On The Totally Geodesic Commensurability Spectrum of Arithmetic Locally Symmetric Spaces

by

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Introduction

1.1. Grasping At Shapes In The Dark.

Imagine you are in a darkened room and two smooth rigid metallic sculptures are placed in front of you. You cannot see them, and have no prior knowledge of what they look like, yet you are asked to determine if the two shapes are the same or different. So in the darkness, what can you do? Perhaps you will walk over to the shapes and start to run your fingers across their surfaces. You might first notice the size of each object. You reason that if one clearly has greater volume, you can say with confidence that they are different shapes. If this is inconclusive, you might next start counting how many distinct holes, like that which appear through a handle of a coffee mug, each have. Again you reason that if the two shapes have a different number of holes, then you may say with confidence the two are different. However, if these two tests are inconclusive, what else might you do? In the darkness, what other tools are at your disposal? You know they are metallic, so you bring over a hammer and strike one, carefully listen to the sound it makes and then repeat with the other sculpture. You reason that if the two make different sounds, then again you can determine they must be different, since the same shape should produce the same sound. If this does not work, you remember all the holes you counted earlier and, all the handles they made, and get your lasso. You toss and tighten your lasso around each handle of the first sculpture, making sure that it has come to rest in a way that if you wiggle it, you cannot make the loop any smaller. You record the lengths of all these loops and then move on to the second sculpture. You reason that if there is a length of a tightened lasso loop on one that is not the length of a tightened lasso loop on the other, then again you can determine they must be different, since the same shape should have the same lasso loop lengths. Yet if this is inconclusive, then what next?

While this scenario may initially seem rather synthetic, and you might just say, "Turn on the lights" this is actually a rather powerful analogy for what geometers do. Often geometers are confronted with a geometric shape in their work, and need to know which shape they are dealing with. However, just like you in the darkened room, the mathematician cannot see this shape, which may exist in, say, 26 dimensions, and instead, like you feeling through the darkness, must instead rely on other data, such as volume and number of holes to identify the shape in front of them. However when these numbers are the same, we are forced to look for ever more refined pieces of data. So we might next record the shape's "sound." Mathematically, sound is a collection of frequencies, so not one number, but a collection of infinitely many numbers. We call such a collection a **spectrum**. In **spectral geometry** we assign (possibly infinite) collections of data to geometric shapes in the hope that, if we can tell apart their spectra, we can then conclude we have two different shapes. So to every shape, in addition to assigning it a volume and number of holes, we may assign to it its "sound spectrum" and its "lasso length spectrum" (which when we make everything more precise will be called the **Laplace spectrum** and **length spectrum**, respectively).

The main point of this thesis is to introduce a new spectrum, a higher dimensional analogue of the lasso spectrum and then examine what this says about certain classes of common spaces. This new spectrum will look at the way higher dimensional nets can be cast and tightened in our space. We will show that this spectrum carries enough data to be able to tell apart many classes of spaces, including spaces for which their sound and lasso spectrum could not tell them apart.

1.2. Can One Hear The Shape Of A Drum?

In this thesis, the smooth rigid shapes being considered are **Riemannian manifolds**. A Riemannian manifold is a pair $(M, \langle -,- \rangle)$, where M is a smooth manifold and $\langle -,- \rangle$ is a map, called a Riemannian metric, that associates with each point $p \in M$ an inner product $\langle -,- \rangle_p$ on the tangent space T_pM. A diffeomorphism $\varphi : M \to M$ is an **isometry** if

$$\langle v, w \rangle_p = \langle (d_p\varphi)v, (d_p\varphi)w \rangle_{\varphi(p)} \quad \text{for all } p \in M \text{ and } v, w \in T_pM.$$

When the metric is clear from context, we will suppress writing it and simply refer to M as a Riemannian manifold. For more on the basic theory of Riemannian manifolds, we refer the reader to [DC]. A Riemannian metric naturally endows a Riemannian manifold with a measure μ, and hence it makes sense to integrate functions on Riemannian manifolds. We define the space of square integrable functions on M to be

$$L^2(M) = \left\{ f : M \to \mathbb{R} \mid \int_M |f|^2 d\mu < \infty \right\}.$$

For such functions, we may define a linear operator called the **Laplace operator**

$$\Delta : L^2(M) \to L^2(M).$$

Example 1.2.1. If $M = \mathbb{R}^3$ and $f \in C^2(\mathbb{R}^3)$, then $\Delta(f(x,y,z)) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$.

The **Laplace spectrum** of a Riemannian manifold M, is the set

$$\mathscr{L}\mathscr{P}(M) := \left\{ (\lambda, n) \in \mathbb{R} \times (\mathbb{Z}_{>0} \cup \{\infty\}) \;\middle|\; \begin{array}{l} \lambda \text{ is an eigenvalue of the Laplace operator } \Delta \\ \text{on } L^2(M) \text{ together with its multiplicity } n. \end{array} \right\}.$$

Two manifolds with the same Laplace spectrum are said to be **isospectral**. The numbers in the Laplace spectrum are related to the frequencies produced if the manifold M were to vibrate. As such, the Laplace spectrum for a wide variety of spaces has been studied for decades. For a detailed treatment of the Laplace spectrum, see [Ch].

A **closed geodesic** on a Riemannian manifold is a locally length minimizing closed loop, which you may think of as our tightened lasso loop. The **weak length spectrum** of a Riemannian manifold M, is the set

$$L(M) := \{\lambda \in \mathbb{R} \mid \lambda \text{ is the length of a closed geodesic in } M\}.$$

For certain spaces, the Laplace spectrum and weak length spectrum are deeply related. McKean outlines this connection in [McK] for compact Riemann surfaces. The following theorem has been known for some time and was finally written down in ([PR09] Theorem 10.1).

Theorem 1.2.2. *Let M_1 and M_2 be two compact locally symmetric spaces with nonpositive sectional curvatures. If $\mathscr{L}\mathscr{P}(M_1) = \mathscr{L}\mathscr{P}(M_2)$, then $L(M_1) = L(M_2)$.*

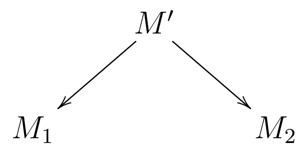
For much more the relationship between $L(M)$, and $\mathscr{L}\mathscr{P}(M)$ and similar spectra, see [LMNR]. Given this data, one may ask the following question.

Question 1: Does $\mathscr{L}\mathscr{P}(M)$ or $L(M)$ determine the isometry class of M?

This question was popularize by Mark Kac's famous question "Can one hear the shape of a drum?" [Kac]. In 1964, Milnor produced two 16-dimensional flat tori which were isospectral but not isometric [Mi]. In 1980, Vignéras produced spaces in arbitrarily high dimension, as well as 2- and 3-dimensional hyperbolic manifolds, which were isometric but not isospectral [Vig]. In 1985, Sunada discovered a more general method for constructing large classes of isospectral nonisometric manifolds [Su]. With all these negative results, we cannot expect these spectra to determine isometry class. However, the procedures used in these three papers always produce manifolds which are *almost* isometric in a sense that they are commensurable, a notion we will discuss in the next section.

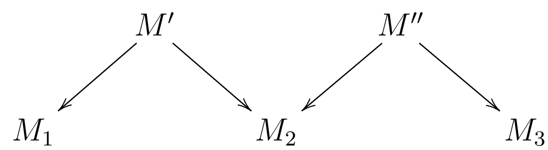
1.3. Length Commensurability

Two manifolds M_1 and M_2 are said to be **commensurable** if there exists a manifold M' which is a finite sheeted cover of both M_1 and M_2.



Proposition 1.3.1. *Commensurability is an equivalence relation among path connected, locally path connected, semilocally simply connected topological spaces.*

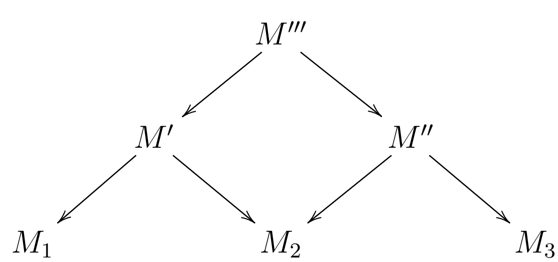
Proof. Commensurability is clearly reflexive and symmetric. It suffices to show it is transitive, which we show using the fundamental group. Our assumptions on the set of spaces considered guarantees a bijection between isomorphism classes of path connected covering spaces and conjugacy classes of subgroups of the fundamental group ([Ha] Theorem 1.38). Suppose M_1 and M_2 are commensurable and M_2 and M_3 are commensurable.



Then $\pi_1(M')$ and $\pi_1(M'')$ are finite index subgroups of $\pi_1(M_2)$. It is a well known result from basic group theory that in such cases

$$[\pi_1(M_2) : \pi_1(M') \cap \pi_1(M'')] \leq [\pi_1(M_2) : \pi_1(M')][\pi_1(M_2) : \pi_1(M'')].$$

Let M''' be the path connected covering space of M_2 correspond to the subgroup $\pi_1(M') \cap \pi_1(M'')$. It follows that M''' is a finite sheeted cover of both M' and M'' and the result follows.



$\square$

In particular, this proposition applies to Riemannian manifolds. Observe that if two Riemannian manifolds M_1 and M_2 are commensurable, then every length of a geodesic on M_1 is a rational multiple of a geodesic on M_2, and vice versa. Motivated by this, we define the **rational length spectrum** to be the set

$$\mathbb{Q}L(M) := \{s\lambda \in \mathbb{R} \mid s \in \mathbb{Q} \text{ and } \lambda \text{ is the length of a closed geodesic in } M\}.$$

If two manifolds have the same rational length spectrum, we say they are **length commensurable**. In particular, commensurable manifolds are length commensurable. Given these new definitions, we may refine our original question.

Question 2: Does $\mathbb{Q}L(M)$ determine the commensurability class of M?

Now we start to get some positive results. Three major results on the topic have been:

Theorem 1.3.2 (Reid [Re92] 1992)**.** *Let M_1 and M_2 be arithmetic hyperbolic 2-manifolds. Then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ implies M_1 and M_2 are commensurable.*

Theorem 1.3.3 (Chinburg, Hamilton, Long, and Reid [CHLR] 2008)**.** *Let M_1 and M_2 be arithmetic hyperbolic 3-manifolds. Then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ implies M_1 and M_2 are commensurable.*

Theorem 1.3.4 (Prasad and Rapinchuk [PR09] 2009, and Garibaldi [Ga] 2012)**.**

1. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from connected absolutely simple real algebraic groups of the same type different from A_n, D_{2n+1}, with $n > 1$, and E_6. Then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ implies M_1 and M_2 are commensurable.*

2. *There exist examples of noncommensurable arithmetic locally symmetric spaces M_1 and M_2 coming from connected absolutely simple real algebraic groups of the same type A_n, D_{2n+1}, with $n > 1$, and E_6 for which $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$*

Corollary 1.3.5. *If M_1 and M_2 are arithmetic hyperbolic n-manifolds where $n \not\equiv 1 \pmod 4$, then $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$ implies that they are commensurable. However, for each positive $n \equiv 1 \pmod 4$ greater than 1, there exist examples of noncommensurable length commensurable arithmetic hyperbolic manifolds.*

It should be noted that these results hold unconditionally for $\mathbb{R}$-rank one groups but the results of [PR09] and [Ga] are conditional upon the truth of Schanuel's conjecture for higher $\mathbb{R}$-rank. We now wish to find some additional geometric data that:

1. can distinguish commensurability classes when $\mathbb{Q}L(M)$ does not,

2. can distinguish commensurability classes coming from nonsimple semisimple groups, and

3. are not conditional upon Schanuel's conjecture.

We shall begin to do so in the next section.

1.4. Totally Geodesic Commensurability

Totally geodesic subspaces of Riemannian manifolds are higher dimensional analogues of geodesics, which are one dimensional. In Chapter 2 we discuss totally geodesic submanifolds in great depth. Results of McReynolds and Reid on totally geodesic surfaces of hyperbolic 3-manifolds [MR] give hope that the collection of totally geodesic subspaces can determine commensurability class. In analogy to the weak length spectrum, we define the **weak totally geodesic spectrum** of a Riemannian manifold to be the set

$$TG(M) = \left\{ \begin{array}{c} \text{Isometry classes of nonflat finite volume} \\ \text{totally geodesic submanifolds of } M \end{array} \right\}.$$

With this definition, we may state McReynolds and Reid's result:

Theorem 1.4.1 (McReynolds & Reid [MR]). *Let M_1 and M_2 be arithmetic hyperbolic 3-manifolds. If $TG(M_1) = TG(M_2)$ then either this set is empty or M_1 and M_2 are commensurable.*

In fact they prove a slightly stronger result, where they only need to assume they have the same isometry classes up to homotopy in the moduli space of isometry classes of the surface. That being said, $TG(M)$ is not an invariant of commensurability class, and hence we define the **totally geodesic commensurability spectrum** to be the set

$$\mathbb{Q}TG(M) = \left\{ \begin{array}{c} \text{Commensurability classes of nonflat finite} \\ \text{volume totally geodesic submanifolds of } M \end{array} \right\}.$$

These are natural analogues of $L(M)$ and $\mathbb{Q}L(M)$. The former is more rigid while the later is an invariant of the commensurability class of M.

The goal of this thesis is to investigate the following question:

Question 3: Does $\mathbb{Q}TG(M)$ determine the commensurability class of M?

We will focus primarily on arithmetic spaces coming from quadratic forms over number fields (see Chapter 4), however we will also address some results on spaces coming from skew hermitian forms over quaternion division algebras over number fields. We begin by showing that $\mathbb{Q}TG(M)$ determines the field of definition.

Theorem A *Let M_1 and M_2 be arithmetic locally symmetric spaces coming quadratic forms of dimension $m \geq 4$ over number fields k_1 and k_2 respectively. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies $k_1 = k_2$.*

We next show that for all spaces coming from quadratic forms, $\mathbb{Q}TG(M)$ determines the commensurability class of M.

Theorem B. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms of dimension $m \geq 5$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies M_1 and M_2 are commensurable.*

Specializing to the even dimensional $\mathbb{R}$-rank 1 case, Theorem B gives Theorem C.

Theorem C. *Let M_1 and M_2 be even dimensional arithmetic hyperbolic manifolds of dimension $n \geq 4$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies M_1 and M_2 are commensurable.*

It is worth noting that Theorem B holds for any $\mathbb{R}$-rank, and unlike [PR09] and [Ga] is not dependent upon the truth of Schanuel's conjecture. It is also worth noting that groups of type B_n and D_n over number fields may produce a Lie group which is not absolutely simple over $\mathbb{R}$, hence these results cover a large class of spaces not covered under the results of Prasad and Rapinchuk.

Though there are considerable obstructions to finishing the analysis for groups coming from skew hermitian forms over division algebra, we do have the following partial result.

Theorem D. *Let M_1 and M_2 be arithmetic locally symmetric spaces where M_1 comes from a quadratic form of dimension $m = 2n$ and M_2 comes from a skew hermitian form of dimension n over a division algebra. Then $\mathbb{Q}TG(M_1) \neq \mathbb{Q}TG(M_2)$.*

To prove these results, we must first establish some geometric and algebraic results. In Chapter 2 we develop the necessary results from Riemannian geometry to define and analyze $\mathbb{Q}TG(M)$. In Chapter 3 we develop the algebraic theory of quadratic forms over local and global fields. In particular, we introduce the classical invariants of forms over local fields and we then state the uniqueness and existence theorems of quadratic forms over local and global fields.

In Chapter 4 we introduce and analyze arithmetic subgroups of $\mathbb{Q}$-groups, arithmetic lattices in semisimple Lie groups, and arithmetic locally symmetric spaces, which allows us to turn our quadratic forms into locally symmetric spaces. While many of the results on arithmetic locally symmetric spaces in this chapter are known, as of the time of writing this, we are unaware of references for them. As such, we state and prove many fundamental results on arithmetic locally symmetric spaces in the hope that this chapter will be a valuable reference for future research.

In Chapter 5 we analyze the Tits index of groups coming from quadratic forms over a number field and create a dictionary between the local indices of such groups and the local invariants of the associated forms. This dictionary enables us in Section 5.4 to rederive Maclachlan's parametrization of commensurability classes of even dimensional arithmetic hyperbolic spaces (see [Mac]). Furthermore, when considering locally symmetric spaces coming from quadratic forms, this dictionary enables us to use the theory of quadratic forms to analyze when one such space sits as a totally geodesic subspace of another such space.

In Chapter 6 we use the results of Chapter 3 and Chapter 5 to construct quadratic subforms whose isometry groups are not isomorphic to subgroups of certain other isometry groups. It is these subforms which enable us to distinguish between the totally geodesic spectra coming from noncommensurable arithmetic locally symmetric spaces coming from quadratic forms. In Chapter 7 we draw upon the results of the previous chapters to prove Theorems A-D.

Chapter 2

Totally Geodesic Subspaces

In this chapter, we establish basic results on totally geodesic subspaces and totally geodesic spectra. In Section 2.1 we give basic results on totally geodesic subspaces. In Section 2.2 we establish basic properties of the totally geodesic commensurability spectrum. In Section 2.3 we develop the theory of locally symmetric spaces, which are the types of spaces we shall focus on for the remainder of this thesis.

2.1. Preliminaries On Totally Geodesic Subspaces

Let M be a Riemannian manifold and let $N \subset M$ be a connected immersed submanifold. We say that N is **geodesic at** $p \in N$ if one of the following equivalent conditions are met:

(G1) The second fundamental form vanishes at p. ([DC] p. 132)

(G2) A geodesic of N starting at p is a geodesic of M. ([DC] p. 132)

(G3) A geodesic of M starting at p and tangent to N at p is a geodesic of N. ([He] p. 79)

We quickly show the equivalence of these definitions. The equivalence of definitions (G1) $\Leftrightarrow$ (G2) is in [DC] p. 132, Proposition 2.9 and (G3) $\Rightarrow$ (G2) is Lemma 14.3, in [He]. For (G2) $\Rightarrow$ (G3), pick some $X \in T_pN \subset T_pM$ and let γ_M be the geodesic of M in the direction of X. Let γ_N be the geodesic of N in the direction of a fixed $X \in T_pN$. Since γ_N is a geodesic in M, it follows by local uniqueness, that $\gamma_M = \gamma_N$ and hence γ_M lies in N.

If N is geodesic at each of its points it is called **totally geodesic**.

Example 2.1.1. We now present some examples of totally geodesic submanifolds of common spaces.

1. **Euclidean space**. Let N be a k-dimensional closed, totally geodesic submanifold of $\mathbb{R}^n$, $k \leq n$, then N is isometric to $\mathbb{R}^k$. In other words, the totally geodesic subspaces are just affine subspaces.

2. **Spheres**. Let N be a k-dimensional closed, totally geodesic submanifold of the n-sphere $\mathbb{S}^n$, $k \leq n$, then N is isometric to $\mathbb{S}^k$. In other words totally geodesic submanifolds of Euclidean space are "great spheres" or more precisely the intersection of $S^n \subset \mathbb{R}^{n+1}$ with hyperplanes through the origin. ([DC] Ch. 6. p. 133)

3. **Hyperbolic space**. Let N be a k-dimensional closed, totally geodesic submanifold of $\mathbb{H}^n$, $k \leq n$, then N is isometric to $\mathbb{H}^k$. ([DC] Ch. 8. ex 2 p. 180)

We now wish to understand the behavior of totally geodesic submanifolds with respect to covers.

Proposition 2.1.2. *Let M be a connected Riemannian manifold, and let $N \subset M$ be a totally geodesic submanifold. Let M' be a connected cover of M and let N' be a connected component of the preimage of N in M'. Then N' is totally geodesic in M'.*

Proof. Let γ be a geodesic of M' going through a point $p' \in N'$ which is tangent to N' at this point. Let $\pi : M' \to M$ denote the covering map. Then $\pi(\gamma)$ is a geodesic in M tangent to N at $\pi(p')$, hence $\pi(\gamma)$ lies in N. We conclude γ lies in N'. By condition (G3), it follows that N' is geodesic at all points and the result follows. □

This next result says that the problem of understanding totally geodesic submanifolds can be restated in terms of understanding totally geodesic submanifolds of the universal cover.

Corollary 2.1.3. *Let M be a connected Riemannian manifold and $\pi : \widetilde{M} \to M$ its universal cover. A connected submanifold $N \subset M$ is totally geodesic if and only if a connected component $\widetilde{N}$ of $\pi^{-1}(N)$ is totally geodesic. Furthermore, N is flat if and only if $\widetilde{N}$ is flat.*

We should note at this time that while being totally geodesic and being flat is preserved under covers, another important property, being of finite volume is not. This will prove to be important in following sections.

In this thesis we will also encounter **Riemannian orbifolds** and will want to make sense of their totally geodesic subspaces. An orbifold is a space where the coordinate charts are defined by the quotient of an open set in $\mathbb{R}^n$ modulo the action of a finite group. For our purposes, we need only consider Riemannian orbifolds which arise a quotients of Riemannian manifolds by discrete groups of isometries. As such, all the orbifolds we consider have a globally symmetric manifold as it orbifold universal cover. For a general treatment of orbifolds, see [Sco]. A subspace of an orbifold is then defined to be totally geodesic if it is the image of a totally geodesic subspace in its universal cover.

Example 2.1.4. Here we present some orbifolds and their totally geodesic submanifolds.

1. Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere and let Γ be a rational rotation about the z-axis. Then $\Gamma \backslash S^2$ is an orbifold with cone points at the north and south poles. Totally geodesic subspaces are the images of the "great circles."

2. Let $\mathbb{H}^2$ be the hyperbolic plane and let $\Gamma = PSL_2(\mathbb{Z})$ acting on $\mathbb{H}^2$ by Möbius transformations. Then $PSL_2(\mathbb{Z})\backslash\mathbb{H}^2$ is an orbifold with two singular points corresponding to the torsion elements $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Totally geodesic subspaces are simply the images of geodesic of the hyperbolic plane.

A Riemannian manifold may admit automorphisms with fixed points. The quotient space is then not a manifold, but an orbifold. Thus the commensurability class will naturally include orbifolds in addition to manifolds.

2.2. Totally Geodesic Spectra

In this section we define spectra of totally geodesic subspaces. There are many collections we could choose and we now explain why we settle on two particular collections. Our motivation is to find a collection of data that is complimentary to length spectra and hence we do not want to include the length spectra. As such, we will only include *nonflat* totally geodesic subspaces. Furthermore, in analogy to looking at closed geodesics, we only want to look at *finite volume* subspaces.

Given these considerations, we define the **weak totally geodesic spectrum** of a Riemannian orbifold to be the set

$$TG(M) = \left\{ \begin{array}{c} \text{Isometry classes of nonflat finite volume} \\ \text{totally geodesic subspaces of } M \end{array} \right\}.$$

Commensurability is well-behaved with respect to totally geodesic subspaces. If M is a Riemannian orbifold, we define its **totally geodesic commensurability spectrum** to be the set

$$\mathbb{Q}TG(M) = \left\{ \begin{array}{c} \text{Commensurability classes of nonflat finite} \\ \text{volume totally geodesic subspaces of } M \end{array} \right\}.$$

This is a natural analogue of the rational length spectrum $\mathbb{Q}L(M)$ defined in the introduction. If two Riemannian orbifolds M and M' have the same totally geodesic commensurability spectrum, we say they are **totally geodesic commensurable**.

Proposition 2.2.1. *Commensurable Riemannian manifolds are totally geodesic commensurable.*

Proof. Let M_1 and M_2 be commensurable and $\widetilde{M}$ be a shared finite sheeted cover with projections π_1 and π_2. Pick a nonflat finite volume totally geodesic submanifold $N_1 \subset M_1$. Then $N_2 := \pi_2(\pi_1^{-1}(N_1))$ is a totally geodesic submanifold of M_2. Since we are dealing with finite covers, N_2 is also nonflat and of finite volume. By symmetry of argument, the result holds. □

2.3. Totally Geodesic Subspaces of Locally Symmetric Spaces

A Riemannian manifold M is a **globally symmetric space** if each point $p \in M$ is an isolated fixed point of an involutive isometry of M. The standard examples of globally symmetric spaces are Euclidean space, spheres, and hyperbolic space. It turns out that the totally geodesic submanifolds of locally symmetric spaces are well behaved.

