Centrally symmetric polytopes with many faces

by

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CHAPTER I

Introduction and main results

A *polytope* is the convex hull of a set of finitely many points in $\mathbb{R}^d$. A polytope $P \subset \mathbb{R}^d$ is *centrally symmetric* (cs, for short) if $P = -P$. A *convex body* is a compact convex set with non-empty interior. A *face* of a polytope (or a convex body) P can be defined as the intersection of P and a closed halfspace H such that the boundary of H contains no interior point of P. The 0-dimensional faces are the *vertices*, and the 1-dimensional faces (called *edges*) are line segments connecting pairs of vertices.

A construction of *cyclic polytopes*, which goes back to Carathéodory [9] and was studied by Motzkin [22] and Gale [15], presents a family of polytopes in $\mathbb{R}^d$ with an arbitrarily large number N of vertices, such that the convex hull of every set of $k \leq d/2$ vertices is a face of P. Such a polytope is obtained as the convex hull of N distinct points on the moment curve $\gamma(t) = \big(t, t^2, \ldots, t^d\big)$ in $\mathbb{R}^d$.

The situation with centrally symmetric polytopes is less understood. A centrally symmetric polytope P is called *k-neighborly* if the convex hull of every set $\{v_1, \ldots, v_k\}$ of k vertices of P, not containing a pair of antipodal vertices $v_i = -v_j$, is a face of P. In contrast with polytopes without symmetry, even 2-neighborly centrally symmetric polytopes cannot have too many vertices: it was shown in [18] that no d-dimensional 2-neighborly centrally symmetric polytope has more than 2^d vertices. Moreover, as was verified in [4], the number $f_1(P)$ of edges (1-dimensional faces) of an arbitrary centrally symmetric polytope $P \subset \mathbb{R}^d$ with N vertices satisfies

$$f_1(P) \ \leq \ \frac{N^2}{2}\left(1 - 2^{-d}\right).$$

In this dissertation, we present constructions of following polytopes

- A d-dimensional 2-neighborly centrally symmetric polytope with roughly $3^{d/2} \approx (1.73)^d$ vertices (Theorem III.2.1.)
- A d-dimensional centrally symmetric polytopes with N vertices and at least $\left(1 - 3^{-\lfloor d/2-1 \rfloor}\right)\binom{N}{2} \approx \left(1 - 0.58^d\right)\frac{N^2}{2}$ edges for an arbitarily large N (Theorem III.2.2.).

These results are published in [6, 7].

For higher-dimensional faces even less is known. It follows from the results of [18] that no centrally symmetric k-neighborly d-polytope can have more than $\lfloor d \cdot 2^{Cd/k} \rfloor$ vertices, where $C > 0$ is some absolute constant. At the same time, the papers [18, 23] used a randomized construction to prove existence of k-neighborly centrally symmetric d-dimensional polytopes with $\lfloor d \cdot 2^{cd/k} \rfloor$ vertices for some absolute constant $c > 0$. However, for $k > 2$, no deterministic construction of a d-dimensional k-neighborly centrally symmetric polytope with $2^{\Omega(d)}$ vertices was known.

Let $f_k(P)$ denote the number of k-dimensional faces of a polytope P. It is proved in [4] that for a d-dimensional centrally symmetric polytope P with N vertices,

$$f_{k-1}(P) \ \leq \ \frac{N}{N-1}\left(1 - 2^{-d}\right)\binom{N}{k}, \quad \text{provided} \quad k \leq d/2.$$

In particular, as the number N of vertices grows while the dimension d of the polytope stays fixed, the fraction of k-tuples $v_1, \ldots, v_k$ of vertices of P that do not form the vertex set of a $(k-1)$-dimensional face of P remains bounded from below by roughly 2^{-d}.

In this dissertation, we present explicit deterministic constructions of following polytopes.

- A d-dimensional centrally symmetric k-neighborly polytope with at least $2^{c_k d}$ vertices where $c_k = 3/20k^2 2^k$ (Theorem III.9).
- A d-dimensional centrally symmetric polytope with N vertices and at least $\left(1 - k^2 \left(2^{-c_k}\right)^d\right) \binom{N}{k}$ faces of dimension $k-1$ for a fixed k and arbitrarily large N and d (Corollary III.11).
- A d-dimensional centrally symmetric polytope with N vertices and at least $\left(1 - (\delta_k)^d\right) \binom{N}{k}$ faces of dimension $k-1$ for a fixed k, any $\delta_k > \left(1 - 5^{-k+1}\right)^{5/(24k+4)}$ and arbitrarily large N and d (Corollary III.7).

These results are published in [6, 7].

Notice that Corollary III.11 improves Corollary III.7. However, while the construction of polytopes in Theorem III.9 and Corollary III.11 uses the notion of k-*independent sets* to improve bounds for explicit constructions, Corollary III.7 can be constructed in a rather simple way.

Our results on cs polytopes provide new bounds on several problems related to strict antipodality. Let $X \subset \mathbb{R}^d$ be a set that affinely spans $\mathbb{R}^d$. A pair of points $u, v \in X$ is called *strictly antipodal* if there exist two distinct parallel hyperplanes H and H' such that $X \cap H = \{u\}$, $X \cap H' = \{v\}$, and X lies in the slab between H and H'. Denote by $A'(d)$ the maximum size of a set $X \subset \mathbb{R}^d$ having the property that every pair of points of X is strictly antipodal, by $A'_d(Y)$ the number of strictly antipodal pairs of a given set Y, and by $A'_d(n)$ the maximum size of $A'_d(Y)$ taken over all n-element subsets Y of $\mathbb{R}^d$. (Our notation follows the recent survey paper [20].)

We observe that an appropriately chosen half of the vertex set of a cs d-polytope with many edges has a large number of strictly antipodal pairs of points. Consequently, our construction of cs d-polytopes with many edges implies — see Theorem III.3 — that

$$A'(d) \geq 3^{\lfloor d/2-1 \rfloor} - 1 \quad \text{and} \quad A'_d(n) \geq \left(1 - \frac{1}{3^{\lfloor d/2-1 \rfloor} - 1}\right)\frac{n^2}{2} - O(n) \quad \text{for all } d \geq 4.$$

Our constructions are based on the *symmetric moment curve.* Barvinok and Novik introduced and studied the symmetric moment curve $U(t) \in \mathbb{R}^{2k}$ in [4], defined by

$$U(t) = \Big(\cos t,\ \sin t,\ \cos 3t,\ \sin 3t, \ldots, \cos(2k-1)t,\ \sin(2k-1)t\Big).$$

In Chapter 2, it is proved that the convex hull $\mathcal{B}_k$ of $U(t)$ is *local k-neighborly.*

Theorem I.1. *For every positive integer k there exists a number*

$$\frac{\pi}{2} \ < \ \phi_k \ < \ \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ of length ϕ_k and arbitrary distinct $n \leq k$ points $t_1, \ldots, t_n \in \Gamma$, the set

$$\operatorname{conv}\Big(U(t_1), \ldots, U(t_n)\Big)$$

is a face of $\mathcal{B}_k$.

It is also verified that the limit of such a ϕ_k as k goes to infinity is $\pi/2$ in Theorem II.2. These results are published in [5]. Note that $\mathcal{B}_k$ is not a polytope but a convex body.

Besides being of intrinsic interest, centrally symmetric polytopes with many faces appear in problems of sparse signal reconstruction, see [13], [23]. Typically, such polytopes are obtained through a randomized construction, for example, as the orthogonal projection of a high-dimensional cross-polytope (octahedron) onto a random subspace, see [18] and [14].

The rest of the dissertation is structured as follows. In chapter 2 we investigate the symmetric moment curves and its properties such as local k-neighborliness. In chapter 3 we provide the deterministic constructions of centrally symmetric polytopes described above.

CHAPTER II

Symmetric moment curve and its neighborliness

The main object of this chapter is the *symmetric moment curve* that for a fixed k lies in $\mathbb{R}^{2k}$ and is defined by

$$U(t) = U_k(t) = \Big(\cos t,\ \sin t,\ \cos 3t,\ \sin 3t, \ldots, \cos(2k-1)t,\ \sin(2k-1)t\Big).$$

We note that

$$U(t+\pi) = -U(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

Since U is periodic, we consider U to be defined on the unit circle $\mathbb{S} = \mathbb{R}/2\pi\mathbb{Z}$. In particular, for every $t \in \mathbb{S}$, the points t and $t+\pi$ are antipodal points on the circle.

We define the convex body $\mathcal{B} \subset \mathbb{R}^{2k}$ as the convex hull of the symmetric moment curve

$$\mathcal{B} = \mathcal{B}_k = \operatorname{conv}\Big(U(t) : \quad t \in \mathbb{S}\Big).$$

Hence $\mathcal{B}$ is symmetric about the origin, $\mathcal{B} = -\mathcal{B}$. We note that $\mathcal{B}_k$ has a non-empty interior in $\mathbb{R}^{2k}$ since $U_k(t)$ does not lie in an affine hyperplane.

In this chapter, we will prove following main results.

Theorem II.1. *For every positive integer k there exists a number*

$$\frac{\pi}{2} \ < \ \phi_k \ < \ \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ *of length* ϕ_k *and arbitrary distinct* $n \leq k$ *points* $t_1, \dots, t_n \in \Gamma$*, the set*

$$\operatorname{conv}\Big(U\left(t_1\right), \dots, U\left(t_n\right)\Big)$$

is a face of $\mathcal{B}_k$*.*

Theorem II.2. *Let* ϕ_k *be the largest number satisfying Theorem 1.1. Then*

$$\lim_{k \longrightarrow +\infty} \phi_k = \frac{\pi}{2} \approx 1.570796327.$$

Theorem II.3. *Let* $\Gamma \subset \mathbb{S}$ *be an open arc with the endpoints* a *and* b *and let* $\overline{\Gamma}$ *be the closure of* Γ*. Let* $t_2, \dots, t_k \in \mathbb{S} \setminus \overline{\Gamma}$ *be distinct points such that the set* $\overline{\Gamma} \cup \big\{t_2, \dots, t_k\big\}$ *lies in an open semicircle in* $\mathbb{S}$*. Suppose that the points* $U(a), U(t_2), \dots, U(t_k)$ *lie in a face of* $\mathcal{B}_k$ *and that the points* $U(b), U(t_2), \dots, U(t_k)$ *lie in a face of* $\mathcal{B}_k$*. Then for all* $t_1 \in \Gamma$ *the set*

$$\operatorname{conv}\Big(U(t_1), \dots, U(t_k)\Big)$$

is a face of $\mathcal{B}_k$*.*

Before we prove the main theorems, we prove technical lemmas in Section 2.1-2.4. We prove Theorem II.1 and II.3 in Section 2.5 and prove II.2 in Section 2.6. In Section 2.7, we discuss local neighborliness of generalized moment curves.

2.1 Raked trigonometric polynomials

We consider *raked trigonometric polynomials* of degree at most $2k-1$:

$$f(t) = c + \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t \quad \text{for} \quad t \in \mathbb{S}, \tag{2.1}$$

where $c, a_j, b_j \in \mathbb{R}$. We say that $\deg f = 2k-1$ if $a_{2k-1} \neq 0$ or if $b_{2k-1} \neq 0$. Equivalently, we can write

$$f(t) = c + \langle C,\ U(t)\rangle,$$

where $C = (a_1, b_1, \ldots, a_k, b_k) \in \mathbb{R}^{2k}$ and $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^{2k}$.

Writing

$$\cos nt = \frac{e^{int} + e^{-int}}{2} \quad \text{and} \quad \sin nt = \frac{e^{int} - e^{-int}}{2i}$$

and substituting $z = e^{it}$, we associate with (2.1) a complex polynomial

$$\mathcal{P}(f)(z) = z^{2k-1}\left(c + \sum_{j=1}^{k} a_j \frac{z^{2j-1} + z^{1-2j}}{2} + \sum_{j=1}^{k} b_j \frac{z^{2j-1} - z^{1-2j}}{2i}\right). \tag{2.2}$$

Hence

$$\deg \mathcal{P}(f) \ \leq\ 4k-2. \tag{2.3}$$

Moreover, if $\deg f = 2k-1$ then for $p = \mathcal{P}(f)$ we have $\deg p = 4k-2$ and $p(0) \neq 0$. Since

$$\cos(t+a) = \cos t \cos a - \sin t \sin a \quad \text{and} \quad \sin(t+a) = \sin t \cos a + \cos t \sin a,$$

for any fixed $a \in \mathbb{S}$ and any raked trigonometric polynomial $f(t)$, the function

$$h(t) = f(t+a) \quad \text{for} \quad t \in \mathbb{S}$$

is also a raked trigonometric polynomial of the same degree.

Definition II.4. We say that a point $t^* \in \mathbb{S}$ is a *root of multiplicity* m (where $m \geq 1$ is an integer) of a trigonometric polynomial f, if

$$f\left(t^*\right) = \ldots = f^{(m-1)}\left(t^*\right) = 0$$

and

$$f^{(m)}\left(t^*\right) \neq 0.$$

Similarly, we say that a number $z^* \in \mathbb{C}$ is a root of multiplicity m of a polynomial $p(z)$ if

$$p\left(z^*\right) = \ldots = p^{(m-1)}\left(z^*\right) = 0$$

and

$$p^{(m)}\left(z^*\right) \neq 0.$$

Remark II.5.

(1) We note that

$$\operatorname{conv}\Big(U\left(t_1\right), \ldots, U\left(t_n\right)\Big)$$

for distinct $t_1, \ldots, t_n \in \mathbb{S}$ is a face of $\mathcal{B}_k$ if and only if there exists a raked trigonometric polynomial $f(t) \not\equiv 0$ of degree at most $2k-1$ such that (i) each t_j, $j = 1, \ldots, n$, is a root of f of an even multiplicity, and (ii) f has no other roots.

(2) We will often use the following observation: if f is a trigonometric polynomial with constant term 1 that does not change sign on $\mathbb{S}$ then $f(t) \geq 0$ for all $t \in \mathbb{S}$, since

$$\frac{1}{2\pi} \int_{\mathbb{S}} f(t)\ dt = 1.$$

2.2 Roots and multiplicities

We consider raked trigonometric polynomials $f(t)$ defined by (2.1). In this section we prove the following main result.

Theorem II.6. *Let $f(t) \not\equiv 0$ be a raked trigonometric polynomial of degree at most $2k-1$, let $t_1, \ldots, t_n \in \mathbb{S}$ be distinct roots of f in $\mathbb{S}$, and let $m_1, \ldots, m_n$ be their multiplicities.*

(1) We have

$$\sum_{i=1}^{n} m_i \;\le\; 4k-2.$$

(2) If the constant term of f is 0 and the set $\{t_1,\dots,t_n\}$ does not contain a pair of antipodal points, then

$$\sum_{i=1}^{n} m_i \;\le\; 2k-1.$$

(3) If $t_1,\dots,t_n$ lie in an open semicircle of $\mathbb{S}$, then

$$\sum_{i=1}^{n} m_i \;\le\; 2k.$$

(4) Suppose that $t_1,\dots,t_n$ lie in an arc $\Gamma \subset \mathbb{S}$ of length less than π, that

$$\sum_{i=1}^{n} m_i = 2k,$$

and that $t^* \in \mathbb{S}\setminus\Gamma$ is yet another root of f. Then $t^* \in \Gamma+\pi$.

To prove II.6, we establish a correspondence between the roots of a trigonometric polynomial $f(t)$ and those of the corresponding complex polynomial $p(z) = \mathcal{P}(f)$ defined by (2.2).

Lemma II.7. *A point $t^* \in \mathbb{S}$ is a root of multiplicity m of $f(t)$ if and only if $z^* = e^{it^*}$ is a root of multiplicity m of $\mathcal{P}(f)$.*

Proof. Let $p = \mathcal{P}(f)$. It follows from (2.2) that

$$p\left(e^{it}\right) = e^{(2k-1)it} f(t). \tag{2.4}$$

Differentiating (2.4), we infer by induction that

$$i^r \sum_{j=1}^{r} d_{j,r} e^{ijt} p^{(j)}\left(e^{it}\right) = \sum_{j=0}^{r} i^{r-j} c_{j,r} e^{(2k-1)it} \cdot f^{(j)}(t) \quad \text{for all} \quad r \ge 1,$$

where the constants $c_{j,r}$, $d_{j,r}$ are positive integers. Thus, if $f^{(r)}(t^*)$ is zero for $r = 0,1,\dots,m-1$ and nonzero for $r=m$, then so is $p^{(r)}\left(e^{it^*}\right)$, and vice versa. The statement now follows. $\square$

Proof of II.6.

