $\mathbb{R}$ denote the trivial representation of G. Then the functor $\Lambda^{\mathbb{R}^n}\Sigma^\infty$ is denoted simply by $\Lambda^n \Sigma^\infty$, and likewise $S^{\mathbb{R}^n}$ is denoted simply as S^n, $\Sigma^{\mathbb{R}^n} =: \Sigma^n$, etc. We can also define the *sphere spectra* $S^n \in G\mathcal{SU}$ (in both positive and negative dimensions) by

$$S^n = \Sigma^\infty S^n = \Sigma^n \Sigma^\infty S^0 \; ; \; S^{-n} = \Lambda^n \Sigma^\infty S^0 \text{ for } n \geq 0.$$

For closed subgroups $H \subset G$, we can define generalized spheres

$$S^n_H = (G/H)_+ \wedge S^n.$$

This allows us to define the homotopy groups, for $H \subset G$, $n \in \mathbb{Z}$, $E \in G\mathcal{SU}$:

$$\pi^H_n E := \pi(S^n_H, E)_G$$

is the homotopy classes of based G-maps from S^n_H to E.

Now we are in a position to define the weak equivalences in $G\mathcal{SU}$. A map $f : D \to E$ in $G\mathcal{SU}$ is a *weak equivalence* if for all $H \subset G$ and $n \in \mathbb{Z}$, the map

$$f_* : \pi^H_n D \to \pi^H_n E$$

is an isomorphism.

Before we can construct the equivariant stable category, we need to define the proper equivariant notion of cell spectra. $E \in G\mathcal{SU}$ is a *G-cell spectrum* if there are subspectra E_n (for $n \geq 0$) of E, wedges J_n of spheres S^q_H (in positive and/or negative dimensions), and maps $j_n : J_n \to E_n$, such that $E_0 = *$, $E_{n+1} = Cj_n$ is the mapping cone, and $E = \cup E_n$. $j_n|_{S^q_H}$ are called *attaching maps*, and $\{E_n\}$ is called the *sequential filtration* of E. A map $f : E \to F$ of G-cell spectra is *sequentially cellular* if $f(E_n) \subset F_n$ for every $n \geq 0$. A *G-cell subspectrum* of a G-cell spectrum E is a subspectrum $A \subset E$ that is itself a G-cell spectrum with sequential filtration $\{A_n\}$, such that $A_n \subset E_n$, and the composite $CS^q_H \to A_n \subset A \subset E$ is a cell of E with image in E_n. This just means that A is a union of cells from E.

A *G-CW spectrum* is a G-cell spectrum such that the attaching maps $S^q_H \to E_n$ factor through only cells of dimension at most q. Thus, if E is a G-CW spectrum, we obtain another filtration $\{E^n\}$ of E called the *skeletal filtration*, where E^n is the *n-skeleton* of E, or the union of cells of E with dimension at most n. A map

$f : E \to F$ of G-CW spectra is *cellular* if it preserves the skeletal filtration, so that $f(E^n) \subset F^n$.

Let $G\mathcal{CU}$ denote the category of G-CW spectra indexed over the complete universe $\mathcal{U}$ and cellular maps. Appropriate analogs of the Cellular Approximation Theorem, Homotopy Extension and Lifting Property, Whitehead's Theorem, and the Brown Representability Theorem all apply (cf. [23], pp. 29-30). We document these results here.

Theorem I.7. *(Cellular Approximation) A map $f : D \to E$ of G-CW spectra that is cellular when restricted to a cell subspectrum A of D is homotopic rel A to a cellular map. Thus, any map of G-CW spectra is homotopic to a cellular map and any two homotopic cellular maps are cellularly homotopic.*

Theorem I.8. *(HELP) Suppose D is a G-cell spectrum, A a cell subspectrum of D, and $e : E \to F$ is a weak equivalance of G-spectra. Suppose $hi_1 = eg$ and $hi_0 = f$ in the diagram*

$$\begin{tikzcd}
A \arrow[rr, "i_0"] \arrow[dd] & & A \wedge I^+ \arrow[dl, "h"'] \arrow[dd] & & A \arrow[ll, "i_1"'] \arrow[dl, "g"] \arrow[dd] \\
& F & & E \arrow[ll, "e"'] & \\
D \arrow[rr, "i_0"'] \arrow[ur, "f"] & & D \wedge I^+ \arrow[ul, dotted, "\tilde{h}"'] & & D. \arrow[ll, "i_1"] \arrow[ul, dotted, "\tilde{g}"']
\end{tikzcd}$$

Then there are maps $\tilde{h}$ and $\tilde{g}$ making the diagram commute, and the inclusion $A \to D$ is a cofibration.

Theorem I.9. *(Whitehead's Theorem) If $e : E \to F$ is a weak equivalence of G-spectra and D is a G-cell spectrum, then $e_* : \pi(D, E)_G \to \pi(D, F)_G$ is a bijection. If E and F are G-cell spectra, then e is an equivalence of spectra.*

Theorem I.10. *(Brown Representability Theorem) A contravariant functor*

$$T : hG\mathcal{CU} \to \textbf{\textit{Set}},$$

where ***Set*** *denotes the category of sets, is representable as* $TE = \pi(E, F)_G$ *for a* G*-CW spectrum* F *if and only if* T *takes wedges to products and homotopy pushouts to weak pullbacks.*

In this context homotopy pushout is a double mapping cylinder and a weak pullback satisfies the existence part of the definition of a pullback, but not necessarily the uniqueness part.

Thus it is possible, given a spectrum $E \in G\mathcal{SU}$, to make a natural choice of a weakly equivalent spectrum in $G\mathcal{CU}$. In other words, working in homotopy categories, there is a functor $\Gamma : hG\mathcal{SU} \to hG\mathcal{CU}$ and a natural weak equivalence $\gamma : \Gamma E \to E$ for $E \in G\mathcal{SU}$. This is Theorem I.5.12 of [23].

The equivariant stable homotopy category $\overline{h}G\mathcal{SU}$ is then defined by formally inverting the weak equivalences in $hG\mathcal{SU}$. The exact procedure is discussed in Lewis-May-Steinberger [23], but the crucial points are as follows. A spectrum $D \in hG\mathcal{SU}$ is *cocomplete* if for every weak equivalence of spectra $e : E \to F$,

$$e_* : \pi(D, E)_G \to \pi(D, F)_G$$

is a bijection. By Whitehead's Theorem, G-CW spectra are cocomplete in the homotopy category of G-spectra, and by the Cellular Approximation Theorem every G-spectrum E admits a cocompletion, which is the weak equivalence $\gamma : \Gamma E \to E$. We can then formally invert the weak equivalences, with maps from $X \to Y$ in the resulting category corresponding to maps $\Gamma X \to \Gamma Y$, with composition carried over from $hG\mathcal{SU}$. Note then that the functor $\Gamma : \overline{h}G\mathcal{SU} \to hG\mathcal{CU}$ is an equivalence.

The process of inverting weak equivalences in this way can be done in more general categories; the appropriate concept is the notion of a model category, as discussed in Dwyer and Spalinski [10]. A *model category* is a category $\mathcal{C}$ with three distinguished classes of morphisms, called *weak equivalences*, *fibrations*, and *cofibrations*, having all the properties than one would expect morphisms so named to have. This is a useful general concept for many applications in homotopy theory.

The equivariant stable category $\bar{h}G\mathcal{SU}$ is a *closed symmetric monoidal category*, meaning it has an associative, commutative, unital smash product functor (with unit the *sphere spectrum* $S = \Sigma^\infty S^0$), and a right adjoint function spectrum functor. $\bar{h}G\mathcal{SU}$ can be called a stable category because of the Desuspension Theorem:

Theorem I.11. *([23], Theorem I.6.1) For all finite dimensional real representations V of G, the natural adjunction maps*

$$\eta : E \to \Omega^V \Sigma^V E,$$

$$\epsilon : \Sigma^V \Omega^V E \to E$$

in $\bar{h}G\mathcal{SU}$ are isomorphisms, so that Σ^V and Ω^V are inverse self-equivalences of $\bar{h}G\mathcal{SU}$.

The analogous result is not true before passage to the stable category. The Desuspension Theorem is proved by constructing inverse adjoint equivalences Λ^V and Λ_V of $\bar{h}G\mathcal{SU}$, and such that Λ^V is naturally equivalent to Ω^V, hence Λ_V is equivalent to Σ^V. The result follows. We will construct Λ^V and Λ_V in the Section 1.4. Theoerem I.11 allows us to write $\Sigma^{-V} E$ for $\Omega^V E$, since we can think of this as desuspension by the representation V.

1.3 Homology and Cohomology

Just as in the nonequivariant case, the stable category $\overline{h}G\mathcal{SU}$ is equivalent to the category of cohomology theories on G-spectra, where such cohomology theories are graded on the free abelian group $RO(G)$ generated by the irreducible G-representations. A *virtual representation* is an element $a = V - W \in RO(G)$, where V and W are sums of distinct irreducible representations. For $a \in RO(G)$ as above, we define a generalized sphere by $S^a := \Sigma^{-W} S^V$. Given a spectrum $E \in \overline{h}G\mathcal{SU}$, we can now define E-homology and E-cohomology by

$$E_a Y = [S^a, Y \wedge E]_G \quad \text{and} \quad E^a Y = [Y, \Sigma^a E]_G \quad \text{for} \quad G\text{-spectra } Y,$$

as should be familiar from the non-equivariant case (cf. Adams [5]). We have not yet defined the smash product of two spectra, but we will do so shortly, thereby making sense of the above definitions. Of cousre, $E^*(Y)$ and $E_*(Y)$ are modules over $\pi_*(E)$. E_* denotes the E-homology of a point, which is isomorphic to $\pi_*(E^G)$. E^* is the E-cohomology of a point, and is also isomorphic to $\pi_*(E^G)$ as non-graded rings. The only difference between E_* and E^* is the grading, and in fact $E^* = E_{-*}$. We will have occasion below for both perspectives.

A spectrum E also gives (co)homology theories at the level of based G-spaces, namely

$$\widetilde{E}_* X = E_*(\Sigma^\infty X) \quad \text{and} \quad \widetilde{E}^* X = E^*(\Sigma^\infty X) \quad \text{for} \quad X \in G\mathcal{T}.$$

All of this is recorded in [23].

We need now to define the smash product of two G-spectra, and this can be done in a coherent manner. In the process we will also define the function spectrum functor. The constructions work as follows. First, we define an *external smash product* on prespectra

$$\wedge : G\mathcal{PU} \times G\mathcal{PU} \to G\mathcal{PU} \oplus \mathcal{U}$$

by

$$(D \wedge E)_{(V \oplus Z)} = D_V \wedge E_Z$$

with structure maps

$$\Sigma^{(W-V)\oplus(Y-Z)} D_V \wedge E_Z \cong \Sigma^{W-V} D_V \wedge \Sigma^{Y-Z} E_Z \to^{\sigma\wedge\sigma} D_W \wedge E_Y.$$

for prespectra D and E. The spectrum level definition is obtained by use of the functor L, namely $D \wedge E = L(lD \wedge lE)$ for spectra D and E.

We can also definite an external function spectrum functor

$$F : G\mathcal{PU} \times G\mathcal{PU} \oplus \mathcal{U} \to G\mathcal{PU}.$$

Here $F(D, E)_V = \mathcal{PU}(D, E[V])$, where $E[V] \in G\mathcal{PU}$ is the prespectrum with Zth space $E[V]_Z = E_{V\oplus Z}$ and structure maps induced by those of E. The structure maps of $F(D, E)$ are given by

$$\tilde{\sigma} : \mathcal{PU}(D, E[V]) \to \mathcal{PU}(D, \Omega^{W-V} E[W]) \cong \Omega^{W-V} \mathcal{PU}(D, E[W]).$$

If E is a G-spectrum, then $F(D, E)$ is also a spectrum, so F restricts to a functor on categories of spectra. The external smash product is left adjoint to the external function spectrum functor ([23], Proposition II.3.4).

Next, for a G-linear isometry $f : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$, one obtains a change of universe functor $f^* : G\mathcal{SU} \to G\mathcal{SU} \oplus \mathcal{U}$ defined by $(f^*E)_V = E_{f(V)}$, with structure maps

$$\sigma : \Sigma^{W-V} E_{f(V)} = E_{f(V)} \wedge S^{W-V} \to^{1\wedge f} E_{f(V)} \wedge S^{f(W)-f(V)} \to^{\sigma} E_{f(W)}.$$

f^* has a left adjoint $f_* : G\mathcal{SU} \oplus \mathcal{U} \to G\mathcal{SU}$, defined as follows.

Definition I.12. For a prespectrum $D \in G\mathcal{SU}\oplus\mathcal{U}$, $(f_*D)_V = D_{f^{-1}(V)} \wedge S^{V-f(f^{-1}(V))}$, with structure maps

$$D_{f^{-1}(V)} \wedge S^{V-f(f^{-1}(V))} \wedge S^{W-V} \cong D_{f^{-1}(V)} \wedge S^{f(f^{-1}(W))-f(f^{-1}(V))} \wedge S^{W-f(f^{-1}(W))}$$
$$\to^{1\wedge f\wedge 1} D_{f^{-1}(V)} \wedge S^{f^{-1}(W)-f^{-1}(V)} \wedge S^{W-f^{-1}(W)} \to^{\sigma\wedge 1} D_{f^{-1}(W)} \wedge S^{W-f(f^{-1}(W))}.$$

We can now definite *internal smash products and function spectra* for spectra $D, E \in G\mathcal{SU}$ by

$$D \wedge E = f_*(D \wedge E) \quad \text{and} \quad F(D, E) = F(D, f^*E),$$

where the smash product and function spectra functors on the right are the external versions. These functors give G-spectra in $G\mathcal{SU}$ as output. While this does depend on the choice of G-linear isometry $f : \mathcal{U} \oplus \mathcal{U} \to \mathcal{U}$. The following theorem from Lewis-May-Steinberger saves us from this potential problem.

Theorem I.13. *([23] Theorem II.1.7) The functors $f_* : GSU \to G \to GSU'$ induced by the various G-linear isometries $f : U \to U'$ induce canonically and coherently naturally equivalent functors on passage to the stable categories $\overline{h}G\mathcal{S}U$ and $\overline{h}G\mathcal{S}U'$.*

Thus, we have constructed coherent smash product and function spectrum functors for the stable category $\overline{h}G\mathcal{SU}$. The (co)homology theories defined above are with respect to this smash product.

1.4 Shift Desuspension and Loop Space of Suspension Spectra

The functors $\Omega^\infty\Lambda_V : G\mathcal{SU} \to G\mathcal{T}$ and $\Lambda^V\Sigma^\infty : G\mathcal{T} \to G\mathcal{SU}$ defined above are actually composites of the functors $\Omega^\infty : G\mathcal{SU} \to G\mathcal{T}$ and $\Sigma^\infty : G\mathcal{T} \to G\mathcal{SU}$ with functors $\Lambda_V, \Lambda^V : G\mathcal{SU} \to G\mathcal{SU}$, respectively. We will define Λ^V and Λ_V for a real G-representation V now, following Lewis-May-Steinberger [23]. The following functors, though constructed on the prespectrum level, preserve spectra, and hence restrict to endofunctors of $G\mathcal{SU}$.