Proposition 2.3.1. *Let M be a Riemannian manifold and N a totally geodesic subspace of M. If M is globally symmetric, the same holds for N.*

This fact is mentioned in [He] Ch. IV §7 Prop 7.1 pg 224 and is an immediate consequence of the definitions. One of the advantages to working with globally symmetric spaces is that questions about the spaces can be translated into questions about its isometry group via the following theorem.

Theorem 2.3.2 ([He] Ch. IV. Theorem 3.3)**.** *Let M be a connected globally symmetric space. Then*

1. $\mathrm{Isom}(M)$ *is a Lie group, as is its identity component* $G := \mathrm{Isom}^\circ(M)$,
2. *For each* $p_0 \in M$, $K := \mathrm{Stab}_G(p_0)$ *is a compact subgroup of* G.
3. G/K *is analytically diffeomorphic to* M *via the mapping* $gK \mapsto gp_0$.

In particular, we may then turn the question of looking at totally geodesic subspaces into the question of looking at Lie subgroups. A semisimple Lie group G may always be written as $G = G_1 G_2 \cdots G_r$ where each G_i is a normal simple[1] Lie subgroup of G and $G_i \cap G_j$ is

[1] Here simple means its Lie algebra is simple. The group can have discrete center.

finite for $i \neq j$. We say the G_i are **factors** of G and that G is **without compact factors** if no G_i is compact. A globally symmetric space is of **noncompact type** (resp. of **compact type**) if the associated Lie group $G = \mathrm{Isom}^\circ(\widetilde{M})$ has no compact factors (resp. is compact.).

Theorem 2.3.3 ([He] Ch V 4.2)**.** *Let M be a simply connected globally symmetric space. Then M is a product*

$$M = M_0 \times M_- \times M_+,$$

where M_0 is a Euclidean space, M_- and M_+ are Riemannian globally symmetric spaces of compact and noncompact type, respectively.

Hence studying globally symmetric spaces can often be reduced to studying individually flat Riemannian manifolds, globally symmetric spaces of compact type, and globally symmetric spaces of noncompact type. We will focus on groups on noncompact type from here on out.

Proposition 2.3.4. *Let M a connected globally symmetric space of noncompact type, $G = \mathrm{Isom}^\circ(M)$ and K a stabilizer of a point $p_0 \in M$.*

1. *Let $H \subset G$ be a semisimple Lie subgroup with no compact factors. Then $N_H := H/(H \cap K)$ is a totally geodesic submanifold of M.*
2. *Let $N \subset M$ be a totally geodesic submanifold of noncompact type such that $p_0 \in N$. Then there exists a semisimple Lie subgroup $H_N \subset G$ with no compact factors such that $H_N/(H_N \cap K) = N$.*

Proof. We begin with 1. Note that N is an immersed submanifold of M. Geodesics of M arise from the exponential map of G. Given an element $X \in \mathrm{Lie}(H)$ we know that $\exp_G(tX) \in H$ for all $t \in \mathbb{R}$, and hence N must be totally geodesic by definition (G3).

We now show 2. Let $\mathrm{Lie}(G) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition. Let $\mathfrak{s} \subset \mathfrak{p}$ be the subspace associated with the tangent space of N. Then $\mathfrak{k}$ acts on $\mathfrak{p}$ by the adjoint representation and let $\mathfrak{k}' = N_{\mathfrak{k}}(\mathfrak{s}) = \{X \in \mathfrak{k} \mid \mathrm{ad}(X)(\mathfrak{s}) \subset \mathfrak{s}\}$. Then $\mathfrak{h} := \mathfrak{k}' \oplus \mathfrak{s}$ is a Lie subalgebra of $\mathrm{Lie}(G)$. Let H_N be the unique connected Lie subgroup of G with Lie algebra $\mathfrak{h}$. It follows that H_N has the desired properties. $\square$

Proposition 2.3.4 shows that understanding totally geodesic submanifolds of noncompact type of globally symmetric spaces of noncompact type can be translated into studying semisimple Lie subgroups of a semisimple group.

A Riemannian manifold M is **locally symmetric** if one of the following three equivalent conditions are met:

(LS1) $\nabla R = 0$ where ∇ is the Levi-Cevita connection and R is the curvature tensor,

(LS2) For each $p \in M$ there exists a normal neighborhood of p on which the geodesic symmetry with respect to p is an isometry.

(LS3) M has universal cover $\widetilde{M}$ which is a globally symmetric space. In which case $M = \Gamma\backslash\widetilde{M}$ where Γ is a discrete torsion-free subgroup of $\mathrm{Isom}^\circ(M)$.

The equivalence of (LS1) and (LS2) is outlined in [DC] Ch. 8. ex 14 p. 190. It is not hard to see that (LS2) and (LS3) are equivalent. For us a **locally symmetric space** is an orbifold which is covered by a locally symmetric manifold. Locally symmetric spaces, which includes complete hyperbolic manifolds, have been a major topic of research and interest over the past century. We say a locally symmetric space is of **noncompact type** (resp. **compact type**, **flat**) if its universal cover is a globally symmetric space of **noncompact type** (resp. **compact type**, **flat**). Definition (LS3) shows that the study of locally symmetric spaces of noncompact type translates to the study of discrete subgroups of semisimple Lie groups with no compact factors, as we shall now record with the following well known proposition.

Proposition 2.3.5. *Let $M_1 = \Gamma_1\backslash G_1/K_1$ and $M_2 = \Gamma_2\backslash G_2/K_2$ be locally symmetric spaces of noncompact type where G_1 and G_2 are connected, adjoint, semisimple Lie groups with no compact factors. Then M_1 and M_2 are isometric if and only if there is a Lie group isomorphism $\varphi : G_1 \to G_2$ such that $\varphi(K_1) = K_2$ and $\varphi(\Gamma_1) = \Gamma_2$*

Since the image of a maximal compact (resp. discrete) subgroup under an automorphism is always a maximal compact (resp. discrete) subgroup, understanding isometry classes of locally symmetric spaces of noncompact type with universal cover G/K reduces to understanding $\mathrm{Aut}(G)$-orbits of discrete subgroups of G.

Let G be a semisimple Lie group and $\Gamma \subset G$ be a discrete subgroup. The Haar measure on G naturally descends to a G-invariant measure on $\Gamma\backslash G$. When the Haar measure on G descends to a measure of finite volume on $\Gamma\backslash G$, Γ is called a **lattice**. When $\Gamma\backslash G$ is compact, Γ is said to be **cocompact** or a **uniform lattice**. Cocompact discrete subgroups are always lattices. The property of being cocompact or being a lattice is an invariant of the commensurability class.

A lattice in a semisimple Lie group G without compact factors is **reducible** if there exists connected normal subgroups H, H' such that $G = HH'$, $H \cap H'$ is finite, and $\Gamma/((\Gamma \cap H)(\Gamma \cap H'))$ is finite. Otherwise, Γ is said to be **irreducible**. Morally speaking, irreducible just means the lattice is not a product of smaller lattices (up to commensurability).

As for globally symmetric spaces, totally geodesic submanifolds of locally symmetric spaces are well behaved.

Proposition 2.3.6. *Let M be a Riemannian orbifold and N a totally geodesic subspace of M. If M is locally symmetric, the same holds for N.*

This is an immediate consequence of definition (LS2). Observe that if N is a totally geodesic subspace of the locally symmetric space $M = \Gamma\backslash G/K$, then N is of finite volume if and only if $\Gamma \cap H_N$ is a lattice in H_N. In Chapter 4 we go on to use quadratic forms over number fields to construct classes semisimple subgroups whose intersection with a lattice is a lattice. To allow us to do this, we must first discuss properties of quadratic forms.

Example 2.3.7. An example of a a locally symmetric space is a hyperbolic manifold. Hyperbolic space $\mathbb{H}^n$ naturally arrises as a connected component of the -1 level set of a real quadratic form q of dimension $n+1$ and signature $n-1$. It follows that the isometry group of this set is $G = PO_0(n,1)$ with maximal compact subgroup $K = P(O(n) \times O(1))$. A **hyperbolic manifold** is a locally symmetric space whose universal cover is hyperbolic space.

Corollary 2.3.8. *Let M be a hyperbolic manifold, let $N \subset M$ be a closed totally geodesic submanifold of dimension greater than one. Then N is a hyperbolic manifold.*

Chapter 3

Local and Global Fields: Quadratic Forms and Class Field Theory

In this chapter we develop the classical theory of quadratic forms over local and global fields which will be used throughout this paper. We set out to give a clear and concise exposition of the uniqueness and existence theorems which will be heavily used throughout this paper. In Section 3.1 we introduce definitions and classical invariants of isometry classes of quadratic forms q over an arbitrary field F of characteristic not 2. The invariants we consider are:

1. **Dimension**: $\dim(q) \in \mathbb{Z}_{\geq 0}$,
2. **Determinant**: $\det(q) \in k^\times/(k^\times)^2$,
3. **Hasse–Minkowski Invariant**: $c(q) \in \mathrm{Br}(k)$, and
4. **Signature**: $\mathrm{sgn}(q) \in \mathbb{Z}$.

In Section 3.2 we discuss basic properties of local and global fields and use a result from class field theory to establish an existence result which will be used heavily in following chapters. In Sections 3.3, 3.4, and 3.5 we discuss how these invariants may be used to complete classify quadratic forms over local fields. In Section 3.6 we introduce the local-to-global results allowing us to completely classify quadratic forms over global fields. To see a more complete treatment of the classical theory of quadratic forms, we refer the reader to [OM].

3.1. Quadratic Forms Over a Field

Fix a field F which is not of characteristic 2 and let V be a vector space over F. A map $b : V \times V \to F$ is said to be

(B1) **Bilinear** if it is F-linear in each coordinate.

(B2) **Symmetric** if $b(v,w) = b(w,v)$ for all $v, w \in V$.

If b satisfies both, it is said to be a **symmetric bilinear form**. For every symmetric bilinear form b, we define a map $q : V \to F$ by $q(v) = b(v,v)$. It is easily shown that q satisfies the following two properties

(Q1) $q(av) = a^2 q(v)$ for all $a \in F$ and $v \in V$,

(Q2) $b(v,w) = \frac{1}{4}(q(v+w) - q(v-w))$ (**Polarization identity**).

Conversely, any map $q : V \to F$ satisfying $(Q1)$ and for which $\frac{1}{4}(q(v+w) - q(v-w))$ is bilinear defines a symmetric bilinear form via (2). We call a map satisfying (Q1) and (Q2) a **quadratic form**. Hence there is a natural bijection between symmetric bilinear forms and quadratic forms. A symmetric bilinear form is analogous to an inner product and its associated quadratic form is analogous to the norm squared map.

A **quadratic space** is the pair (V, q) where q is a quadratic form. The **dimension** of q, denoted $\dim q$, is the dimension of V as an F-vector space. Let $m = \dim q$. In what follows, wherever possible we shall reserve the symbol m to denote the dimension of a quadratic form. Upon fixing a basis $\{e_1, e_2, \dots e_m\}$, V may be identified with F^m. Then to each symmetric bilinear form b we may associate to it a symmetric $m \times m$ matrix as follows: Let $B_{ij} := b(e_i, e_j)$ and B be the $m \times m$ matrix with coordinates $\{B_{ij}\}$. Then for all $v, w \in V$, $v = \sum v_i e_i$ and $w = \sum w_j e_j$ where $v_i, w_j \in F$ we have

$$b(v,w) = b\left(\sum_{i=1}^{m} v_i e_i, \sum_{j=1}^{m} w_j e_j\right) = \sum_{i=1}^{m}\sum_{j=1}^{m} v_i w_j \; b(e_i, e_j) = \sum_{i=1}^{m}\sum_{j=1}^{m} v_i B_{ij} w_j = {}^t v B w,$$

where ${}^t v$ denotes the transpose of v. Having made these identifications, where have the following identification.

Proposition 3.1.1. *Let F be a field of characteristic other than 2. Then there exists a bijection*

$$\{\textit{quadratic forms}\} \leftrightarrow \{\textit{symmetric matrices with coefficients in } F\}.$$

Two quadratic forms q_1 and q_2 of dimension m are **isometric** if there exists $T \in GL_m(F)$ such that $q_1(Tv) = q_2(v)$ for all $v \in V$. Two matrices $A, B \in M_m(F)$ are **similar** if there exists a $T \in GL_m(F)$ such that $A = {}^t TBT$ where ${}^t T$ is the transpose of T.

Proposition 3.1.2. *Let F be a field of characteristic other than 2. Then there exists a bijection*

$$\{\textit{isometry classes of quadratic forms}\} \leftrightarrow \left\{ \begin{array}{c} \textit{similarity classes of symmetric} \\ \textit{matrices with coefficients in } F \end{array} \right\}.$$

We say two quadratic forms q_1 and q_2 are **similar** is there exists some $a \in F^\times$ such that $q_1 = aq_2$. We say a quadratic form r is a **subform** of a quadratic form q if there is some third form t such that $r \oplus t$ is isometric to q. We say a symmetric bilinear form b is **nondegenerate** when $b(v, w) = 0$ for all $w \in V$ implies that $v = 0$. A quadratic form corresponding to a nondegenerate symmetric bilinear form is said to be **regular**. In this paper, all quadratic forms will be assumed to be regular unless explicitly stated otherwise. We define the **determinant** of q to be the determinant of some $Q \in GL_m(F)$ representing q. Note however that since $\det({}^tTQT) = \det Q(\det T)^2$, the determinant is only well defined up to square class, and hence we view $\det q \in F^\times/(F^\times)^2$. Though the determinant is a square class, we will often write a representative of this class (i.e., $\det q = a$ as opposed to $\det q = a(F^\times)^2$). A common renormalization of the determinant is the **discriminant** defined by $\mathrm{disc}(q)$ where $\mathrm{disc}(q) = (-1)^{\dim(q)(\dim(q)-1)/2}\det(q)$. It contains the same information as the determinant if one knows the dimension, but often results in simpler expressions.

Proposition 3.1.3. *Let $Q \in GL_m(F)$ be symmetric. Then Q is similar (over F) to a diagonal matrix.*

Proof. This is a result of the Gram–Schmidt process. □

It follows that every isometry class of quadratic form can be represented by a diagonal matrix. Choosing such a representation we write $q = \langle a_1, a_2, \ldots, a_m \rangle$ where the associated diagonal matrix is $\mathrm{diag}(a_1, a_2, \ldots, a_m)$. This representation will allow us to define the Hasse–Minkowski invariant and signature of a quadratic form.

Let $a, b \in F^\times$. Then the **Hilbert symbol** $\left(\frac{a,b}{F}\right) = (a, b)_F$ denotes the isomorphism class of the quaternion algebra defined by $F[i, j]$ such that $i^2 = a, j^2 = b$, and $ij = -ji$. When the field F is understood, we simply write (a, b). The Hilbert symbol satisfies some algebraic properties which we now state (see [M] Chp III, Thm 4.4) :

1. **Defined up to square class**: $(a, bc^2) = (a, b)$
2. **Symmetry**: $(a, b) = (b, a)$
3. **Multiplicativity**: $(a_1a_2, b) = (a_1, b)(a_2, b)$

4. **Nondegeneracy**: For $a \in F^\times$ not a square, there exists a $b \in F^\times$ such that $(a,b) \neq 1$

5. $(a,-a) = 1$ for all $a \in F^\times$

6. $(a,a) = (a,-1)$ for all $a \in F^\times$

Given a quadratic form $q = \langle a_1, a_2, \ldots, a_m \rangle$, we define the **Hasse–Minkowski invariant**[1] $c(q)$ by

$$c(q) := \begin{cases} \prod_{i<j}(a_i, a_j) & \text{if } m \geq 2 \text{, and} \\ 1 & \text{if } m = 1. \end{cases}$$

As a consequence of the definition and the above properties of the Hilbert symbol, the Hasse–Minkowski invariant satisfies the following properties:

1. **Product formula** $c(q_1 \oplus q_2) = c(q_1)c(q_2)(\det q_1, \det q_2)$

2. **Hyperbolic planes** $c(\bigoplus_n \langle 1, -1 \rangle) = (-1,-1)^{n(n-1)/2}$

It turns out that $c(q)$ is independent of the choice of representative of the isometry class of q and hence a well defined invariant of the isometry class of q ([Lam] V.3.18). Furthermore $c(q)$ behaves nicely with respect to similar forms as we now show.

Lemma 3.1.4. *Let F be a any field not of characteristic 2 and let q be a quadratic form over F of dimension m. Then*

$$c(\lambda q) = \left(\lambda, (-1)^{\frac{m(m-1)}{2}} (\det q)^{m-1}\right) c(q).$$

In particular this reduces to

$$c(\lambda q) = \begin{cases} (\lambda, \operatorname{disc}(q)) \; c(q) & \text{when } m \text{ is even,} \\ \left(\lambda, (-1)^{\frac{m(m-1)}{2}}\right) \; c(q) & \text{when } m \text{ is odd.} \end{cases}$$

Proof. In general

$$\begin{aligned} c(\lambda q) &= \prod_{i<j} (\lambda a_i, \lambda a_j) \\ &= \prod_{i<j} (\lambda, \lambda)(\lambda, a_i)(\lambda, a_j)(a_i, a_j) \\ &= (\lambda, -1)^{\frac{m(m-1)}{2}} (\lambda, \det q)^{m-1} c(q) \\ &= \left(\lambda, (-1)^{\frac{m(m-1)}{2}} (\det q)^{m-1}\right) c(q). \end{aligned}$$

The reduction when m is even and odd immediately follows. □

[1]Unfortunately there is a lack of uniformity in the literature when it comes to this invariant. In some texts and papers, this invariant is simply referred to as the Hasse invariant. Additionally some authors use the symbol $s(q)$ instead of $c(q)$. Lastly some references call $\prod_{i \leq j}(a_i, a_j) = c(q)(-1, \det q)$ the Hasse invariant.

In general this invariant is difficult to compute, however, when F is a nonarchimedian local field or $\mathbb{R}$, then $c(q)$ can only take values ± 1, and over $\mathbb{C}$, $c(q)$ is identically 1.

Example 3.1.5. Let $q = \langle 1, 1, \ldots, 1, -1 \rangle$ over $\mathbb{R}$, then $c(q) = (1,1)^{m(m-1)/2}(1,-1)^m = 1$. This example is important in the study of hyperbolic manifolds, since hyperbolic manifolds arise as the locally symmetric spaces corresponding to the isometry group of q.

Suppose that F is an ordered field, for example $\mathbb{R}$. Then any quadratic form q over F, can be represented $\langle a_1, a_2, \ldots, a_m \rangle$ where there first m_+ terms are positive and the remainder $m_- := m - m_+$ terms are negative. The **signature** of q is the number $\operatorname{sgn}(q) = m_+ - m_-$. However, sometimes we will refer to the pair (m_+, m_-) as the signature of q since the two pairs (m, s) and (m_+, m_-) contain equivalent information. It turns out that this value is independent of the choice of representative of the isometry class of q and hence a well defined invariant of the isometry class of q.

Let E/F be a field extension. Then every quadratic space (V, q) over F gives a quadratic space of E by tensoring over F. Namely let $V_E := V \otimes_F E$ and let q_E denote the extension q to V_E. When it will not cause confusion, we will sometimes denote the extended form by the symbol q as well.

3.2. Local and Global Fields and Some Class Field Theory

In this section we discuss some basic facts from algebraic number theory and class field theory that we use throughout this paper. The fields that we are interested in are what are called local and global fields. While we are only interested in characteristic 0 local and global fields, many results hold independently of characteristic and hence we give results in full generality. A **global field** is either a finite extension of $\mathbb{Q}$, which is called an algebraic number field, or a function field in one variable over a finite field. A **multiplicative valuation** on a field F is a map $|\cdot| : F \to \mathbb{R}_{\geq 0}$ satisfying the following three properties.

(V1) $|x| = 0$ if and only if $x = 0$,

(V2) $|xy| = |x|\ |y|$ for all $x, y \in F$, and

(V3) $|x + y| \leq |x| + |y|$ for all $x, y \in F$.

The **trivial valuation** is the valuation $|x| = 1$ for all $x \in F^\times$. A multiplicative valuation naturally endows F with a metric $d(x, y) := |x - y|$. A **local field** is a field which is locally compact with respect to the metric induced by a nontrivial valuation. These fields are $\mathbb{C}$, $\mathbb{R}$, finite extensions of $\mathbb{Q}_p$ or formal Laurent series over a finite field.

A valuation $|\cdot|$ is called **nonarchimedean** if it satisfies the following strengthening of condition (V3).

(V3′) $|x+y| \leq \max\{|x|, |y|\}$ for all $x, y \in F$.

If $|\cdot|$ does not satisfy (V3′), it is called **archimedean**. A **nonarchimedean local field** (resp. **archmimedean local field**) is a local field F whose valuation is nonarchimedean (resp. archimedean). For example, both $\mathbb{R}$ and $\mathbb{C}$ are archimedean, while finite extensions of $\mathbb{Q}_p$ are nonarchimedean. If F is a nonarchimedean local field, then its **valuation ring** is the set $\mathcal{O}_F = \{x \in F \mid |x| \leq 1\}$. By (V3′), this set is a ring. Two nontrivial valuations $|\cdot|_1$ and $|\cdot|_2$ on a field F are **equivalent** if one of the following equivalent conditions holds

(E1) $|\cdot|_1$ and $|\cdot|_2$ define the same topology on F,

(E2) $|x|_1 < 1$ if an only if $|x|_2 < 1$ for all $x \in F$, and

(E3) $|\cdot|_1 = |\cdot|_2^a$ for some $a \in \mathbb{R}_{\geq 0}$.

It is not hard to see that this in fact determines an equivalence relation on the set of valuations.

Global fields admit many inequivalent nontrivial multiplicative valuations. If k is a global field, an equivalence class of multiplicative valuations is called a **place of** k. Let V_k denote the set of all places of k. If $|\cdot|$ is a multiplicative valuation on k, let $k_{|\cdot|}$ denote the metric space completion of k with respect to the metric induced by $|\cdot|$. Since k is a global, it follows that in fact $k_{|\cdot|}$ is locally compact, and hence a local field. Furthermore k naturally embeds into $k_{|\cdot|}$, and by definition (E1) above, there is a well defined map $v \mapsto k_{|\cdot|} =: k_v$ where $v \in V_k$ and $|\cdot|$ is any valuation in the equivalence class v.

When k is a number field, then $v \in V_k$ is called:

- a **complex place** if $k_v = \mathbb{C}$,
- a **real place** if $k_v = \mathbb{R}$,
- a **infinite place** if $k_v = \mathbb{C}$ or $\mathbb{R}$,
- a **finite place** if $k_v = L$ where L is a finite extension of $\mathbb{Q}_p$ for some prime p.

For more on the theory of algebraic number fields, see [AT] or [M2].

Example 3.2.1. Ostrowski's Theorem ([M] Thm 7.12) states that over $\mathbb{Q}$ the finite places are in bijective correspondence with primes $p \geq 2$ and $\mathbb{Q}$ has precisely one infinite place, the usual real place.

The **adèles** over k are the subring $\mathbb{A}_k$ of $\prod_{v \in V_k} k_v$ defined as follows:

$$\mathbb{A}_k := \left\{ \{x_v\}_{v \in V_k} \in \prod_{v \in V_k} k_v \;\middle|\; \text{all but finitely many } x_v \in \mathcal{O}_{k_v} \right\}.$$

When given the standard restricted direct product topology, the adèles are in fact a locally compact abelian group with respect to addition. For detailed treatment we refer the reader to [La] Chapter VII. The adèles are a valuable tool in the study of class field theory, algebraic groups, and hence arithmetic locally symmetric spaces. Some interesting geometric results which use the adèles include volume computations [Pr89] and cusp computations [St]. We may now state and use the famous Grunwald–Wang Theorem.

