Topics in Singularities and Jet Schemes

by

Zhixian Zhu

A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2013

Doctoral Committee:

Professor Mircea Mustaţă, Chair
Professor William Fulton
Professor Henriette Elvang
Professor Robert Lazarsfeld
Professor Karen Smith

To my parents and Cheng

ACKNOWLEDGEMENTS

First and foremost, I am indebted to my adviser, Mircea Mustaţă, for his support and patience in guiding my research and career. His invaluable advice and shared intuition have helped me choose research topics, provided ideas for many proofs. Moreover, he has helped me understand the concepts and workings of a good mathematical presentation both written and in front of live audience.

I owe William Fulton, Robert Lazarsfeld, and Karen Smith for several useful courses and the vast amount knowledge they have willingly and enthusiastically shared with me. I also would like to thank all other professors and post-docs that served as my instructors during my graduate years at the University of Michigan: Bhargav Bhatt, Igor Dolgachev, Mel Hochster, Jesse Kass, Jeffrey Lagarias and Wenliang Zhang.

Fellow graduate students-both those at Michigan and colleagues from afar-have also helped me to learn many things in my time at Michigan. Although many of them deserve mention, let me thank Linquan Ma, Juan Perez, Yefeng Shen and Xiaolei Zhao in particular for sharing in years of mathematical conversations while becoming close friends.

I am extremely grateful for many semesters of undisturbed research provided by the financial support of my adviser Mircea Mustaţă, Yongbin Ruan, Kartik Prasanna, Karen Smith and of the department.

Last, but not least, I would also like to thank my friends and family for their

helpful advice and constant encouragement.

TABLE OF CONTENTS

CHAPTER I

Introduction

The fundamental objects of study in algebraic geometry are algebraic varieties, that is, sets of points defined by systems of polynomial equations. A large class of algebraic varieties are smooth varieties, such as projective complex manifolds. In order to study even the smooth varieties, we have to extend the category of smooth projective varieties to the one consisting of varieties with some specified singularities.

The study of singularities got a prominent role starting with the so-called Minimal Model Program (MMP), a program initiated in the beginning of the 1980s, and which aims at a classification of algebraic varieties of higher dimension. It has been noticed already in the early stages of this program that in order to study smooth algebraic varieties, one has also to allow varieties with mild singularities. One distinguishes in this process several classes of singularities, such as *terminal*, *rational*, *canonical*, *log terminal* and *log canonical*, which are defined as follows.

If $f : Y \to X$ is a birational morphism of normal varieties, then there is a divisor supported on the exceptional locus of f, denoted by $K_{Y/X}$, called the relative canonical divisor. If X and Y are smooth varieties, then $K_{Y/X}$ is an effective divisor, locally defined by the Jacobian determinant of f. For every exceptional divisor E on Y, $\mathrm{ord}_E(K_{Y/X})$ is the coefficient of E in $K_{Y/X}$.

We say X has *terminal, canonical, log terminal, log canonical* singularities if the coefficient of E in $K_{Y/X}$ is greater than (or equal to) some specified constants for all exceptional divisors E of all birational morphisms $f : Y \to X$. For details, see Definition II.1. The category of $\mathbf{Q}$-factorial projective varieties with terminal singularities is the smallest category that contains the category of smooth projective varieties and in which MMP works. Canonical singularities appear on canonical models of varieties of general type. More generally, such singularities can be defined for pairs (see chapter II). Given a pair (X, Y), where Y is a subscheme of X, there is a numerical invariant, called log canonical threshold and denoted $\mathrm{lct}(X, Y)$, which measures how far (X, Y) is from being log canonical.

In this dissertation, we study such singularities, and in particular, invariants such as the log canonical threshold, via jet schemes and spaces of arcs. The jet schemes X_m are higher order analogue of tangent spaces. (The familiar case is $m = 1$, when X_1 parametrizes tangent vectors to X). The arcs space parameterizes germs of formal arcs on X.

We now describe the setting for jet schemes and arc spaces. Let k be a field of arbitrary characteristic. If X is a scheme of finite type over k and m is a non-negative integer, then the jet scheme X_m of X parameterizes m-jets on X, that is, morphisms $\operatorname{Spec} k[t]/(t^{m+1}) \to X$. Note that $X_0 = X$ and X_1 is the total tangent space of X. For every $m \geq i$, we have a canonical projection $\rho_i^m \colon X_m \to X_i$ induced by truncation of jets. We denote by π_m the projection $\rho_0^m \colon X_m \to X$. For every point $x \in X$, we write $X_{m,x}$ for the fiber of π_m at x, the m–jets of X centered at x. If $f : X \to Y$ is a morphism of schemes, then we get a corresponding morphism $f_m : X_m \to Y_m$, taking $\gamma : \operatorname{Spec} k[t]/(t^{m+1}) \to X$ to $f \circ \gamma$.

The space of arcs X_∞ is the projective limit of the X_m, equipped with projection morphisms

$$\psi_m : X_\infty \to X_m.$$

It parameterizes all formal arcs on X, that is, morphisms $\operatorname{Spec} k[\![t]\!] \to X$. If $f : X \to Y$ is a morphism of schemes, by taking the projective limit of the morphisms f_m, we get a morphism $f_\infty : X_\infty \to Y_\infty$. X_∞ is in general an infinite-dimensional space, but one is typically interested in subsets that are given as inverse images of constructible subsets by the canonical projections $X_\infty \to X_m$; these are called *cylinders*. Interesting examples of such subsets arise as follows. Consider a non-zero ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ defining a subscheme $Y \subset X$. If $\gamma : \operatorname{Spec} k[\![t]\!] \to X$ is an arc on X, then the inverse image of Y by γ is defined by an ideal in $k[\![t]\!]$. If the ideal is generated by t^e, then we put $\operatorname{ord}_\gamma(Y) = \operatorname{ord}_\gamma(\mathfrak{a}) = e$. If the ideal is zero, then we put $\operatorname{ord}_\gamma Y = \operatorname{ord}_\gamma \mathfrak{a} = \infty$. For every $p \geq 0$, the *contact locus* of order p of $\mathfrak{a}$ is the cylinder

$$\operatorname{Cont}^p(Y) = \operatorname{Cont}^p(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(\mathfrak{a}) = p\}.$$

An irreducible cylinder $C \subset X_\infty$ that does not dominate X determines a nontrivial valuation ord_C on the function field $k(X)$, given by the order of vanishing along a general element in C. We will refer below to these valuations as *cylinder valuations.*

Jet schemes and arc spaces have recently attracted a lot of attention in connection with both motivic integration, due to Kontsevich [Kon] and Denef and Loeser [DL], and applications to singularities. Work of Denef, Loeser, Ein, Ishii, Lazarsfeld, Mustaţă, and others allows the translation of geometric properties of the jet schemes X_m (such as dimension or number of irreducible components) into properties of the singularities of X. Roughly speaking, X has "worse" singularities when the corresponding jet schemes X_m have larger dimensions or have more irreducible components. Furthermore, using the central result of this theory, the Change of Variable formula, one can show that there is a close link between certain invariants of singularities defined in terms of divisorial valuations and the geometry of the contact loci in arc spaces. This link was first explored by Mustaţă in [Mus1] and [Mus2], and then further studied in [EMY], [Ish], [ELM] and [FEI].

In this dissertation, we prove results in two directions. The first one is concerned with the study of singularities of Brill-Noether loci $W^r_d(C)$ (in particular, the theta divisor Θ) in terms of their jet schemes. The other concerns the generalization of the correspondence between divisorial valuations and closed irreducible cylinders in the arc space in [ELM] to arbitrary characteristics.

We now describe the setting for our study of Brill-Noether loci. Let k be an algebraically closed field of characteristic zero and C a smooth projective curve of genus g over k. Recall that $\mathrm{Pic}^d(C)$ parameterizes line bundles of degree d on C and $W^r_d(C)$ is the subscheme of $\mathrm{Pic}^d(C)$ parameterizing line bundles L of with $h^0(L) \geq r+1$. The theta divisor Θ is $W^0_{g-1}(C)$. Riemann's Singularity Theorem says that for every line bundle L of degree $g-1$ in the theta divisor Θ, the multiplicity of Θ at L is $h^0(C, L)$.

Kempf [Kem] described the tangent cone of $W^0_d(C)$ at every point. In particular, he generalized Riemann's multiplicity result to the $W^0_d(C)$ locus. In his paper, he described the singularities of $W^0_d(C)$ and its tangent cone as follows. Let L be a point of $W^0_d(C)$, with $d < g$ and $l = \dim H^0(L)$. The tangent cone $\mathcal{T}_L(W^0_d(C))$ has rational singularities and therefore $W^0_d(C)$ has rational singularities. One can identify the tangent space of $\mathrm{Pic}^d(C)$ at L with the vector space $H^0(C, K)^*$. Kempf showed that the multiplicity of $W^0_d(C)$ at L, which is equal to the degree of $\mathbf{P}\mathcal{T}_L(W^0_d(C))$ as a subscheme of $\mathbf{P}H^0(C, K)^*$, is the binomial coefficient

$$\binom{h^1(L)}{l-1} = \binom{g-d+l-1}{l-1}.$$

Following the work of Riemann and Kempf, there has been much interest in the singularities of general theta divisors. For instance, using vanishing theorems, Ein and Lazarsfeld [EL] showed that if Θ is an irreducible theta divisor on an abelian variety A, then Θ is normal and has rational singularities.

Our first main result in this direction is on the study of the singularities of the theta divisor from the point of view of its jet schemes. Using a similar idea to that used by Kempf, we reprove Riemann's Singularity Theorem using jet schemes. We also compute the dimension of the space of m–jets centered at the singular locus of Θ for each m. Recall that for every $m \geq 1$, we have a truncation morphism $\pi_m^{\Theta} : \Theta_m \to \Theta$. We denote by Θ_{sing} the singular locus of the theta divisor.

Theorem I.1. *For every smooth projective curve C of genus $g \geq 3$ over k, and every integer $m \geq 1$, we have $\dim(\pi_m^{\Theta})^{-1}(\Theta_{sing}) = (g-1)(m+1)-1$ if C is a hyperelliptic curve. For nonhyperelliptic curves, we have $\dim(\pi_m^{\Theta})^{-1}(\Theta_{sing}) = (g-1)(m+1)-2$.*

Applying [EMY, Theorem 3.3] and [Mus1, Theorem 3.3] to the theta divisor, we obtain the following result concerning the singularities of the theta divisor.

Corollary I.2. *Let C be a smooth projective curve of genus $g \geq 3$ over k. The theta divisor has terminal singularities if C is a nonhyperelliptic curve. If C is hyperelliptic, then the theta divisor has canonical non-terminal singularities.*

A result of Elkik in [Elk] implies that for a divisor D in a smooth variety, D has rational singularities if and only if D has canonical singularities. One thus recovers the classical result that the theta divisor has rational singularities.

Using similar ideas, we are able to estimate the dimensions of the jet schemes of the Brill-Noether locus $W_d^r(C)$ for generic curves. Using Mustaţă's formula from [Mus2] describing the log canonical threshold in terms of dimensions of jet schemes, we obtain the following formula for the log canonical threshold of the pair $(\mathrm{Pic}^d(C), W^r_d(C))$.

Theorem I.3. *For a general projective smooth curve C of genus g, let L be a line bundle of degree d with $d \leq g-1$ and $l = h^0(L)$. The log canonical threshold of $(\mathrm{Pic}^d(C), W^r_d(C))$ at $L \in W^r_d(C)$ is*

$$\mathrm{lct}_L(\mathrm{Pic}^d(C), W^r_d(C)) = \min_{1\leq i\leq l-r}\left\{\frac{(l+1-i)(g-d+l-i)}{l+1-r-i}\right\}.$$

Recall that one can locally define a map from $\mathrm{Pic}^d(C)$ to a matrix space such that $W^r_d(C)$ is the pull back of a suitable generic determinantal variety. It follows from the above theorem that for generic curves, the local log canonical threshold of $(\mathrm{Pic}^d(C), W^r_d(C))$ at L is equal to the local log canonical threshold of that generic determinantal variety at the image of L (for the formula for the log canonical threshold of a generic determinantal variety, see Theorem 3.5.7. in [Doc]).

As we alluded in the previous paragraphs, there is a formula relating the log canonical threshold of a pair to the asymptotic behavior of the dimension of the jet schemes, see [Mus2] and [ELM]. The key ingredients in the proofs are the Change of Variable formula developed in the theory of motivic integration and the existence of log resolutions. While a version of the Change of Variable formula also holds in positive characteristic, the use of log resolutions in the proofs in [Mus2] and [ELM] restricted the result to ground fields of characteristic zero. More generally, the approach in [ELM] gave a general correspondence between cylinders in the space of arcs of X and divisorial valuations of the function field of X, which takes a cylinder to the corresponding cylinder valuation. Via this correspondence, the codimension of the cylinder is related to the log discrepancy of the corresponding divisorial valuation, the key invariant that appears, for example, in the definition of the log canonical threshold.

In chapter V, we show by induction on the codimension of cylinders and only using the change of variable formula for blow-ups along smooth centers, that the above mentioned correspondence between divisorial valuations and cylinders holds in arbitrary characteristic. Furthermore, via this correspondence the log discrepancy of the valuation corresponds to the codimension of the cylinder.

Recall that a prime divisor E on a normal variety X', having a birational morphism to X, is called a *divisor over* X. We identify such divisors if they give the same valuation. It is easy to see that $\mathrm{ord}_E(K_{X'/X})$ does not depend on a particular choice of X', hence we write it as $\mathrm{ord}_E(K_{-/X})$ instead. Given a cylinder $C = (\psi_m)^{-1}(S)$ for some m and some constructible subset $S \subseteq X_m$, we define $\operatorname{codim} C := \operatorname{codim}(S, X_m)$, which is independent of the choice of S and m.

Theorem I.4. *Let X be a smooth variety of dimension n over a perfect field k. There is a correspondence between irreducible closed cylinders $C \subset X_\infty$ that do not dominate X and divisorial valuations, as follows:*

(1) If C is an irreducible closed cylinder which does not dominate X, then there is a divisor E over X and a positive integer q such that

$$\mathrm{ord}_C = q \cdot \mathrm{ord}_E\,.$$

Furthermore, we have $\operatorname{codim} C \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X}))$.

(2) To every divisor E over X and every positive integer q, we can associate an irreducible closed cylinder C that does not dominate X such that

$$\mathrm{ord}_C = q \cdot \mathrm{ord}_E \ \textit{ and } \ \operatorname{codim} C = q \cdot (1 + \mathrm{ord}_E(K_{-/X})).$$

We thus are able to prove the log canonical threshold formula avoiding the use of log resolutions. We recall that log canonical threshold is defined by

$$\operatorname{lct}(X,Y) = \inf_{E/X} \frac{\operatorname{ord}_E(K_{-/X})+1}{\operatorname{ord}_E(Y)}$$

where E varies over all divisors over X. If the field k is of characteristic zero, then $\operatorname{lct}(X,Y)$ can be determined by the divisors on a single birational morphism to X, namely on a so-called log resolution of the pair (X,Y). If the field is of characteristic p, in the absence of a result giving existence of log resolutions, we have to deal with divisors on all birational morphisms.

Theorem I.4 easily implies the following formula for the log canonical threshold of (X,Y).

Theorem I.5. *Let X be a smooth variety of dimension n defined over a perfect field k, and Y be a closed subscheme. Then*

$$\operatorname{lct}(X,Y) = \inf_{C\subset X_\infty} \frac{\operatorname{codim} C}{\operatorname{ord}_C(Y)} = \inf_{m\geq 0} \frac{\operatorname{codim}(Y_m,X_m)}{m+1}$$

where C varies over the irreducible closed cylinders which do not dominate X.

We now turn to a more detailed overview of the content of the different chapters and of the proofs of the main results. In Chapter II, we recall the formalism of log singularities and divisorial valuations. We proceed to review various classes of singularities in birational geometry, such as log terminal, log canonical, and canonical singularities.

Jet schemes and arc spaces are defined in Chapter III. We refer the reader to [EM] for a more detailed introduction to these spaces. Since Theorem I.5 is proved for pairs with smooth ambient varieties, we recall some basic results on cylinders in arc spaces of smooth varieties over a perfect field. The remainder of Chapter III is devoted to reviewing some results on jet schemes and singularities that we will use in the last two chapters.

Chapter IV is entirely devoted to singularities of Brill-Noether loci. Let us preview the techniques and terminology used there. In this chapter, k is an algebraically closed field of characteristic 0. Let C be a smooth projective curve over k. Our goal is to estimate the dimension of the jet schemes of $W_d^r(C)$. Here we take Θ_m as an example and briefly describe the proof of Riemann's Singularity Theorem that we give using jet schemes (see proof of Theorem IV.5). Let L be a point in Θ. Recall that $\Theta_{m,L}$ is the fiber of $\pi_m : \Theta_m \to \Theta$ at L. By the definition of $\mathrm{Pic}^{g-1}(C)$, an element $\mathcal{L}_m \in \mathrm{Pic}^{g-1}(C)_m$ is identified with a line bundle on $C \times \operatorname{Spec} k[t]/t^{(m+1)}$. Using the description of the theta divisor as a determinantal variety, we partition the scheme $\Theta_{m,L}$ into constructible subsets $C_{\lambda,m}$, indexed by partitions λ of length $h^0(C,L)$ with sum $\geq m+1$. Several invariants of $\mathcal{L}_m \in \Theta_{m,L}$ are determined by the corresponding partition λ. For instance, λ determines the dimension of the kernel of the truncation map $H^0(C \times \operatorname{Spec} k[t]/t^{(m+1)}, \mathcal{L}_m) \to H^0(C,L)$. In this way, λ determines for each $j \leq m$ the dimension of the subspace of sections in $H^0(L)$ that can be extended to sections of $\mathcal{L}_j$, where $\mathcal{L}_j$ is the image of $\mathcal{L}_m$ under the truncation map $\mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C)_j$. Since the inequality $\mathrm{mult}_L \Theta \geq l := h^0(L)$ follows from the determinantal description of Θ, we focus on the opposite inequality. In order to show that $\mathrm{mult}_L \Theta \leq l$, it is enough to prove that $\Theta_{l,L} \neq \mathrm{Pic}^{g-1}(C)_{l,L}$. If this is not the case, then the image of $\Theta_{l,L}$ in $\Theta_{1,L} = \mathrm{Pic}^{g-1}(C)_{1,L}$ is $\Theta_{1,L}$. Using the partition associated to any $\mathcal{L}_l \in \Theta_{l,L}$, we show that the restriction map $H^0(\mathcal{L}_1) \to H^0(L)$ is nonzero. On the other hand, we can identify $\mathcal{L}_1$ in $\mathrm{Pic}^{g-1}(C)_{1,L}$ to a Čech cohomolgy class in $H^1(C, \mathcal{O}_C)$. Furthermore, the obstruction to lifting a section $s \in H^0(C,L)$ to a section of $\mathcal{L}_1$ can be described using the pairing

$$H^0(C,L) \otimes H^1(C,\mathcal{O}_C) \xrightarrow{\nu} H^1(C,L),$$

that is, s lifts if and only if $\nu(s \otimes \mathcal{L}_1) = 0$. Since the set of elements in $H^1(C, \mathcal{O}_C)$

for which there is a nonzero such s is of codimension one, this gives a contradiction, proving that $\mathrm{mult}_L \Theta \leq l$.

The remainder of Chapter IV is largely devoted to the proof of Theorems I.1 and I.3. Let L be a point in $W^r_d(C)$. Using the local description of $W^r_d(C)$ as a determinantal variety, we partition the scheme $W^r_d(C)_{m,L}$ into constructible subsets $C_{\lambda,m}$ indexed by partitions λ. We give a criterion on the partition λ associated to $\mathcal{L}_m \in \mathrm{Pic}^d(C)_m$ to have $\mathcal{L}_m \in W^r_d(C)_m$. We also prove a formula for $h^0(\mathcal{L}_m)$ in terms of the partition λ associated to $\mathcal{L}_m$. Recall that Gieseker and Petri proved that if C is general in the moduli space of curves, then the natural pairing

$$\mu_L : H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$$

is injective for every line bundle L on C. For general curves in the sense of Gieseker and Petri, we use the injectivity of the morphism μ_L to prove Theorems I.1 and I.3 by estimating the dimensions of $C_{\lambda,m}$ for every $L \in W^r_d(C)$.

Chapter V concerns the aforementioned correspondence between divisorial valuations and irreducible closed cylinders. In this chapter, k is a perfect field of arbitrary characteristic. Our techniques build upon the work of Ein, Lazarsfeld and Mustaţă in [ELM], which establishes such correspondence for smooth varieties over a field of characteristic zero. Let us briefly explain the techniques we use. Given a divisorial valuation ν, it is known that $\nu = q \cdot \mathrm{ord}_E$ for a positive integer q and a prime divisor E on Y, where $Y \xrightarrow{\pi} X$ is the composition of a sequence of blow ups along smooth centers. (Strictly speaking, we need to restrict to an open subset before each blow up. We leave the details to the proof in Chapter V.) This was first observed by Artin, and is also proved in [KM, Lemma 2.45]. Let C be the closure of the image of $\mathrm{Cont}^q(E)$ via the map $\pi_\infty : Y_\infty \to X_\infty$. It is easy to check that $\mathrm{ord}_C = q \cdot \mathrm{ord}_E = \nu$. In order to prove the relation between the codimension of C and the discrepancy

$q \cdot (1 + \mathrm{ord}_E(K_{Y/X}))$, we need a version of the Change of Variable Theorem. This theorem, due to Kontsevish [Kon] and Denef and Loeser [DL], plays an important role in Motivic Integration. We only need a version for blow ups along smooth centers that we state below. Although the proof is well known, since it is so elementary, we give a proof in Chapter V for completeness,.

Lemma I.6. *Let X be a smooth variety of dimension n over k and Z a smooth irreducible subvariety of codimension $c \geq 2$. Let $f : X' \to X$ be the blow up of X along Z and E the exceptional divisor.*

(a) For every nonnegative integer e and every $m \geq 2e$, the induced morphism

$$\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})) \to f_m(\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})))$$

is a piecewise trivial $\mathbf{A}^e$–fibration.

(b) For every $m \geq 2e$, the fiber of f_m over a point $\gamma_m \in f_m(\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})))$ is contained in a fiber of $X'_m \to X'_{m-e}$.

In particular, part (b) of Lemma I.6 implies that if C' is an irreducible cylinder in X'_∞ dominating a cylinder C in X_∞, then $\mathrm{codim}\, C - \mathrm{codim}\, C' = \mathrm{ord}_C K_{X'/X}$. Applying part (b) iteratively in the above settting, we are able to relate $\mathrm{codim}\, C$ to $\mathrm{ord}_C(K_{Y/X})$. The other direction of the proof is similar. Given an irreducible closed cylinder C which does not dominate X, we consider the center of ord_C. If the center is already a divisor E, then we can check that E is the desired divisor. Otherwise, after possibly replacing X by an open subset, we blow up X along the center and find a cylinder C' in X'_∞ which dominates C. It follows from the construction of C' that $\mathrm{ord}_{C'} = \mathrm{ord}_C$ and $\mathrm{codim}(C') < \mathrm{codim}(C)$. We may and will replace X and C by X' and C' and run the above argument again. The upshot is that after finitely many blow ups, since the codimension of a cylinder is a positive integer, the center of ord_C is a divisor E. Therefore $\mathrm{ord}_C = q\,\mathrm{ord}_E$ for some positive integer q, hence the cylinder valuation ord_C is a divisorial valuation.

We conclude Chapter V by giving some applications of this correspondence. The log canonical threshold formula in Theorem I.5 easily follows from Theorem I.4. We apply this formula and obtain a comparison theorem via reduction modulo p, as well as a version of Inversion of Adjunction in positive characteristic.

CHAPTER II

Divisorial Valuations and Singularities

2.1 Log Resolutions and Discrepancies

In this section, we work with pairs (X, cZ), where X is a normal scheme over a field k, Z is a proper closed subscheme of X and c is a nonnegative real number. Assume that X is a smooth variety and D_i are prime divisors on X. We say that $E = \sum D_i$ has *simple normal crossings* (abbreviated as *snc*) if E is reduced, each D_i is smooth and they intersect everywhere transversally, i.e. E is defined in a neighborhood of any point by an equation of type $x_1 \cdot x_2 \cdots x_k = 0$ for some $k \leq \dim X$, where $x_1, x_2, \ldots, x_n$ are local coordinates. A divisor $D = \sum_i d_i D_i$ has *simple normal crossing support* (*snc support* for short) if the underlying reduced divisor $\sum D_i$ has simple normal crossings.

Suppose $f : Y \to X$ is a birational morphism between varieties. Let U be the largest open subset of X such that f is an isomorphism over U. The closed subset $Y \setminus f^{-1}(U)$ is the *exceptional locus* of f, denoted by $\mathrm{Exc}(f)$. We say that a Weil divisor on Y is exceptional if its support is contained in $\mathrm{Exc}(f)$.

If (X, cZ) is a pair, a *log resolution* of (X, cZ) is a proper birational morphism $f : Y \to X$ such that

(1) Y is smooth exceptional locus $Exc(f)$ is a divisor.