Write $\mathcal{U} = U \oplus V^\infty$ (pull out all the copies of V for bookkeeping purposes). The finite dimensional representations in $\mathcal{U}$ may be written as $W = Z + V^n$ for some $Z \subset U$ and some $n \geq 0$. For $D \in G\mathcal{PU}$, define

$$\tilde{\sigma}_+ : D_{(Z+V^{n-1})} \to \Omega^V D_{(Z+V^n)}$$

to be $\tilde{\sigma}$ composed with the homeomorphism $\Omega^V D_{(Z+V^n)} \to \Omega^V D_{(Z+V^n)}$ which reinterprets the loop coordinate as the $(Z + V^{n+1}) - (Z + V^n)$ coordinate rather than the $(Z + V^n) - (Z + V^{n-1})$ coordinate.

We can now define $\Lambda^V D$ as a prespectrum by $\Lambda^V D_{(Z+V^n)} = \Omega^V D_Z$ if $n = 0$ and $\Lambda^V D_{(Z+V^n)} = D_{(Z+V^{n-1})}$ if $n \geq 1$. We define

$$\tilde{\sigma} : \Lambda^V D_{(Z+V^n)} \to \Omega^{W-Z} \Lambda^V D_{(W+V^n)}$$

and

$$\tilde{\sigma} : \Lambda^V D_{(Z+V^n)} \to \Omega^V \Lambda^V D_{(Z+V^{n+1})}$$

in terms of the original $\tilde{\sigma}$ corresponding to D to be $\tilde{\sigma} = \Omega^V \tilde{\sigma}$ if $n = 0$, and $\tilde{\sigma} = \tilde{\sigma}_+$ if $n \geq 1$.

Similarly, we define $\Lambda_V D_{(Z+V^n)} = D_{(Z+V^{n+1})}$, with structural maps defined in an analogous way using the map

$$\tilde{\sigma}_- : D_{(Z+V^n)} \to \Omega^V D_{(Z+V^{n+1})},$$

which is the composition of the original $\tilde{\sigma}$ corresponding to D and the inverse of the homeomorphism used to construct $\tilde{\sigma}_+$. Define $\widetilde{\sigma}$ to be the ordinary $\widetilde{\sigma}$ for D for $Z + V^n \subset W + V^n$ and $\widetilde{\sigma}_-$ for $Z + V^n \subset Z + V^{n+1}$. Λ_V and Λ^V are inverse adjoint equivalences of $G\mathcal{SU}$ ([23], Lemma I.7.2).

Now evidently shift desuspension Λ^V and loop space Ω^V are isomorphic functors on passage to the stable homotopy category $\overline{h}G\mathcal{SU}$, though they are not in general the same before inverting the weak equivalences. Nevertheless the present author and Kriz show in [3] that shift desuspension and loop space do correspond for *suspension spectra* - that is, spectra of the form $\Sigma^\infty X$ for a based G-space X- before passage to the stable category, and that is the main result of the paper referenced. This result should perhaps have been obtained earlier, except that it was generally assumed to be false. We present the theorem and its proof here.

Theorem I.14. *(Abram and Kriz, [3]) For any $X \in G\mathcal{T}$, there is a natural isomorphism*

$$\Lambda^V \Sigma^\infty X \cong \Omega^V \Sigma^\infty X$$

in $G\mathcal{SU}$.

Proof. Let $\mathcal{A}(V)$ denote the indexing set of all finite-dimensional real G-representations containing V. Let $E(V)$ be defined as a spectrum in $G\mathcal{SA}(V)$ by defining, for $W \in \mathcal{A}(V)$,

(1.4)
$$E(V)_W = E_{W-V}.$$

To define the structure maps, we need some notation. Let ${}_E\tilde{\sigma}_Z^W$ denote the E-structure map $\tilde{\sigma} : E_Z \to \Omega^{W-Z} E_W$. Then $E(V)$ has structure maps

(1.5)
$${}_{E(V)}\tilde{\sigma}_W^{W'} = {}_E\, \tilde{\sigma}_{W-V}^{W'-V}.$$

That the diagram (1.2) commutes for $E(V)$ follows from its commutativity for E, so $E(V)$ is a G-spectrum after application of the spectrification functor L.

The spectrum $E(V)$ just constructed is in fact naturally isomorphic to $\Omega^V E$. To see this, note that for $W \in \mathcal{A}(V)$, the structure maps

(1.6)
$${}_E\tilde{\sigma}_{W-V}^{W'-V} : E_{W-V} \to \Omega^V E_W$$

give an isomorphism

$$E(V)_W \to \Omega^V E_W. \tag{1.7}$$

That this gives an isomorphism on the level of spectra follows from the evident diagram

$$\begin{tikzcd} E_{W-V} \arrow[r, "\tilde{\sigma}^{W'-V}_{W-V}"] \arrow[d, "\tilde{\sigma}^{W}_{W-V}"'] & \Omega^{W'-W} E_{W'-V} \arrow[d, "\Omega^{W'-W}\tilde{\sigma}^{W}_{W'-V}"] \\ \Omega^V E_W \arrow[r] & \Omega^{W'-W}\Omega^V E_{W'}. \end{tikzcd} \tag{1.8}$$

But by (1.2) both compositions in (1.8) are equal to $\nu^{-1}\tilde{\sigma}^{W'}_{W-V}$, where ν is the natural isomorphism $\Omega^{W-V}\Omega^{W'-W}E_{W'} \to \Omega^{W'-V}E_{W'}$. In more detail, composing $\nu_{W'-W,V} \circ \Omega^{W'-W}\tilde{\sigma}^{W'}_{W'-V} \circ \tilde{\sigma}^{W'-V}_{W-V}$, which is $\tilde{\sigma}^{W'}_{W-V}$ by (1.2). Composing $\nu_{W'-W,V}$ with the bottom row with the left column gives $\nu_{V,W'-W} \circ \Omega^V\tilde{\sigma}^{W'}_{W} \circ \tilde{\sigma}^{W}_{W-V}$, which is $\tilde{\sigma}^{W'}_{W-V}$ by (1.2).

By cofinality, $\Lambda^V\Sigma^\infty X$ is naturally isomorphic to LD, where $D \in G\mathcal{PA}(V)$ is the prespectrum with W-th space

$$D_W = \Sigma^{W-V}X,$$

with the obvious structure maps. This gives a functor from $G\mathcal{PU}$ to $G\mathcal{PA}(V)$, and this commutes with spectrification (the functor L) on inclusion prespectra. We obtain an isomorphism

$$LD \cong E(V),$$

i.e.

$$\Lambda^V\Sigma^\infty X \cong \Omega^V\Sigma^\infty X, \tag{1.9}$$

which is what was to be proved. $\square$

Despite Theorem I.14, it is widely expected that shift desuspension and loop space should not correspond in general, since the obvious spectrum level maps fail to commute with the structure maps due to a switch of isomorphic representation summands in the definition of Λ^V. Surprisingly, this issue has not yet been settled.

1.5 Equivariant Complex Cobordism

Having developed the relevant foundations of equivariant stable homotopy theory, we are now in a position to define our key object of study, namely, the homotopical equivariant complex cobordism spectrum MU_G. First we will recall some geometrical history and definitions. An almost complex structure on a smooth real manifold M of even dimension is a complex structure on its tangent bundle τ_M, i.e. a map $J : E(\tau_M) \to E(\tau_M)$ which maps each fiber $\mathbb{R}$-linearly into itself, and such that $J(J(v)) = -v$ for every $v \in E(\tau_M)$. An almost complex structure J on M is a complex structure if every point x of M has a neighborhood U such that there is a diffeomorphism $h : U \cong \mathbb{C}^n$ whose derivative is complex linear. A smooth real manifold M of arbitrary dimension (not necessarily even) is stably complex if there is a $k \geq 0$ such that $M \oplus \mathbb{R}^k$ admits a complex structure. Similarly M admits a stably almost complex structure if $M \oplus \mathbb{R}^k$ admits a complex structure on its tangent bundle. A manifold admits a stably complex structure if and only if it admits a stably almost complex structure (cf. [27]). Two closed stably complex n-manifolds are *cobordant* if together they comprise the boundary of an $(n+1)$-dimensional stably complex manifold. Cobordism is an equivalence relation, and the set of equivalence classes is a graded ring under disjoint union and Cartesian product. Call it the *complex cobordism ring*, and denote it Ω_*^U. Let $MU(n)$ be the Thom space of the universal complex n-bundle $\gamma_n : EU(n) \to BU(n)$. With structure maps induced by the classifying map of the bundle $\gamma_n \oplus \underline{1}$, where $\underline{1}$ is a trivial line bundle, we get a nonequivariant spectrum MU. MU has a very nice description.

Theorem I.15. *(Milnor) The coefficient ring*

$$MU_* \cong \mathbb{Z}[x_1, x_2, x_3, \ldots]$$

for elements $x_k \in \pi_{2k}MU$*.*

Ω^U_* and MU^* are isomorphic rings. The Pontrjagin-Thom construction - to which we will return - provides the map $\Omega^U_* \to MU^*$, and the map $MU^* \to \Omega^U_*$ is given by smooth approximation and a transversality argument. This works as follows, following Milnor and Stasheff [27]. Let $f : MU^* \to \pi_*(MU)$ be an isomorphism of rings (non-graded), and suppose $x \in MU^*$. Then $f(x) \in \pi_*(MU)$ is represented by some map $g : S^m \to MU$ for some n. S^m is compact, so g has image inside some $MU(n)$. Now $MU(n)$ is the Thom space of the canonical bundle $\gamma_n : EU(n) \to BU(n)$. $MU(n) - \{\infty\}$ can be given the structure of a smooth manifold, and we can approximate g by a homotopyic map $\tilde{g}$ such that $g(z) = \tilde{g}(z)$ for every $z \in g^{-1}(\infty)$ and such that $\tilde{g}$ is smooth on $g^{-1}(MU(n) - \{\infty\})$. It can also be arranged for $\tilde{g}$ to be transverse to the zero cross-section $BU(n)$. Then $\tilde{g}^{-1}(BU(n))$ is a smooth manifold, and its cobordism class depends only on the homotopy class of g. This cobordism class is the image of $x \in MU^*$ in Ω^U_*. Complex cobordism gives a generalized (co)homology theory via the spectrum MU.

Equivariantly (for compact Lie groups G) one can also define a geometric notion of cobordism. Ω^U_{G*} is the cobordism ring of smooth closed G-manifolds M equipped with an equivalence class of embeddings

$$M \to V \oplus \mathbb{R}^n,$$

where V is a complex G-representation, and the equivalence relation on embeddings is defined in a standard way. In the end, the cobordism groups defined geometrically can be characterized as the homotopy groups of an equivariant spectrum that is not indexed over the complete universe. The resulting equivariant cohomology theory is not graded by representations. These naive equivariant complex cobordism groups are very difficult to compute explicitly, perhaps as hard as the stable homotopy groups of spheres.

One needs therefore a new notion, and the correct idea was first studied by tom Dieck [33]. tom Dieck's equivariant homotopical bordism spectrum MU_G is what I call *equivariant complex cobordism*, following Kriz [21], and we define this presently. First, we must recall some basic definitions and geometric constructions. If X is an inner product space on which G acts by isometries, and n is a non-negative integer, the *G-equivariant Grassmannian* $\mathrm{Gr}(n, X)$ is the set of all linear G-subspaces of X of dimension n. The Grassmannian $\mathrm{Gr}(n, X)$ can itself be given the structure of a G-space.

We will also need the Pontryagin-Thom construction, which works as follows. Suppose $p : E \to B$ is a real vector bundle over the paracompact base space B. The *Thom space* $T(E)$ with respect to the bundle E is defined as follows. Since B is paracompact, the vector bundle $p : E \to B$ can be metrized. Let $\mathbb{D}(E)$ be the subspace of elements of E with norm at most 1, and let $\mathbb{S}(E)$ be the subspace of E with norm exactly 1. There is a corresponding "disk bundle" with total space $\mathbb{D}(E)$ and a "sphere bundle" with total space $\mathbb{S}(E)$. The *Thom space* $T(E)$ is defined by

$$T(E) := \mathbb{D}(E)/\mathbb{S}(E).$$

One can also define the Thom space without reference to a metric. The sphere bundle $\mathbb{S}(E)$ is formed by taking the one-point compactification of each fiber. The Thom space is then given by joining all the points of $\mathbb{S}(E)$ which were added when forming the sphere bundle and using this as the basepoint.

We will define $MU_G = LD$ for a suitable prespectrum D. Define D by

$$D_V = Gr(n, \mathcal{U} \oplus V)^{\gamma_n},$$

where on the right-hand side $Gr(n, W)^{\gamma_n}$ denotes the Thom space by the canonical n-dimensional complex line bundle of the Grassmannian G-space of n-dimensional $\mathbb{C}$-subspaces of W. Here n is the complex dimension of the representation V. The structure maps are induced by the classifying maps for $\gamma \oplus (W - V)$.

MU_G is a ring spectrum, but it is not Noetherian. This has proven to be an obstacle to calculation (cf. Greenlees [13]), so nice descriptions of MU_G^* are difficult to obtain in general, but for G a finite abelian group useful descriptions may be obtained, and this is what we develop in the next chapter. I will note also that the homotopy groups $\pi_*(MU_G)$ have a geometric interpretation as stabilized cobordism groups, given as the direct limit under suspension. This was done by Bröcker and Hook [6].

The relationship between geometric and homotopical equivariant cobordism is not as simple in the equivariant case. The Pontrjagin-Thom construction still works, but transversality is false, so

$$MU_G^* \ncong \Omega^U_{G*}.$$

This was studied by Wasserman [35].

We recall now some recent results pertaining to equivariant complex cobordism. In [31], Strickland gives an explicit and simple description of $MU_{\mathbb{Z}/2}$ via generators and relations:

Theorem I.16. *(Strickland) Let $L = \{\sum a_{ij} x^i y^j\}$ denote the Lazard ring. Let R be generated over L by elements s_{ij} $(i, j \geq 0)$ and t_i $(i \geq 0)$ subject to the following* *relations:*

$$t_0 = 0;$$

$$s_{10} = 1;$$

$$s_{i0} = 0 \text{ for } i > 1;$$

$$t_k - s_{0k} = s_{00} t_{k+1};$$

$$s_{jk} - a_{jk} = s_{00} s_{j+1,k}.$$

Give R a grading by $|s_{ij}| = |a_{ij}| = 2(1-i-j)$ and $|t_k| = 2(1-k)$. Then there is an isomorphism $R \cong MU^*_{\mathbb{Z}/2}$ as graded MU^*-algebras.

Such explicit descriptions for more general MU_G would be valuable, and this is a worthy area for further research. To obtain his result, Strickland makes use of a pullback diagram of Kriz [21] describing $MU_{\mathbb{Z}/p}$, to which we will return. A good first step toward a more general result would be a similar diagram for MU_G, for more general G. In Chapter II we give such a description for G finite abelian.

In [13], Greenlees proves the following theorem:

Theorem I.17. *(Greenlees) For a finite abelian group G, MU^*_G classifies G-equivariant formal group laws over Noetherian rings, in the since that there is a homomorphism of rings from the G-equivariant analogue L_G of the Lazard ring to MU^*_G that is surjective and whose kernel is Euler-torsion, Euler-divisible and $\mathbb{Z}$-torsion.*

Euler-torsion and Euler-divisibility refer to being torsion and divisible with respect to equivariant Euler classes. We will return to the concept of equivariant Euler classes soon. Greenlees conjectures that MU^*_G classifies G-equivariant formal group laws in general for any abelian compact Lie group G. This motivates a study of the algebraic structure of the equivariant formal group law of equivariant complex cobordism, and this is discussed in Chapter III.

1.6 Sinha's Computation

The goal of the present thesis is to give an explicit description of the equivariant complex cobordism ring $(MU_G)_*$ of a finite abelian group. Another computation of $(MU_G)_*$ is due to Sinha [30]. Sinha's computation relies heavily upon the following theorem of Comezaña [8].