Theorem 3.2.2 (Grunwald–Wang)**.** *Let*

1. *k be a number field,*
2. *S be a finite set of places of k, and*
3. *for each $v \in S$, let χ_v be a character of $k_v^\times$.*

Then there exists a character χ of $GL_1(\mathbb{A}_k)/GL_1(k)$ whose restriction to $k_v^\times$ is χ_v, for all $v \in S$. Let n_v be the order of χ_v and $n = lcm(n_v)$. Then it is possible to choose χ to have order n, except possibly when $2^t | n$ for some t with $k[\zeta_{2^t}]$ not cyclic over k.

For more on this theorem, see ([M] Chp VIII Thm 2.4). We use it to prove the following existence result:

Corollary 3.2.3. *Let*

1. *k be a number field,*
2. *S be a finite set of places of k, and*
3. *for each $v \in S$, let α_v be a square class in $k_v^\times$.*

Then there exists an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

Proof. Each nontrivial (resp. trivial) square class α_v corresponds to a unique quadratic (resp. trivial) extension L_v/k_v. By local class field theory, this corresponds to a character χ_v of $k_v^\times$ of order 2 (resp. order 1). Then by Theorem 1.1, there exists a character of $GL_1(\mathbb{A}_k)/GL_1(k)$. Since $n = 2$ and $k[\zeta_2] = k$ is trivially cyclic, we conclude χ has order 2. Hence by global class field theory, this gives a quadratic extension L/k where $L = k(s)$. Then $s \in \alpha_v$ for all $v \in S$. □

Being able to choose elements with prescribed local behavior will prove to be valuable in Chapter 6 and 7.

3.3. Quadratic Forms Over $\mathbb{C}$

In this section we discuss quadratic forms over $\mathbb{C}$. Due to the fact that $\mathbb{C}$ is algebraically closed, isometry classes quadratic forms over $\mathbb{C}$ are easily classified.

Proposition 3.3.1.

1. *(Uniqueness.) Let q and q' be quadratic forms over $\mathbb{C}$. Then $q \cong q'$ if and only if* $\dim q = \dim q'$.
2. *(Existence.) For each $m \in \mathbb{Z}_{\geq 1}$, there exists a quadratic form over $\mathbb{C}$ such that* $\dim q = m$.

Proof. We begin with uniqueness. Clearly isometric forms have the same dimension. We now show forms of the same dimension are isometric. Let $Q \in GL_n(\mathbb{C})$ be symmetric. It suffices to show that Q is similar to the identity I_m. By the Gram–Schmidt process, Q is similar to a diagonal matrix $D = \operatorname{diag}(a_1, \ldots, a_m)$. Then let $T = \operatorname{diag}\left(\frac{1}{\sqrt{a_1}}, \frac{1}{\sqrt{a_2}}, \ldots, \frac{1}{\sqrt{a_m}}\right)$ for $a_i \in \mathbb{C}$ and an appropriate branch cut of $f(z) = \sqrt{z}$ missing these m values. Then $I_m = {}^tTDT$ and the result follows.

We now show existence. For each $m \geq 1$, let q be defined by the diagonal matrix I_m. $\square$

3.4. Quadratic Forms Over $\mathbb{R}$

In this section we discuss quadratic forms over $\mathbb{R}$. This was originally worked out by Sylvester in 1852 [Sy].

Proposition 3.4.1.

1. *(Uniqueness.) Let q and q' be quadratic forms over $\mathbb{R}$. Then $q \cong q'$ if and only if* $\dim q = \dim q'$ *and* $\operatorname{sgn}(q) = \operatorname{sgn}(q')$.
2. *(Existence.) For each pair $(m, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$ there exists a quadratic form q over $\mathbb{R}$ such that* $\dim q = m$ *and* $\operatorname{sgn}(q) = s$ *so long as $-m \leq s \leq m$ and* $m - s \equiv 0 \pmod 2$.

Proof. We begin with uniqueness. Clearly isometric forms have the same dimension. The fact that signature is preserved under isometry goes back to an old argument of Sylvester's using Descartes' rule to count signs in the characteristic polynomial of the matrix associated with forms [Sy]. Conversely, given two forms with the same dimension and signature we now show to be isometric. Let

$$I_{m_+,m_-} = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{m_+}, \underbrace{-1, -1, \dots - 1}_{m_-}).$$

where $m_+ + m_- = m$. It suffices to show that any $Q \in GL_m(\mathbb{R})$ symmetric matrix of signature (m_+, m_-) is similar to I_{m_+,m_-}. By the Gram–Schmidt process, Q is similar to a diagonal matrix $D = \mathrm{diag}(a_1, \dots, a_m)$, which has of signature (m_+, m_-). Then let $T = \mathrm{diag}\left(\frac{1}{\sqrt{|a_1|}}, \frac{1}{\sqrt{|a_2|}}, \dots, \frac{1}{\sqrt{|a_m|}}\right)$. Then $I_{m_+,m_-} = {}^tTDT$ and the result follows.

We now show existence. For each $(m, s) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}$, let q be defined by the diagonal matrix I_{m_+,m_-} where $m_+ + m_- = m$ and $m_+ - m_- = s$. □

Furthermore, if q is a quadratic form over $\mathbb{R}$, it follows that its dimension and signature uniquely determine its determinant.

$$\det q = (-1)^{\frac{\dim(q) - \mathrm{sgn}(q)}{2}} \tag{3.1}$$

3.5. Quadratic Forms Over Nonarchimedean Local Fields

In this section we discuss quadratic forms over nonarchimedean local fields. This turns out to require the most sophisticated analysis of all local fields and we refer the reader to [OM] for proofs. Nonarchimedean local fields are either finite extensions of the p-adic numbers or formal Laurent series $F((T))$ over a finite field F. We shall primarily be concerned with the characteristic zero case, however we state these results in full generality.

Theorem 3.5.1. *Let L denote a nonarchimedean local field.*

1. *(Uniqueness.) Let q and q' be quadratic forms over L. Then $q \cong q'$ if and only if $\dim q = \dim q'$, $\det(q) = \det(q')$, and $c(q) = c(q')$.*

2. *(Existence.) For each triple $(m, d, c) \in \mathbb{Z}_{\geq 1} \times L^\times/(L^\times)^2 \times \{\pm 1\}$, there exists a quadratic form q over L such that $\dim q = m$, $\det q = d$ and $c(q) = c$, with the only constrains that when $m = 1$ we must have $c = 1$ and when $m = 2$ and $d = -1$ we must have $c = 1$.*

Proof. See [OM] Chapter VI, Theorem 63:23. The statement in the text looks slightly different because [OM] uses the alternate definition of the Hasse–Minkowski Invariant mentioned above and hence the values for this invariant differ by a factor of $(-1, \det q)$. □

While the exceptional restrictions on Hasse–Minkowski invariants in the cases $m = 1$ and $m = 2$ may seem inconsequential, they will play an integral role in the construction of subforms in Chapter 6.

3.6. Quadratic Forms Over Number Fields

In this section, we will study quadratic forms over number fields using the results from earlier sections. The main tool will be Theorem 3.6.1 and Theorem 3.6.2.

Let (V, q) be an n-dimensional quadratic space over k. Then for each $v \in V_k$, we have a quadratic space defined over k_v by $(V \otimes_k k_v, q \otimes k_v)$ where $q \otimes k_v$ denotes the natural extension of q to $V \otimes_k k_v$.

Theorem 3.6.1 (Local-to-Global Uniqueness)**.** *Let k be a number field and q and q' be quadratic forms over k. Then $q \cong q'$ if and only if $q \otimes k_v \cong q' \otimes k_v$ for all $v \in V_k$.*

Proof. See [OM] Chapter VI, Theorem 66:4. □

Theorem 3.6.2 (Local-to-Global Existence)**.** *Let k be a number field and let*

- $m \in \mathbb{Z}_{\geq 1}$,
- $d \in k^\times/(k^\times)^2$, *and*
- $S \subset V_k$ *be a finite subset of even cardinality.*

Given a family $\{q_v\}_{v \in V_k}$ where q_v is a quadratic form over k_v satisfying

- $\dim q_v = m$,
- $\det q_v = d$, *and*
- $c_v(q_v) = -1$ *if and only if* $v \in S$.

Then there exists a quadratic form q over k such that $q \otimes k_v = q_v$ for all $v \in V_k$.

Proof. See [OM] Chapter VII, Theorem 72:1. □

Chapter 4

Arithmetic Groups and Arithmetic Locally Symmetric Spaces

The theory of arithmetic groups lies at the intersection of geometry and number theory. Over the past century, there has been considerable research on finding and understanding discrete subgroups of Lie groups. In the 1960's the work of Borel, Harish-Chandra, Mostow, Tamagawa, and others began using arithmetic techniques to study certain discrete subgroups of matrix groups ([B63], [BoHC], [MT]). With great success, these "arithmetic" groups allow us to apply the power of algebraic number theory to study discrete subgroups. In this chapter, we discuss the general theory of arithmetic groups and how it applies to quadratic forms. In Section 4.1, we recall well known results from the theory of arithmetic groups of algebraic $\mathbb{Q}$-groups. In Section 4.2, we relate the theory of arithmetic groups to the theory of arithmetic lattice in semisimple Lie groups without compact factors. In Section 4.3, we state and prove results in the theory of arithmetic locally symmetric spaces that are known, but for which we cannot find a reference. In Section 4.4 and Section 4.5, we explicitly show the connection between arithmetic groups and quadratic forms over number fields and skew hermitian forms over division algebras over number fields.

In this and following chapters, we reserve bold font to denote algebraic groups. We will reserve the symbol F to denote an abstract field and k to denote a number field.

4.1. Arithmetic Subgroups of Algebraic $\mathbb{Q}$-Groups

In this section, we discuss the basic definitions and useful theorems about arithmetic subgroups of algebraic $\mathbb{Q}$-groups. Let $\mathbf{G}$ be an algebraic group defined over $\mathbb{Q}$. Then there exists a faithful $\mathbb{Q}$-rational embedding $\rho : \mathbf{G} \to \mathbf{GL}(V)$ for some $\mathbb{Q}$-vector space V ([B1], Prop 1.10). Let $L \subset V$ be a $\mathbb{Z}$-lattice of V, i.e., a free $\mathbb{Z}$-module such that $L \otimes_{\mathbb{Z}} \mathbb{Q} = V$. Define the group

$$G_{\rho,L} := \{g \in \mathbf{G}(\mathbb{Q}) \mid \rho(g)(L) = L\}.$$

Any subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ commensurable[1] with $G_{\rho,L}$ is an **arithmetic subgroup** of $\mathbf{G}(\mathbb{Q})$. Were we to chose a different embedding, ρ', and different $\mathbb{Z}$-lattice, L', we would have obtained a different group $G_{\rho',L'}$, however, any such $G_{\rho',L'}$ is commensurable with $G_{\rho,L}$ because $G_{\rho,L}$ and $G_{\rho',L'}$ contain $G_{\rho_0,L\cap L'}$ with finite index. It follows that the commensurability class of an arithmetic group is independent of the choices of ρ and L (see [B2] Proposition 7.12 and preceding discussion). In other words, the $\mathbb{Q}$-isomorphism class of $\mathbf{G}$ determines a commensurability class of arithmetic groups.

Example 4.1.1.

1. **The prototypical examples.** Let $V = \mathbb{Q}^n$ and let $L = \mathbb{Z}^n \subset \mathbb{Q}^n$ be the standard $\mathbb{Z}$-lattice. It follows that $SL_n(\mathbb{Z})$ is an arithmetic subgroup of $\mathbf{SL_n}(\mathbb{Q})$ and $GL_n(\mathbb{Z})$ is an arithmetic subgroup of $\mathbf{GL_n}(\mathbb{Q})$.

2. **Principle congruence subgroups of $\mathbf{G}(\mathbb{Z})$.** For $l \in \mathbb{Z}_{\geq 1}$, define the quotient map $\varphi_l : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}/l\mathbb{Z})$ and let $GL_n(\mathbb{Z}, l) := \ker(\varphi_l)$. Since $GL_n(\mathbb{Z}, l)$ has finite index in $GL_n(\mathbb{Z})$, it is arithmetic. Fix a $\mathbb{Q}$-rational embedding $\rho : \mathbf{G} \to \mathbf{GL}_n$ and let L denote the standard $\mathbb{Z}$-lattice, $\mathbb{Z}^n$. Then $\Gamma_l := G_{\rho,L} \cap GL_n(\mathbb{Z}, l)$ has finite index in $G_{\rho,L}$ and hence is an arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$. We call such groups **principle congruence subgroups**. There has been considerable research studying these groups, and in particular, determining which $\mathbb{Q}$-groups $\mathbf{G}$ have the property that every arithmetic subgroup of $\mathbf{G}(\mathbb{Q})$ contains a principle congruence subgroup. Groups with this property are said to have an affirmative answer to the Congruence Subgroup Problem (CSP). For more on CSP, see [BMS] and [Se] for some of the foundational work and [PR] for recent developments.

3. **Arithmetic groups in division algebras**. Say for example we have the quaternion division algebra $D = \left(\frac{3,5}{\mathbb{Q}}\right) = \mathbb{Q}[i, j]$ where $i^2 = 3$, $j^2 = 5$, and $ij = -ji$. It turns out that the multiplicative group

$$SL_1(D) = \{\gamma \in D \mid \mathrm{Nrm}(\gamma) = 1\}$$

is the $\mathbb{Q}$-points of a $\mathbb{Q}$-group which we will denote $\mathbf{SL}_1(D)$. There is a natural embedding $D \subset M_2(\mathbb{Q}(\sqrt{5})) \subset M_2(\mathbb{R})$, via

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i \mapsto \begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}, \quad j \mapsto \begin{pmatrix} \sqrt{5} & 0 \\ 0 & -\sqrt{5} \end{pmatrix}, \quad ij \mapsto \begin{pmatrix} 0 & -3\sqrt{5} \\ \sqrt{5} & 0 \end{pmatrix}.$$

[1]Two subgroups Λ, Λ' of a group G are said to be **commensurable** if $\Lambda \cap \Lambda'$ is finite index in both Λ and Λ'. It can be shown that commensurability determines an equivalence relation among subgroups of G.

It follows that $SL_1(D) \subset SL_2(\mathbb{R})$ and it is not hard to see that $\mathbf{SL}_1(D)$ is in fact a $\mathbb{Q}$-form of $\mathbf{SL_2}$. Let $\mathcal{O}_D$ be the $\mathbb{Z}$-span of $1, i, j, ij$ in D. Then $SL_1(\mathcal{O}_D) := \mathcal{O}_D \cap SL_1(D)$ is an arithmetic subgroup of $SL_1(D) \subset SL_2(\mathbb{R})$.

Often we will assume the existence of some embedding ρ and lattice L, and we will denote $\mathbf{G}(\mathbb{Z}) := G_{\rho,L}$. Note however that not all arithmetic groups arise as the stabilizer of a lattice. This follows from the fact that every lattice stabilizer contains a congruence subgroup ([B2] Proposition 7.12) but there are groups for which there is a negative answer to CSP ([PR], §2.1).

One way to construct algebraic $\mathbb{Q}$-groups is to start with a k-group, where k is a number field, and then apply the restriction of scalars functor $R_{k/\mathbb{Q}}$ ([PlRa] §2.1.2, or [MaR2] §10.3)). This functor has the property that if $\mathbf{G}$ is an algebraic k-group, then $R_{k/\mathbb{Q}}\mathbf{G}$ is an algebraic $\mathbb{Q}$-group and there is an abstract group isomorphism between $\mathbf{G}(k)$ and $(R_{k/\mathbb{Q}}\mathbf{G})(\mathbb{Q})$. With this identification, it makes sense to talk about arithmetic subgroups of $\mathbf{G}(k)$. Furthermore, it is not hard to see that arithmetic subgroups of $\mathbf{G}(k)$ are precisely the groups commensurable with the stabilizer of a $\mathcal{O}_k$-lattice of a k-vector space V where there is a k-rational embedding of $\mathbf{G}$ into $\mathbf{GL}(V)$.

An **absolutely (resp. absolutely almost) simple algebraic F-group** is an algebraic F-group which, after extending scalars to $\overline{F}$, is (resp. isogenous to) a simple semisimple algebraic $\overline{F}$-group. For example the $\mathbb{C}$-group $\mathbf{SL}_n$ is absolutely almost simple but not absolutely simple since it has nontrivial center equal to the n^{th} roots of unity. The semisimple $\mathbb{R}$-group $R_{\mathbb{C}/\mathbb{R}}\mathbf{SL}_2$, which is related to the study of hyperbolic 3-manifolds, is not absolutely almost simple, because after extending to $\mathbb{C}$, the group is isomorphic to $\mathbf{SL}_2 \times \mathbf{SL}_2$. If we start with an absolutely almost simple k-group, where k is a number field, then $R_{k/\mathbb{Q}}(\mathbf{G})$ is always a semisimple $\mathbb{Q}$-group. An **F-simple F-group** is an algebraic F-group which, up to isogeny, does not contain a proper nontrivial normal F-subgroup. Absolutely almost simple F-groups are F-simple and $R_{\mathbb{C}/\mathbb{R}}\mathbf{SL}_2$ is $\mathbb{R}$-simple. As we now record, all semisimple $\mathbb{Q}$-groups are built from absolutely almost simple groups over number fields.

Proposition 4.1.2.

1. *Let $\mathbf{G}$ be a $\mathbb{Q}$-simple $\mathbb{Q}$-group. Then there is a number field k and an absolutely almost simple k-group $\mathbf{H}$ such that $\mathbf{G}$ and $R_{k/\mathbb{Q}}\mathbf{H}$ are $\mathbb{Q}$-isomorphic.*

2. *Let $\mathbf{G}$ be a semisimple $\mathbb{Q}$-group. Then there is number fields k_i, $1 \le i \le r$ and absolutely almost simple k_i-groups $\mathbf{H}_i$ such that up to isogeny $\mathbf{G}$ and $\prod_{i=1}^{r} R_{k_i/\mathbb{Q}}\mathbf{H}_i$ are $\mathbb{Q}$-isomorphic.*

Arithmetic groups are often Zariski dense, as given by Borel's density theorem

Theorem 4.1.3 (Borel's Density Theorem [B65])**.** *Let* $\mathbf{G}$ *be a semisimple* $\mathbb{Q}$*-group with no* $\mathbb{R}$*-anisotropic* $\mathbb{Q}$*-simple factor* $\mathbf{H}$*. Let* $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be an arithmetic subgroup. Then* Γ *is Zariski dense.*

Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{C}$ and let $\Gamma \subset \mathbf{G}(\mathbb{C})$ be a Zariski dense subgroup. A **field of definition**[2] for Γ is a field $F \subset \mathbb{C}$ for which there exits an F-form $\mathbf{G}'$ of $\mathbf{G}$ and an isomorphism $\varphi : \mathbf{G} \to \mathbf{G}'$ defined over a finite extension of F such that $\varphi(\Gamma) \subset \mathbf{G}'(F)$ ([MaR2] Definition 10.3.10). Vinberg showed that for Zariski dense groups, there is a unique minimal field of definition.

Theorem 4.1.4 (Vinberg, [Vin])**.** *Let* $\mathbf{G}$ *be a semisimple algebraic* $\mathbb{C}$*-group and* $\Gamma \subset \mathbf{G}(\mathbb{C})$ *be a Zariski dense subgroup. Then there exists a minimal field of definition of* Γ *which is an invariant of the commensurability class of* Γ*. Furthermore, it is the field*

$$k(\Gamma) := \mathbb{Q}(\mathrm{Tr}(\mathrm{Ad}(\gamma)) \mid \gamma \in \Gamma),$$

where Ad *is the adjoint representation of* $\mathbf{G}$*.*

In general, the field of definition of Γ can be strictly smaller then the field that $\mathbf{G}$ is defined over. As we shall now see, for absolutely almost simple groups, the field of definition of an arithmetic group coincides with the field of definition of the group.

Lemma 4.1.5 ([PR09], Lemma 2.6)**.** *Let* k *be a number field and let* $\mathbf{G}$ *be an almost absolutely simple algebraic* k*-group. Let* $\Gamma \subset \mathbf{G}(k)$ *be a Zariski dense arithmetic subgroup. Then* $k = k(\Gamma)$*.*

We now move on to some fundamental results on how arithmetic groups of a $\mathbb{Q}$-group $\mathbf{G}$ behave inside the Lie group $\mathbf{G}(\mathbb{R})$.

Proposition 4.1.6. *Let* $\mathbf{G}$ *be an algebraic* $\mathbb{Q}$*-group and let* $\Gamma \subset \mathbf{G}(\mathbb{R})$ *be an arithmetic subgroup. Then* Γ *is discrete in* $\mathbf{G}(\mathbb{R})$*.*

Proof. Let L be a $\mathbb{Z}$-lattice in $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$. Since L is discrete in $V_{\mathbb{R}}$, then $\mathrm{End}(L) \subset \mathrm{End}(V_{\mathbb{R}})$ is discrete. It follows that that $GL(L) \subset GL(V_{\mathbb{R}})$ is discrete, and hence G_L is discrete. ☐

The following is an incredibly important result in the area, which determines which an arithmetic group is a lattice or cocompact.

Theorem 4.1.7 ([BoHC] 7.8)**.** *Let* $\mathbf{G}$ *be a connected algebraic* $\mathbb{Q}$*-group and let* $\Gamma \subset \mathbf{G}(\mathbb{R})$ *be an arithmetic subgroup. Let* $X(\mathbf{G})_{\mathbb{Q}}$ *denote the set of* $\mathbb{Q}$*-rational characters of* $\mathbf{G}$*.*

[2] There are many equivalent definitions for "field of definition" or more generally a "ring of definition." For one such definition, we refer the reader to [Vin].

1. *Suppose $X(\mathbf{G})_{\mathbb{Q}} = \{1\}$. Then Γ is a lattice.*

2. *Suppose $X(\mathbf{G})_{\mathbb{Q}} = \{1\}$ and $\mathbf{G}(\mathbb{Q})$ contains no unipotent elements. Then Γ is cocompact.*

Applying this theorem to semisimple groups, we obtain the following results.

Corollary 4.1.8. *Let $\mathbf{G}$ be a connected semisimple algebraic $\mathbb{Q}$-group and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be an arithmetic group. Then*

1. *Γ is a lattice in $\mathbf{G}(\mathbb{R})$.*

2. *Γ is cocompact in $\mathbf{G}(\mathbb{R})$ if and only if $\mathbf{G}$ is $\mathbb{Q}$-anisotropic.*

Example 4.1.9. We now determine which of the arithmetic groups described in Example 4.1.1 are lattices and which are cocompact.

1. Since $\mathbf{GL}_n$ admits a nontrivial $\mathbb{Q}$-rational character, namely the determinant, $GL_n(\mathbb{Z})$ is discrete but not a lattice in $GL_n(\mathbb{R})$. It is easy to see that the quotient is not finite volume. It is topologically $SL_n(Z)\backslash SL_n(R)$ cross the ray (± 1) $(\mathbb{R} \setminus \{0\})$.

2. Since $\mathbf{SL}_n$ does not admit a nontrivial $\mathbb{Q}$-rational characters (since it is semisimple), $SL_n(\mathbb{Z})$ is a lattice in $SL_n(\mathbb{R})$. However, $SL_n(\mathbb{Z})$ is not cocompact because there exist unipotent elements in $SL_n(\mathbb{Q})$ (or equivalently since $\mathbf{SL}_n$ is $\mathbb{Q}$-isotropic).