(2) $f^{-1}(Z)$ is a divisor and $f^{-1}(Z) \cup \mathrm{Exc}(f)$ is a divisor with snc support.

Log resolutions exist for pairs over a field of characteristic zero by the main Theorems of [Hir]. One can construct map f such that it is isomorphism over $X \setminus (X_{\text{sing}} \cup Z)$.

If X is a smooth variety, the canonical line bundle of top differentials is denoted by ω_X. If $\omega_X \cong \mathcal{O}_X(K_X)$, K_X is called a canonical divisor of X. Given a normal variety X, we have a Weil divisor K_X on X, uniquely defined up to linear equivalence, such that its restriction to the smooth locus $U = X \setminus X_{sing}$ of X is equal to a canonical divisor on U. X is called $\mathbf{Q}$–Gorenstein if there is a positive integer m such that mK_X is Cartier. From now on, we always assume that X is $\mathbf{Q}$–Gorenstein for any pair (X, cZ). Moreover, we may and will fix an integer m such that mK_X is a Cartier divisor.

We now review some definitions in the theory of singularities of pairs (X, cZ). We refer the reader to [KM, Section 2.3] for a more detailed introduction. Suppose that X' is a normal variety over k and $f : X' \to X$ is a birational (not necessarily proper) map. Let E be a prime divisor on X'. Any such E is called a divisor *over* X. The local ring $\mathcal{O}_{X',E} \subset k(X')$ is a DVR which corresponds to a divisorial valuation ord_E on $k(X) = k(X')$. The closure of $f(E)$ in X is called the *center* of E, denoted by $c_X(E)$. If $f' : X'' \to X$ is another birational morphism and $F \subset X''$ is a prime divisor such that $\text{ord}_E = \text{ord}_F$ as valuations of $k(X)$, then we consider E and F to define the same divisor over X.

Let E be a prime divisor over X as above. If Z is a closed subscheme of X, then we define $\text{ord}_E(Z)$ as follows. We may assume that E is a divisor on X' and that the scheme-theoretic inverse image $f^{-1}(Z)$ is an effective Cartier divisor on X'. Then $\text{ord}_E(Z)$ is the coefficient of E in $f^{-1}(Z)$. Recall that the *relative canonical divisor* $K_{X'/X}$ is the unique $\mathbf{Q}$–divisor supported on the exceptional locus of f such that $mK_{X'/X}$ is linearly equivalent with $mK_{X'} - f^*(mK_X)$. When X is smooth, we can alternatively describe $K_{X'/X}$ as follows. Let U be a smooth open subset of X' such that $U \cap E \neq \emptyset$. The restriction of f to U is a birational morphism of smooth varieties that we denote by g. In this case, the relative canonical divisor $K_{U/X}$ is the effective Cartier divisor defined by $\det(dg)$ on U.

We also define $\mathrm{ord}_E(K_{-/X})$ as the coefficient of E in $K_{U/X}$. Note that both $\mathrm{ord}_E(Z)$ and $\mathrm{ord}_E(K_{-/X})$ do not depend on the particular choice of f, X' and U.

For every real number $c > 0$, the *log discrepancy* of the pair (X, cZ) with respect to E is

$$a(E; X, cZ) := \mathrm{ord}_E(K_{-/X}) + 1 - c \cdot \mathrm{ord}_E Z.$$

Similarly, the *discrepancy* of the pair (X, cZ) with respect to E is

$$b(E; X, cZ) := \mathrm{ord}_E(K_{-/X}) - c \cdot \mathrm{ord}_E Z.$$

It is clear that $a(E; X, cZ) = b(E; X, cZ) + 1$ for every E.

2.2 Singularities and Log Canonical Threshold

Definition II.1. Let X be a normal $\mathbf{Q}$–Gorenstein n–dimensional variety and let Z be a proper closed subscheme of X. If Y is a closed subset of X, then the *discrepancy* of (X, cZ) along Y is defined by

$$\mathrm{discrep}(Y; X, cZ) := \inf\{b(E; X, cZ) \mid E \text{ is an exceptional divisor over } X,\ c_X(E) \cap Y \neq \emptyset\}.$$

The *total discrepancy* of (X, cZ) along Y is defined by

$$\mathrm{totaldiscrep}(Y; X, cZ) := \inf\{b(E; X, cZ) \mid E \text{ divisor over } X,\ c_X(E) \cap Y \neq \emptyset\}.$$

If $Y = X$, we write $\mathrm{discrep}(X, cZ)$ and $\mathrm{totaldiscrep}(X, cZ)$ for simplicity. If $Z = \emptyset$, we simply write $\mathrm{discrep}(X)$ and $\mathrm{totaldiscrep}(X)$.

Remark II.2. If X is smooth and E is a prime divisor on X such that it is not contained in Z, then $b(E; X, cZ) = 0$. Hence $\mathrm{totaldiscrep}(X, cZ) \leq 0$. Similarly, if E is the exceptional divisor obtained by blowing up a codimension 2 smooth subvariety which is not contained in Z, then $b(E; X, cZ) = 1$. Therefore $\mathrm{discrep}(X, cZ) \leq 1$ if $\dim X \geq 2$. One can show that if $\mathrm{totaldiscrep}(Y; X, cZ) < -1$, then

$$\mathrm{totaldiscrep}(Y; X, cZ) = \mathrm{discrep}(Y; X, cZ) = -\infty,$$

see [KM, Corollary 2,31]. Hence $\mathrm{discrep}(X, cZ) = -\infty$ or $-1 \leq \mathrm{discrep}(X, Z) \leq 1$ and $\mathrm{totaldiscrep}(X, cZ) = -\infty$ or $-1 \leq \mathrm{totaldiscrep}(X, cZ) \leq 0$.

Definition II.3. Let X be a $\mathbf{Q}$-Gorenstein normal scheme. We say that X has *terminal singularities* (respectively *canonical singularities*) if $\mathrm{discrep}(Y; X) > 0$ (respectively $\mathrm{discrep}(Y; X) \geq 0$).

A pair (X, cZ) has *Kawamata log terminal* singularities along Y (or *klt* for short) if

$$\mathrm{totaldiscrep}(Y; X, cZ) > -1.$$

We say that (X, cZ) has *log canonical* (or *lc* for short) singularities if

$$\mathrm{totaldiscrep}(Y; X, cZ) \geq -1.$$

We say that X has *log terminal* (respectively *log canonical*) singularities if the pair $(X, \emptyset)$ is klt (respectively lc).

If a pair (X, Z) is not log canonical, then Remark II.2 implies that

$$\mathrm{totaldiscrep}(X, Z) = -\infty.$$

Hence in this case the discrepancy does not provide much information about the the singularities of the pair. We recall the definition of another invariant that describes how far the pair (X, Z) is from being log canonical. The *log canonical threshold* of (X, Z) at Y, denoted by $\mathrm{lct}_Y(X, Z)$, is defined as follows: if $Z = X$, we set $\mathrm{lct}_Y(X, Z) = 0$, otherwise

$$\mathrm{lct}_Y(X, Z) = \sup\{c \in \mathbf{R}_{\geq 0} \mid (X, cZ) \text{ is klt around } Y\}.$$

In particular, $\mathrm{lct}_Y(X, Z) = \infty$ if and only if $Z \cap Y = \emptyset$. If $Y = X$, we simply write $\mathrm{lct}(X, Z)$ for $\mathrm{lct}_Y(X, Z)$.

By the definition of $a(E; X, Z)$, we obtain that

$$\begin{aligned} \mathrm{lct}_Y(X, Z) &= \sup\left\{c \in \mathbf{R}_{\geq 0} \mid c \cdot \mathrm{ord}_E(Z) < \mathrm{ord}_E(K_{-/X}) + 1 \textit{ for every } E\right\} \\ &= \inf_{E/X} \frac{\mathrm{ord}_E(K_{-/X}) + 1}{\mathrm{ord}_E Z} \end{aligned}$$

where E varies over all divisors over X such that $c_X(E) \cap Y \neq \emptyset$.

Remark II.4. The definition of the above classes of singularities and of the log canonical threshold involves all divisors over X. If the ground field k is of characteristic 0, they are determined by those primes divisors on a single log resolution. For example, see [Kol, Corollary 3.12 and Proposition 8.5] or [EM, Proposition 7.2]. In particular, log canonical thresholds are positive rational numbers if X has klt singularities. Log canonical thresholds can also be described as the first jumping number of multiplier ideals. We refer to [Lar] for the study of the log canonical threshold in connection with multiplier ideals.

For varieties over a field of positive characteristic, one has to use all proper birational morphisms to obtain the log canonical threshold. We will see in Chapter V, as a corollary of inversion of adjunction that we have, as in characteristic zero, for smooth varieties X, $\mathrm{lct}_x(X, Z) \geq 1/\mathrm{ord}_x(Z) > 0$, for every point $x \in Z$. Here $\mathrm{ord}_x(Z)$ is the maximal integer value q such that $I_{Z,x} \subseteq m_{X,x}^q$, where $I_{Z,x}$ is the ideal of Z at x and $m_{X,x}$ the ideal defining x. However, we still don't know whether the log canonical threshold is a rational number in positive characteristics.

CHAPTER III

Jet schemes and arc spaces

3.1 Jet Schemes

In this section, we first recall the definition and basic properties of jet schemes and arc spaces. For a more detailed discussion of jet schemes, see [EM] or [Mus1].

We start with the absolute setting and explain the relative version of jet schemes later. Let k be a field of arbitrary characteristic. A variety is an integral scheme, separated and of finite type over k. Given a scheme X of finite type over k and an integer $m \geq 0$, the m^{th} order jet scheme X_m of X is a scheme of finite type over k satisfying the following adjunction

$$\operatorname{Hom}_{\mathrm{Sch}/k}(Y, X_m) \cong \operatorname{Hom}_{\mathrm{Sch}/k}(Y \times \operatorname{Spec} k[t]/(t^{m+1}), X) \tag{3.1}$$

for every scheme Y of finite type over k. It follows that if X_m exists, then it is unique up to a canonical isomorphism. We will show the existence in Proposition III.5.

Let L be a field extension of k. A morphism $\operatorname{Spec} L[t]/(t^{m+1}) \to X$ is called an L–valued m–jet of X. If γ_m is a point in X_m, we call it an m–jet of X. If κ is the residue field of γ_m, then γ_m induces a morphism $(\gamma_m)_\kappa : \operatorname{Spec} \kappa[t]/(t^{m+1}) \to X$.

It is easy to check that $X_0 = X$. For every $j \leq m$, the natural ring homomorphism $k[t]/(t^{m+1}) \to k[t]/(t^{j+1})$ induces a closed embedding

$$\operatorname{Spec} k[t]/(t^{j+1}) \to \operatorname{Spec} k[t]/(t^{m+1})$$

and the adjunction (3.1) induces a truncation map $\rho_j^m : X_m \to X_j$. For simplicity, we usually write π_m^X or simply π_m for the projection $\rho_0^m : X_m \to X = X_0$. A morphism of schemes $f : X \to Y$ induces morphisms $f_m \colon X_m \to Y_m$ for every m. At the level of L–valued points, this takes an $L[t]/(t^{m+1})$–valued point γ of X_m to $f \circ \gamma$. For every point $x \in X$, we write $X_{m,x}$ for the fiber of π_m at x, the m–jets of X centered at x.

Example III.1. X_1 is the total tangent space $T_X = \mathrm{Spec}(\mathrm{Sym}(\Omega_{X/k}))$. By Lemma III.6, we will see that it is enough to show the assertion when X is affine. Let $X = \mathrm{Spec}\, A$, where A is a k–algebra. We show that for every k–algebra B,

$$\mathrm{Hom}_{k-alg}(\mathrm{Sym}(\Omega_{A/k}), B) \cong \mathrm{Hom}_{k-alg}(A, B[t]/(t^2)). \tag{3.2}$$

Recall that a k–algebra morphism $f : \mathrm{Sym}(\Omega_{A/k}) \to B$ is equivalent to a k–algebra morphism $g : A \to B$ and a k–module morphism $\Omega_{A/k} \to B$, which corresponds to a k-derivation $d : A \to B$. Note that giving g and d is equivalent to giving a k–algebra morphism $\psi_f : A \to B[t]/(t^2)$ where $\psi_f(a) = g(a) + t \cdot d(a)$, we obtain equation (3.2).

Example III.2. If $X = \mathbf{A}^n = \mathrm{Spec}\, k[x_1, \ldots, x_n]$, then X_m is isomorphic to

$$\mathbf{A}^{n(m+1)} = \mathrm{Spec}\, k[a_{i,j}]$$

for $1 \leq i \leq n$ and $0 \leq j \leq m$. Furthermore, for $m \geq j \geq 0$, the truncation morphism $\rho_j^m : \mathbf{A}^{n(m+1)} \to \mathbf{A}^{n(j+1)}$ is the projection onto the first $n(j+1)$ coordinates. To see this, we may assume $Y = \mathrm{Spec}\, B$ for some k–algebra B and check that $\mathbf{A}^{n(m+1)}$ satisfies the adjunction (3.1) for every $m \geq 0$. Indeed, a B–valued point γ of X_m corresponds to a k–algebra homomorphism $\gamma^* : k[x_1, \ldots, x_n] \to B[t]/(t^{m+1})$, which is uniquely determined by the image of x_i. We write $\gamma^*(x_i) = \sum_{j=0}^{m} b_{i,j} t^j$ for each i, with $b_{i,j} \in B$. It is clear that γ corresponds to the B–valued point of $\mathbf{A}^{n(m+1)} = \mathrm{Spec}\, k[a_{i,j}]$ which maps $a_{i,j}$ to $b_{i,j}$.

More generally, if X is a smooth variety of dimension n, then all projections $\rho^m_{m-1} : X_m \to X_{m-1}$ are locally trivial with fiber $\mathbf{A}^n$. In particular, X_m is a smooth variety of dimension $n(m+1)$.

In Chapter V, we will use the relative version of jet schemes. We now recall some basic facts about this context.

We work over a fixed separated scheme S of finite type over a noetherian ring R. Let $f : W \to S$ be a scheme of finite type over S. If s is a point in S, we denote by W_s the fiber of f over s.

Definition III.3. The m^{th} relative jet scheme $(W/S)_m$ satisfies the following adjunction

$$\text{(3.3)} \qquad \mathrm{Hom}_{\mathrm{Sch}/S}(Y \times_R \operatorname{Spec} R[t]/(t^{m+1}), W) \cong \mathrm{Hom}_{\mathrm{Sch}/S}(Y, (W/S)_m),$$

for every scheme of finite type Y over S.

As in the absolute setting, we have $(W/S)_0 \cong W$. If $(W/S)_m$ and $(W/S)_j$ exist with $m \geq j$, then there is a canonical projection $\rho^m_j : (W/S)_m \to (W/S)_j$. For simplicity, we usually write π_m for the projection $\rho^m_0 : (W/S)_m \to W$.

Example III.4. If $W = \mathbf{A}^n_S$ with the natural projection $f : W \to S$, then for every $m \geq 0$, $(W/S)_m \cong \mathbf{A}^{n(m+1)}_S$. Furthermore, for every $m \geq j$, the truncation morphism $\rho^m_j : (\mathbf{A}^n_S)_m \to (\mathbf{A}^n_S)_j$ is the projection onto the first $n(j+1)$ coordinates. The proof is similar to that of the absolute case in Example III.2.

We now prove the existence of the relative jet schemes, which is similar to that of the absolute case. For details, see [Mus2].

Proposition III.5. *If $f : W \to S$ is a scheme of finite type over S, the m^{th} order relative jet scheme $(W/S)_m$ exists for every $m \in \mathbf{N}$.*

Before proving the proposition, we first show that the construction of relative jet scheme is compatible with open embeddings.

Lemma III.6. *Let U be an open subset of a scheme W over S. If $(W/S)_m$ exists, then $(U/S)_m$ exists and $(U/S)_m \cong (\pi_m^W)^{-1}(U)$.*

Proof. We denote by $g : U \to S$ the composition of the open embedding $U \hookrightarrow W$ and $f : W \to S$. We have to show that for every S–scheme Y,

$$\text{Hom}_{\text{Sch}\,/S}(Y \times_R \operatorname{Spec} R[t]/(t^{m+1}), U) \cong \text{Hom}_{\text{Sch}\,/S}(Y, (\pi_m^W)^{-1}(U)). \tag{3.4}$$

We first assume that $S = \operatorname{Spec} A$, where A is a finitely generated R–algebra. It is enough to show that the above adjunction holds for every affine scheme Y. Let B be an A–algebra and $Y = \operatorname{Spec} B$. If $\gamma : \operatorname{Spec} B[t]/(t^{m+1}) \to W$ is a B–valued jet of W, let $\gamma_0 = \pi_m(\gamma)$ be the induced morphism $\operatorname{Spec} B \to W$. It is clear that γ factors through U if and only if γ_0 factors through U. Applying adjointness of $(W/S)_m$ in (3.3), we deduce that $(\pi_m^X)^{-1}(U)$ satisfies (3.4).

Given an arbitrary scheme S of finite type over R, let $(S_\alpha)_{\alpha \in I}$ be an affine covering of S. Let $W_\alpha = f^{-1}(S_\alpha)$ and $U_\alpha = g^{-1}(S_\alpha)$. For each $\alpha, \beta \in I$, let $S_{\alpha\beta} = S_\alpha \cap S_\beta$, $W_{\alpha\beta} = f^{-1}(S_{\alpha\beta})$ and $U_{\alpha\beta} = g^{-1}(S_{\alpha\beta})$. Since S is separated, each $S_{\alpha\beta}$ is affine. The above argument showed that for every α, there is a canonical isomorphism $(U_\alpha/S_\alpha)_m \cong (\pi_m^{W_\alpha})^{-1}(U_\alpha)$. Furthermore, these isomorphisms agree on the overlaps $S_{\alpha\beta}$. We conclude that $(U/S)_m = (\pi_m^W)^{-1}(U)$. □

We now prove Proposition III.5 by first constructing the relative jet scheme locally and gluing the schemes along overlaps.

Proof. By covering S by affine open subschemes, we may and will assume S is an affine scheme. Let $S = \operatorname{Spec} A$, where A is a finitely generated R–algebra. We first construct $(W/S)_m$ when W is an affine scheme over S. Let $W = \operatorname{Spec} B$ for some A–algebra B. Consider a closed embedding $W \to \mathbf{A}^n_S$ such that W is defined by the ideal $I = (f_1, \ldots, f_r) \subseteq A[x_1, \ldots, x_n]$. An S–morphism $\varphi : \operatorname{Spec} B[t]/(t^{m+1}) \to W$ is given by $\varphi^*(x_i) = \sum_{j=0}^{m} b_{i,j} t^j$ with $b_{i,j} \in B$ such that $f_l(\varphi^*(x_1), \ldots, \varphi^*(x_n)) = 0$ in $B[t]/(t^{m+1})$ for every l.

Given any $u_i = \sum_{j=0}^{m} a_{i,j} t^j$ in $A[t]/(t^{m+1})$ for $1 \leq i \leq n$, we can write

$$f_l(u_1, \ldots, u_n) = \sum_{p=0}^{m} g_{l,p}(a_{i,j}) t^p, \tag{3.5}$$

for some polynomials $g_{l,p}$ in $A[a_{i,j}]$ with $1 \leq i \leq n$ and $0 \leq j \leq m$. Let Z be the closed subscheme of $\mathbf{A}^{n(m+1)}_S = \operatorname{Spec} A[a_{i,j}]$ defined by $(g_{l,p})$ for $1 \leq l \leq r$ and $0 \leq p \leq m$. It is clear that φ is a $B[t]/(t^{m+1})$–valued point of (W/S) if and only if the corresponding $(b_{i,j})$ defines a B–valued point of Z. Hence $(X/S)_m \cong Z$.

Given W an arbitrary S–scheme of finite type, we consider an affine open cover $W = \bigcup_\alpha W_\alpha$. We have seen that $(W_\alpha/S)_m$ exists for every $m \geq 0$. Let $\pi^\alpha_m : (W_\alpha/S)_m \to W_\alpha$ be the canonical projection. For every α and β, we write $W_{\alpha\beta} = W_\alpha \cap W_\beta$. The inverse image $(\pi^\alpha_m)^{-1}(W_{\alpha\beta})$ and $(\pi^\beta_m)^{-1}(W_{\alpha\beta})$ are canonically isomorphic since they are isomorphic to $(W_{\alpha\beta}/S)_m$. Hence we can construct a scheme $(W/S)_m$ by gluing the schemes $(W_\alpha/S)_m$ along their overlaps. Moreover, the projections π^α_m glue to give an S–morphism

$$\pi_m : (W/S)_m \to W.$$

It is clear that $(W/S)_m$ is the m^{th} relative jet scheme of W over S. $\square$

For every scheme morphism $S' \to S$ and every W/S as above, we denote by W' the fiber product $W \times_S S'$. By the functorial definition of relative jet schemes, we can check that

$$(W'/S')_n \cong (W/S)_n \times_S S'$$

for every n. In particular, for every $s \in S$, we conclude that the fiber of $(W/S)_n \to S$ over s is isomorphic to $(W_s)_n$.

Recall that $\pi_n : (W/S)_n \to W$ is the canonical projection. We now show that there is an S–morphism, called the zero-section map, $\sigma_n : W \to (W/S)_n$ such that $\pi_n \circ \sigma_n = \mathrm{id}_W$ for every n. We have a natural map $g_n : W \times \operatorname{Spec} R[t]/(t^{n+1}) \to W$, the projection onto the first factor. By (3.3), g_n induces a morphism $\sigma_n^W : W \to (W/S)_n$, the *zero-section* of π_n. For simplicity, we usually write σ_n for σ_n^W. Note that for every n and every scheme W over S, there is a natural action:

$$\Gamma_n : \mathbf{A}^1_S \times_S (W/S)_n \to (W/S)_n$$

of the affine group $\mathbf{A}^1_S$ on the jet schemes $(W/S)_n$ defined as follows. For an A–valued point (a, γ_n) of $\mathbf{A}^1_S \times_S (W/S)_n$ where $a \in A$ and $\gamma_n : \operatorname{Spec} A[t]/(t^{n+1}) \to W$, we define $\Gamma_n(a, \gamma_n)$ as the composition map $\operatorname{Spec} A[t]/(t^{n+1}) \xrightarrow{a^*} \operatorname{Spec} A[t]/(t^{n+1}) \xrightarrow{\gamma_n} X$, where a^* corresponds to the A–algebra homomorphism $A[t]/(t^{n+1}) \to A[t]/(t^{n+1})$ mapping t to at. One can check that the image of the zero section σ_n is equal to $\Gamma_n(\{0\} \times (W/S)_n)$.

Lemma III.7. *Let $f : W \to S$ be a family of schemes and $\tau : S \to W$ a section of f. For every $m \geq 1$, the function*

$$d(s) = \dim(\pi_m^{W_s})^{-1}(\tau(s))$$

is upper semi-continuous on S.

Proof. Due to the local nature of the assertion, we may assume that $S = \operatorname{Spec} A$ is an affine scheme. Given a point $s \in S$, we denote by $w = \tau(s)$ in W. Let W' be an open affine neighborhood of w in W. Consider the restriction map $f' : W' \to S$ of f, one can show that there is an nonzero element $a \in A$ such that such that

τ maps the affine neighborhood $S' \cong \operatorname{Spec} A_h$ of s into W'. Let W'' be the affine neighborhood $g^{-1}(S')$ of w and $f'' : W'' \to S'$. The restriction of τ defines a section $\tau' : S' \to W''$. Replacing f by f'' and τ by τ', we may and will assume that both W and S are affine schemes. Let $W = \operatorname{Spec} B$, where B is a finitely generated A–algebra. The section τ induce a ring homomorphism $\tau^* : B \twoheadrightarrow A$. Choose A–algebra generators u_i of B such that $\tau^*(u_i) = 0$. Let C be the polynomial ring $A[x_1, \ldots, x_n]$. We define a ring homomorphism $\varphi : C \to B$ which maps x_i to u_i for every i. Let $I = (f_1, \ldots, f_r)$ be the kernel of φ. One can check that $f_l \in (x_1, \ldots, x_n)$ for every l with $1 \leq l \leq r$. Hence W is a closed subscheme of $\mathbf{A}^n_S = \operatorname{Spec} A[x_1, \ldots, x_n]$ defined by the system of polynomials (f_l) and the zero section $o : S \to \mathbf{A}^n_S$ factors through τ. It is clear that $(\mathbf{A}^n_S)_m = \operatorname{Spec} A[a_{i,j}] \cong \mathbf{A}^{n(m+1)}_S$ for $1 \leq i \leq n$ and $0 \leq j \leq m$ and $\sigma_m^{\mathbf{A}^n_S} \circ o : S \to \mathbf{A}^{n(m+1)}_S$ is the zero-section.