Part (1) follows from II.7 and bound (2.3).

If f has a zero constant term, then f satisfies

$$f(t+\pi) = -f(t) \quad \text{for all} \quad t \in \mathbb{S}.$$

Then $t_i + \pi$ is a root of $f(t)$ of multiplicity m_i and the proof of Part (2) follows from Part (1).

To prove Part (3), let $g(t) = f'(t)$. Then g has a zero constant term. If

$$\sum_{i=1}^{n} m_i > 2k,$$

then by Rolle's Theorem, the total number of roots of $g(t)$ in the semicircle, counting multiplicities, is at least $2k$, and so $g(t) \equiv 0$ by Part (2), which is a contradiction.

To prove Part (4), we assume without loss of generality that $t_1, \ldots, t_n$ is the order of the roots on the arc Γ and let $\tilde{\Gamma}$ be the closed arc with the endpoints t_1 and t_n. By Rolle's Theorem, the total number of roots of $g(t)$, counting multiplicities, in $\tilde{\Gamma}$ is at least $2k-1$, and hence the total number of roots of $g(t)$, counting multiplicities, in $\tilde{\Gamma} \cup \left(\tilde{\Gamma} + \pi\right)$ is at least $4k-2$. See Figure 2.1 for $k = 3$. If $t^* \notin \tilde{\Gamma} \cup \left(\tilde{\Gamma} + \pi\right)$, then by Rolle's theorem there is a root of $g(t)$ outside of $\tilde{\Gamma} \cup \left(\tilde{\Gamma} + \pi\right)$, and hence the total number of roots of $g(t)$ in $\mathbb{S}$, counting multiplicities, is at least $4k-1$. Thus, by Part (1), $g(t) \equiv 0$, which is a contradiction. $\square$

Lemma II.8. *Let $t_1, \ldots, t_n \in \mathbb{S}$ be distinct points lying in an open semicircle and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

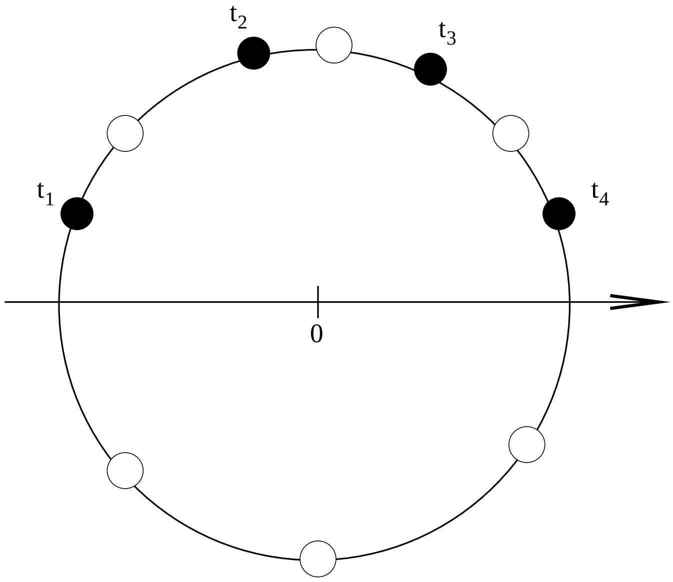


Figure 2.1: The roots of $f(t)$ (black dots) and the roots of $g(t) = f'(t)$ (white dots) for $k = 3$.

Then the following $2k$ vectors

$$U(t_i) - U(t_n) \quad \textit{for} \quad i = 1, \ldots, n-1,$$
$$\frac{d^j}{dt^j} U(t)\Big|_{t=t_i} \quad \textit{for} \quad j = 1, \ldots, m_i - 1 \quad \textit{if} \quad m_i > 1 \quad \textit{and} \quad i = 1, \ldots, n,$$
$$\frac{d^{m_1}}{dt^{m_1}} U(t)\Big|_{t=t_1}$$

are linearly independent in $\mathbb{R}^{2k}$.

Proof. Seeking a contradiction, we assume that the vectors are not linearly independent. Then there exists a non-zero vector $C \in \mathbb{R}^{2k}$ orthogonal to all these $2k$ vectors. Consider the raked trigonometric polynomial

$$f(t) = \langle C,\ U(t) - U(t_n) \rangle \quad \text{for} \quad t \in \mathbb{S}.$$

Then $t_1, \ldots, t_n$ are roots of $f(t)$. Moreover, the multiplicity of t_i is at least m_i for $i > 1$ and at least $m_1 + 1$ for $i = 1$. It follows from Part (3) of II.6 that $f(t) \equiv 0$, which contradicts that $C \neq 0$. $\square$

Finally, we prove that a raked trigonometric polynomial is determined, up to a constant factor, by its roots of the total multiplicity $2k$ provided those roots lie in an open semicircle.

Corollary II.9. *Let $t_1, \ldots, t_n \in \mathbb{S}$ be distinct points lying in an open semicircle, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

Then there exists a unique raked trigonometric polynomial $f(t)$ of degree at most $2k-1$ and with constant term 1, such that t_i is a root of $f(t)$ of multiplicity m_i for all $i = 1, \ldots, n$. Moreover, f depends analytically on $t_1, \ldots, t_n$.

Proof. Such a polynomial $f(t)$ can be written as

$$f(t) = \langle C,\ U(t) - U(t_n) \rangle \quad \text{for} \quad t \in \mathbb{S},$$

where $C \in \mathbb{R}^{2k}$ is orthogonal to the $2k-1$ vectors

$$\begin{aligned} &U(t_i) - U(t_n) \quad \text{for} \quad i = 1, \ldots, n-1, \\ &\frac{d^j}{dt^j} U(t)\Big|_{t=t_i} \quad \text{for} \quad j = 1, \ldots, m_i - 1 \quad \text{if} \quad m_i > 1 \quad \text{and} \quad i = 1, \ldots, n. \end{aligned}$$

By II.8, these $2k-1$ vectors span a hyperplane in $\mathbb{R}^{2k-1}$ and hence, up to a scalar, there is a unique choice of C. By Part (2) of II.6, f has a non-zero constant term if $C \neq 0$. Therefore, there is a unique choice of C that makes the constant term of $f(t)$ equal 1. By Part (3) of II.6 the multiplicities of the roots t_i are exactly m_i for $i = 1, \ldots, n$. □

We note that in fact $\deg f = 2k-1$. This follows from Part (3) of II.6.

We will also need the following *deformation construction*.

Lemma II.10. *Let $f(t)$ be a raked trigonometric polynomial of degree $2k-1$ such that $f(-t) = f(t)$, and let $p = \mathcal{P}(f)$ be the corresponding complex polynomial associated* *with f via (2.2). Then $p(0) \neq 0$ and the multiset M of roots of p can be split into $2k-1$ unordered pairs $\{\zeta_j, \zeta_j^{-1}\}$ for $j = 1, \ldots, 2k-1$. Moreover, for any real $\lambda \neq 0$, the multiset M_λ consisting of $2k-1$ unordered pairs $\{\xi_j, \xi_j^{-1}\}$ defined by*

$$\xi_j + \xi_j^{-1} = \lambda\left(\zeta_j + \zeta_j^{-1}\right) \quad \text{for} \quad j = 1, \ldots, 2k-1$$

is the multiset of roots of a certain complex polynomial p_λ such that $p_\lambda = \mathcal{P}(f_\lambda)$ for a raked trigonometric polynomial $f_\lambda(t)$ of degree $2k-1$ satisfying $f_\lambda(-t) = f_\lambda(t)$.

Proof. This is Lemma 5.1 of [4]. □

We call $f_\lambda(t)$ a λ-*deformation* of f.

2.3 Parametric families of trigonometric polynomials

Let $\Gamma \subset \mathbb{S}$ be an open arc. We consider raked trigonometric polynomials

$$f_s(t) = 1 + \sum_{j=1}^{k} a_j(s)\cos(2j-1)t + \sum_{j=1}^{k} b_j(s)\sin(2j-1)t \quad \text{for} \quad t \in \mathbb{S}, \tag{2.5}$$

where $a_j(s)$ and $b_j(s)$ are real analytic functions of $s \in \Gamma$. We define

$$g_s(t) = \frac{\partial}{\partial s} f_s(t),$$

and so

$$g_s(t) = \sum_{j=1}^{k} a_j'(s)\cos(2j-1)t + \sum_{j=1}^{k} b_j'(s)\sin(2j-1)t. \tag{2.6}$$

The goal of this section is to prove the following result.

Theorem II.11. *Let $\Gamma \subset \mathbb{S}$ be an open arc, let $t_2, \ldots, t_n \in \mathbb{S} \setminus \Gamma$ be distinct points such that the set $\Gamma \cup \{t_2, \ldots, t_n\}$ lies in an open semicircle, and let $m_1, \ldots, m_n$ be positive integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

For every $s \in \Gamma$, let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 such that for $i = 2, \ldots, n$ the point t_i is a root of $f_s(t)$ of multiplicity m_i and s is a root of $f_s(t)$ of multiplicity m_1, cf. II.9. Define

$$g_s(t) = \frac{\partial}{\partial s} f_s(t).$$

Then

$$g_s(t) \not\equiv 0 \quad \textit{for all} \quad s \in \Gamma.$$

To prove II.11, we use the notion of the wedge product.

Given linearly independent vectors $V_1, \ldots, V_{2k-1} \in \mathbb{R}^{2k}$ we define their wedge product

$$W = V_1 \wedge \ldots \wedge V_{2k-1}$$

as the unique vector W orthogonal to the hyperplane spanned by $V_1, \ldots, V_{2k-1}$ whose length is the volume of the $(2k-1)$-dimensional parallelepiped spanned by $V_1, \ldots, V_{2k-1}$ and such that the basis $V_1, \ldots, V_{2k-1}, W$ is co-oriented with the standard basis of $\mathbb{R}^{2k}$. If vectors $V_1, \ldots, V_{2k-1}$ are linearly dependent, we let

$$V_1 \wedge \ldots \wedge V_{2k-1} = 0.$$

Suppose that vectors $V_1(s), \ldots, V_{2k-1}(s)$ depend smoothly on a real parameter s. We will use the following standard fact:

$$\begin{aligned} &\frac{d}{ds}\Big(V_1(s) \wedge \ldots \wedge V_{2k-1}(s)\Big) \\ &\quad = \sum_{j=1}^{2k-1} V_1(s) \wedge \ldots \wedge V_{j-1}(s) \wedge \frac{d}{ds} V_j(s) \wedge V_{j+1}(s) \wedge \ldots \wedge V_{2k-1}(s). \end{aligned} \tag{2.7}$$

Proof of II.11.

For $s \in \Gamma$, consider the following ordered set of $2k-1$ vectors:

$$
\begin{aligned}
&U(t_i) - U(t_n) \quad \text{for} \quad i = 2, \ldots, n-1, \\
&\frac{d^j}{dt^j} U(t)\Big|_{t=t_i} \quad \text{for} \quad j = 1, \ldots, m_i - 1 \quad \text{if} \quad m_i > 1 \quad \text{and} \quad i = 2, \ldots, n, \\
&U(s) - U(t_n), \\
&\frac{d^j}{dt^j} U(t)\Big|_{t=s} \quad \text{for} \quad j = 1, \ldots, m_1 - 1 \quad \text{if} \quad m_1 > 1.
\end{aligned}
\tag{2.8}
$$

Let $C(s)$ be the wedge product of vectors of (2.8). By II.8, the vectors of (2.8) are linearly independent for all $s \in \Gamma$, and hence $C(s) \neq 0$ for all $s \in \Gamma$.

For $s \in \Gamma$, define a raked trigonometric polynomial

$$F_s(t) = \big\langle C(s),\ U(t) - U(t_n) \big\rangle. \tag{2.9}$$

We note that $F_s(t) \not\equiv 0$ for all $s \in \Gamma$. For $i = 2, \ldots, n$, the point t_i is a root of $F_s(t)$ of multiplicity at least m_i and s is a root of $F_s(t)$ of multiplicity at least m_1. By Part (3) of II.6 the multiplicities are exactly m_i. Let $\alpha(s)$ be the constant term of $F_s(t)$. Then

$$\alpha(s) = -\big\langle C(s), U(t_n) \big\rangle.$$

By Part (2) of II.6

$$\alpha(s) \neq 0 \quad \text{for all} \quad s \in \Gamma.$$

Therefore,

$$f_s(t) = \frac{F_s(t)}{\alpha(s)}.$$

Seeking a contradiction, let us assume that $g_s(t) \equiv 0$ for some $s \in \Gamma$.

We have

$$g_s(t) = \frac{\partial}{\partial s} f_s(t) = \frac{\alpha(s) \frac{\partial}{\partial s} F_s(t) - \alpha'(s) F_s(t)}{\alpha^2(s)}.$$

If $g_s(t) \equiv 0$, then

$$\alpha(s)\frac{\partial}{\partial s}F_s(t) - \alpha'(s)F_s(t) \equiv 0,$$

and (2.9) yields that

$$\alpha(s)C'(s) - \alpha'(s)C(s) = 0 \tag{2.10}$$

for some $s \in \Gamma$. Let us consider $C'(s)$, the derivative of the wedge product of (2.8). Applying formula (2.7) we note that all of the $2k-1$ terms of (2.7) except the last one are zeros since the corresponding wedge product either contains a zero vector or two identical vectors. Hence $C'(s)$ is the wedge product of the following ordered set of vectors

$$\begin{aligned}
&U(t_i) - U(t_n) \quad \text{for} \quad i = 2, \ldots, n-1, \\
&\frac{d^j}{dt^j}U(t)\Big|_{t=t_i} \quad \text{for} \quad j = 1, \ldots, m_i - 1 \quad \text{if} \quad m_i > 1 \quad \text{and} \quad i = 2, \ldots, n, \\
&U(s) - U(t_n), \\
&\frac{d^j}{dt^j}U(t)\Big|_{t=s} \quad \text{for} \quad j = 1, \ldots, m_1 - 2 \quad \text{if} \quad m_1 > 2 \quad \text{and} \quad j = m_1.
\end{aligned} \tag{2.11}$$

The wedge products (2.8) for $C(s)$ and (2.11) for $C'(s)$ differ in two vectors,

$$A(s) = \frac{d^{m_1-1}}{dt^{m_1-1}}U(t)\Big|_{t=s} \quad \text{and} \quad B(s) = \frac{d^{m_1}}{dt^{m_1}}U(t)\Big|_{t=s} \quad \text{if} \quad m_1 > 1$$

and

$$A(s) = U(s) - U(t_n) \quad \text{and} \quad B(s) = \frac{d}{dt}U(t)\Big|_{t=s} \quad \text{if} \quad m_1 = 1.$$

The vector $A(s)$ is present in (2.8) and absent in (2.11) while the vector $B(s)$ is absent in (2.8) and present in (2.11). Therefore, (2.10) implies that the set consisting of the vector

$$\alpha(s)A(s) - \alpha'(s)B(s)$$

and the $2k-2$ vectors common to wedges (2.8) and (2.11) is linearly dependent. However, as $\alpha(s)\neq 0$, this contradicts II.8. $\square$

We will need the following result.

Lemma II.12. *Let $f_s(t)$ and $g_s(t)$ be trigonometric polynomials (2.5) and (2.6) respectively and let m be a positive integer.*

(1) If $t^*\in\mathbb{S}$ is a root of $f_s(t)$ of multiplicity at least m for all $s\in\Gamma$, then t^* is a root of $g_s(t)$ of multiplicity at least m for all $s\in\Gamma$.

(2) If $m>1$ and s is a root of $f_s(t)$ of multiplicity at least m for all $s\in\Gamma$, then s is a root of $g_s(t)$ of multiplicity at least $m-1$ for all $s\in\Gamma$.

Proof. Suppose that

$$f_s(t^*)=\ldots=\frac{\partial^{m-1}}{\partial t^{m-1}}f_s(t)\Big|_{t=t^*}=0.$$

Differentiating with respect to s yields Part (1).

Suppose that

$$f_s(s)=\frac{\partial^j}{\partial t^j}f_s(t)\Big|_{t=s}=0 \quad \text{for} \quad j=1,\ldots,m-1.$$

Differentiating with respect to s we obtain

$$\begin{aligned}0=&\frac{\partial}{\partial s}f_s(t)\Big|_{t=s}+\frac{\partial}{\partial t}f_s(t)\Big|_{t=s}=\frac{\partial}{\partial s}\frac{\partial^j}{\partial t^j}f_s(t)\Big|_{t=s}+\frac{\partial^{j+1}}{\partial t^{j+1}}f_s(t)\Big|_{t=s}\\ &\text{for} \quad j=1,\ldots,m-1.\end{aligned}$$

Therefore,

$$g_s(s)=\frac{\partial^j}{\partial t^j}g_s(t)\Big|_{t=s}=0 \quad \text{for} \quad j=1,\ldots,m-2,$$

and the proof of Part (2) follows. $\square$

2.4 Critical arcs

This section is devoted to verifying the following result.