3. Since $\mathbf{SL}_1(D)$ is $\mathbb{Q}$-anisotropic, $SL_1(\mathcal{O}_D)$ is in fact a cocompact lattice in $SL_2(\mathbb{R})$.

4.2. Arithmetic Lattices in Semisimple Lie Groups

In this section, we discuss how arithmetic groups yield lattices in semisimple Lie groups with no compact factors. Let G be a connected, adjoint, semisimple Lie group with no compact factors. Let $\Gamma \subset G$ be a lattice. Then Γ is **arithmetic** if there exists a semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and a surjective analytic homomorphism $\pi : \mathbf{G}(\mathbb{R})^\circ \to G$ with compact kernel such that $\pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)$ and Γ are commensurable. In what follows, we shall say that $\mathbf{G}$ **gives rise to** Γ. If $\mathbf{H} \subset \mathbf{G}$ is a $\mathbb{Q}$-simple factor, we may and will always assume that it is $\mathbb{R}$-isotropic, since otherwise $\mathbf{H}(\mathbb{R})^\circ \subset \ker(\pi)$, and we may just replace $\mathbf{G}$ with $\mathbf{G}/\mathbf{H}$.

Example 4.2.1. We now write down some arithmetic subgroups of $PSL_2(\mathbb{R})$.

1. The groups $SL_n(\mathbb{Z})$, $SL_n, (\mathbb{Z}, l)$ and $SL_n(\mathcal{O}_D)$ from Example 4.1.1 are arithmetic groups of $SL_2(\mathbb{R})$, hence upon taking the quotient of $SL_n(\mathbb{R})$ by its center, gives an arithmetic subgroup of $PSL_2(\mathbb{R})$.

2. More generally, let k be a totally real number field and let D be a quaternion algebra with center k such that D ramifies at all but one real place. Our splitting conditions restrictions on D gives

$$(R_{k/\mathbb{Q}}\mathbf{SL}_1(D))(\mathbb{R}) \cong SL_2(\mathbb{R}) \times \prod SU_3,$$

hence it surjects onto $PSL_2(\mathbb{R})$ with compact kernel. Let $\mathcal{O}_D \subset D$ be an order. Then $SL_1(\mathcal{O}_D)$ is an arithmetic subgroup of $(R_{k/\mathbb{Q}}\mathbf{SL}_1(D))(\mathbb{Q})$ which surjects onto a lattice in $PSL_2(\mathbb{R})$.

3. Let φ be an analytic automorphism of $PSL_2(\mathbb{R})$, then $\varphi(PSL_2(\mathbb{Z}))$ is also an arithmetic lattice. In particular, something like $\begin{pmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{pmatrix} PSL_2(\mathbb{Z}) \begin{pmatrix} \pi^{-1} & 0 \\ 0 & \pi \end{pmatrix}$ is an arithmetic lattice in $PSL_2(\mathbb{R})$, even though it is not commensurable with $PSL_2(\mathbb{Z})$.

It may appear as though arithmetic lattices are rather specific and potentially rare type of lattice. However, thanks to Margulis's amazing arithmeticity theorem [Mar] and the work of Gromov and Schoen [GS], irreducible lattices in groups not locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$ are always arithmetic. In particular, irreducible lattices in $\mathbb{R}$-rank 2 and higher are all automatically arithmetic.

Two subgroups Λ, Λ' of a G are **commensurable up to G-automorphism** if there exists an analytic $\varphi \in \mathrm{Aut}(G)$ such that Λ and $\varphi(\Lambda')$ are commensurable. It can be shown that commensurability up to G-automorphism is an equivalence relation among subgroups of G. It is not hard to so that if $\Gamma, \Gamma' \subset G$ are two subgroups which are commensurable up to G-automorphism and one in an arithmetic lattice, then so is the other.

In this thesis, we will primarily concern ourselves with arithmetic lattices in semisimple Lie groups coming from the restriction of scalars of absolutely almost simple groups arising from quadratic forms over number fields. These cover all groups of Cartan–Killing type B_n and "half" groups of Cartan–Killing type D_n. For the sake of completeness, we will also briefly discuss arithmetic lattices coming from the restriction of scalars of absolutely almost simple groups (of Cartan–Killing type D_n) arising from skew hermitian forms over division algebras over number fields. These two cases cover all $\mathbb{Q}$-simple groups coming from simple groups over number field of type B_n , $n \geq 1$ and D_n, $n \geq 5$.

4.3. Arithmetic Locally Symmetric Spaces

An **arithmetic locally symmetric space of noncompact type** is a space M of the form $\Gamma\backslash G/K$ where G is a connected, adjoint, semisimple Lie group with no compact factors, $K \subset G$ is a maximal compact subgroup, and Γ is arithmetic as defined in the previous section. Hence, lying beneath this definition, there is a semisimple algebraic $\mathbb{Q}$-group $\mathbf{G}$ and a projection $\pi : \mathbf{G}(\mathbb{R})^\circ \to G$ with compact kernel such that $\pi(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)$ is commensurable up to G-automorphism with Γ. Let K' denote a maximal compact subgroup of $\mathbf{G}(\mathbb{R})^\circ$ containing $\pi^{-1}(K)$, then $\ker(\pi) \subset K'$ and hence $(\mathbf{G}(\mathbb{Z}) \cap \mathbf{G}(\mathbb{R})^\circ)\backslash\mathbf{G}(\mathbb{R})^\circ/K'$ is commensurable with $\Gamma\backslash G/K$.

Since K and $\ker(\pi)$ are compact, and an algebraic group over $\mathbb{R}$ has only finitely many connected components, we note that M is compact (resp. has finite volume) if and only if $\mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R})$ is compact (resp. has finite volume). As such, Corollary 4.1.8 is an essential tool for determining whether a given arithmetic locally symmetric space is compact (resp. has finite volume).

By Borel's Density Theorem [B60], it follows that lattices in real semisimple algebraic groups without compact factors are Zariski dense, and hence thanks to Lemma 4.1.5 and Theorem 4.1.4, we may define the field of definition of a commensurability class of finite volume locally symmetric spaces which come from absolutely almost simple groups over a number field. Let $M = \Gamma\backslash G/K$ be a locally symmetric space. Then **the field of definition** of M is $k(M) := k(\Gamma)$.

The following is an immediate consequence of the definitions.

Lemma 4.3.1. *Let M_1 and M_2 be commensurable locally symmetric spaces of noncompact type. If M_1 is arithmetic, so is M_2.*

The following two theorem are known, but as of the writing of this paper, we are unaware of references. As such, we provide a proofs here.

Theorem 4.3.2. *Let M be an arithmetic locally symmetric space of noncompact type. Let $N \subset M$ be a nonflat finite volume totally geodesic subspace. Then N is arithmetic.*

Proof. Let $M = \Gamma'\backslash G/K'$ for G a connected, adjoint, semisimple Lie group with no compact factors. By Proposition 2.3.4, there exists a connected, semisimple Lie subgroup $H' \subset G$ with no compact factors such that $N = \Lambda'\backslash H/(K' \cap H')$ where $\Lambda' := \Gamma' \cap H'$ is a lattice in H'.

By our arithmeticity assumption, there exists a semisimple $\mathbb{Q}$-group $\mathbf{G}$, an arithmetic group $\Gamma \subset \mathbf{G}(\mathbb{Q})$ such that there is a projection $\pi : \mathbf{G}(\mathbb{R})^\circ \to G$ with compact kernel and $\pi(\Gamma)$ is commensurable up to G-automorphism with Γ'. Let $\varphi \in \mathrm{Aut}(G)$ be such that $\pi(\Gamma)$ and $\varphi(\Gamma')$ are commensurable. Let H denote the connected componant of the intersection of $\pi^{-1}(\varphi^{-1}(H'))$ with the noncompact factors of $\mathbf{G}(\mathbb{R})$. (This group can also be viewed as the unique connected Lie subgroup of $\mathbf{G}(\mathbb{R})$ with Lie algebra $\mathrm{Lie}(\varphi^{-1}(H'))$.) Now let $K \subset \mathbf{G}(\mathbb{R})^\circ$ be a maximal compact containing $\pi^{-1}(K')$. It follows that M is commensurable to $M' := (\Gamma \cap \mathbf{G}(\mathbb{R})^\circ)\backslash \mathbf{G}(\mathbb{R})^\circ / K$ and hence $N' := \Lambda \backslash H/(K \cap H)$, where $\Lambda := \Gamma \cap H$, is commensurable with N. By Lemma 4.3.1 it suffices to show the arithmeticity of N'. The result then follows by Proposition 4.3.3 below. □

Proposition 4.3.3. *Let*

1. $\mathbf{G}$ *be an semisimple* $\mathbb{Q}$*-group,*

2. $\Gamma \subset \mathbf{G}(\mathbb{Q})$ *be an arithmetic subgroup,*

3. $H \subset \mathbf{G}(\mathbb{R})$ *be a connected semisimple Lie subgroup with no compact factors, and*

4. $\Lambda \subset \Gamma$ *be a subgroup which is also a lattice in* H*.*

Then $H = \mathbf{H}(\mathbb{R})^\circ$ *where* $\mathbf{H} \subset \mathbf{G}$ *is a semisimple* $\mathbb{Q}$*-subgroup and* $\Lambda \subset \mathbf{H}(\mathbb{Q})$ *is arithmetic.*

Proof. Since H is a semisimple Lie group sitting inside the real points of a linear group, H is the connected component of the real points of some semisimple $\mathbb{R}$-subgroup $\mathbf{H} \subset \mathbf{G}$. By Borel's Density Theorem [B60] Λ is Zariski dense in $\mathbf{H}$. The Zariski closure of an abstract subgroup sitting inside the $\mathbb{Q}$-points of a group is also a $\mathbb{Q}$-group ([B1] Chapter 1 Proposition 1.3 (b)). Hence $\mathbf{H}$ is defined over $\mathbb{Q}$. Furthermore, let $V := \mathrm{Lie}(\mathbf{G})$ and $W := \mathrm{Lie}(\mathbf{H})$. Now the adjoint representation $\mathrm{Ad} : \mathbf{G} \to \mathbf{GL}(V)$ is defined over $\mathbb{Q}$. There is a lattice $L \subset V$ which Γ stabilizes ([B2] Proposition 7.12). Since Λ stabilizes W, it stabilizes $L \cap W$ and hence Λ is an arithmetic subgroup of H. □

Theorem 4.3.4. *Let* M_1 *and* M_2 *be finite volume arithmetic locally symmetric spaces arising from the semisimple* $\mathbb{Q}$*-groups* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *respectively. Then* M_1 *and* M_2 *are commensurable if and only if* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are* $\mathbb{Q}$*-isomorphic.*

Proof. First suppose $\mathbf{G}_1$ and $\mathbf{G}_2$ are $\mathbb{Q}$-isomorphic via the $\mathbb{Q}$-isomorphism φ. By assumption, there are $\Gamma_i' \subset \mathbf{G}_i(\mathbb{Q})$ arithmetic such that M_i is commensurable with $\Gamma_i \backslash \mathbf{G}_i(\mathbb{R})^\circ / K_i$, and the result then immediately follows from the fact that $\varphi(\Gamma_1)$ and Γ_2 are commensurable.

Now suppose M_1 and M_2 are commensurable. By assumption, there exists a connected adjoint semisimple Lie group with no compact factors, G, and two arithmetic lattices $\Gamma_1, \Gamma_2 \subset G$ which are commensurable up to G-automorphism, such that $M_i = \Gamma_i \backslash G/K$ where K is a maximal compact subgroup. The result then follows from Proposition 4.3.5 below. □

Proposition 4.3.5. *Let* G *be a connected adjoint semisimple Lie group with no compact factors. Let* $\Gamma_1, \Gamma_2 \subset G$ *be arithmetic lattices which are commensurable up to* G*-automorphism. Let* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *be the connected adjoint semisimple* $\mathbb{Q}$*-groups giving rise to* Γ_1 *and* Γ_2 *respectively. Then* $\mathbf{G}_1$ *and* $\mathbf{G}_2$ *are* $\mathbb{Q}$*-isomorphic.*

Proof. Let φ be an analytic automorphism of G for which Γ_1 and $\varphi(\Gamma_2)$ are commensurable. Pick finite index $\Lambda_i \subset \Gamma_i$ which are isomorphic via φ. Let $\mathbf{H}_i \subset \mathbf{G}_i$ be the product of the connected $\mathbb{R}$-simple $\mathbb{R}$-isotropic components of $\mathbf{G}_i$. Then $\pi_i|_{\mathbf{H}_i(\mathbb{R})^\circ} : \mathbf{H}_i(\mathbb{R})^\circ \to G$ is an isomorphism. It follows that $\Lambda_i \subset \mathbf{H}_i(\mathbb{Q})$.

Since π_i induces an $\mathbb{R}$-rational isomorphism between $\mathbf{H}_i$ and $\mathbf{Aut}(\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C})$, and φ induces an $\mathbb{R}$-rational automorphism on $\mathbf{Aut}(\mathrm{Lie}(G) \otimes_{\mathbb{R}} \mathbb{C})$, it follows that there is an $\mathbb{R}$-rational isomorphism, which we also denote φ, from $\mathbf{H}_1$ to $\mathbf{H}_2$ which sends Λ_1 to Λ_2.

For each i, $\mathbf{G}_i \cong \prod_{j=1}^{r_i} R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$ where $\mathbf{S}_j$ is an absolutely simple group over a number field $k_{i,j}$. Furthermore, $\Lambda_{i,j} := \Lambda_i \cap (R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j})(\mathbb{Q})$ is an arithmetic group in $(R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j})(\mathbb{Q}) = \mathbf{S}_{i,j}(k_{i,j})$ ([BoHC], 6.11). Since $\mathbf{G}_i$ give rise to locally symmetric spaces of noncompact type, we may assume that no $\mathbb{Q}$-simple normal subgroup of either $\mathbf{G}_i$ is $\mathbb{R}$-anisotropic. Hence Borel's Density Theorem [B65] implies that $\Lambda_{i,j}$ is Zariski dense in $\mathbf{S}_{i,j}$. Since each $\Lambda_{i,j}$ is a normal irreducible subgroup of Λ_i, the isomorphism φ must send each $\Lambda_{1,j}$ to some $\Lambda_{2,j'}$, from which we conclude $r_1 = r_2 := r$ and φ induces a permutation also denoted $\varphi \in S_r$. Our assumption on $\mathbb{Q}$-simple factors implies that each $R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$ contains an $\mathbb{R}$-simple $\mathbb{R}$-isotropic factor. Since φ sends $\mathbb{R}$-isotropic $\mathbb{R}$-simple factors of $R_{k_{1,j}/\mathbb{Q}} \mathbf{S}_{1,j}$ to $\mathbb{R}$-isotropic $\mathbb{R}$-simple factors of $R_{k_{2,\varphi(j)}/\mathbb{Q}} \mathbf{S}_{2,\varphi(j)}$, we conclude $\mathbf{S}_{2,j}$ and $\mathbf{S}_{2,\varphi(j)}$ have the same Cartan-Killing type. Let $\mathbf{H}_{i,j}$ be a fixed $\mathbb{R}$-simple $\mathbb{R}$-isotropic component of $R_{k_{i,j}/\mathbb{Q}} \mathbf{S}_{i,j}$. Then φ induces an F-isomorphism between $\mathbf{S}_{1,j}$ and $\mathbf{S}_{2,\varphi(j)}$, where $F = \mathbb{R}$ when $\mathbf{H}_{1,j}$ is absolutely simple, and $F = \mathbb{C}$ otherwise. Furthermore, this isomorphism sends $\Lambda_{1,j}$ to $\Lambda_{2,\varphi(j)}$, hence by [PR09] Proposition 2.5, $k_{1,j} = k_{2,\varphi(j)} =: k_j$ and $\mathbf{S}_{1,j}$ and $\mathbf{S}_{2,\varphi(j)}$ are k_j-isomorphic. The conclusion follows. □

In the next sections, we will explicitly go through the constructions arithmetic locally symmetric spaces coming from $\mathbb{Q}$-simple groups arising from quadratic forms over number fields and skew hermitian forms over division algebras over number fields.

4.4. Arithmetic Locally Symmetric Spaces Coming From Quadratic Forms

In this section, we give an explicit construction of the irreducible arithmetic lattices of semisimple Lie groups of the form

$$G = \prod_1^r SO(p_i, q_i) \times \prod_1^s SO_m(\mathbb{C}),$$

where $1 \leq p_i, q_i \leq m-1$ and $p_i + q_i = m$, which come from quadratic forms. These lattices arise from the $\mathbb{Q}$-simple groups that are formed by applying the restriction of scalars functor to the isometry group of a quadratic form over a number field. When $m > 3$ is odd, all irreducible arithmetic lattices arise from this construction.

A quadratic space (V, q) over a number field k gives rise to an algebraic k-group $\mathbf{SO}(q)$, namely it is the algebraic k-group $\mathbf{G}$ having the property that for any field extension E/k,

$$\mathbf{G}(E) = \{T \in GL(V_E) \mid \forall v \in V_E, q(Tv) = q(v)\}.$$

Restriction of scalars this gives rise to an algebraic $\mathbb{Q}$-group $R_{k/\mathbb{Q}}\mathbf{SO}(q)$, and hence arithmetic groups. In this section, we look at this construction more carefully so as to give us insight into subgroups coming from subforms.

We begin by showing how these forms give absolutely almost simple k-groups, which then give semisimple Lie groups. We then show how certain totally geodesic submanifolds of these spaces arise.

1. Let k be a number field with infinite places $V_k^\infty = \{v_1, \ldots, v_l\}$.

2. Let (V, q) an m-dimensional quadratic space over k such that at each real place v_i, the quadratic form $q_{v_i} := q \otimes k_{v_i}$ on $V \otimes_k k_{v_i}$ has signature $(m_+^{(v_i)}, m_-^{(v_i)})$.

3. Let $\mathbf{G} = \mathbf{SO}(V, q)$ be the absolutely almost simple algebraic k group defined by (V, q) and let $SO(q) := \mathbf{G}(k)$ be its group of k-points.

4. For each $v_i \in V_k^\infty$, let $\mathbf{G}_{v_i}$ denote the algebraic group defined by $(V_{k_{v_i}}, q_{v_i})$. If v_i is a real place, then $\mathbf{G}_{v_i}(\mathbb{R}) \cong SO(m_+^{(v_i)}, m_-^{(v_i)})$. If v_i is a complex place, then $\mathbf{G}_{v_i}(\mathbb{R}) \cong SO_m(\mathbb{C})$.

5. Let $\mathbf{G}' := R_{k/\mathbb{Q}}\mathbf{G}$ be the algebraic $\mathbb{Q}$-group formed by restriction of scalars. Then $\mathbf{G}'$ is a semisimple $\mathbb{Q}$-group and $\mathbf{G}'(\mathbb{R}) = \prod \mathbf{G}_{v_i}(\mathbb{R})$ is a semisimple Lie group which has compact factors at precisely the real places where q is anisotropic. By the construction of restriction of scalars, there is an isomorphism $\mathbf{G}(k) \cong \mathbf{G}'(\mathbb{Q})$. Hence there is a natural embedding

$$SO(q) \to \left(\prod_{\substack{v_i \text{ real} \\ q_i \text{ anisotropic}}} SO(m) \times \prod_{\substack{v_i \text{ real} \\ q_i \text{ isotropic}}} SO(m_+^{(v_i)}, m_-^{(v_i)}) \times \prod_{v_1 \text{ complex}} SO_m(\mathbb{C}) \right)$$

6. Let G be the projection of $\mathbf{G}'(\mathbb{R})$ onto its noncompact factors and denote the projection map by $\pi : \mathbf{G}'(\mathbb{R}) \to G$.

7. Fix an $\mathcal{O}_k$-lattice $L \subset V$ and let $G_L = \{T \in \mathbf{G}(k) \mid T(L) \subset L\}$. By the theory of arithmetic groups, G_L sits as a discrete arithmetic subgroup of the semisimple Lie group $\mathbf{G}'(\mathbb{R})$.

8. Define Γ_L to be the projection of G_L to G. Then Γ_L sits as a discrete arithmetic lattice of the semisimple Lie group without compact factors G. Let $K \subset G$ its maximal compact subgroup and let $M_L := \Gamma_L \backslash G/K$. This space M_L is an **arithmetic locally symmetric space coming from a quadratic form.**

A choice of another $\mathcal{O}_k$-lattice $L' \subset V$ will produce a space $M_{L'}$ which is commensurable with M_L. Hence choosing q uniquely determines a commensurability class which we denote by M_q.

Let (W, r) be a quadratic k-subspace of (V, q). Then $\mathbf{H} = \mathbf{SO}(W, r)$ is the absolutely almost simple k-subgroup of $\mathbf{G}$ having the property that for any field extension E/k,

$$\mathbf{H}(E) = \{T \in GL(W_E) \mid \forall w \in W_E, q(Tw) = q(w)\}.$$

Let $\mathbf{H}' := R_{k/\mathbb{Q}}\mathbf{H}$. Then $\mathbf{H}'$ is a semisimple $\mathbb{Q}$-subgroup of $\mathbf{G}'$. It follows that $L \cap W$ is an $\mathcal{O}_k$-lattice of W, hence $G_L \cap \mathbf{H}'(\mathbb{R})$ is an arithmetic subgroup of $\mathbf{H}'(\mathbb{R})$. Let H be the image of $\mathbf{H}'(\mathbb{R})$ under the projection map π onto the noncompact factors of $\mathbf{G}'(\mathbb{R})$. Then $\pi(G_L \cap \mathbf{H}'(\mathbb{R}))$ is an arithmetic subgroup of H. Note that H may be trivial. By the remarks after Proposition 2.3.6, $N_r := \pi(G_L \cap \mathbf{H}'(\mathbb{R})) \backslash H/(H \cap K)$ is a totally geodesic submanifold of M_L.

We have just proven the following result:

Proposition 4.4.1. *Let k be a number field and q a quadratic form over k. Every quadratic subform r of q produces a commensurability class of totally geodesic submanifolds $N_r \subset M_q$. Furthermore, if $\dim r > 2$ and r is isotropic at a real place of k, then N_r is a commensurability class of nontrivial, nonflat, finite volume, locally symmetric spaces of noncompact type.*

Remark 4.4.2. An arithmetic hyperbolic space coming from a quadratic form of dimension greater than 4 is compact if and only if its field of definition k is strictly larger than $\mathbb{Q}$. This can easily be deduced from the fact that if $|k : \mathbb{Q}| > 1$, then there is more than one real place. By the above construction, the form must is anisotropic over all but one place, and hence the form must be anisotropic over k. By Theorem 4.1.7, the result follows. Conversely, if $k = \mathbb{Q}$, then the form must be isotropic, and again by Theorem 4.1.7 the result follows.

Proposition 4.4.3. *All even dimensional arithmetic hyperbolic manifolds come from quadratic forms.*

Sketch of proof. Let M be a $2n$-dimensional hyperbolic manifold. Then M is a locally symmetric space coming from a group of Cartan–Killing type B_n. For $n \geq 2$, then by the classification of groups over number fields [Ti], the result immediately follows. For $n = 1$, then there is an exceptional isomorphism $A_1 = B_1$, hence M could come from a number field, or be an $\mathbb{R}$-rank 1 group of type A_1. However in fact, there is nothing new. Suppose we have a group of type A_1 coming from the division algebra $D = \left(\frac{a,b}{k}\right)$ with k a totally real number field over which D ramifies at all but one infinite place. The isomorphism class is determined by the 4-dimensional quadratic norm form $\varphi := \langle 1, -a, -b, ab \rangle$. Note that at each of these real places, this form is anisotropic if and only if the 3-dimensional subform $q := \langle -a, -b, ab \rangle$ is anisotropic. Let $D_0 \subset D$ be the set of pure quaternions. Then (D_0, q) is a 3-dimensional quadratic space and $SL_1(D)$ is its set of isometries, since $Nrd(\mu\delta) = Nrd(\delta)$ for all $\delta \in D_0$ and $\mu \in SL_1(D)$. Conversely, let q be a 3-dimensional form which is anisotropic at all but one real place. It follows that q is similar to a quadratic subform of a 4-dimensional form φ with diagonal representation $\langle 1, -a, -b, ab \rangle$ for some $a, b \in k^\times$ (see ([Lam] X.4.21) and ([Lam] X.4.16)). Hence the group of isometries of q is contains the norm 1 group of the division algebra $D = \left(\frac{a,b}{k}\right)$. □

4.5. Arithmetic Locally Symmetric Spaces Coming From Skew Hermitian Forms Over Division Algebras Over Number Fields

While in this thesis we do not need the specific of the construction of the other type of arithmetic lattice arising in groups of type D_n, we recall it for completeness. These arithmetic groups arise from skew Hermitian forms over a quaternion division algebra over a number field. For more on these lattices, we refer the reader to [LM].