We thus obtain an embedding $(W/S)_m \subset \mathbf{A}^{(m+1)n}_S$ which induces an embedding $(\pi^W_m)^{-1}(\tau(S)) \subset (\pi_m^{\mathbf{A}^n_S})^{-1}(o(S)) \cong \mathbf{A}^{mn}_S$ such that $\sigma^W_m \circ \tau$ corresponding to the zero-section of $\mathbf{A}^{mn}_S = \operatorname{Spec} A[a_{i,j}]$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Recall that $(W/S)_m$ as a subscheme of $\mathbf{A}^{n(m+1)}_S$ is defined by the polynomials $g_{l,p}$ in equation (3.5). Let $\deg a_{i,j} = j$. Since f_l has no constant terms, we can check that each $g_{l,p}$ is homogenous of degree p. Hence the coordinate ring of $(\pi^W_m)^{-1}(\tau(S))$, denoted by T, is a graded A–algebra.

For every $s \in S$ corresponding to a prime ideal $\mathfrak{p}$ of A, we obtain that

$$d(s) = \dim(\pi_m^{W_s})^{-1}(\tau(s)) = \dim(T \otimes_A A/\mathfrak{p}).$$

Our assertion follows from a semi-continuity result on the dimension of fibers of a projective morphism (see [Eis, Theorem 14.8]). $\square$

Remark III.8. Let X be a smooth variety over a field k and Y a closed subscheme of X. If T is an irreducible component of Y_m for some m, then T is invariant under the action of $\mathbf{A}^1$. Since $\pi_m(T) = \sigma_m^{-1}(T \cap \sigma_m(X))$, it follows that $\pi_m(T)$ is closed in X.

3.2 Arc Spaces and Contact Loci

We now turn to the projective limit of jet schemes. It follows from the description in the proof of Proposition III.5 that the projective system

$$\cdots \to X_m \to X_{m-1} \to \cdots \to X_0$$

consists of affine morphisms. Hence the projective limit exists in the category of schemes over k. This is called the *space of arcs* of X, denoted by X_∞. Note that in general, it is not of finite type over k. There are natural projection morphisms $\psi_m : X_\infty \to X_m$. It follows from the projective limit definition and the functorial description of the jet schemes that for every k–field extension A we have

$$\mathrm{Hom}(\mathrm{Spec}(A), X_\infty) \simeq \varprojlim \mathrm{Hom}(\mathrm{Spec}\ A[t]/(t^{m+1}), X) \simeq \mathrm{Hom}(\mathrm{Spec}\ A[\![t]\!], X)$$

In particular, for every field extension L of k, an L–valued point of X_∞, called an *L–valued arc*, corresponds to a morphism from Spec $L[\![t]\!]$ to X. We denote the closed point of Spec $L[\![t]\!]$ by 0 and by η the generic point. A point in X_∞ is called an *arc in* X. If γ is a point in X_∞ with residue field κ, γ induces a κ–valued arc, i.e. a morphism $\gamma_\kappa : \mathrm{Spec}\ \kappa[\![t]\!] \to X$. If $f : X \to Y$ is a morphism of schemes of finite type, by taking the projective limit of the morphisms $f_m : X_m \to Y_m$ we get a morphism $f_\infty : X_\infty \to Y_\infty$.

For every scheme X, a *cylinder* in X_∞ is a subset of the form $C = \psi_m^{-1}(S)$, for some m and some constructible subset $S \subseteq X_m$. If X is a smooth variety of pure dimension n over k, then all truncation maps ρ^m_{m-1} are locally trivial with fiber $\mathbf{A}^n$. In particular, all projections $\psi_m : X_\infty \to X_m$ are surjective and $\dim X_m = (m+1)n$.

From now on, we will assume that X is smooth and of pure dimension n. We say that a cylinder $C = \psi_m^{-1}(S)$ is *irreducible (closed, open, locally closed)* if so is S. It is clear that all these properties of C do not depend on the particular choice of m and S. We define the *codimension of* C by

$$\operatorname{codim} C := \operatorname{codim}(S, X_m) = (m+1)n - \dim S.$$

Since the truncation maps are locally trivial, $\operatorname{codim} C$ is independent of the particular choice of m and S.

For a closed subscheme Z of a scheme X defined by the ideal sheaf $\mathfrak{a}$ and for an L–valued arc $\gamma : \operatorname{Spec} L[\![t]\!] \to X$, the inverse image of Z by γ is defined by a principal ideal in $L[\![t]\!]$. If this ideal is generated by t^e with $e \geq 0$, then we define the vanishing order of γ along Z to be $\operatorname{ord}_\gamma(Z) = e$. On the other hand, if this is the zero ideal, we put $\operatorname{ord}_\gamma(Z) = \infty$. If γ is a point in X_∞ with residue field L, then we define $\operatorname{ord}_\gamma(Z)$ by considering the corresponding morphism $\operatorname{Spec} L[\![t]\!] \to X$. The *contact locus of order* e with Z is the subset of X_∞

$$\operatorname{Cont}^e(Z) = \operatorname{Cont}^e(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(Z) = e\}.$$

We similarly define

$$\operatorname{Cont}^{\geq e}(Z) = \operatorname{Cont}^{\geq e}(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(Z) \geq e\}.$$

For $m \geq e$, we can define constructible subsets $\operatorname{Cont}^e(Z)_m$ and $\operatorname{Cont}^{\geq e}(Z)_m$ of X_m in the obvious way. (In fact, the former one is locally closed, while the latter one is closed.) By definition, we have

$$\operatorname{Cont}^e(Z) = \psi_m^{-1}(\operatorname{Cont}^e(Z)_m) \text{ and } \operatorname{Cont}^{\geq e}(Z) = \psi_m^{-1}(\operatorname{Cont}^{\geq e}(Z)_m).$$

This implies that $\operatorname{Cont}^{\geq e}(Z)$ is a closed cylinder and $\operatorname{Cont}^e(Z)$ is a locally closed cylinder in X_∞.

CHAPTER IV

Singularities of Brill-Noether Loci

4.1 Introduction to varieties of special linear series on a curve

In this section, k is an algebraically closed field of characteristic 0. Let C be a smooth projective curve of genus g over field k. We now recall the definition of $\mathrm{Pic}^d(C)$. For every scheme S, let p and q be the projections of $S \times C$ onto S and C respectively. *A family of degree d line bundles on C parameterized by a scheme S* is a line bundle on $C \times S$ which restricts to a degree d line bundle on $C \times \{s\}$, for every s in S. We say that two such families $\mathcal{L}$ and $\mathcal{L}'$ are *equivalent* if there is a line bundle $\mathcal{R}$ on S such that $\mathcal{L}' \cong \mathcal{L} \otimes q^*\mathcal{R}$. $\mathrm{Pic}^d(C)$ parameterizes degree d line bundles on C; more precisely, it represents the functor

$$F : \mathrm{Sch}\,/k \to \mathrm{Set}$$

where $F(S)$ is the set of equivalence classes of families of degree d line bundles on C parameterized by S. A universal line bundle $\mathcal{P}$ on $C \times \mathrm{Pic}^d(C)$ is a *Poincaré line bundle* of degree d for C.

Recall now that $W^r_d(C)$ is the closed subset of $\mathrm{Pic}^d(C)$ parameterizing line bundles L of degree d with $\dim |L| \geq r$:

$$W^r_d(C) = \{L \in \mathrm{Pic}^d(C) : \deg L = d, h^0(L) \geq r+1\}.$$

In particular, we have the theta divisor

$$\Theta := \{L \in \mathrm{Pic}^{g-1}(C) : h^0(C, L) \neq 0\} = W^0_{g-1}(C).$$

Each $W^r_d(C)$ has a natural scheme structure as a degeneracy locus we now describe.

Let E be any effective divisor on C of degree $e \geq 2g - d - 1$ and let $\mathcal{E} = \mathcal{O}(E)$. The following facts are standard (see [ACGH, §IV.3]. For every family of degree d line bundles $\mathcal{L}$ on $S \times C$, the sheaves $p_*(\mathcal{L} \otimes q^*(\mathcal{E}))$ and $p_*(\mathcal{L} \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E})$ are locally free of ranks $d + e + 1 - g$ and e, respectively. Moreover, there is an exact sequence on S

$$0 \to p_*\mathcal{L} \to p_*(\mathcal{L} \otimes q^*(\mathcal{E})) \xrightarrow{\Phi_{\mathcal{L}}} p_*(\mathcal{L} \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E}) \to R^1p_*(\mathcal{L}) \to 0. \tag{4.1}$$

With the above notation, $W^r_d(C)$ represents the functor $\mathrm{Sch}/k \to \mathrm{Set}$ given by

$$S \mapsto \left\{ \begin{array}{c} \text{equivalence classes of families } \mathcal{L} \text{ of degree } d \text{ line bundles on} \\ S \times C \xrightarrow{p} S \text{ such that } \operatorname{rank}(\Phi_{\mathcal{L}}) \leq d + e - g - r \end{array} \right\}.$$

It can be shown that the above condition $\operatorname{rank}(\Phi_{\mathcal{L}}) \leq d + e - g - r$ does not depend on the particular choice of e and E.

In particular, the line bundle $L \in \mathrm{Pic}^d(C)$ is in $W^r_d(C)$ if and only if locally all the $e + d + 1 - g - r$ minors of Φ_L vanish. Therefore $W^r_d(C)$ is a determinantal variety.

Let T_m be the scheme $\operatorname{Spec} k[t]/(t^{m+1})$. We now discuss the jet schemes of the theta divisor Θ_m for all m. By the definition of Θ, we have Θ_m consists of line bundles $\mathcal{L}_m \in \mathrm{Pic}(T_m \times C)$ such that $\deg(\mathcal{L}_m|_{\{0\}\times C}) = g - 1$ and $\det(\Phi_{\mathcal{L}_m}) = 0$ in $k[t]/(t^{m+1})$.

Given a positive integer n, we recall that a *partition of* n is a weakly increasing sequence $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_l$ such that $\lambda_1 + \cdots + \lambda_l = n$. The number l of integers in the sequence is called the length of the partition, and the value λ_l is the largest term. The set of partitions with length l is denoted by Λ_l, and the set of partitions with length l and largest term at most m is denoted by $\Lambda_{l,m}$. For every i with $1 \leq i \leq m$, if $\lambda \in \Lambda_{l,m}$, we define $\overline{\lambda} \in \Lambda_{l,i}$ by putting $\overline{\lambda}_k = \min\{\lambda_k, i\}$ for every k with $1 \leq k \leq l$. We thus obtain a natural map $\Lambda_{l,m} \to \Lambda_{l,i}$.

Fix an effective divisor E of degree $e \geq 2g - d - 1$ on C. We now associate a partition to every $\mathcal{L}_m \in \mathrm{Pic}^d(C)_m$. $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}))$ and $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E})$ are locally free sheaves on T_m, hence they are finitely generated free modules over $k[t]/(t^{m+1})$.

Definition IV.1. A family of line bundles $\mathcal{L}_m$ of degree d on C over T_m is called of type $\lambda \in \Lambda_{l,m+1}$ if there are bases of $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}))$ and $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E})$ in which $\Phi_{\mathcal{L}_m}$ is represented by the matrix in $M_{(d+e+1-g)\times e}(k[t]/(t^{m+1}))$

$$\begin{pmatrix} 1 & & & & & & 0 & 0 \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ & & & t^{\lambda_1} & 0 & & \vdots & \vdots \\ & & & & \ddots & & & \\ & & 0 & & & t^{\lambda_l} & 0 & 0 \end{pmatrix}$$

Definition IV.2. Given a partition λ , let $r_i(\lambda)$ be the number of k such that $\lambda_k = i$ and let $n_i(\lambda)$ be the number of k such that $\lambda_k \geq i$.

It is easy to see that the partition λ in Definition IV.1 does not depend on the choice of bases. If L is the image of $\mathcal{L}_m$ under the truncation map

$$\pi_m : \mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C),$$

then we will see below that the length of the partition associated to $\mathcal{L}_m$ is $h^0(C, L)$.

We now give a criterion to decide whether an element $\mathcal{L}_m \in \mathrm{Pic}^{g-1}(C)_m$ is a jet of Θ in terms of the partition λ.

Lemma IV.3. *For every family of line bundles $\mathcal{L}_m \in \mathrm{Pic}^{g-1}(C)_m$ centered at $L \in \mathrm{Pic}^{g-1}(C)$ and of type $\lambda \in \Lambda_{l,m+1}$, the following are equivalent:*

(i) $\mathcal{L}_m \in \Theta_{m,L}$.

(i)′ $\det(\Phi_{\mathcal{L}_m}) = 0$ *in* $k[t]/(t^{m+1})$.

(ii) $\sum_{i=1}^{l} \lambda_i \geq m+1$.

(ii)′ $\sum_{j=1}^{m+1} r_j(\lambda) \cdot j \geq m+1$.

(ii)′′ $\sum_{k=1}^{m+1} n_k(\lambda) \geq m+1$.

Proof. Recall that $\Theta = W^0_{g-1} \subset \mathrm{Pic}^{g-1}(C)$. With the above notation, for every family of line bundles $\mathcal{L}_m$ in Θ_m, the sheaves $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}))$ and $p_*(\mathcal{L}_m \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E})$ are locally free of rank e. The definition shows that the theta divisor parameterizes the line bundles $\mathcal{L}_m$ for which $\det(\Phi_{\mathcal{L}_m}) = 0$. This proves the equivalence between (i) and (i)′. It is clear that with the choice of basis in Definition IV.1,

$$\det(\Phi_{\mathcal{L}_m}) = t^{\lambda_1 + \cdots + \lambda_l} \in k[t]/(t^{m+1}).$$

Therefore the determinant vanishes if and only if $\sum_{i}^{l} \lambda_i \geq m+1$.

In order to complete the proof of the lemma, it suffices to show that

$$\sum_{i=1}^{l} \lambda_i = \sum_{j=1}^{m+1} r_j(\lambda) \cdot j = \sum_{k=1}^{m+1} n_k(\lambda).$$

The first equality is clear by the definition of $r_j(\lambda)$. The second equality follows from $n_k(\lambda) = \sum_{j\geq k} r_j(\lambda)$. Indeed, $\sum_{k=1}^{m+1} n_k(\lambda) = \sum_{k=1}^{m+1}\sum_{j\geq k} r_j(\lambda) = \sum_{j=1}^{m+1} r_j(\lambda)\cdot j$. $\square$

Using the definition of $W^r_d(C)$, we have the following description of $W^r_d(C)_{m,L}$, which gives a generalization of Lemma IV.3.

Lemma IV.4. *Let $\mathcal{L}_m \in \mathrm{Pic}^d(C)_m$ have type $\lambda = (1 \leq \lambda_1 \leq \cdots \leq \lambda_l \leq m+1)$. The following are equivalent:*

(i) $\mathcal{L}_m \in W^r_d(C)_{m,L}$.

(i') *All the $(e+d+1-g-r)$ minors of $\Phi_{\mathcal{L}_m}$ vanish in $k[t]/(t^{m+1})$.*

(ii) $\sum_{i=1}^{l-r} \lambda_i \geq m+1$.

(ii') $\sum_{i=1}^{l-r}(l-i-r+1)(\lambda_i - \lambda_{i-1}) \geq m+1$, *where* $\lambda_0 = 0$.

The proof of this lemma is very similar to that of Lemma IV.3, so we leave it to the reader.

Our first goal is to recover Riemann's Singularity Theorem using jet schemes.

Theorem IV.5. *For every $L \in \Theta$, we have* $\mathrm{mult}_L\Theta = h^0(C,L)$.

Remark IV.6. Note that the multiplicity of a divisor at a point is one if and only if the divisor is smooth at that point, hence Theorem IV.5 implies in particular that a line bundle $L \in \Theta$ is a smooth point if and only if $h^0(C,L) = 1$.

Before proving the theorem we need some preparations. For every degree d line bundle L, we shall first describe the fiber of $\rho^m_{m-1} : \mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{m-1,L}$.

Let E be the effective divisor of degree $e \geq 2g-d-1$ in Definition IV.1. By the universal property of $\mathrm{Pic}^d(C)$, every $\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L}$ is identified with a line bundle on $C \times T_m$. Let us fix a line bundle $L \in \mathrm{Pic}^d(C)$ and a family of line bundles

$\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L}$ lying over L. For every $0 \le i \le m$, we denote by $\mathcal{L}_i$ the image of $\mathcal{L}_m$ in $\mathrm{Pic}^d(C)_{i,L}$ under the truncation map $\mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C)_i$. By the short exact sequence (4.1), $H^0(\mathcal{L}_i)$ is the kernel of the morphism

$$\Phi_{\mathcal{L}_i} : M_i = H^0(\mathcal{L}_i \otimes q^*(\mathcal{E})) \to N_i = H^0(\mathcal{L}_i \otimes q^*(\mathcal{E}) \otimes \mathcal{O}_{q^{-1}E}).$$

There is a $k[t]/(t^{m+1})$-module map $\pi_i^m : H^0(\mathcal{L}_m) \to H^0(\mathcal{L}_i)$ induced by restriction of sections. This can be described as follows. Applying the Base-change Theorem to the morphism $T_i \hookrightarrow T_m$, we obtain the following commutative diagram

$$\begin{array}{ccccc}
H^0(\mathcal{L}_m) & \hookrightarrow & M_m & \xrightarrow{\Phi_{\mathcal{L}_m}} & N_m \\
\downarrow{\scriptstyle \pi_i^m} & & \downarrow{\scriptstyle \rho_M} & & \downarrow{\scriptstyle \rho_N} \\
H^0(\mathcal{L}_i) & \hookrightarrow & M_i & \xrightarrow{\Phi_{\mathcal{L}_i}} & N_i
\end{array}$$

Clearly $M_i = M_m \otimes_{k[t]/(t^{m+1})} k[t]/(t^{i+1})$ and $N_i = N_m \otimes_{k[t]/(t^{m+1})} k[t]/(t^{i+1})$ and the vertical maps are induced by the quotient map $k[t]/(t^{m+1}) \to k[t]/(t^{i+1})$.

Lemma IV.7. *For every $0 \le i \le m$, there is an embedding of $k[t]/(t^{m+1})$-modules*

$$v_i^m : H^0(\mathcal{L}_i) \hookrightarrow H^0(\mathcal{L}_m)$$

such that the image is the kernel of $\pi_{m-i-1}^m : H^0(\mathcal{L}_m) \to H^0(\mathcal{L}_{m-i-1})$.

Proof. The multiplication with t^{m-i} defines a linear map of $k[t]/(t^{m+1})$-modules

$$k[t]/(t^{i+1}) \to k[t]/(t^{m+1})$$

and induces embeddings of $k[t]/(t^{m+1})$ modules $M_i \xrightarrow{u_i^m} M_m$ and $N_i \xrightarrow{w_i^m} N_m$. Therefore it induces an injective $k[t]/(t^{m+1})$-module morphism $v_i^m : H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_m)$.

It is clear that the image of the embedding $u_i^m : M_i \to M_m$ is $\mathrm{Ann}_{M_m}(t^{i+1})$. By definition, we have $H^0(\mathcal{L}_m) \cap \mathrm{Ann}_{M_m}(t^{i+1}) = \mathrm{Ann}_{H^0(\mathcal{L}_m)}(t^{i+1})$. The multiplication map $w_i^m : N_i \to N_m$ is injective, and one deduces easily that the image of v_i is $\mathrm{Ann}_{H^0(\mathcal{L}_m)}(t^{i+1})$. Since $\ker \pi_{m-i-1}^m = \mathrm{Ann}_{H^0(\mathcal{L}_m)}(t^{i+1})$, this completes our proof. $\square$

Lemma IV.8. *For every family of line bundles $\mathcal{L}_m \in \mathrm{Pic}^d(C)_m$ of type $\lambda \in \Lambda_{l,m+1}$, we have*

$$h^0(\mathcal{L}_m) = \sum_{k=1}^{m+1} n_k(\lambda).$$

Proof. Choose bases $\{e_j\}$ and $\{f_h\}$ for the free modules M_m and N_m such that $\Phi_{\mathcal{L}_m}$ is represented by the matrix

$$\begin{pmatrix} 1 & & & & & & 0 & 0 \\ & \ddots & & & & & & \\ & & 1 & & 0 & & & \\ & & & t^{\lambda_1} & & & \vdots & \vdots \\ & & & & \ddots & & & \\ & & 0 & & & t^{\lambda_l} & 0 & 0 \end{pmatrix} = A_0 + A_1 \cdot t + \cdots + A_m \cdot t^m.$$

All A_i are $(d+e+1-g)\times e$ matrices over the field k. For every $0 \le i \le m$ the image of $\{e_j\}$ under the map $\rho_M : M_m \to M_i$ gives a basis of M_i over $k[t]/(t^{i+1})$. Similarly, the image of $\{f_h\}$ under $\rho_N : N_m \to N_i$ gives a basis of N_i. With respect to these bases, the homomorphism $\Phi_{\mathcal{L}_i}$ is represented by the matrix $A_0 + A_1 \cdot t + \cdots + A_i \cdot t^i$.

We first consider the case $m = 0$. Φ_L is represented by A_0, which is a diagonal matrix with 1 showing up on the first $e + d + 1 - g - l$ rows, hence

$$h^0(L) = \dim_k \ker \Phi_L = l = n_1(\lambda).$$

Let λ' be the type of $\mathcal{L}_{m-1}$. One can check easily that λ' is the image of λ under the natural map $\Lambda_{l,m+1} \to \Lambda_{l,m}$. For $k \le m$, we have $n_k(\lambda') = n_k(\lambda)$. Now it suffices to show that $h^0(\mathcal{L}_m) - h^0(\mathcal{L}_{m-1}) = n_{m+1}(\lambda)$ for $m \ge 1$. For each $i > 0$, A_i is a diagonal matrix with entries 0 or 1, with 1's in the rows $(e+d+1-g)-l+r_1+\cdots+r_{i-1}+j$, with $1 \le j \le r_i$, where $r_i = r_i(\lambda)$ (See Definition IV.2). We now consider $\{t^k \cdot e_j\}$

and $\{t^k \cdot f_h\}$ where $0 \le k \le m$ to be the bases of M_m and N_m, respectively, as linear spaces over k. The matrix associated to $\Phi_{\mathcal{L}_m}$ as a morphism of $k-$linear spaces has the upper triangular form

$$\Psi_{\mathcal{L}_m} = \begin{pmatrix} A_0 & A_1 & A_2 & \cdots & A_{m-1} & A_m & 0 & 0 \\ & A_0 & A_1 & \cdots & A_{m-2} & A_{m-1} & 0 & 0 \\ & & A_0 & \ddots & & \vdots & \vdots & \vdots \\ & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ & & & & A_0 & A_1 & 0 & 0 \\ & 0 & & & & A_0 & 0 & 0 \end{pmatrix}$$

Therefore the associated matrix $\Psi_{\mathcal{L}_{m-1}}$ of $\Phi_{\mathcal{L}_{m-1}}$ as a k-linear map is the bottom right corner submatrix of the associated matrix of $\Phi_{\mathcal{L}_m}$, obtained by omitting the rows and columns containing the left upper corner A_0.

In each row and column of the matrix $\Psi_{\mathcal{L}_m}$, there is at most one nonzero element. Therefore $\operatorname{rank} \Psi_{\mathcal{L}_m} = \operatorname{rank} \Psi_{\mathcal{L}_{m-1}} + \sum_{i=0}^{m} \operatorname{rank} A_i$. Since $\operatorname{rank}(A_0) = d + e + 1 - g - l$ and $\operatorname{rank}(A_i) = r_i(\lambda)$ for $1 \le i \le m$, we deduce that

$$\dim_k \ker \Phi_{\mathcal{L}_m} - \dim_k \ker \Phi_{\mathcal{L}_{m-1}} = n_{m+1}(\lambda).$$

Therefore $h^0(\mathcal{L}_m) - h^0(\mathcal{L}_{m-1}) = n_{m+1}(\lambda)$. $\square$

Remark IV.9. For every j with $0 \le j \le m$, Lemmas IV.7 and IV.8 imply that the image of the morphism $\pi_0^j : H^0(\mathcal{L}_j) \to H^0(L)$ has dimension equal to

$$h^0(\mathcal{L}_j) - \dim_k \ker(\pi_0^j) = h^0(\mathcal{L}_j) - h^0(\mathcal{L}_{j-1}) = n_{j+1}(\lambda).$$

Therefore π_0^j is a zero map if and only if $n_{j+1}(\lambda) = 0$.