Theorem I.18. *(Comezaña) For any abelian group G, $(MU_G)_*$ is a free MU_*-module concentrated in even degrees.*

To state Sinha's result, we will need to know what is meant by an equivariant Euler class. Let $\mathbb{C}P^\infty_G = \mathbb{C}P(\mathcal{U})$ be the complex projective space on the complete G-universe, and let γ_G be the corresponding universal bundle. Since MU_G admits a canonical complex orientation, we can let $e \in \widetilde{MU}^2_G T(\gamma_G)$ be the orientation class, where T denotes the Thom space. Let V be a finite dimensional complex representaiton of G of real dimension $2k$. Then there is a corresponding Euler class $e_V \in \widetilde{MU}^{2k}_G T(\gamma_G \otimes V)$, where in $\gamma_G \otimes V$, the V is thought of as a G-gundle over a point. This is given as follows, following Kriz [20]. If L is an irreducible complex G-representation, then $e_L \in \widetilde{MU}^2_G T(\gamma_G \otimes L)$ is the image of e under the map

$$\widetilde{MU}^2_G T(\gamma_G) \to \widetilde{MU}^2_G T(\gamma_G \otimes L) \tag{1.10}$$

induced by the classifying map $\phi : \mathbb{C}P^\infty_G \to \mathbb{C}P^\infty_G$ of the bundle $\gamma_G \otimes L$. Now if $V = L_1 \oplus L_2 \oplus \cdots \oplus L_k$, then

$$e_V = \prod_{i=1}^{k} e_{L_i}, \tag{1.11}$$

where the product on the right hand side of 1.11 is calculated by the Thom diagonal. We will discuss equivariant complex orientation more in Chapter III, where they will be used to compute equivariant formal group laws.

Let $(S^1)^n$ be a torus, let V be a non-trivial irreducible complex representation of $(S^1)^n$, and let e_V be the Euler class associated to V. Let $K(V) \lhd (S^1)^n$ be the subgroup that acts trivially on V, and for all subgroups H of $(S^1)^n$ let

$$\mathrm{res}_H^{(S^1)^n} : (MU_{(S^1)^n})_* \to (MU_H)_*$$

be the homomorphism of algebras obtained by restriction to the subgroup H. By Comezaña's Theorem, we may fix a splitting s_V of $\mathrm{res}_{K(V)}^{(S^1)^n}$ as MU_*-modules, though this splitting need not be unique nor a ring homomorphism. Let β_V serve as an abbreviation for the composition $s_v \circ \mathrm{res}_{K(V)}^{(S^1)^n}$. Sinha defines an MU_*-linear operation $\Gamma_V : (MU_{(S^1)^n})_* \to (MU_{(S^1)^n})_*$ as follows:

Definition I.19. (Sinha) For $x \in (MU_{(S^1)^n})_*$, $\Gamma_V(x) \in (MU_{(S^1)^n})_*$ is the unique class such that

$$e_V \cdot \Gamma_V(x) = x - \beta_V(x).$$

Sinha's computation in [30] is accomplished by the following theorem.

Theorem I.20. *(Sinha) As an MU_*-algebra, $(MU_{(S^1)^n})_*$ is generated over the operations Γ_V by the classes e_V and $[\mathbb{C}P(\underline{m} \oplus V)]$, where m ranges over the natural numbers and V ranges over the non-trivial irreducible complex representations of $(S^1)^n$, subject to the following relations:*

(a) $e_V\Gamma_V(x) = x - \beta_V(x)$

(b) $\Gamma_V(\beta_V(x)) = 0$

(c) $\Gamma_V(e_V) = 1$

(d) $\Gamma_V(xy) = \Gamma_V(x)y + \beta_V(x)\Gamma_V(y) + \Gamma_V(\beta_V(x)\beta_V(y))$

(e)

$$\Gamma_V(\Gamma_W(x)) = \Gamma_W(\Gamma_V(x))\Gamma_W(\Gamma_V(\beta_W(x)))$$
$$- \Gamma_W(\Gamma_V(e_W))\beta_V(\Gamma_W(x)) - \Gamma_W(\Gamma_V(\beta_V(e_W)\beta_V(\Gamma_W(x))),$$

for any classes x and y in $(MU_{(S^1)^n})_*$.

Sinha also gives a surprising short exact sequence

$$0 \to (MU_{(S^1)^n})_* \to^{\cdot e_V} (MU_{(S^1)^n})_* \to^{\mathrm{res}^{(S^1)^n}_{K(V)}} (MU_{K(V)})_* \to 0.$$

This short exact sequence can be used to recover generators of $(MU_G)_*$ for any abelian group G. Due to the highly non-constructive proof of Comezaña's Theorem I.18, Sinha's generators are therefore necessarily non-explicit. What is remarkable about Theorem I.20 applied to $(MU_{S^1})_*$ is that changing generators within the choices allowed leads to an isomorphism of rings with relations of the same form.

In the next chapter, we will develop a more explicit algebraic description of $(MU_G)_*$ for a finite abelian group G. In the sense that we both make heavy use of localization by inverting Euler classes, our techniques are similar to that of [30]. However, utilizing the well-known Tate diagram, we obtain a more explicit description of $(MU_G)_*$ as an MU_*-module. In the case $G = \mathbb{Z}/p^n$ our description takes the form of an n-fold pullback diagram, generalizing the result of Kriz [21] for $G = \mathbb{Z}/p$. The description for a general finite abelian group is as the limit of a diagram obtained from several multifold pullbacks, one for each maximal series in G. Thus, we rely on induction arguments, and our result does not extend beyond the finite case. Nonetheless, we believe this work will facilitate computations in equivariant stable homotopy theory that were not possible before.

CHAPTER II

Computing The Equivariant Complex Cobordism Ring

Our goal is to build on the techniques of Kriz [21] to compute coefficients of the G-equivariant complex cobordism ring MU_G. We will develop this theory for G a finite abelian group.

2.1 Previous Work: The Case $G = \mathbb{Z}/p$

Let F denote Lazard's universal formal group law, and let V^G denote the G-fixed points of V. In [21], Kriz proves the following theorem.

Theorem II.1.

$$\begin{tikzcd}
(MU_{\mathbb{Z}/p})_* \arrow[r] \arrow[d] & MU_*[u_k, u_k^{-1}, b_k^{(i)} | i > 0, k \in (\mathbb{Z}/p)^\times] \arrow[d, "\phi"] \\
MU_*[[u]]/([p]_F u) \arrow[r, "\iota"] & MU_*[[u]]/([p]_F u)[u^{-1}]
\end{tikzcd} \tag{2.1}$$

is a perfect square and a pullback of rings, where ι is localization and $\phi(b_k^{(i)} u_k)$ is the coefficient of x^i in $x +_F [k]_F u$, where $b_k^{(0)} = 1$. This determines ϕ, since its target is a domain.

A *perfect square* is a diagram that is both a pushout and pullback of abelian groups. To prove this theorem, Kriz considered the Tate diagram

(2.2)

$$
\begin{tikzcd}
(E\mathbb{Z}/p_+ \wedge MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} \arrow[r] \arrow[d, "\simeq"'] & (MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} \arrow[r] \arrow[d] & \Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p} \arrow[d] \\
(E\mathbb{Z}/p_+ \wedge F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}))^{\mathbb{Z}/p} \arrow[r] & F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p})^{\mathbb{Z}/p} \arrow[r] & \widehat{MU}_{\mathbb{Z}/p}.
\end{tikzcd}
$$

The bottom right corner of the diagram denotes the *Tate spectrum*

$$\widehat{E}_G = (\widetilde{EG} \wedge F(EG_+, E_G))^G, \tag{2.3}$$

when E_G is a G-spectrum. The top right corner of the diagram is the *geometric fixed points* of $MU_{\mathbb{Z}/p}$, which are defined as follows.

Definition II.2. For a G-spectrum E over a complete universe $\mathcal{U}$, $D := lE$, $l : G\mathcal{SU} \to G\mathcal{PU}$ the forgetful functor from spectra to prespectra, the geometric fixed points are defined by

$$\Phi^G E = L(\varinjlim \Sigma^{-V^G} D_V^G).$$

Lewis-May-Steinberger observe that, for $E \in G\mathcal{S}U$, $N \trianglelefteq G$, and $J = G/N$, $\mathcal{U}^N$ the J-universe obtained by taking N-fixed points of each finite dimensional subrepresentation of $\mathcal{U}$, there is a natural weak equivalence in $J\mathcal{SU}^N$:

$$\Phi^N E \simeq (\widetilde{E\mathcal{F}[N]} \wedge E)^N, \tag{2.4}$$

where $\mathcal{F}[N] = \{H \subseteq G : N \text{ is not contained in } H\}$. Thus, these spectra have the same homotopy groups. The notation of familes $\mathcal{F}$ and their classifying spaces $E\mathcal{F}$ will be revisited and explained in Section 2.2.

Proposition 2.5 of [21] states that, after passage to $\mathbb{Z}/p$-equivariant homotopy groups, the right-most square of the diagram (2.2) becomes the diagram (2.1). The hardest part of the proof of Theorem II.1 is then to compute the map ϕ, for which Kriz used tom Dieck's calculation of the geometric fixed points $\Phi^{\mathbb{Z}/p} MU_{\mathbb{Z}/p}$ in [34]. One useful property is that $\widetilde{E\mathbb{Z}/p}$ has an *infinite sphere model.* Namely, if V is the direct sum of the nontrivial irreducible real representations of $\mathbb{Z}/p$, then

$$\widetilde{E\mathbb{Z}/p} = S^{\infty V} = ho\varinjlim(S^0 \subset S^V \subset S^{2V} \subset \cdots). \tag{2.5}$$

We will see the relevant techniques again, so we will not repeat Kriz's argument here.

2.2 The General Case

The main result of the present thesis is the extension of Kriz's computation, recorded in Section 2.1, to the case of a finite abelian group. This result is also recorded in the paper [2]. Before we can state the result, we need to recall some definitions from Lewis-May-Steinberger [23].

Definition II.3. A *family* $\mathcal{F}$ of subgroups of a finite group G is a collection of subgroups that is closed under subconjugation. Given a family $\mathcal{F}$, a based G-space E is an $\mathcal{F}$*-space* if the isotropy group of each of its points other than the basepoint is in $\mathcal{F}$. An $\mathcal{F}$*-CW complex* is a G-CW complex such that the domain of every attaching map is of the form S^n_H for some $H \in \mathcal{F}$. An $\mathcal{F}$-space E is said to be *universal* if, for any (unbased) $\mathcal{F}$-CW complex X, there is a unique homotopy class of G-maps $X \to E\mathcal{F}$. $E\mathcal{F}$ must have the homotopy type of a G-CW complex, and this restriction ensures that $E\mathcal{F}$ is unique up to equivalence. An alternative characterization is that $(E\mathcal{F})^H$ be empty if $H \notin \mathcal{F}$ and nonempty and contractible if $H \in \mathcal{F}$. EG is universal for the trivial family $\{e\}$.

Having defined $E\mathcal{F}$, $\widetilde{E\mathcal{F}}$ is defined via a cofibration sequence

$$E\mathcal{F}_+ \to S^0 \to \widetilde{E\mathcal{F}}.$$

Therefore $\widetilde{E\mathcal{F}}^H \simeq S^0$ if $H \notin \mathcal{F}$ and $\widetilde{E\mathcal{F}}^H \simeq *$ if $H \in \mathcal{F}$. The main families needed for this work are the family $\mathcal{F}(H)$ of subgroups contained in a given subgroup $H \subseteq G$, and the family $\mathcal{F}[H]$ of subgroups not containing H. As is standard, we write EG/H for the universal space $E\mathcal{F}(H)$, and this is consistent with the definition of EG above.

Let G be a finite abelian group. Let $P(G)$ denote the partially ordered set of all nonempty sets $S = \{H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_k\}$ of subgroups of G which are totally ordered by inclusion. The ordering on $P(G)$ is given by $S \leq T$ if and only if $S \subseteq T$; that is, every subgroup in S is also in T.

Let X be a G-equivariant spectrum indexed on the complete G-universe $\mathcal{U}$. We define a functor

$$\Gamma = \Gamma_{G,X} : P(G) \to G\mathcal{SU}$$

as follows:

(2.6)

$$\Gamma(S) = F(EG/H_{k+}, \widetilde{E\mathcal{F}[H_k]} \wedge F(EG/H_{k-1+}, \ldots \wedge F(EG/H_{1+}, \widetilde{E\mathcal{F}[H_1]} \wedge X)\ldots)).$$

There is a canonical natural morphism of G-spectra

$$Y \to F(EG/H_+, \widetilde{E\mathcal{F}[H]} \wedge Y), \tag{2.7}$$

and the effect of Γ on arrows is defined by iterating these maps. Iterating the maps (2.7) also gives a canonical natural transformation

$$\mathrm{Const}_X \to \Gamma, \tag{2.8}$$

where Const_X is the constant functor on $P(G)$ with value X. The following theorem is our main result:

Theorem II.4. *For a finite abelian group G and $X = MU_G$ the G-equivariant complex cobordism spectrum, applying the homotopy groups functor π_* to (2.8) gives an isomorphism*

$$MU_{G*} \to^{\cong} \lim_{\leftarrow} \Gamma(S)_*. \tag{2.9}$$

By a transitivity argument, it suffices to take the inverse limit on the right side of (2.9) over the restriction of Γ to the subset $P'(G) \subset P(G)$ of sets S consisting of either a single group or a pair of groups $H_1 \subsetneq H_2$ which do not contain any intermediate subgroups. Also note that the validity of the isomorphism (2.9) in the category of abelian groups automatically implies its validity in the category of commutative rings, since the isomorphism concerns an inverse limit.

Before we prove Theorem II.4, we will compute the functor Γ_{G,MU_G}. Together with Theorem II.4, this computation will constitute an explicit description of the ring $(MU_G)_*$ for the finite abelian group G. By (2.4), $(\widetilde{E\mathcal{F}[H_1]} \wedge MU_G)_*^{H_1}$ is the geometric fixed points $\Phi^{H_1} MU_G$. It follows by Corollary 10.4 of [33] that

$$(\widetilde{E\mathcal{F}[H_1]} \wedge MU_G)_*^{H_1} = MU_{G*}[u_L^{\pm 1}, u_L^{(i)} | i > 0, L \in \overline{H_1^*}], \tag{2.10}$$

where $A^* = \mathrm{Hom}(A, S^1)$ and $\overline{A} = A - \{0\}$. We set $u_L^{(0)} = u_L$. Under the canonical map of (2.10) into

$$(\widetilde{E\mathcal{F}[H_1]} \wedge F(EG_+, MU_G))_*^{H_1} = MU_{G*}[[u_L | L \in \overline{H_1^*}]]/(u_L +_F u_M = u_{LM})[u_L^{-1} | L \in \overline{H_1^*}],$$

we have

$$u_L^{(i)} \mapsto \text{ the coefficient of } x^i \text{ in } x +_F u_L. \tag{2.11}$$

We will proceed via an induction argument. Assume we have calculated the coefficients of the H_{j-1}-spectrum

$$MU_{S,j-1} = (\widetilde{E\mathcal{F}[H_{j-1}]} \wedge F(EG/H_{j-2}, \ldots \widetilde{E\mathcal{F}[H_1]} \wedge MU_G) \ldots))^{H_{j-1}}. \tag{2.12}$$

$MU_{S,j-1}$ is a split H_j/H_{j-1}-spectrum only for $j=2$. Here a G-spectrum E is *split* if there is a nonequivariant map $\zeta : E \to E^G$ such that the composite $E \to E^G \to E$ with the inclusion $E^G \to E$ is homotopic to the identity. In any case, the Borel cohomology spectral sequence associated with

$$(2.13) \qquad F(EG/H_{j-1_+}, MU_{S,j-1})_*^{H_j}$$

collapses by evenness. It follows that (2.13) has an associated graded object isomorphic to

$$(2.14) \qquad (MU_{S,j-1})_*^{H_{j-1}} BH_j/H_{j-1}.$$

The coefficients

$$(2.15) \qquad (\widetilde{E\mathcal{F}[H_j]} \wedge F(EG/H_{j-1_+}, MU_{S,j-1}))_*^{H_j}$$

are computed from (2.13) by inverting the Euler classes u_L of the irreducible complex representations L of H_j which are non-trivial on H_j.