1. Let k be a number field.
2. Let D be a quaternion division algebra with center k.
3. Let (V, h) an n-dimensional skew Hermitian space over D.
4. Let $\mathbf{G} = \mathbf{SU}(V, h)$, $G' := (R_{k/\mathbb{Q}}\mathbf{G})(\mathbb{R})$, where R denotes restriction of scalars.
5. Let G be the projection of G' and onto its noncompact factors and $K \subset G$ its maximal compact subgroup.
6. Fix an order $\mathcal{O}_D$ in D and an $\mathcal{O}_D$-lattice $L \subset V$, and let $G_L = \{T \in \mathbf{G}(k) \mid T(L) \subset L\}$.
7. Define Γ_L to be the projection of G_L to G and let $M_L = \Gamma_L \backslash G/K$.

A choice of another order in D and another lattice $L' \subset V$ will produce a space $M_{L'}$ which is commensurable with M_L. Hence choosing q uniquely determines a commensurability class which we denote by M_q.

Chapter 5

The Index of Isometry Groups of Quadratic Forms

The Tits index is combinatorial data assigned to an algebraic group. In Section 5.1 we introduce the definition of the index and some of its basic properties. In Section 5.2 we study the types of indices coming from isometry groups of quadratic forms over local fields. In particular we establish a dictionary between the indices of such groups and the classical invariants of the associated forms. In Section 5.3 we study how different two forms can be if they have isomorphic isometry groups. In Section 5.4, we use our results to rederive Maclachlan's parametrization of commensurability classes of even dimensional arithmetic hyperbolic spaces. In Section 5.5 we establish a dictionary between the invariants of skew hermitian forms over division algebras over local fields and the indices of their unitary groups.

5.1. The Tits Index of an Algebraic Group

Semisimple algebraic groups defined over a separably closed field are classified (up to isogeny) by their Cartan–Killing type (i.e., Dynkin diagram). However, over a general field k, there can be many semisimple k-groups which are not k-isogenous but have the same Cartan–Killing type over k^{sep}, the separable closure of k. For example, the Lie groups $SL_2(\mathbb{R})$ and $SU(2)$ are the real points of $\mathbb{R}$-groups which over $\mathbb{C}$ have Cartan–Killing type A_1, yet are not $\mathbb{R}$-isogenous.

The Tits index of a semisimple groups is a collection of combinatorial data, analogous to the Dynkin diagram, that largely encapsulates the k-isogeny classes of a semisimple k-group. A full exposition of the theory can be found in [Ti]. In this section we define the the Tits index and present the Tits classification theorem.

Let $\mathbf{G}$ be a semisimple algebraic group defined over a field k. Let $\mathbf{S} \subset \mathbf{G}$ be a maximal k-split torus and let $\mathbf{T}$ be a maximal torus containing $\mathbf{S}$ which is defined over k. Then $\mathbf{T}$

splits over k^{sep}, the separable closure of k, hence there is an action

$$\rho : \mathrm{Gal}(k^{sep}/k) \to \mathrm{Aut}(\mathbf{G}).$$

Let $\mathbf{B}$ be a Borel subgroup defined over k^{sep} containing $\mathbf{T}$. For each $\gamma \in \mathrm{Gal}(k^{sep}/k)$, the map ρ_γ sends $\mathbf{B}$ to some other Borel subgroup $\rho_\gamma(\mathbf{B})$.

We can associate to the pair $(\mathbf{G}, \mathbf{T})$ its (absolute) root system $\Phi(\mathbf{G}, \mathbf{T})$ in which case ρ induces an action

$$\rho : \mathrm{Gal}(k^{sep}/k) \to \mathrm{Aut}(\Phi(\mathbf{G}, \mathbf{T})).$$

Let $\Delta(\mathbf{B}, \mathbf{T}) \subset \Phi(\mathbf{G}, \mathbf{T})$ be the set of simple roots associated to $\mathbf{B}$. A simple root $\alpha \in \Delta(\mathbf{B}, \mathbf{T})$ which is nontrivial when restricted to $\mathbf{S}$ is called a **distinguished root**. For each $\gamma \in \mathrm{Gal}(k^{sep}/k)$, the map ρ_γ sends $\Delta(\mathbf{B}, \mathbf{T})$ to $\Delta(\rho_\gamma(\mathbf{B}), \mathbf{T})$. For each such set of simple roots, there exists a unique element w_γ of the Weyl group $W(\mathbf{G}, \mathbf{T})$ of $\Phi(\mathbf{G}, \mathbf{T})$ which sends $\Delta(\rho_\gamma(\mathbf{B}), \mathbf{T})$ to $\Delta(\mathbf{B}, \mathbf{T})$. This defines a **the $*$-action**

$$* : \mathrm{Gal}(k^{sep}/k) \to \mathrm{Aut}(\Delta(\mathbf{B}, \mathbf{T}))$$

$$\gamma \mapsto w_\gamma \circ \rho_\gamma.$$

For the triple $(\mathbf{G}, \mathbf{T}, \mathbf{S})$ we can assign its **Tits index** which is the aggregation of the following data:

1. The Dynkin diagram associated to the absolute root system $\Phi(\mathbf{G}, \mathbf{T})$.
2. Circle the distinguished orbits.
3. The $*$-action of $\mathrm{Gal}(k^{sep}/k)$.

We often denote the Tits index of an absolutely almost simple k-group $\mathbf{G}$ by ${}^g X_{n,r}^{(d)}$ where:

- X_n is the Cartan–Killing type of $\mathbf{G} \otimes k^{sep}$,
- n is the k^{sep}-rank of $\mathbf{G}$
- g is the order of the image of the $*$-action map[1],
- r is the k-rank of $\mathbf{G}$, and
- d is is an additional invariant[2].

[1] In the cases we are analyzing in this thesis, $g = 1$ or $g = 2$ depending on whether $\mathbf{G}$ in an inner or outer form respectively.

[2] In the cases we are analyzing in this thesis, d is the degree of the division algebra associated with the group. In particular, for quadratic forms this is always 1. When d is 1, we often leave the spot blank.

Example 5.1.1. Let $\mathbf{G}$ be a group coming from an $2n+1$ dimensional quadratic form q defined over a field k and Witt index r. Then $\mathbf{G} = \mathbf{SO}(q)$ and has Cartan–Killing type B_n. There are no nontrivial automorphism of the diagram, so the $*$-action is trivial. By assumption, q has Witt index r, and hence $q = q' \oplus q_0$ where q' is hyperbolic of dimension $2r$ and q_0 is anisotropic of dimension $2(n-r)+1$.

Thus the Tits index of type ${}^1B_{n,n-1}$ can be represented:

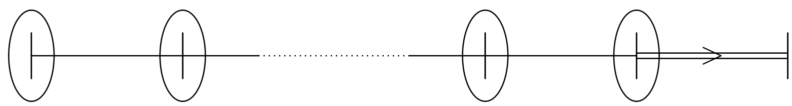


Figure 5.1: Tits index of the nonsplit group of type B_n over a nonarchimedean local field.

The **semisimple anisotropic kernel** of $\mathbf{G}$ is the group $\mathscr{D}(Z_{\mathbf{G}}(\mathbf{S}))$ where $Z_{\mathbf{G}}(\mathbf{S})$ is the centralizer of $\mathbf{S}$ in $\mathbf{G}$ and $\mathscr{D}(Z_{\mathbf{G}}(\mathbf{S}))$ denotes its derived group. Observe that by construction, this group is always an anisotropic k-subgroup of $\mathbf{G}$.

Theorem 5.1.2 (Tits Classification)**.** *A semisimple group $\mathbf{G}$ defined over k is determined up to k-isomorphism by its k^{sep}-isomorphism class, its index, and its semisimple anisotropic kernel.*

Proof. See [Ti] Theorem 2.7.1. □

Corollary 5.1.3. *If two semisimple k-groups have nonisomorphic indices, they cannot be isomorphic.*

5.2. The Dictionary Between the Index of $SO(q)$ and the Invariants of q

In this section we relate the classical invariants of a quadratic form to the Tits index of its corresponding isometry group. We begin by recalling some basic facts relating a form's invariants and whether or not it is isotropic.

Proposition 5.2.1. *Let L be a nonarchimedean local field.*

1. *Let q' be a 3-dimensional quadratic form over L. Then q' is isotropic if and only if $c(q') = (-1, -\det q')$.*

2. *Let q' be a 4-dimensional quadratic form over L. Then q' is anisotropic if and only if $\operatorname{disc}(q') = 1$ and $c(q') = -(-1,-1)$.*

Proposition 5.2.2. *Let L be a nonarchimedean local field and let q be an m-dimensional quadratic form over L where $m \geq 5$. Then q is isotropic.*

For proofs, see [Ca] Chp 4 Lemmas 2.5, 2.6, and 2.7 p. 59-60. Though the proofs are explicitly written with $k = \mathbb{Q}$, they are generalizable to an arbitrary number field. We now use these results to relate a form's invariants to its index. See the Table 5.1 below to see the summary of this section's results.

Proposition 5.2.3. *Let k be a number field. Let q be a quadratic form of dimension $2n+1$. Then the Tits index of $\mathbf{SO}(q_v)$ at a finite place $v \in V_k$ is $B_{n,n}$ if and only if*

$$c_v(q_v) = (-1,-1)_v^{\frac{n(n-3)}{2}} (-1, \det(q_v))_v^n. \tag{5.1}$$

Remark 5.2.4. Note that this shows things are not as simple as just on-off whether or not c is trivial. However it is clear that two forms with the **same determinant** which differ at a c have different diagrams, though it is not immediately obvious which one is the split one from the c data.

Proof. We will show that the following statements are equivalent:

1. $\mathbf{SO}(q_v)$ is of type $B_{n,n}$.
2. $q_v \cong \langle 1, -1\rangle^{n-1} \oplus q'_v$ where q'_v is an isotropic 3-dimensional form.
3. $q_v \cong \langle 1, -1\rangle^{n-1} \oplus q'_v$ where $c_v(q'_v) = (-1, -\det(q'_v))$.
4. $c_v(q_v) = (-1,-1)^{\frac{n(n-3)}{2}} (-1, \det(q_v))^n$.

First 1 is equivalent to 2 by the classification of algebraic k-groups in [Ti]. Next, 2 is equivalent to 3 by Proposition 5.2.1 (1). Lastly 3 is equivalent to 4 by the following computation:

$$\begin{aligned}
c_v(q_v) &= c_v\left(\langle 1, -1\rangle^{n-1} \oplus q'_v\right) \\
&= c_v(\langle 1, -1\rangle^{n-1})\ c_v(q'_v)\ ((-1)^{n-1}, \det(q'_v)) \\
&= (-1,-1)^{\frac{(n-1)(n-2)}{2}}\ (-1, -\det(q'_v))\ (-1, \det(q'_v))^{n-1} \\
&= (-1,-1)^{\frac{(n-1)(n-2)}{2}+1}\ (-1, \det(q'_v))^n \\
&= (-1,-1)^{\frac{(n^2-3n+2+2)}{2}}\ (-1, (-1)^{n-1}\det(q_v))^n \\
&= (-1,-1)^{\frac{n^2-3n}{2}+2}\ (-1, \det(q_v))^n \\
&= (-1,-1)^{\frac{n(n-3)}{2}}\ (-1, \det(q_v))^n.
\end{aligned}$$

$\square$

Proposition 5.2.5. *Let k be a number field. Let q be a quadratic form of dimension $2n$. Then the Tits index at a finite place $v \in V_k$ of $\mathbf{SO}(q_v)$ is ${}^1D_{n,n-2}$ if and only if*

$$\operatorname{disc}(q_v) = 1 \qquad c_v(q_v) = -(-1,-1)^{\frac{n(n-1)}{2}}. \tag{5.2}$$

Proof. We will show that the following statements are equivalent:

1. $\mathbf{SO}(q_v)$ is of type ${}^1D_{n,n-2}$.

2. $q_v = \langle 1, -1\rangle^{n-2} \oplus q'_v$ where q'_v is an anisotropic 4-dimensional form.

3. $q_v = \langle 1, -1\rangle^{n-2} \oplus q'_v$ where $\operatorname{disc}(q'_v) = 1$ and $c_v(q'_v) = -(-1,-1)$.

4. $\operatorname{disc}(q_v) = 1$ and $c_v(q_v) = -(-1,-1)^{\frac{n(n-1)}{2}}$.

First 1 is equivalent to 2 by the classification of algebraic k-groups in [Ti]. Next, 2 is equivalent to 3 by Proposition 5.2.1 (2). Lastly 3 is equivalent to 4 by the following computations:

$$\begin{aligned}
\operatorname{disc}(q_v) &= \operatorname{disc}\left(\langle 1,-1\rangle^{n-2} \oplus q'_v\right)\\
&= \operatorname{disc}(\langle 1,-1\rangle^{n-2}) \operatorname{disc}(q'_v)\\
&= 1
\end{aligned}$$

$$\begin{aligned}
c_v(q_v) &= c_v\left(\langle 1,-1\rangle^{n-2} \oplus q'_v\right)\\
&= c_v(\langle 1,-1\rangle^{n-2})\ c_v(q'_v)\ ((-1)^{n-2}, \det(q'_v))\\
&= (-1,-1)^{\frac{(n-2)(n-3)}{2}}\ -(-1,-1)\\
&= -(-1,-1)^{\frac{(n-2)(n-3)}{2}+1}\\
&= -(-1,-1)^{\frac{(n^2-5n-6+2)}{2}}\\
&= -(-1,-1)^{\frac{(n^2-n)}{2}-2(n-1)}\\
&= -(-1,-1)^{\frac{n(n-1)}{2}}.
\end{aligned}$$

□

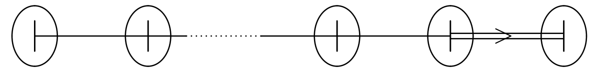
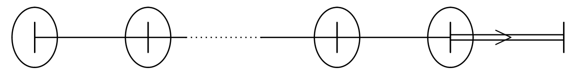
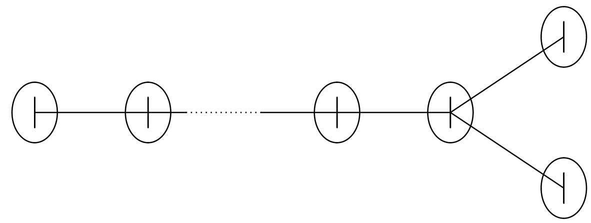
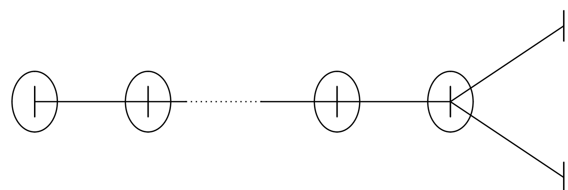
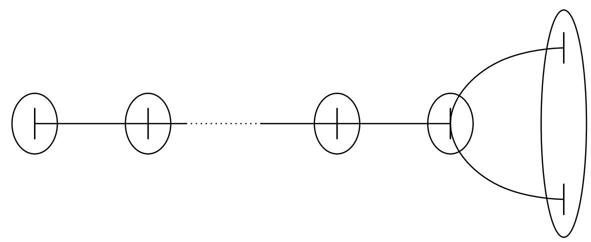
Type	Classical Invariants	Tits Index
$B_{n,n}$	$\dim(q) = 2n+1$ $\det(q) =$anything $c(q) = (-1,-1)^{\frac{n(n-3)}{2}}(-1,\det(q))^n$	
$B_{n,n-1}$	$\dim(q) = 2n+1$ $\det(q) =$anything $c(q) = -(-1,-1)^{\frac{n(n-3)}{2}}(-1,\det(q))^n$	
${}^1D^{(1)}_{n,n}$	$\dim(q) = 2n$ $\det(q) = (-1)^n$ (i.e. $\mathrm{disc}(q) = 1$) $c(q) = (-1,-1)^{\frac{n(n-1)}{2}}$	
${}^1D^{(1)}_{n,n-2}$	$\dim(q) = 2n$ $\det(q) = (-1)^n$ (i.e. $\mathrm{disc}(q) = 1$) $c(q) = -(-1,-1)^{\frac{n(n-1)}{2}}$	
${}^2D^{(1)}_{n,n-1}$	$\dim(q) = 2n$ $\det(q) \neq (-1)^n$ (i.e. $\mathrm{disc}(q) \neq 1$) $c(q) =$ anything	

Table 5.1: The classical invariants of a quadratic form over a nonarchimedean local field and the index of its associated isometry group.

5.3. Quadratic Forms Representing an Orthogonal Group

Let **G** be an algebraic group coming from a quadratic form. In general there are many isometry classes of forms that give the same group. Any quadratic form giving the group **G** shall be said to **represent G**. To translate our problems into studying quadratic forms, we will need to carefully pick the forms which represent the group.

We now begin exploring what types of forms represent the same group.

Lemma 5.3.1.

1. *Isometric forms represent the same group.*

2. Similar forms represent the same group.

Proof. We begin showing 1. Let q_1 and q_2 be isometric forms. Represent q_i with the symmetric m by m matrix Q_i and let $T \in GL_n(k)$ be the isometry between them. Then $SO(q_i) \subset SL_n(k)$ and since ${}^tTQ_1T = Q_2$, if $g \in SO(q_1)$, then $TgT^{-1} \in SO(q_2)$,

$$q_2(v) = q_1(T^{-1}v) = q_1(gT^{-1}v) = q_2(TgT^{-1}v)$$

hence $SO(q_i)$ are conjugate groups in $GL_n(k)$. We now show 2. Suppose $q_2 = aq_1$. Then pick $g \in SO(q_1)$.

$$q_2(v) = aq_1(v) = aq_1(gv) = q_2(gv)$$

Therefore $g \in SO(q_2)$, and hence $SO(q_i)$ are equal. □

Proposition 5.3.2. *Let q be an m-dimensional quadratic form where m is odd. Then q' represents $\mathbf{G} := \mathbf{SO}(q)$ if and only if q' is similar to q.*

Proof. In Lemma 5.3.1, we showed similar forms represent the same group. Now suppose q' represents $\mathbf{G}$. Let $a \in k^\times/(k^\times)^2$ such that $\det q' = a \det q$. We shall show aq and q' are isometric. Note that aq also represents $\mathbf{G}$, and since m is odd, $\det(aq) = a \det q = \det q'$.

- At each complex place $v \in V_k$, aq and q' have the same dimension, and hence are isometric by Proposition 3.3.1 (a).
- At each real place $v \in V_k$, since m is odd, Equation 3.1 shows that the index of $\mathbf{G}$ together with the determinant $\det q'$ uniquely determines the signature of $q' \otimes k_v$. Hence at each finite place, $\mathrm{sgn}(q') = \mathrm{sgn}(aq)$. Hence they are isometric by Proposition 3.4.1 (a).
- At each finite place $v \in V_k$, since m is odd, Equation 5.1 shows that the index $\mathbf{G}$ together with $\det q'$ uniquely determines $c(q')$. Hence at each finite place, $c(q') = c(aq)$. Hence they are isometric by Proposition 3.5.1 (a).

Hence by Theorem 3.6.1, aq and q' are isometric and the result follows. □

Using similar techniques, we prove the following theorem.

Theorem 5.3.3. *Let k be a number field, q and q' be $m = 2n+1$-dimensional quadratic forms over k, and $\mathbf{G}_i = \mathbf{SO}(q_i)$. Then $\mathbf{G}_1$ and $\mathbf{G}_2$ are k-isomorphic if and only if the groups $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ have the same index for all $v \in V_k$.*

In particular, if q is an $m = 2n+1$-dimensional quadratic forms over k, then the k-isomorphism class of $\mathbf{G} := \mathbf{SO}(q)$ is determined by its index at all places.

Proof. If $\mathbf{G}_1$ and $\mathbf{G}_2$ are k-isomorphic, then $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ are k_v-isomorphic for all $v \in V_k$, and hence by Theorem 5.1.2, they have the same index at every place.

We now prove the other direction and suppose that $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ have the same index for all $v \in V_k$. We may replace q_2 with the similar form $\frac{\det q_1}{\det q_2} q_2$, and since m is odd, we may now assume $\det q_1 = \det q_2$. As we observed in the proof of the previous proposition, at local places the index and the determinant determine the isometry class of a representing form. Therefore $q_1 \otimes k_v$ and $q_2 \otimes k_v$ are isometric for all $v \in V_k$, and hence by Theorem 3.6.1, q_1 and q_2 are isometric. The result follows from Lemma 5.3.1 (a). □

Remark 5.3.4. Theorem 5.3.3 says that the local index determines groups over number fields of Cartan-Killing type B_n. Unfortunately, a similar result cannot hold for groups of type D_n. In particular, there exists a number field k and k-groups $\mathbf{G}_1$ and $\mathbf{G}_2$ of type D_n, $n \equiv 1 \pmod 4$, which have the same index at every place $v \in V_k$, yet are not k-isomorphic. The existence of such examples is related to the existence of noncommensurable length-commensurable arithmetic locally symmetric spaces of type D_n for n odd. See ([PR09], 9.15) for details.

5.4. Parametrizing Commensurability Classes of Hyperbolic Manifolds

In this section, we show how the computations in Section 5.2 may be used to parametrize even dimensional arithmetic hyperbolic manifolds. In so doing, we shall provide another proof of the results of Maclachlan in [Mac]. For the reader's convenience we recall Maclachlan's parametrization here.

Theorem 5.4.1 (Maclachlan [Mac] Theorem 1.1)**.** *The commensurability classes of discrete arithmetic subgroups of* $\mathrm{Isom}(\mathbb{H}^{2n})$, $n \geq 1$, *are parametrized for each totally real number field* k *by sets of the form*

$$(v, \{p_1, p_2, \ldots, p_r\})$$

where $v \in V_K$ *is a real place and* $\{p_1, p_2, \ldots, p_r\}$ *are set of prime ideals in the ring of integers* $\mathcal{O}_k$ *where*

$$r \equiv \begin{cases} 0 \pmod 2 & \text{if } n \equiv 0 \pmod 4, \\ [k:\mathbb{Q}] - 1 \pmod 2 & \text{if } n \equiv 1 \pmod 4, \\ [k:\mathbb{Q}] \pmod 2 & \text{if } n \equiv 2 \pmod 4, \\ 1 \pmod 2 & \text{if } n \equiv 3 \pmod 4. \end{cases} \tag{5.3}$$

Maclachlan's method uses the theory of quaternion algebras and Clifford algebras. We will now use equation 5.1 of the previous section to quickly rederive his results. We will need the following lemma. Note that a place $v \in V_k$ is called **dyadic** if k_v is nonarchimedean with residue field of characteristic 2. For example, the place associated with the prime 2 is dyadic over $\mathbb{Q}$ since $\mathbb{Q}_2$ is nonarchimedean with residue field $\mathbb{Z}/2\mathbb{Z}$.

Lemma 5.4.2. *Let $k/\mathbb{Q}$ be a totally real number field. Let*

$$\delta(k) := \left\{ \text{number of dyadic places where } \left(\tfrac{-1,-1}{\mathbb{Q}}\right) \text{ ramifies} \right\}.$$

Then $\delta(k) \equiv [k:\mathbb{Q}] \pmod 2$.