We now fix a line bundle $L \in \mathrm{Pic}^d(C)$ and describe the fibers of the truncation maps $\rho^m_{m-1} : \mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{m-1,L}$ for every m. Let $\{U_\alpha\}$ be an affine covering of C which trivializes the line bundle L by isomorphisms $\gamma_\alpha : L|_{U_\alpha} \cong \mathcal{O}_{U_\alpha}$. Let $\{g_{\alpha\beta} = \gamma_\beta \circ \gamma_\alpha^{-1}\}$ be the corresponding transition functions. For every scheme U_α and $i \geq 1$, we have short exact sequence of sheaves on $U_\alpha \times T_i$ as follows:

$$0 \to \mathcal{O}_{U_\alpha} \to \mathcal{O}^*_{U_\alpha \times T_i} \to \mathcal{O}^*_{U_\alpha \times T_{i-1}} \to 0,$$

where the embedding morphism maps $x \in \mathcal{O}_{U_\alpha}$ to $1 + x \cdot t^i$. Since U_α is affine, $H^j(\mathcal{O}_{U_\alpha})$ vanishes for every $j \geq 1$. We thus obtain an isomorphism

$$H^1(\mathcal{O}_{U_\alpha \times T_i}) \cong H^1(\mathcal{O}_{U_\alpha \times T_{i-1}}).$$

In other words, we have $\mathrm{Pic}(U_\alpha \times T_i) \cong \mathrm{Pic}(U_\alpha \times T_{i-1})$ for every α. By induction on i with $0 \leq i \leq m$, we deduce that $\{U_\alpha \times T_m\}$ is an affine covering of $C \times T_m$ which trivializes every line bundle $\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L}$. In particular, for every line bundle $\mathcal{L}_1 \in \mathrm{Pic}^d(C)_{1,L}$ on $C \times T_1$, there is a trivialization for $\mathcal{L}_1$ on the covering $\{U_\alpha \times T_1\}$ with the transition functions $\{g_{\alpha\beta}(1 + t\varphi^{(1)}_{\alpha\beta})\}$. This gives a bijection $\xi : \mathrm{Pic}^d(C)_{1,L} \to H^1(C, \mathcal{O}_C)$ via $\xi(\mathcal{L}_1) = [\varphi^{(1)}_{\alpha\beta}]$.

In general, we fix a family of line bundles $\mathcal{L}_{m-1} \in \mathrm{Pic}^d(C)_{m-1,L}$. After we also fix a point $\mathcal{M}$ in the fiber of ρ^m_{m-1} over $\mathcal{L}_{m-1}$, we get an isomorphism

$$(\rho^m_{m-1})^{-1}(\mathcal{L}_{m-1}) \cong H^1(C, \mathcal{O}_C).$$

Since we will use later the description in terms of Čech cohomology classes, we describe this isomorphism as follows. We choose a trivialization of $\mathcal{L}_{m-1}$ with the transition functions $g^{m-1}_{\alpha\beta} := g_{\alpha\beta}(1 + t\varphi^{(1)}_{\alpha\beta} + \cdots + t^{m-1}\varphi^{(m-1)}_{\alpha\beta})$. It is easy to see that there is a trivialization for $\mathcal{M}$ with transition functions

$$g^m_{\alpha\beta} =: g_{\alpha\beta}(1 + t\varphi^{(1)}_{\alpha\beta} + \cdots + t^{m-1}\varphi^{(m-1)}_{\alpha\beta} + t^m\varphi^{(m)}_{\alpha\beta}).$$

Every point $\mathcal{L}_m \in (\rho^m_{m-1})^{-1}(\mathcal{L}_{m-1})$ has transition functions

$$g_{\alpha\beta}(1 + t\varphi^{(1)}_{\alpha\beta} + \cdots + t^{m-1}\varphi^{(m-1)}_{\alpha\beta} + t^m(\varphi^{(m)}_{\alpha\beta} + \psi_{\alpha\beta}))$$

where $[\psi_{\alpha\beta}] \in H^1(C, \mathcal{O}_C)$. We thus obtain an isomorphism

$$\xi : (\rho^m_{m-1})^{-1}(\mathcal{L}_{m-1}) \to H^1(C, \mathcal{O}_C)$$

given by $\xi(\mathcal{L}_m) = [\psi_{\alpha\beta}]$. Abusing the notation, we write $[\mathcal{L}_m]$ for the cohomology class corresponding to $\mathcal{L}_m$. Note, however, that this depends on the choice of $\mathcal{M}$.

Let $s_{m-1} \in H^0(\mathcal{L}_{m-1})$ be a nonzero section. The obstruction to extending s_{m-1} to a section of $\mathcal{L}_m$ can be described as follows. We have a short exact sequence of sheaves on $C \times T_m$,

$$0 \to L \to \mathcal{L}_m \to \mathcal{L}_{m-1} \to 0.$$

Let $\delta_{\mathcal{L}_m}$ be the connecting map $H^0(\mathcal{L}_{m-1}) \to H^1(C, L)$. The long exact sequence on cohomology implies that s_{m-1} can be extended to a section s_m of $\mathcal{L}_m \in (\rho^m_{m-1})^{-1}(\mathcal{L}_{m-1})$ if and only if $\delta_{\mathcal{L}_m}(s_{m-1}) = 0$.

With the above notation, we get the following more explicit obstruction to extending a section of $\mathcal{L}_{m-1}$ in terms of Čech cohomology.

Lemma IV.10. *Fix a line bundle $\mathcal{M}$ in the fiber of ρ^m_{m-1} over $\mathcal{L}_{m-1}$. For a fixed section $s_{m-1} = (\sum_{j=0}^{m-1} c^{(j)}_\alpha t^j) \in H^0(\mathcal{L}_{m-1})$, let s_0 be the its image under*

$$\pi^m_0 : H^0(\mathcal{L}_{m-1}) \to H^0(L).$$

The section s_{m-1} has an extension to a section of $\mathcal{L}_m$ if and only if

(†) $\nu(s_0 \otimes [\mathcal{L}_m])$ is the cohomology class corresponding to $(-\gamma_\alpha^{-1}(\sum_{j=1}^{m} \varphi^{(j)}_{\alpha\beta} c^{(m-j)}_\alpha))$

where ν is the natural pairing $H^0(C, L) \otimes H^1(C, \mathcal{O}) \to H^1(C, L)$.

Proof. Assume there is an extension of $s_{m-1} \in H^0(\mathcal{L}_{m-1})$ to a section $s_m \in H^0(\mathcal{L}_m)$. Locally s_{m-1} is given by functions $\sum_{j=0}^{m-1} c_\alpha^{(j)} t^j \in \Gamma(U_\alpha \times T_{m-1}, \mathcal{O}_{U_\alpha \times T_{m-1}})$. We write s_m as $\sum_{j=0}^{m-1} c_\alpha^{(j)} t^j + c_\alpha^{(m)} t^m$. Let $\gamma_\alpha^m : \mathcal{L}_m|_{U_\alpha \times T_m} \to \mathcal{O}_{U_\alpha \times T_m}$ be a trivialization of $\mathcal{L}_m$ on $U_\alpha \times T_m$. We thus have the following equality on $(U_\alpha \cap U_\beta) \times T_m$:

$$(\gamma_\alpha^m)^{-1}(\sum_{j=0}^{m} c_\alpha^{(j)} t^j) = (\gamma_\beta^m)^{-1}(\sum_{j=0}^{m} c_\beta^{(j)} t^j).$$

Since $(\gamma_\alpha^m)^{-1} = (\gamma_\beta^m)^{-1} \circ g_{\alpha\beta}^m$, we have $(\gamma_\beta^m)^{-1} \circ g_{\alpha\beta}^m(\sum_{j=0}^{m} c_\alpha^{(j)} t^j) = (\gamma_\beta^m)^{-1}(\sum_{j=0}^{m} c_\beta^{(j)} t^j)$. More explicitly, we obtain

$$g_{\alpha\beta}(1 + t\varphi_{\alpha\beta}^{(1)} + \cdots + t^{m-1}\varphi_{\alpha\beta}^{(m-1)} + t^m(\varphi_{\alpha\beta}^{(m)} + \psi_{\alpha\beta}^{(m)}))(\sum_{j=0}^{m} c_\alpha^{(j)} t^j) = (\sum_{j=0}^{m} c_\beta^{(j)} t^j)$$

in $\mathcal{O}((U_\alpha \cap U_\beta) \times T_m)$. We now expend this equation and take the coefficient of t^i for i with $0 \le i \le m$. If $i < m$, the equation we obtain from the coefficient of t^i always holds since s_{m-1} is a section of $\mathcal{L}_{m-1}$. For $i = m$, we obtain

$$g_{\alpha\beta}(\psi_{\alpha\beta} \cdot c_\alpha^{(0)} + \sum_{j=1}^{m} \varphi_{\alpha\beta}^{(j)} \cdot c_\alpha^{(m-j)} + c_\alpha^{(m)}) = (c_\beta^{(m)})$$

in $\mathcal{O}((U_\alpha \cap U_\beta) \times T)$. Note that the restriction to the trivialization γ_α^m to the subsheaf L of $\mathcal{L}_m$ is exactly the trivialization γ_α, we have

$$(\gamma_\beta)^{-1} \circ g_{\alpha\beta}(\psi_{\alpha\beta} \cdot c_\alpha^{(0)} + \sum_{j=1}^{m} \varphi_{\alpha\beta}^{(j)} \cdot c_\alpha^{(m-j)} + c_\alpha^{(m)}) = (\gamma_\beta)^{-1}(c_\beta^{(m)})$$

as sections of L on $(U_\alpha \cap U_\beta)$.

Clearly $(\gamma_\beta)^{-1} \circ g_{\alpha\beta}(c_\alpha^{(m)}) - (\gamma_\beta)^{-1}(c_\beta^{(m)})$ gives the zero cohomology class in $H^1(C, L)$. We obtain that $\nu(s_0 \otimes [\mathcal{L}_m])$, the cohomology class corresponding to $(\gamma_\alpha^{-1}(\psi_{\alpha\beta} \cdot c_\alpha^{(0)}))$ is equal to the cohomology class corresponding to $(-\gamma_\alpha^{-1}(\sum_{j=1}^{m} \varphi_{\alpha\beta}^{(j)} c_\alpha^{(m-j)}))$. By reversing the argument, we also obtain the converse. □

Remark IV.11. The identification between the fiber of $\mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{m-1,L}$ and $H^1(C, \mathcal{O}_C)$ is not canonical. In particular, the expression for $\gamma(s_0 \otimes [\mathcal{L}_m])$ in (†)

does depend on $\mathcal{M}$. However, for any fixed nonzero section s_{m-1}, the dimension of the subset

$$\{\mathcal{L}_m \in (\rho^m_{m-1})^{-1}(\mathcal{L}_{m-1}) \mid H^0(\mathcal{L}_m) \to H^0(\mathcal{L}_{m-1}) \text{ has nonempty fiber over } s_{m-1}\}$$

is independent of $\mathcal{M}$.

We now prove Theorem IV.5. The idea is similar to that in Kempf's proof of Riemann's multiplicity formula.

Proof of Theorem IV.5. For every effective Cartier divisor D on a smooth variety X and a point $x \in D$, the multiplicity of D at x is equal to the minimal positive integer m such that $D_{m,x}$ is a proper subset of $X_{m,x}$.

Let $L \in \Theta$ be a line bundle with $l = h^0(L)$. We first show that

$$\Theta_{m,L} = \mathrm{Pic}^{g-1}(C)_{m,L}$$

for every $m < l$. This follows from the description of Θ as a determinantal variety. Indeed, let $\mathcal{L}_m \in \mathrm{Pic}^{g-1}(C)_{m,L}$ be a line bundle of type $\lambda \in \Lambda_{l,m}$, then $\sum_{i=1}^{l} \lambda_i \geq l > m$. By Lemma IV.3, we have $\mathcal{L}_m \in \Theta_{m,L}$. Hence $\Theta_{m,L} = \mathrm{Pic}^{g-1}(C)_{m,L}$ for every $m < l$.

We now show that $\Theta_{m,L} \neq \mathrm{Pic}^{g-1}(C)_{m,L}$ for $m = l$. Let $\mathcal{Z}_1$ be the image of $\Theta_{m,L}$ under $\mathrm{Pic}^{g-1}(C)_m \to \mathrm{Pic}^{g-1}(C)_1$. It suffices to show that $\mathcal{Z}_1 \neq \mathrm{Pic}^{g-1}(C)_{1,L}$. For every $\mathcal{L}_m \in \Theta_{m,L}$ of type $\lambda = (1 \leq \lambda_1 \leq \cdots \leq \lambda_l)$, Lemma IV.3 implies that $\sum_{i=1}^{l} \lambda_i \geq m + 1$. Hence $\lambda_l \geq 2$ and $n_2(\lambda) \geq 1$. By Lemma IV.8, we have $h^0(\mathcal{L}_1) - h^0(L) = n_2(\lambda) \geq 1$. By Remark IV.9, we see that the map

$$\pi^1_0 : H^0(\mathcal{L}_1) \to H^0(L)$$

is not zero. Equivalently, there is a nonzero section $s_0 \in H^0(L)$ which can be extended to a section of $H^0(\mathcal{L}_1)$. Let $\mathcal{Z}_2$ be the subset

$$\{\mathcal{L}_1 \in \mathrm{Pic}^{g-1}(C)_1 \mid \pi^1_0 : H^0(\mathcal{L}_1) \to H^0(L) \text{ is not zero}\}.$$

We have seen that $\mathcal{Z}_1$ is a subset of $\mathcal{Z}_2$, hence it suffice to show that $\mathcal{Z}_2 \neq \mathrm{Pic}^{g-1}(C)_{1,L}$.

We now apply Lemma IV.10 with $m = 1$. Let $\mathcal{M}$ be the trivial deformation of L, i.e. $\mathcal{M}$ represents the zero tangent vector at L. To compute the dimension of $\mathcal{Z}_2$, we consider the proper subset

$$\mathcal{Z} = \{(W, \mathcal{L}_1) \mid \nu(s_0 \otimes [\mathcal{L}_1]) = 0 \text{ for every } s_0 \in W\}$$

of $\mathbf{P}(H^0(C,L)) \times H^1(C, \mathcal{O}_C)$. Here $\mathbf{P}(H^0(C,L))$ stands for the projective space of one dimensional subspaces of $H^0(C,L)$. Let W be an element in $\mathbf{P}(H^0(C,L))$ and s_0 a nonzero element of W. The induced map $H^1(C,\mathcal{O}_C) \to H^1(C,L)$ taking $[\mathcal{L}_1]$ to $\nu(s_0 \otimes [\mathcal{L}_1])$ is surjective. Hence each fiber of the first projection map

$$\mathcal{Z} \to \mathbf{P}(H^0(C,L))$$

is a codimension l vector space of $H^1(C,\mathcal{O}_C)$. We obtain $\dim \mathcal{Z} = g - 1$. Since $\mathcal{Z}_2$ is a subset of the image of the second projection map $\mathcal{Z} \to H^1(C,\mathcal{O}_C)$, we obtain $\dim \mathcal{Z}_2 \leq g - 1$. Hence $\mathcal{Z}_2 \neq \mathrm{Pic}^{g-1}(C)_{1,L}$. This completes the proof. $\square$

For smooth projective curves of genus $g \leq 2$, Riemann's Singularity Theorem implies that the theta divisor is smooth. We consider the singularities of the theta divisor for curves of genus $g \geq 3$ in the next section.

4.2 Singularities of the Theta divisor and of the W_d^r loci

Our first goal in this section is to give an upper bound for $\dim W_d^r(C)_{m,L}$ for each $L \in \mathrm{Pic}^d(C)$ and $m \geq 0$. We fix a line bundle L of degree d with $l = h^0(L)$.

For every partition $\lambda \in \Lambda_{l,m+1}$, we denote by $C_{\lambda,m}$ the subset

$$\{\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L} \mid \mathcal{L}_m \text{ is of type } \lambda\}.$$

It is easy to see that locally $C_{\lambda,m}$ is the pull back of a locally closed subset of the m-th jet scheme of the variety of $(d+e+1-g) \times e$ matrices. Therefore $C_{\lambda,m}$ is a constructible subset of $\mathrm{Pic}^d(C)_{m,L}$. By Lemma IV.4, we have $W^r_d(C)_{m,L} = \bigcup_\lambda C_{\lambda,m}$, where λ varies over the partitions in $\Lambda_{l,m+1}$ satisfying $\sum_{i=1}^{l-r} \lambda_i \geq m+1$. In particular, we have a finite union $\Theta_{m,L} = \bigcup_\lambda C_{\lambda,m}$, where λ varies over all elements in $\Lambda_{l,m+1}$ with $\sum_{i=1}^{l} \lambda_i \geq m+1$. In order to estimate the dimension of $\Theta_{m,L}$, it is enough to bound the dimension of $C_{\lambda,m}$ for every $\lambda \in \Lambda_{l,m+1}$. The idea is to describe the image of $C_{\lambda,m}$ under the truncation map $\rho^m_i : \mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{i,L}$ for every $i \leq m$.

Definition IV.12. *A weak flag of $H^0(C,L)$ of signature $\kappa = (\kappa_i)$* with $\kappa_1 \geq \cdots \geq \kappa_n$ is a sequence of subspaces of $H^0(C,L)$,

$$\mathbf{V} : \; H^0(C,L) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{n-1} \supseteq V_n$$

such that $\dim V_i = \kappa_i$ for every $1 \leq i \leq n$. Here n is called the *length* of the weak flag $\mathbf{V}$. Given a weak flag $\mathbf{V}$ of $H^0(C,L)$ of length n, for every $i \leq n$ we denote by $\mathbf{V}_{(i)}$ the truncated weak flag of length i:

$$\mathbf{V}_{(i)} : H^0(C,L) = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_{i-1} \supseteq V_i.$$

For every $\mathcal{L}_m \in C_{\lambda,m}$ and every j with $0 \leq j \leq m$, we denote by $\mathcal{L}_j$ the image of $\mathcal{L}_m$ under $\rho^m_j : \mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C)_j$. Lemma IV.8 implies that the function $C_{\lambda,m} \to \mathbf{Z}$ which takes $\mathcal{L}_m$ to $h^0(\mathcal{L}_j) = \sum_{k=1}^{j+1} n_k(\lambda)$ is constant. For a fixed $\mathcal{L}_m \in C_{\lambda,m}$, the images V_j of the morphisms $\pi^j_0 : H^0(\mathcal{L}_j) \to H^0(L)$ give a weak flag $\mathbf{V}_{\mathcal{L}_m}$ of $H^0(L)$ of length m. Remark IV.9 implies that $\dim V_j = \dim H^0(\mathcal{L}_j) - \dim H^0(\mathcal{L}_{j-1}) = n_{j+1}(\lambda)$. Hence the signature κ of the weak flag $\mathbf{V}_{\mathcal{L}_m}$, with $\kappa_j = n_{j+1}(\lambda)$, only depends on the partition λ.

Lemma IV.7 shows that there is a short exact sequence

$$0 \to H^0(\mathcal{L}_{m-1}) \xrightarrow{v^m_{m-1}} H^0(\mathcal{L}_m) \twoheadrightarrow V_m \to 0.$$

We now choose a splitting of this short exact sequence, which gives a decomposition $H^0(\mathcal{L}_m) = H^0(\mathcal{L}_{m-1}) \oplus \widetilde{V}_m$, with $\widetilde{V}_m$ mapping isomorphically onto V_m. The restriction map $\pi^m_{m-1} : H^0(\mathcal{L}_m) \to H^0(\mathcal{L}_{m-1})$ maps $\widetilde{V}_m$ isomorphic to its image. For the short exact sequence

$$0 \to H^0(\mathcal{L}_{m-2}) \xrightarrow{v^{m-1}_{m-2}} H^0(\mathcal{L}_{m-1}) \twoheadrightarrow V_{m-1} \to 0,$$

we can choose a splitting $H^0(\mathcal{L}_{m-1}) = H^0(\mathcal{L}_{m-2}) \oplus \widetilde{V}_{m-1}$ such that the restriction map π^m_{m-1} maps $\widetilde{V}_m$ into $\widetilde{V}_{m-1}$. By descending induction on i with $0 \leq i \leq m$, we can find a subspace $\widetilde{V}_i \subset H^0(\mathcal{L}_i)$ for each i such that

1. The restriction of the truncation map $\pi^i_0 : H^0(\mathcal{L}_i) \to H^0(L)$ to $\widetilde{V}_i$ induces an isomorphism onto V_i.

2. The truncation map $\pi^i_{i-1} : H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_{i-1})$ takes $\widetilde{V}_i$ into $\widetilde{V}_{i-1}$

Definition IV.13. A weak flag $\mathbf{V}$ of $H^0(C, L)$ of length m is *extended compatibly* to the line bundle $\mathcal{L}_m$ if there are linear subspaces $\widetilde{V}_i \subset H^0(C, \mathcal{L}_i)$ for each $i \leq m$ such that (1) and (2) above hold.

In this case, the set of linear subspaces $\{\widetilde{V}_i\}_i$ as above is called a *compatible extension* of $\mathbf{V}$ to line bundle $\mathcal{L}_m$. The above argument shows that every weak flag $\mathbf{V}_{\mathcal{L}_m}$ associated to a line bundle $\mathcal{L}_m$ can be extended compatibly to the line bundle $\mathcal{L}_m$.

For every i with $1 \leq i \leq m$, recall that $\overline{\lambda}$ is the image of $\lambda \in \Lambda_{l,m+1}$ under the map $\Lambda_{l,m+1} \to \Lambda_{l,i+1}$. Given a weak flag $\mathbf{V}$ of $H^0(L)$ of length m, we denote by $S^\lambda_{i,\mathbf{V}}$ the set of line bundles $\mathcal{L}_i \in \mathrm{Pic}^d(C)_{i,L}$ such that $\mathcal{L}_i \in C_{i,\overline{\lambda}}$ and $\mathbf{V}_{(i)}$ can be extended compatibly to $\mathcal{L}_i$. For a fixed non-increasing sequence κ, we define $S^\lambda_{i,\kappa} = \bigcup_{\mathbf{V}'} S^\lambda_{i,\mathbf{V}'}$, where $\mathbf{V}'$ varies over all weak flags of $H^0(L)$ of signature κ. For convenience, we set $S^\lambda_{0,\mathbf{V}} = S^\lambda_{0,\kappa} = \{L\}$.

Standard arguments show that $S^{\lambda}_{i,\mathbf{V}}$ and $S^{\lambda}_{i,\kappa}$ are constructible subsets of $\mathrm{Pic}^d(C)_{i,L}$. For the benefit of the reader, we give the details in the appendix. The truncation map $\rho^m_i : \mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{i,L}$ maps $C_{\lambda,m}$ to the set $S^{\lambda}_{i,\kappa}$ with $\kappa_j = n_{j+1}(\lambda)$. In order to estimate the dimension of $C_{\lambda,m}$, we only need to estimate $\dim S^{\lambda}_{i,\kappa}$ for a suitable $i \leq m$.

Definition IV.14. For a fixed weak flag $\mathbf{V}$ of $H^0(L)$ of length m, for every i and j with $1 \leq i \leq j \leq m$, we define $\widetilde{S}^{\lambda}_{i,j,\mathbf{V}}$ to be the set of pairs $(\mathcal{L}_i, W)$ such that

1. $\mathcal{L}_i \in S^{\lambda}_{i,\mathbf{V}}$ and W is a subspace of $H^0(\mathcal{L}_i)$ of dimension κ_j.
2. There is a compatible extension $\{\widetilde{V}_l\}_{l\leq i}$ of $\mathbf{V}_{(i)}$ to $\mathcal{L}_i$ such that W is the inverse image of V_j in $H^0(\mathcal{L}_i)$ under the isomorphism $\widetilde{V}_i \to V_i$.

We call the W in a pair $(\mathcal{L}_i, W)$ as above a lifting of V_j to $\mathcal{L}_i$. For any element $s \in V_j$, the preimage of s via the isomorphism $W \to V_j$ is called a lifting of s to the level i.

In the appendix we also show that $\widetilde{S}^{\lambda}_{i,j,\mathbf{V}}$ is a constructible subset of a suitable Grassmann bundle. For the convenience, we set $\widetilde{S}^{\lambda}_{0,j,\mathbf{V}} = \{(L, V_j)\}$ for every λ.

Lemma IV.15. *Let X_1 and Y_1 be constructible subsets of algebraic varieties X and Y respectively. Let $f : X_1 \to Y_1$ be the restriction of a morphism $g : X \to Y$. If all the fibers of f are of dimension $d \geq 0$, then* $\dim X_1 = \dim Y_1 + d$.

Proof. Since Y_1 is a constructible subset of Y, we write Y_1 as a finite disjoint union of locally closed subset V_k of Y. We may assume that all subsets V_k are irreducible. For every k, the inverse image $f^{-1}(V_k)$, as the intersection of $g^{-1}(V_k)$ with X_1, is a constructible subset of X. We thus have

$$\dim Y_1 = \max_k\{\dim V_k\} \text{ and } \dim X_1 = \max_k\{\dim f^{-1}(V_k)\}.$$

Hence it is enough to show the statement for the map $f^{-1}(V_k) \to V_k$ for every k. We may thus assume that Y_1 is an irreducible algebraic variety.

Consider the stratification $X_1 = \coprod_{l=1}^{m}(W_l)$, where each W_l is a locally closed subset of X. For every l, the morphism $W_l \to Y_1$ has fibers of dimension $\leq d$. We get

$$\dim W_l \leq \dim f(W_l) + d \leq \dim Y_1 + d.$$

This implies that $\dim X_1 = \max_l \{\dim W_l\} \leq \dim Y_1 + d$.