Theorem II.13.

(1) For every $k \geq 1$ there exists a non-empty open arc $\Gamma \subset \mathbb{S}$ with the following property: if $t_1, \ldots, t_n \in \Gamma$ are distinct points and $m_1, \ldots, m_n$ are positive even integers satisfying

$$\sum_{i=1}^{m} m_i = 2k,$$

then the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has each point t_i as a root of multiplicity m_i, has no other roots in $\mathbb{S}$. Moreover, $f(t) \geq 0$ for all $t \in \mathbb{S}$.

(2) Let $\Gamma \subset \mathbb{S}$ be an open arc as in Part (1) of the maximum possible length and let a and b be the endpoints of Γ. Then there are positive even integers m_a and m_b such that $m_a + m_b = 2k$ and such that the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has a root at $t = a$ of multiplicity m_a and a root at $t = b$ of multiplicity m_b also has a root of an even multiplicity in the arc $\Gamma + \pi$.

(3) Fix positive even integers m_a and m_b such that $m_a + m_b = 2k$. Let $\Gamma \subset \mathbb{S}$ be an open arc of length less than π and let a be an endpoint of Γ. For $b \in \Gamma$ let $f_b(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root at $t = a$ of multiplicity m_a and a root at $t = b$ of multiplicity m_b. Let $x, y, z \in \Gamma$ be distinct points such that y lies between a and z and x lies between a and y (See Figure 2.2). Suppose that $f_y(t) \geq 0$ for all $t \in \mathbb{S}$ and that f_y has a root (of necessarily even multiplicity) in the arc $\Gamma + \pi$. Then $f_x(t)$ is positive for all $t \in \mathbb{S} \setminus \{a, x\}$ while $f_z(t)$ is negative for some $t \in \mathbb{S}$.

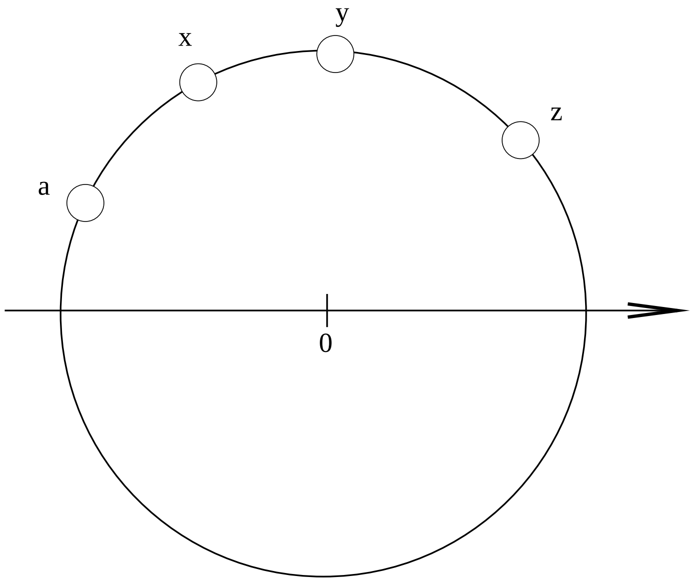


Figure 2.2: Possible place for a, x, y, z for Theorem II.13(3)

Let us denote for a moment the maximum possible length of an arc Γ satisfying Part (1) of II.13 by ψ_k. In II.17 below we prove that $\psi_k = \phi_k$, the maximum length of an arc with the neighborliness property of II.1.

Example II.14. Suppose that $k = 2$. The only possible set of multiplicities in Part (2) of II.13 is $m_a = 2$ and $m_b = 2$. The polynomial $f(t) = 1 - \cos 3t$ has roots at $t = \pm 2\pi/3$ and a root at $t = 0$, all of multiplicity 2, while remaining non-negative on $\mathbb{S}$. Combining Parts (3) and (2) of II.13 we conclude that

$$\psi_2 = \frac{2\pi}{3} \approx 2.094395103.$$

Suppose that $k = 3$. The only possible set of multiplicities in Part (2) of II.13 is $m_a = 2$ and $m_b = 4$. The polynomial $f(t) = 1 - \cos 5t$ has roots at $t = 0, \pm 2\pi/5$, and $t = \pm 4\pi/5$, all of multiplicity 2, while remaining non-negative on $\mathbb{S}$. Applying to $f(t)$ the deformation of II.10 with $\lambda = 1/\cos(\pi/5)$ results in the polynomial $f_\lambda(t)$ that has a root of multiplicity 4 at $t = \pi$, roots of multiplicity 2 at the points $\pm\alpha$

such that

$$\cos\alpha = \frac{\cos(2\pi/5)}{\cos(\pi/5)} = \frac{3-\sqrt{5}}{2},$$

and no other roots (See Figure 2.3 for the roots of $f(t)$ and $f_\lambda(\mathrm{t})$). Hence $f_\lambda(t)$ does not change its sign on $\mathbb{S}$. Scaling f_λ, if necessary, to make the constant term 1, we ensure that $f_\lambda(t)$ is non-negative on $\mathbb{S}$. It follows by II.13 that

$$\psi_3 = \pi - \alpha = \pi - \arccos\frac{3-\sqrt{5}}{2} \approx 1.962719003.$$

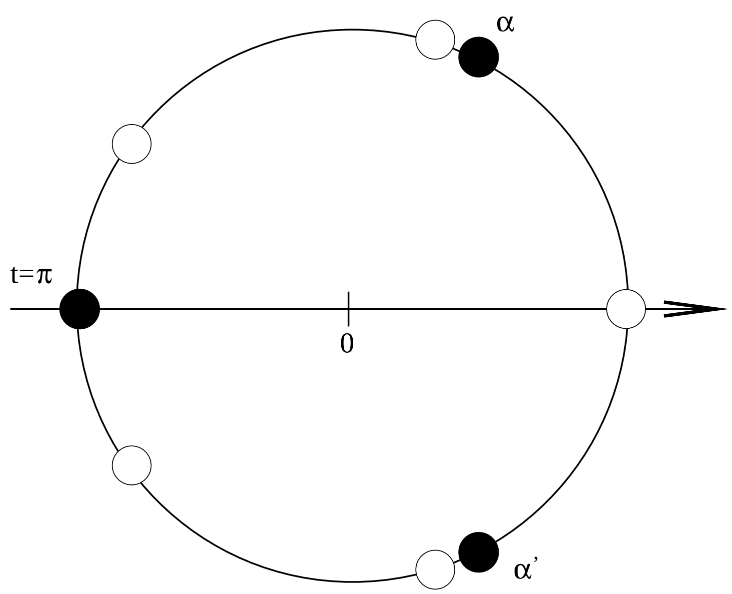


Figure 2.3: The roots of $f(t)$ (white dots) and the roots of $f_\lambda(t)$ (black dots) for $k = 3$.

Suppose that $k = 4$. There are two possibilities for multiplicities m_a and m_b in Part (2) of II.13. We have either $m_a = 2$ and $m_b = 6$ or $m_a = m_b = 4$. It turns out that the arc satisfying the latter conditions is shorter. As follows from II.24 below, we have $\psi_4 = 2\alpha$, where $\alpha > 0$ is the smallest positive root of the equation

$$\cos\alpha + 1 - \frac{1}{2}\tan^2\alpha + \frac{3}{8}\tan^4\alpha - \frac{5}{16}\tan^6\alpha = 0.$$

Computations show that

$$\begin{aligned}\psi_4 =&\pi - \arccos\left(-\frac{1}{384}\left(4768281 + 2688000\sqrt{15}\right)^{1/3}\right.\\&\left.\qquad + \frac{14693}{128\left(4768281 + 2688000\sqrt{15}\right)^{1/3}} + \frac{61}{128}\right)\\ \approx&1.870658533.\end{aligned}$$

In this case, the raked trigonometric polynomial f of degree 7 that has roots of multiplicity 4 at $t = \pm\psi_4/2$, also has a root of multiplicity 2 at $t = \pi$.

In general, our computations suggest that in Part (2) of II.13 one should always choose $m_a = m_b = k$ if k is even and $m_a = k + 1$ and $m_b = k - 1$ if k is odd, but I have been unable to prove that.

To prove II.13, we need some technical results on convergence of trigonometric polynomials.

All raked trigonometric polynomials (2.1) of degree at most $2k - 1$ form a real $(2k + 1)$-dimensional vector space, which we make into a normed space by letting

$$\|f\| = \max_{t \in \mathbb{S}} |f(t)|$$

for a trigonometric polynomial f. For a complex polynomial p of degree at most $4k - 2$ we define

$$\|p\| = \max_{z:\ |z|=1} |p(z)|.$$

We note that

$$\|\mathcal{P}(f)\| = \|f\|$$

for any trigonometric polynomial f. We define the convergence of trigonometric and complex polynomials with respect to the norm $\| \cdot \|$.

Lemma II.15. *Fix a positive integer m. For a positive integer j, let $A_j \subset \mathbb{S}$ be a non-empty closed set and let $f_j(t)$ be a trigonometric polynomial of degree at most $2k-1$ that has at least m roots, counting multiplicities, in A_j. Suppose that $A_{j+1} \subset A_j$ for all j, and let*

$$B = \bigcap_{j=1}^{\infty} A_j.$$

Suppose further that for some trigonometric polynomial f we have

$$f = \lim_{j \longrightarrow +\infty} f_j.$$

Then f has at least m roots, counting multiplicities, in B.

Suppose, in addition, that $f \not\equiv 0$, $m = 2k$, B lies in an open semicircle, and that for every j the multiplicities of all roots of f_j in A_j are even. Then the multiplicities of all roots of f in B are even.

Proof. Let $p_j = \mathcal{P}(f_j)$. By II.7, p_j is a complex polynomial that can be written as

$$p_j(z) = (z - z_{1j}) \cdots (z - z_{mj})\, q_j(z), \tag{2.12}$$

where $q_j(z)$ is a complex polynomial of degree at most $4k - 2 - m$ and $z_{1j}, \ldots, z_{mj}$ are not necessarily distinct complex numbers of modulus 1 whose arguments lie in A_j. In addition,

$$\lim_{j \longrightarrow +\infty} p_j = p,$$

where $p(z) = \mathcal{P}(f)$. We infer from (2.12) that the numbers

$$\max_{z:\ |z| = \frac{1}{2}} |q_j(z)|$$

are uniformly bounded from above, and since all norms on the finite-dimensional space of complex polynomials of degree at most $4k-2$ are equivalent, it follows that the norms $\|q_j\|$ are uniformly bounded from above. Hence we can find a subsequence $\{j_n\}$ such that

$$\lim_{n \longrightarrow +\infty} q_{j_n} = q$$

for some complex polynomial q and

$$\lim_{n \longrightarrow +\infty} z_{ij_n} = z_i^*, \quad \text{where} \quad z_i^* \in B \quad \text{for} \quad i = 1, \ldots, m.$$

Then, necessarily

$$p(z) = (z - z_1^*) \ldots (z - z_m^*)\, q(z).$$

Hence by II.7, the raked trigonometric polynomial $f(t)$ has at least m roots in B, counting multiplicities. If $m = 2k$ and $p \not\equiv 0$, Part (3) of II.6 implies that $z_1^*, \ldots, z_m^*$ are the only roots of $p(z)$ in B. The result follows. $\square$

The following lemma plays the crucial role in our proof of II.13

Lemma II.16. *Let $\Gamma \subset \mathbb{S}$ be an open arc with the endpoints a and b and let $\overline{\Gamma}$ be its closure. Let $t_2, \ldots, t_n \in \mathbb{S} \setminus \overline{\Gamma}$ be distinct points such that the set $\overline{\Gamma} \cup \{t_2, \ldots, t_n\}$ lies in an open semicircle, and let $m_1, \ldots, m_n$ be positive even integers such that*

$$\sum_{i=1}^{n} m_i = 2k.$$

For $s \in \overline{\Gamma}$, let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root of multiplicity m_i at t_i for $i = 2, \ldots, n$ and a root of multiplicity m_1 at $t = s$. If both $f_a(t)$ and $f_b(t)$ are non-negative on $\mathbb{S}$, then for every $s \in \Gamma$, the trigonometric polynomial $f_s(t)$ is positive on $\mathbb{S} \setminus \{s, t_2, \ldots, t_n\}$.

Proof. Let us consider

$$g_s(t) = \frac{\partial}{\partial s} f_s(t)$$

as in II.11. By II.12, for all $s \in \Gamma$, the point t_i is a root of $g_s(t)$ of multiplicity at least m_i for $i = 2, \ldots, n$ and s is a root of $g_s(t)$ of multiplicity at least $m_1 - 1$. Let $\mathbb{S}_+$ be an open semicircle containing $\overline{\Gamma}$ and the points $t_2, \ldots, t_n$.

Seeking a contradiction, let us assume that $f_{t_1}(t^*) = 0$ for some $t_1 \in \Gamma$ and some $t^* \in \mathbb{S} \setminus \{t_1, t_2, \ldots, t_n\}$. By Part (4) of II.6, $t^* \in \mathbb{S}_+ + \pi$. We have $f_a(t^*) \geq 0$ and $f_b(t^*) \geq 0$. Therefore, the function

$$s \longmapsto f_s(t^*)$$

attains a local minimum in Γ at some point s^*. Then

$$g_{s^*}(t^*) = 0 \quad \text{and} \quad f_{s^*}(t^*) \leq 0.$$

Since $f_{s^*}(t)$ has a constant term of 1, we obtain

$$f_{s^*}(t) + f_{s^*}(t + \pi) = 2 \quad \text{for all} \quad t \in \mathbb{S},$$

and hence

$$t^* + \pi \neq s^*, t_2, \ldots, t_n.$$

Since the constant term of $g_{s^*}(t)$ is 0, Part (2) of II.6 implies that $g_{s^*}(t) \equiv 0$. This however contradicts II.11.

Hence for every $s \in \Gamma$ the trigonometric polynomial $f_s(t)$ has no roots other than $s, t_2, \ldots, t_n$. By Remark II.5(2), we have $f_s(t) > 0$ for all $t \in \mathbb{S} \setminus \{s, t_2, \ldots, t_n\}$. $\square$

Proof of II.13.

To prove Part (1), let us choose a point $t^* \in \mathbb{S}$ and let us assume, seeking a contradiction, that there is a nested sequence of open arcs

$$\Gamma_1 \supset \Gamma_2 \supset \ldots \supset \Gamma_i \supset \ldots \tag{2.13}$$

such that

$$\bigcap_{j=1}^{\infty} \Gamma_j = \{t^*\},$$

and such that for every j there is a raked trigonometric polynomial $f_j(t)$ of degree $2k-1$, with constant term 1, with $2k$ roots, counting multiplicities, in Γ_j and a root somewhere else on the circle. By Part (4) of II.6, that additional root must lie in $\Gamma_j + \pi$. Let $h_j(t)$ be the scaling of f_j to a trigonometric polynomial of norm 1. Then there is a subsequence of the sequence $h_j(t)$ converging to a raked trigonometric polynomial h. In particular, $\|h\| = 1$, and hence $h(t) \not\equiv 0$. It follows from II.15 that t^* is a root of h of multiplicity at least $2k$ and that $t^* + \pi$ is a root of h. Since both t^* and $t^* + \pi$ are roots of $h(t)$, we obtain that $h(t)$ has a zero constant term and that $t^* + \pi$ is, in fact, a root of $h(t)$ of multiplicity at least $2k$. Hence Part (1) of II.6 implies that $h(t) \equiv 0$, which is a contradiction.

By Remark II.5(2), a trigonometric polynomial with constant term 1 that does not change its sign on $\mathbb{S}$ is non-negative on $\mathbb{S}$. Finally, the example of polynomial $1 - \cos(2k-1)t$ shows that the length of an arc Γ in Part (1) is less than π.