Proof. Over $\mathbb{Q}$, Hamilton's quaternions ramify at precisely 2 and ∞. Now the following diagram of Brauer groups commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Br(\mathbb{Q}) & \longrightarrow & \bigoplus Br(\mathbb{Q}_p) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Br(k) & \longrightarrow & \bigoplus Br(k_v) & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \end{array}$$

Hence over k, Hamilton's quaternions ramify at precisely $\delta(k)$ places over 2 and $[k:\mathbb{Q}]$ places over ∞. Since a quaternion algebra ramifies at an even number of places, the result follows. □

We now prove Theorem 5.4.1.

Proof of Theorem 5.4.1. As we showed in Proposition 5.3.2, similarity classes of quadratic forms of dimension $2n+1$ parametrize the groups of Cartan–Killing type B_n over k. Picking the determinant 1 representative of each similarity class, we have that the set

$$\mathcal{F} := \{q \mid \dim q = 2n+1, \det q = 1, \text{ and } q \text{ gives rise to a hyperbolic manifold}\}$$

parametrizes commensurability classes of $2n$-dimensional arithmetic hyperbolic spaces.

For $q \in \mathcal{F}$, there is a unique real place v_q where q is isotropic, and at all other real places, is anisotropic. Let $v_1, \ldots v_l$ denote the real embeddings of k. We now fix $v := v_i$ for $1 \le i \le l$ and analyze all forms in $\mathcal{F}_i := \{q \in \mathcal{F} \mid q \text{ is isotropic at } v_i\}$.

For $q \in \mathcal{F}_i$, the fact that $\det q = 1$ now implies that q has signature $(1, 2n)$ at v and signature $(2n+1, 0)$ at all other real places. A basic computation shows that the Hasse–Minkowski invariants at the real places are then

$$c_{v_j}(q) = \begin{cases} (-1)^n & i = j \\ 1 & i \neq j. \end{cases}$$

Let $V_k^s = \{v \in V_k \mid (-1,-1)_v = +1\}$ and $V_k^r = \{v \in V_k \mid (-1,-1)_v = -1\}$. These sets correspond to the finite places where Hamilton's quaternions split[3] and ramify, respectively.

[3] This includes all finite nondyadic places as well as some dyadic places.

For $q \in \mathcal{F}_i$, let $e_s(q)$ (resp. $e_r(q)$) denote the number of finite places in V_k^s (resp. V_k^r) where $\mathbf{SO}(q)$ is not split. Clearly $r(q) := e_s(q) + e_r(q)$ is the total number of finite places where $\mathbf{SO}(q)$ is not split. (Note that this is always finite because any k-group is quasisplit at all but finitely many places and quasisplit groups of type B_n are split.)

We now use equation 5.1 to relate $r(q)$ to the local Hasse-Minkowski invariants of q. Since q has determinant 1, equation 5.1 may be simplified to state that $\mathbf{SO}(q)$ splits over v if and only if

$$c_v(q) = (-1,-1)_v^{\frac{n(n-3)}{2}}.$$

Let $f_s(q)$ (resp. $f_r(q)$) denote the number of finite places v in V_k^s (resp. V_k^r) where $c_v(q) = -1$. If as in Lemma 5.4.2, $\delta(k)$ is the number of dyadic places where $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ ramifies, then it follows that:

- $f_s(q) = e_s(q)$, and
- $f_r(q) = \begin{cases} e_r(q) & \text{if } n \equiv 0, 3 \pmod 4, \\ \delta(k) - e_r(q) & \text{if } n \equiv 1, 2 \pmod 4. \end{cases}$

By Theorem 3.6.2, the local Hasse-Minkowski invariants of q must satisfy the compatibility condition that $\prod_{v \in V_k} c_v(q) = 1$. It follows that

$$(-1)^n(-1)^{f_s(q)}(-1)^{f_r(q)} = 1$$

and hence

$$n + f_s(q) + f_r(q) \equiv 0 \pmod 2. \tag{5.4}$$

Putting the pieces together, we now have the following four cases:

- **Case 1**: $n \equiv 0 \pmod 4$
  Equation 5.4 immediately gives $r(q) \equiv 0 \pmod 2$.

- **Case 2**: $n \equiv 1 \pmod 4$
  Equation 5.4 gives
  $$n + e_s(q) + \delta(k) - e_r(q) \equiv 0 \pmod 2.$$
  By Lemma 5.4.2 and simplifying,
  $$1 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod 2,$$
  and hence
  $$r(q) \equiv [k : \mathbb{Q}] - 1 \pmod 2.$$

- **Case 3**: $n \equiv 2 \pmod 4$
  Again using Lemma 5.4.2, equation 5.4 gives

$$0 + e_s(q) + [k : \mathbb{Q}] - e_r(q) \equiv 0 \pmod 2,$$

  and hence

$$r(q) \equiv [k : \mathbb{Q}] \pmod 2.$$

- **Case 4**: $n \equiv 3 \pmod 4$
  Equation 5.4 immediately gives $r(q) \equiv 1 \pmod 2$.

We have shown that every form $q \in \mathcal{F}$ uniquely determines a set $(v_q, \{v_1, v_2, \ldots, v_{r(q)}\})$ where v_q is the unique real place where q is isotropic, and $\{v_1, v_2, \ldots, v_{r(q)}\}$ is precisely the set of finite places where $\mathbf{SO}(q)$ is not split over k_v, where $r(q)$ satisfies equation 5.3.

We now show that any collection $(v_0, \{v_1, v_2, \ldots, v_r\})$ where $v_0 \in V_k$ is a real place, $\{v_1, v_2, \ldots, v_r\}$ is a set of finite places, and r satisfies equation 5.3, determines a form in $\mathcal{F}$. Let $\{q_v\}_{v \in V_k}$ be a family of $(2n+1)$-dimensional forms of determinant 1 satisfying the following:

- q_{v_0} has signature $(1, 2n)$,
- q_v has signature $(2n+1, 0)$ at all other real places,
- for $v \in V_k$ finite, $\mathbf{SO}(q_v)$ is not split if and only if $v \in \{v_1, v_2, \ldots, v_r\}$, and hence $c_v(q_v)$ is determined by equation 5.1.

The above computations show that this family satisfies the compatibility condition of Theorem 3.6.2, and hence there exists a global form $q \in \mathcal{F}$ with localizations q_v.

It follows that sets of the form $(v_0, \{v_1, v_2, \ldots, v_r\})$ where $v_0 \in V_k$ is a real place, $\{v_1, v_2, \ldots, v_r\}$ is a set of finite places, and r satisfies equation 5.3, parametrize $\mathcal{F}$ and hence the theorem follows.

□

Hence we have reconstructed Maclachlan's results using the Tits index and classical theory of quadratic forms. With proper modification, these techniques may be used to rederive Maclachlan's parametrization of commensurability classes of odd dimensional arithmetic hyperbolic spaces coming from quadratic forms (see [Mac] Cor. 7.5). Again with proper modification, these techniques are generalizable to give parametrizations of commensurability classes of certain higher rank locally symmetric spaces.

5.5. The Dictionary Between the Index of $SU(h)$ and the Invariants of h

In this section we relate the classical invariants of a skew hermitian form over a division algebra over local fields to the Tits index of its corresponding isometry group. First we recall the following two results relating classical invariants of skew hermitian forms to whether they are isotropic.

Theorem 5.5.1 (Tsukamoto's Theorem, [Sch] Chapter 10, 3.6)**.** *Let L be a nonarchimedean local field and D the unique nonsplit quaternion algebra over L. For skew hermitian forms over $(D, *)$ the following statements hold:*

(i) Every form of dimension > 3 is isotropic.

(ii) In dimension 1 all regular forms are anisotropic: there are forms of any determinant $\neq 1$.

(iii) For any dimension > 1 there are forms of any determinant. In dimension 2 exactly the forms of discriminant[4] 1 are isotropic. In dimension 3 exactly the forms of discriminant 1 are anisotropic.

By the same sort of analysis we did for quadratic forms, we use this theorem to relate the invariants to the Tits indices in Table 5.2. Let h be an $n-$dimensional skew hermitian form over a division algebra over a local field L. Let λ denote a pure quaternion. First suppose n is even. Then by Theorem 5.5.1 (i):

$$h = \langle \lambda, -\lambda \rangle^{\frac{n-2}{2}} \oplus h' \text{ where } \dim h' = 2$$

Now by Theorem 5.5.1 (iii), it follows that h' is isotropic if and only if $\mathrm{disc}(h) = 1$. Now suppose n is odd. Then again by Theorem 5.5.1 (i):

$$h = \langle \lambda, -\lambda \rangle^{\frac{n-2}{2}} \oplus h' \text{ where } \dim h' = 3$$

Now by Theorem 5.5.1 (iii), it follows that h' is anisotropic if and only if $\mathrm{disc}(h) = 1$. Combining this we obtain the following table:

Lastly we see that over real closed fields, only one index can arise.

Theorem 5.5.2 ([Sch] Chapter 10, 3.7)**.** *Let L be a real closed field and D the unique nonsplit quaternion algebra over L. Every skew hermitian form of dimension > 1 is isotropic and forms of equal dimension are isometric.*

[4]The **discriminant** of an n-dimensional skew hermitian form h is $(-1)^n \det h$.

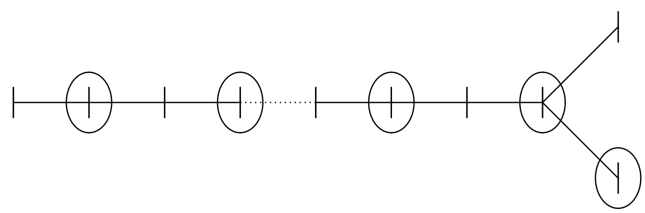
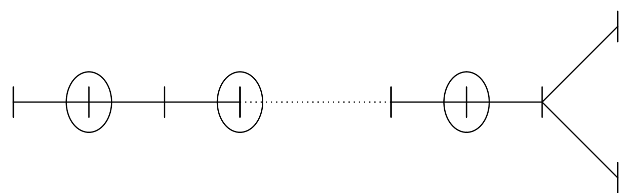
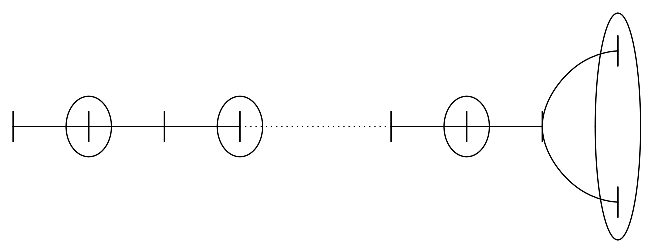
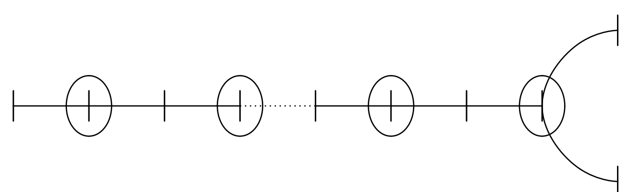
Type	Classical Invariants	Tits Index
${}^1D_{n,r}^{(2)}$	$\dim(h) = n = 2r$ $\det(h) = (-1)^n$ (i.e. $\operatorname{disc}(h) = 1$)	
${}^1D_{n,r}^{(2)}$	$\dim(h) = n = 2r + 3$ $\det(h) = (-1)^n$ (i.e. $\operatorname{disc}(h) = 1$)	
${}^2D_{n,r}^{(2)}$	$\dim(h) = n = 2r + 1$ $\det(h) \neq (-1)^n$ (i.e. $\operatorname{disc}(h) \neq 1$)	
${}^2D_{n,r}^{(2)}$	$\dim(h) = n = 2r + 2$ $\det(h) \neq (-1)^n$ (i.e. $\operatorname{disc}(h) \neq 1$)	

Table 5.2: The classical invariants of a skew hermitian form over a division algebra over a nonarchimedean local field and its associated index.

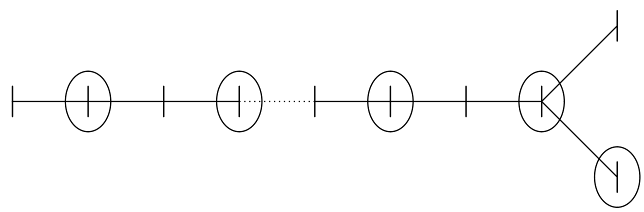
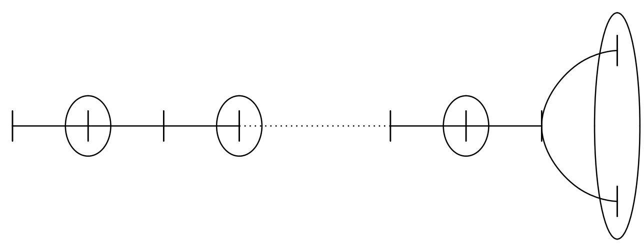
Type	Classical Invariants	Tits Index
${}^1D_{n,r}^{(2)}$	$\dim(h) = n = 2r$	
${}^2D_{n,r}^{(2)}$	$\dim(h) = n = 2r + 1$	

Table 5.3: The classical invariants of a skew hermitian form over a division algebra over a real closed field and its associated index.

Chapter 6

Construction of Subforms of Quadratic Forms

This chapter is dedicated to showing that over number fields, certain nonisometric forms cannot have the same subforms up to certain equivalences. Specifically, if q_1 and q_2 are two nonisometric quadratic forms defined over a number field, then we will answer the following question: *Is there a subform of one whose isometry group cannot be represented by a subform of the other.*

Toward these ends, we construct proper quadratic subforms with very specific local properties. These local properties will exploit the exceptional restrictions on the Hasse–Minkowski invariant in dimensions 1 and 2 to force the desired results on isometry groups. In Sections 6.1 and 6.2 we show that forms representing different groups have subforms that represent different subgroups. We will use these results in Chapter 7 to prove Theorems A, B, and C. In Section 6.3, we show that the constructions in the previous sections are the strongest results we can hope for.

6.1. Constructing Nonrepresentable Subforms 1: Nonisometric At a Real Place

In this section we investigate what happens to forms and groups when there are differences at some infinite place.

Let k be a number field and let q be an m-dimension quadratic form over k. If $v \in V_k$ is a real place, then we shall say q is **ordered at** v if the signature $(m_+^{(v)}, m_-^{(v)})$ of $q \otimes k_v$ satisfies $m \geq m_+^{(v)} \geq m_-^{(v)} \geq 0$. We call q **ordered** if it is ordered at all real places. Every form is similar to an ordered field as we shall now see.

Lemma 6.1.1. *Let k be a number field and q a quadratic form over k. Then there exists an $a \in k^\times$ so that aq is ordered.*

Proof. Let $S \subset V_k$ denote the set of all real places and let $S_0 \subset S$ denote the set of all real places where q is not ordered. For each $v \in S$, let

$$\alpha_v = \begin{cases} -(k_v^\times)^2 & \text{if } v \in S_0, \\ (k_v^\times)^2 & \text{if } v \notin S_0. \end{cases}$$

By Corollary 3.2.3, there exists $a \in k^\times$ such that $a(k_v^\times)^2 = \alpha_v$ for all $v \in S$ and hence aq is ordered. □

Note that two quadratic forms over $\mathbb{R}$ represent the same $\mathbb{R}$-group if an only if they are similar. Using this fact we begin constructing subform of one form which are not similar to a subform of the other.

Lemma 6.1.2. *Let q_1 and q_2 be nonisometric m-dimensional quadratic forms over $\mathbb{R}$ with signatures (m_1, n_1) and (m_2, n_2) respectively such that $m_1 > m_2 \geq n_2 > n_1$. Then for all $j \in \mathbb{Z}_{\geq 1}$ such that*

$$n_1 + n_2 < j < m$$

there exists an isotropic j-dimensional form dividing q_2 that is not similar to a form dividing q_1. Furthermore, this form can be realized by deleting $m - j$ entries in a diagonal representation of q_2.

Proof. The idea of the proof is that we pick a subform r of q_2 such that neither r nor $-r$ divides q_1. We may represent

$$q_1 = \langle \underbrace{a_1, \ldots, a_{m_1}}_{>0}, \underbrace{a_{m_1+1}, \ldots, a_m}_{<0} \rangle \quad \text{and} \quad q_2 = \langle \underbrace{b_1, \ldots, b_{m_2}}_{>0}, \underbrace{b_{m_2+1}, \ldots, b_m}_{<0} \rangle,$$

with $a_i, b_j \in \mathbb{R}$. The desired subform may be obtained by deleting the first $m - j$ entries of q_2, namely let

$$r := \langle b_{m-j+1}, b_{m-j+2}, \ldots b_{m-1}, b_m \rangle.$$

By construction, r has signature $(j - n_2, n_2)$ from which we can see that r is always isotropic and both

- $j - n_2 > n_1 + n_2 - n_2 = n_1$, and
- $n_2 > n_1$.

Hence neither r nor $-r$ is a subform of q_1. □

Remark 6.1.3. The more isotropic both forms are, the fewer subforms arise from this construction. In particular, there are no subforms precisely when m is even and the two forms have signatures

$$\left(\frac{m}{2}-1, \frac{m}{2}+1\right) \quad \text{and} \quad \left(\frac{m}{2}, \frac{m}{2}\right)$$

In the end, our goal is to construct locally symmetric spaces of noncompact type, and hence want isotropic subforms. Hence Lemma 6.1.2 largely succeeds, but we need to address the case in the above remark.

Lemma 6.1.4. *Let q_1 and q_2 be nonisometric m-dimensional quadratic forms over $\mathbb{R}$ with signatures (m_1, n_1) and (m_2, n_2) respectively such that $m_1 > m_2 \geq n_2 > n_1 > 0$. Then for all $j \in \mathbb{Z}_{\geq 1}$ such that*

$$m_1 < j < m$$

there exists an isotropic j-dimensional form dividing q_1 that is not similar to a form dividing q_2. Furthermore, this form can be realized by deleting $m-j$ entries in a diagonal representation of q_1.

Proof. Again we may represent

$$q_1 = \langle \underbrace{a_1, \ldots, a_{m_1}}_{>0}, \underbrace{a_{m_1+1}, \ldots, a_m}_{<0} \rangle \quad \text{and} \quad q_2 = \langle \underbrace{b_1, \ldots, b_{m_2}}_{>0}, \underbrace{b_{m_2+1}, \ldots, b_m}_{<0} \rangle,$$

with $a_i, b_j \in \mathbb{R}$. This time the desired subform may be obtained by deleting the last $m-j$ entries of q_1, namely let

$$r := \langle a_1, a_2, \ldots, a_j \rangle.$$

By construction, r has signature $(m_1, n_1 - m + j)$ from which we can see that r is always isotropic and by our initial assumptions, both

- $m_1 > m_2$.
- $m_1 > n_2$, and

Hence neither r nor $-r$ is a subform of q_2. ☐

Remark 6.1.5. The more anisotropic q_1 is, the fewer subforms arise from this construction. In particular, there are no subforms arising from this construction precisely when $m_1 = m-1$.

Combining Lemma 6.1.2 and Lemma 6.1.4 we obtain the following corollary.

Corollary 6.1.6. *Let q_1 and q_2 be nonisometric quadratic forms over $\mathbb{R}$ of dimension $m \geq 5$. Then there exists an isotropic $(m-1)$-dimensional subform of one which is not similar to a subform of the other. Furthermore, this form can be realized by deleting one entry in a diagonal representation of either q_1 or q_2.*

We should note that the bound $m \geq 5$ is strict because neither Lemma 6.1.2 nor Lemma 6.1.4 may be applied to the nonisometric 4-dimensional real forms q_1 and q_2 with signatures $(3,1)$ and $(2,2)$ respectively. It is not hard to see that every isotropic subform of one is a subform of the other. This failure is related to subtleties related to comparing the groups $SO(3,1)$ and $SO(2,2)$.

We now apply these results on quadratic forms over $\mathbb{R}$ to obtain the following result over number fields.

Theorem 6.1.7. *Let*

1. *k be a number field,*
2. *$m \geq 5$,*
3. *q_1 and q_2 be ordered m-dimensional quadratic forms over k such that there is a real place v_0 where q_1 and q_2 are not isometric over k_{v_0},*

Then there exists an $(m-1)$-dimensional quadratic k-form r which is isotropic at v_0 and divides one form, but for which no subform dividing the other form represents $\mathbf{SO}(r)$.

Proof. Begin by representing $q_1 = \langle a_1, \ldots, a_m \rangle$ and $q_2 = \langle b_1, \ldots, b_m \rangle$, $a_i, b_j \in k$. Then by Corollary 6.1.6, we may delete one entry to get an $(m-1)$-dimensional subform which over k_{v_0} is not similar to a subform of the other, and the result then follows. □

6.2. Constructing Nonrepresentable Subforms 2: Isometric At All Real Places

In this section we complete the analysis of forms by assuming they are the same at all infinite places but differ at a finite place. While the set of hypothesis on the forms will seem restrictive, in the next chapter we will see that we may always choose forms that represent a group that has these properties. Hence the we will see that these results will be sufficient to prove the main theorems of this paper.

Theorem 6.2.1. *Let*

1. *k be a number field,*

2. $m = 2n + 1$ *for* $n \geq 2$*,*

3. q_1 *and* q_2 *be nonisometric ordered* m*-dimensional quadratic forms over* k *such that* $q_{1,v} \cong q_{2,v}$ *for each infinite place* $v \in V_k$*, and*

4. there be some finite place $v_0 \in V_k$ *where:*

   a) $\det_{v_0} q_1 = 1 = \det_{v_0} q_2$*,*

   b) $c_{v_0}(q_1) \neq c_{v_0}(q_2)$*.*

Then there exist $(m-1)$*-dimensional quadratic forms* r_i *dividing* q_i *such that no subform of* q_j *represents* $\mathbf{SO}(r_i)$ *for* $i \neq j$*. Furthermore if the* q_i *are isotropic at a real place, then the* r_i *can be chosen to be isotropic at that real place as well.*

Proof. The basic idea of the proof is that we are going to construct the desired forms locally and then use the existence, uniqueness, and local-to-global results of Chapter 3 to create the desired global forms. Let

$$S = \{v_0\} \cup \{\text{infinite real places of } k\}$$

For each $v \in S$, we pick square classes $\alpha_v \in k_v^\times/(k_v^\times)^2$ as follows:

- For v_0, let α_{v_0} be such that $\alpha_{v_0} = (-1)^n$.
- For each infinite $v \in S$ let $\alpha_v = \det q_1 (k_v^\times)^2 (= \det q_2 (k_v^\times)^2)$.

By Corollary 3.2.3 above, we may choose an $s \in k^\times$ such that $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define $t_{1,v}, t_{2,v}, r_{1,v}, r_{2,v}$ to be the quadratic k_v-forms with invariants given by:

$$\begin{aligned} \dim t_{i,v} &= 1 & \dim r_{i,v} &= m-1 \\ \det t_{i,v} &= \frac{\det q_i}{s} & \det r_{i,v} &= s \\ c_v(t_{i,v}) &= 1 & c_v(r_{i,v}) &= c_v(q_i)\left(s, \frac{\det q_i}{s}\right)_v. \end{aligned}$$

We know such forms exist by Theorem 3.5.1 (2).