We now prove the other direction of the inequality. Let $\{W_l\}_{l=1,\dots,m_0}$ be the collection of those W_l that dominates Y_1. (This collection is nonempty since otherwise $f^{-1}(y)$ would be empty for a general point $y \in Y_1$.) We choose an open subset $V \subset Y_1$ such that $\dim(W_l \cap f^{-1}(y))$ is constant for $y \in V$ and $l \leq m_0$. There is a subset W_l with $l \leq m_0$ such that $\dim(f^{-1}(y) \cap W_l) = \dim f^{-1}(y) = d$. We obtain that $\dim W_l = d + \dim Y_1$. We thus have

$$\dim X \geq \dim W_1 = \dim Y_1 + d.$$

This completes the proof. $\square$

Lemma IV.16. *For a fixed point $L \in \mathrm{Pic}^d(C)$ with $l = h^0(C, L)$ and a partition $\lambda \in \Lambda_{l,m+1}$, let $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_m)$ be a signature of length m, with $\kappa_j \leq n_{j+1}(\lambda)$ for every $j \leq m$, and $\boldsymbol{V}$ a weak flag of $H^0(L)$ of signature κ. For every i with $1 \leq i \leq m$, we write d_i for the dimension of the kernel of*

$$\mu_{V_i} : V_i \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K).$$

Then the the following holds:

(1) $\dim S^{\lambda}_{i,\boldsymbol{V}} - \dim S^{\lambda}_{i-1,\boldsymbol{V}} \leq g + d_i - \kappa_i \cdot (g - d - 1 + n_i(\lambda))$,

(2) $\dim S^{\lambda}_{i,\boldsymbol{V}} \leq gi - \sum_{j=1}^{i} \{(\kappa_j \cdot (g - d - 1 + n_j(\lambda)) - d_j\}$.

Proof. Since we fix the partition λ, we may and will omit the superscript λ in the proof. We apply Lemma IV.10 to compute the dimension of $S_{i,\mathbf{V}}$ inductively on i. $S_{0,\mathbf{V}} = \{L\}$ implies that $\dim S_{0,\mathbf{V}} = 0$. Consider the following commutative diagram:

$$\begin{array}{ccc} \widetilde{S}_{i,i,\mathbf{V}} & \xrightarrow{h} & \widetilde{S}_{i-1,i,\mathbf{V}} \\ \downarrow_{\rho_1} & & \downarrow_{\rho_2} \\ S_{i,\mathbf{V}} & \rightarrow & S_{i-1,\mathbf{V}} \end{array}$$

The horizontal map h maps $(\mathcal{L}_i, W')$ to $(\mathcal{L}_{i-1}, W)$, where W is the image of W' under the truncation map $\pi^i_{i-1} : H^0(C, \mathcal{L}_i) \to H^0(C, \mathcal{L}_{i-1})$. The vertical map ρ_1 is given by mapping $(\mathcal{L}_i, W')$ to $\mathcal{L}_i$ and ρ_2 is defined similarly.

Let $\mathcal{L}_i$ be a fixed point in $S_{i,\mathbf{V}}$. The fiber of ρ_1 over the point $\mathcal{L}_i$ is the set of linear subspaces $W' \subset H^0(\mathcal{L}_i)$ that map isomorphically onto V_i via $\pi^i_0 : H^0(\mathcal{L}_i) \to H^0(L)$. Let $\{s_{0,k}\}_k$ be a basis of V_i. A lifting W' of V_i is determined by the preimage of $s_{0,k}$ in W' for each k. By Lemma IV.7, we see that for every $s \in V_i$, any two liftings of s to the level i differ up to an element of $H^0(\mathcal{L}_{i-1})$. Therefore, the relative dimension of the map ρ_1 is $h^0(\mathcal{L}_{i-1}) \cdot \kappa_i = (\sum_{j=1}^{i} n_j(\lambda)) \cdot \kappa_i$. Similarly the relative dimension of the second vertical map ρ_2 is $(\sum_{j=1}^{i-1} n_j(\lambda)) \cdot \kappa_i$.

Consider the horizontal map h. For every element $(\mathcal{L}_{i-1}, W) \in \widetilde{S}_{i-1,i,\mathbf{V}}$, we now give a criterion to decide whether it is in the image of h or not. Fix an element $\mathcal{M}$ in the fiber of $\rho^i_{i-1} : \mathrm{Pic}^d(C)_i \to \mathrm{Pic}^d(C)_{i-1}$ over $\mathcal{L}_{i-1}$. We identify the fiber $(\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1})$ with $H^1(C, \mathcal{O}_C)$. Let $\{s_{0,k}\}_k$ be a basis of W. We donote by $s_{i-1,k}$ the lifting of $s_{0,k}$ to the level $i-1$ in W. With the notation in Lemma IV.10, every element $s_{i-1,k} = (\sum_{j=0}^{i-1} c^{(j)}_{k,\alpha} t^j) \in H^0(\mathcal{L}_{i-1})$ has an extension to a section of $\mathcal{M}' \in (\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1})$ if and only if the following equation holds

$$(\dagger_k)\ \ \nu(s_{0,k} \otimes [\mathcal{M}']) = \text{the cohomology class corresponding to } (-\gamma_\alpha^{-1}(\sum_{j=1}^{i} \varphi^{(j)}_{\alpha\beta} c^{(i-j)}_{k,\alpha})).$$

Hence $(\mathcal{L}_{i-1}, W)$ is in the image of h if and only if there is a point $\mathcal{M}' \in (\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1})$ such that the above identity $(\dagger_k)$ holds for every k. We now assume that $(\mathcal{L}_{i-1}, W)$ is in the image of h and fix an element $(\mathcal{M}', W')$ in the fiber of h over $(\mathcal{L}_{i-1}, W)$. The above argument implies that

$$\rho_1(h^{-1}(\mathcal{L}_{i-1}, W)) = \{[\mathcal{L}_i] \in (\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1}) \mid \nu(s_{0,k} \otimes ([\mathcal{L}_i] - [\mathcal{M}'])) = 0 \text{ for every } k\}.$$

By taking the dual linear spaces, we now deduce that $\rho_1(h^{-1}(\mathcal{L}_{i-1}, W))$ is an affine space consisting of the elements in $H^1(C, \mathcal{O}_C)$ that annihilate the image of the pairing

$$\mu_{V_i} : V_i \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K).$$

It follows that $\dim \rho_1(h^{-1}(\mathcal{L}_{i-1}, W)) = g - (\kappa_i \cdot (l - d - 1 + g) - d_i)$.

If $\mathcal{L}_i$ is an element in $\rho_1(h^{-1}(\mathcal{L}_{i-1}, W))$, then a pair $(\mathcal{L}_i, W')$ is in the fiber of h over $(\mathcal{L}_{i-1}, W)$ if and only if the truncation map $H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_{i-1})$ takes W' into W. A lifting W' of W is determined by the preimage of $\{s_{i-1,k}\}_k$ in W'. By Lemma IV.7, we see that any two liftings only differ by an element of $H^0(L)$. Hence we deduce that $h^{-1}(\mathcal{L}_{i-1}, W) \cap \rho_1^{-1}(\mathcal{L}_i)$ is an affine space of dimension $\kappa_i \cdot l$. Thus the dimension of every nonempty fiber of the horizonal map h is $g + d_i - \kappa_i \cdot (l - d - 1 + g) + \kappa_i \cdot l$.

By Lemma IV.15, we have:

$$\dim \widetilde{S}_{i,i,\mathbf{V}} = \dim S_{i,\mathbf{V}} + \kappa_i \cdot (\sum_{j=1}^{i} n_j(\lambda))$$
$$\dim \widetilde{S}_{i,i,\mathbf{V}} \leq \widetilde{S}_{i-1,i,\mathbf{V}} + g + d_i - \kappa_i \cdot (l - d - 1 + g) + \kappa_i \cdot l$$
$$\dim \widetilde{S}_{i-1,i,\mathbf{V}} = \dim S_{i-1,\mathbf{V}} + \kappa_i \cdot (\sum_{j=1}^{i-1} n_j(\lambda)))$$

It follows that

$$\dim S_{i,\mathbf{V}} - \dim S_{i-1,\mathbf{V}} \leq g - \kappa_i \cdot (g - d - 1) + d_i - \kappa_i \cdot n_i(\lambda).$$

This proves (1).

Part (2) follows from (1) by induction on i, using $\dim S_{0,\mathbf{V}} = 0$. $\square$

Remark IV.17. From the proof, we know that the equality in (1) can be achieved if the map $h : \widetilde{S}^{\lambda}_{i,i,\mathbf{V}} \to \widetilde{S}^{\lambda}_{i-1,i,\mathbf{V}}$ is a surjection. In fact, we will see that equality can be achieved when we apply the above lemma in the proofs of the main theorems.

We now prove our first main result.

Proof of Theorem **A**. Let C be a curve of genus $g \geq 3$. Since we fix the curve C, we may and will write W^r_d for $W^r_d(C)$ for every r and d. By Remark IV.6, we know that $\Theta_{\text{sing}} = W^1_{g-1} = \bigcup_{l\geq 2}(W^{l-1}_{g-1} \smallsetminus W^l_{g-1})$. To bound the dimension of $(\pi^{\Theta}_m)^{-1}(\Theta_{\text{sing}})$, it is enough to bound the dimension of $(\pi^{\Theta}_m)^{-1}(W^{l-1}_{g-1} \smallsetminus W^l_{g-1})$ for each $l \geq 2$.

Let L be a point in $W^{l-1}_{g-1} \smallsetminus W^l_{g-1}$. We have seen in the proof of Theorem IV.5 that $\Theta_{m,L} = \mathrm{Pic}^{g-1}(C)_{m,L}$ for $m < l$. Hence $\dim \Theta_{m,L} = mg$ for $m < l$. We now assume that $m \geq l$. Recall that we put $C_{\lambda,m} = \{\mathcal{L}_m \in \Theta_{m,L} \mid \mathcal{L}_m \text{ is of type } \lambda\}$, where λ is a partition in $\Lambda_{l,m+1}$. By Lemma IV.3, $\Theta_{m,L}$ is a finite union of $C_{\lambda,m}$, with λ satisfying $\sum_{i=1}^{l} \lambda_i \geq m+1$. In order to prove the theorem, we first bound the dimension of each $C_{\lambda,m}$.

We now fix a partition $\lambda \in \Lambda_{l,m+1}$ with $\sum_{i=1}^{l} \lambda_i \geq m+1$. Let κ be the signature with $\kappa_i = 1$ for every $i \leq \lambda_l - 1$ and $\kappa_i = 0$ for $i \geq \lambda_l$. If $\mathcal{L}_m \in C_\lambda$, we denote by $\mathcal{L}_i$ the image of $\mathcal{L}_m$ under $\rho^m_i : \mathrm{Pic}^m(C)_m \to \mathrm{Pic}^i(C)_i$ for every $i \leq m$. The definition of $n_k(\lambda)$ implies that λ_l is the largest index k such that $n_k(\lambda) \neq 0$. Remark IV.9 implies that the map $\pi^{\lambda_l - 1}_0 : H^0(\mathcal{L}_{\lambda_l - 1}) \to H^0(L)$ is nonzero while the map $\pi^{\lambda_l}_0 : H^0(\mathcal{L}_{\lambda_l}) \to H^0(L)$ is zero. Let $W \subset H^0(C, L)$ be the 1–dimensional subspace in the image of $\pi^{\lambda_l-1}_0$. Consider a weak flag of $H^0(L)$ of signature κ,

$$\mathbf{V}_W : H^0(C, L) = V_0 \supset V = V_1 = \cdots V_{\lambda_l - 1} = W \supset V_{\lambda_l} = \cdots = V_m = 0.$$

Hence $\mathcal{L}_i$ is in $S^{\lambda}_{i,\mathbf{V}}$ for each $i \leq \lambda_l - 1$. We thus conclude that the truncation map

$\rho^m_{\lambda_l-1} : \mathrm{Pic}^{g-1}(C)_{m,L} \to \mathrm{Pic}^{g-1}(C)_{\lambda_l-1,L}$ maps $C_{\lambda,m}$ into $S^\lambda_{\lambda_l-1,\kappa}$.

Let Flag_κ be the variety parameterizing all weak flags of signature κ. Let W be a 1-dimensional subspace of $H^0(C, L)$. It defines a weak flag $\mathbf{V}_W = \{V_i\}$ of signature κ. We thus have a bijection between Flag_κ and $\mathbf{P}(H^0(C, L))$. We now compute the dimension of $S^\lambda_{\lambda_l-1,\mathbf{V}_W}$. Let s_0 be a nonzero element in W. The multiplication map

$$m_{s_0} : H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$$

is always injective. We thus conclude that $W \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$ is injective. Recall that d_i is the dimension of the kernel of map

$$\mu_{V_i} : V_i \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C).$$

We conclude that $d_i = 0$ for every i with $0 \leq i \leq m$. Moreover, the dual map of m_{s_0}, denoted by $m^*_{s_0} : H^1(C, \mathcal{O}_C) \to H^1(C, L)$, is a surjection. Lemma IV.10 implies that for every $i \leq \lambda_l - 1$ and every $s_{i-1} \in H^0(\mathcal{L}_{i-1})$ which is a lifting of s_0, there are line bundles $\mathcal{L}_i$ over $\mathcal{L}_{i-1}$ such that s_{i-1} can be extended as a section of $\mathcal{L}_i$. Therefore, the horizontal map $h : \widetilde{S}^\lambda_{i,i,\mathbf{V}_W} \to \widetilde{S}^\lambda_{i-1,i,\mathbf{V}_W}$ is a surjection. By Lemma IV.16 and Remark IV.17, we obtain that for every weak flag $\mathbf{V}_W$

$$\dim S^\lambda_{\lambda_l-1,\mathbf{V}_W} = (\lambda_l - 1)g - \sum_{k=1}^{\lambda_l-1} n_k(\lambda).$$

By Lemma IV.15, we obtain that $\dim S^\lambda_{\lambda_l-1,\kappa} \leq \max_W\{\dim S^\lambda_{\lambda_l-1,\mathbf{V}_W}\} + \dim \mathbf{P}H^0(L)$, where $W \in H^0(C, L)$. We consider $C_{\lambda,m}$ as a subset of the preimage of $S^\lambda_{\lambda_l-1,\kappa}$ under the map $\rho^m_{\lambda_l-1} : \mathrm{Pic}^{g-1}(C)_m \to \mathrm{Pic}^{g-1}(C)_{\lambda_l-1}$.

Hence

$$\begin{aligned}
\dim C_{\lambda,m} &\leq g \cdot (m - \lambda_l + 1) + \max_W\{\dim S_{\lambda_l-1,\mathbf{V}_W}\} + \dim \mathbf{P}(H^0(L)) \\
&= mg - \sum_{j=1}^{\lambda_l-1} n_j + l - 1 \\
&= mg - (\sum_{i=1}^{l} \lambda_i - r_{\lambda_l}) + l - 1
\end{aligned}$$

Martens' theorem says that for every smooth curve of genus $g \geq 3$, and every d and r with $2 \leq d \leq g-1$ and $0 < 2r \leq d$, we have $\dim W^r_d(C) \leq d-2r$. (See [Kem]). For $1 \leq m < l$, we have

$$\begin{aligned} \dim(\pi^{\Theta}_m)^{-1}(W^{l-1}_{g-1} \smallsetminus W^{l}_{g-1}) &= \dim(W^{l-1}_{g-1} \smallsetminus W^{l}_{g-1}) + mg \\ &\leq g-1-2(l-1)+mg \\ &= (m+1)(g-1)+(m-2(l-1)) \\ &\leq (m+1)(g-1)-m \end{aligned}$$

For $m \geq l$, we have

(4.2)

$$\begin{aligned} \dim(\pi^{\Theta}_m)^{-1}(W^{l-1}_{g-1} \smallsetminus W^{l}_{g-1}) &\leq \max_{\lambda} \{\dim(C_\lambda)+g-2l+1\} \\ &\leq \max_{\lambda} \left\{(m+1)(g-1)-(\sum_{i=1}^{l}\lambda_i - m-1)-(l-r_{\lambda_l}(\lambda))\right\} \end{aligned}$$

where λ varies over partitions in $\Lambda_{l,m+1}$ with $\sum_{i=1}^{l}\lambda_i \geq m+1$. We conclude that for every m,we have $\dim(\pi^{\Theta}_m)^{-1}(W^{l-1}_{g-1} \smallsetminus W^{l}_{g-1}) \leq (m+1)(g-1)$. Furthermore, if the equality is achieved for some m, then there is $\lambda \in \Lambda_{l,m+1}$ such that $\sum_{i=1}^{l}\lambda_i = m+1$ and $l = r_{\lambda_l}(\lambda)$, i.e. $\lambda_1 = \cdots = \lambda_l$. It follows that for m such that $m+1$ is not divisible by any integer $l \in [2, g-1]$, the set $(\pi^{\Theta}_m)^{-1}(\Theta_{\text{sing}}) = \bigcup_{l\geq 2}(\pi^{\Theta}_m)^{-1}(W^{l-1}_{g-1} \smallsetminus W^{l}_{g-1})$ has dimension smaller than $(m+1)(g-1)$. Hence Θ_m is irreducible for arbitrarily large m, which implies that Θ_m is irreducible for all m. (See [Mus1, Proposition 1.6].) This implies that

$$\dim(\pi^{\Theta}_m)^{-1}(\Theta_{\text{sing}}) \leq (m+1)(g-1)-1$$

for every m.

In order to get the lower bound for $\dim(\pi^{\Theta}_m)^{-1}(\Theta_{\text{sing}})$, we need the following lemma, see [Mus1, proposition 1.6].

Lemma IV.18. *If X is a locally complete intersection variety of dimension n and*

$Z \subset X$ is a closed subscheme, then $\dim(\pi_{m+1}^X)^{-1}(Z) \geq \dim(\pi_m^X)^{-1}(Z) + n$ *for every* $m \geq 1$.

If C is a hyperelliptic curve, we show that $\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) = (m+1)(g-1)-1$ by induction on $m \geq 1$. By [ACGH, §VI.4], we know that Θ_{sing} has dimension equal to $g-3$, which implies that $\dim(\pi_1^\Theta)^{-1}(\Theta_{\mathrm{sing}}) = g-3+g = 2(g-1)-1$. We thus have the assertion for $m = 1$. Assume now that the assertion holds for $m-1$. A repeated application of Lemma IV.18 implies that for every $m \geq 1$,

$$\dim(\pi_m^\Theta)^{-1}(L) \geq (m-1)(g-1) + \dim(\pi_1^{-1})(L).$$

Hence for every $L \in \Theta_{\mathrm{sing}}$, $\dim(\pi_m^\Theta)^{-1}(L) \geq (m-1)(g-1)+g$. Therefore

$$\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) \geq \dim \Theta_{\mathrm{sing}} + (m-1)(g-1) + g = (m+1)(g-1) - 1.$$

This completes the proof of the theorem for hyperelliptic curves.

We now assume that C is a nonhyperelliptic curve of genus g, and show that $\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) = (m+1)(g-1)-2$ by induction on m. By [ACGH, §VI.4], we have $\dim \Theta_{\mathrm{sing}} = g-4$, hence $\dim(\pi_1^\Theta)^{-1}(\Theta_{\mathrm{sing}}) = g-4+g = 2(g-1)-2$. This proves the assertion for $m = 1$. A repeated application of Lemma IV.18 implies that for every $L \in \Theta_{\mathrm{sing}}$ and every $m \geq 1$, $\dim(\pi_m^\Theta)^{-1}(L) \geq (m-1)(g-1)+\dim(\pi_1^\Theta)^{-1}(L)$. We thus have $\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) \geq g-4+(m-1)(g-1)+g = (m+1)(g-1)-2$.

In order to finish the proof, it is enough to show that

$$\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) < (m+1)(g-1)-1$$

for every m. Assume there is some m_0 such that

$$\dim(\pi_{m_0}^\Theta)^{-1}(\Theta_{\mathrm{sing}}) \geq (m_0+1)(g-1)-1.$$

A repeated application of Lemma IV.18 implies that for every $m > m_0$,

$$\dim(\pi_m^\Theta)^{-1}(\Theta_{\mathrm{sing}}) \geq (m-m_0)(g-1) + \dim(\pi_{m_0}^\Theta)^{-1}(\Theta_{\mathrm{sing}}) \geq (m+1)(g-1)-1.$$

On the other hand, for nonhyperelliptic curves, Martens' theorem has a better bound, namely $\dim W_d^r \leq d - 2r - 1$. By applying it for the theta divisor, we have

$$\dim(W_{g-1}^{l-1} \smallsetminus W_{g-1}^{l}) \leq g - 2l.$$

Arguing as in (†), we obtain

$$\dim(\pi_m^{\Theta})^{-1}(W_{g-1}^{l-1} \smallsetminus W_{g-1}^{l}) \leq \max_{\lambda} \left\{ (m+1)(g-1) - (\sum_{i=1}^{l} \lambda_i - m - 1) - (l - r_{\lambda_l}(\lambda)) - 1 \right\}$$

where λ varies over partitions in $\Lambda_{l,m=1}$ with $\sum_{i=1}^{l} \lambda_i \geq m+1$. It follows that unless there is a $\lambda \in \Lambda_{l,m+1}$ with $\sum\limits_{i=1}^{l} \lambda_i = m+1$ and $r_{\lambda_l}(\lambda) = l$, we have

$$\dim(\pi_m^{\Theta})^{-1}(\Theta_{\text{sing}}) < (m+1)(g-1) - 1.$$

Therefore this holds for every m such that $m+1$ is not divisible by any integer $2 \leq l \leq g-1$. Since there are arbitrarily large such m, we obtain a contradiction. □

In [Mus1], Mustaţă describes complete intersection rational singularities in terms of jet schemes as follows. If X is a local complete intersection variety of dimension n over k, then the following are equivalent:

(i) X has rational singularities.

(ii) X has canonical singularities.

(iii) X_m is irreducible for each m.

(iv) $\dim \pi_m^{-1}(X_{\text{sing}}) < n(m+1)$ for every m.

The equivalence of the first two parts is due to Elkik, see [Elk]. Note also that by Theorem 3.3 in [EMY], for a reduced irreducible divisor D on a smooth variety X of dimension n, the following are equivalent,

(i) The jet scheme D_m is a normal variety for every m.

(ii) D has terminal singularities.

(iii) For every m, $\dim(\pi^D_m)^{-1}(D_{\mathrm{sing}}) \leq (m+1)(n-1)-2$.

Applying these two results to the theta divisor, we obtain the following result concerning the singularities of this variety.

Corollary IV.19. *Let C be a smooth projective curve of genus $g \geq 3$ over k. The theta divisor has terminal singularities if C is a nonhyperelliptic curve. If C is hyperelliptic, then the theta divisor has canonical non-terminal singularities. In particular, the theta divisor has rational singularities for every smooth curve.*

We now apply the above ideas to compute the log canonical threshold of the pair $(\mathrm{Pic}^d(C), W^r_d(C))$ at a point $L \in W^r_d(C)$, where C is general in the moduli space of curves.

In [Mus2, Corollary 3.6], one gives the following formula for the log canonical threshold of a pair in terms of the dimensions of the jet schemes. If $Y \subset X$ is a closed subscheme and $Z \subset X$ is a nonempty closed subset, then the log canonical threshold of the pair (X, Y) at Z is given by

$$\mathrm{lct}_Z(X,Y) = \dim X - \sup_{m \geq 0} \frac{\dim(\pi^Y_m)^{-1}(Y \cap Z)}{m+1}.$$

For every $L \in W^r_d(C)$, the above formula implies that

$$\mathrm{lct}_L(\mathrm{Pic}^d(C), W^r_d(C)) = g - \sup_{m \geq 0} \frac{\dim W^r_d(C)_{m,L}}{m+1}.$$

Our main goal is to estimate the dimension of $W^r_d(C)_{m,L}$ for each m.

We now turn to the proof of Theorem **B**. Let C be a general smooth projective curve of genus g and let L be a line bundle on C. The generality assumption on C implies that the natural pairing

$$\mu_0 : H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$$

is injective for every L. This was stated by Petri and first proved by Gieseker [Gie].

Before proving the theorem, we need to prove an identity for every partition as preparation.

Lemma IV.20. *Let $\lambda \in \Lambda_{l,m+1}$ and $\lambda_0 = 0$. We now prove that*

$$\sum_{i=1}^{\lambda_l} n_i^2(\lambda) = \sum_{i=1}^{l} (l-i+1)^2(\lambda_i - \lambda_{i-1}).$$

Proof. Given a partition $\lambda \in \Lambda_{l,m+1}$, we may write it as:

$$1 \leq \lambda_1 = \cdots = \lambda_{m_1} < \lambda_{m_1+1} = \cdots = \lambda_{m_2} < \lambda_{m_2+1} \cdots \lambda_{m_k} < \lambda_{m_k+1} = \cdots = \lambda_l.$$

For simplicity, we write n_i for $n_i(\lambda)$. It is easy to see that

$$\begin{aligned}
n_1 &= \cdots = n_{\lambda_{m_1}} = l \\
n_{\lambda_{m_1}+1} &= \cdots = n_{\lambda_{m_2}} = l - m_1 \\
\cdots \\
n_{\lambda_{m_k}+1} &= \cdots = n_{\lambda_l} = l - m_{k-1}
\end{aligned}$$

This implies that

$$\begin{aligned}
\sum_{i=1}^{l} (l-i+1)^2(\lambda_i - \lambda_{i-1}) &= l^2(\lambda_1) + (l-m_1)^2(\lambda_{m_1+1} - \lambda_{m_1}) + \cdots + (l-m_k)^2(\lambda_{m_k+1} - \lambda_{m_k}) \\
&= l^2(\lambda_{m_1}) + (l-m_1)^2(\lambda_{m_2} - \lambda_{m_1}) + \cdots + (l-m_k)^2(\lambda_l - \lambda_{m_k}) \\
&= \sum_{i=1}^{\lambda_{m_1}} l^2 + \sum_{i=\lambda_{m_1+1}}^{\lambda_{m_2}} n_i^2 + \cdots + \sum_{i=\lambda_{m_k+1}}^{\lambda_l} n_i^2 \\
&= \sum_{i=1}^{\lambda_l} n_i^2 = \sum_{i=1}^{\lambda_l} n_i^2(\lambda)
\end{aligned}$$

□

Proof of Theorem B. Let C be a general smooth projective curve in the sense of Petri and Gieseker. Let L be a line bundle in $W_d^r(C)$ with $l = h^0(C, L) \geq r+1$.