Let R_j, for $j \in \{0, \ldots, k\}$, be G/H_j-representatives of the irreducible non-trivial complex H_{j+1}/H_j-representations, where $H_0 = \{e\}$ and $H_{k+1} = G$. Define a ring

(2.16)

$$\mathcal{A}_S = MU_*[u_L, u_M^{-1}, u_N^{(i)} | i > 0, L \in R_0 \coprod \cdots \coprod R_k, M \in R_0 \coprod \cdots \coprod R_{k-1}, N \in R_0].$$

We define a topology $\mathcal{T}_S$ on the ring $\mathcal{A}_S$ as follows. A sequence of monomials

$$a_t \prod_{L \in R_1 \coprod \cdots \coprod R_k} u_L^{n(L,t)} \in \mathcal{A}_S$$

with

$$0 \neq a_t \in MU_*[u_L^{\pm 1}, u_L^{(i)} | i > 0, L \in R_0]$$

converges to 0 if and only if there exists a $j \in \{1, \ldots, k\}$ such that $n(L,t)$ is eventually constant in t for $L \in R_i$, $i > j$, and $n(L,t) \to^t \infty$ for $L \in R_j$. A sequence $\langle p_t \rangle$ from

$\mathcal{A}_S$ converges to 0 if and only if for all choices of monomial summands m_t of p_t, the sequence of monomials $\langle m_t \rangle$ converges to 0 in t. A set $T \subset \mathcal{A}_S$ is closed with respect to $\mathcal{T}_S$ if and only if the limit of every sequence in T convergent in $\mathcal{A}_S$ is in T.

We can define a closed ideal $I_S \subset \mathcal{A}_S$ as follows. Let I_S be generated by the relations

$$u_{L_1} +_F u_{L_2} = \left(\sum_{i=1}^{m} \right)_F u_{M_i},$$

where

$$L_1 L_2 \cong \prod_{i=1}^{m} M_i$$

and there is a $j \in \{1, \dots k\}$ such that

$$L_1, L_2 \in R_j$$

and

$$M_i \in R_{j+1} \coprod \cdots \coprod R_k.$$

Since $\mathcal{A}_S$, $\mathcal{T}_S$, and I_S depend on the group G, we will sometimes denote them by $\mathcal{A}_{G,S}$, $\mathcal{T}_{G,S}$, and $I_{G,S}$, respectively, when there is a possibility of ambiguity. We are now able to describe the coefficients of the spectrum $\Gamma(S)_*$.

Theorem II.5. *For $S \in P(G)$, the coefficients of the image of S under the functor $\Gamma : P(G) \to G\mathcal{SU}$ are*

$$\Gamma(S)_* = (\mathcal{A}_S)_{\widehat{\mathcal{T}_S}} / I_S, \tag{2.17}$$

i.e. the completion of $\mathcal{A}_S$ with respect to $\mathcal{T}_S$, modulo the closed ideal I_S.

What is meant by the completion of $\mathcal{A}_S$ at $\mathcal{T}_S$ is the following. With $\mathcal{T}_S$, $\mathcal{A}_S$ is a topological ring. There is a topological ring R containing $\mathcal{A}_S$ as a dense subring enducing $\mathcal{T}_S$ as the subspace topology, and such that R is complete. R can be constructed by Cauchy sequences in the standard way.

Proof. We will proceed by induction on the order of G and the order k of S. If $|G| = 1$ the statment is obvious, and for $k = 1$ the computation (2.10) of tom Dieck is already sufficient. Suppose $k > 1$, and assume $H_k \neq G$. Filter the ring

$$(\mathcal{A}_{G,S})_{\mathcal{T}_{G,S}}/I_{G,S}$$

by powers of the ideal

$$(u_L | L \in R_k).$$

The associated graded ring is by definition

$$((\mathcal{A}_{H_k,S})_{\widehat{\mathcal{T}}_{H_k,S}}/I_{H_k,S})[[u_L | L \in R_k]]/(u_L +_F u_M = u_{LM}),$$

which coincides with (2.14) by the inductive hypothesis. The filtration coincides with the Borel cohomology spectral sequence, so the statement follows.

Remaining is the case $H_k = G$. In this case we have, by definition,

$$(\mathcal{A}_{G,S})_{\widehat{\mathcal{T}}_{G,S}}/I_{G,S} = (\mathcal{A}_{G,S-\{G\}})_{\widehat{\mathcal{T}}_{G,S-\{G\}}}/I_{G,S-\{G\}}[u_L^{-1} | L \in R_k].$$

This is $\Gamma(S)_*$ by the induction hypothesis and (2.10). $\square$

We still need to compute Γ on arrows. This is determined by

$$u_L \mapsto u_L$$

and

$$u_L^{(i)} \mapsto \mathrm{coeff}_{x^i}(x +_F u_L).$$

Since the description of $\Gamma(S)_*$ in Theorem II.5 depends on choices of G/H_j-representatives of the irreducible complex H_{j+1}/H_j-representations, we need also to understand the nature of this dependence. Thus, we must specify how our description changes when different representatives are chosen. For $j > 1$, replacing L

by

$$L' = L \coprod_{i=1}^{m} M_i,$$

with $M_i \in R_{j+1} \coprod \cdots \coprod R_k$, we use the relation

$$u_{L'} = u_L +_F u_{M_1} +_F \cdots +_F u_{M_m}.$$

For $j = 1$, we use the relation

$$u_{L'} +_F x = u_L +_F (u_{M_1} +_F \cdots +_F u_{M_m} +_F x).$$

This is done by comparing coefficients at x^i, where

$$u_{M_1} +_F \cdots +_F u_{M_m} +_F x$$

is expanded as a power series in x.

Having completed our description of the functor Γ, it remains is to prove Theorem II.4. Before we proceed to the proof, we will prove an easier derived statement. Note that the natural transformation (2.8) induces a canonical morphism of G-spectra

$$\eta_X : X \to \operatorname{ho}\lim_{\leftarrow} \Gamma. \tag{2.18}$$

Theorem II.6. *η_X is an equivalence of G-spectra for any G-spectrum X.*

Proof. We will proceed by induction on the order of G. If $|G| = 1$, the statement is obvious. Assume inductively that the statement is true for G'-spectra whenever $|G'| < |G|$. Let

$$\check{\mathrm{P}}(G) = \{S \in P(G) | G \notin S\}.$$

Let $\mathcal{D}$ be the diagram

$$\begin{tikzcd} & \widetilde{E\mathcal{F}[G]} \wedge X \arrow[d] \\ \mathrm{ho}\lim_{\leftarrow} \Gamma|_{\check{\mathrm{P}}(G)} \arrow[r] & \widetilde{E\mathcal{F}[G]} \wedge \mathrm{ho}\lim_{\leftarrow} \Gamma|_{\check{\mathrm{P}}(G)}. \end{tikzcd} \tag{2.19}$$

Transitivity of homotopy limits gives an equivalence

$$\mathrm{ho}\lim_{\leftarrow} \Gamma \to \mathrm{ho}\lim_{\leftarrow} \mathcal{D}. \tag{2.20}$$

Recall that $EG/G = *$. Clearly if $H \subsetneq G$, there is a canonical inclusion $P(H) \subseteq \check{\mathrm{P}}(G)$. Let

$$Q = \{H \subsetneq G\}$$

be partially ordered by inclusion. Consider the functor

$$H \to \mathrm{ho}\lim_{\leftarrow} \Gamma|_{P(H)}$$

on Q.

There is a canonical equivalence

$$\mathrm{ho}\lim_{\leftarrow}(\mathrm{ho}\lim_{\leftarrow} \Gamma_{P(H)}) \to^{\cong} \mathrm{ho}\lim_{\leftarrow} \Gamma|_{\check{\mathrm{P}}(G)}, \tag{2.21}$$

where the left-most homotopy limit is taken over Q.

Since by the induction hypothesis

$$F(EG/H_+, X) \to \mathrm{ho}\lim_{\leftarrow} \Gamma_{P(H)}$$

is an equivalence for every $H \in Q$, (2.21) gives a canonical equivalence

$$F(E\mathcal{F}[G]_+, X) = \mathrm{ho}\lim_{\leftarrow} F(EG/?_+, X)|_Q \to^{\cong} \mathrm{ho}\lim_{\leftarrow} \Gamma|_{\check{\mathrm{P}}(G)}. \tag{2.22}$$

It follows that if we let $\mathcal{E}$ be the diagram

$$
\begin{tikzcd}
 & \widetilde{E\mathcal{F}[G]} \wedge X \arrow[d] \\
F(E\mathcal{F}[G]_+, X) \arrow[r] & \widetilde{E\mathcal{F}[G]} \wedge F(E\mathcal{F}[G]_+, X),
\end{tikzcd}
\tag{2.23}
$$

the canonical map

$$\operatorname{ho}\lim_{\leftarrow} \mathcal{E} \to \operatorname{ho}\lim_{\leftarrow} \mathcal{D} \tag{2.24}$$

is an equivalence. This commutes with the canonical morphism

$$X \to \operatorname{ho}\lim_{\leftarrow} \mathcal{E}. \tag{2.25}$$

The fiber of the canonical morphism

$$X \to \widetilde{E\mathcal{F}[G]}$$

maps to the fiber of the bottomw row of $\mathcal{E}$ by the canonical equivalence

$$E\mathcal{F}[G]_+ \wedge X \to E\mathcal{F}[G]_+ \wedge F(E\mathcal{F}[G]_+, X),$$

hence the canonical morphism (2.25) is an equivalence, which proves the theorem. □

While the derived Theorem II.6 was proved for any G-spectrum X, the isomorphism (2.9) of the main Theorem II.4 concerns the equivariant complex cobordism spectrum MU_G, and relies on its specific properties. We will prove Theorem II.4 by induction. Extending the methods of Section 2.1, the computation of MU_G requires an isotropy separation argument, with computations at various fixed point spectra. The definition (2.6) of Γ comes from iterating the Tate diagram construction and computing geometric fixed points, and this entails a discussion of a slightly more general class of spectra. Namely, we have the following definition.

Definition II.7. The *generalized equivariant complex cobordism class* $\mathcal{MU} = \bigcup \mathcal{MU}_G$ is the smallest class of G-equivariant spectra for all G finite abelian such that:

(1) $MU_G \in \mathcal{MU}_G$;

(2) If $R \in \mathcal{MU}_G$ and $H \subsetneq G$, then the geometric fixed points

$$\Phi^H R = (\widetilde{E\mathcal{F}[H]} \wedge R)^H \in \mathcal{MU}_{G/H};$$

(3) If $R \in \mathcal{MU}_G$, then $F(EG_+, R) \in \mathcal{MU}_G$.

Elements of $\mathcal{MU}_G$ are called *generalized MU_G's.*

A crucial feature is the following.

Proposition II.8. *The completion theorem of [15] and the statements of Section 7 of [13] remain valid with MU_G replaced by R, where R is any generalized MU_G.*

Before proving Proposition II.8, it is timely to restate the cited results. We give partial versions of these results to avoid the unecessary digression that would be required to introduce the various definitions and constructions required to state the precise versions of these results. For complete versions, the reader is therefore referred to the cited work. The augmentation ideal of J_G of a ring R_* with respect to a group G is the kernel $J_G = \ker(R_*^G \to R_*)$.

Theorem II.9. *(Greenlees and May, [15] Theorem 1.3) Suppose G is a finite group or a finite extension of a torus. For any sufficiently large finitely generated ideal $I \subset J_G$, J_G the augmentation ideal of $(MU_G)_*$,*

$$EG_+ \wedge M_G \to \Gamma_I(M_G) \textit{ and } (M_G)_{\widehat{I}} \to F(EG_+, M_G)$$

are equivalences for any MU_G-module M_G, where

$$\Gamma_I(M_G) = \{x \in M_G | I^N x = 0 \textit{ for some } N\},$$

and the subscript $\widehat{I}$ denotes completion at the ideal I.

Lemma II.10. *(Greenlees, [13] Lemma 7.1) If $H_j^I(M_G) = 0$ for $i \geq 0$ then $M \to H_0^I(M_G)$ induces an isomorphism in local cohomology $H_I^*(\cdot)$, hence also an isomorphism $\check{H}_I^i(\cdot)$ for $i \geq 1$. Here M_G is an MU_G-module, and we are referring to local (co)homology and local Čech cohomology.*

Here Čech cohomology is a theory based on open covers, but a rigorous review would take us too far afield.

Lemma II.11. *(Greenlees, [13] Lemma 7.2) $K(MU_G) \cong K(F(EG_+, MU_G))$, where for an MU_G-module X, $K(X)$ is defined by the exact sequence*

$$0 \to K(X) \to X_*^G(\widetilde{EG}) \to \check{H}_I^0(X_*^G).$$

Corollary II.12. *(Greenlees, [13] Corolllary 7.3) $MU_G \wedge \widetilde{EG}$ and $F(EG_+, MU_G) \wedge \widetilde{EG}$ are isomorphic in odd degrees.*

We can now prove Proposition II.8.

Proof. All generalized MU_G's arise by beginning with MU_A for some finite abelian group A, and successively applying the functor Φ^H or $F(EK_+, ?)$ for subquotients H, K of A. If only geometric fixed point functors are applied, we obtain an MU_G-algebra R where R_* is flat over $(MU_G)_*$ by Corollary 10.4 of Greenlees [13]. In this case the proofs of Theorem II.9, Lemma II.10, Lemma II.11, and Corollary II.12 apply without alteration, replacing MU_G by R.

If the construction of R from MU_A requires the application of function spectrum functors of the form $F(EK_+, ?)$, then the coefficients of R_* are still known by the tom Dieck style computations used when computing $\Gamma(S)_*$. We see that the Euler classes of representations still generate the augmentation ideal of R_*, and the proofs of the cited results still carry over. □

We are now ready to prove Theorem II.4.

Proof. The statement of Theorem II.4 is actually valid with MU_G replaced by any element $R \in \mathcal{MU}_G$, and we will prove this. We argue by induction on the order of G. For $|G| = 1$, the statement is clear. Now suppose $|G| > 1$ for a given G. If $\{e\} \neq H \subseteq G$, let $\mathcal{M}_H$ be the subdiagram of Γ given by

$$\begin{tikzcd} & \dots F(EG/H'_+, \widetilde{E\mathcal{F}[H']} \wedge R) \arrow[d] \\ F(EG/\{e\}_+, \widetilde{E\mathcal{F}[\{e\}]} \wedge R) \arrow[r] & \dots \widetilde{E\mathcal{F}[H']} \wedge F(EG/\{e\}_+, \widetilde{E\mathcal{F}[\{e\}]} \wedge R), \end{tikzcd} \tag{2.26}$$

where H' ranges over all subgroups $H \subseteq H' \subseteq G$. The dots in the diagram are an abbreviated notation, indicating that the top and bottom right corners of the diagram actually consist of diagrams indexed over the subset of $P(G)$ containing only sets S with $H \subseteq H_1$, the smallest subgroup from S. This subset is isomorphic to $P(G/H)$. Recall also that $\widetilde{E\mathcal{F}[\{e\}]} = S^0$.