To prove Part (2), we construct a nested sequence of open arcs (2.13) such that

$$\bigcap_{j=1}^{\infty} \Gamma_j = \overline{\Gamma},$$

where $\overline{\Gamma}$ is the closure of Γ. By our assumption, for every j there is a raked trigonometric polynomial $f_j(t)$ of degree at most $2k-1$ that has $2k$ roots counting multiplicity in Γ_j and a root elsewhere, necessarily in $\Gamma_j + \pi$. As in the proof of Part (1), let us scale $f_j(t)$ to a trigonometric polynomial $h_j(t)$ such that $\|h_j\| = 1$ and construct the limit trigonometric polynomial h. Then $h \not\equiv 0$, and by II.15, h has roots $t_1, \ldots, t_n \in \overline{\Gamma}$ of even multiplicities $m_1, \ldots, m_n$ such that $m_1 + \ldots + m_n = 2k$, and a root $t^* \in \overline{\Gamma} + \pi$. By Part (2) of II.6, h has a non-zero constant term.

We rescale h to a raked trigonometric polynomial $f(t)$ with constant term 1. Then each t_i is a root of $f(t)$ of multiplicity m_i and $f(t^*) = 0$.

It is easy to see that the endpoints a and b of Γ are roots of $f(t)$. Our goal is to show that for every $i = 1, \ldots, n$ we have either $t_i = a$ or $t_i = b$, that is, that there are no roots inside Γ.

Seeking a contradiction, let us assume that $t_1 \in \Gamma$. We choose an open arc $\tilde{\Gamma} \subset \Gamma$ containing t_1 and such that $t_i \notin \tilde{\Gamma}$ for $i = 2, \ldots, n$. For $s \in \tilde{\Gamma}$, let $f_s(t)$ be the raked trigonometric polynomial of II.11 that has a root at $t = s$ of multiplicity m_1 and a root at t_i of multiplicity m_i for $i = 2, \ldots, n$. In particular,

$$f_s = f \quad \text{if} \quad s = t_1.$$

We observe that

$$f_s(t) \geq 0 \quad \text{for all} \quad t \in \mathbb{S} \quad \text{and all} \quad s \in \tilde{\Gamma}.$$

Indeed, if $f_s(t_0) < 0$ for some $t_0 \in \mathbb{S}$ then a trigonometric polynomial $\hat{f}$ with constant term 1 that has a root of multiplicity m_1 at s and roots of multiplicity m_i at some points $\hat{t}_i \in \Gamma$ sufficiently close to t_i will also satisfy $\hat{f}(t_0) < 0$, which contradicts the definition of Γ. Hence $f_x(t) \geq 0$ for all $t \in \mathbb{S}$ and $f_y(t) \geq 0$ for all $t \in \mathbb{S}$. II.16 then implies that $f(t^*) = f_{t_1}(t^*) > 0$, which is a contradiction.

To prove Part (3), we note that for any $b \in \Gamma$ sufficiently close to a, by Part (1) of the theorem we have $f_b(t) > 0$ for all $t \in \mathbb{S} \setminus \{a, b\}$. We can choose such a point b so that x lies between b and y and then $f_x(t) > 0$ for all $t \in \mathbb{S} \setminus \{a, x\}$ by II.16. Assume now that $f_z(t) \geq 0$ for all $t \in \mathbb{S}$. Then by II.16 we have $f_y(t) > 0$ for all $t \in \mathbb{S} \setminus \{a, y\}$, which is a contradiction. $\square$

Lemma II.17. *Let ψ_k be the maximum length of an open arc Γ in II.13 and let ϕ_k be the maximum length of an open arc Γ in II.1. Then $\psi_k = \phi_k$.*

Proof. From Remark II.5(1) it follows immediately that $\phi_k \geq \psi_k$.

Let $\Gamma \subset \mathbb{S}$ be an open arc of length ψ_k with the endpoints a and b and let $\tilde{\Gamma} \supset \Gamma$ be a closed arc with the endpoints a and c strictly containing Γ and lying in an open semicircle. By Part (2) of II.13 there exist positive even integers m_a and m_b such that $m_a + m_b = 2k$ and a raked trigonometric polynomial $f(t)$ of degree $2k-1$ and with constant term 1 that has a root at $t = a$ of multiplicity m_a, a root at $t = b$ of multiplicity m_b, and some other root $t^* \in \Gamma + \pi$. For $s \in \tilde{\Gamma}$ let $f_s(t)$ be the unique raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root of multiplicity m_a at $t = a$ and a root of multiplicity m_b at $t = s$. Seeking a contradiction, let us assume that for any distinct $t_1, \ldots, t_k \in \tilde{\Gamma}$, the unique raked trigonometric polynomial of degree $2k-1$ and with constant term 1 that has roots of multiplicity two at $t_1, \ldots, t_k$ remains non-negative on the entire circle $\mathbb{S}$. As in the proof of II.13, using the limit argument, we conclude that $f_c(t) \geq 0$ for all $t \in \mathbb{S}$. This, however, contradicts Part (3) of Theorem 5.1 since f_b has a root in $\Gamma + \pi$.

In view of Remark II.5, it follows that for some distinct $t_1, \ldots, t_k \in \tilde{\Gamma}$, the convex hull

$$\operatorname{conv}\Big(U(t_1), \ldots, U(t_k)\Big)$$

is not a face of $\mathcal{B}_k$. Hence $\phi_k \leq \psi_k$. □

2.5 Neighborliness of the symmetric moment curve

In this section we prove Theorems II.1 and II.3. Our proofs are based on the following main result.

Theorem II.18. *For every positive integer k there exists a number $\pi > \phi_k > \pi/2$ such that if $\Gamma \subset \mathbb{S}$ is an open arc of length ϕ_k, $t_1, \ldots, t_n \in \Gamma$ are distinct points, and*

$m_1, \ldots, m_n$ are positive even integers such that

$$\sum_{i=1}^{n} m_i = 2k,$$

then the unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has a root of multiplicity m_i at t_i for $i = 1, \ldots, n$ is positive everywhere else on the circle $\mathbb{S}$.

The proof is based on II.13 and the following lemma.

Lemma II.19. *Let $f(t)$ be the raked trigonometric polynomial of degree $2k-1$ with constant term 1 that has a root of multiplicity $2m$ at $t = 0$ and a root of multiplicity $2n$ at $t = \pi/2$, where m and n are positive integers such that $m + n = k$. Then $f(t)$ has no other roots in the circle $\mathbb{S}$.*

Proof. We have

$$f(t) = 1 + \sum_{j=1}^{k} a_j \cos(2j-1)t + \sum_{j=1}^{k} b_j \sin(2j-1)t$$

for some real a_j and b_j. In addition,

$$f'(0) = \ldots = f^{(2m-1)}(0) = 0 \quad \text{and} \quad f'(\pi/2) = \ldots = f^{(2n-1)}(\pi/2) = 0. \tag{2.14}$$

Let

$$a(t) = \sum_{j=1}^{k} a_j \cos(2j-1)t \quad \text{and} \quad b(t) = \sum_{j=1}^{k} b_j \sin(2j-1)t,$$

so that

$$f(t) = 1 + a(t) + b(t) \quad \text{and} \quad f'(t) = a'(t) + b'(t). \tag{2.15}$$

Observe that

$$\frac{d^{2r-1}}{dt^{2r-1}} a(t)\Big|_{t=0} = 0 \quad \text{and} \quad \frac{d^{2r}}{dt^{2r}} b(t)\Big|_{t=0} = 0 \tag{2.16}$$

for any positive integer r, and

$$\frac{d^{2r}}{dt^{2r}}a(t)\Big|_{t=\pi/2} = 0 \quad \text{and} \quad \frac{d^{2r-1}}{dt^{2r-1}}b(t)\Big|_{t=\pi/2} = 0 \tag{2.17}$$

for any positive integer r.

Combining (2.14) – (2.17) we conclude that $t = 0$ is a root of $a'(t)$ of multiplicity at least $2m - 1$ and a root of $b'(t)$ of multiplicity at least $2m$. Similarly, $t = \pi/2$ is a root of $a'(t)$ of multiplicity at least $2n$ and a root of $b'(t)$ of multiplicity at least $2n - 1$. Since $f(0) = 0$, we obtain that $a(t) \not\equiv 0$, and hence $a'(t) \not\equiv 0$. Also since $f(\pi/2) = 0$, it follows that $b(t) \not\equiv 0$, and hence $b'(t) \not\equiv 0$. By Part (2) of II.6, the trigonometric polynomial $a'(t)$ has a root of multiplicity $2m - 1$ at $t = 0$, a root of multiplicity $2n$ at $t = \pi/2$ and no other roots in the circle $\mathbb{S}$, while the trigonometric polynomial $b'(t)$ has a root of multiplicity $2m$ at $t = 0$, a root of multiplicity $2n - 1$ at $t = \pi/2$ and no other roots in the circle.

We conclude that the functions $a(t)$ and $b(t)$ are monotone on the interval $0 < t < \pi/2$. Since $a(0) = -1$ and $a(\pi/2) = 0$, we infer that $a(t)$ is monotone increasing for $0 < t < \pi/2$, and hence $a(t) < 0$ for all $0 < t < \pi/2$. Since $b(0) = 0$ and $b(\pi/2) = -1$, we obtain that $b(t)$ is monotone decreasing for $0 < t < \pi/2$, and therefore $b(t) < 0$ for all $0 < t < \pi/2$. As

$$a(t + \pi) = -a(t) \quad \text{and} \quad b(t + \pi) = -b(t),$$

it follows that $a(t) > 0$ for $\pi < t < 3\pi/2$ and $b(t) > 0$ for $\pi < t < 3\pi/2$. Therefore,

$$f(t) \geq 1 \quad \text{for all} \quad \pi \leq t \leq 3\pi/2. \tag{2.18}$$

The latter equation yields the result, as by Part (4) of II.6, a root t^* of $f(t)$ distinct from 0 and $\pi/2$, if exists, must satisfy $\pi \leq t^* \leq 3\pi/2$. $\square$

Proof of II.18. By Part (1) of II.13, there exists a number $\phi_k > 0$ such that if a raked trigonometric polynomial $f(t)$ of degree $2k - 1$ and with a constant term 1 has roots at $t = 0$ and $t = \phi_k$ with positive even multiplicities summing up to $2k$, then $f(t)$ is positive everywhere else. It follows from II.19 and II.16 that the same remains true for all $0 < \phi_k \leq \pi/2$. Using the shift $f(t) \longmapsto f(t+a)$ of raked trigonometric polynomials, we conclude that for every arc $\Gamma \subset \mathbb{S}$ of length not exceeding $\pi/2$, a raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has roots of even multiplicities summing up to $2k$ at the endpoints of Γ remains positive everywhere else in $\mathbb{S}$. The proof now follows from Part (2) of II.13. □

Proof of II.1 and II.3.

II.1 follows from II.18 and Remark II.5, while II.3 follows from Remark II.5 and II.19. □

2.6 The limit of neighborliness

In this section, we prove Theorem II.2. Our goal is to construct a raked trigonometric polynomial $f_k(t)$ of degree $2k-1$ such that $f_k(t)$ has a root of multiplicity $2k-2$ at $t=0$, roots of multiplicity 2 each at $t = \pm\beta_k$ for some $\pi/2 < \beta_k < \pi$, and such that $f_k(t) \geq 0$ for all $t \in \mathbb{S}$. It then follows from II.13 and II.17 that $\phi_k \leq \beta_k$, and establishing that $\beta_k \longrightarrow \pi/2$ as k grows, we complete the proof.

Lemma II.20. *The function*

$$f(t) = \sin^{2k-1} t$$

is a raked trigonometric polynomial of degree $2k-1$.

Proof. We have

$$\sin^{2k-1} t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^{2k-1} = \frac{1}{(-4)^{k-1}} \frac{1}{2i} \sum_{j=0}^{2k-1} \binom{2k-1}{j} (-1)^j e^{i(2k-2j-1)t}$$
$$= \frac{1}{(-4)^{k-1}} \sum_{j=0}^{k-1} \binom{2k-1}{j} \left(\frac{(-1)^j e^{i(2k-2j-1)t} + (-1)^{2k-1-j} e^{i(2j-2k+1)t}}{2i}\right)$$
$$= \frac{1}{(-4)^{k-1}} \sum_{j=0}^{k-1} \binom{2k-1}{j} (-1)^j \sin(2k-2j-1)t.$$

□

Lemma II.21. *For $k \geq 1$ let*

$$h_k(t) = \int_0^t \sin^{2k-1}(\tau)\ d\tau.$$

Then $h_k(t)$ is a raked trigonometric polynomial of degree $2k-1$ and $t = 0$ is a root of $h_k(t)$ of multiplicity $2k$. Moreover,

$$h_k(t) = \frac{(2k-2)!!}{(2k-1)!!} \left(1 - (\cos t) \sum_{j=0}^{k-1} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right),$$

where we agree that $0!! = (-1)!! = 1$.

Proof. From II.20, $h_k(t)$ is a raked trigonometric polynomial of degree $2k-1$. Moreover, $h_k(0) = 0$ and $h'_k(t) = \sin^{2k-1} t$, from which it follows that $t = 0$ is a root of $h_k(t)$ of multiplicity $2k$. Since

$$h_1(t) = \int_0^t \sin\tau\ d\tau = 1 - \cos t,$$

and since for $n > 1$,

$$\int_0^t \sin^n \tau\ d\tau = -\frac{1}{n}\left(\sin^{n-1} t\right)(\cos t) + \frac{n-1}{n} \int_0^t \sin^{n-2} \tau\ d\tau,$$

we obtain by induction that

$$\int_0^t \sin^{2k-1} \tau\ d\tau = \frac{(2k-2)!!}{(2k-1)!!} \left(1 - \cos t \sum_{j=0}^{k-1} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t\right),$$

as claimed. □

Lemma II.22. *Let $h_k(t)$ be the trigonometric polynomial defined in II.21 and let*

$$F_k(t) = \sin^2(t) h_{k-1}(t) - h_k(t).$$

Then there exists a unique

$$\frac{\pi}{2} \;<\; \beta_k \;<\; \pi$$

such that

$$F_k\left(\beta_k\right) = 0.$$

In addition,

$$\lim_{k \longrightarrow +\infty} \beta_k = \frac{\pi}{2}.$$

Proof. From II.21, we deduce

$$F_k\left(\frac{\pi}{2}\right) = h_{k-1}\left(\frac{\pi}{2}\right) - h_k\left(\frac{\pi}{2}\right) = \frac{(2k-4)!!}{(2k-3)!!} - \frac{(2k-2)!!}{(2k-1)!!} \;>\; 0 \quad \text{and}$$
$$F_k(\pi) = -h_k(\pi) = -2\frac{(2k-2)!!}{(2k-1)!!} \;<\; 0.$$

Moreover,

$$F_k'(t) = 2(\sin t)(\cos t) h_{k-1}(t) - h_k'(t) + \sin^2(t) h_{k-1}'(t) = 2(\sin t)(\cos t) h_{k-1}(t).$$

In particular, $F_k'(t) < 0$ for $\pi/2 < t < \pi$, and hence $F_k(t)$ is decreasing on the interval $\pi/2 < t < \pi$. Since $F_k(\pi/2) > 0$ and $F_k(\pi) < 0$, there is a unique $\pi/2 < \beta_k < \pi$ such that $F_k\left(\beta_k\right) = 0$.

To find the limit behavior of β_k, we use the expansion

$$(1-x)^{-1/2} = \sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} x^j \quad \text{for real} \quad -1 < x < 1.$$

Substituting $x = \sin^2 t$ we obtain

$$\sum_{j=0}^{\infty} \frac{(2j-1)!!}{(2j)!!} \sin^{2j} t = -\frac{1}{\cos t} \quad \text{provided} \quad \pi/2 < t < \pi.$$

Hence from II.21, for $\pi/2 < t < \pi$ we have

$$\begin{aligned} h_k(t) =& \frac{(2k-2)!!}{(2k-1)!!}\left(1-(\cos t)\left(\sum_{j=0}^{\infty}\frac{(2j-1)!!}{(2j)!!}\sin^{2j}t - \sum_{j=k}^{\infty}\frac{(2j-1)!!}{(2j)!!}\sin^{2j}t\right)\right) \\ =& \frac{(2k-2)!!}{(2k-1)!!}\left(2+(\cos t)\sum_{j=k}^{\infty}\frac{(2j-1)!!}{(2j)!!}\sin^{2j}t\right) \end{aligned}$$

and

$$\frac{(2k-3)!!}{(2k-4)!!}F_k(t) =$$
$$2\sin^2 t - 2\frac{2k-2}{2k-1} + (\cos t)\sum_{j=k}^{\infty}\left(\frac{(2j-3)!!}{(2j-2)!!} - \frac{2k-2}{2k-1}\frac{(2j-1)!!}{(2j)!!}\right)\sin^{2j}t.$$

It follows that $F_k(t) < 0$ for every $\pi/2 < t < \pi$ such that $\sin^2 t \leq (2k-2)/(2k-1)$. Since $F_k(t)$ is decreasing for $\pi/2 < t < \pi$, we conclude that

$$\sin^2\beta_k > \frac{2k-2}{2k-1}, \tag{2.19}$$

and hence

$$\lim_{k\longrightarrow+\infty}\beta_k = \frac{\pi}{2},$$

as desired. □

Lemma II.23. *Let $h_k(t)$ be the trigonometric polynomial defined in II.21 and let β_k be the number defined in II.22. Let*

$$f_k(t) = \sin^2(\beta_k)\, h_{k-1}(t) - h_k(t).$$

Then $f_k(t)$ is a raked trigonometric polynomial of degree $2k-1$ such that $t=0$ is a root of $f_k(t)$ of multiplicity $2k-2$, $t=\pm\beta_k$ are the roots of multiplicity 2 each and $f_k(t)\geq 0$ for all $t\in\mathbb{S}$.