Since we are only interested in the asymptotic behavior of $W^r_d(C)_{m,L}$, we can assume that $m \geq l$. By Lemma IV.4 we have a stratification $W^r_d(C)_{m,L} = \bigcup C_{\lambda,m}$, where $\lambda \in \Lambda_{l,m+1}$ are taken over all length l partitions satisfying $\sum_{i=1}^{l-r} \lambda_i \geq m+1$.

We now fix such a partition λ. Let κ be a signature with $\kappa_i = n_{i+1}(\lambda)$ for every i with $1 \leq i \leq m$. For every $\mathcal{L}_m \in C_{\lambda,m}$, we denote by $\mathcal{L}_i$ the image of $\mathcal{L}_m$ under the truncation $\rho^m_i : \mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C)_i$. The images V_i of the maps $\pi^i_0 : H^0(\mathcal{L}_i) \to H^0(C, L)$ give a weak flag $\mathbf{V}_{\mathcal{L}_m}$. By Remark IV.9, we obtain

$$\dim V_i = h^0(\mathcal{L}_i) - h^0(\mathcal{L}_{i-1}) = n_{i+1}(\lambda).$$

Therefore $\mathbf{V}_{\mathcal{L}_m}$ is a weak flag of signature κ. The image of $\mathcal{L}_m$ in $\mathrm{Pic}^d(C)_{\lambda_l-1}$ is in $S^\lambda_{\lambda_l-1,\mathbf{V}_{\mathcal{L}_m}}$. Hence the truncation map

$$\rho^m_{\lambda_l-1} : \mathrm{Pic}^d(C)_m \to \mathrm{Pic}^d(C)_{\lambda_l-1}$$

maps $C_{\lambda,m}$ to $S^\lambda_{\lambda_l-1,\kappa} = \bigcup_{\mathbf{V}} S^\lambda_{\lambda_l-1,\mathbf{V}}$, where $\mathbf{V}$ varies over all weak flags of signature κ. The key step is to compute the dimension of $S^\lambda_{\lambda_l-1,\mathbf{V}}$ for each $\mathbf{V}$. We keep the notation in the proof of Lemma IV.16.

The fact that the canonical pairing

$$\mu_0 : H^0(C, L) \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$$

is injective implies that all restrictions $\mu_{V_i} : V_i \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$ are injective. Hence $d_i = \dim \ker \mu_{V_i}$ is zero for every weak flag $\mathbf{V}$ of $H^0(L)$ of signature κ.

We now show that if the canonical pairing μ_0 is injective, then all horizontal maps $h : \widetilde{S}^\lambda_{i,i,\mathbf{V}} \to \widetilde{S}^\lambda_{i-1,i,\mathbf{V}}$ in the proof of Lemma IV.16 are surjective. Let $(\mathcal{L}_{i-1}, W)$ be an element in $\widetilde{S}^\lambda_{i-1,i,\mathbf{V}}$. Given a point $\mathcal{M}$ in the fiber of $\rho^i_{i-1} : \mathrm{Pic}^d(C)_i \to \mathrm{Pic}^d(C)_{i-1}$ over $\mathcal{L}_{i-1}$, we get an isomorphism $(\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1}) \cong H^1(C, \mathcal{O}_C)$. Let $\{s_{0,p}\}_p$ be a basis of

V_i, and $s_{i-1,p}$ the lifting of $s_{0,p}$ to the level $i-1$ in W. It is easy to see that $(\mathcal{L}_{i-1}, W)$ is in the image of h if and only if there is an element $\mathcal{L}_i \in (\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1})$ such that for every p, the section $s_{i-1,p}$ has an extension to a section of $\mathcal{L}_i$. By Lemma IV.10, we deduce that for every p, $s_{i-1,p}$ has an extension to a section of $\mathcal{L}_i$ if and only if the equation of the following form holds:

$$(\diamond_p) \qquad \nu(s_{0,p} \otimes [\mathcal{L}_i]) = \tau_p.$$

where τ_p is a cohomology class in $H^1(C, L)$ determined by the section $s_{i-1,p}$. In order to prove that $(\mathcal{L}_{i-1}, W)$ is in the image of h, it suffices to show the existence of an element $\mathcal{L}_i \in (\rho^i_{i-1})^{-1}(\mathcal{L}_{i-1})$ such that the equation $(\diamond_p)$ holds for every p.

Recall that $H^1(C, \mathcal{O}_C)$ is the dual space of $H^0(C, K_C)$, hence we identify $[\mathcal{L}_i]$ with a linear map $H^0(C, K_C) \to k$. By the duality between $H^1(C, L)$ and $H^0(C, K_C \otimes L^{-1})$, we identify τ_p with a linear map $H^0(C, K_C \otimes L^{-1}) \to k$. For every p, there is a map $m_{s_{0,p}} : H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$ taking $\gamma \in H^0(C, K_C \otimes L^{-1})$ to $\mu_0(s_{0,p} \otimes \gamma)$. Hence the equation $(\diamond_p)$ holds for $\mathcal{L}_i$ for all p if and only if the composition map

$$H^0(C, K_C \otimes L^{-1}) \overset{m_{s_{0,p}}}{\hookrightarrow} H^0(C, K_C) \overset{[\mathcal{L}_i]}{\to} k$$

is equal to τ_p for all p.

Let A_p be the image of $m_{s_{0,p}}$. The fact that $V_i \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$ is injective implies that the sum $\sum_p A_p$ in $H^0(C, K_C)$ is a direct sum. We conclude that there is a Čech cohomology classes $[\mathcal{L}_i]$ satisfying $(\diamond_p)$ for all p. Therefore $(\mathcal{L}_{i-1}, W)$ is in the image of h.

Applying Lemma IV.16 and Remark IV.17 to the case $i = \lambda_l - 1$, we obtain

$$\dim S^{\lambda}_{\lambda_l - 1, \mathbf{V}} = g(\lambda_l - 1) - \sum_{i=2}^{\lambda_l} n_i(\lambda)(g - d - 1) - \sum_{i=1}^{\lambda_l - 1} n_{i+1}(\lambda) n_i(\lambda).$$

Recall that Flag_κ is the variety parameterizing all weak flag variety of signature κ. We denote by D_κ the dimension of Flag_κ. It is easy to see that Flag_κ is exactly the usual flag variety of signature κ' where κ' is the longest decreasing subsequence of κ. Since $k_1 = n_2(\lambda) \leq n_1(\lambda) = l = H^0(C,L)$, there are only finitely many ways to get strictly decreasing sequence with values $\leq l$ and length $\leq l$. There are thus only finitely many integers D_κ. Let K_1 be the maximal value among these numbers. Clearly K_1 only depends on l. In particular, it is independent on m, hence $\lim\limits_{m\to\infty} \frac{K_1}{m} = 0$.

Note that $S^\lambda_{\lambda_l-1,\kappa} = \bigcup\limits_{\mathbf{V}} S^\lambda_{\lambda_l-1,\mathbf{V}}$, where $\mathbf{V}$ varies over all weak flags in Flag_κ. We thus have $\dim S^\lambda_{\lambda_l-1,\kappa} \leq \max\limits_{\mathbf{V}}\{\dim S^\lambda_{\lambda_l-1,\mathbf{V}}\} + K_1$. We have seen that $\rho^m_{\lambda_l-1}(C_{\lambda,m}) \subset S^\lambda_{\lambda_l-1,\kappa}$, hence $\dim C_{\lambda,m} \leq (m-\lambda_l+1)g + \dim S^\lambda_{\lambda_l-1,\kappa}$. For each $m \geq l$, we thus have

$$\begin{aligned}
\operatorname{codim}(\mathrm{Pic}^d(C)_{m,L}, W^r_d(C)_{m,L}) &= \min_\lambda \{mg - \dim C_{\lambda,m}\} \\
&\geq \min_\lambda \left\{ \sum_{i=2}^{\lambda_l} n_i(\lambda)(g-d-1) + \sum_{i=1}^{\lambda_l-1} n_i(\lambda)n_{i+1}(\lambda) - K_1 \right\} \\
&= \min_\lambda \left\{ (\sum_{i=1}^{l} \lambda_i - l)(g-d-1) + \sum_{i=1}^{\lambda_l-1} n_i(\lambda)n_{i+1}(\lambda) - K_1 \right\}
\end{aligned}$$

where λ varies over the partitions in $\Lambda_{l,m+1}$ with $\sum\limits_{i=1}^{l-r} \lambda_i \geq m+1$.

Note that $\sum\limits_{i=1}^{\lambda_l-1} n_i(\lambda)n_{i+1}(\lambda) \geq \sum\limits_{i=1}^{\lambda_l} n_i^2(\lambda) - l^2$, and since $\lim\limits_{m\to\infty} \frac{l^2}{m} = 0 = \lim\limits_{m\to\infty} \frac{K_1}{m}$, we obtain

$$\begin{aligned}
&\inf_{m\to\infty} \frac{\operatorname{codim}(\mathrm{Pic}^d(C)_{m,L}, W^r_d(C)_{m,L})}{m+1} \\
\geq\ &\inf_{m\to\infty} \min_\lambda \left\{ \frac{1}{m+1} \left((\sum_{i=1}^{l} \lambda_i - l)(g-d-1) + \sum_{i=1}^{\lambda_l-1} n_{i+1}(\lambda)n_i(\lambda) - K_1 \right) \right\} \\
\geq\ &\inf_{m\to\infty} \min_\lambda \left\{ \frac{1}{m+1} \left((\sum_{i=1}^{l} \lambda_i)(g-d-1) + \sum_{i=1}^{\lambda_l} n_i^2(\lambda) \right) \right\}
\end{aligned}$$

By Lemma IV.20, we thus obtain

$$\begin{aligned}
&\inf_{m\to\infty} \frac{\operatorname{codim}(\operatorname{Pic}^d(C)_{m,L}, W_d^r(C)_{m,L})}{m+1} \\
\geq\ & \inf_{m\to\infty} \min_{\lambda} \left\{ \frac{1}{m+1} \left(\sum_{i=1}^{l} (l-i+1)(\lambda_i - \lambda_{i-1})(g-d-1) + \sum_{i=1}^{l} (l-i+1)^2 (\lambda_i - \lambda_{i-1}) \right) \right\} \\
=\ & \inf_{m\to\infty} \min_{\lambda} \left\{ \frac{1}{m+1} \sum_{i=1}^{l} (\lambda_i - \lambda_{i-1})(g-d+l-i)(l-i+1) \right\}
\end{aligned}$$

For every i with $1 \leq i \leq l$, let $x_i = \lambda_i - \lambda_{i-1}$. Consider a linear function of the form $\sum_{i=1}^{l} b_i x_i$ with $b_i \geq 0$, defined over the region

$$\{(x_1, \cdots, x_l) \in \mathbb{R}^l \mid x_i \geq 0 \text{ for every } i,\ \sum_{i=1}^{l-r} (l-i-r+1)x_i \geq m+1\}.$$

The minimum value of this function is achieved at the vertices of this region, i.e. the points with all the x_i but one equal to 0 and $\sum_{i=1}^{l-r}(l-i-r+1)x_i = m+1$.

We thus have

$$(\sharp) \qquad \operatorname{lct}_L(\operatorname{Pic}^d(C), W_d^r(C)) \geq \min_{i=1}^{l-r} \left\{ \frac{(l+1-i)(g-d-i+l)}{l+1-r-i} \right\}$$

On the other hand, recall that one can locally define a map from $\operatorname{Pic}^d(C)$ to a variety of matrices $M_{(d+e+1-g)\times e}$ such that $W_d^r(C)$ is the pull back of a suitable generic determinantal variety Y defined by $e+d+1-g-r$ minors. Let Φ_L be the image of L. The right hand side in ($\sharp$) is the log canonical threshold of the pair $(M_{(d+e+1-g)\times e}, Y)$ at the point Φ_L (for the formula of log canonical threshold of a generic determinantal variety, see [Doc, Theorem 3.5.7]). We thus have $\operatorname{lct}_L(\operatorname{Pic}^d(C), W_d^r(C)) \leq \operatorname{lct}_{\Phi_L}(M_{(d+e+1-g)\times e}, Y)$, by [Lar, Example 9.5.8], which completes the proof. □

4.3 Appendix

Let L be a line bundle in $\operatorname{Pic}^d(C)$ with $l = h^0(C, L)$. In this section, we are going to show that the subsets $S_{i,\mathbf{V}}^{\lambda}$ and $S_{i,\kappa}^{\lambda}$ of $\operatorname{Pic}^d(C)_{i,L}$ defined in section 2 are constructible subsets of $\mathrm{Pic}^d(C)_{i,L}$. The key point is to realize $\widetilde{S}^{\lambda}_{i,j,\mathbf{V}}$ as a constructible subset of a suitable product of Grassmann bundles.

Let X be a scheme and E a vector bundle of rank n over X. For every $d \leq n$, we denote by $Gr(d,E)$ the Grassmann bundle of d-dimensional subspaces in E and by π the projection morphism from $Gr(d,E)$ to X. We write elements in $Gr(d,E)$ as pairs (x,W) where x is a point in X and W is a dimension d subspace of E_x.

Lemma IV.21. *If $\Phi : E \to F$ is a homomorphism of vector bundles on the scheme X, then we have*

1. *The subset $I_\Phi := \{x \in X \mid \Phi_x : E_x \to F_x$ is an injection$\}$ is an open subset of X.*

2. *If H is a subbundle of F, then the set $M^\Phi_H := \{x \mid \Phi_x(E_x) \subset H_x\}$ is a closed subset of X.*

The proof of Lemma IV.21 is standard, so we leave it to the reader.

Recall that $\mathcal{P}$ is a Poincaré line bundle on $\mathrm{Pic}^d(C) \times C$. From the definition of jet schemes, we have $\mathrm{Pic}^d(C)_m \times C_m \cong (\mathrm{Pic}^d(C) \times C)_m \cong \mathrm{Hom}(T_m, \mathrm{Pic}^d(C) \times C)$. By the adjunction (3.1) in section 1 for $Y = \mathrm{Pic}^d(C)_m \times C_m$ and $X = \mathrm{Pic}^d(C) \times C$, the identity map of $\mathrm{Pic}^d(C)_m \times C_m$ gives an evaluation morphism

$$\mathrm{Pic}^d(C)_m \times C_m \times T_m \xrightarrow{\Xi} \mathrm{Pic}^d(C) \times C.$$

For every m, we also have a morphism $C \xrightarrow{\gamma_m} C_m$ that takes a point to the corresponding constant jet. We have the composition map

$$\eta : \mathrm{Pic}^d(C)_m \times C \times T_m \xrightarrow{\mathrm{id} \times \gamma_m \times \mathrm{id}} \mathrm{Pic}^d(C)_m \times C_m \times T_m \xrightarrow{\Xi} \mathrm{Pic}^d(C) \times C.$$

We denote by $\mathcal{B}_m$ the pull back of the line bundle $\mathcal{P}$ to $\mathrm{Pic}^d(C)_m \times C \times T_m$ via η.

Recall that for every partition λ in $\Lambda_{l,m+1}$, $C_{\lambda,m}$ is the locally closed subset

$$\{\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L} \mid \mathcal{L}_m \text{ is of type } \lambda\}.$$

For every $0 \leq i \leq m$, there is a natural map $\Lambda_{l,m+1} \to \Lambda_{l,i+1}$ mapping λ to $\overline{\lambda}$ where $\overline{\lambda}_k = \min\{\lambda_k, i+1\}$ for each $k \leq l$. We have seen that $\rho_i^m : \mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{i,L}$ maps $C_{\lambda,m}$ to $C_{\overline{\lambda},i}$.

We now fix a partition $\lambda \in \Lambda_{l,m+1}$. We denote by $\mathcal{B}_{\lambda,m}$ the restriction of $\mathcal{B}_m$ to the subscheme $C_{\lambda,m} \times C \times T_m$, where on $C_{\lambda,m}$ we consider the reduced scheme structure. We denote by p_1 the projection to the first factor $\mathrm{Pic}^d(C)_{m,L} \times C \times T_m \to \mathrm{Pic}^d(C)_{m,L}$. It is easy to check that for every $\mathcal{L}_m \in \mathrm{Pic}^d(C)_{m,L}$ corresponding to a morphism $f : T_m \to \mathrm{Pic}^d(C)$, the restriction of $\mathcal{B}_m$ to the fiber of $p_1^{-1}(\mathcal{L}_m) \cong C \times T_m$ is $(f \times \mathrm{id}_C)^*(\mathcal{P}) \cong \mathcal{L}_m$.

Recall that for every i with $0 \leq i \leq m$, there is a closed embedding $\iota_i^m : T_i \hookrightarrow T_m$. Let

$$\nu_i^m : C_{\lambda,m} \times C \times T_i \hookrightarrow C_{\lambda,m} \times C \times T_m$$

be the induced embedding. Let $\mathcal{D}_{\lambda,i}$ be the sheaf $p_{1*}(\nu_i^m)_*(\nu_i^m)^*(\mathcal{B}_{\lambda,m})$ on $C_{\lambda,m}$. Consider the function $C_{\lambda,m} \to \mathbf{Z}$ that takes $\mathcal{L}_m$ to $h^0(C \times T_i, \mathcal{L}_i)$, where $\mathcal{L}_i$ is the image of $\mathcal{L}_m$ in $\mathrm{Pic}^d(C)_i \cong \mathrm{Pic}^d(C \times T_i)$. Lemma IV.8 implies that this function is constant on $C_{\lambda,m}$. By the Base Change Theorem, we deduce that $\mathcal{D}_{\lambda,i}$ is a locally free sheaf of rank $\sum_{j=1}^{i+1} n_j(\lambda)$ on $C_{\lambda,m}$, whose fiber over a point $\mathcal{L}_m$ is $H^0(C, L_i)$. For every i and j with $0 \leq j \leq i \leq m$, the embedding map ν_j^m factors through ν_i^m. We thus have a natural morphism of sheaves

$$(\nu_i^m)_*(\nu_i^m)^*(\mathcal{B}_{\lambda,m}) \to (\nu_j^m)_*(\nu_j^m)^*(\mathcal{B}_{\lambda,m})$$

on $C_{\lambda,m} \times C \times T_m$. Applying $(p_1)_*$ to it, we obtain a vector bundle map

$$\Phi_j^i : \mathcal{D}_{\lambda,i} \to \mathcal{D}_{\lambda,j}$$

on $C_{\lambda,m}$ whose restriction to the fiber over $\{\mathcal{L}_m\}$ is the truncation map

$$\pi^i_j : H^0(\mathcal{L}_i) \to H^0(\mathcal{L}_j).$$

For a fixed partition $\lambda \in \Lambda_{l,m+1}$, we consider $\kappa = (\kappa_1, \cdots, \kappa_m)$ a signature with $k_j \leq n_{j+1}(\lambda)$ for every $j \leq m$. For every $i \leq m$, a point in the fiber product of Grassmann bundles

$$\mathcal{G}_{\lambda,i,\kappa} := Gr(\kappa_1, \mathcal{D}_{\lambda,1}) \times_{C_{\lambda,m}} \cdots \times_{C_{\lambda,m}} Gr(\kappa_i, \mathcal{D}_{\lambda,i})$$

over $C_{\lambda,m}$ is written as an $(m+1)$–tuple $(\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i)$, where $\mathcal{L}_m \in C_{\lambda,m}$ and $\widetilde{V}_j$ is a dimension κ_j subspace of $(\mathcal{D}_{\lambda,j})|_{\mathcal{L}_m} \cong H^0(\mathcal{L}_j)$ for every $j \leq i$. For every weak flag $\mathbf{V}$ of $H^0(C, L)$ of signature κ, we denote by $\mathcal{P}^\lambda_{m,i,\mathbf{V}}$ the subset of points $(\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i) \in \mathcal{G}_{\lambda,i,\kappa}$, where $\mathcal{L}_m \in C_{\lambda,m}$ and $\{\widetilde{V}_1, \ldots, \widetilde{V}_i\}$ is a compatible extension of $\mathbf{V}_{(i)}$ to the line bundle $\mathcal{L}_i$. We also write $\mathcal{P}^\lambda_{m,i,\kappa}$ for $\bigcup_{\mathbf{V}} \mathcal{P}^\lambda_{m,i,\mathbf{V}}$, where $\mathbf{V}$ varies over all weak flags of $H^0(C, L)$ of signature κ.

Recall that Flag_κ is the variety parameterizing weak flags of $H^0(C, L)$ of signature κ. We denote by $\widetilde{\mathcal{P}}^\lambda_{m,i,\kappa}$ the subset of points

$$(\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i; \mathbf{V}') \in \mathcal{G}_{\lambda,i,\kappa} \times \mathrm{Flag}_\kappa$$

where $\mathbf{V}' \in \mathrm{Flag}_\kappa$ and $(\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i) \in \mathcal{P}^\lambda_{m,i,\mathbf{V}'}$.

Lemma IV.22. *Let $\lambda \in \Lambda_{l,m+1}$ and κ be a signature of length m with $\kappa_j \leq n_{j+1}(\lambda)$ for every $1 \leq j \leq m$. Then for every i with $1 \leq i \leq m$, $\widetilde{\mathcal{P}}^\lambda_{m,i,\kappa}$ is a constructible subsets of $\mathcal{G}_{\lambda,i,\kappa} \times$ Flag$_\kappa$.*

Proof. For simplicity, we write X for the scheme $\mathcal{G}_{\lambda,i,\kappa} \times \mathrm{Flag}_\kappa$. For j with $1 \leq j \leq i$, we denote by p_j the projection X onto $Gr(\kappa_j, \mathcal{D}_{\lambda,j})$ and by p_{i+1} the projection of X onto Flag_κ. For a fixed j with $1 \leq j \leq i$, we denote by $q_j : Gr(\kappa_j, \mathcal{D}_{\lambda,j}) \to C_{\lambda,m}$.

The composition map

$$X \xrightarrow{p_j} Gr(\kappa_j, \mathcal{D}_{\lambda,j}) \xrightarrow{q_j} C_{\lambda,m}$$

does not depend on a particular choice of j for $j \leq i$. We denote it by χ.

For every j with $1 \leq j \leq i$, we denote by T_j the tautological subbundle of $q_j^*(\mathcal{D}_{\lambda,j})$ on $Gr(\kappa_j, \mathcal{D}_{\lambda,j})$. Let $\mathcal{T}_j = p_j^* T_j$ and $\mathcal{F}_j$ be the vector bundle $p_j^* q_j^*(\mathcal{D}_{\lambda,j}) = \chi^*(\mathcal{D}_{\lambda,j})$. Hence $\mathcal{T}_j$ is a subbundle of $\mathcal{F}_j$ for each j. Over a point $x = (\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i; \mathbf{V}') \in X$, we have $\mathcal{T}_{j,x} = \widetilde{V}_j$ and $\mathcal{F}_{j,x}$ is $H^0(\mathcal{L}_j)$ where $\mathcal{L}_j$ is the image of $\mathcal{L}_m$ under $\mathrm{Pic}^d(C)_{m,L} \to \mathrm{Pic}^d(C)_{j,L}$. For every k and j with $0 \leq k \leq j \leq i$, we write Ψ_k^j for the composition $\mathcal{T}_j \hookrightarrow \mathcal{F}_j \to \mathcal{F}_k$.

Let $R_1 \supseteq R_2 \supseteq \cdots \supseteq R_m$ be the tautological flag bundles on Flag_κ, where the fiber of R_j over a point $\mathbf{V}' = \{V_n'\}_n$ in Flag_κ is V_j'. We write $\mathcal{R}_j$ for the pull back of R_j via $p_{i+1} : X \to \mathrm{Flag}_\kappa$. Over a point $x = (\mathcal{L}_m; \widetilde{V}_1, \cdots, \widetilde{V}_i; \mathbf{V}') \in X$, where $\mathbf{V}' = \{V_j'\} \in \mathrm{Flag}_\kappa$ we have $\mathcal{R}_{j,x} = V_j'$. Note that $\mathcal{D}_0$ is the trivial vector bundle on $C_{\lambda,m}$ with fiber $H^0(C, L)$. Hence $\mathcal{F}_0$ is a trivial bundle on X with fiber $H^0(C, L)$. It implies that $\mathcal{R}_j$ is a subbundle of $\mathcal{F}_0$.