For each infinite place $v \in V_k$, define forms $t_{1,v}, t_{2,v}$ by:

$$t_{i,v} = \left\langle \frac{\det q_i}{s} \right\rangle.$$

At each complex place $t_{i,v}$ divides $q_i \otimes k_v$. By assumption, q_1 and q_2 are ordered at each real place $v \in V_k$, and hence $t_{i,v} (= \langle 1 \rangle)$ is a subform of $q_i \otimes k_v$. Therefore at each infinite place it makes sense to take the compliment of $t_{i,v}$ in $q_i \otimes k_v$ and we may define forms $r_{1,v}, r_{2,v}$ by

$$r_{i,v} = t_{i,v}^{\perp}.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_{i,v}) = 1 = c_v(q_i)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_i \otimes k_v$. Observe that $r_{i,v}$ has signature $(m_+^{(v)} - 1, m_-^{(v)})$, and hence is isotropic whenever $q_i \otimes k_v$ is isotropic. Also note that

$$c_v(r_{i,v}) = (-1)^{\frac{m_-^{(v)}(m_-^{(v)}-1)}{2}} = c_v(q_i).$$

We shall now show that for each place $v \in V_k$, $t_{i,v} \oplus r_{i,v} \cong q_i \otimes k_v$. This is true by construction at the infinite places. Now suppose v is finite. Clearly

$$\dim(t_{i,v} \oplus r_{i,v}) = 1 + (n-1) = n = \dim(q_i \otimes k_v)$$

$$\det(t_{i,v} \oplus r_{i,v}) = (\det q_i / s)s = \det(q_i \otimes k_v)$$

and by the product formula for the Hasse–Minkowski invariant

$$c(t_{i,v} \oplus r_{i,v}) = c(t_{i,v})c(r_{i,v}) \left(\frac{\det q_i}{s}, s \right) = c_v(q_i) \left(\frac{\det q_i}{s}, s \right)^2 = c(q_i \otimes k_v),$$

and hence by Theorem 3.5.1 (1), they are isomorphic.

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 3.6.2. Observe that $c_v(t_{i,v}) = 1$ for all $v \in V_k$ and hence $\prod_{v \in V_k} c_v(t_{i,v}) = 1$. Next observe that by our choice of s, $\left(s, \frac{\det q_i}{s}\right)_v = 1$ at each infinite place, and hence

$$\begin{aligned}
\prod_{v \in V_k} c_v(r_{i,v}) &= \left(\prod_{v \in V_k \text{ finite}} c_v(r_{i,v}) \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(r_{i,v}) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(r_{i,v}) \right) \\
&= \left(\prod_{v \in V_k \text{ finite}} c_v(q_i) \left(s, \frac{\det q_i}{s} \right)_v \right) \times \left(\prod_{v \in V_k \text{ real}} c_v(q_i) \right) \times \left(\prod_{v \in V_k \text{ complex}} c_v(q_i) \right) \\
&= \prod_{v \in V_k} c_v(q_i) \times \prod_{v \in V_k} \left(s, \frac{\det q_i}{s} \right)_v \\
&= 1. \qquad (6.1)
\end{aligned}$$

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula.

By Theorem 3.6.2, there exist quadratic forms t_i and r_i over k such that for all $v \in V_k$, $t_i \otimes k_v \cong t_{i,v}$ and $r_i \otimes k_v \cong r_{i,v}$. Furthermore, for each $v \in V_k$, we have shown that $t_{i,v} \oplus r_{i,v} \cong q_i \otimes k_v$ so by Theorem 3.6.1 we conclude $t_i \oplus r_i \cong q_i$, and hence r_i is a subform of q_i.

Let $\mathbf{H}_i = \mathbf{SO}(r_i)$. We must show that $\mathbf{H}_i \subset \mathbf{G}_j$ if and only if $i = j$, and hence this reduces to showing that there are no representatives r_i' of $\mathbf{H}_i$ such that $r_i' \subset q_j$ for $j \neq i$. Now $\mathbf{H}_i$ is a group of type D_n over k. Let r_i' be any representative of $\mathbf{H}_i$. Then $\mathbf{H}_i$ determines the following invariants of r':

1. $\dim(r_i') = 2n = \dim(r_i)$.
2. $\mathrm{disc}_v(r_i') = 1$ at precisely the places $v \in V_k$ where $\mathbf{H}_i \otimes k_v$ is a group of inner type (i.e., the $*$-action is trivial). This means that $\mathrm{disc}_v(r_i') = 1$ if and only if $\mathrm{disc}_v(r_i) = 1$, or in other words, at the places where $\mathrm{disc}_v(r_i) = 1$, then $\det r' = \det r$.
3. $c_v(r_i') = c_v(r_i)$ at each place v where $\mathrm{disc}_v(r_i) = 1$ (see equation 5.2).

Now let r_i' be any quadratic form satisfying these three. Suppose there exists some form t_i' such that $r_i' \oplus t_i' \cong q_j$ for $i \neq j$. It immediately follows that $\dim t_i' = 1$, $\det t_i' = \det q_j / \det r_i'$ and by the exceptional restriction, $c(t_i') = 1$.

Observe that our choice of s implies that

$$\mathrm{disc}_v(r_i) = (-1)^n \det r_i = (-1)^{2n} = 1$$

Hence at v_0, we have $\det r_i = \det r_i'$ and $c_{v_0}(r_i) = c_{v_0}(r_i')$, which we use in the following computation of $c_{v_0}(q_j)$:

$$\begin{aligned}
c_{v_0}(q_j) &= c_{v_0}(r_i' \oplus t_i') \\
&= c_{v_0}(r_i')c_{v_0}(t_i') \left(\det r_i', \frac{\det q_j}{\det r_i'}\right)_{v_0} \\
&= c_{v_0}(r_i) \left(\det r_i, \frac{\det q_j}{\det r_i}\right)_{v_0} \\
&= \left(c_{v_0}(q_i) \left(\det r_i, \frac{\det q_i}{\det r_i}\right)_{v_0}\right) \left(\det r_i, \frac{\det q_j}{\det r_i}\right)_{v_0} \\
&= c_{v_0}(q_i) \left(\det r_i, \frac{\det q_i \det q_j}{(\det r_i)^2}\right)_{v_0} \\
&= c_{v_0}(q_i) \left(\det r_i, 1\right)_{v_0} \\
&= c_{v_0}(q_i).
\end{aligned}$$

However this contradicts our initial assumption that $c_{v_0}(q_i) \neq c_{v_0}(q_j)$ and the conclusion follows. □

Example 6.2.2. Consider the following 5-dimensional quadratic forms over $\mathbb{Q}$:

$$q_1 = \langle 1, 1, 1, 1, -5 \rangle \qquad \text{and} \qquad q_2 = \langle 1, 1, 3, 3, -5 \rangle.$$

Observe that $\det q_1 = -5 = \det q_2$, which in $\mathbb{Q}_3$ is a square. Furthermore, a quick computation shows $c_3(q_1) = 1$ and $c_3(q_2) = -1$. Hence by Theorem 6.2.1, there exists a 4-dimensional quadratic form $r \subset q_1$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_1)$ but $\mathbf{H}$ is not k-isomorphic to a subgroup of $\mathbf{SO}(q_2)$.

It is not hard to check that $r = \langle 1, 1, 1, -5 \rangle$ is such a form.

Theorem 6.2.3. *Let*

1. *k be a number field,*
2. *$m = 2n$ for $n \geq 2$,*
3. *q_1 and q_2 be nonisometric ordered m-dimensional quadratic forms over k such that*
   a) $\det q_1 = \det q_2$ *(and hence* $\operatorname{disc}(q_1) = \operatorname{disc}(q_2)$*),*
   b) *$q_{1,v} \cong q_{2,v}$ at each infinite place v,*
4. *there be some finite place $v_0 \in V_k$ where:*
   a) $\operatorname{disc}_{v_0}(q_1) = 1 = \operatorname{disc}_{v_0}(q_2)$,
   b) $c_{v_0}(q_1) = (-1,-1)_{v_0}^{\frac{n(n-1)}{2}} \neq -(-1,-1)_{v_0}^{\frac{n(n-1)}{2}} = c_{v_0}(q_2)$.

Then there exists an $(m-1)$-dimensional quadratic form r dividing q_1 such that no subform of q_2 represents $\mathbf{SO}(r)$. Furthermore if the q_1 is isotropic at a real place, then the r can be chosen to be isotropic at that real place as well.

Proof. Again we are going to construct the desired forms locally and then use the existence, uniqueness, and local-to-global results of Chapter 3 to create the desired global forms.

Let $S = \{\text{infinite real places of } k\}$. For each $v \in S$, we pick the trivial square class $\alpha_v \in k_v^\times/(k_v^\times)^2$. By Corollary 3.2.3 above, we may choose an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define t_v, r_v to be the quadratic k_v-forms with invariants given by:

$$\begin{aligned} \dim t_v &= 1 & \dim r_v &= m-1 \\ \det t_v &= \frac{\det q_1}{s} & \det r_v &= s \\ c_v(t_v) &= 1 & c_v(r_v) &= c_v(q_1)\ \left(s, \frac{\det q_1}{s}\right)_v. \end{aligned}$$

We know such forms exist by Theorem 3.5.1 (2).

For each infinite place $v \in V_k$, define form t_v by:

$$t_v = \left\langle \frac{\det q_1}{s} \right\rangle.$$

At each complex place t_v divides $q_1 \otimes k_v$. By assumption, q_1 is ordered at each real place $v \in V_k$, and hence $t_v(= \langle 1 \rangle)$ is a subform of $q_1 \otimes k_v$. Therefore at each infinite place it makes sense to take the compliment of t_v in $q_1 \otimes k_v$ and we may define forms r_v by

$$r_v = t_v^{\perp}.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_1)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_1 \otimes k_v$. Observe that r_v has signature $(m_+^{(v)} - 1, m_-^{(v)})$, and hence is isotropic whenever $q_1 \otimes k_v$ is isotropic. Also note that

$$c_v(r_v) = (-1)^{\frac{m_-^{(v)}(m_-^{(v)}-1)}{2}} = c_v(q_1).$$

Just as in the proof of Theorem 6.2.1, we have:

- The families $\{t_v\}_{v \in V_k}$ and $\{r_v\}_{v \in V_k}$ satisfy the global compatibility conditions (see 6.1), and hence by Theorem 3.6.2, there exist quadratic forms t and r over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$ and $r \otimes k_v \cong r_v$.
- By Theorem 3.5.1 (1), $t_v \oplus r_v$ and $q_1 \otimes k_v$ are isometric at each place $v \in V_k$.
- By Theorem 3.6.1 we conclude $t \oplus r \cong q_1$, and hence r is a subform of q_1.

We claim that $\mathbf{H} := \mathbf{SO}(r)$ is the split group $B_{n-1,n-1}$ at v_0. By equation 5.1, this means we must show

$$c_{v_0}(r) = (-1,-1)_{v_0}^{\frac{(n-1)(n-4)}{2}} (-1,s)_{v_0}^{n-1} = (-1,-1)_{v_0}^{\frac{n(n-1)}{2}} (-1,s)_{v_0}^{n-1}$$

A direct computation yields

$$\begin{aligned} c_{v_0}(r) &= c_{v_0}(q_1) \left(s, \frac{\det q_1}{s} \right)_{v_0} \\ &= (-1,-1)_{v_0}^{\frac{(n-2)(n-3)}{2}+1} (s, -\det q_1)_{v_0} \\ &= (-1,-1)_{v_0}^{\frac{(n-2)(n-3)}{2}+1} (s, -(-1)^n)_{v_0} \\ &= (-1,-1)_{v_0}^{\frac{(n-2)(n-3)}{2}+1} (s,-1)_{v_0}^{n-1} \\ &= (-1,-1)_{v_0}^{\frac{n^2-5n+6+2}{2}} (s,-1)_{v_0}^{n-1} \\ &= (-1,-1)_{v_0}^{\frac{n(n-1)}{2}} (s,-1)_{v_0}^{n-1}. \end{aligned}$$

As we have just seen, $\mathbf{H}$ is split at k_{v_0}, hence

$$\operatorname{rank}_{k_{v_0}}(\mathbf{H}) = \frac{(m-1)-1}{2} = \frac{(2n-1)-1}{2} = n-1 > n-2 = \operatorname{rank}_{k_{v_0}}(\mathbf{G_2}).$$

We have just shown that $\mathbf{H} \otimes k_{v_0}$ cannot be a subgroup of $\mathbf{G}_2 \otimes k_{v_0}$, and hence $\mathbf{H}$ cannot be a subgroup of $\mathbf{G}_2$. □

For more details on the rank of a semisimple algebraic group, we refer the reader to [B1].

Remark 6.2.4. An interesting consequence of the proof is that over a local field, the split group of type ${}^1D_{n,n}^{(1)}$ cannot contain a subgroup of type $B_{n-1,n-2}$.

Example 6.2.5. Consider the following 4-dimensional quadratic forms over $\mathbb{Q}$:

$$q_1 = \langle 1, 1, 5, -1 \rangle \qquad \text{and} \qquad q_2 = \langle 3, 3, 5, -1 \rangle.$$

Observe that $\det q_1 = -5 = \det q_2$, which in $\mathbb{Q}_3$ is a square. Hence these have discriminant 1 in $\mathbb{Q}_3$. Furthermore, a quick computation shows $c_3(q_1) = 1$ and $c_3(q_2) = -1$. Hence by Theorem 6.2.3, there exists a 3-dimensional quadratic form $r \subset q_1$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_1)$ but $\mathbf{H}$ is not k-isomorphic to a subgroup of $\mathbf{SO}(q_2)$.

It is not hard to check that $r = \langle 1, 1, -1 \rangle$ is such a form.

Theorem 6.2.6. . *Let*

1. *k be a number field,*
2. *$m = 2n$ for $n \geq 3$,*
3. *q_1 and q_2 be nonisometric ordered m-dimensional quadratic forms over k such that*
   a) $\det q_1 \neq \det q_2$ *(and hence* $\operatorname{disc}(q_1) \neq \operatorname{disc}(q_2)$*),*
   b) $q_{1,v} \cong q_{2,v}$ *at each infinite place v,*
4. *there be some finite place $v_0 \in V_k$ where:*
   a) $\operatorname{disc}_{v_0} q_1 = 1$,
   b) $\operatorname{disc}_{v_0} q_2 \neq 1$,
   c) $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \operatorname{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$

Then there exists an $(m-2)$-dimensional quadratic form r dividing q_2 such that no subform of q_1 represents $\mathbf{SO}(r)$. Furthermore if the q_2 is isotropic at a real place, then the r can be chosen to be isotropic at that real place as well.

Proof. As we did in Theorems 6.2.1 and 6.2.6 we construct the desired forms locally and use the results of Chapter 3 to create global forms. Let

$$S = \{v_0\} \cup \{\text{infinite real places of } k\}$$

For each $v \in S$, we pick square classes $\alpha_v \in k_v^\times/(k_v^\times)^2$ as follows:

- For v_0, let α_{v_0} be such that $\alpha_{v_0} = (-1)^{\frac{m-2}{2}}(k_{v_0}^\times)^2$.
- For infinite $v \in S$ let $\alpha_v = \det q_2 (k_v^\times)^2$.

By Corollary 3.2.3 above, we may choose an $s \in k^\times$ for which $s \in \alpha_v$ for all $v \in S$.

For each finite place $v \in V_k$, define t_v, r_v to be the quadratic k_v-forms with invariants given by:

$$\begin{aligned} \dim t_v &= 2 & \dim r_v &= m-2 \\ \det t_v &= \frac{\det q_2}{s} & \det r_v &= s \\ c_v(t_v) &= 1 & c_v(r_v) &= c_v(q_2)\left(s, \frac{\det q_2}{s}\right)_v. \end{aligned}$$

We know such forms exist by Theorem 3.5.1 (2).

For each infinite place $v \in V_k$, define form t_v by:

$$t_v = \left\langle 1, \frac{\det q_2}{s} \right\rangle.$$

At each complex place t_v divides $q_2 \otimes k_v$. By assumption, q_2 is ordered at each real place $v \in V_k$, and hence $t_v (= \langle 1, 1\rangle)$ is a subform of $q_2 \otimes k_v$. Therefore at each infinite place it makes sense to take the compliment of t_v in $q_2 \otimes k_v$ and we may define forms r_v by

$$r_v = t_v^\perp.$$

At each complex place $v \in V_k$, we trivially have $c_v(r_v) = 1 = c_v(q_2)$. For each real $v \in V_k$, let $(m_+^{(v)}, m_-^{(v)})$ denote the signature of $q_2 \otimes k_v$. Observe that r_v has signature $(m_+^{(v)} - 2, m_-^{(v)})$, and hence is isotropic whenever $q_2 \otimes k_v$ is isotropic. Also note that

$$c_v(r_v) = (-1)^{\frac{m_-^{(v)}(m_-^{(v)}-1)}{2}} = c_v(q_2).$$

We shall now show that for each place $v \in V_k$, $t_v \oplus r_v \cong q_2 \otimes k_v$. This is true by construction at the infinite places. Now suppose v is finite. Clearly

$$\dim(t_v \oplus r_v) = 1 + (n-1) = n = \dim(q_2 \otimes k_v),$$

$$\det(t_v \oplus r_v) = (\det q_2/s)s = \det(q_2 \otimes k_v),$$

and by the product formula for the Hasse–Minkowski invariant

$$c(t_v \oplus r_v) = c(t_v)c(r_v)\left(\frac{\det q_2}{s}, s\right) = c_v(q_2)\left(\frac{\det q_2}{s}, s\right)^2 = c(q_2 \otimes k_v),$$

and hence by Theorem 3.5.1 (1), they are isomorphic.

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 3.6.2. Observe that $c_v(t_v) = 1$ for all $v \in V_k$ and hence $\prod_{v\in V_k} c_v(t_v) = 1$. Next observe that by our choice of s, $\left(s, \frac{\det q_2}{s}\right)_v = 1$ at each infinite place, and hence

$$\begin{aligned}\prod_{v\in V_k} c_v(r_v) &= \left(\prod_{v\in V_k \text{ finite}} c_v(r_v)\right) \times \left(\prod_{v\in V_k \text{ real}} c_v(r_v)\right) \times \left(\prod_{v\in V_k \text{ complex}} c_v(r_v)\right)\\ &= \left(\prod_{v\in V_k \text{ finite}} c_v(q_2)\ \left(s, \frac{\det q_2}{s}\right)_v\right) \times \left(\prod_{v\in V_k \text{ real}} c_v(q_2)\right) \times \left(\prod_{v\in V_k \text{ complex}} c_v(q_2)\right)\\ &= \prod_{v\in V_k} c_v(q_2) \times \prod_{v\in V_k} \left(s, \frac{\det q_2}{s}\right)_v\\ &= 1. \end{aligned} \tag{6.2}$$

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula.

By Theorem 3.6.2, there exist quadratic forms t and r over k such that for all $v \in V_k$, $t\otimes k_v \cong t_v$ and $r\otimes k_v \cong r_v$. Furthermore, for each $v \in V_k$, we have shown that $t_v\oplus r_v \cong q_2\otimes k_v$ so by Theorem 3.6.1 we conclude $t \oplus r \cong q_2$, and hence r is a subform of q_2.

Let $\mathbf{H} = \mathbf{SO}(r)$. We will show that $\mathbf{H} \not\subset \mathbf{G}_1 = \mathbf{SO}(q_1)$, and hence that there are no representatives r' of $\mathbf{H}$ such that $r' \subset q_1$. Again $\mathbf{H}$ is a group of type D_n over k. Let r' be any representative of $\mathbf{H}$. As in the proof of Theorem 6.2.1, the group $\mathbf{H}$ determines the following invariants of r':

1. $\dim(r') = 2n - 2 = \dim(r)$.

2. $\mathrm{disc}_v(r') = 1$ at precisely the places $v \in V_k$ where $\mathbf{H} \otimes k_v$ is a group of inner type (i.e., the $*$-action is trivial). This means that $\mathrm{disc}_v(r') = 1$ if and only if $\mathrm{disc}_v(r) = 1$, or in other words, at the places where $\mathrm{disc}_v(r) = 1$, then $\det r' = \det r$.

3. $c_v(r') = c_v(r)$ at each place v where $\mathrm{disc}_v(r) = 1$ (see equation 5.2).

Let r' be any quadratic form satisfying these three properties. Suppose there exists some form t' such that $r' \oplus t' \cong q_1$. It follows that $\dim t' = 2$, $\det t' = \det q_1/\det r'$.

Observe that our choice of s implies that

$$\mathrm{disc}_v(r) = (-1)^{(n-1)} \det r = (-1)^{2n-2} = 1$$

Hence at v_0, we have $\det r = \det r'$ and $c_{v_0}(r) = c_{v_0}(r')$. Furthermore we have

$$\begin{aligned}
\det{}_{v_0} t' &= \frac{\det_{v_0} q_1}{\det_{v_0} r'} \\
&= \frac{(-1)^{\frac{m}{2}} \mathrm{disc}(q_1)}{(-1)^{\frac{m-2}{2}} \mathrm{disc}(r')} \\
&= \frac{(-1)^{\frac{m}{2}}}{(-1)^{\frac{m-2}{2}}} \\
&= -1,
\end{aligned}$$

and thus by the exceptional restriction, $c_{v_0}(t') = 1$. Now the product formula at v_0 yields the following contradiction:

$$\begin{aligned}
c_{v_0}(q_1) &= c_{v_0}(r' \oplus t') \\
&= c_{v_0}(r') \left(\det r', \frac{\det q_1}{\det r'} \right)_{v_0} \\
&= c_{v_0}(q_2) \left(\det r', \frac{\det q_2}{\det r'} \right)_{v_0} \left(\det r', \frac{\det q_1}{\det r'} \right)_{v_0} \\
&= c_{v_0}(q_2) \left((-1)^{\frac{m-2}{2}}, \frac{\det q_1 \det q_2}{(\det r')^2} \right)_{v_0} \\
&= c_{v_0}(q_2) \left((-1)^{\frac{m-2}{2}}, (-1)^{\frac{m}{2}} \det q_2 \right)_{v_0} \\
&= c_{v_0}(q_2) \left(-1, \mathrm{disc}(q_2)\right)_{v_0}^{\frac{m-2}{2}} .
\end{aligned}$$

Hence no representative of $\mathbf{H}$ can be a subform of q_1, concluding the proof. □

Example 6.2.7. Consider the following 6-dimensional quadratic forms over $\mathbb{Q}$:

$$q_1 = \langle 1, 1, 1, 3, 3, -1 \rangle \qquad \text{and} \qquad q_2 = \langle 1, 1, 1, 1, 1, -5 \rangle.$$

Observe that $\det q_1 = -1 \neq -5 = \det q_2$. Furthermore, $\mathrm{disc}_3(q_1) = 1$, but $\mathrm{disc}_3(q_2) = 5$ which is not a square in $\mathbb{Q}_3$. Furthermore, a quick computation shows $c_3(q_1) = -1$ and $c_3(q_2) = 1$. Hence by Theorem 6.2.6, there exists a 4-dimensional quadratic form $r \subset q_2$ so that $\mathbf{H} := \mathbf{SO}(r) \subset \mathbf{SO}(q_2)$ but $\mathbf{H}$ is not k-isomorphic to a subgroup of $\mathbf{SO}(q_1)$.

It is not hard to check that $r = \langle 1, 1, 1, -5 \rangle$ is such a form.

6.3. Constructing Subforms In Codimension > 2

We have shown that given certain nonisometric forms, we may find codimension 1 or 2 subforms of one that are not represented in the other. In this section we show that this is the best we can hope for.

Proposition 6.3.1. *Let k be a number field and let q_1 and q_2 be m-dimensional quadratic forms over k, $m \geq 4$, which are isometric at each infinite place. If r is a j-dimensional subform of q_1, where $0 < j < m-2$, then r is also a subform of q_2.*

Proof. As usual, we construct forms locally from which we will obtain a global form. For each finite $v \in V_k$, let t_v be the k_v form uniquely determined by

- $\dim t_v = n - m$,
- $\det t_v = \dfrac{\det q_2}{\det r}$, and
- $c_v(t_v) = c_v(q_2)\ c_v(r)\ \left(\det(r), \dfrac{\det(q_2)}{\det(r)}\right)_v$.