Proof. It follows by II.21 that $f_k(t)$ is a trigonometric polynomial of degree $2k-1$ and that $t=0$ is a root of $f_k(t)$ of multiplicity at least $2k-2$. From the definition of β_k in II.23, we conclude that $t = \beta_k$ is a root of $f_k(t)$. Moreover, since

$$f_k'(t) = \sin^{2k-1} t - \left(\sin^2 \beta_k\right) \sin^{2k-3} t,$$

we have $f'(\beta_k) = 0$, so the multiplicity of the root at $t = \beta_k$ is at least 2. By Part (3) of II.6, the multiplicities of the roots at $t = 0$ and $t = \beta_k$ are $2k-2$ and 2 respectively and there are no other roots of $f_k(t)$ in the open arc $0 < t < \pi$. Also, by II.21 and (2.19), we have

$$f_k(\pi) = 2\sin^2(\beta_k)\frac{(2k-4)!!}{(2k-3)!!} - 2\frac{(2k-2)!!}{(2k-1)!!} > 0.$$

Since $f_k(-t) = f_k(t)$, we conclude that $t = -\beta_k$ is a root of $f_k(t) \geq 0$ of multiplicity 2 and that $f_k(t) > 0$ for all $t \neq 0, \pm\beta_k$. $\square$

Proof of II.2. Let ψ_k be the maximum length of an open arc satisfying Part (1) of II.13. It follows from II.23 that $\psi_k \leq \beta_k$, and hence from II.17 that $\phi_k \leq \beta_k$. II.22 then yields the proof. $\square$

All available computational evidence suggests that for even k the smallest length of the arc in Part (2) of II.13 is achieved when the multiplicites m_a and m_b are equal: $m_a = m_b = k$. The following results provides an explicit equation for the length of such an arc.

Proposition II.24. *Suppose that k is even. Let $\alpha_k > 0$ be the smallest number such that the necessarily unique raked trigonometric polynomial $f(t)$ of degree $2k-1$ with constant term 1 that has roots at $t = \alpha_k$ and $t = -\alpha_k$ of multiplicity k each also has a root t^* elsewhere in $\mathbb{S}$. Then $t^* = \pi$ and α_k is the smallest positive root of the equation $F(\alpha) = 0$ where*

$$F(\alpha) = \cos\alpha + 1 + \sum_{j=1}^{k-1}(-1)^j \frac{(2j-1)!!}{(2j)!!}\tan^{2j}\alpha. \tag{2.20}$$

Proof of II.24. We note that the raked trigonometric polynomial $\tilde{f}(t) = f(-t)$ also has a root of multiplicity k at $t = \alpha_k$ and a root of multiplicity k at $t = -\alpha_k$. By II.9, we must have $\tilde{f}(t) = f(t)$, and hence

$$f(t) = 1 + \sum_{j=1}^{k} a_j \cos(2j-1)t \tag{2.21}$$

for some real $a_1, \ldots, a_k$. Then the raked trigonometric polynomial $f'(t)$ has roots at $t = \alpha_k$, $-\alpha_k$, $\alpha_k + \pi$, and $-\alpha_k + \pi$ of multiplicity $k-1$ each as well as roots at $t = 0$ and $t = \pi$. By Part (1) of II.6, $f'(t)$ has no other roots and $a_k \neq 0$. By Part (4) of II.6, the root t^* must lie in an open arc Γ with the endpoints $\alpha_k + \pi$ and $-\alpha_k + \pi$. From the definition of α_k, it follows that $f(t) \geq 0$ for all $t \in \Gamma$, and hence t^* is a local minimum of $f(t)$. Thus $f'(t^*) = 0$, and so $t^* = \pi$. Moreover, $t^* = \pi$ is a root of $f(t)$ of multiplicity 2.

We choose

$$\lambda = \frac{1}{\cos \alpha_k}$$

in II.10 and consider the λ-deformation $f_\lambda(t)$ of $f(t)$. Let

$$p = \mathcal{P}(f) \quad \text{and} \quad p_\lambda = \mathcal{P}(f_\lambda).$$

Since α_k and $-\alpha_k$ are roots of $f(t)$ of multiplicity k each, the complex numbers $e^{i\alpha_k}$ and $e^{-i\alpha_k}$ are roots of p of multiplicity k each. Then $z = 1$ is a root of $p_\lambda(z)$ of multiplicity $2k$, and hence $t = 0$ is a root of f_λ of multiplicity $2k$.

As $t = \pi$ is a root of f of multiplicity 2, it follows that $z = -1$ is a root of $p(z)$ of multiplicity 2. Thus,

$$\frac{-1 + \sin \alpha_k}{\cos \alpha_k} \quad \text{and} \quad \frac{-1 - \sin \alpha_k}{\cos \alpha_k} \tag{2.22}$$

are roots of $p_\lambda(z)$.

Since $t=0$ is a root of multiplicity $2k$ of $f_\lambda(t)$ we must have

$$f_\lambda(t)=\beta h_k(t) \quad \text{for some real} \quad \beta\neq 0,$$

where $h_k(t)$ is the trigonometric polynomial of II.24. Therefore,

$$p_\lambda(z)=\beta z^{2k-1}\left(1-\left(\frac{z+z^{-1}}{2}\right)\left(1+\sum_{j=1}^{k-1}\frac{(2j-1)!!}{(2j)!!}\left(\frac{z-z^{-1}}{2i}\right)^{2j}\right)\right). \tag{2.23}$$

Substituting either of the roots of (2.22) in (2.23), we obtain the desired equation

$$F\left(\alpha_k\right)=0.$$

Suppose now that some number $0<\alpha<\pi/2$ also satisfies the equation $F(\alpha)=0$. Then

$$\frac{-1+\sin\alpha}{\cos\alpha} \quad \text{and} \quad \frac{-1-\sin\alpha}{\cos\alpha} \tag{2.24}$$

are roots of polynomial $q=\mathcal{P}\left(h_k\right)$, where $h_k(t)$ is the trigonometric polynomial of II.21. Let us choose $\lambda=\cos\alpha$ and let $g_\lambda(t)$ be the λ-deformation of $h_k(t)$ as in II.10. Let $q_\lambda=\mathcal{P}\left(g_\lambda\right)$. Since the numbers introduced in (2.24) are roots of q, we conclude that $z=-1$ is a root of multiplicity 2 of $q_\lambda(z)$, and hence $t=\pi$ is a root of multiplicity 2 of $g_\lambda(t)$. Similarly, since $t=0$ is a root of multiplicity $2k$ of $h_k(t)$, we conclude that $z=1$ is a root of multiplicity $2k$ of $q(z)$, and hence the numbers $e^{i\alpha}$ and $e^{-i\alpha}$ are roots of q_λ, each of multiplicity k. Therefore, $t=\alpha$ and $t=-\alpha$ are roots of $g_\lambda(t)$, each of multiplicity k. It then follows, by minimality of α_k, that $\alpha\geq\alpha_k$, which completes the proof. □

2.7 Neighborliness for generalized moment curves

In this section, we consider local neighborliness for generalized moment curves in $\mathbb{R}^{2k}$:

$$\begin{aligned} M(t)&=M(p_1,\cdots,p_k)(t)\\ &=\{(\cos 2\pi p_1 t,\ \sin 2\pi p_1 t,\ \cdots,\cos 2\pi p_k t,\ \sin 2\pi p_k t):t\in\mathbb{R}\} \end{aligned} \tag{2.25}$$

where p_i's are positive integers. We assume that p_i's are increasing and have the greatest common divisor 1. A generalized moment curve $M(t)$ in $\mathbb{R}^{2k}$ has *local neighborliness property* if there exists a number

$$0 < \ \phi \ < \ \pi$$

such that for an arbitrary open arc $\Gamma \subset \mathbb{S}$ of length ϕ and arbitrary distinct $n \leq k$ points $t_1, \ldots, t_n \in \Gamma$, the set

$$\operatorname{conv}\Big(M\left(t_1\right), \ldots, M\left(t_n\right)\Big)$$

is a face of the convex hull of the curve $M(t)$.

Smilansky [24] studied the generalized moment curves for $k = 2$ and described the facial structure of the convex hull of the curves for all p_1 and p_2. From Theorem 1 in [24], one can check that the convex hull of $M(p_1, p_2)$ has local neighborliness property if and only if $p_1 = 1$.

On the other hand, it is well known that the classical moment curves $M(1, 2, \cdots, n)$ has neighborliness property for a positive integer n (see for example, [9]). For the symmetric moment curves $M(1, 3, \cdots, 2n-1)$, Theorem II.1 showed that the symmetric moment curves has local neighborliness property.

At this point, it is natural to ask when the generalized moment curves have local neighborliness property. For $k > 2$, is it still true that $M(p_1, \cdots, p_k)$ has local neighborliness property if $p_1 = 1$? In this section, we prove that the answer for the above question is no, by presenting a counterexample when $k = 5$ and $(p_1, p_2, p_3, p_4, p_5) = (1, 12, 13, 14, 15)$.

Consider the *M-trigonometric polynomial*

$$f(t) = c + \sum_{j=1}^{k} a_j \cos p_j t + \sum_{j=1}^{k} b_j \sin p_j t.$$

Note that for distinct points $t_1, \cdots, t_n \in \mathbb{S}$, the convex hull of $M(t_1), \cdots, M(t_n)$ is a face of the convex hull of the generalized moment curve $M(p_1, \cdots, p_k)(t)$ if and only if there exists M-trigonometric polynomial $f(t)$ such that $f(t) \geq 0$ for $t \in \mathbb{S}$ and equality holds if and only if t is one of t_i's.

We need the following lemma.

Theorem II.25. *Let $F(t)$ be a M-polynomial having a root at 0 with multiplicity $2k$. Then such F is unique up to constants and it is given by*

$$F(t) = 1 + \sum_{i=1}^{k} a_i \cos p_i t$$

where the (column) vector $v = (a_1, \cdots, a_k)$ satisfies $Av = (-1, 0, \cdots, 0)$ for $A = \left(p_j^{2(i-1)}\right)_{i,j=1}^{k}$.

Proof. Let $F(t)$ be a M-polynomial

$$F(t) = 1 + \sum_{i=1}^{k} a_i \cos p_i t + \sum_{i=1}^{k} b_i \sin p_i t$$

having a root at 0 with multiplicity $2k$. By simplifying

$$F^{(k)}(0) = 0 \quad \text{for } k = 0, \ldots, 2k-1,$$

we have

$$1 + \sum_{i=1}^{k} a_i = 0$$

$$\sum_{i=1}^{k} a_i p_i^{2i} = 0 \quad \text{for } i = 1, \ldots, k-1$$

$$\sum_{i=1}^{k} b_i p_i^{2i+1} = 0 \quad \text{for } i = 0, \ldots, k-1.$$

Therefore, $b_i = 0$ for all i and $Av = (-1, 0, \cdots, 0)$ where A and v are defined above. $\square$

Theorem II.26. *Assume that $M(p_1, \cdots, p_k)$ has local neighborliness property. Then for a unique M-polynomial $F(t)$ with a constant term 1 having a root at 0 with multiplicity $2k$, we have $F(t) \geq 0$ for all $t \in \mathbb{S}$.*

Proof. Since $M(p_1, \cdots, p_k)$ has local neighborliness property, for any positive integer n, there exists an arc $\Omega_n \subset \mathbb{S}$ of length at most $1/n$ containing $M(0)$, distinct points $t_{1,n}, \cdots, t_{k,n} \in \Omega_n$ and a supporting hyperplane H_n which intersects $M(t)$ at $M(t_{i,n})$ for $i = 1, \cdots, k$. The set of all affine hyperplanes intersecting the compact set $M(\mathbb{S})$ is compact in natural topology; for example, if we view the set of affine hyperplanes in $\mathbb{R}^{2k}$ as a subset of the Grassmannian of all linear hyperplanes in $\mathbb{R}^{2k+1}$. Therefore, the sequence of hyperplanes H_n has a limit hyperplane H and corresponding M-polynomial $F(t)$. By using a similar argument in the proof of Lemma II.15, one can show that $F(t)$ has a root at 0 with multiplicity at least $2k$ and therefore such $F(t)$ is unique up to constants by Theorem II.25. Since H_n are supporting hyperplanes of the curve $M(t)$, the limit $H(t)$ is also the support hyperplane of $M(t)$. Therefore, we can choose $F(t)$ to be

$$F(t) = 1 + \sum_{i=1}^{k} a_i \cos p_i t$$

such that $F(t) \geq 0$ for $t \in \mathbb{S}$. $\square$

In the appendix, equations and graphs of $F(t)$ are given for $k = 5$, $(p_1, p_2, p_3, p_4, p_5) = (1, 2, 3, 4, 5), (1, 3, 5, 7, 9), (1, 12, 13, 14, 15)$. It is shown that $F(t)$ for $(p_1, p_2, p_3, p_4, p_5) = (1, 12, 13, 14, 15)$ has at least 8 zeroes other than 0 and it turns negative at some point. Therefore, $M(1, 12, 13, 14, 15)$ does not have local neighborliness property.

CHAPTER III

Centrally symmetric polytopes with many faces

3.1 Introduction

3.1.1 Cs neighborliness

What is the maximum number of k-dimensional faces that a centrally symmetric d-dimensional polytope with N vertices can have? While the answer in the class of all polytopes is classic by now [21], very little is known in the centrally symmetric case. Here we present several constructions that significantly improve existing lower bounds on this number.

It was proved in [18] that a cs 2-neighborly d-dimensional polytope cannot have more than 2^d vertices. In Theorem III.2(1) we present a construction of a cs 2-neighborly d-polytope with about $3^{d/2} \approx (1.73)^d$ vertices.

More generally, it was verified in [4] that a cs d-dimensional polytope with N vertices cannot have more than $\left(1 - 0.5^d\right) \frac{N^2}{2}$ edges. In Theorem III.2(2), we construct a cs d-dimensional polytope with N vertices (for an arbitrarily large N) and at least $\left(1 - 3^{-\lfloor d/2-1 \rfloor}\right) \binom{N}{2} \approx \left(1 - 0.58^d\right) \frac{N^2}{2}$ edges.

For higher-dimensional faces even less is known. It follows from the results of [18] that no cs k-neighborly d-polytope can have more than $\lfloor d \cdot 2^{Cd/k} \rfloor$ vertices, where $C > 0$ is some absolute constant. At the same time, the papers [18, 23] used a randomized construction to prove existence of k-neighborly cs d-dimensional polytopes with $\lfloor d \cdot 2^{cd/k} \rfloor$ vertices for some absolute constant $c > 0$. However, for $k > 2$ no deterministic construction of a d-dimensional k-neighborly cs polytope with $2^{\Omega(d)}$ vertices is known. In Theorem III.9 and Remark III.10 we present a deterministic construction of a cs k-neighborly d-polytope with at least $2^{c_k d}$ vertices where $c_k = 3/20k^2 2^k$. We then use this result in Corollary III.11 to construct for a fixed k and arbitrarily large N and d, a cs d-polytope with N vertices that has a record number of $(k-1)$-dimensional faces. Our construction relies on the notion of k-independent families [1, 12] (see also [2]).

Through Gale duality m-dimensional subspaces of $\mathbb{R}^N$ correspond to $(N-m)$-dimensional cs polytopes with $2N$ vertices (For more details, See [18]). If the subspace is "almost Euclidean" (meaning that the ratio of the ℓ^1 and ℓ^2 norms of nonzero vectors of the subspace remains within certain bounds, see [18] for technical details), then the corresponding polytope turns out to be k-neighborly. Despite considerable efforts, see for example [17], no explicit constructions of "almost Euclidean" subspaces is known for m anywhere close to N. Our polytopes give rise to subspaces of $\mathbb{R}^N$ of codimension $O(\log N)$ and it would be interesting to find out if the resulting subspaces are indeed "almost Euclidean".