With the notation in Lemma IV.21, we have

$$\widetilde{\mathcal{P}}^\lambda_{m,i,\kappa} = \bigcap_{j=1}^{i} (I_{\Psi_0^j} \cap M_{\mathcal{T}_{j-1}}^{\Psi_{j-1}^j} \cap M_{\mathcal{R}_j}^{\Psi_0^j})$$

This completes the proof. □

Corollary IV.23. *With the notation in Lemma IV.22, let* $\boldsymbol{V}$ *be a weak flag of* $H^0(C, L)$ *of signature* κ*. For every* i *with* $1 \leq i \leq m$*,* $\mathcal{P}^\lambda_{m,i,\kappa}$ *and* $\mathcal{P}^\lambda_{m,i,\boldsymbol{V}}$ *are both constructible subsets of* $\mathcal{G}_{\lambda,i,\kappa}$*.*

Proof. We denote by pr_1 and pr_2 the projections of $\mathcal{G}_{\lambda,i,\kappa} \times \mathrm{Flag}_\kappa$ onto $\mathcal{G}_{\lambda,i,\kappa}$ and Flag_κ, respectively. We thus deduce that $\mathcal{P}^\lambda_{m,i,\kappa}$, as the image of $\widetilde{\mathcal{P}}^\lambda_{m,i,\kappa}$ under pr_1, is a constructible subset of $\mathcal{G}_{\lambda,i,\kappa}$. It is clear that $\mathcal{P}^\lambda_{m,i,\mathbf{V}}$ is the image of $\widetilde{\mathcal{P}}^\lambda_{m,i,\kappa} \cap pr_2^{-1}(\mathbf{V})$

under pr_1. Lemma IV.22 implies that $\widetilde{\mathcal{P}}^{\lambda}_{m,i,\kappa} \cap pr_2^{-1}(\mathbf{V})$ is a constructible subset of $\mathcal{G}_{\lambda,i,\kappa} \times \mathrm{Flag}_{\kappa}$. This completes the proof. □

Corollary IV.24. *Let κ be a signature of length m with $k_j \leq n_{j+1}(\lambda)$ for every $j \leq m$, and $\boldsymbol{V} \in Flag_{\kappa}$. For every i with $1 \leq i \leq m$, the subsets $S^{\lambda}_{i,\kappa}$ and $S^{\lambda}_{i,\boldsymbol{V}}$ are constructible subset of* $\mathrm{Pic}^d(C)_{i,L}$.

Proof. For a fixed i, let $\overline{\kappa}$ be a signature of length i such that $\overline{\kappa}_j = \kappa_j$ for every $j \leq i$. Recall that $\overline{\lambda}$ is the image of λ under $\Lambda_{l,m+1} \to \Lambda_{l,i+1}$. By the definition of $S^{\lambda}_{i,\kappa}$, we have $S^{\lambda}_{i,\kappa} = S^{\overline{\lambda}}_{i,\overline{\kappa}}$ for every $i \leq m$. It suffice to prove the assertion in case $i = m$.

Recall that χ is the morphism of projection $\mathcal{G}_{\lambda,m,\kappa} \to C_{\lambda,m}$. The fact that $S^{\lambda}_{m,\kappa}$ is the image of $\mathcal{P}_{m,m,\kappa}$ under χ and Corollary IV.23 shows that $S^{\lambda}_{m,\kappa}$ is a constructible subset of $\mathrm{Pic}^d(C)_{m,L}$. The assertion for $S^{\lambda}_{i,\mathbf{V}}$ is proved similarly. □

Lemma IV.25. $\widetilde{S}^{\lambda}_{i,j,\boldsymbol{V}}$ *is a constructible subset of the Grassmann bundle $Gr(\kappa_j, \mathcal{D}_i)$ on $C_{\lambda,i}$.*

The proof of this lemma is similar to those of Lemma IV.22 and Corollary IV.23, hence we leave it to the reader.

CHAPTER V

Divisorial Valuations via Arc Spaces

5.1 Cylinder Valuations and Divisorial Valuations

The main goal of this section is to establish the correspondence between cylinders and divisorial valuations described in the introduction. Let X be a variety over a field k. Recall that a subset C of X_∞ is *thin* if there is a proper closed subscheme Z of X such that $C \subset Z_\infty$.

Lemma V.1. *Let X be a smooth variety over k. If C is a nonempty cylinder in X_∞, then C is not thin.*

For the proof of Lemma V.1, see [ELM][Proposition 1].

Lemma V.2. *Let $f : X' \to X$ be a proper birational morphism of schemes over k. Let Z be a closed subset of X and $F = f^{-1}(Z)$. If f is an isomorphism over $X \setminus Z$, then the restriction map of f_∞*

$$\varphi : X'_\infty \setminus F_\infty \to X_\infty \setminus Z_\infty$$

is bijective on the L–valued points for every field extension L of k. In particular, φ is surjective.

Proof. Since f is proper, the Valuative Criterion for properness implies that an arc $\gamma : \operatorname{Spec} L[\![t]\!] \to X$ lies in the image of f_∞ if and only if the induced morphism

$\gamma_\eta : \operatorname{Spec} L((t)) \to X$ can be lifted to X'. γ is not contained in Z_∞ implies that γ_η factor through $X \setminus Z \hookrightarrow X$. Since f is an isomorphism over $X \setminus Z$, hence there is a unique lifting of γ_η to X'. This shows that φ is surjective. The injectivity of φ follows from the Valuative Criterion for separatedness of f. The last assertion follows from the fact that a morphism of schemes (not necessary to be of finite type) over k is surjective if the induced map on L–valued points is surjective for every field extension L. $\square$

The Change of Variable Theorem due to Kontsevish [Kon] and Denef and Loeser [DL] will play an important role in our arguments. We now state a special case of this theorem as Lemma I.6.

Lemma V.3. *Let X be a smooth variety of dimension n over k and Z a smooth irreducible subvariety of codimension $c \geq 2$. Let $f : X' \to X$ be the blow up of X along Z and E the exceptional divisor.*

(a) For every positive integer e and every $m \geq 2e$, the induced morphism

$$\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})) \to f_m(\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})))$$

is a piecewise trivial $\mathbf{A}^e$ fibration.

(b) For every $m \geq 2e$, the fiber of f_m over a point $\gamma_m \in f_m(\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})))$ is contained in a fiber of $X'_m \to X'_{m-e}$.

Although Lemma I.6 is well-known, we give a proof for completeness. The idea is similar to that of [Bli, Theorem 3.3].

Proof. Suppose that γ' is an element in $\mathrm{Cont}^e(K_{X'/X}) \subset X'_\infty$. We denote by γ the image of γ' in X_∞. Let $\gamma_m = \psi_m^X(\gamma)$ and $\gamma'_m = \psi_m^{X'}(\gamma')$. We denote by L the residue field of γ_m and L' the residue field of γ'_m. We now describe the fiber of f_m over γ_m.

Let $x' = \psi_\infty^{X'}(\gamma')$ and $x = \psi_\infty^X(\gamma)$. The residue field of x is a subfield of L. Since $e \geq 1$, then $x \in Z$. Let U be an affine open neighborhood of x in X such that Z is defined by c regular local parameters, denoted by $z_1, \ldots, z_c$.

Due to the local nature of the question, we can replace X by U and X' by the blow up of U along $Z \cap U$. It follows that $X' \subset \mathbf{P}_U^{c-1}$, defined by the 2×2 minors of the matrix

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_c \\ z_1 & z_2 & \cdots & z_c \end{pmatrix}$$

where the y_i are the homogeneous coordinates of $\mathbf{P}^{c-1}$. Suppose that the point x' is in the affine patch U' of X' such that y_1 is not zero. We set $y_1 = 1$, then the above equations are $y_i z_1 = z_i$ for every $i \geq 2$. The morphism $U' \to U$ induces the ring map

$$f^* : \mathcal{O}(U) \hookrightarrow \mathcal{O}(U)[y_2, \ldots, y_c] = \mathcal{O}(U').$$

The exceptional divisor E is defined by $z_1 = 0$ in $\mathcal{O}(U')$. Since $\operatorname{codim} Z = c$, it follows that $K_{X'/X}$ is defined by z_1^{c-1}. Hence $\operatorname{ord}'_\gamma E = \frac{\operatorname{ord}_{\gamma'} K_{X'/X}}{c-1} = \frac{e}{c-1}$. For simplicity, we write α for $\frac{e}{c-1}$. Note that $m \geq 2e \geq e+\alpha$. Consider the induced ring homomorphism $(\gamma'_m)^* : \mathcal{O}(U') \to L'[t]/(t^{m+1})$, since $f_m(\gamma'_m) = \gamma_m$, we obtain

$$\begin{aligned} (\gamma_m)^*(z_1) &= (\gamma'_m)^*(z_1) &= t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i} t^i \\ (\gamma_m)^*(z_2) &= (\gamma'_m)^*(z_2) &= \sum_{i=0}^{m} a_{2,i} t^i \\ &\cdots \\ (\gamma_m)^*(z_c) &= (\gamma'_m)^*(z_c) &= \sum_{i=0}^{m} a_{c,i} t^i \end{aligned} \tag{5.1}$$

where the coefficients $a_{i,j} \in L$ with $a_{1,0} \neq 0$.

We now fix a finitely generated L–algebra M and show that the M–valued points of $f_m^{-1}(\gamma_m)$ are the affine space $\mathbf{A}^e$ over M. Let γ''_m be an M–valued m–jet in the fiber of $f_m : \mathrm{Cont}^e(K_{X'/X})_m \to X_m$ over γ_m. For every $1 \le j \le c$, we get

$$(\gamma''_m)^*(y_j) = \sum_{i=0}^{m} b_{j,i}t^i$$

with $b_{j,i} \in M$.

The equality $f_m(\gamma''_m) = \gamma_m$ implies that

$$\begin{array}{lllll} & (\gamma_m)^*(z_1) & = (\gamma''_m)^*(z_1) & = t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i}t^i & \mod t^{m+1} \\ & (\gamma_m)^*(z_2) & = (\gamma''_m)^*(y_2 z_1) & = (\sum_{i=0}^{m} b_{2,i}t^i)(t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i}t^i) & \mod t^{m+1} \\ (5.2) & & & \cdots & \\ & (\gamma_m)^*(z_c) & = (\gamma''_m)^*(y_c z_1) & = (\sum_{i=0}^{m} b_{c,i}t^i)(t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i}t^i) & \mod t^{m+1} \end{array}$$

Comparing the first rows in equations (1) and (2), we get

$$\sum_{i=0}^{m} b_{1,i}t^i = (\gamma_m)^*(z_1) = t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i}t^i \quad \mod t^{m+1}.$$

This implies that all coefficients $b_{1,j}$ for $1 \le j \le m$ are determined in terms of $\{a_{1,i}\}$.

For the coefficients $b_{2,j}$, we compare the second rows in equations (5.1) and (5.2), i.e.

$$(\sum_{i=0}^{m} b_{2,i}t^i)(t^\alpha \sum_{i=0}^{m-\alpha} a_{1,i}t^i) = \sum_{i=0}^{m} a_{2,i}t^i \quad \mod t^{m+1}.$$

Expanding the product of two sums, we observe that all the coefficients $b_{2,m-\alpha+1}, \ldots, b_{2,m}$ do not show up. On the other hand, the coefficients of t^j with $j \ge \alpha$ give equations as follows:

$$\begin{array}{lrcl} & a_{2,\alpha} & = & b_{2,0}a_{1,0} \\ & a_{2,\alpha+1} & = & b_{2,0}a_{1,1} + b_{2,1}a_{1,0} \\ (5.3) & & \cdots & \\ & a_{2,m} & = & b_{2,0}a_{m-\alpha} + \cdots + b_{2,m-\alpha}a_{1,0} \end{array}$$

Note that $a_{1,0} \ne 0$, by induction on the second index of b, we can solve $b_{2,0}, \ldots, b_{2,m-\alpha}$ in terms of $\{a_{1,i}\}$ and $\{a_{2,j}\}$. We do the similar computations on $z_3, \ldots, z_c$. It is clear that the set of M–valued points of the fiber of f_m over γ_m is an M–affine space with coordinates $b_{j,i}$ for $2 \leq j \leq c$ and $m - \alpha + 1 \leq i \leq m$. This implies that the fiber of f_m at γ_m is $\mathbf{A}^e_{\operatorname{Spec} L}$. This complete the proof of the part (a).

Let $\gamma_m \in \psi_m(\operatorname{Cont}^e(K_{X'/X}))$. The proof of Part (a) implies that any two jets γ'_m and γ''_m in the set $f_m^{-1}(\gamma_m)$ only differ in the last α coordinates. Hence they have the same image via the truncation map $\rho^m_{m-\alpha}$. In particular, this implies part (b). $\square$

Let X be a smooth variety of dimension n over k. For every irreducible cylinder C which does not dominate X, we define a discrete valuation as follows. Let γ be the generic point of C with residue field L. We thus have an induced ring homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(0)} \to \operatorname{Spec} L[\![t]\!]$. Lemma V.1 implies that $\ker \gamma^*$ is zero. Hence γ^* extends to an injective homomorphism $\gamma^* : k(X) \to L(\!(t)\!)$. We define a map

$$\operatorname{ord}_C : k(X)^* \to \mathbf{Z}$$

by $\operatorname{ord}_C(f) := \operatorname{ord}_\gamma(f) = \operatorname{ord}_t(\gamma^*(f))$. If C does not dominate X, then ord_C is a discrete valuation. If C' is a dense subcylinder of C, then they define the same valuation. Given an element $f \in k(X)$, we can check that $\operatorname{ord}_C(f) = \operatorname{ord}_{\gamma'}(f)$ for general point γ' in C.

From now on we assume that k is a perfect field. We first prove that every valuation defined by a cylinder is a divisorial valuation.

Lemma V.4. *If C is an irreducible closed cylinder in X_∞ which does not dominate X, then there exist a divisor E over X and a positive integer q such that*

$$\operatorname{ord}_C = q \cdot \operatorname{ord}_E . \tag{5.4}$$

Furthermore, we have $\operatorname{codim}(C) \geq q \cdot (1 + \operatorname{ord}_E(K_{-/X}))$.

Proof. We will prove that such divisor E can be reached by a sequence of blow ups of smooth centers after shrinking to suitable open subsets. Let (R, m) be the valuation ring associated to the valuation ord_C. Suppose that C is $\psi_m^{-1}(S)$ for some closed irreducible subset S in X_m. Chevalley's Theorem implies that the image of the cylinder C by the projection $\psi_0(C) = \pi_m(S)$ is a constructible set. We denote its closure in X by Z. This is *the center of* ord_C. If C is irreducible and does not dominate X, then Z is a proper reduced irreducible subvariety of X. The generic smoothness theorem implies that there is an nonempty open subset U of X such that $U \cap Z$ is smooth. Since U contains the the generic point of Z, then $C \cap U_\infty$ is an open dense subcylinder of C. Note that U_∞ is an open subset of X_∞, we have $\mathrm{codim}(C, X_\infty) = \mathrm{codim}(C \cap U_\infty, U_\infty)$ and $C \subseteq X_\infty$ and $C \cap U_\infty \subseteq U_\infty$ define the same valuation. This implies that we can replace X by U and C by $C \cap U_\infty$. As a consequence, we may and will assume that Z is a smooth subvariety of X.

If Z is a prime divisor on X, then the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring of $k(X)$ with maximal ideal $m_{X,Z}$. Given two local rings (A, p) and (B, q) of $k(X)$, we denote by $(A, p) \preceq (B, q)$ if $A \subseteq B$ is a local inclusion, i.e. $p = q \cap A$. This defines a partial order on the set of local rings of $k(X)$. By the definition of Z, we deduce that

$$(\mathcal{O}_{X,Z}, m_{X,Z}) \preceq (R, m).$$

Since every valuation ring is maximal with respect to the partial order $\preceq$, it follows that $\mathcal{O}_{X,Z}$ is equal to the valuation ring R of ord_C, and $\mathrm{ord}_C = q \cdot \mathrm{ord}_Z$ for some integer $q > 0$. Therefore we may take $X' = X$ and $E = Z$, in which case $\mathrm{ord}_E(K_{-/X}) = 0$. The equality $\mathrm{ord}_C Z = q \cdot \mathrm{ord}_Z Z = q$ implies that C is a subcylinder of $\mathrm{Cont}^{\geq q}(E)$. Since E is a smooth divisor, we obtain that $\mathrm{codim}\,\mathrm{Cont}^{\geq q}(E) = q$. This proves the inequality

$$\operatorname{codim}(C) \geq q \cdot (1 + \operatorname{ord}_E(K_{-/X})) = q.$$

We now assume that Z is not a divisor, i.e. $\operatorname{codim} Z \geq 2$. Let $f : X' \to X$ be the blow up of X along Z. We claim that there exists an irreducible closed cylinder C' in X'_∞ such that the morphism f_∞ maps C' into C dominantly.

Let e be the vanishing order $\operatorname{ord}_C(K_{X'/X})$. We can assume that $C = (\psi_m^X)^{-1}(S)$ for some closed irreducible subset S in X_m with $m \geq 2e$. The smoothness of X implies that $C \setminus Z_\infty$ is a dense subset of C. Let $F = f^{-1}(Z)$ be the exceptional divisor on X'. It is clear that $f_\infty^{-1}(Z_\infty) = F_\infty$. We denote by

$$\varphi : X'_\infty \setminus F_\infty \to X_\infty \setminus Z_\infty$$

the restriction of f_∞. Let γ be the generic point of C and L the residue field of γ. Hence γ induces a morphism

$$\gamma_L : \operatorname{Spec} L[\![t]\!] \to X.$$

Lemma V.1 implies that $\gamma \in X_\infty \setminus Z_\infty$. By Lemma V.2, we deduce that ψ is bijective on L–valued piont, hence there is a unique L–valued point of X'_∞ mapping to γ_L via φ. We denote by γ' its underlying point in X'_∞. It is clear that $f_\infty(\gamma') = \gamma$. For simplicity we write γ_m for $\psi_m^X(\gamma)$ and γ'_m for $\psi_m^{X'}(\gamma')$. By Lemma I.6 part (a), we deduce that $f_m^{-1}(\gamma_m)$ is an affine space of dimension e over the residue field of γ_m. Hence the image of $f_m^{-1}(\gamma_m)$ in X'_∞, denoted by T, is irreducible. Since γ_m is the generic point of S, there is a unique component of $f_m^{-1}(S)$ which contains T. Let S' be this component and C' the cylinder $(\psi_m^{X'})^{-1}(S')$ in X'_∞. We now check that the closed irreducible cylinder C' satisfies the above conditions. The fact

$$f_m(\gamma'_m) = f_m \circ \psi_\infty^{X'}(\gamma') = \psi_m^X(\gamma) = \gamma_m$$

implies that $\gamma'_m \in T$. We deduce that $\gamma' \in C'$. It follows that f_∞ maps C' into C dominantly.

The fact that the center of ord_C on X is Z implies that $\mathrm{ord}_C(F) > 0$, hence $e = \mathrm{ord}_C(K_{X'/X}) > 0$. Lemma I.6 implies that $f_m : S' \to S$ is dominant with general fibers of dimensional e. We thus have $\dim S' = \dim S + e$, hence

$$\operatorname{codim} C' = \dim X'_m - \dim S' = \dim X_m - (\dim S + e) = \operatorname{codim} C - e$$

We now set $X^{(0)} = X, X^{(1)} = X', C^{(0)} = C$ and $C^{(1)} = C'$. Since C' dominates C, we get that ord_C and $\mathrm{ord}_{C'}$ are equal as valuations of $k(X)$. If the center of $\mathrm{ord}_{C'}$ on X' is not a divisor, we blow up this center again (we may need to shrink X' to make the center to be smooth). We now run the above argument for the variety $X^{(1)}$ and $C^{(1)}$ and obtain $X^{(2)}$ and $C^{(2)}$. Since every such blow up decreases the codimension of the cylinder, which is an non-negative integer, we deduce that after s blow ups, the center of the valuation $\mathrm{ord}_{C^{(s)}}$ on $X^{(s)}$ is a divisor, denoted by E. We have

$$\mathrm{ord}_C = \mathrm{ord}_{C^{(1)}} = \cdots = \mathrm{ord}_{C^{(s)}} = q \cdot \mathrm{ord}_E .$$

We now check the inequality $\operatorname{codim} C \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X}))$. At each step, we have

$$\operatorname{codim}(C) = \operatorname{codim}(C^{(1)}) + \mathrm{ord}_C(K_{X^{(1)}/X})$$
$$\operatorname{codim}(C^{(1)}) = \operatorname{codim}(C^{(2)}) + \mathrm{ord}_C(K_{X^{(2)}/X^{(1)}})$$
$$\cdots$$
$$\operatorname{codim}(C^{(s-1)}) = \operatorname{codim}(C^{(s)}) + \mathrm{ord}_C(K_{X^{(s)}/X^{(s-1)}})$$

We thus obtain that

$$\begin{aligned}\operatorname{codim}(C) &= \operatorname{codim}(C^{(s)}) + \sum_{i=1}^{s} \mathrm{ord}_C(K_{X^{(i)}/X^{(i-1)}}) \\ &= \operatorname{codim}(C^{(s)}) + \mathrm{ord}_C(K_{X^{(s)}/X})\end{aligned}$$

It is clear that $\mathrm{ord}_C E = q \cdot \mathrm{ord}_E(E) = q$, hence $C^{(s)} \subseteq \mathrm{Cont}^{\geq q}(E)$, and therefore $\operatorname{codim} C^{(s)} \geq \operatorname{codim} \mathrm{Cont}^{\geq q}(E) = q$. This complete the proof. $\square$

Lemma V.5. *Let X be a smooth variety and S a constructible subset of X_m for some m.*

(a) $\overline{\psi_m^{-1}(S)} = \psi_m^{-1}(\overline{S})$.

(b) If U is an open subset of X and C is a cylinder in U_∞, then the closure $\overline{C}$ in X_∞ is a closed cylinder in X_∞.

Proof. We first prove part (a). Since ψ_m is continuous with respect to the Zariski topologies, we deduce that $\psi_m^{-1}(\overline{S})$ is closed. We thus have $\overline{\psi_m^{-1}(S)} \subseteq \psi_m^{-1}(\overline{S})$. If $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an arc $\gamma \in \psi_m^{-1}(\overline{S}) \setminus \overline{\psi_m^{-1}(S)}$. Let U be an affine neighborhood of $\psi_0(\gamma)$ in X and $W = S \cap U_m$. It is clear that

$$\gamma \in (\psi_m^U)^{-1}(\overline{W}) \setminus \overline{(\psi_m^U)^{-1}(W)}.$$

In order to get a contradiction, we can replace X by U and S by W. We thus may assume that X is an affine variety. It follows from the construction of jet schemes that X_m are smooth affine varieties. Let $X_m = \operatorname{Spec} A_m$ for every $m \geq 0$. Hence $X_\infty = \operatorname{Spec} A$ where $A = \bigcup_m A_m$. We claim that if $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an integer $n \geq m$ such that

$$\overline{\psi_n(\overline{\psi_m^{-1}(S)})} \neq \psi_n(\psi_m^{-1}(\overline{S})).$$

Since $\psi_n(\psi_m^{-1}(\overline{S})) = (\rho_m^n)^{-1}(\overline{S})$ and $\psi_n(\psi_m^{-1}(S)) = (\rho_m^n)^{-1}(S)$, we deduce that

$$\overline{(\rho_m^n)^{-1}(S)} = \overline{\psi_n(\psi_m^{-1}(S))} \subseteq \overline{\psi_n(\overline{\psi_m^{-1}(S)})} \subsetneq (\rho_m^n)^{-1}(\overline{S}).$$

On the other hand, since ρ_m^n is a locally trivial affine bundle with fiber $\mathbf{A}^{\dim X(n-m)}$, we have $\overline{(\rho_m^n)^{-1}(S)} = (\rho_m^n)^{-1}(\overline{S})$. We thus get an contraction.

We now prove the claim. Let I be the radical ideal defining $\overline{\psi_m^{-1}(S)}$ in X_∞ and J the radical ideal defining $\psi_m^{-1}(\overline{S})$. If $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an element

$f \in I \setminus J$. There exist an integer $n \geq m$ such that $f \in A_n$. Let $I_n = I \cap A_n$ and $J_n = J \cap A_n$. It is clear that $\overline{\psi_n(\psi_m^{-1}(\overline{S}))}$ is the closed subset of X_n defined by J_n. Similarly $\psi_n(\overline{\psi_m^{-1}(S)}) = (\rho_m^n)^{-1}(\overline{S})$ is the closed subset of X_n defined by the ideal I_n. Since $f \in I_n \setminus J_n$, we thus have the assertion of the claim. This completes the proof of part (a).