For a given $H \neq \{e\}$, we can take homotopy limits at the top and bottom right corners of (2.26) to get a diagram

$$\begin{tikzcd} & \widetilde{E\mathcal{F}[H]} \wedge R \arrow[d] \\ F(EG_+, MU) \arrow[r] & \widetilde{E\mathcal{F}[H]} \wedge F(EG_+, R). \end{tikzcd} \tag{2.27}$$

We can combine the diagrams (2.27) for the various $H \neq \{e\}$ by putting in the canonical arrows between the corresponding upper right and lower right corners induced by inclusions of the subgroups H. This is equivalent to the homotopy limit of the diagram formed by taking the union of the diagrams $\mathcal{M}_H$, and this is isomorphic to Γ.

Alternatively, we can take homotopy limits over $H \neq \{e\}$ in the upper and lower right corners of (2.27) to obtain the ordinary Tate square for R:

(2.28)
$$\begin{tikzcd} & \widetilde{EG} \wedge R \arrow[d] \\ F(EG_+, R) \arrow[r] & \widetilde{EG} \wedge F(EG_+, R). \end{tikzcd}$$

By the induction hypothesis, the coefficients of the upper right and lower right corners of (2.27) equal the inverse limits of the coefficient functor applied to the corresponding parts of the diagram (2.26). Consider the spectral sequences corresponding to the homotopy limits of the upper right and lower right corners of (2.27). The vertical arrows of (2.27) give isomorphisms in filtration degrees at least 1 of the E_2-terms of these spectral sequences, and hence these terms may be ignored. This follows from the first sentence of the proof of Lemma II.11 in [13], which is valid for R by Proposition II.8. It follows that the corners of the diagram (2.28) for R are obtained as the non-derived limits of the corresponding parts of Γ.

The homotopy limit of the Tate square can only have a derived term in filtration degree 1, but such a term cannot exist because it would create odd degree elements in $(MU_G)_*$, which cannot exit by [8]. This concludes the proof of the main theorem. $\square$

2.3 An Illustration: The Case $G = \mathbb{Z}/p^n$

Having proved our main theorem, we will now illustrate our result in the case $G = \mathbb{Z}/p^n$, and will provide a lower-level argument for our result in this case.

Theorem II.13. *Let $u_{[k]}$ denote $[p^k]_F u$, and*

$$R_k = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]}),$$

$$S_k = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]})[u_{[k]}^{-1}],$$

$$R^n = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^n - 1\}].$$

Then $(MU_{\mathbb{Z}/p^n})_*$ is the n-fold pullback of the diagram of rings

(2.29)

$$
\begin{tikzcd}
 & & & & R^n \arrow[d, "\phi_{n-1}"] \\
 & & & R_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d, "\phi_{n-2}"] & S_{n-1} \\
 & & R_2 \arrow[r, "\cdots"] \arrow[d, "\phi_1"] & S_{n-2} & \\
 & R_1 \arrow[r, "\psi_1"] \arrow[d, "\phi_0"] & S_1 & & \\
R_0 \arrow[r, "\psi_0"] & S_0. & & &
\end{tikzcd}
$$

The maps ψ_k are localization by inverting $u_{[k]}$, and the maps ϕ_k are determined by the properties of sending $u_{[k+1]}$ to $[p]_F u_{[k]}$ and $b_j^{(i)} u_j$ to the coefficient of x^i in $x +_F [j]_F u_{[k]}$. ϕ^{n-1} is determined by the property of sending $b_j^{(i)} u_j$ to the coefficient of x^i in $x +_F [j]_F u_{[k]}$.

Proof. Beginning with the cofibration of $\mathbb{Z}/p^n$-spaces

$$E\mathbb{Z}/p^n{}_+ \to S^0 \to \widetilde{E\mathbb{Z}/p^n},$$

smash with $MU_{\mathbb{Z}/p^n}$ and $F(E\mathbb{Z}/p^n{}_+, MU_{\mathbb{Z}/p^n})$ to obtain the Tate diagram, whose right square is

(2.30)

$$
\begin{tikzcd}
(MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n} \arrow[r] \arrow[d] & (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n} \arrow[d] \\
F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n} \arrow[r] & \widehat{MU}_{\mathbb{Z}/p^n}.
\end{tikzcd}
$$

This is a pullback square, since the left vertical map

$$(E\mathbb{Z}/p^n{}_+ \wedge MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n} \to (E\mathbb{Z}/p^n{}_+ \wedge F(E\mathbb{Z}/p^n{}_+, MU_{\mathbb{Z}/p^n}))^{\mathbb{Z}/p^n}$$

of the Tate diagram is an isomorphism.

We will be able to compute the bottom row of (2.30), but to compute the top right we will have to express it as a pullback. This is done by considering another Tate diagram. In particular, consider the exact sequence $0 \to P_1 \to \mathbb{Z}/p^n \to Q_1 \to 0$, where $P_1 \cong \mathbb{Z}/p$, and $Q_1 \cong \mathbb{Z}/p^{n-1}$. We first take the P_1-fixed points of $\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n}$. Beginning with the cofibration of Q_1-spaces

$$EQ_{1+} \to S^0 \to \widetilde{EQ_1}$$

and smashing with $(\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1}$ and $F(EQ_{1+}, (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1})$, then taking Q_1-fixed points gives a Tate diagram, whose left vertical map is an isomorphism and whose right square is therefore a pullback of rings:

(2.31)

$$\begin{tikzcd}
((\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1})^{Q_1} \arrow[r] \arrow[d] & (\widetilde{EQ_1} \wedge (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1})^{Q_1} \arrow[d] \\
F(EQ_{1+}, (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1})^{Q_1} \arrow[r] & (\widetilde{EQ_1} \wedge F(EQ_{1+}, (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1}))^{Q_1}.
\end{tikzcd}$$

Notice that the top left corner of (2.31) is

$$((\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1})^{Q_1} = (\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n},$$

which is also the top right corner of (2.30). We will show later how to compute the bottom row of (2.31). The difficulty is again to compute the top right corner.

For the sake of notation, it is convenient to note now that

$$(\widetilde{E\mathbb{Z}/p^n} \wedge MU_{\mathbb{Z}/p^n})^{P_1} = \Phi^{P_1} MU_{\mathbb{Z}/p^n}. \tag{2.32}$$

This follows from (2.4) and the fact that $\widetilde{E\mathbb{Z}/p^n} = \widetilde{E\mathcal{F}[P_1]}$, which is true because both sides of this equation have fixed point set $*$ with respect to the trivial subgroup $\{e\}$, and fixed point set S^0 with respect to any other subgroup of $\mathbb{Z}/p^n$. The top right corner of (2.31) can then be written as $(\widetilde{EQ_1} \wedge \Phi^{P_1} MU_{\mathbb{Z}/p^n})^{Q_1}$.

As may be anticipated, we proceed by considering the exact sequence $0 \to P_2 \to Q_1 \to Q_2 \to 0$, where $P_2 \cong \mathbb{Z}/p$ and $Q_2 \cong \mathbb{Z}/p^{n-2}$. Rather than taking Q_1-fixed points all at once, we begin by taking P_2-fixed points $(\widetilde{EQ_1} \wedge \Phi^{P_1} MU_{\mathbb{Z}/p^n})^{P_2}$. A useful computation is the following.

Lemma II.14. *Let P^2 denote the subgroup of $\mathbb{Z}/p^n$ isomorphic to $\mathbb{Z}/p^2$. Then*

$$(\widetilde{EQ_1} \wedge \Phi^{P_1} MU_{\mathbb{Z}/p^n})^{P_2} \cong \Phi^{P^2} MU_{\mathbb{Z}/p^n}. \tag{2.33}$$

Proof. $\widetilde{EQ_1} = \widetilde{E\mathcal{F}[P_2]}$, so by (2.4) the left hand side of (2.33) is $\Phi^{P_2}\Phi^{P_1} MU_{\mathbb{Z}/p^n}$. The equation we have to verify is

$$(\widetilde{E\mathcal{F}[P_2]} \wedge (\widetilde{E\mathcal{F}[P_1]} \wedge MU_{\mathbb{Z}/p^n})^{P_1})^{P_2} = (\widetilde{E\mathcal{F}[P^2]} \wedge MU_{\mathbb{Z}/p^n})^{P^2}. \tag{2.34}$$

But both sides of (2.34) are isomorphic to $(\widetilde{E\mathcal{F}[P_1]} \wedge MU_{\mathbb{Z}/p^n})^{P^2}$. $\square$

Now we are ready to give the induction argument. For $1 \le k < n$, let P^k denote the subgroup of $\mathbb{Z}/p^n$ isomorphic to $\mathbb{Z}/p^k$. Suppose we have already considered the exact sequence $0 \to P_k \to Q_{k-1} \to Q_k \to 0$, for $P_k \cong \mathbb{Z}/p$, $Q_{k-1} \cong \mathbb{Z}/p^{n-k+1}$, and hence $Q_k \cong \mathbb{Z}/p^{n-k}$. We are trying to compute $(\widetilde{EQ_k} \wedge \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{Q_k}$. We consider an exact sequence $0 \to P_{k+1} \to Q_k \to Q_{k+1}$, with $P_{k+1} \cong \mathbb{Z}/p$ and $Q_{k+1} \cong \mathbb{Z}/p^{n-k-1}$, and rather than taking Q_k-fixed points all at once, we begin by taking P_{k+1}-fixed points. Now $(\widetilde{EQ_k} \wedge \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{P_{k+1}} = \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n}$, and the proof is the same as that for Lemma II.14. Taking the cofiber sequence of Q_{k+1}-spaces

$$EQ_{k+1_+} \to S^0 \to \widetilde{EQ_{k+1}},$$

and smashing with $\Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n}$ and $F(EQ_{k+1_+}, \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n})$, then taking Q_{k+1}-fixed points, gives a Tate diagram whose left vertical map is an isomorphism and whose right square is the pullback of rings:

(2.35)

$$\begin{tikzcd}
(\Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n})^{Q_{k+1}} \arrow[r] \arrow[d] & (\widetilde{EQ_{k+1}} \wedge \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n})^{Q_{k+1}} \arrow[d] \\
F(EQ_{k+1+}, \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n})^{Q_{k+1}} \arrow[r] & (\widetilde{EQ_{k+1}} \wedge F(EQ_{k+1+}, \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n}))^{Q_{k+1}}.
\end{tikzcd}$$

Of course, if $k = n-1$ then $\Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n} = \Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n}$, $Q_{k+1} = \{e\}$, and there is no need to consider a pullback diagram such as (2.35), since tom Dieck's method is in that case already sufficient for computation. The result of this induction is a description of $(MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n}$ as an n-fold pullback diagram as in (2.29).

Since $\pi_*(MU_{\mathbb{Z}/p^n})^{\mathbb{Z}/p^n} = \pi_*^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n} = (MU_{\mathbb{Z}/p^n})_*$, it remains only to compute, on homotopy, the maps

$$\psi_k : F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{Q_k} \to (\widetilde{EQ_k} \wedge F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n}))^{Q_k}, \tag{2.36}$$

$$\phi_k : F(EQ_{k+1+}, \Phi^{P^{k+1}} MU_{\mathbb{Z}/p^n})^{Q_{k+1}} \to (\widetilde{EQ_k} \wedge F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n}))^{Q_k}, \tag{2.37}$$

and

$$\phi^{n-1} : \Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n} \to (\widetilde{EQ_{n-1}} \wedge F(EQ_{n-1+}, \Phi^{P^{n-1}} MU_{\mathbb{Z}/p^n}))^{Q_{n-1}}. \tag{2.38}$$

Lemma II.15.

$$\pi_*(F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{Q_k}) =$$
$$MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \dots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]}).$$

Proof. First, observe that

$$\Phi^{P^k} MU_{\mathbb{Z}/p^n} = (\widetilde{E\mathcal{F}[P^k]} \wedge MU_{\mathbb{Z}/p^n})^{P^k} = (S^{\infty V} \wedge MU_{\mathbb{Z}/p^n})^{P^k}, \tag{2.39}$$

where V is the direct sum of infinitely many copies of each irreducible representation of $\mathbb{Z}/p^n$ which is nontrivial on the subgroup P^k.

Recall that $MU_{\mathbb{Z}/p^n} = LD$ for a prespectrum D over the complete universe $\mathcal{U}$ such that $D_W = Gr(n, \mathcal{U} \oplus W)^{\gamma_n}$, where $Gr(n, W)$ is the $\mathbb{Z}/p^n$-space of n-dimensional complex subspaces of W, n the dimension of W, and the superscript γ_n denotes Thom space. Also recall that

$$\Phi^{P^k} MU_{\mathbb{Z}/p^n} = \lim_{\rightarrow} \Sigma^{-W^{P^k}} D_W^{P^k},$$

following Definition II.2. Let V_j be the irreducible representation of $\mathbb{Z}/p^n$ on which the generator acts by $e^{2\pi i j/p^n}$. For each $i \in \{1, 2, \ldots, p^k - 1\}$, let $\mathcal{U}^i$ be the direct sum of infinitely copies of each V_j such that $j \equiv i (\mod p^k)$. Now

$$(Gr(n, \mathcal{U} \oplus W)^{\gamma_n})^{P^k} \cong (Gr(n, \mathcal{U})^{\gamma_n})^{P^k} =$$

$$\bigvee_{0 \leq m, k_j \leq n; \sum_{j \in P^{k \times}} k_j = n-m} Gr(m, \mathcal{U}^{P^k})^{\gamma_n} \wedge Gr(k_1, \mathcal{U}^1)_+ \wedge \cdots \wedge Gr(k_{p^k-1}, \mathcal{U}^{p^k-1})_+.$$

Then if $\dim(W \cap \mathcal{U}^i) = l_i$ and $\dim(W \cap \mathcal{U}^{P^k}) = m'$, $m' - m + \sum_{j=1}^{p^k-1} l_j - k_j = 0$, and

$$\Sigma^{-W^{P^k}} D_W^{P^k} \cong$$

$$\bigvee \Sigma^{-2m} Gr(m, \mathcal{U}^{P^k})^{\gamma^m} \wedge \Sigma^{2k_1 - 2l_1} Gr(k_1, \mathcal{U}^1)_+ \wedge \cdots \wedge \Sigma^{2k_{p^k-1} - 2l_{p^k-1}} Gr(k_{p^k-1}, \mathcal{U}^{p^k-1})_+.$$

Taking the colimit, the right hand side becomes

$$\bigvee MU \wedge \Sigma^{2k_1 - 2l_1} BU_+ \wedge \cdots \wedge \Sigma^{2k_{p^k-1} - 2l_{p^k-1}} BU_+. \tag{2.40}$$

As was done in [21] for $G = \mathbb{Z}/p$, we wish to write $(\Phi^{P^k} MU_{\mathbb{Z}/p^n})_*$ as a ring of Laurent series over $(MU \wedge BU_+ \wedge \cdots \wedge BU_+)_*$ in variables u_j. For $j \in \{1, 2, \ldots, p^k - 1\}$, select $j_0 \equiv j (\mod p^k)$, and let u_j denote the element of $\pi_{V_{j_0} - 2} MU_{\mathbb{Z}/p^n}$ multiplication by which gives $(V_{j_0} - 2)$-periodicity of $(MU_{\mathbb{Z}/p^n})_*$. Define $b_j^{(i)}$ so that $b_j^{(i)} u_j$ corresponds to the homotopy class $b_j \in MU_*BU(1)_+ = MU_*\{b_0, b_1, b_2, \ldots, b_{p^k-1}\}$ in the wedge summand of (2.40) with $m > 0$, $k_i = \delta_{ij}$ (the Kronecker delta), $l_i = 0$ for all i. Then

$$(2.41) \qquad (\Phi^{P^k} MU_{\mathbb{Z}/p^n})_* = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^k - 1\}].$$