3.1.2 Antipodal points

Our results on cs polytopes provide new bounds on several problems related to strict antipodality. Let $X \subset \mathbb{R}^d$ be a set that affinely spans $\mathbb{R}^d$. A pair of points $u, v \in X$ is called *strictly antipodal* if there exist two distinct parallel hyperplanes H and H' such that $X \cap H = \{u\}$, $X \cap H' = \{v\}$, and X lies in the slab between H and H'. Denote by $A'(d)$ the maximum size of a set $X \subset \mathbb{R}^d$ having the property that every pair of points of X is strictly antipodal, by $A'_d(Y)$ the number of strictly antipodal pairs of a given set Y, and by $A'_d(n)$ the maximum size of $A'_d(Y)$ taken over all n-element subsets Y of $\mathbb{R}^d$. (Our notation follows the recent survey paper [20].)

The notion of strict antipodality was introduced in 1962 by Danzer and Grünbaum [10] who verified that $2d-1 \leq A'(d) \leq 2^d$ and conjectured that $A'(d) = 2d-1$. However, twenty years later, Erdős and Füredi [11] used a probabilistic argument to prove that $A'(d)$ is exponential in d. Their result was improved by Talata (see [8, Lemma 9.11.2]) who found an explicit construction showing that for $d \geq 3$,

$$A'(d) \geq \lfloor (\sqrt[3]{3})^d/3 \rfloor.$$

Talata also announced that $(\sqrt[3]{3})^d/3$ in the above formula can be replaced with $(\sqrt[4]{5})^d/4$. (It is worth remarking that Erdős and Füredi established existence of an *acute set* in $\mathbb{R}^d$ that has an exponential size in d. As every acute set has the property that all of its pairs of vertices are strictly antipodal, their result implied an exponential lower bound on $A'(d)$. A significant improvement of the Erdős–Füredi bound on the maximum size of an acute set in $\mathbb{R}^d$ was recently found by Harangi [16].)

Regarding the value of $A'_d(n)$, Makai and Martini [19] showed that for $d \geq 4$,

$$\left(1 - \frac{\text{const}}{(1.0044)^d}\right)\frac{n^2}{2} - O(1) \leq A'_d(n) \leq \left(1 - \frac{1}{2^d-1}\right)\frac{n^2}{2}.$$

Here we observe that an appropriately chosen half of the vertex set of a cs d-polytope with many edges has a large number of strictly antipodal pairs of points. Consequently, our construction of cs d-polytopes with many edges implies — see Theorem III.3 — that

$$A'(d) \geq 3^{\lfloor d/2-1 \rfloor} - 1 \quad \text{and} \quad A'_d(n) \geq \left(1 - \frac{1}{3^{\lfloor d/2-1 \rfloor} - 1}\right)\frac{n^2}{2} - O(n) \quad \text{for all } d \geq 4.$$

The rest of the chapter is structured as follows. In Section 2, we present our construction of a cs 2-neighborly d-polytope with many vertices as well as that of a cs d-polytope with arbitrarily many vertices and a record number of edges. Section 3 is devoted to applications of these results to problems on strict antipodality. In Section 4, we construct centrally symmetric polytopes with many faces of given dimension which generalizes results for the number of edges in Section 2. Finally, in Section 5 we provide a deterministic construction of a cs k-neighborly d-polytope and of a cs d-polytope with arbitrarily many vertices and a record number of $(k-1)$-faces.

We also frequently use the following well-known fact about polytopes: if $T : \mathbb{R}^{d'} \longrightarrow \mathbb{R}^{d''}$ is a linear transformation and $P \subset \mathbb{R}^{d'}$ is a polytope, then $Q = T(P)$ is also a polytope and for every face F of Q the inverse image of F,

$$T^{-1}(F) = \{x \in P : \quad T(x) \in F\},$$

is a face of P.

3.2 Centrally symmetric polytopes with many edges

In this section we provide a construction of a cs 2-neighborly polytope of dimension d and with about $3^{d/2} \approx (1.73)^d$ vertices as well a construction of a cs d-polytope with N vertices (for an arbitrarily large N) that has about $\left(1 - 3^{-d/2}\right)\binom{N}{2} \approx \left(1 - 0.58^d\right)\binom{N}{2}$ edges. Our trick allows us to halve the dimension of the polytope from [6] while keeping the number of vertices almost the same as before.

For an integer $m \geq 1$, consider the curve $\Phi_m : \mathbb{S} \longrightarrow \mathbb{R}^{2(m+1)}$ where

$$\Phi_m(t) := \big(\cos t,\, \sin t,\, \cos 3t,\, \sin 3t, \ldots, \cos(3^m t),\, \sin(3^m t)\big). \tag{3.1}$$

Note that $\Phi_1 = U_2$. The key to our construction is the following observation.

Lemma III.1. *For an integer $m \geq 1$ and a finite set $C \subset \mathbb{S}$, define*

$$P(C, m) = \operatorname{conv}\left(\Phi_m(t) :\, t \in C\right).$$

Then $P(C,m)$ is a polytope of dimension at most $2(m+1)$ that has $|C|$ vertices. Moreover, if the elements of C satisfy

$$(3.2) \qquad \begin{aligned} &3^i t_1 \not\equiv 3^i t_2 \mod 2\pi \quad \textit{for all } t_1, t_2 \in C \textit{ such that } t_1 \neq t_2, \\ &\textit{and all } i = 1, 2, \ldots, m-1, \end{aligned}$$

then for every pair of distinct points $t_1, t_2 \in C$ that lie on an open arc of length $\pi(1 - \frac{1}{3^m})$, the interval $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(C,m)$.

Proof. To show that $P(C,m)$ has $|C|$ vertices, we consider the projection $\mathbb{R}^{2(m+1)} \longrightarrow \mathbb{R}^4$ that forgets all but the first four coordinates. Since $\Phi_1 = U_2$, the image of $P(C,m)$ is the polytope

$$P(C,1) = \operatorname{conv}\left(U_2(t) : t \in C\right).$$

By Theorem II.1, the polytope $P(C,1)$ has $|C|$ distinct vertices: $U_2(t)$ for $t \in C$. Furthermore, the inverse image of each vertex $U_2(t)$ of $C(m,1)$ in $P(C,m)$ consists of a single vertex $\Phi_m(t)$ of $P(C,m)$. Therefore, $\Phi_m(t)$ for $t \in C$ are all the vertices of P_m without duplicates.

To prove the statement about edges, we proceed by induction on m. As $\Phi_1 = U_2$, the $m = 1$ case follows from [24] (see Theorem II.1 and II.14).

Suppose now that $m \geq 2$. Let t_1, t_2 be two distinct elements of C that lie on an open arc of length $\pi(1 - \frac{1}{3^m})$. There are two cases to consider.

Case I: t_1, t_2 lie on an open arc of length $2\pi/3$. In this case, the above projection of $\mathbb{R}^{2(m+1)}$ onto $\mathbb{R}^4$ maps $P(C,m)$ onto $P(C,1)$, and according to the base of induction, $[\Phi_1(t_1), \Phi_2(t_2)]$ is an edge of $P(C,1)$. Since the inverse image of a vertex $\Phi_1(t)$ of $P(C,1)$ in $P(C,m)$ consists of a single vertex $\Phi_m(t)$ of $P(C,m)$, we conclude that $[\Phi_m(t_1), \ \Phi_m(t_2)]$ is an edge of $P(C,m)$.

Case II: t_1, t_2 lie on an open arc of length $\pi(1 - \frac{1}{3^m})$, but not on an arc of length $2\pi/3$. (Observe that since $3t_1 \not\equiv 3t_2 \mod 2\pi$, the points t_1 and t_2 may not form an arc of length exactly $2\pi/3$.) Then $3t_1$ and $3t_2$ do not coincide and lie on an open arc of length $\pi(1 - \frac{1}{3^{m-1}})$. Consider the projection of $\mathbb{R}^{2(m+1)}$ onto $\mathbb{R}^{2m}$ that forgets the first two coordinates. The image of $P(C, m)$ under this projection is

$$P(3C, m-1), \quad \text{where} \quad 3C := \{3t \mod 2\pi : \ t \in C\} \subset \mathbb{S},$$

and since the pair $(3C, m-1)$ satisfies eq. (3.2), by the induction hypothesis, the interval

$$[\Phi_{m-1}(3t_1),\ \Phi_{m-1}(3t_2)]$$

is an edge of $P(3C, m-1)$. By eq. (3.2), the inverse image of a vertex $\Phi_{m-1}(3t)$ of $P(3C, m-1)$ in $P(C, m)$ consists of a single vertex $\Phi_m(t)$ of $P(C, m)$, and hence we infer that $[\Phi_m(t_1),\ \Phi_m(t_2)]$ is an edge of $P(C, m)$. $\square$

We are now in a position to state and prove the main result of this section. We follow the notation of Lemma III.1.

Theorem III.2. *Fix integers $m \geq 2$ and $s \geq 2$. Let $A_m \subset \mathbb{S}$ be the set of $2(3^m - 1)$ equally spaced points:*

$$A_m = \left\{ \frac{\pi(j-1)}{3^m - 1} : \quad j = 1, \ldots, 2(3^m - 1) \right\},$$

and let $A_{m,s} \subset \mathbb{S}$ be the set of $2(3^m - 1)$ clusters of s points each, chosen in such a way that for all $j = 1, \ldots, 2(3^m - 1)$, the j-th cluster lies on an arc of length 10^{-m} that contains the point $\frac{\pi(j-1)}{3^m-1}$, and the entire set $A_{m,s}$ is centrally symmetric. Then

1. *The polytope $P(A_m, m)$ is a centrally symmetric 2-neighborly polytope of dimension $2(m+1)$ that has $2(3^m - 1)$ vertices.*

2. *The polytope $P(A_{m,s}, m)$ is a centrally symmetric $2(m+1)$-dimensional polytope that has $N := 2s(3^m - 1)$ vertices and at least $N(N-s-1)/2 > (1-3^{-m})\binom{N}{2}$ edges.*

Proof. To see that $P(A_m, m)$ is centrally symmetric, note that the transformation

$$t \mapsto t + \pi \mod 2\pi$$

maps A_m onto itself and also that $\Phi_m(t+\pi) = -\Phi_m(t)$. The same argument applies to $P(A_{m,s}, m)$.

We now show that the dimension of $P(A_m, m)$ is $2(m+1)$. If not, then the points $\Phi_m(t) : t \in A_m$ are all in an affine hyperplane in $\mathbb{R}^{2(m+1)}$, and hence the $2(3^m-1)$ elements of A_m are roots of a trigonometric polynomial of the form

$$f(t) = c + \sum_{j=0}^{m} a_j \cos(3^j t) + \sum_{j=0}^{m} b_j \sin(3^j t).$$

Moreover, a_m and b_m cannot both be zero as by our assumption $f(t)$ has at least $2(3^m-1)$ roots, and so the degree of $f(t)$ is at least $3^m - 1 > 3^{m-1}$. Thus the complex polynomial $\mathcal{P}(f)$ defined by eq. (2.2) is of the form

$$\mathcal{P}(f)(z) = d_m z^{2\cdot 3^m} + d_{m-1} z^{3^m+3^{m-1}} + d_{m-2} z^{3^m+3^{m-2}} + \cdots + c z^{3^m} + \cdots + \overline{d_m}, \quad \text{where } d_m \neq 0.$$

Note that since $m > 1$, $3^m + 3^{m-1} < 2\cdot 3^m - 2$. In particular, the coefficients of $z^{2\cdot 3^m - 1}$ and $z^{2\cdot 3^m - 2}$ are both equal to 0. Therefore, the sum of all the roots (counted with multiplicities) of $\mathcal{P}(f)$ as well as the sum of their squares is 0. As $\deg \mathcal{P}(f) = 2\cdot 3^m$, the (multi)set of roots of $\mathcal{P}(f)$ consists of $\{e^{it} : t \in A_m\}$ together with two additional roots, denote them by ζ_1 and ζ_2. The complex numbers $e^{it} : t \in A_m$ form a geometric progression, and it is straightforward to check that

$$\sum_{t\in A_m} e^{it} = 0 \quad \text{and} \quad \sum_{t\in A_m} e^{2it} = 0.$$

Hence for the sum of all the roots of $\mathcal{P}(f)$ and for the sum of their squares to be zero, we must have

$$\zeta_1 + \zeta_2 = 0 \quad \text{and} \quad \zeta_1^2 + \zeta_2^2 = 0.$$

Thus $\zeta_1 = \zeta_2 = 0$, and so the constant term of $\mathcal{P}(f)$ is zero. This however contradicts the fact that the constant term of $\mathcal{P}(f)$ equals $\overline{d_m}$, where $d_m \neq 0$. Therefore, the polytope $P(A_m, m)$ is full-dimensional.

Finally, to see that $P(A_m, m)$ is 2-neighborly, observe that it follows from the definition of A_m that if $t_1, t_2 \in A_m$ are not antipodes, then they lie on a closed arc of length $\pi(1 - \frac{1}{3^m - 1})$, and hence also on an an open arc of length $\pi(1 - \frac{1}{3^m})$. In addition, since $3^m - 1$ is relatively prime to 3, we obtain that for every two distinct elements t_1, t_2 of A_m, $3^i t_1 \not\equiv 3^i t_2 \mod 2\pi$ (for $i = 1, \ldots, m-1$). Part (1) of the theorem is then immediate from Lemma III.1.

To compute the dimension of $P(A_{m,s}, m)$, note that if it is smaller than $2(m+1)$, then $P(A_{m,s}, m)$ is a subset of an affine hyperplane in $\mathbb{R}^{2(m+1)}$. As all vertices of this polytope lie on the curve Φ_m, such a hyperplane corresponds to a trigonometric polynomial of degree 3^m that has at least $N = 2s(3^m - 1) \geq 4(3^m - 1) > 2 \cdot 3^m$ roots. This is however impossible, as no nonzero trigonometric polynomial of degree D has more than $2D$ roots.

To finish the proof of Part (2), note that since each cluster of $A_{m,s}$ lies on an open arc of length

$$10^{-m} < \frac{\pi}{2}\left(\frac{1}{3^m - 1} - \frac{1}{3^m}\right)$$

that contains the corresponding element of A_m, and since multiplication by 3^i modulo 2π maps A_m bijectively onto itself, it follows that

- $3^i t_1 \not\equiv 3^i t_2 \mod 2\pi$ (for $i = 1, \ldots, m-1$) holds for all distinct $t_1, t_2 \in A_{m,s}$. (Indeed, for t_1, t_2 from the same cluster, the points $3^i t_1$ and $3^i t_2$ of $\mathbb{S}$ do not coincide as $3^m/10^m < 2\pi$, and for t_1, t_2 from different clusters, $3^i t_1$ and $3^i t_2$ do not coincide as the distance between them along $\mathbb{S}$ is at least $\frac{\pi}{3^m-1} - \frac{2\cdot 3^m}{10^m} > 0$.)

- Every two points $t_1, t_2 \in A_{m,s}$ lie on an open arc of length $\pi(1-\frac{1}{3^m})$ as long as they do not belong to a pair of opposite clusters.

Thus Lemma III.1 applies and shows that the interval $[\Phi_m(t_1), \Phi_m(t_2)]$ is an edge of $P(A_{m,s}, m)$ for all $t_1, t_2 \in A_{m,s}$ that are not from opposite clusters. In other words, each vertex of $P(A_{m,s}, m)$ is incident with at least $N - s - 1$ edges. This yields the promised bound on the number of edges of $P(A_{m,s}, m)$ and completes the proof of Part (2). □

3.3 Applications to strict antipodality problems

In this section we observe that an appropriately chosen half of the vertex set of any cs $2k$-neighborly d-dimensional polytope has a large number of pairwise strictly antipodal $(k-1)$-simplices. The results of the previous section then imply new lower bounds on questions related to strict antipodality. Specifically, in the following theorem we improve both Talata's and Makai–Martini's bounds.

Theorem III.3.