For the proof of part (b), let $C = (\psi_m^U)^{-1}(S)$ for some integer $m \geq 0$ and some constructible subset S of U_m. We now consider S as a constructible subset of X_m and apply part (a), we thus obtain $\overline{C} = \overline{\psi_m^{-1}(S)} = (\psi_m^X)^{-1}(\overline{S})$. This completes the proof. $\square$

Lemma V.6. *Let X and X' be smooth varieties over a field k, and $f : X' \to X$ a blow up with smooth center. If C' is a closed cylinder of X', then the closure of the image $f_\infty(C')$, denoted by C, is a cylinder in X'. We also have*

$$\operatorname{ord}_C = \operatorname{ord}_{C'};\ \operatorname{codim} C = \operatorname{codim} C' + \operatorname{ord}_{C'} K_{X'/X}.$$

Proof. Let $e = \operatorname{ord}_{C'} K_{X'/X}$. For simplicity, we write ψ'_m for $\psi_m^{X'}$ and ψ_m for ψ_m^X for every $m \geq 0$. We first show that C is a closed cylinder. We choose an integer $p \geq e$ and a constructible subset T' of X'_p such that $C' = (\psi'_p)^{-1}(T')$. Let $m = e + p$. We denote by S' the inverse image of T' by the canonical projection $\rho_p^m : {X'}_m \to {X'}_p$. Let $S = f_m(S')$. Lemma I.6 part (b) implies that $f_m^{-1}(f_m(S')) \subseteq (\rho_p^m)^{-1}(T') = S'$. We thus have $f_m^{-1}(f_m(S')) = S'$. It follows that $f_\infty(C') = \psi_m^{-1}(S)$. Hence

$$C = \overline{f_\infty(C')} = \overline{\psi_m^{-1}(S)} = \psi_m^{-1}(\overline{S})$$

is an irreducible closed cylinder in X_∞. Here the last equality follows from Lemma V.5 part (a). Since C' dominates C, we have $\operatorname{ord}_C = \operatorname{ord}_{C'}$. The codimension equality follows from the fact that $\dim S' = \dim S + e$ by Lemma I.6. $\square$

Lemma V.7. *Let X be a smooth variety over a perfect field k. If $f : Y \to X$ is a birational morphism from a normal variety Y and E is a prime divisor, then for every positive integer q, there exist an irreducible cylinder $C \subset X_\infty$ such that* $\mathrm{ord}_C = q \cdot \mathrm{ord}_E$ *and*

$$\operatorname{codim}(C) = q \cdot (1 + \mathrm{ord}_E(K_{Y/X})) \tag{5.5}$$

Proof. Let ν be the divisorial valuation $q \cdot \mathrm{ord}_E$ on the function field $K(X)$. We define a sequence of varieties and maps as follows. Let Z_0 be the center of ν on X and $X^{(0)} = X$. We choose an open subset $U^{(0)}$ of $X^{(0)}$ such that $Z_0 \cap U^{(0)}$ is a nonempty smooth subvariety of $U^{(0)}$. If $Z_0 \cap U^{(0)}$ is not a divisor, then let $f_1 : X^{(1)} \to U^{(0)}$ be the blow up of $U^{(0)}$ along $Z_0 \cap U^{(0)}$ and $h_1 : X^{(1)} \to X$ the composition of f_1 with the embedding $U^{(0)} \hookrightarrow X$. If $f_i : X^{(i)} \to U^{(i-1)}$ and $h_i : X^{(i)} \to X^{(i-1)}$ are already defined, then we denote by Z_i the center of ν on $X^{(i)}$. We pick an open subset $U^{(i)} \subset X^{(i)}$ such that $Z_i \cap U^{(i)}$ is a smooth subvariety of $U^{(i)}$. If Z_i is not a divisor, then we denote by $f_{i+1} : X^{(i+1)} \to U^{(i)}$ the blow up of U_i along $Z_i \cap U^{(i)}$ and $h_{i+1} : X^{(i+1)} \to X^{(i)}$ the composition of f_{i+1} with the embedding $U^{(i)} \to X^{(i)}$. By [KM][Lemma 2.45], we know there is an integer $s \geq 0$ such that Z_s is a prime divisor on $U^{(s)}$ and $\mathrm{ord}_{Z_s} = \mathrm{ord}_E$. Hence we can replace Y by a smooth variety $U^{(s)}$ and $E = Z_s \cap U^{(s)}$. We write $g_i : Y \to X^{(i)}$ for the composition of morphisms h_j for j with $i < j \leq s$ and the embedding $U^{(s)} \subset X^{(s)}$.

Let C_s be the locally closed cylinder $\mathrm{Cont}^q(E)$ in Y_∞ and C_0 the closure of its image $(g_0)_\infty(C_s)$ in X_∞. It is clear that $\operatorname{codim} C_s = q$. We now show that $C = C_0$ is a cylinder that satisfies our conditions. For every i with $1 \leq i \leq s$, we denote by C_i the closure of the image of C_s in $X^{(i)}_\infty$ under the map $(g_i)_\infty : Y_\infty \to X^{(i)}_\infty$. Similarly, we denote by D_i the closure of the image of C_s in $U^{(i)}_\infty$. It is clear that D_i is the closure of the image of C_{i+1} in $U^{(i)}_\infty$ under the map $(f_{i+1})_\infty : X^{(i+1)}_\infty \to U^{(i)}_\infty$ and C_i is the closure of D_i in $X_\infty^{(i)}$. By Lemma V.6 and Lemma V.5 part (b), using descending induction on $i < s$, we deduce that D_i is a cylinder in $U_\infty^{(i)}$ and C_i is a cylinder in $(X_i)_\infty$. We also deduce that $\mathrm{ord}_{C_i} = \mathrm{ord}_{D_i} = \mathrm{ord}_{C_{i+1}}$ and

$$\operatorname{codim} C_i = \operatorname{codim} D_i = \operatorname{codim} C_{i+1} + \mathrm{ord}_{C_i} K_{X^{(i+1)}/X^{(i)}}.$$

We thus obtain $\mathrm{ord}_C = \mathrm{ord}_{C_1} = \cdots = \mathrm{ord}_{C_s} = q \cdot \mathrm{ord}_E$ and

$$\begin{aligned}\operatorname{codim} C \quad &= \operatorname{codim} C_1 + \mathrm{ord}_C(K_{X^{(1)}/X})\\ &\dots\\ &= \operatorname{codim} C_s + \sum_{i=0}^{s-1} \mathrm{ord}_C(K_{X^{(i+1)}/X^{(i)}}) = q + q \cdot \mathrm{ord}_E(K_{Y/X}).\end{aligned}$$

□

It is clear that Theorem I.4 follows from Lemma V.4 and Lemma V.7. We now prove Theorem I.5.

Proof. If $Y = X$, the assertion is trivial. Hence we may and will assume Y is a closed subscheme of X and $Y \neq X$. By Theorem I.4, we deduce that

$$\mathrm{lct}(X, Y) := \inf_E \frac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(Y)} = \inf_C \frac{\operatorname{codim} C}{\mathrm{ord}_C(Y)}.$$

where C varies over the irreducible closed cylinders which do not dominate X.

We first show that

$$(\dagger) \qquad \mathrm{lct}(X, Y) \leq \inf_{m \geq 0} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}$$

For every $m \geq 0$, let S_m be an irreducible component of Y_m which computes the codimension of Y_m in X_m and C_m the closed irreducible cylinder $\psi_m^{-1}(S_m)$ in X_∞. We thus obtain

$$\operatorname{codim}(C_m) = \operatorname{codim}(S_m, X_m) = \operatorname{codim}(Y_m, X_m).$$

The image $\psi_0(C_m) = \rho_0^m(S_m)$ is contained in Y, which implies that C_m does not dominate X. By the definition of contact loci, we know that $Y_m = \mathrm{Cont}^{\geq m+1}(Y)_m$ in X_m. This implies that $\mathrm{ord}_{C_m}(Y) \geq m+1$. We conclude that

$$\mathrm{lct}(X,Y) \leq \frac{\mathrm{codim}(C_m)}{\mathrm{ord}_{C_m}(Y)} = \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

Taking infimum over all integers $m \geq 0$, we now have the inequality (†).

We now prove the reverse inequality. Given an irreducible closed cylinder C which does not dominate X. If $\mathrm{ord}_C(Y) = 0$, then $\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} = \infty$. Hence

$$\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

From now on, we may and will assume that $\mathrm{ord}_C(Y) > 0$. Let $m = \mathrm{ord}_C(Y) - 1$. Since C is a subcylinder of the contact locus $\mathrm{Cont}^{\geq m+1}(Y) = \psi_m^{-1}(Y_m)$, we have

$$\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \frac{\mathrm{codim}(Y_m, X_m)}{m+1} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

We now take infimum over all cylinders C which do not dominate X and obtain

$$\mathrm{lct}(X,Y) = \inf_{C} \frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

□

Let X be a smooth variety over a perfect field k, Y a closed subscheme of X and Z a closed subset of X. Recall that

$$\mathrm{lct}_Z(X,Y) = \inf_{E/X} \frac{\mathrm{ord}_E(K_{-/X}) + 1}{\mathrm{ord}_E Y}$$

where E varies over all divisors over X whose center in X intersects Z. By the correspondence in Theorem I.4(2), we deduce that for every such divisor E over X, the corresponding closed irreducible cylinder C satisfies

$$\overline{\psi_0^X(C)} \cap Z \neq \emptyset.$$

Applying the argument in the proof of Theorem I.5, we can show the following generalized log canonical threshold formula in terms of jet schemes.

Proposition V.8. *Let (X,Y) be a pair over a perfect field k and Z a closed subset of X. We have*

$$\operatorname{lct}_Z(X,Y) = \inf_{C\subset X_\infty} \frac{\operatorname{codim} C}{\operatorname{ord}_C(\mathfrak{a})} = \inf_{m\geq 0} \frac{\operatorname{codim}_Z(Y_m, X_m)}{m+1}$$

where C varies over all irreducible closed cylinders with $\overline{\psi_0(C)} \cap Z \neq \emptyset, \overline{\psi_0(C)} \neq X$ and $\operatorname{codim}_Z(Y_m, X_m)$ *is the minimum codimension of an irreducible component T of Y_m such that $\overline{\pi_m(T)} \cap Z \neq \emptyset$.*

Remark V.9. We have seen that

$$\operatorname{lct}(X,Y) := \inf_E \frac{1+\operatorname{ord}_E(K_{-/X})}{\operatorname{ord}_E(Y)} = \inf_C \frac{\operatorname{codim} C}{\operatorname{ord}_C(Y)} = \inf_{m\geq 0} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}.$$

If one of the infumums can be achieved, then so are the other two. In particular, when the base field k is of characteristic 0, log resolutions of (X,Y) exist. Hence the log canonical threshold $\operatorname{lct}(X,Y)$ can be computed at some exceptional divisor E in the log resolution. In this case, all the infimums can be replaced by minimuns.

Remark V.10. Let k be an algebraically closed field of characteristic 0 and $K = k(s)$ the function field of $\mathbf{A}^1_k$. Hence K is not a perfect field. There are examples of pairs (X,Z) over K such that the formula in Theroem I.5 does not hold. For instance, let $X = \operatorname{Spec} K[x]$ and Y be a prime divisor on X defined by a single equation $(x^p - s)$. It is easy to check that $\operatorname{lct}(X,Y) = 1$. On the other hand, for every m, $X_m = \mathbf{A}^{m+1}$ and $Y_m = \mathbf{A}^{m-\lfloor \frac{m}{p} \rfloor}$. Hence $\inf\limits_{m\geq 0} \frac{\operatorname{codim}(Y_m,X_m)}{m+1} = 1/p$. We thus have $\operatorname{lct}(X,Y) \neq \inf\limits_{m\geq 0} \frac{\operatorname{codim}(Y_m,X_m)}{m+1}$.

5.2 The log canonical threshold via jets

In this section, we apply Theorem I.5 to deduce properties of log canonical threshold for pairs. Our first corollary of Theorem I.5 is the following comparison result in the setting of reduction to prime characteristic. Suppose that X is the affine variety $\mathbf{A}^n_{\mathbf{Z}}$ over the ring $\mathbf{Z}$ and Y is a subscheme of X defined by an ideal $\mathfrak{a} \subset \mathbf{Z}[x_1, \cdots, x_n]$ that contained in the ideal $(x_1, \cdots, x_n)$. For every prime number p, let $X_p = \mathbf{A}^n_{\mathbf{F}_p}$ and Y_p be the subscheme of X_p defined by $\mathfrak{a} \cdot \mathbf{F}_p[x_1, \cdots, x_n]$. Note that a log resolution of $(X_{\mathbf{Q}}, Y_{\mathbf{Q}})$ induces a log resolution of the pair (X_p, Y_p) for p large enough. It follows that $\mathrm{lct}_0(Y_{\mathbf{Q}}, X_{\mathbf{Q}}) = \mathrm{lct}_0(Y_p, X_p)$ for all but finitely many p. We now prove the following inequality for every prime p.

Corollary V.11. *If (X, Y) is a pair as above, then for every prime integer p, we have*

$$\mathrm{lct}_0(X_{\mathbf{Q}}, Y_{\mathbf{Q}}) \geq \mathrm{lct}_0(X_p, Y_p),$$

where the log canonical thresholds are computed at the origin.

Proof. Using [Mus2][Corollary 3.6], we obtain

$$\mathrm{lct}_0(Y_{\mathbf{Q}}, X_{\mathbf{Q}}) = \inf_{m \geq 0} \frac{\mathrm{codim}((Y_{\mathbf{Q}})_{m,0}, (X_{\mathbf{Q}})_m)}{m+1}.$$

By Proposition V.8, for every integer $m \geq 0$, we have

$$\mathrm{lct}_0(X_p, Y_p) \leq \frac{\mathrm{codim}_0((Y_p)_m, (X_p)_m)}{m+1} \leq \frac{\mathrm{codim}((Y_p)_{m,0}, (X_p)_m)}{m+1}.$$

In order to complete the proof, it is enough to show that for every $m \geq 1$ and every prime p,

$$\mathrm{codim}((Y_p)_{m,0}, (X_p)_m) \leq \mathrm{codim}((Y_{\mathbf{Q}})_{m,0}, (X_{\mathbf{Q}})_m).$$

Since $\dim(X_{\mathbf{Q}})_m = \dim(X_p)_m = n(m+1)$, it suffices to show that

$$\dim(Y_p)_{m,0} \geq \dim(Y_{\mathbf{Q}})_{m,0}$$

for every p.

Let S be $\operatorname{Spec}\mathbf{Z}$. Recall that $(Y/S)_m$ is the m^{th} relative jet scheme of Y/S. Since $\mathfrak{a}\subset(x_1,\cdots,x_n)$, the zero map $\operatorname{Spec}\mathbf{Z}\to X$ factors through Y. Let $\tau:\operatorname{Spec}\mathbf{Z}\to Y$ be the zero section. By Lemma III.7, we deduce that for every $m\geq 1$, the function $f(s)=\dim(Y_s)_{m,\tau(s)}=\dim(Y_s)_{m,0}$ is upper semi-continuous on S. Hence we have $\dim(Y_p)_{m,0}\geq\dim(Y_{\mathbf{Q}})_{m,0}$ for every m and p. This completes the proof. $\square$

This in turn has an application to an open problem about the connection between log canonical thresholds and F-pure thresholds. Recall that in positive characteristic Takagi and Watanabe [TW] introduced an analogue of the log canonical threshold, the F-pure threshold. With the above notation, it follows from [HW] that $\operatorname{lct}_0(X_p,Y_p)\geq\operatorname{fpt}_0(X_p,Y_p)$ for every prime p, where $\operatorname{fpt}_0(X_p,Y_p)$ is the F–pure threshold of the pair (X_p,Y_p) at 0. By combining this with Corollary V.11, we obtain the following result, which seems to have been an open question.

Corollary V.12. *With the above notation, we have* $\operatorname{lct}_0(X_{\mathbf{Q}},Y_{\mathbf{Q}}))\geq\operatorname{fpt}_0(X_p,Y_p)$ *for every prime* p.

Let k be a perfect field and $\overline{k}$ be the algebraic closure of k. For every scheme X over k, we denote by $\overline{X}$ the fiber product $X\times_k\operatorname{Spec}\overline{k}$.

Corollary V.13. *Let X be a smooth variety over a perfect field k and Y a closed subscheme of X. We have*

$$\operatorname{lct}(X,Y)=\operatorname{lct}(\overline{X},\overline{Y}).$$

Proof. For every scheme Z over field k, we know that $\dim Z=\dim\overline{Z}$. We thus have for every $m\geq 0$,

$$\operatorname{codim}(Y_m,X_m)=\operatorname{codim}(\overline{Y}_m,\overline{X}_m).$$

Our assertion follows from Theorem I.5. $\square$

Remark V.14. Corollary is not true if the base field is not perfect. For instance, let k be an algebraically closed field and $K = k(s)$ the function field of $\mathbf{A}^1_k$. Let $X = \operatorname{Spec} K[x]$ and Y be the closed subscheme of X defined by $x^p - s$. We have seen that $\operatorname{lct}(X, Y) = 1$. Let $\overline{K}$ be the algebraic closure of K. We thus have $X_{\overline{K}} = \mathbf{A}^1_{\overline{K}}$ and $Y_{\overline{K}}$ is a nonreduced subscheme of $X_{\overline{K}}$ defined by $(x - s^{1/p})^p$. One can check that $\operatorname{lct}(X_{\overline{K}}, Y_{\overline{K}}) = 1/p$.

Corollary V.15. *Let X be a smooth variety over a perfect field k and Y a closed subscheme of X. If H is a smooth irreducible divisor on X which intersects Y and $Z \subset H$ is a nonempty closed subset, then*

$$\operatorname{lct}_Z(X, Y) \geq \operatorname{lct}_Z(H, H \cap Y).$$

Proof. The case $H \cap Y = H$ is trivial since $\operatorname{lct}_Z(H, H \cap Y) = 0$. We may thus assume $Y \cap H \neq H$. Similarly, if $Z \cap Y = \emptyset$, then both $\operatorname{lct}_Z(X, Y)$ and $\operatorname{lct}_Z(H, H \cap Y)$ are equal to ∞. We should assume $Z \cap Y \neq \emptyset$ from now on.

By Proposition V.8, we only have to prove that for every $m \geq 0$,

$$\operatorname{codim}_Z(Y_m, X_m) \geq \operatorname{codim}_Z((H \cap Y)_m, H_m).$$

Let T be an irreducible component of Y_m such that

$$\pi_m(T) \cap Z \neq \emptyset \text{ and } \operatorname{codim} T = \operatorname{codim}_Z(Y_m, X_m).$$

Since H is a Cartier divisor on X, $H \cap Y$ is defined locally in Y by one equation. This implies that $(H \cap Y)_m = H_m \cap Y_m$ is defined locally in Y_m by $m+1$ equations. If $\pi_m(T \cap H_m) \cap Z \neq \emptyset$, then there is a component of $T \cap H_m$, denoted by S, such that $\pi_m(S) \cap Z \neq \emptyset$ and $\dim S \geq \dim T - (m+1)$. Note that $\dim X_m = \dim H_m + m + 1$

and we conclude that

$$\operatorname{codim}_Z((H \cap Y)_m, H_m) \leq \operatorname{codim}(S, H_m) \leq \operatorname{codim}(T \cap H_m, H_m) \leq \operatorname{codim}(T, X_m).$$

We now prove that $\pi_m(T \cap H_m) \cap Z \neq \emptyset$. Let $\gamma_m \in T$ such that $\pi_m(\gamma_m) \in Z$. Recall that $\sigma_m : Y \to Y_m$ is the zero section. Since T is invariant under the action of $\mathbf{A}^1$, the orbit of γ_m is a subset of T. In particular, $\sigma_m(\pi_m(\gamma_m)) \in T$. Since the zero section is functorial by its construction, we get $\sigma_m(Y \cap H) \subset Y_m \cap H_m$. In particular, $\sigma_m(\pi_m(\gamma_m))$ is in $T \cap H_m$ and its image under π_m is in Z. This completes our proof. □

Corollary V.16. *If X is a smooth complex variety and $Y \subset X$ is a proper closed subscheme, then for we have* $\operatorname{lct}(X, Y) > 0$*.*

Proof. Since log canonical thresholds can be computed after passing to an algebraic closure of k, we can assume k is algebraically closed. It follows from the definition that $\operatorname{lct}(X, Y) = \inf_{x \in Y} \operatorname{lct}_x(X, Y)$. For every $x \in Y$, we will show that

$$\operatorname{lct}_x(X, Y) \geq 1/\operatorname{ord}_x(Y). \tag{5.6}$$

We thus have $\operatorname{lct}_x(X, Y) \geq 1/d$ where $d = \max_{x \in Y}\{\operatorname{ord}_x Y\}$. Here $\operatorname{ord}_x Y$ is the maximal value q such that $I_{Y,x} \subseteq m_{X,x}^q$, where $m_{X,x}$ is the ideal defining x.

We prove the inequality (5.6) by induction on $\dim(X)$. If X is a smooth curve, then it follows from definition that $\operatorname{lct}_x(X, Y) = \operatorname{ord}_x Y$. We now assume that $\dim X \geq 2$. After replacing X by an open neighborhood of x, we may find H, a smooth divisor passing through x, such that $\operatorname{ord}_x(H \cap Y) = \operatorname{ord}_x Y$. By Corollary V.15, we have

$$\operatorname{lct}_x(X, Y) \geq \operatorname{lct}_x(H, H \cap Y) \geq 1/\operatorname{ord}_x(H \cap Y) = 1/\operatorname{ord}_x Y.$$

This completes the proof. □

BIBLIOGRAPHY

BIBLIOGRAPHY

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths and J.Harris, Geometry of Algebraic Curves I, Grundlehren der mathematischen Wissenschaften, A series of Comprehensive Studies in Mathematics, vol **267**, Springer-Verlag, New York, 1985.

[Bli] M. Blickle, A short course on geometric motivic integration, Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I, 189C243, London Math. Soc. Lecture Note Ser., **383**, Cambridge Univ. Press, Cambridge, (2011).

[Doc] R. Docampo Álvarez, Arcs on Determinantal Varieties, Ph.D. Thesis, Univeristy of Illinois at Chicago, 2009, arxiv: math/1011.1930v2.

[DL] J. Denef and F. Loeser, Germs of arcs on singular algebraic varieties an motivic integration, Invent. Math. **135** (1999), 201–232.

[Eis] D. Eisenbud, Commutative Algebra with a View Torward Algebraic Geoemtry, Graduate Texts in Mathematics **150**, Springer Verlag, (1995).

[Elk] R. Elkik, Rationalité des singularités canoniques, Invent. Math. **64**, 1981, 1-6.

[EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular vatieties, J. Amer. Math. Soc. 10 (1997), 243–258.

[EM] L. Ein, M. Mustaţă, Jet schemes and singularities, Algebraic geometry–Seattle 2005, Part **2**, 505–546, Proc. Sympos. Pure Math., **80**, Part **2**, Amer. Math. Soc., Providence, RI, 2009.

[ELM] L. Ein, R. Lazarsfeld, and M. Mustaţă, Contact loci in arc spaces, Compos. Math. **140** (2004), 1229–1244.

[EMY] L. Ein, M. Mustaţă and T. Yasuda, Jet schemes, log discrepancies and inversion of adjunction, Invent. Math. **153**, (2003),519–535.

[FEI] T. De Fernex, L. Ein and S. Ishii, Divisorial valuation via arcs, Publ. RIMS **44** (2008), 425–448.

[Gie] D. Gieseker, Stable curves and special divisors, Invent. Math. **66** (1982), 251–275.

[Har] R. Hartshorne, Algebraic Geometry, Graduate Texts in Mathematics, **52**, Springer-Verlag, New York, 1977.

[Hir] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II, Ann. of Math. (2) **79** (1964), 109C326.

[HW] N. Hara and K.-i. Watanabe, F-regular and F-pure rings vs. log terminal and log canonical singularities, J. Algebraic Geom. **11** (2002), 363–392.

[Ish] S. Ishii, Arcs, valuatinons and the Nash map, J. reine angew. Math. **588**(2005), 71-92

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. (2) **79** (1964), 109–326.

[Kem] G. Kempf, On the geometry of a theorem of Riemann, Ann. of Math. **98** (1973), 178–185.

[Kon] M. Kontsevich, Motivic integration. Lecture at Orsay, 1995.

[Kol] J. Kollár, Singularities of pairs, Algebraic geometrySanta Cruz 1995, Proc. Sympos. Pure Math., vol. **62**, Amer. Math. Soc., Providence, RI, 1997, pp. 221C287.

[KM] J. Kollár and S. Mori, Birational Geometry of Algebraic Varieties, Cambridge University Press, 1998.

[Lar] R. Lazarsfeld, Positivity in Algebraic Geometry II, Positivity for Vector Bundles, and Multiplier Ideals,Ergebnisse der Mathematik und Grenzgebiete, **3**, Folge, Springer-Verlag, Berlin, 2004.

[Mum] D. Mumford, Lectures on Curves on an Algebraic Surface, Annals of Mathematics Studies, **59**, Princeton University Press, New Jersey, 1966.

[Mus1] M. Mustaţă, Jet schemes of locally complete intersection canonical singularities, with an appendix by David Eisenbud and Edward Frenkel, Invent. Math., **145** (2001), 397–424.

[Mus2] M. Mustaţă, Singularities of pairs via jet schemes, J. Amer. Math. Soc. **15** (2002), 599–615.

[TW] S. Takagi and K.-i. Watanabe, On F-pure thresholds, J. Algebra **282** (2004), 278–297.