We are interested in $F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{Q_k}$. Because $\Phi^{P^k} MU_{\mathbb{Z}/p^n}$ is a split spectrum,

$$F(EQ_{k+}, \Phi^{P^k} MU_{\mathbb{Z}/p^n})^{Q_k} = F(BQ_{k+}, MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^k - 1\}]).$$

The homotopy groups are calculated by considering the Gysin cofibration sequence

$$BQ_{k+} \to \mathbb{C}P^\infty_+ \to (\mathbb{C}P^\infty)^{(\gamma_1)^{p^{n-k}}},$$

where the right hand side is the Thom space, γ_1 the canonical line bundle on $\mathbb{C}P^\infty$. This gives the lemma. $\square$

Greenlees and May [14] then give that the map ψ_k is localization by inverting the Euler class. At the top right of the diagram (2.29), it is clear by the computations of tom Dieck that

$$(2.42) \qquad (\Phi^{\mathbb{Z}/p^n} MU_{\mathbb{Z}/p^n})_* = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^n - 1\}].$$

It is important to note that our construction in non-canonical, since for u_j we require a choice of representative $j_0 \cong j (\mathrm{mod}\ \ p^k)$. Nonetheless the other choices must lie in the image of ϕ_k. Now that the map ϕ_k sends $b_j^{(i)} u_j$ to the coefficient at x^i of $x +_F [j]_F u_{[k]}$ follows from the same proof method used by Kriz (Lemma 2.14, [21]). The computation of the map ϕ^{n-1} is also the same. We will give the argument for the map

$$\phi_k : R_{k+1} \to S_k,$$

recalling that

$$R_{k+1} = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^{k+1} - 1\}][[u_{[k+1]}]]/([p^{n-k-1}]_F u_{[k+1]}),$$

and

$$S_k = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, 2, \ldots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]})[u_{[k]}^{-1}].$$

Due to the similarity with Kriz [21], we will follow his notation and narrative for the remainder of this section. From this point, I have nothing to add to Kriz's argument, and include it just for completeness. First, consider the $\mathbb{Z}/p^n$-equivariant complex line bundle $\xi = \gamma_1 \otimes \alpha^{j_0}$ on $\mathbb{C}P^\infty \times E\mathbb{Z}/p^n$, where $\mathbb{Z}/p^n$ acts trivially on $\mathbb{C}P^\infty$, γ_1 is the canonical line bundle on $\mathbb{C}P^\infty$, and α^{j_0} is the equivariant line bundle on a point arising from the representation $\alpha^{j_0} : \mathbb{Z}/p^n \to \mathbb{C}^*$ for which the generator acts by $e^{2\pi i j_0/p^n}$, pulled back to $E\mathbb{Z}/p^n$, j_0 representatives chosen earlier. To make the theorem true, we should choose $j_0 = j$. Recall that $\mathbb{C}P_G^\infty = \mathbb{C}P\{\mathcal{U}\}$ refers to the space of complex lines in the complete G-universe $\mathcal{U}$, and $\gamma_{1,G}$ is the canonical G-equivariant complex line bundle on this space. There is a map of line bundles

$$\omega : \xi \to \gamma_{1,\mathbb{Z}/p^n}$$

obtained by considering the inclusion $\mathbb{C}P^\infty \subset \mathbb{C}P^\infty_{\mathbb{Z}/p^n}$, which gives a map

$$\rho : \gamma_1 \to \gamma_{1,\mathbb{Z}/p^n}.$$

There is also a composition

$$E\mathbb{Z}/p^n \to^{/\mathbb{Z}/p^n} B\mathbb{Z}/p^n \to \mathbb{C}P^\infty,$$

inducing a map

$$\lambda : \alpha^j \to \gamma_1.$$

The map ω is defined to be the composite map

$$\xi = \gamma_1 \otimes \alpha^j \to^{Id\otimes\lambda} \gamma_1 \to^{\rho} \gamma_{1,\mathbb{Z}/p^n}.$$

There is another map

$$\psi : \xi \to \gamma_{1,\mathbb{Z}/p^n}, \tag{2.43}$$

given as follows: Let $\mathbb{C}P^\infty_j = \mathbb{C}P\{\mathcal{U}^j\}$. Consider the map on based spaces

$$\mathbb{C}P^\infty \times E\mathbb{Z}/p^n \to \mathbb{C}P^\infty \cong \mathbb{C}P^\infty_j \subset \mathbb{C}P^\infty_{\mathbb{Z}/p^n}.$$

The pullback of $\gamma_{1,\mathbb{Z}/p^n}$ under this map is $\xi = \gamma_1 \otimes \alpha^j$, and this therefore defines the map ψ of (2.43).

By the classification of $\mathbb{Z}/p^n$-complex line bundles, the maps ω and ψ are homotopic through maps of equivariant line bundles. Therefore, the map

$$\mathbb{C}P^\infty_+ \wedge E\mathbb{Z}/p^n_+ \to \mathbb{C}P^\infty_+ \to^{\cong} \mathbb{C}P^\infty_{j\ +} \to (\mathbb{C}P^\infty_{\mathbb{Z}/p^n\ +})^{\gamma_{1,\mathbb{Z}/p^n}} \tag{2.44}$$

is $\mathbb{Z}/p^n$-homotopic to

(2.45)

$$\mathbb{C}P^\infty_+ \wedge E\mathbb{Z}/p^n{}_+ \to^{/\mathbb{Z}/p^n} (\mathbb{C}P^\infty \times B\mathbb{Z}/p^n)_+ \to \mathbb{C}P^\infty_+ \to \mathbb{C}P^\infty_{\mathbb{Z}/p^n\ +} \to (\mathbb{C}P^\infty_{\mathbb{Z}/p^n\ +})^{\gamma_{1,\mathbb{Z}/p^n}}.$$

The target of (2.44) and (2.45) maps to $\Sigma^2 MU_{\mathbb{Z}/p^n}$. Since $MU_{\mathbb{Z}/p^n}$ is a ring spectrum, and by adjunction, we get two maps

$$\mathbb{C}P^\infty_+ \wedge MU \to F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n}),$$

which on coefficients are maps

$$MU_*\mathbb{C}P^\infty \to \pi^{\mathbb{Z}/p^n}_{*-2}(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n}).$$

The map coming from (2.44) sends b_i to the image of $b_j^{(i)} u_j$.

To calculate the image of b_i under the map coming from (2.45), recall that the splitting $MU \to MU_{\mathbb{Z}/p^n}$ induces an isomorphism

$$\pi_* F(B\mathbb{Z}/p^n_+, MU) = \pi_*^{\mathbb{Z}/p^n} F(E\mathbb{Z}/p^n_+, MU) \to \pi_*^{\mathbb{Z}/p^n} F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n}). \tag{2.46}$$

The map coming from (2.45) is the composition of (2.46) with the coefficients of

$$MU \wedge \mathbb{C}P^\infty_+ \to \Sigma^{-2} F(B\mathbb{Z}/p^n_+, MU),$$

which is adjoint to

$$\begin{aligned} &MU \wedge (B\mathbb{Z}/p^n \times \mathbb{C}P^\infty)_+ \to \\ &\qquad MU \wedge \mathbb{C}P^\infty_+ \to MU \wedge (\mathbb{C}P^\infty_+)^{\gamma_1} \to MU \wedge \Sigma^{-2} MU \to \Sigma^{-2} MU. \end{aligned}$$

The map

$$B\mathbb{Z}/p^n \times \mathbb{C}P^\infty_+ \to \mathbb{C}P^\infty_+ \to (\mathbb{C}P^\infty_+)^{\gamma_1} \to \Sigma^{-2} MU$$

is the class

$$x +_F [j]_F u_{[k]} \in MU^*(B\mathbb{Z}/p^n \otimes \mathbb{C}P^\infty), \tag{2.47}$$

where u generates $MU^* B\mathbb{Z}/p^n$ and x generates $MU^* \mathbb{C}P^\infty$. The map (2.46) is, on coefficients, the slant product with (2.47). Since b_i and x^i are dual,

$$b_i / x +_F [j]_F u_{[k]}$$

is the coefficient of $x +_F [j]_F u_{[k]}$ at x^i. That ϕ_k sends $u_{[k+1]}$ to $[p]_F u_{[k]}$ is clear. This concludes the proof of Theorem II.13. □

To summarize, we have described the equivariant complex cobordism ring $(MU_G)_*$ as the limit of a diagram Γ. When G is the cyclic p-group $G = \mathbb{Z}/p^n$, Γ takes the form of an n-fold pullback diagram, obtained by successively expanding the top right corner of Tate diagrams by other Tate diagrams. For G finite abelian, Γ is obtained by gluing together several n-fold pullback diagrams $\Gamma(S)$ arising from maximal chains $S = \{H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_n\}$ of subsgroups of G (i.e. no intermediate subgroups). The diagrams $\Gamma(S)$ can also be obtained by expanding the top right corners of a succession of Tate diagrams. $(MU_G)_*$ is obtained by taking homotopy groups on Γ and taking a limit. The argument of Section 2.2 gives a new Isotropy Separation Spectral Sequence (ISSS) for computing the coefficients of equivariant spectra E by looking at the various fixed point spectra. The computation goes through because the ISSS collapses for the spectrum $E = MU_G$, since $(MU_G)_*$ is in even degrees.

CHAPTER III

Equivariant Formal Group Laws

3.1 Introduction

Before defining equivariant formal group laws, we will need some preliminaries from algebra and character theory. If R is a ring and I an ideal of R, R_I refers to the localization of R at I, which is

$$R_I = \{\frac{a}{b} | a \in R \text{ and } b \notin I\}, \tag{3.1}$$

informally speaking. Now an ideal I in a ring R also determines a topology, called the *Krull topology*, on R. A basis of neighborhoods of $0 \in R$ for the Krull topology is given by the powers I^n of the ideal I. The completion of R with respect to the Krull topology is given by

$$R_{\widehat{I}} = \lim_{\leftarrow} R/I^n. \tag{3.2}$$

If $R_{\widehat{I}} = R$, R is called *complete* at the ideal I. Let G be a finite abelian group. Let $\widehat{G}$ denote the character group of G, which is the group of group homomorphisms $f : G \to \mathbb{C}^\times$. An abelian group of order n has n characters $f_1, \ldots, f_n$, and these are the irreducible complex representations of the group.

Let E be a G-equivariant spectrum. Kriz [20] makes the following definition.

Definition III.1. A G-equivariant formal group law consists of:

(1) A ring R complete at an ideal I and a cocommutative, coassociative, counital comultiplication

$$\Delta : R \to R\hat{\otimes}R \quad (R\hat{\otimes}R = (R \otimes R)_{\hat{I}\otimes R \oplus R \otimes I}),$$

(2) An I-continuous map of rings $\epsilon : R \to E^*[\hat{G}]\check{}$ compatible with compultiplication:

$$\begin{tikzcd} R \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & R\hat{\otimes}R \arrow[d, "\epsilon\hat{\otimes}\epsilon"] \\ E^*[\hat{G}]\check{} \arrow[r, "\psi"'] & E^*[\hat{G}]\check{} \otimes E^*[\hat{G}]\check{}, \end{tikzcd}$$

(3) A system of elements $x_L \in R$, $L \in \hat{G}$ such that

$$R/(x_L | L \in \widehat{G}) \cong E_*$$

and

$$I = \prod_{L \in \hat{G}} (x_L)$$

and

$$x_L = (\epsilon(L) \otimes 1)\Delta(x_1) \text{ for } L \in \hat{G}.$$

Here the $\check{}$ in $E^*[\widehat{G}]\check{}$ refers to discretization. The cocommutativity, coassociativity, and counitality of a comultiplication are determined by diagrams dual to those which determine commutativity, associativity, and unitality of a product. ψ is the natural comultiplication.

Greenlees [13] makes an equivalent definition of equivariant formal group laws for compact Lie groups, but I adopt Kriz's definition because I believe its explicit nature adds to the clarity of our discussion.

Kriz [20] shows that a complex orientation on a G-equivariant spectrum E over the complete universe $\mathcal{U}$ specifies a G-equivariant formal group law with $R = E^*\mathbb{C}P^\infty_G$.

Here $\mathbb{C}P^\infty_G = \mathbb{C}P(\mathcal{U})$ is the complex projective space on the complete G-universe. A complex orientation on a G-spectrum E is a class $x \in E^2(\mathbb{C}P^\infty_G)$ which satisfies the appropriate equivariant analogues of the Thom theorems, but we will not admit of this digression. The interested reader is referred to [27] for an historical perspective on Thom's theory. Kriz's proof is constructive, and our goal is to trace this construction to compute the equivariant formal group law corresponding to the equivariant complex cobordism spectrum MU_G, for the case where G is a finite abelian group. We can do this because MU_G has a canonical complex orientation. In fact, MU_G is the universal complex-oriented G-spectrum, in the sense that the complex orientations of a ring spectrum E are in bijective correspondence with maps of ring spectra $\phi : MU_G \to E$, which maps send the orientation class of MU_G to that of E. Here a ring spectrum E is a spectrum with a homotopy associative and homotopy unital product $\mu : E \wedge E \to E$, with unit map $\eta : S \to E$, S the sphere spectrum. It is well known that MU_G is a ring spectrum.

The computation of the equivariant formal group law of MU_G is of interest because of Conjecture 2.4 of [13], which asks whether the coefficient ring of equivariant complex cobordism classifies equivariant formal group laws in the same way that non-equivariant cobordism classifies traditional formal group laws, vis á vis Quillen's Theorem ([29], Theorem 1.3.4). Greenlees [13] shows that the equivariant complex cobordism ring classifies equivariant formal group laws over Noetherian rings, but the general result is still unknown. Allow us to formulate Greenlees' Conjecture in this way:

Conjecture III.2. *For any complex oriented G-equivariant spectrum E there is a unique homomorphism of rings $\theta : MU_G^* \to E^*$ such that θ induces maps that send the structures* (1)*,* (2)*, and* (3) *for the canonical equivariant formal group law* *corresponding to MU_G to the corresponding structure for E.*

Our goal at present is to describe the equivariant formal group law corresponding to MU_G for G a finite abelian group, with an eye toward Greenlees' Conjecture III.2.

3.2 The Case $G = \mathbb{Z}/p$

Let us begin by considering the case $G = \mathbb{Z}/p$. Then we may write

$$\hat{G} = \{1, \alpha, \alpha^2, \ldots, \alpha^{p-1}\}.$$

Let γ_G be the canonical line bundle on $\mathbb{C}P^\infty_G$. Let $E = MU_G$. Then E has a canonical complex orientation, hence gives an equivariant formal group law by Theorem 3 of [20]. We will match notation with Definition III.1 to describe this equivariant formal group law. [20] gives $R = E^*\mathbb{C}P^\infty_G = MU^*_{\mathbb{Z}/p}\mathbb{C}P^\infty_{\mathbb{Z}/p}$, and

$$I = \left(\prod_{L\in\hat{G}} x_L\right) = \left(\prod_{k=0}^{p-1} x_{\alpha^k}\right) = \left(\prod_{k=0}^{p-1} x_k\right) \text{ (defining } x_k := x_{\alpha^k}). \tag{3.3}$$

The elements x_k are Thom classes $x_k \in \widetilde{E}^2T(\gamma_G \otimes \alpha^k)$, where T denotes Thom space. These are obtained as follows. Let $x_0 \in \widetilde{E}^2T(\gamma_G)$ be the orientation class. Now let $\phi : \mathbb{C}P^\infty_G \to \mathbb{C}P^\infty_G$ classify $\gamma_G \otimes \alpha^k$, i.e. $\phi^*(\gamma_G) = \gamma_G \otimes \alpha^k$. Then we define

$$x_k = Im(\widetilde{E}^2T(\gamma_G) \to^{\widetilde{E}^2T\phi} \widetilde{E}^2T(\gamma_G \otimes \alpha^k)).$$

Let $U = \bigotimes_{k=0}^{p-1} \alpha^k$. Now the product on the right hand side of (3.3) is to be calculated by the Thom diagonal $\Delta_t : T(\gamma_G \otimes U) \to \bigwedge_{k=0}^{p-1} T(\gamma_G \otimes \alpha^k)$.