1. *For every $m \geq 1$, there exists a set $X_m \subset \mathbb{R}^{2(m+1)}$ of size $3^m - 1$ that affinely spans $\mathbb{R}^{2(m+1)}$ and such that each pair of points of X_m is strictly antipodal. Thus, $A'(d) \geq 3^{\lfloor d/2-1 \rfloor} - 1$ for all $d \geq 4$.*

2. *For all positive integers m and s, there exists a set $Y_{m,s} \subset \mathbb{R}^{2(m+1)}$ of size $n := s(3^m - 1)$ that has at least*

$$\left(1 - \frac{1}{3^m - 1}\right) \cdot \frac{n^2}{2}$$

pairs of antipodal points. Thus, $A'_d(n) \geq \left(1 - \frac{1}{3^{\lfloor d/2-1 \rfloor}-1}\right) \cdot \frac{n^2}{2} - O(n)$ *for all* $d \geq 4$ *and* n.

One can generalize the notion of strictly antipodal points in the following way: for a set $X \subset \mathbb{R}^d$ that affinely spans $\mathbb{R}^d$, we say that two simplices, σ and σ', spanned by the points of X are strictly antipodal if there exist two distinct parallel hyperplanes H and H' such that X lies in the slab defined by H and H', $H \cap \mathrm{conv}(X) = \sigma$, and $H' \cap \mathrm{conv}(X) = \sigma'$. Makai and Martini [19] asked about the maximum number of pairwise strictly antipodal $(k-1)$-simplices in $\mathbb{R}^d$. The following result gives a lower bound to their question.

Theorem III.4. *There exists a set of* $\lfloor (d/2) \cdot 2^{cd/k} \rfloor$ *points in* $\mathbb{R}^d$ *with the property that every two disjoint* k*-subsets of* X *form the vertex sets of strictly antipodal* $(k-1)$*-simplices. In particular, there exists a set of* $\lfloor \frac{d}{2k} \cdot 2^{cd/k} \rfloor$ *pairwise strictly antipodal* $(k-1)$*-simplices in* $\mathbb{R}^d$*. Here* $c > 0$ *is an absolute constant.*

The key to our proofs are the results of Section 2 and paper [18] along with the following observation.

Lemma III.5. *Let* $P \subset \mathbb{R}^d$ *be a full-dimensional cs polytope on the vertex set* $V = X \sqcup (-X)$*. If* U_1*,* U_2 *are subsets of* X *such that* $U_1 \cup (-U_2)$ *is the vertex set of a* $(|U_1| + |U_2| - 1)$*-face of* P*, then* $\sigma_1 := \mathrm{conv}(U_1)$ *and* $\sigma_2 := \mathrm{conv}(U_2)$ *are strictly antipodal simplices spanned by points of* X*. In particular, if* P *is* 2*-neighborly, then every pair of vertices of* X *is strictly antipodal, and, more generally, if* P *is* $2k$*-neighborly, then every two disjoint* k*-subsets of* X *form a pair of strictly antipodal* $(k-1)$*-simplices.*

Proof. Since $\tau_1 := \mathrm{conv}(U_1 \cup (-U_2))$ is a face of P, there exists a supporting hyperplane H_1 of P that defines τ_1: specifically, P is contained in one of the closed half-spaces bounded by H_1 and $P \cap H_1 = \tau_1$. As P is centrally symmetric, the hyperplane $H_2 := -H_1 = \{x \in \mathbb{R}^d : -x \in H_1\}$ is a supporting hyperplane of P that defines the opposite face, $\tau_2 := \operatorname{conv}((-U_1) \cup U_2)$. Thus P, and hence also X, is contained in the slab between H_1 and H_2. Moreover, since U_1, U_2 are subsets of X, it follows that $-U_1$ and $-U_2$ are contained in $-X$, and hence disjoint from X. Therefore,

$$H_i \cap \operatorname{conv}(X) = H_i \cap P \cap \operatorname{conv}(X) = \tau_i \cap \operatorname{conv}(X) = \operatorname{conv}(U_i) = \sigma_i \quad \text{for } i = 1, 2.$$

The result follows. $\square$

Proof of Theorem III.3: Consider the sets A_m and $A_{m,s}$ of Theorem III.2. Define

$$A_m^+ = \{t \in A_m : 0 \le t < \pi\},$$

and define $A_{m,s}^+$ by taking the union of those clusters of $A_{m,s}$ that lie on small arcs around the points of A_m^+. In particular, $|A_m^+| = 3^m - 1$ and $|A_{m,s}^+| = s(3^m - 1)$. Let

$$X_m := \left\{\Phi_m(t) : t \in A_m^+\right\} \subset \mathbb{R}^{2(m+1)} \quad \text{and} \quad Y_{m,s} := \left\{\Phi_m(t) : t \in A_{m,s}^+\right\} \subset \mathbb{R}^{2(m+1)}.$$

Theorem III.2 and Lemma III.5 imply that each pair of points of X_m is strictly antipodal, and each pair of points of $Y_{m,s}$ that are not from the same cluster is strictly antipodal. The claim follows. $\square$

Proof of Theorem III.4: It was proved in [18, 23] (by using a probabilistic construction) that if k, d, and N satisfy

$$k \le \frac{cd}{1 + \log \frac{N}{d}},$$

where $c > 0$ is some absolute constant, then there exists a d-dimensional cs polytope on $2N$ vertices that is $2k$-neighborly. Solving this inequality for N, implies existence of a d-dimensional cs polytope on $\lfloor d \cdot 2^{cd/k} \rfloor$ vertices that is $2k$-neighborly. This together with Lemma III.5 yields the result. $\square$

3.4 Cs polytopes with many faces of given dimension

In this section, we prove the following theorem.

Theorem III.6. *Fix an integer $k \geq 1$. For a non-negative integer m, consider the map $\Psi_{k,m} : \mathbb{S} \longrightarrow \mathbb{R}^{6k(m+1)}$ defined by*

$$\Psi_{k,m}(t) = \Big(U_{3k}(t),\ U_{3k}(5t),\ \ldots,\ U_{3k}\left(5^m t\right)\Big).$$

For a positive even integer n, let $A_{m,n} \subset \mathbb{S}$ be the set of $n5^m$ equally spaced points,

$$A_{m,n} = \left\{ \frac{2\pi j}{n5^m} : \quad j = 0, \ldots, n5^m - 1 \right\},$$

and let

$$P = P_{k,m,n} = \operatorname{conv}\Big(\Psi_{k,m}(t) : \quad t \in A_{m,n}\Big).$$

Then

1. *The polytope $P \subset \mathbb{R}^{6k(m+1)}$ is a centrally symmetric polytope with $n5^m$ distinct vertices:*

   $$\Psi_{k,m}(t) \quad \text{for } t \in A_{m,n}$$

   and of dimension $d \leq 6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$; moreover, if $n > 2(6k-1)$, then the dimension of P is equal to $6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$.

2. *Let $t_1, \ldots, t_k$ be points chosen independently at random from the uniform distribution in $A_{m,n}$ (in particular, some of t_i may coincide). Then the probability that*

   $$\operatorname{conv}\Big(\Psi_{k,m}(t_1), \ldots, \Psi_{k,m}(t_k)\Big)$$

   is not a face of P does not exceed

   $$\left(1 - 5^{-k+1}\right)^m.$$

We obtain the following corollary.

Corollary III.7. *Let $P_{k,m,n}$ be the polytope of Theorem III.6 with $N = n5^m$ vertices and dimension $d \leq 6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$. Then*

$$f_{k-1}\left(P_{k,m,n}\right) \ \geq \ \binom{N}{k} - \left(1 - 5^{-k+1}\right)^m \frac{N^k}{k!}.$$

The construction of Theorem III.6 produces a family of centrally symmetric polytopes of an increasing dimension d and with an arbitrarily large number of vertices such that for any fixed $k \geq 1$, the probability $p_{d,k}$ that k randomly chosen vertices of the polytope do not span a face decreases exponentially in d. However, it does not start doing so very quickly: for instance, to make $p_{d,k} < 1/2$ we need to choose d as high as $2^{\Omega(k)}$.

Proof of Theorem III.6. We observe that the transformation

$$t \longmapsto t + \pi \mod 2\pi$$

maps the set $A_{m,n}$ onto itself and that

$$\Psi_{k,m}(t+\pi) = -\Psi_{k,m}(t) \quad \text{for all} \quad t \in \mathbb{S}.$$

Hence P is centrally symmetric. Consider the projection $\mathbb{R}^{6k(m+1)} \longrightarrow \mathbb{R}^{6k}$ that forgets all but the first $6k$ coordinates. Then the image of $P_{k,m,n}$ is the polytope

$$Q_{k,m,n} = \operatorname{conv}\Big(U_{3k}(t): \quad t \in A_{m,n}\Big). \tag{3.3}$$

By Theorem II.1, the polytope $Q_{k,m,n}$ has $n5^m$ distinct vertices: $U_{3k}(t)$ for $t \in A_{m,n}$. Furthermore, the inverse image of each vertex $U_{3k}(t)$ of $Q_{k,m,n}$ in $P_{k,m,n}$ consists of a single vertex $\Psi_{k,m}(t)$ of $P_{k,m,n}$. Therefore, $\Psi_{k,m,n}(t)$ for $t \in A_{m,n}$ are all the vertices of $P_{k,m,n}$ without duplicates.

To estimate the dimension of $P = P_{k,m,n}$, we observe that for all $t \in \mathbb{S}$, the fifth coordinate of $U_{3k}(t)$ coincides with the first coordinate of $U_{3k}(5t)$ while the sixth coordinate of $U_{3k}(t)$ coincides with the second coordinate of $U_{3k}(5t)$, etc. Taking into account all coincidences of coordinates, the polytope P lies in a subspace of dimension $6k(m+1)-2m\lfloor(3k+2)/5\rfloor$, and hence $\dim P \leq 6k(m+1)-2m\lfloor(3k+2)/5\rfloor$. Moreover, if $n > 2(6k-1)$, then an argument identical to the one used in the proof of Theorem III.2 (by counting roots of trigonometric polynomials) shows that $\dim P = 6k(m+1) - 2m\lfloor(3k+2)/5\rfloor$.

We prove Part (2) by induction on m. The statement trivially holds for $m = 0$. Let us assume that $m \geq 1$ and consider the map $\phi : A_{m,n} \longrightarrow A_{m-1,n}$ defined by

$$\phi(t) = 5t \mod 2\pi.$$

Then

$$\phi\left(A_{m,n}\right) = A_{m-1,n}$$

and for every $t \in A_{m-1,n}$, the inverse image $\phi^{-1}(t)$ of t consists of 5 equally spaced points from $A_{m,n}$. We note that if t is a random point uniformly distributed in $A_{m,n}$, then $\phi(t)$ is uniformly distributed in $A_{m-1,n}$. The proof of the theorem will follow from the following two claims.

Claim I. Let $t_1, \ldots, t_k \in A_{m,n}$ be arbitrary, not necessarily distinct, points. If

$$\operatorname{conv}\Big(\Psi_{k,m-1}\left(5t_i\right), \quad i = 1, \ldots, k\Big) \tag{3.4}$$

is a face of $P_{k,m-1,n}$ then

$$\operatorname{conv}\Big(\Psi_{k,m}\left(t_i\right), \quad i = 1, \ldots, k\Big) \tag{3.5}$$

is a face of $P_{k,m,n}$.

Claim II. Let $s_1, \ldots, s_k \in A_{m-1,n}$ be arbitrary, not necessarily distinct, points. Then the conditional probability that

$$\operatorname{conv}\Big(\Psi_{k,m}(t_i): \quad i = 1, \ldots, k\Big)$$

is not a face of $P_{k,m,n}$ given that

$$\phi\left(t_i\right) = s_i \quad \text{for} \quad i = 1, \ldots, k$$

does not exceed $1 - 5^{-k+1}$.

To prove Claim I, we consider the projection $\mathbb{R}^{6k(m+1)} \longrightarrow \mathbb{R}^{6km}$ that forgets the first $6k$ coordinates. The image of $P_{k,m,n}$ under this projection is $P_{k,m-1,n}$ and if (3.4) is a face of $P_{k,m-1,n}$ then

$$\begin{aligned} \operatorname{conv}\Big(\Psi_{k,m}\left(x_{ij}\right): \quad \phi\left(x_{ij}\right) = \phi\left(t_i\right) \quad &\text{for} \quad i = 1, \ldots, k \\ &\text{and} \quad j = 1, 2, 3, 4, 5\Big) \end{aligned} \tag{3.6}$$

is a face of $P_{k,m,n}$ as it is the inverse image of (3.4) under this projection. The face (3.6) is the convex hull of at most $5k$ distinct points and no two points x_{ij} in (3.6) are antipodal. Since a set of up to $6k$ distinct points $U_{3k}\left(x_{ij}\right)$ no two of which are antipodal is linearly independent, the face (3.6) is a simplex. Therefore, the set (3.5) is a face of (3.6), and hence also a face of $P_{k,m,n}$. Claim I now follows.

To prove Claim II, we fix a sequence $s_1, \ldots, s_k \in A_{m-1,n}$ of not necessarily distinct points. Then there are exactly 5^k sequences $t_1, \ldots, t_k \in A_{m,n}$ of not necessarily distinct points such that $\phi\left(t_i\right) = s_i$ for $i = 1, \ldots, k$. Choose an arbitrary t_1 subject to the condition $\phi\left(t_1\right) = s_1$. Let $\Gamma \subset \mathbb{S}$ be a closed arc of length $2\pi/5$ centered at t_1. Then for $i = 2, \ldots, k$ there is at least one $t_i \in \Gamma$ such that $\phi(t_i) = s_i$. By Theorem II.1, for such a choice of $t_2, \ldots, t_k$, the set

$$\operatorname{conv}\Big(U_{3k}\left(t_i\right): \quad i = 1, \ldots, k\Big) \tag{3.7}$$

is a face of the polytope $Q_{k,m,n}$ defined by (3.3). Considering the projection

$$P_{k,m,n} \longrightarrow Q_{k,m,n}$$

as above, we conclude that (3.5) is a face of $P_{k,m,n}$ as it is the inverse image of (3.7).

Hence the conditional probability that (3.5) is not a face is at most

$$\frac{5^{k-1}-1}{5^{k-1}} = 1 - 5^{-k+1}.$$

□

3.5 Constructing k-neighborly cs polytopes

The goal of this section is to present a deterministic construction of a cs k-neighborly d-polytope with at least $2^{c_k d}$ for $c_k = 3/20k^2 2^k$ vertices. This requires the following facts and definitions.

A family $\mathcal{F}$ of subsets of $[m] := \{1, 2, \ldots, m\}$ is called *k-independent* if for every k distinct subsets $I_1, \ldots, I_k$ of $\mathcal{F}$ all 2^k intersections

$$\bigcap_{j=1}^{k} J_j, \quad \text{where } J_j = I_j \text{ or } J_j = I_j^c := [m] \setminus I_j, \text{ are non-empty.}$$

The crucial component of our construction is a deterministic construction of k-independent families of size larger than $2^{m/5(k-1)2^k}$ given in [12].

For a subset I of $[m]$ and a given number $a \in \{0, 1\}$, we (recursively) define a sequence $x(I, a) = (x_0, x_1, \ldots, x_m)$ of **zeros and ones** according to the following rule:

$$x_0 = x_0(I, a) := a \quad \text{and}$$

$$x_n = x_n(I, a) \equiv \begin{cases} \sum_{j=0}^{n-1} x_j & \text{if } n \notin I \\ 1 + \sum_{j=0}^{n-1} x_j & \text{if } n \in I \end{cases} \mod 2 \quad \text{for } n \geq 1. \tag{3.8}$$

We also set

$$t(I,a) := \pi \sum_{j=0}^{m} \frac{x_j}{3^j} \in \mathbb{S}. \tag{3.9}$$

A few observations are in order. First, it follows from (3.8) that $x(I,a) \neq x(J,a)$ if $I \neq J$, and that $x(I,a)$ and $x(I^c, 1-a)$ agree in all but the 0-th component, where they disagree. Hence

$$t(I,a) = t(I^c, 1-a) + \pi \mod 2\pi.$$

Second, since $\sum_{j=1}^{\infty} \frac{1}{3^j} = \frac{1}{2}$ and since all components of $x(I,a)$ are zeros and ones, we infer from eq. (3.9) that for all $1 \leq n \leq m$ and all $0 \leq \epsilon \leq 1/3^{m+1}$, the point $3^n \cdot (t(I,a) + \pi\epsilon)$ of $\mathbb{S}$ either lies on the arc $[0, \pi/2)$ or on the arc $[\pi, 3\pi/2)$ depending on the parity of

$$\sum_{j=0}^{n} 3^{n-j} x_j(I,a) \equiv \sum_{j=0}^{n} x_j(I,a) \mod 2.$$

As, by (3.8), $\sum_{j=0}^{n} x_j(I,a)$ is even if $n \notin I$ and is odd if $n \in I$, we obtain that

$$3^n \cdot (t(I,a) + \pi\epsilon) \in [\pi, 3\pi/2) \mod 2\pi \quad \text{for all } n \in I \text{ and } a \in \{0,1\}. \tag{3.10}$$

The relevance of k-independent sets to cs k-neighborly polytopes is explained by the following lemma along with Theorem II.1.