We know such forms exist by Theorem 3.5.1 (2). Since $q_1 \otimes k_v$ and $q_2 \otimes k_v$ are isometric at each infinite $v \in V_k$, then $r \otimes k_v$ is a subform of $q_2 \otimes k_v$, and hence it makes sense to take its compliment. We therefore define

- $t_v := (r \otimes k_v)^{\perp}$,

From this definition, it immediately follows that at each infinite place

$$c_v(t_v) = c_v(q_2)c_v(r)\left(\det(r), \frac{\det(q_2)}{\det(r)}\right)_v.$$

We now wish to build a global form, and hence must check that our forms satisfy the compatibility criteria of Theorem 3.6.2. This can be seen with the following computation:

$$\begin{aligned}\prod_{v \in V_k} c_v(t_v) &= \left(\prod_{v \in V_k} c_v(q_2)c_v(r)\left(\det(r), \frac{\det(q_2)}{\det(r)}\right)_v\right) \\ &= \left(\prod_{v \in V_k} c_v(q_2)\right) \times \left(\prod_{v \in V_k} c_v(r)\right) \times \left(\prod_{v \in V_k}\left(\det(r), \frac{\det(q_2)}{\det(r)}\right)_v\right) \\ &= 1.\end{aligned}$$

Where we know the final product is trivial because both the Hasse–Minkowski invariant and the Hilbert symbol of global objects satisfy the product formula. Hence we may now use

Theorem 3.6.2 to obtain a quadratic form t over k such that for all $v \in V_k$, $t \otimes k_v \cong t_v$. Furthermore, for each $v \in V_k$, $t_v \oplus r_v$ and $q_2 \otimes k_v$ have the same local invariants so by Theorem 3.5.1 they are isometric, and by Theorem 3.6.1 we conclude $t \oplus r \cong q_2$, and hence r is a subform of q_2. □

Chapter 7

Main results

In this chapter, we bring together the results of the earlier chapters to prove the main results of this paper. In Section 7.1 we will prove results about seimsimple subgroups of algebraic groups over number fields. In so doing, we will introduce the notion of the semisimple subgroup spectrum. In Sections 7.2 and 7.3 we will prove Theorems A - D which were mentioned in the introduction. In Section 7.4 we study what the condition $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ says about two arithmetic hyperbolic spaces M_1 and M_2. In particular, we analyze when this implies that, up to commensurability, M_1 is a totally geodesic subspace of M_2. In Section 7.5 we conclude this thesis with some final remarks.

7.1. The Semisimple Subgroup Spectrum

If $\mathbf{G}$ is an algebraic group defined over a field k, let its **semisimple subgroup spectrum** be the set

$$SS_k(\mathbf{G}) = \left\{ \begin{array}{c} \text{Isomorphism classes of proper} \\ \text{semisimple } k\text{-subgroups of } \mathbf{G} \end{array} \right\}.$$

The goal of this section is to prove the following algebraic result.

Theorem 7.1.1. *Let k be a number field and $\mathbf{G}_1$ and $\mathbf{G}_2$ be semisimple k-groups of the same type, either B_n or D_n, such that either*

1. *both come from quadratic forms of dimension $m \geq 5$, or*
2. *one comes from a quadratic form of dimension $m \geq 3$ and the other from a skew hermitian form.*

Then $SS_k(\mathbf{G}_1) = SS_k(\mathbf{G}_2)$ implies $\mathbf{G}_1$ and $\mathbf{G}_2$ are k-isomorphic.

The proof of Theorem 7.1.1 will require many parts, the first of which is the following lemma.

Lemma 7.1.2. *Let k be a number field and let $\mathbf{G}_1$ and $\mathbf{G}_2$ be k-groups. If $\mathbf{G}_1$ and $\mathbf{G}_2$ are k-isomorphic, then for all $v \in V_k$, $\mathbf{G}_1 \otimes k_v$ and $\mathbf{G}_2 \otimes k_v$ are k_v-isomorphic.*

Proof. Any k-isomorphism $\varphi : \mathbf{G}_1 \to \mathbf{G}_2$ extends to a k_v-isomorphism $\varphi_v : \mathbf{G}_1 \otimes k_v \to \mathbf{G}_2 \otimes k_v$ for all $v \in V_k$. □

It follows that if a group is not isomorphic at just one place, they are are not k-isomorphic. Hence there is no k-isomorphism of groups that would permute signatures at the real places.

Example 7.1.3. Consider the forms $q_1 = \langle 1, 1, \sqrt{2} \rangle$ and $q_2 = \langle 1, 1, -\sqrt{2} \rangle$ over $\mathbb{Q}(\sqrt{2})$. Then even though there is an automophsm of $\mathbb{Q}(\sqrt{2})$ sending $\sqrt{2} \mapsto -\sqrt{2}$, and hence q_1 to q_2, the groups $\mathbf{SO}(q_1)$ and $\mathbf{SO}(q_2)$ are not k-isomorphic. Hence the arithmetic lattices they define in $SO(2, 1)$ are not commensurable.

To prove Theorem 7.1.1, we will in fact prove the contrapositive. We assume we have two nonisomorphic groups which give nonisometric forms. We then chose certain subforms and, using the classical invariants, check the Tits index at local places to guarantee these forms give rise to the desired subgroups. This process is carried out in Theorem 7.1.4 and Proposition 7.1.5.

Theorem 7.1.4. *Let k be a number field and $\mathbf{G}_1$ and $\mathbf{G}_2$ be semisimple k-groups coming come from quadratic forms of dimension $m \geq 5$. If $\mathbf{G_1}$ and $\mathbf{G_2}$ are not k-isomorphic, then there exists a semisimple k-subgroup $\mathbf{H}$ which is a subgroup of one but not the other. Furthermore, if either $\mathbf{G}_1$ or $\mathbf{G}_2$ is isotropic at a real place, the $\mathbf{H}$ can be chosen to be isotropic at a real place.*

Proof. Let q_1 and q_2 represent $\mathbf{G}_1$ and $\mathbf{G}_2$ respectively such that at each real infinite place v the signature $(m_{+,i}^{(v)}, m_{-,i}^{(v)})$ of q_i. By Lemma 6.1.1, we may assume that $m \geq m_{+,i}^{(v)} \geq m_{-,i}^{(v)} \geq 0$ for all real places v. If q_1 and q_2 are not the isometric at every real place, then by Theorem 6.1.7 the result follows.

Now suppose q_1 and q_2 are isometric at all infinite places. Since the groups $\mathbf{G}_1$ and $\mathbf{G}_2$ are not k-isomorphic, the Hasse principle for special orthogonal groups ([PlRa] p. 348) implies that there exists some finite place v_0 where $\mathbf{G}_1 \otimes k_{v_0}$ and $\mathbf{G}_2 \otimes k_{v_0}$ are not k_{v_0}-isomorphic. Since the groups are not isometric at v_0, the forms are not isometric. If m is odd, then by using Corollary 3.2.3, may replace q_1 and q_2 with similar forms as necessary to guarantee that $\det_{v_0} q_1 = \det_{v_0} q_2 = 1$ while not altering the signatures at the infinite places. Hence $c_{v_0}(q_1) \neq c_{v_0}(q_2)$ and then by Theorem 6.2.1 the result follows.

Now suppose $m = 2n$ is even. If $\det q_1 = \det q_2$ but $\operatorname{disc}_{v_0}(q_i) \neq 1$, then by Lemma 3.1.4 and Corollary 3.2.3, we may replace q_2 with a similar form while not altering the signatures at the infinite place and for which $c_{v_0}(q_1) = c_{v_0}(q_2)$. This would imply q_1 and q_2 are isomorphic over k_{v_0}, contradicting our choice of v_0. Hence if $\det q_1 = \det q_2$, then after possibly relabelling, their invariants must satisfy both of the following:

1. $\operatorname{disc}_{v_0}(q_1) = 1 = \operatorname{disc}_{v_0}(q_2)$, and
2. $c_{v_0}(q_1) = (-1,-1)_{v_0}^{\frac{n(n-1)}{2}} \neq -(-1,-1)_{v_0}^{\frac{n(n-1)}{2}} = c_{v_0}.(q_2)$

By Theorem 6.2.3 the result follows. Otherwise, if $\det q_1 \neq \det q_2$ then in terms of their forms this means, after possible relabeling:

1. $\operatorname{disc}_{v_0} q_1 = 1$,
2. $\operatorname{disc}_{v_0} q_2 \neq 1$,

Furthermore, if $c_{v_0}(q_1) = c_{v_0}(q_2)(-1, \operatorname{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$, then we will replace q_2 with a similar form an the following way. Let $S = \{v_0\} \cup \{\text{infinite real places of } k\}$ and for each $v \in S$, we pick a square class $\alpha_v \in k_v^\times/(k_v^\times)^2$ as follows:

- at v_0, $(\alpha_{v_0}, \operatorname{disc}(q_2))_{v_0} = -1$ (note that such a class exists by the the nondegeneracy of the Hilbert symbol and the fact that $\operatorname{disc}(q_2) \neq 1$), and
- for all $v \in S$ real, α_v is trivial.

Then by Lemma 3.1.4, it follows that $c_{v_0}(\lambda q_2) = -c_{v_0}(q_2)$ and replacing q_2 by λq_2, it follows that $c_{v_0}(q_1) \neq c_{v_0}(q_2)(-1, \operatorname{disc}(q_2))_{v_0}^{\frac{m-2}{2}}$. Then by Theorem 6.2.6 the result follows. $\square$

Proposition 7.1.5. *Let $\mathbf{G}_i$ be algebraic k_i-groups, $i \in \{1,2\}$, such that $\mathbf{G}_1$ comes from a $2n$-dimensional quadratic form over k_1, $n \geq 2$, and $\mathbf{G}_2$ comes from an n-dimensional skew hermitian forms over a division algebra D over k_2. Then there exists a semisimple k_1-group $\mathbf{H}$ which is a subgroup of $\mathbf{G}_1$ but not of $\mathbf{G}_2$. Furthermore, if $\mathbf{G}_1$ is isotropic at a real place then $\mathbf{H}$ can be chosen to be isotropic at a real place.*

Proof. Choose a form q to represent $\mathbf{G}_1$ such that $\langle a_1, a_2, \ldots, a_{2n}\rangle$ is a diagonal representation of q. Let $q' = \langle a_1, a_2, \ldots a_j\rangle$ for $\frac{n}{2} + 2 < j < 2n$ and let $\mathbf{H} = \mathbf{SO}(q) \subset \mathbf{G}_1$. We shall show that $\mathbf{H}$ cannot be a subgroup of $\mathbf{G}_2$. Let $v \in V_k$ be a finite place where D ramifies.

$$\operatorname{rank}_{k_v}(\mathbf{G}_2) \leq \frac{n}{2} \leq j - 2 \leq \operatorname{rank}_{k_v}(\mathbf{H}).$$

Hence by rank considerations $\mathbf{H}$ cannot be a subgroup of $\mathbf{G}_2$. Furthermore, if q is isotropic at a real place, then we may pick a q' which is also isotropic and the result follows. $\square$

The question still remains when both groups come from skew hermitian forms over a number field. What we can say is the following.

Proposition 7.1.6. *Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be algebraic k-groups coming from n-dimensional skew hermitian forms over division algebras D_1 and D_2 respectively. If D_1 and D_2 are not isomorphic, then $SS(\mathbf{G}_1) \neq SS(\mathbf{G}_2)$.*

Proof. Let $\langle a_1, a_2, \ldots, a_n \rangle$ be any diagonal representation of h_1. Let $h_1' = \langle a_1, a_2, \ldots a_j \rangle$ for $\frac{n}{2} + 2 < j < n$. Let $\mathbf{H} = \mathbf{SU}(h) \subset \mathbf{G}_1$. We shall show that $\mathbf{H}$ cannot be a subgroup of $\mathbf{G}_2$. Since D_1 and D_2 are not isomorphic, then there is a finite place $v \in V_k$ where one splits and the other ramifies. After relabeling if necessary, we may assume D_1 splits and D_2 ramifies.

$$\mathrm{rank}_{k_v}(\mathbf{G}_2) \leq \frac{n}{2} \leq j - 2 \leq \mathrm{rank}_{k_v}(\mathbf{H}).$$

Hence by rank considerations $\mathbf{H}$ cannot be a subgroup of $\mathbf{G}_2$. □

What remains open is the following conjecture.

Conjecture 7.1.7. Let $\mathbf{G}_1$ and $\mathbf{G}_2$ be algebraic k-groups coming from n-dimensional skew hermitian forms over the same division algebras D. If $\mathbf{G}_1$ and $\mathbf{G}_2$ are not k-isomorphic, then $SS(\mathbf{G}_1) \neq SS(\mathbf{G}_2)$.

This proves to be difficult to address due to the lack of local and global existence theorems for skew hermitian forms over division algebras.

7.2. Proof of Theorems A, B, and C

In this section, we answer the following question.

Question: *If M is an arithmetic locally symmetric space arising from a quadratic form, to what extent does $\mathbb{Q}TG(M)$ determine its commensurability class?*

To be able to perform the necessary arithmetic analysis, we need to make certain that the spaces we are considering have a common field of definition. Fortunately, totally geodesic subspaces can carry the field of definition as the next theorem shows.

Theorem 7.2.1. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms of dimension ≥ 4. Then $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ implies $k(M_1) \subset k(M_2)$.*

Proof. By assumption, M_1 arises from an absolutely almost simple algebraic k_1-group $\mathbf{G}_1$ where k_1 is a number field. Let q_1 represent $\mathbf{G}_1$. Let r be a subform of q_1 defined over k_1 of dimension $j \geq 3$ such that which is isotropic at at least one real place where q_1 is isotropic. Then the almost absolutely simple k_1-group $\mathbf{H} := \mathbf{SO}(r)$ gives rise to a nonflat finite volume totally geodesic submanifold $N_1 \subset M_1$. Since $\mathbf{H}$ is an almost absolutely simple k-group, it follows that $k(N_1) = k_1$. By assumption there exists a nonflat totally geodesic submanifold $N_2 \subset M_2$ which is commensurable to N_1, hence by Theorem 4.1.4 and Lemma 4.1.5, $k(N_2) = k(N_1) = k_1$. Since the minimal field of definition is generated by traces of the adjoint map, it follows that $k(N_2) \subset k(M_2)$, and hence $k(M_1) \subset k(M_2)$. □

By symmetry of argument, Theorem A follows as a corollary.

Theorem A. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming quadratic forms of dimension $m \geq 4$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies $k(M_1) = k(M_2)$.*

Notice that in fact all that matters is that M_1 and M_2 have the same set of commensurability classes of forms arising from some fixed dimension j. In particular, for arithmetic hyperbolic spaces, to have the same field of definition it suffices that they contain the same totally geodesic surfaces, which we record in the following proposition.

Proposition 7.2.2. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms of dimension ≥ 4. Suppose every totally geodesic surface in one is commensurable to a totally geodesic surface in the other. Then $k(M_1) \subset k(M_2)$.*

Now that we know $\mathbb{Q}TG(M)$ determines the field of definition for these spaces, we can check that it also determines the dimension of the quadratic forms giving rise to these spaces.

Proposition 7.2.3. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms. Suppose that q_1 and q_2 are quadratic forms representing M_1 and M_2 respectively. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies $\dim q_1 = \dim q_2$.*

Proof. We shall prove the contrapositive. By Theorem A, q_1 and q_2 are both quadratic forms over the same number field $k := k(M_i)$. If $\dim q_1 \neq \dim q_2$, then maybe after reordering, $\dim q_1 > \dim q_2$. Let $v_0 \in V_k$ be a real place where $q_1 \otimes k_{v_0}$ is isotropic. Then by deleting one entry in a diagonal representation of q_1 we have a $(\dim q_1 - 1)$-dimensional form r which is isotropic at v_0, and by dimensional considerations, there is no which no proper subform of q_2 which can represent $\mathbf{H} := \mathbf{SO}(r)$. Hence r gives rise to a finite volume totally geodesic submanifold N of M_1 which N cannot be a proper totally geodesic submanifold of M_2. The result then follows. □

Theorem B. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms of dimension $m \geq 5$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies M_1 and M_2 are commensurable.*

Proof. By Theorem A, there exists quadratic forms q_1 and q_2 over $k := k(M_i)$ such that M_i arises from the absolutely almost simple k-groups $\mathbf{G}_i := \mathbf{SO}(q_i)$. By Proposition 7.2.3, $\dim q_1 = \dim q_2$. Now suppose M_1 and M_2 are not commensurable. By Theorem 7.1.4 there exists an $i \in \{1, 2\}$ where $\mathbf{G}_i$ contains a semisimple k-subgroup $\mathbf{H}$ which is isotropic at a real place and which is not contained in $\mathbf{G}_j$, $j \neq i$. Hence M_i contains a totally geodesic submanifold not commensurable to a totally geodesic submanifold of M_j and the resulting contradiction shows M_1 and M_2 are commensurable. □

Specializing to the even dimensional $\mathbb{R}$-rank 1 case, Theorem B gives Theorem C.

Theorem C. *Let M_1 and M_2 be even dimensional arithmetic hyperbolic manifolds of dimension $n \geq 4$. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies M_1 and M_2 are commensurable.*

Unravelling the proof of Theorem B and Theorem 7.1.4, we see that we can tell apart noncommensurable even dimensional arithmetic hyperbolic spaces using only totally geodesic hypersurfaces, as we record in the following proposition.

Proposition 7.2.4. *Let M_1 and M_2 be even dimensional arithmetic hyperbolic manifolds of dimension $n \geq 4$. Suppose every totally geodesic hypersurface in one is commensurable to a totally geodesic hypersurface in the other. Then M_1 and M_2 are commensurable.*

Note that our constructions show noncommensurable arithmetic locally symmetric spaces coming from quadratic forms have different small codimension totally geodesic subspaces, while (due to Proposition 6.3.1) these spaces have the more or less the same totally geodesic subspaces in high codimensions.

7.3. Proof of Theorem D

While this thesis is primarily concerned with groups coming from quadratic forms, we may also apply our techniques to groups coming from skew hermitian forms over division algebras.

Theorem D. *Let M_1 and M_2 be arithmetic locally symmetric spaces where M_1 comes from a quadratic form of dimension $m = 2n$ and M_2 comes from a skew hermitian form of dimension n over a division algebra. Then $\mathbb{Q}TG(M_1) \neq \mathbb{Q}TG(M_2)$.*

Proof. The result follows from Theorem 7.1.5. □

Similar to the case for groups, what remains open is the following conjecture.

Conjecture 7.3.1. Let M_1 and M_2 be arithmetic locally symmetric spaces coming from skew hermitian forms over a division algebra. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies M_1 and M_2 are commensurable.

Proving this would complete the analysis of $\mathbb{Q}TG(M)$ for arithmetic spaces of Cartan–Killing type D_n for all $n \geq 5$, and all those of type D_n, $2 \leq n \leq 4$, not arising from exceptional isomorphisms (e.g., D_4 triality). The obstruction to proving this is the lack of local and global existence theorems for skew hermitian forms over division algebras.

7.4. Does $\mathbb{Q}TG(M)$ "See" Totally Geodesic Subspaces?

The following question was posed to us by Jean-François Lafont:

Question: Let M_1 and M_2 be Riemannian manifolds. When is it the case that $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ implies $M_1 \subset M_2$?

It turns out that for arithmetic hyperbolic spaces, we can largely answer this question. When the difference $\dim M_2 - \dim M_1$ is large, we have a positive result as we shall now see.

Proposition 7.4.1. *Let M_1 and M_2 be arithmetic hyperbolic spaces. Suppose that $3 \leq \dim M_1 \leq \dim M_2 - 3$ and $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$. Then up to commensurability $M_1 \subset M_2$.*

Proof. Since M_1 and M_2 contain the same commensurability classes of totally geodesic surfaces, Proposition 7.2.2 implies that $k(M_1) = k(M_2)$. Let $k := k(M_i)$ and q_i be quadratic forms over k which give rise to M_i. Our assumption on $\mathbb{Q}TG$ shows that q_1 and q_2 are isotropic at the same real place of k. Then by Proposition 6.3.1, it follows that q_1 is a subform of q_2 and the result follows. □

However when $\dim M_2 - \dim M_1$ is small we can have negative results. Hence there do exist counterexamples to the above question.

Example 7.4.2. Consider following quadratic forms over $\mathbb{Q}$ described in Example 6.2.7:

$$q_1 = \langle 1, 1, 1, -5 \rangle \qquad \text{and} \qquad q_2 = \langle 1, 1, 1, 3, 3, -1 \rangle.$$

By Theorem 6.2.6, up to commensurability, the 3-dimensional hyperbolic space M_{q_1} does not sit as a totally geodesic subspace of the five dimensional space M_{q_2}, yet by Proposition 6.3.1 and Proposition 4.4.3, they contain precisely the same totally geodesic surfaces.

Hence we have proven the following.

Proposition 7.4.3. *There exist arithmetic hyperbolic manifolds M_1 and M_2 for which $\mathbb{Q}TG(M_1) \subset \mathbb{Q}TG(M_2)$ but M_1 is not commensurable to a totally geodesic submanifold of M_2.*

7.5. Final Remarks

One of the main goals of this paper was to introduce and show the strength of the totally geodesic commensurability spectrum $\mathbb{Q}TG(M)$ in determining commensurability classes of locally symmetric spaces. When paired with the length commensurability spectrum, $\mathbb{Q}L(M)$, these spectra become a powerful tool in analyzing commensurability classes of locally symmetric spaces.

It is worth noting that Theorems A, B, C, and D hold for any $\mathbb{R}$-rank, and unlike the result of [PR09] and [Ga], are not dependent upon the truth of Schanuel's conjecture. It is also worth noting that groups of type B_n and D_n over number fields may produce many semisimple groups with no compact factors which are not absolutely simple over $\mathbb{R}$, hence these result cover a large class of spaces not covered under the results of [PR09].

Theorem B shows that for spaces coming from quadratic forms, the totally geodesic commensurability spectrum determines the commensurability class which in turn determines the rational length spectrum $\mathbb{Q}L(M)$. We have just shown the following theorem.

Theorem 7.5.1. *Let M_1 and M_2 be arithmetic locally symmetric spaces coming from quadratic forms of dimension ≥ 5. Then $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ implies $\mathbb{Q}L(M_1) = \mathbb{Q}L(M_2)$.*

Hence the set of totally geodesic subspaces determines the rational multiples of the lengths of all closed geodesics, even though there exist closed geodesics which do not lie in any proper nonflat totally geodesic subspace. (The existence of such geodesics follows from the existence of $\mathbb{R}$-regular elements in these arithmetic lattices. See [Pr94] for an elementary proof of this fact.)

Just as we saw the limitations of $\mathbb{Q}L(M)$ in [PR09], there are limitations of $\mathbb{Q}TG(M)$. There exist noncommensurable locally symmetric spaces M_1 and M_2 for which $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$. For example, the following theorem of Alan Reid shows there are locally symmetric spaces with no nonflat finite volume totally geodesic submanifolds.

Theorem 7.5.2 ([Re87]). *There exists infinitely many commensurability classes of compact hyperbolic 3-manifolds which have no immersed totally geodesic surfaces. These may be chosen to have the same trace field.*

While this theorem shows the existence of noncommensurable locally symmetric spaces M_1 and M_2 for which $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$, as of the time of writing this thesis, we are unaware of similar results for which there are nontrivial totally geodesic subspaces. As such, in future research we hope to use our techniques to be able to answer to following question.

Question 7.5.3. Are there arithmetic locally symmetric spaces M_1 and M_1 such that $\mathbb{Q}TG(M_1) = \mathbb{Q}TG(M_2)$ is *nonempty* but M_1 and M_2 are noncommensurable?

Furthermore, in addition to analyzing Conjecture 7.3.1, we plan to continue pursuing this manner of analysis for groups coming from hermitian and skew hermitian forms over number fields, allowing us to obtain similar results for groups and spaces of Cartan–Killing type A_n and C_n. In the end, we hope that this thesis will be an important stepping stone for future research on commensurability classes of totally geodesic subspaces of arithmetic locally symmetric space.

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