Lemma 7 of [20] gives that $R = E^*\mathbb{C}P^\infty_G$ is complete at I, and

$$E^*(\mathbb{C}P^\infty_G \times \mathbb{C}P^\infty_G) \cong R\hat{\otimes}R.$$

The comultiplication Δ is induced by the map classifying the $\otimes$-multiplication of line bundles:

$$\mu : \mathbb{C}P^\infty_G \times \mathbb{C}P^\infty_G \to \mathbb{C}P^\infty_G,$$

i.e. for line bundles $\xi = f^*\gamma_G$, $\omega = g^*\gamma_G$, $\xi \otimes \omega = (\mu(f \times g))^*$.

Now we need to define the map $\epsilon : R = MU^*_{\mathbb{Z}/p}\mathbb{C}P^\infty_{\mathbb{Z}/p} \to E^*[\hat{G}]^\vee = MU^*_{\mathbb{Z}/p}[\widehat{\mathbb{Z}/p}]^\vee$. This is done by choosing a basepoint $*_k$ in each connected component of

$$(\mathbb{C}P^\infty_G)^G = \coprod_{\alpha^k \in \widehat{\mathbb{Z}/p}} \mathbb{C}P^\infty.$$

ϵ is the induced map in cohomology of the G-equivariant map

$$\coprod_{L \in \hat{G}} *_L \to G.$$

All of the above is documented in [20]. Our goal is to understand better the algebraic structure of the ring $MU^*_{\mathbb{Z}/p}\mathbb{C}P^\infty_{\mathbb{Z}/p}$. Now by Lemma 7 of [20], we have the following:

$$(3.4) \qquad MU^*_{\mathbb{Z}/p}\mathbb{C}P^\infty_{\mathbb{Z}/p} \cong MU^*_{\mathbb{Z}/p}\{\{x_0, x_0x_1, x_0x_1x_2, \ldots\}\}.$$

In (3.4) the terms on the right hand side are a flag basis of the complete universe $\mathcal{U}$, and $MU^*_{\mathbb{Z}/p}\{\{x_0, x_0x_1, x_0x_1x_2, \ldots\}\}$ denotes

$$\left\{ \sum_{i=0}^{\infty} a_i x_0 x_1 \cdots x_i \,\middle|\, a_i \in MU^*_{\mathbb{Z}/p} \right\}$$

The Borel cohomology

$$\begin{aligned} \pi_*^{\mathbb{Z}/p}(F(E\mathbb{Z}/p_+, MU^*_{\mathbb{Z}/p}\mathbb{C}P^\infty_{\mathbb{Z}/p}) &\to^{\cong} \pi_*^{\mathbb{Z}/p}(F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p})^*\mathbb{C}P^\infty_{\{e\}}) \\ &= \pi_*^{\mathbb{Z}/p}(F(E\mathbb{Z}/p_+, MU_{\mathbb{Z}/p}))[[x]]. \end{aligned}$$

The elements x_k are in $MU_*[[u]]/([p]_F u)[[x_0]]$, with $MU_*[[u]]/([p]_F u)$ as in [21]. The x_k are given by the relation $x_k = x_0 +_F [k]_F u$, which corresponds to the final relation of (3) in the definition of equivariant formal group laws above. Recall the pullback diagram

$$\begin{tikzcd}
(MU_{\mathbb{Z}/p})_* \arrow[r] \arrow[d] & MU_*[u_k, u_k^{-1}, b_k^{(i)} | i > 0, k \in (\mathbb{Z}/p)^\times] \arrow[d, "\phi"] \\
MU_*[[u]]/([p]_F u) \arrow[r, "\iota"] & MU_*[[u]]/([p]_F u)[u^{-1}].
\end{tikzcd}$$

The element $\phi(b_k^{(i)} u_k)$ is the coefficient of x^i in $x +_F [k]_F u$, so we can use this to compute the ring $MU^*_{\mathbb{Z}/p}\{\{x_0, x_0x_1, x_0x_1x_2, \ldots\}\} \cong R$. In particular, clearly we can write $x_k \in MU_*[[u]]/([p]_F u)[[x]]$ as a sum in the terms $x = x_0, x_0x_1, x_0x_1x_2, \ldots$ with coefficients in $MU^*_{\mathbb{Z}/p}$, but by applying the obvious map

$$MU_*[u_k, u_k^{-1}, b_j^{(i)} | i > 0, k \in (\mathbb{Z}/p)^\times][[x]] \to^{\phi} MU_*[[u]]/([p]_F u)[u^{-1}][[x]],$$

we are able to write x_k in terms of the elements $u_j, u_j^{-1}, b_j^{(i)}$, and $x = x_0$.

In particular, $x_k = x_0 +_F [k]_F u$, so the coefficient $c_i \in MU_*[[u]]/([p]_F u)$ of x_0^i in the expansion of x_k is $\phi(u_k b_k^{(i)})$. Thus x_k is the image under ϕ of the element

$$\sum_{i=0}^{\infty} u_k b_k^{(i)} x^i,$$

where $b_j^{(0)} = 1$.

Theorem 11.2 of Greenlees [13] allows us to write

$$\Phi^{\mathbb{Z}/p} MU^*_{\mathbb{Z}/p} \mathbb{C}P^\infty_{\mathbb{Z}/p} = \prod_{k=0}^{p-1} \Phi^{\mathbb{Z}/p} MU^*_{\mathbb{Z}/p}[[x_k]] = \prod_{k=0}^{p-1} \Phi^{\mathbb{Z}/p} MU^*_{\mathbb{Z}/p}[[x +_F [k]_F u]]. \tag{3.5}$$

Now the elements $b_j^{(i)}$ lend themselves the following description, also following Greenlees [13]: form a flag basis for $\Phi^{\mathbb{Z}/p} MU^*_{\mathbb{Z}/p} \mathbb{C}P(\bigoplus_\infty \alpha^k))$ for a fixed k. Here "${}_\infty\alpha^k$" denotes the direct sum of infinitely many copies of the representation α^k. Then take the dual basis of $\Phi^{\mathbb{Z}/p}(MU_{\mathbb{Z}/p})_* \mathbb{C}P(\bigoplus_\infty \alpha^k)$. This dual basis consists of the elements $b_k^{(i)}$.

The equation (3.5) allows us to obtain $R = MU^*_{\mathbb{Z}/p} \mathbb{C}P^\infty_{\mathbb{Z}/p}$ as the pullback of the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & \prod_{k=0}^{p-1} MU^*[u_j, u_j^{-1}, b_j^{(i)} | i>0, j \in (\mathbb{Z}/p)^\times][[x_k]] \arrow[d, "\phi"] \\ MU^*[[u]]/([p]_F u)[[x]] \arrow[r, "\psi"'] & \prod_{k=0}^{p-1} MU^*[[u]]/([p]_F u)[u^{-1}][[x_k]]. \end{tikzcd} \tag{3.6}$$

Now we need to describe the maps ϕ and ψ. Certainly $\psi(u) = \prod_{k=0}^{p-1} u$, and $\psi(x) = \prod_{k=0}^{p-1}(x_k -_F [k]_F u)$. The map $\phi = \prod_{j=0}^{p-1} \phi_j$, where $\phi_j(u_k b_k^{(i)})$ is the coefficient of $(x_j -_F [j]_F u)^i$ in an expansion of $x_j +_F [k-j]_F u$ in terms of $x_j -_F [j]_F u$, and $\phi_j(x_j) = x_j$.

To complete the description of the equivariant formal group law corresponding to equivariant complex cobordism, we must describe $R \widehat{\otimes} R$ and the coproduct $R \to R\widehat{\otimes} R$.

$R\widehat{\otimes}R$ also has a description via a pullback diagram of rings:

(3.7)

$$\begin{tikzcd} R\widehat{\otimes}R \arrow[r] \arrow[d] & \prod_{k=0}^{p-1}\prod_{r=0}^{p-1} MU^*[u_j, u_j^{-1}, b_j^{(i)} | i>0, j \in (\mathbb{Z}/p)^\times][[x_k, y_r]] \arrow[d, "\phi'"] \\ MU^*[[u]]/([p]_F u)[[x,y]] \arrow[r, "\psi'"'] & \prod_{k=0}^{p-1}\prod_{r=0}^{p-1} MU^*[[u]]/([p]_F u)[u^{-1}][[x_k, y_r]]. \end{tikzcd}$$

ψ' is the obvious map $x \mapsto \prod_{k=0}^{p-1}\prod_{r=0}^{p-1}(x_k -_F [k]_F u)$, $y \mapsto \prod_{k=0}^{p-1}\prod_{r=0}^{p-1}(y_r -_F [r]_F u)$, $u \mapsto \prod_{j=0}^{p-1}\prod_{r=0}^{p-1} u$. The map $\phi' = \prod_{j=0}^{p-1}\prod_{r=0}^{p-1} \phi'_{jr}$, where $\phi'_{jr}(x_j) = x_j$, $\phi'_{jr}(y_r) = y_r$, and $\phi'_{jr}(u_k b_k^{(i)})$ is the coefficient of $(x_j -_F [j]_F u)^i$ in an expansion of $x_j +_F [k-j]_F u$ in terms of $x_j -_F [j]_F u$.

To define the coproduct $R \to R\widehat{\otimes}R$, it suffices to define maps from the top right, bottom right, and bottom left of (3.6) to the corresponding rings in (3.7), compatible with those diagrams. The map $MU^*[[u]]/([p]_F u)[[x]] \to MU^*[[u]]/([p]_F u)[[x,y]]$ is determined by $u \mapsto u$ and $x \mapsto x +_F y$. The map

$$\prod_{k=0}^{p-1} MU^*[[u]]/([p]_F u)[u^{-1}][[x_k]] \to \prod_{k=0}^{p-1}\prod_{r=0}^{p-1} MU^*[[u]]/([p]_F u)[u^{-1}][[x_k, y_r]]$$

is determined on the kth component by $u \mapsto \prod_{k_1+k_2=k} u$ and

$x_k \mapsto \prod_{k_1+k_2=k}(x_{k_1} +_F y_{k_2})$, and the other factors zero. By $k_1 + k_2 = k$ we of course

mean $k_1 + k_2 \equiv k(\mod p)$. Finally, the map

$$\prod_{k=0}^{p-1} MU^*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in (\mathbb{Z}/p)^\times][[x_k]]$$

$$\to \prod_{k=0}^{p-1}\prod_{r=0}^{p-1} MU^*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in (\mathbb{Z}/p)^\times][[x_k, y_r]]$$

is determined by the kth component factor maps $u_j \mapsto \prod_{k_1+k_2=k} u_j$, $b_j^{(i)} \mapsto \prod_{k_1+k_2=k} b_j^{(i)}$,

and $x_k \mapsto \prod_{k_1+k_2=k}(x_{k_1} +_F y_{k_2})$. This completes our description of the equivariant formal group law corresponding to $MU_{\mathbb{Z}/p}$.

3.3 The General Case

The description of the equivariant formal group law for $MU_{\mathbb{Z}/p}$ in the previous section followed immediately from Kriz's papers [21] and [20]. It is not surprising, then, that our description of MU_G for G a finite abelian group in Section 2.2 allows for an analagous description of the equivariant formal group laws in this case. We adopt here the notation of Section 2.2.

To describe the formal group law for MU_G, we need to describe the rings R and $R\widehat{\otimes}R$, the ideal I, the system of elements $x_L \in R$ for $L \in \widehat{G}$, the coproduct $\Delta : R \to R\widehat{\otimes}R$, and the map $\epsilon : R \to E^*[\widehat{G}]^\vee$. It follows from [20] that

$$R = MU_G^*\mathbb{C}P_G^\infty \cong MU_G^*\{\{1, x_{L_1}, x_{L_1 \oplus L_2}, \ldots\}\}, \tag{3.8}$$

where $L_1 \oplus L_2 \oplus \cdots$ is any splitting of the complete G-universe $\mathcal{U}$. The elements $x_L \in \widetilde{MU}_G^2 T(\gamma_G \otimes L)$ are Thom classes, computed just as in [20], where γ_G is the canonical line bundle on $\mathbb{C}P_G^\infty$. The ideal I is

$$I = \left(\prod_{L \in \widehat{G}} x_L\right), \tag{3.9}$$

where the product on the right is computed by the Thom diagonal just as in Section 3.2. We define

$$x_{L_1\oplus L_2\oplus\cdots\oplus L_m} = \prod x_{L_1}x_{L_2}\cdots x_{L_m}, \tag{3.10}$$

and now the right hand side of (3.8) is well-defined.

Now MU_G^* is a split MU^*-module, and the splitting map $MU \to MU_G$ induces an isomorphism

$$\pi_*^G(F(EG_+, MU)) \cong \pi_*^G(F(EG_+, MU_G)), \tag{3.11}$$

and it follows that

$$\pi_*^G(F(EG_+, MU_G)^*\mathbb{C}P_G^\infty) \cong MU_*[[u_L|L \in \overline{G^*}]]/(u_L +_F u_M = u_{LM})[[x]]. \tag{3.12}$$

We are now able to give a better description of the elements x_L. Clearly,

$$x_0 = x \in MU_*[[u_L|L \in \overline{G^*}]]/(u_L +_F u_M = u_{LM})[[x]], \tag{3.13}$$

while

$$x_L = x_0 +_F u_L. \tag{3.14}$$

Theorem 11.2 of [13] gives

$$\Phi^G MU_G^*\mathbb{C}P_G^\infty \cong \prod_{L\in\overline{G^*}} \Phi^G MU_G^*[[x_L]] = \prod_{L\in\overline{G^*}} \Phi^G MU_G^*[[x +_F u_L]]. \tag{3.15}$$

We now turn our attention to the ring R, seeking an explicit description. In fact, following Theorem II.4 and (3.8), such a description is not difficult to obtain. Let $S = \{H_1 \subsetneq H_2 \subsetneq \cdots \subsetneq H_k\} \in P(G)$, and recall that we may assume that there are no intermediate subgroups between H_{j-1} and H_j for all j. Then $\Gamma(S)$ is the k-fold pullback of a diagram $\mathcal{N}_S$ of the form

(3.16)
$$\begin{tikzcd}
 & & & & Q^k \arrow[d, "\phi_{k-1}"] \\
 & & & Q_{k-1} \arrow[r, "\psi_{k-1}"] \arrow[d, "\phi_{k-2}"] & S_{k-1} \\
 & & Q_2 \arrow[r, "\cdots"] \arrow[d, "\phi_1"] & S_{k-2} & \\
 & Q_1 \arrow[r, "\psi_1"] \arrow[d, "\phi_0"] & S_1 & & \\
Q_0 \arrow[r, "\psi_0"] & S_0. & & &
\end{tikzcd}$$

Here

$$Q_j = \prod_{L \in \overline{H_j^*}} (A_{H_j,S})_{\widehat{\mathcal{T}}_{H_j,S}} / I^*_{H_j,S}[[u_L | L \in R_j]]/(u_L +_F u_M = u_{LM})[[x_L]];$$

$$S_j = \prod_{L \in \overline{H_{j+1}^*}} (A_{H_j,S})_{\widehat{\mathcal{T}}_{H_j,S}} / I^*_{H_j,S}[[u_L | L \in R_j]]/(u_L +_F u_M = u_{LM})[[x_L]].$$

Q^k is defined similarly to the rings Q_j, also using the computations in the proof of Theorem II.5. The maps of $\mathcal{N}_S$ are determined by the maps of the diagram $\Gamma(S)$, together with the conditions

(3.17)
$$x_L \mapsto \prod_{M \equiv L (\bmod H_j)} x_M +_F (u_L - u_M).$$

for the horizontal maps and

(3.18)
$$x_L \mapsto x_L$$

for the vertical maps. Taking the union of the diagrams $\mathcal{N}_S$ over the various series S gives a diagram Γ', and R is the limit of Γ', i.e.