Lemma III.8. *Let $\mathcal{F}$ be a k-independent family of subsets of $[m]$, let $\epsilon_I \in [0, 1/3^{m+1}]$ for $I \in \mathcal{F}$, and let*

$$V^{\epsilon}(\mathcal{F}) = \bigcup_{I \in \mathcal{F}} \{t(I,0) + \pi\epsilon_I,\ t(I^c,1) + \pi\epsilon_I\} \subset \mathbb{S}.$$

Then for every k distinct points $t_1, \ldots, t_k$ of $V^{\epsilon}(\mathcal{F})$ no two of which are antipodes, there exists an integer $n \in [m]$ such that the subset $\{3^n t_1, \ldots, 3^n t_k\}$ of $\mathbb{S}$ is entirely contained in $[\pi, 3\pi/2)$.

Proof. As $t_1, \dots, t_k$ are elements of $V^\epsilon(\mathcal{F})$, by relabeling them if necessary, we can assume that

$$t_j = \begin{cases} t(I_j, 0) + \pi\epsilon_{I_j} & \text{if } 1 \leq j \leq q \\ t(I_j^c, 1) + \pi\epsilon_{I_j} & \text{if } q < j \leq k \end{cases}$$

for some $0 \leq q \leq k$ and $I_1, \dots, I_k \in \mathcal{F}$. Moreover, the sets $I_1, \dots, I_k$ are distinct, since $t_1, \dots, t_k$ are distinct and no two of them are antipodes. As $\mathcal{F}$ is a k-independent family, the intersection $(\cap_{j=1}^q I_j) \cap (\cap_{j=q+1}^k I_j^c)$ is non-empty. The result follows, since by eq. (3.10), for any element n of this intersection, $\{3^n t_1, \dots, 3^n t_k\} \subset [\pi, 3\pi/2)$. $\square$

For $I \in \mathcal{F}$, define $\epsilon_I = \epsilon_{I^c} := \sum_{i \in I} 10^{-i-m}$. Then

$$\begin{aligned} &3^n t_1 \not\equiv 3^n t_2 \mod 2\pi \quad \text{for all } t_1, t_2 \in V^\epsilon(\mathcal{F}) \text{ such that } t_1 \neq t_2, \\ &\text{and all } 1 \leq n \leq m. \end{aligned} \tag{3.11}$$

Indeed, if t_1 and t_2 are antipodes, then so are $3^n t_1$ and $3^n t_2$, and (3.11) follows. If t_1 and t_2 are not antipodes, then there exist two distinct and not complementary subsets I, J of $[m]$ such that $t_1 = t(I, a) + \pi\epsilon_I$ and $t_2 = t(J, b) + \pi\epsilon_J$ for some $a, b \in \{0, 1\}$. Hence, by definition of ϵ_I and ϵ_J,

$$\pi/10^{2m} < 3^n \cdot \pi|\epsilon_I - \epsilon_J| < \pi(3/10)^m,$$

while by definition of $t(I, a)$ and $t(J, b)$, the distance between the points $3^n \cdot t(I, a)$ and $3^n \cdot t(J, b)$ of $\mathbb{S}$ along $\mathbb{S}$ is either 0 or at least $\pi/3^m$. In either case, it follows that the distance between $3^n\,(t(I, a) + \pi\epsilon_I)$ and $3^n\,(t(J, b) + \pi\epsilon_J)$ is positive, yielding eq. (3.11).

We are now in a position to present our construction of k-neighborly cs polytopes. The construction is similar to that in Theorem III.2, except that it is based on the set $V^\epsilon(\mathcal{F}) \subset \mathbb{S}$, where $\mathcal{F}$ is a k-independent family of subsets of $[m]$, instead of $A_m \subset \mathbb{S}$, and on a modification of Φ_m to a curve that involves U_k instead of U_2.

Let $U_k : \mathbb{S} \longrightarrow \mathbb{R}^{2k}$ be the symmetric moment curve. In analogy with the curve Φ_m (see eq. (3.1)), for integers $m \geq 0$ and $k \geq 3$, define the curve $\Psi_{k,m} : \mathbb{S} \longrightarrow \mathbb{R}^{2k(m+1)}$ by

$$\Psi_{k,m}(t) := \left(U_k(t), U_k(3t), U_k(3^2 t), \ldots, U_k(3^m t)\right). \tag{3.12}$$

Thus, $\Psi_{k,0} = U_k$ and $\Psi_{k,m}(t+\pi) = -\Psi_{k,m}(t)$.

The following theorem is the main result of this section. We use the same notation as in Lemma III.8. Also, mimicking the notation of Lemma III.1, for a subset C of $\mathbb{S}$, we denote by $P_k(C, m)$ the polytope $\text{conv}(\Psi_{k,m}(t) : \ t \in C)$.

Theorem III.9. *Let $m \geq 1$ and $k \geq 3$ be fixed integers, let $\mathcal{F}$ be a k-independent family of subsets of $[m]$, and let $\epsilon_I = \sum_{i \in I} 10^{-i-m}$ for $I \in \mathcal{F}$. Then the polytope*

$$P_k(V^\epsilon(\mathcal{F}), m) := \text{conv}\left(\Psi_{k,m}(t) : \quad t \in V^\epsilon(\mathcal{F})\right)$$

is a cs k-neighborly polytope of dimension at most $2k(m+1) - 2m\lfloor (k+1)/3 \rfloor$ that has $2|\mathcal{F}|$ vertices.

Remark III.10. For a fixed k and an arbitrarily large m, a deterministic algorithm from [12] produces a k-independent family $\mathcal{F}$ of subsets of $[m]$ such that $|\mathcal{F}| > 2^{m/5(k-1)2^k}$. Combining this with Theorem III.9 results in a cs neighborly polytope of dimension $d \approx \frac{4}{3}km$ and more than $2^{3d/20k^2 2^k}$ vertices. Of a special interest is the case of $k = 3$: the algorithm from [12] provides a 3-independent family of size $\approx 2^{0.092m}$, which together with Theorem III.9 yields a deterministic construction of a cs 3-neighborly polytope of dimension $\leq d$ and with about $2^{0.023d}$ vertices.

Proof of Theorem III.9: As in the proof of Theorem III.2, the polytope $P_k(V^\epsilon(\mathcal{F}), m)$ is centrally symmetric since $V^\epsilon(\mathcal{F})$ is a cs subset of $\mathbb{S}$ and since $\Psi_{k,m}(t+\pi) = -\Psi_{k,m}(t)$.

Also as in the proof of Theorem III.2, the fact that $P_k(V^\epsilon(\mathcal{F}), m)$ has $2|\mathcal{F}|$ vertices follows by considering the projection $\mathbb{R}^{2k(m+1)} \longrightarrow \mathbb{R}^{2k}$ that forgets all but the first $2k$ coordinates. Indeed, the image of $P_k(V^\epsilon(\mathcal{F}), m)$ under this projection is the polytope

$$P_k(V^\epsilon(\mathcal{F}), 0) = \operatorname{conv}\left(U_k(t) : t \in V^\epsilon(\mathcal{F})\right),$$

and this latter polytope has $2|\mathcal{F}|$ vertices (by Theorem II.1).

To prove k-neighborliness of $P_k(V^\epsilon(\mathcal{F}), m)$, let $t_1, \ldots, t_k \in V^\epsilon(\mathcal{F})$ be k distinct points no two of which are antipodes. By Lemma III.8, there exists an integer $1 \leq n \leq m$ such that the points $3^n t_1, \ldots, 3^n t_k$ of $\mathbb{S}$ are all contained in the arc $[\pi, 3\pi/2)$. Consider the projection $\mathbb{R}^{2k(m+1)} \longrightarrow \mathbb{R}^{2k(m+1-n)}$ that forgets the first $2kn$ coordinates followed by the projection $\mathbb{R}^{2k(m+1-n)} \longrightarrow \mathbb{R}^{2k}$ that forgets all but the first $2k$ coordinates. The image of $P_k(V^\epsilon(\mathcal{F}), m)$ under this composite projection is

$$P_k(3^n V^\epsilon(\mathcal{F}), 0) = \operatorname{conv}(U_k(3^n t) : \quad t \in V^\epsilon(\mathcal{F})),$$

and, since $\{3^n t_1, \ldots, 3^n t_k\} \subset [\pi, 3\pi/2)$, Theorem II.1 implies that the set $\{U_k(3^n t_i) : i = 1, \ldots, k\}$ is the vertex set of a $(k-1)$-face of this latter polytope. As, by eq. (3.11), the inverse image of a vertex $U_k(3^n t)$ of $P_k(3^n V^\epsilon(\mathcal{F}), 0)$ in $P_k(V^\epsilon(\mathcal{F}), m)$ consists of a single vertex $\Psi_{k,m}(t)$ of $P_k(V^\epsilon(\mathcal{F}), m)$, we obtain that $\{\Psi_{k,m}(t_i) : i = 1, \ldots, k\}$ is the vertex set of a $(k-1)$-face of $P_k(V^\epsilon(\mathcal{F}), m)$. This completes the proof of k-neighborliness of $P_k(V^\epsilon(\mathcal{F}), m)$.

To bound the dimension of $P_k(V^\epsilon(\mathcal{F}), m)$, observe that the third coordinate of $U_k(t)$ coincides with the first coordinate of $U_k(3t)$ while the fourth coordinate of $U_k(t)$ coincides with the second coordinate of $U_k(3t)$, etc. Thus $P_k(V^\epsilon(\mathcal{F}), m)$ is in a subspace of $\mathbb{R}^{2k(m+1)}$, and to bound the dimension of this subspace we must account for all repeated coordinates. This can be done exactly as in [6, Lemma 2.3]. $\square$

Fix $s \geq 2$, and let $V^\epsilon(\mathcal{F}, s)$ be a centrally symmetric subset of $\mathbb{S}$ obtained by replacing each point $t \in V^\epsilon(\mathcal{F})$ (in Theorem III.9) with a cluster of s points that all lie on a sufficiently small open arc containing t. Then the proof of Theorem III.9 implies that the polytope $P_k(V^\epsilon(\mathcal{F}, s), m)$ is a cs polytope with $N := 2s|\mathcal{F}|$ vertices, of dimension at most $2k(m+1) - 2m\lfloor (k+1)/3 \rfloor$, and such that every k vertices of this polytope no two of which are from opposite clusters form the vertex set of a $(k-1)$-face. Choose a k-element set from the union of these $2|\mathcal{F}|$ clusters (of s points each) at random from the uniform distribution. Then the probability that this set has no two points from opposite clusters is at least

$$\prod_{i=0}^{k-1} \frac{(2|\mathcal{F}| - i)s - i}{2|\mathcal{F}|s - i} \geq \prod_{i=0}^{k-1} \left(1 - \frac{i}{|\mathcal{F}|}\right) \geq 1 - \frac{k^2}{|\mathcal{F}|}.$$

Thus, the resulting polytope has at least

$$\left(1 - \frac{k^2}{|\mathcal{F}|}\right) \binom{N}{k}$$

$(k-1)$-faces. Combining this estimate with Remark III.10, we obtain

Corollary III.11. *For a fixed k and arbitrarily large N and d, there exists a cs d-dimensional polytope with N vertices and at least*

$$\left(1 - k^2 \left(2^{-3/20k^2 2^k}\right)^d\right) \binom{N}{k}$$

$(k-1)$-faces.

This corollary improves Corollary III.7 asserting existence of cs d-polytopes with N vertices and at least $\left(1 - (\delta_k)^d\right)\binom{N}{k}$ faces of dimension $k-1$, where $\delta_k \approx (1 - 5^{-k+1})^{5/(24k+4)}$.

APPENDICES

This appendix illustrates graphs and equations of $F(t)$. See Chapter 2.7.

k:=5

M[p_]:=Table[p[[j]]^(2(i − 1)), {i, 1, k}, {j, 1, k}]

M[{1, 2, 3, 4, 5}]//MatrixForm

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 9 & 16 & 25 \\ 1 & 16 & 81 & 256 & 625 \\ 1 & 64 & 729 & 4096 & 15625 \\ 1 & 256 & 6561 & 65536 & 390625 \end{pmatrix}$$

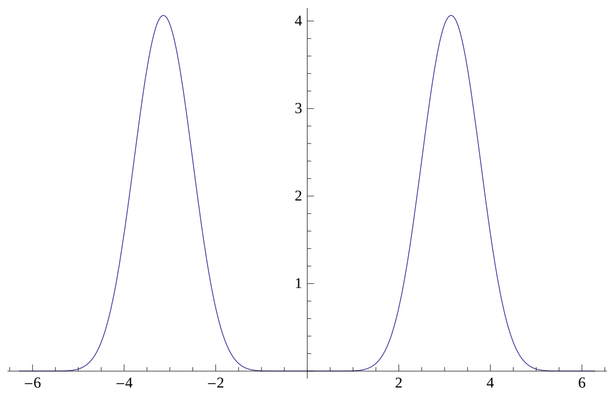
B[p_, t_]:=Table[Cos[p[[i]] ∗ t], {i, 1, 5}]

F[p_, t_]:=1 + B[p, t].Inverse[M[p]].Table[If[i == 1, −1, 0], {i, 1, k}]

F[{1, 2, 3, 4, 5}, t]//TraditionalForm

$$-\frac{5\cos(t)}{3} + \frac{20}{21}\cos(2t) - \frac{5}{14}\cos(3t) + \frac{5}{63}\cos(4t) - \frac{1}{126}\cos(5t) + 1$$

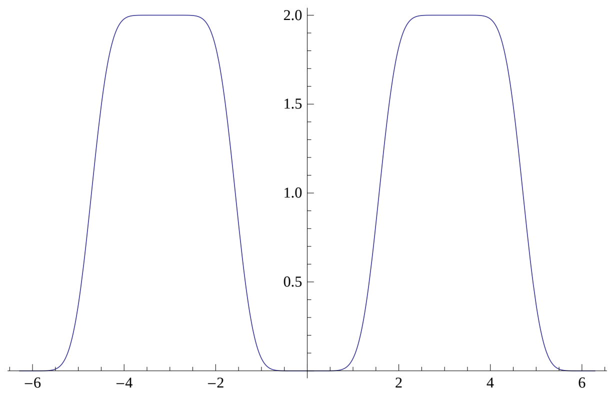
Plot[F[{1, 2, 3, 4, 5}, t]==0, {t, −2 ∗ Pi, 2 ∗ Pi}]



F[{1, 3, 5, 7, 9}, t]//TraditionalForm

$$-\frac{19845\cos(t)}{16384} + \frac{2205\cos(3t)}{8192} - \frac{567\cos(5t)}{8192} + \frac{405\cos(7t)}{32768} - \frac{35\cos(9t)}{32768} + 1$$

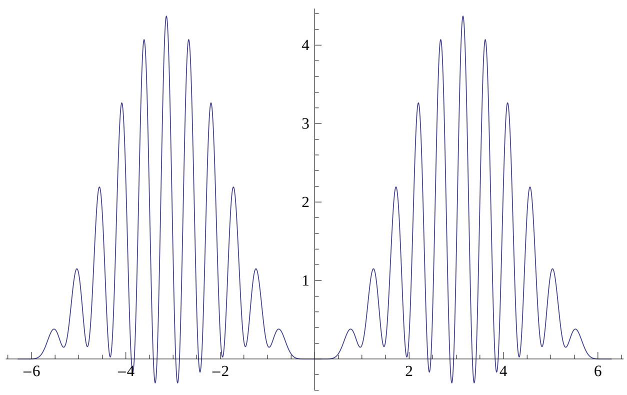
Plot[F[{1, 3, 5, 7, 9}, t]==0, {t, −2 ∗ Pi, 2 ∗ Pi}]



F[{1, 12, 13, 14, 15}, t]//TraditionalForm

$$-\frac{45\cos(t)}{44}+\frac{49}{99}\cos(12t)-\cos(13t)+\frac{20}{29}\cos(14t)-\frac{169\cos(15t)}{1044}+1$$

Plot[F[{1, 12, 13, 14, 15}, t]==0, {t, −2 ∗ Pi, 2 ∗ Pi}]



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