(3.19)
$$R = \operatorname{ho}\lim_{\leftarrow} \Gamma'_S.$$

Having computed R, we must compute $R\widehat{\otimes}R$, though this is now fairly standard. Let $\mathcal{N}'_S$ differ from the diagram $\mathcal{N}_S$ only insofar as whenver we would adjoin $[[x_L]]$, instead adjoin $[[x_L, y_L]]$. The maps are determined by the maps of $\mathcal{N}_S$ and the corresponding mappings for the elements y_L. The union of the diagrams $\mathcal{N}'_S$ gives a diagram Γ'', of which $R\widehat{\otimes}R$ is the limit, so that

$$R\widehat{\otimes}R = \operatorname{ho}\lim_{\leftarrow} \Gamma'''_S. \tag{3.20}$$

Now the coproduct $\Delta : R \to R\widehat{\otimes}R$ is not difficult to describe at the level of the diagrams Γ' and Γ''. The corresponding map $\Gamma' \to \Gamma''$ is determined by the identity map $\Gamma \to \Gamma$, and the condition

$$x_L \mapsto \prod_{MN \equiv L (\bmod H_j)} (x_M +_F y_N) \tag{3.21}$$

for the maps $Q_j \to Q'_j$ and $S_{j-1} \to S'_{j-1}$.

Finally the map $\epsilon : R \to E^*[\widehat{G}]^{\check{}}$ is the induced map in cohomology of the G-equivariant map

$$\coprod_{L \in \widehat{G}} *_L \to G, \tag{3.22}$$

where $*_L$ is a chosen basepoint in the appropriate connected componenet of

$$(\mathbb{C}P^\infty_G)^G = \coprod_{L \in \widehat{G}} \mathbb{C}P^\infty.$$

This works exactly the same as for the case $G = \mathbb{Z}/p$ above. This completes our description of the equivariant formal group law corresponding to the equivariant complex cobordism spectrum MU_G of a finite abelian group.

3.4 The Case $G = \mathbb{Z}/p^n$

There is intricate structure hiding beneath the surface of our description of the equivariant formal group law for MU_G in the previous section. To illucidate the discussion, and as a special case of Section 3.3, we now compute the equivariant formal group law given by the complex orientation on $MU_{\mathbb{Z}/p^n}$. Much of the structure follows immediately from Kriz [20]. Namely,

$$R = MU^*_{\mathbb{Z}/p^n}\mathbb{C}P^\infty_{\mathbb{Z}/p^n} \cong MU^*_{\mathbb{Z}/p^n}\{\{1, x_{L_1}, x_{L_1}x_{L_2}, \ldots\}\}, \tag{3.23}$$

for $L_1 \oplus L_2 \oplus \cdots$ any splitting of the complete universe $\mathcal{U}$. The ideal I is

$$I = \left(\prod_{L\in\widehat{\mathbb{Z}/p^n}} x_L\right), \tag{3.24}$$

where the elements x_L are Thom classes, and the product computed by the Thom diagonal as before. We define

$$x_{L_1\oplus L_2\oplus\cdots\oplus L_m} = \prod x_{L_1}x_{L_2}\cdots x_{L_m}, \tag{3.25}$$

so the right hand side of (3.23) is now well-defined.

We want to understand better the structure of the ring R via the description of (3.23). Just as for the case $G = \mathbb{Z}/p$, the splitting map $MU \to MU_{\mathbb{Z}/p^n}$ induces an isomorphism

$$\pi_*^{\mathbb{Z}/p^n}(F(E\mathbb{Z}/p^n_+, MU)) \cong \pi_*^{\mathbb{Z}/p^n}(F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n})),$$

so that $\pi_*^{\mathbb{Z}/p^n}(F(E\mathbb{Z}/p^n_+, MU_{\mathbb{Z}/p^n})^*\mathbb{C}P^\infty_{\mathbb{Z}/p^n}) \cong MU_*[[u]]/([p^n]_F u)[[x]]$. For $0 \le j \le p^{n-1}$, let $x_j := x_{\alpha^j}$ be the Thom class in $\widetilde{MU}^2_{\mathbb{Z}/p^n}T(\gamma_{\mathbb{Z}/p^n}\otimes\alpha^j)$ given as before, where $\gamma_{\mathbb{Z}/p^n}$ is the canonical line bundle on $\mathbb{C}P^\infty_{\mathbb{Z}/p^n}$. Thus the ideal $I = (\prod_{j=0}^{p^{n-1}} x_j)$. It is clear that the elements x_j may be written as series in $x_0, x_0x_1, x_0x_1x_2, \ldots$ with coefficients in $MU^*_{\mathbb{Z}/p^n}$. We give a more useful description of the elements x_j arising from the diagram of Theorem II.13. Of course $x_0 = x \in MU_*[[u]]/([p^n]_F u)[[x]]$, and $x_j = x_0 +_F [j]_F u$. Let R_k, S_k, $0 \le k \le n-1$, and R^n be as in Theorem II.13, and refer to that theorem for notation. Then the element $u_j b_j^{(i)}$ of R^n maps to an element of S_{n-1} that does not include the term $u_{[n-1]}^{-1}$, so this element really lives in R_{n-1}. For $0 < k < n$, the resulting element of R_k maps to an element of S_{k-1} which does not include the term $u_{[k-1]}^{-1}$, so it really lives in R_{k-1}. This allows us to map the elements $u_j b_j^{(i)}$ of R^n to $R_0 = MU_*[[u]]/([p^n]_F u)$; call this map ϕ. Then there is an implied map $\phi : MU_*[u_j b_j^{(i)} | i > 0, 1 \leq j \leq p^n - 1][[x]] \to MU_*[[u]]/([p^n]_F u)[[x]]$. Since $u_j b_j^{(i)}$ maps to the coefficient of x^i in $x +_F [j]_F u$, x_j is the image under ϕ of the element

$$\sum_{i=0}^{\infty} u_j b_j^{(i)} x^i,$$

just as was true for the case $G = \mathbb{Z}/p$. We would also like a nice description of the ring

$$R = MU^*_{\mathbb{Z}/p^n}(\mathbb{C}P^\infty_{\mathbb{Z}/p^n}) = MU^*_{\mathbb{Z}/p^n}\{\{\mathcal{U}\}\}$$

as a product, as we had for the case $G = \mathbb{Z}/p$. Greenlees' Theorem 11.2 [13] gives us the following:

(3.26)

$$\Phi^{\mathbb{Z}/p^n} MU^*_{\mathbb{Z}/p^n}\{\{\mathcal{U}\}\} = \prod_{j=0}^{p^n-1} \Phi^{\mathbb{Z}/p^n} MU^*_{\mathbb{Z}/p^n}[[x_j]] = \prod_{j=0}^{p^n-1} \Phi^{\mathbb{Z}/p^n} MU^*_{\mathbb{Z}/p^n}[[x +_F [j]_F u]].$$

Moreover, we obtain R as an n-fold pullback, using Theorem II.13. Let us recall the notation of that theorem:

$$u_{[k]} = [p^k]_F u;$$

$$R_k = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, \ldots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]});$$

$$S_k = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i > 0, j \in \{1, \ldots, p^k - 1\}][[u_{[k]}]]/([p^{n-k}]_F u_{[k]})[u_{[k]}^{-1}];$$

$$R^n = MU_*[u_j, u_j^{-1}, b_j^{(i)} | i.0, j \in \{1, \ldots, p^n - 1\}].$$

Also, the map $\psi_k : R_k \to S_k$ is localization by inverting $u_{[k]}$, and the map $\phi_k : R_{k+1} \to S_k$ is determined by the property of sending $b_j^{(i)} u_j$ to the coefficent of x^i in

$x +_F [j]_F u_{[k]}$, and $u_{[k+1]} \mapsto [p]_F u_{[k]}$. The map $\phi^{n-1} : R^n \to S_{n-1}$ is defined similarly. Recall then that $MU_{\mathbb{Z}/p^n}$ is the pullback of the diagram

(3.27)

$$\begin{tikzcd}
 & & & & R^n \arrow[d, "\phi_{n-1}"] \\
 & & & R_{n-1} \arrow[r, "\psi_{n-1}"] \arrow[d, "\phi_{n-2}"] & S_{n-1} \\
 & & R_2 \arrow[r, "\cdots"] \arrow[d, "\phi_1"] & S_{n-2} & \\
 & R_1 \arrow[r, "\psi_1"] \arrow[d, "\phi_0"] & S_1 & & \\
R_0 \arrow[r, "\psi_0"] & S_0. & & &
\end{tikzcd}$$

The various powers of the Euler class which are invertible on this diagram allow for certain product decompositions of the ring $R = MU^*_{\mathbb{Z}/p^n}\mathbb{C}P^\infty_{\mathbb{Z}/p^n}$. Let R^n, S_k, R_k stand for the cohomology rings now, rather than homology. Then R is the pullback of the following diagram of rings:

(3.28)

$$\begin{tikzcd}
 & & & \prod_{k\in(\mathbb{Z}/p^n)^*} R^n[[x_k]] \arrow[d, "\phi_{n-1}"] \\
 & & \prod_{k\in(\mathbb{Z}/p^{n-1})^*} R_{n-1}[[x_k]] \arrow[r, "\psi_{n-1}"] \arrow[d, "\cdots"] & \prod_{k\in(\mathbb{Z}/p^n)^*} S_{n-1}[[x_k]] \\
 & \prod_{k\in(\mathbb{Z}/p)^*} R_1[[x_k]] \arrow[r, "\psi_1"] \arrow[d, "\phi_0"] & \prod_{k\in(\mathbb{Z}/p^2)^*} S_1[[x_k]] & \\
R_0[[x]] \arrow[r, "\psi_0"] & \prod_{k\in(\mathbb{Z}/p)^*} S_0[[x_k]]. & &
\end{tikzcd}$$

The horizontal maps, as implied, are induced by the maps ψ_k and the condition $x_j \mapsto \prod_{r \equiv j(\mod p^k)}(x_r +_F [j-r]_F u_{[k]})$. The vertical maps are induced by the maps ϕ_k and the condition $x_j \mapsto x_j$ for all j.

There is a similar description of $R\hat{\otimes}R$ as a pullback:

(3.29)
$$
\begin{tikzcd}
 & & \prod_{k,r\in(\mathbb{Z}/p^n)^*} R^n[[x_k,y_r]] \arrow[d, "\phi_{n-1}"] \\
 & \prod_{k,r\in(\mathbb{Z}/p^{n-1})^*} R_{n-1}[[x_k,y_r]] \arrow[r, "\psi_{n-1}"] \arrow[dr, "\cdots"] & \prod_{k,r\in(\mathbb{Z}/p^n)^*} S_{n-1}[[x_k,y_r]] \\
 & \prod_{k,r\in(\mathbb{Z}/p)^*} R_1[[x_k,y_r]] \arrow[r, "\psi_1"] \arrow[d, "\phi_0"] & \prod_{k,r\in(\mathbb{Z}/p^2)^*} S_1[[x_k,y_r]] \\
R_0[[x,y]] \arrow[r, "\psi_0"] & \prod_{k,r\in(\mathbb{Z}/p)^*} S_0[[x_k,y_r]]. &
\end{tikzcd}
$$

The maps are determined by the maps of (3.28) and the corresponding conditions for y_r. Namely, under the horizontal maps, $y_r \mapsto \prod_{s\equiv r(\mod p^k)}(y_s +_F [r-s]_F u_{[k]})$. Under the vertical maps, $y_r \mapsto y_r$.

It remains to specify the coproduct $\Delta : R \to R\hat{\otimes}R$ on the terms of the diagrams (3.28) and (3.29). The map $\prod_{k\in(\mathbb{Z}/p^j)^*} R_j[[x_k]] \to \prod_{k,r\in(\mathbb{Z}/p^j)^*} R_j[[x_k,y_r]]$ is determined by the identity map on R_j and the condition $x_k \mapsto \prod_{k_1+k_2=k}(x_{k_1} +_F y_{k_2})$, where by $k_1+k_2=k$ we of course mean $k_1+k_2 \equiv k(\mod p^j)$. The map on the top right of the diagrams is defined similarly. The map

$$\prod_{k\in(\mathbb{Z}/p^j)^*} S_{j-1}[[x_k]] \to \prod_{k,r\in(\mathbb{Z}/p^j)^*} S_{j-1}[[x_k,y_r]]$$

is determined by the identity map on S_{j-1} and the condition

$$x_k \mapsto \prod_{k_1+k_2=k} x_{k_1} +_F y_{k_2}.$$

Finally, the map $\epsilon : R \to E^*[\widehat{G}]^\vee$ is constructed just as in the general case. This completes our description of the equivariant formal group law corresponding to $MU_{\mathbb{Z}/p^n}$.

3.5 Conclusion

We have given explicit algebraic descriptions of the equivariant complex cobordism ring MU_G and its corresponding equivariant formal group law for G a finite abelian group. The corresponding description of $MU_{\mathbb{Z}/p}$ by Kriz [21] was used by Strickland [31] to give generators and relations for $MU_{\mathbb{Z}/2}$. The present work should allow Strickland's computations to be carried over for more general groups G. Moreover, the description of the equivariant formal group laws in Section 3.3 provide an avenue of approach toward Greenlees' Conjecture III.2. The resolution of Greenlees' Conjecture could allow for a lexicon to be built up between the topology of equivariant stable spectra and the algebra of equivariant formal group laws, as was done non-equivariantly following Quillen's Theorem I.1.

There is also reason to believe that the Isotropy Separation Spectral Sequence used in Section 2.2 to compute the equivariant complex cobordism ring can be applied to the study of other important equivariant spectra. The $RO(G)$-graded coefficients of the equivariant Eilenberg-MacLane spectra, for instance, are still unknown beyond the case $G = \mathbb{Z}/p^n$, which case was resolved by Hu and Kriz [18]. The Isotropy Separation Spectral Sequence could very well facilitate this computation over a finite abelian group.

Other worthy computations include that of spectra constructed from real cobordism $M\mathbb{R}$, such as the spectrum Ω defined by Hill, Hopkins and Ravenel [16], and discussed in Section 1.1. Hill, Hopkins, and Ravenel obtain their results without computing explicitly the coefficients of Ω, and this computation may yet enhance the power of their methods.

In conclusion, the methods and results of this thesis clear the path for much future research in and around equivariant stable homotopy theory, and it is the author's hope to continue this research program to the expansion of the empire of mathematical knowledge.

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